Chapter 4
Brouwer in Higher Dimensions

THE BROUWER FIXED-POINT THEOREM IN ALL FINITE DIMENSIONS

Overview. Having discussed the Brouwer Fixed-Point Theorem (Chap. 1) and proved it for triangles (Chap. 2), we're ready to prove it in every dimension for closed balls and even for compact, convex sets. Our proof will be quite different from that of Chap. 2, with the combinatorics of Sperner's Lemma replaced by methods of analysis.

Prerequisites. Undergraduate-level real analysis, especially calculus of functions of several variables. Some metric-space theory may be helpful, but is not required; all the action takes place in $\mathbb{R}^N$.

4.1 Fixed Points and Retractions

To say that a metric space (S,d) has the "fixed-point property" means that every continuous mapping of the space into itself has a fixed point. Thus the Brouwer Fixed-Point Theorem can be restated:

Theorem 4.1. *For every positive integer N, the closed unit ball of* $\mathbb{R}^N$ *has the fixed-point property.*

Our proof of Brouwer's theorem will involve reduction to an equivalent result about an important class of mappings called *retractions.* Suppose S is a metric space and A is a subset of S. To say that a continuous mapping $P\colon S \to A$ is a *retraction* of S onto A means that $P(S) = A$ and the restriction of P to A is the identity map on A. When this happens we'll call A a "retract" of S.

Exercise 4.1. A continuous mapping P is a retraction onto its image if and only if $P \circ P = P$.

Perhaps the most familiar example of a retraction is a linear projection taking $\mathbb{R}^N$ onto a subspace. Here are two such examples, where $S = \mathbb{R}^2$, A is the horizontal axis, and $x = (\xi_1, \xi_2)$ is a typical vector in $\mathbb{R}^2$.

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(a) Let $P(x) = (\xi_1, 0)$. Here P is the *orthogonal* projection of $\mathbb{R}^2$ onto the horizontal axis.
(b) Let $P(x) = (\xi_1 + \xi_2, 0)$. Now P is a 45° projection onto the horizontal axis.

Here's an example more relevant to our immediate purposes. Consider a closed annulus in $\mathbb{R}^2$ centered at the origin, having outer radius 1 and some positive inner radius. For x in this annulus let $P(x) = x/|x|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$. Then P is a continuous map taking the annulus onto its outer boundary, the unit circle, upon which its restriction is the identity map. Thus the unit circle is a retract of the annulus. This example is of interest to us because no such mapping exists for the unit disc:

The unit circle *not* a retract of the closed unit disc.

This follows immediately from the $N = 2$ version of the Brouwer Fixed-Point Theorem (Chap. 2). Indeed, if there *were* a retraction P taking the closed unit disc onto the unit circle, then $Q = -P$ would be a continuous mapping of the disc into itself (more precisely: onto the unit circle), that has no fixed point.

This argument for the disc works just as well for the closed unit ball of $\mathbb{R}^N$ so: *The Brouwer Fixed-Point Theorem for dimension N will show that no closed ball in $\mathbb{R}^N$ can be retracted onto its boundary.* It is the *converse* of this result that will concern us for the rest of this chapter. Our strategy will be to prove, independent of Brouwer's Theorem:

Theorem 4.2 (The No-Retraction Theorem)**.** *For each positive integer N: There is no retraction taking the closed unit ball of $\mathbb{R}^N$ onto its boundary.*

We'll show in the next section that the No-Retraction Theorem implies the Brouwer Fixed-Point Theorem, after which we'll give our "Brouwer-independent" proof of the No-Retraction Theorem.

4.2 "No-Retraction" ⇒ "Brouwer"

We've already noted (Chap. 1, Sect. 1.6) that for $N = 1$ Brouwer's Theorem follows from the Intermediate Value Theorem of elementary calculus, so now we'll work in $\mathbb{R}^N$ with $N > 1$, employing the notation $|\cdot|$ for the Euclidean norm in that space.

Suppose the closed unit ball B of $\mathbb{R}^N$ *does not* have the fixed-point property, i.e., that there is a continuous map $f\colon B \to B$ that has no fixed point. We'll show that f can be used to construct a retraction of B onto its boundary, thus establishing (the contrapositive equivalent of) the result we want to prove.

To visualize this retraction, note that since we're assuming f fixes no point of B we can draw, for each x in B, the directed half-line that starts at $f(x)$ and passes through x. Let $P(x)$ be the point at which this line intersects ∂B, noting that $P(x) = x$ if $x \in \partial B$. Thus P will be the retraction we seek—once we prove its continuity. It seems intuitively clear from Fig. 4.1 that P should be continuous. To prove this without recourse to pictures we need to represent P analytically:

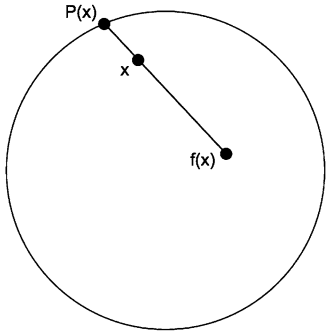

Fig. 4.1 The retraction $P : B \to \partial B$

$$P(x) = x + \lambda(x)u(x) \qquad (x \in B) \tag{4.1}$$

where $u(x)$ is the unit vector in the direction from $f(x)$ to x:

$$u(x) = \frac{x - f(x)}{|x - f(x)|} \qquad (x \in B) \tag{4.2}$$

and $\lambda(x)$ is the scalar ≥ 0 chosen to make $|P(x)| = 1$ (so $\lambda(x) = 0$ if $x \in \partial B$).

Since $x - f(x)$ is continuous on B and never zero there, u inherits the continuity of f. As for $\lambda = \lambda(x)$, it is the non-negative solution to the equation

$$0 = |P(x)|^2 - 1 = |x + \lambda u(x)|^2 - 1 = \lambda^2 + 2b\lambda - c \tag{4.3}$$

where $c = 1 - |x|^2$ and $b = \langle x, u(x) \rangle$, the dot product of the vectors x and $u(x)$. The quadratic equation (4.3) yields solutions $-b \pm \sqrt{b^2 + c}$; since $c \geq 0$ we know that these solutions are real. Since $\sqrt{b^2 + c} \geq \sqrt{b^2} = |b|$ we know that the non-negative solution is the one with the plus sign. Thus

$$\lambda(x) = -\langle x, u(x) \rangle + \sqrt{\langle x, u(x) \rangle^2 + (1 - |x|^2)} \qquad (x \in B), \tag{4.4}$$

which establishes the desired continuity of λ, and therefore of P. □

Exercise 4.2 (More on the “fictitious” unit vector[1] $u(x)$). In the argument above we defined $\lambda(x)$ to take the value zero for $x \in \partial B$, a fact reflected in Eq. (4.3). Note that, thanks to Eq. (4.4) this implies $\langle x, u(x) \rangle \geq 0$ whenever $|x| = 1$. Prove that for all $x \in \partial B$ and $y \in B$ we must have $\langle x, x - y \rangle \geq 0$, with equality if and only if $y = x$. Conclude that in the argument above, $\langle x, u(x) \rangle > 0$ whenever $x \in \partial B$, hence the quantity under the radical sign on the right-hand side of (4.4) is > 0 for every point of B.

[1] …“fictitious” because its existence stems from our assumption that there exists a retraction of B onto its boundary, which we’re in the process of proving cannot exist. Fictitious or not, we will need the result of this exercise in the next section!

4.3 Proof of the Brouwer Fixed-Point Theorem

We know now that the Brouwer Fixed-Point Theorem (henceforth "BFPT") is equivalent to the No-Retraction Theorem (henceforth "NRT") in the sense that each implies the other. In this section we'll show that the BFPT follows from a "C^1 version" of NRT, which we'll then proceed to establish. Here is an outline of the argument. First we'll show that:

$$C^1\text{-NRT} \implies C^1\text{-BFPT} \implies \text{BFPT} \qquad (*)$$

where, the prefix "C^1-" means that the result is being claimed only for maps whose (real-valued) coordinate functions have continuous first order partial derivatives on some open set that contains B. Then we'll get down to business and prove C^1-NRT.

(a) C^1-BFPT $\implies$ BFPT. The key is the following approximation theorem:

> Given $f: B \to \mathbb{R}^N$ continuous and $\varepsilon > 0$ there exists a C^1 map $g: B \to \mathbb{R}^N$ with $|f(x) - g(x)| \le \varepsilon$ for every $x \in B$.

See Appendix A.2 for a proof.[2] Now suppose $f: B \to B$ is a continuous map. To show that f has a fixed point, let $\varepsilon > 0$ be given and choose—by the above-mentioned approximation result—a C^1 map $f_\varepsilon: B \to \mathbb{R}^N$ with

$$|f_\varepsilon(x) - (1-\varepsilon)f(x)| \le \varepsilon \qquad (x \in B). \tag{4.5}$$

By the "reverse triangle inequality" we have $|f_\varepsilon(x)| - (1-\varepsilon)|f(x)| \le \varepsilon$ for every $x \in B$, i.e.,

$$|f_\varepsilon(x)| \le \varepsilon + (1-\varepsilon)|f(x)| \le \varepsilon + (1-\varepsilon) = 1.$$

Thus f_ε maps B into itself, so by our assumption that the C^1-BFPT holds, f_ε has a fixed point $p_\varepsilon \in B$. By the (ordinary) triangle inequality, for every $x \in B$:

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= |f_\varepsilon(x) - (1-\varepsilon)f(x) - \varepsilon f(x)| \\ &\le |f_\varepsilon(x) - (1-\varepsilon)f(x)| + \varepsilon|f(x)| \\ &\le \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

so in particular

$$|f(p_\varepsilon) - p_\varepsilon| = |f(p_\varepsilon) - f_\varepsilon(p_\varepsilon)| \le 2\varepsilon \qquad (k \in \mathbb{N}), \tag{4.6}$$

i.e., p_ε is a "2ε-approximate fixed point" for f. Since ε is an arbitrary positive number, the Approximate-Fixed-Point Lemma (Lemma 2.2, page 24) guarantees that f has a fixed point. □

[2] More is true: the Stone–Weierstrass Theorem (see, e.g., [101, Theorem 7.6, page 159]) guarantees that the coordinate functions of g can even be chosen to be polynomials (in n variables).

(b) *C^1-NRT $\implies$ C^1-BFPT.* Suppose C^1-BFPT fails, so there exists a C^1 map $f\colon B \to B$ with no fixed point. We'll show that in this case the retraction P given by Eqs. (4.1)–(4.4) on page 43 is C^1 on B. In the defining Eq. (4.2) for the unit vector u, the function $x - f(x)$ is C^1 and never zero, hence the denominator $|x - f(x)|$ is C^1 and (thanks to the compactness of B) bounded away from zero on B. Thus u is C^1 on B. The only issue left is the C^1 nature of the parameter $\lambda(x)$ on the right-hand side of Eq. (4.1), but this follows immediately from Eq. (4.4) and the fact that, on the right-hand side of that equation, the quantity under the radical sign is C^1 and—thanks to Exercise 4.2—strictly positive for each $x \in B$. □

(c) *Proof of C^1-NRT.* This is the heart of our proof of the BFPT. Suppose C^1-NRT is false, i.e., suppose there exists a C^1 retraction P taking B onto its boundary. We will show that this leads to a contradiction. The argument takes place in several steps.

STEP I: *A bridge from the identity map to P.*
For $0 \le t \le 1$ define the map $P_t\colon B \to \mathbb{R}^N$ by

$$P_t(x) = (1-t)x + tP(x) \qquad (x \in B). \tag{4.7}$$

Directly from this definition it follows that:

(a) P_0 is the identity map on B, while $P_1 = P$.
(b) Each P_t is a C^1 map that—since each of its values is a convex combination of two elements of B—takes B into itself.
(c) Each map P_t fixes every point of ∂B.

For the next step let B° denote the interior of B, i.e., the open unit ball of $\mathbb{R}^N$.

STEP II: *There exists $t_0 \in (0,1]$ such that for all $t \in [0,t_0]$,*

(a) $\det P_t'(x) > 0$ *for all $x \in B$.*
(b) *P_t is a homeomorphism of B° onto itself.*

Here $P_t'(x)$ is the derivative of P_t evaluated at $x \in B^\circ$ (see Appendix A.1); we view $P_t'(x)$ as an $N \times N$ matrix whose entries are continuous, real-valued functions on some open set that contains B. We're claiming that for t sufficiently close to zero, P_t inherits the salient properties of the identity map P_0. Let's defer the proof of this statement until we've seen how it leads to the desired contradiction.

STEP III: *Deriving the contradiction.* Define $h\colon [0,1] \to \mathbb{R}$ by the multiple Riemann integral:

$$h(t) = \int_{B^\circ} \det P_t{}'(x)\,dx \qquad (0 \le t \le 1).$$

By STEP II and the Change-of-Variable Theorem (Theorem A.4):

$$h(t) = \int_{P_t(B^\circ)} dx = \text{Volume of } B^\circ \qquad (0 \le t \le t_0). \tag{4.8}$$

Now $\det P_t'$ is a polynomial in t with continuous real-valued coefficients, so by (4.8) $h(t)$ is a polynomial in t that, on the interval $[0,t_0]$, takes the constant value "volume of B°," and so has that constant value for *all* $t \in [0,1]$. In particular, $h(1) > 0$. But we're assuming that $P_1 = P$ maps B° into the unit sphere ∂B, a subset of $\mathbb{R}^N$ that has no interior, so by the Inverse-Function Theorem (Appendix A, Theorem A.3) its derivative matrix $P'(x)$ is singular for every $x \in B^\circ$. Thus for $t = 1$ the integrand on the right-hand side of (4.8) is identically zero, i.e., $h(1) = 0$. Contradiction!

PROOF OF STEP II. This takes place in several stages, each of which expresses the fact that as we restrict t to increasingly smaller values, P_t inherits successively more properties of the identity map P_0.

STEP IIA: *For all t sufficiently small, P_t is a homeomorphism of B onto $P_t(B)$.*

Because P is a C^1 map on the compact set B, the Mean-Value Inequality (Appendix A, Theorem A.2, page 184) provides a positive constant L such for each pair x, y of points in B,

$$|P(x) - P(y)| \le L|x - y|,$$

i.e., P satisfies a "Lipschitz condition" on B. Thus for $x, y \in B$ and $0 \le t \le 1$:

$$\begin{aligned}
|P_t(x) - P_t(y)| &= |(1-t)(x-y) + t[P(x) - P(y)]| \\
&\ge (1-t)|x-y| - t|P(x) - P(y)| \\
&\ge (1-t)|x-y| - tL|x-y| \\
&= [1 - t(1+L)]\,|x-y|\,.
\end{aligned}$$

Conclusion: For $0 \le t < 1/(1+L)$ the mapping P_t takes B one-to-one into itself, and $(P_t)^{-1}$ satisfies a Lipschitz condition, hence is continuous. In other words, for all sufficiently small t, the mapping P_t is a homeomorphism taking B onto some subset of B.

Our goal now is to show that, at least for t sufficiently small, this subset is all of B. Since P_t is the identity map on ∂B, it will be enough to show that $P_t(B^\circ) = B^\circ$ for all sufficiently small t.

STEP IIB: *For all t sufficiently small, $P_t(B^\circ)$ is an open subset of B°.*

From the definition (4.7) of P_t we see that for each $t \in [0,1]$:

$$P_t'(x) = (1-t)I + tP'(x) \qquad (x \in B)$$

where I denotes the $N \times N$ identity matrix. Thus the "C^1-ness" of the retraction P translates into continuity for the map $(t,x) \to P_t'(x)$ as it takes the compact product space $[0,1] \times B$ into the space of $N \times N$ real matrices endowed with the metric of $\mathbb{R}^{N^2}$. Since continuous functions on compact metric spaces are uniformly continuous, the function $(t,x) \to \det P_t'(x)$ is a uniformly continuous real-valued function on $[0,1] \times B$. Since $P_0'(x)$ is the $N \times N$ identity matrix for each $x \in B$ this uniform continuity implies (exercise) that there exists $0 < t_0 < 1/C$ (C being the constant of Step IIa) such that $\det P_t'(x) \geq 1/2$ for each $(t,x) \in [0,t_0] \times B$. This justifies our application of the change-of-variable formula in STEP III, and shows that $P_t'(x)$ is invertible for all $t \in [0,t_0]$ and all $x \in B$. Thus if $0 \leq t \leq t_0$ the Inverse-Function Theorem (Appendix A, Theorem A.3) shows that P_t maps open sets to open sets; in particular $P_t(B^\circ)$ is open in B°.

STEP IIC: *For all t as promised by* STEP IIB, $P_t(B^\circ) = B^\circ$.

Fix such a t, so P_t is a homeomorphism of B^0 onto $P_t(B^0)$. Suppose $P_t(B^\circ) \neq B^\circ$. Then there is a point $y_0 \in B^\circ$ that belongs to the boundary of $P_t(B^\circ)$. One can therefore choose a sequence (y_k) of points in $P_t(B^\circ)$ with $y_k \to y_0$. Thus there exists a sequence (x_k) in B° with $P_t(x_k) = y_k$ for each index k. Thanks to the compactness of B we may assume, upon replacing (x_k) by an appropriate subsequence, that $\lim_k x_k = x_0 \in B$. Thus $y_0 = P_t(x_0)$ by the continuity of P_t. It follows that $x_0 \in B^\circ$; otherwise x_0 would belong to ∂B so, because P_t is the identity map on ∂B, the point $y_0 = P_t(x_0)$ would equal x_0, and so would lie on ∂B, contradicting our assumption that y_0 lies in B°.

This completes the proof of STEP II, and with it, the proof of the Brouwer Fixed-Point Theorem. □

4.4 Retraction and Convexity

So far the work of this chapter has concentrated on the equivalence of the Brouwer Fixed-Point Theorem and the No-Retraction Theorem. Here is a different (and very useful) connection between fixed points and retractions.

Theorem 4.3. *Every retract of a space with the fixed-point property has the fixed-point property.*

Proof. Suppose S is a metric space with the fixed-point property, A is a subset of S, and $P\colon S \to A$ is a retraction of S onto A. Let $f\colon A \to A$ be a continuous map. We need to show that f has a fixed point. Since $g = f \circ P$ maps S into itself it has a fixed point. Since g maps S into A this fixed point, call it a, belongs to A. But the restriction of P to A is the identity map, so $a = g(a) = f(P(a)) = f(a)$. □

Which spaces have the fixed-point property? Every one-point space has it (trivially), and for each positive integer N the closed unit ball of $\mathbb{R}^N$ has it (The Brouwer Fixed-Point Theorem). No circle has it (nontrivial rotations have no fixed point), hence no closed curve (homeomorphic image of a circle) has it.

The extension of Brouwer's theorem provided by Theorem 4.3 allows us to exhibit more examples. Here is one that is "one dimensional," but not homeomorphic to a closed interval (exercise).

Example 4.4. *The letter "X" has the fixed-point property.*

Proof. Here "the letter 'X' " is the union in $\mathbb{R}^2$ of those parts of the lines $y = x$ and $y = -x$ that lie in B, the closed unit ball of $\mathbb{R}^2$ (a.k.a "the closed unit disc"). Then $X \subset B$, so by Brouwer's theorem and Theorem 4.3 above we need only show that X is a retract of B. We'll accomplish this by modifying the "non-orthogonal" projection introduced above in Sect. 4.1. The set X divides B into four quadrants, each bisected by the coordinate half-axes. Project each point in B onto X by moving it parallel to the closest coordinate axis. Thus, each point of a coordinate axis goes to the origin, each point of X stays fixed, each point of the region above X goes straight down onto X, each point to the right of X goes horizontally onto X, etc. The result is a map P that takes B onto X, and whose restriction to X is the identity. I leave it to you to convince yourself that P is continuous. □

Exercise 4.3. Which capital letters of the English alphabet have the fixed-point property?

So much for amusing examples. Here's the result we're really after.

Theorem 4.5 (The "Convex" Brouwer Fixed-Point Theorem)**.** *Every compact convex subset of* $\mathbb{R}^N$ *has the fixed-point property.*

Proof. Let C be a compact convex subset of $\mathbb{R}^N$.

Claim. C is a retract of $\mathbb{R}^N$.

Even though $\mathbb{R}^N$ does not have the fixed-point property, this will prove our result. Indeed, since C is compact it is contained in a closed ball B (not necessarily the unit ball now) which, by Brouwer's theorem, has the fixed-point property. The Claim will give us a retraction P of $\mathbb{R}^N$ onto C, and the restriction of P to B will be a retraction of B onto C. The result will then follow from Theorem 4.3.

Proof of the Claim. The retraction we're about to produce—important in its own right—is the *Closest-Point Map.* Suppose $x \in \mathbb{R}^N$. Since C is compact there is at least one point $\kappa \in C$ with $|x - \kappa| = \inf\{|x - c| : c \in C\}$ (Proof: There is a sequence (c_j) of points in C for which $|x - c_j|$ converges to the infimum in question. By the compactness of C, this sequence has a convergent subsequence, whose limit κ is a point that achieves the infimum).

The convexity of C guarantees that κ is the *unique* closest point in C to x. To prove this, suppose $k \in C$ is another point "closest to x." For convenience let

$$d = \inf\{|x - c| : c \in C\} = |x - \kappa| = |x - k|$$

Let $v = x - \kappa$ and $w = x - k$. By the Parallelogram Law:

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2 = 4d^2 .$$

On the other hand, the convexity of C guarantees that $(\kappa + k)/2 \in C$, hence

$$|(v + w)/2| = |(\kappa + k)/2 - x| \geq d.$$

The last two displays yield

$$4d^2 + |v-w|^2 \le 4d^2$$

so $0 = |v-w| = |k-\kappa|$, hence $\kappa = k$, as desired.

Now that we know there's a unique closest point in C to x, let's give it a name: $P(x)$. Thus P maps $\mathbb{R}^N$ onto C, and fixes each point of C. To show that P retracts $\mathbb{R}^N$ onto C we need only verify its continuity. The result below shows that this follows from the "closest-point uniqueness" from which the mapping P owes its definition. □

Proposition 4.6. *Suppose (X,d) is a metric space and A is a compact subset of X such that every $x \in X$ has a unique closest point $P(x)$ in A. Then P is a retraction of X onto A.*

Proof. Define the function "distance to A" by

$$d_A(x) = \inf\{a \in A\colon d(x,a)\} \qquad (x \in X).$$

Note first that $d_A\colon X \to [0,\infty)$ is continuous; in fact, it is "non-expansive" in the sense that

$$|d_A(x) - d_A(y)| \le d(x,y) \qquad (x,y \in X). \tag{4.9}$$

In fact this is true for every $A \subset X$. To see why, fix x and y in X; suppose (without loss of generality) that $d_A(x) \ge d_A(y)$. Then $d_A(x) \le d(x,a) \le d(x,y) + d(y,a)$ for every $a \in A$, from which follows (thanks to the fact that a was an arbitrary element of A) that $d_A(x) \le d(x,y) + d_A(y)$, which is another way of stating (4.9).

Now let's return to our compact subset A that *does* have the "unique closest point" property, and the map $P(x) =$ "closest point in A to x." We're trying to show that P is continuous, so fix $x_0 \in X$ and suppose (x_n) is a sequence in X that converges to x_0. Our goal is to show that $P(x_n) \to P(x_0)$. Since A is compact, the sequence $(P(x_n))$ of closest points has a subsequence convergent to a point—call it y_0—of A. To keep notation under control, let's replace (temporarily) the whole sequence by this subsequence, so that $P(x_n) \to y_0$. Then:

$$\begin{aligned} d_A(x_0) &= \lim_n d_A(x_n) && \text{(continuity of } d_A) \\ &= \lim_n d(x_n, P(x_n)) && \text{(definition of } P) \\ &= d(x_0, y_0) && \text{(definition of } y_0) \end{aligned}$$

so y_0 is a closest point in A to x_0, hence by uniqueness, $y_0 = P(x_0)$. This argument actually proves that if x_0 is a point of X and (x_n) is a sequence that converges to x_0, then every subsequence of (x_n) has a further subsequence whose image under P converges to $P(x_0)$. Thus $P(x_n) \to P(x_0)$, as desired. This completes the proof of the Proposition, and with it the proof that the closest-point mapping of $\mathbb{R}^N$ onto the compact convex subset C is continuous, hence a retraction. □

Notes

Proof of the Brouwer Fixed-Point Theorem. The argument given here is C.A. Rogers' modification [99] of an argument due to John Milnor [78]. In [67] Peter Lax proves an "oriented" version of the change-of-variable formula for multiple integrals, and uses this result to provide a more direct proof of Brouwer's Theorem. For a differential-forms interpretation of Lax's change-of-variable argument see [53], which also gives a valuable survey of papers that offer analytic proofs of Brouwer's Theorem.

More on proofs of the Brouwer Fixed-Point Theorem. We've seen two proofs of the Brouwer Fixed-Point Theorem: the one in Chap. 2 (for $N = 2$) based on the Sperner Lemma, and the one in this chapter. There are many others; see [112] for a nice survey. Brouwer's original proof [18], published in 1912, used methods of (what has since become known as) algebraic topology. Simultaneously, and for the rest of his life, Brouwer thought deeply about the foundations of mathematics—a pursuit that ultimately led him, 40 years later, to renounce this proof of his theorem [19].

The "Closest Point Property" of convex sets. With a little more care we can weaken the compactness hypothesis on the convex set C to just "closed-ness." The idea is that an application, similar to the one above, of the Parallelogram Law shows that the "minimizing sequence" (c_j) discussed above is actually a Cauchy sequence, and therefore converges, its limit being the unique closest point in C to x. In case C is a linear subspace of $\mathbb{R}^N$ this closest point turns out to be the orthogonal projection of x onto C. These arguments generalize, with no essential changes, to the setting of infinite dimensional Hilbert space (see [125, Sect. 3.2, p. 26 ff.], for example).

Non-expansiveness of the closest-point map. For a closed convex subset of $\mathbb{R}^N$ (or more generally of a Hilbert space) the "closest point map" P is more than just continuous: it is "non-expansive" in the sense that

$$|P(x) - P(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^N$; see, for example, [50, Theorem 3.13, p. 118] for the details.