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Matrix Monotone Functions and the Generalized Powers–Størmer Inequality

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Abstract. In this note a generalization of Powers–Størmer inequality for operator monotone functions on $[0, +\infty)$ and for positive linear functional on general C^*-algebras will be introduced and be shown that the generalized Powers–Størmer inequality characterizes the tracial functionals on C^*-algebras and the monotonicity for a given function.

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1. Introduction

Let $n \in \mathbf{N}$ and M_n be the algebra of $n \times n$ matrices. We call a function f matrix convex of order n or n-convex in short whenever the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B),\ \lambda \in [0,1]$$

holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of A and B are contained in an interval I ($\subset \mathbf{R}$). Matrix monotone functions on I are similarly defined as the inequality

$$A \leq B \Longrightarrow f(A) \leq f(B)$$

for an arbitrary selfadjoint matrices $A, B \in M_n$ such that $A \leq B$ and all eigenvalues of A and B are contained in I.

We denote the spaces of operator monotone functions and of operator convex functions by $P_\infty(I)$ and $K_\infty(I)$ respectively. The spaces for n-monotone functions

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and n-convex functions are written as $P_n(I)$ and $K_n(I)$. We have then

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)$$
$$K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I).$$

Here we meet the facts that $\cap_{n=1}^\infty P_n(I) = P_\infty(I)$ and $\cap_{n=1}^\infty K_n(I) = K_\infty(I)$. We regard these two decreasing sequences as noncommutative counterpart of the classical piling sequence $\{C^n(I), C^\infty(I), \mathrm{Anal}(I)\}$, where the class $\mathrm{Anal}(I)$ denotes the set of all analytic functions over I. We could understand that the class of operator monotone functions $P_\infty(I)$ corresponds to the class $\{C^\infty(I), \mathrm{Anal}(I)\}$ by the famous characterization of those functions by Loewner as the restriction of Pick functions.

In these circumstances, it will be well recognized that we should not stick our discussions only to those classes $P_\infty(I)$ and $K_\infty(I)$, that is, the class of operator monotone functions and that of operator convex functions. Those classes $\{P_n(I)\}$ and $\{K_n(I)\}$ are not merely optional ones to $P_\infty(I)$ and $K_\infty(I)$. They should play important roles in the aspect of noncommutative calculus as the ones $\{C^n(I)\}$ play in usual (commutative) calculus.

The first basic question is whether $P_{n+1}(I)$ (resp. $K_{n+1}(I)$) is strictly contained in $P_n(I)$ (resp. $K_n(I)$) for every n. In [31] the gap for $n = 2$, that is, $P_3([0,\infty)) \subsetneq P_2([0,\infty))$, was pointed out. This gap problem for arbitrary n, however, has been solved only recently ([9], [25], [12]). (See Section 2.)

On the other hand, there are basic equivalent assertions known only at the level of operator monotone functions and operator convex functions by [10], [11]. We shall discuss those (equivalent) assertions as the correlation problem between two kinds of piling structures $\{P_n(I)\}$ and $\{K_n(I)\}$, that is, we are planning at first to discuss relations between those assertions at each level n.

In [26] (resp. [17]) we discussed about the following 3 assertions at each level n among them in order to see clear insight of the aspect of the problems:

(i) $f(0) \leq 0$ and f is n-convex (resp. n-concave) in $[0,\alpha)$,

(ii) For each matrix A with its spectrum in $[0,\alpha)$ and a contraction C in the matrix algebra M_n,

$$f(C^*AC) \leq C^*f(A)C,$$

(resp. $f(C^*AC) \leq C^*f(A)C$)

(iii) The function $\frac{f(t)}{t}$ (resp. $\frac{t}{f(t)}$) ($= g(t)$) is n-monotone in $(0,\alpha)$.

Then we showed that for each n the condition (ii) is equivalent to the condition (iii) and the assertion that f is n-convex with $f(0) \leq 0$ implies that $g(t)$ is $(n-1)$-monotone (resp. f is n-concave with $f(0) \geq 0$ implies that $g(t)$ is $(n-1)$-monotone). (See Section 3.)

Powers–Størmer inequality (see, for example, [29, Lemma 2.4], [28, Theorem 11.19]) asserts that for $s \in [0,1]$ the following inequality

$$2\,\mathrm{Tr}(A^sB^{1-s}) \geq \mathrm{Tr}(A + B - |A - B|) \tag{1.1}$$

holds for any pair of positive matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [1]. This inequality was first proven in [1], using an integral representation of the function t^s. After that, N. Ozawa gave a much simpler proof for the same inequality, using fact that for $s \in [0,1]$ function $f(t) = t^s$ ($t \in [0,+\infty)$) is an operator monotone ([18, Proposition 1.1]). Recently, Y. Ogata in [24] extended this inequality to standard von Neumann algebras. The motivation for the present paper is to investigate whether replacing the function $f(t) = t^s$ by another operator monotone function (this class is intensively studied, see [9][25]) can yield a smaller upper bound for $\mathrm{Tr}(A + B - |A - B|)$ than what is used in quantum hypothesis testing. Based on N. Ozawa's proof we formulate Powers–Størmer's inequality for an arbitrary operator monotone function on $[0,+\infty)$ in the context of general C^*-algebras. (See Section 4.)

As applications, the generalized Powers–Størmer inequality characterizes the trace property for a normal linear positive functional on a von Neumann algebras and for a linear positive functional on a C^*-algebra. (See Section 5.) It also characterizes the monotonicity of a given function in this inequality. (See Section 6.)

2. Preliminary

We shall sometimes use the standard regularization procedure, cf. for example Donoghu [6, p11]. Let ϕ be a positive and even C^∞-function defined on the real axis, vanishing outside the closed interval $[-1,1]$ and normalized such that

$$\int_{-1}^{1} \phi(x) = 1.$$

For any locally integrable function f defined in an open interval (a, b) we form its regularization

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\int_a^b \phi(\frac{t-s}{\varepsilon}) f(s)ds, \quad t \in \mathbf{R}$$

for small $\varepsilon > 0$, and realize that it is infinitely many times differentiable. For $t \in (a + \varepsilon, b - \varepsilon)$ we may also write

$$f_\varepsilon(t) = \int_{-1}^{1} \phi(s) f(t - \varepsilon s)ds.$$

If f is continuous, then f_ε converges uniformly to f on any compact subinterval of (a, b). If in addition f is n-convex (or n-monotone) in (a, b), then f_ε is n-convex (or n-monotone) in the slightly smaller interval $(a + \varepsilon, b - \varepsilon)$. Since the pointwise limit of a sequence of n-convex (or n-monotone) functions is again n-convex (or n-monotone), we may therefore in many applications assume that an n-convex or n-monotone function is sufficiently many times differentiable.

For a sufficiently smooth function $f(t)$ we denote its nth divided difference for n-tuple of points $\{t_1, t_2, \ldots, t_n\}$ defined as, when they are all different,

$$[t_1, t_2]_f = \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \quad \text{and inductively}$$

$$[t_1, t_2, \ldots, t_n]_f = \frac{[t_1, t_2, \ldots, t_{n-1}]_f - [t_2, t_3, \ldots, t_n]_f}{t_1 - t_n}.$$

And when some of them coincides such as $t_1 = t_2$ and so on, we put as

$$[t_1, t_1]_f = f'(t_1) \quad \text{and inductively} \quad [t_1, t_1, \ldots, t_1]_f = \frac{f^{(n-1)}(t_1)}{(n-1)!}.$$

When there appears no confusion we often skip the referring function f. We notice here the most important property of divided differences is that it is free from permutations of $\{t_1, t_2, \ldots, t_n\}$ in an open interval I.

Proposition 2.1.

(1) (Ia) *Monotonicity* (*Loewner* 1934 [21])

$$f \in P_n(I) \Longleftrightarrow ([t_i, t_j]) \geq 0 \textit{ for any } \{t_1, t_2, \ldots, t_n\}$$

(IIa) *Convexity* (*Kraus* 1936 [20])

$$f \in K_n(I) \Longleftrightarrow ([t_1, t_i, t_j]) \geq 0 \textit{ for any } \{t_1, t_2, \ldots, t_n\},$$

where t_1 can be replaced by any (fixed) t_k.

(2) (Ib) *Monotonicity* (*Loewner* 1934 [21], *Dobsch* 1937 [5]-*Donoghue* 1974 [6]) *For $f \in C^{2n-1}(I)$*

$$f \in P_n(I) \Longleftrightarrow M_n(f;t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!}\right) \geq 0 \ \forall t \in I.$$

(IIb) *Convexity* (*Hansen–Tomiyama* 2007 [12]) *For $f \in C^{2n}(I)$*

$$f \in K_n(I) \Longrightarrow K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!}\right) \geq 0 \ \forall t \in I.$$

In particular, for $n = 2$ the converse is also true.

We remind that to prove the implication $M_n(f;t) \geq 0 \Rightarrow f \in P_n(I)$ in (Ib) the local property for the monotonicity plays an essential role. Similarly to prove the converse implication in the criterion of convexity in (IIb) in the above proposition we need **the local property conjecture for the convexity**, that is, if f is n-convex in the intervals (a, b) and (c, d) $(a < c < b < d)$, then f is n-convex on (a, d).

Now we have only a partial sufficiency, that is, if $K_n(f;t_0)$ is positive, then there exists a neighborhood of t_0 on which f is n-convex. (See [12, Theorem 1.2] for example.)

The method for the implication (II_b) $\Rightarrow$ (IIa) under the assumption of the local property theorem for the convexity may be familiar for some specialist.

Proposition 2.2. *Let* $f \in C^{2n}(I)$ *such that* $K_n(f;t) = \left(\frac{f^{(i+j)}(t)}{(i+j)!}\right) \geq 0\ \forall t \in I$. *Suppose that* n*-convexity has the local property. Then* $f \in K_n(I)$.

3. Double piling structure

As we have mentioned in the introduction, there are basic equivalent assertions known for operator monotone functions and operator convex functions (cf. [10]). Namely we have

Theorem A. *For* $0 < \alpha \leq \infty$, *the following assertions for a real-valued continuous function* f *in* $[0, \alpha)$ *are equivalent:*

(1) f *is operator convex and* $f(0) \leq 0$.

(2) *For an operator* A *with its spectrum in* $[0, \alpha)$ *and a contraction* C,

$$f(C^*AC) \leq C^*f(A)C.$$

(3) *For two operators* A, B *with their spectra in* $[0, \alpha)$ *and two contractions* C, D *such that* $C^*C + D^*D \leq 1$ *we have the inequality*

$$f(C^*AC + D^*BD) \leq C^*f(A)C + D^*f(B)D.$$

(4) *For an operator* A *with its spectrum in* $[0, \alpha)$ *and a projection* P *we have the inequality,*

$$f(PAP) \leq Pf(A)P$$

(5) *The function* $g(t) = \frac{f(t)}{t}$ *is operator monotone in the open interval* $(0, \alpha)$.

In this section, we shall discuss mutual relationships of the above assertions when we restrict the property of the function f at each fixed level n, that is, when f and g are assumed to be only n-matrix convex and n- matrix monotone. We regard the problem as the problem of double piling structure of those decreasing sequences $\{P_n(I)\}$ and $\{K_n(I)\}$ down to $P_\infty(I)$ and $K_\infty(I)$ respectively. In this sense, standard double piling structure known for these assertions before is the following. We describe these implications using the following convention: if the fact that the statement (A) holds for the matrix algebra M_m implies that statement (B) holds for the matrix algebra M_n, then we write $(A)_m \to (B)_n$.

Theorem A is proved in the following way.

$$(1)_{2n} \to (2)_n \to (5)_n \to (4)_n, (2)_{2n} \to (3)_n \to (4)_n, \text{and } (4)_{2n} \to (1)_n.$$

Therefore, those assertions become equivalent when f is operator convex and g is operator monotone by the piling structure.

Thus, the basic problem for double piling structure is to find the minimum difference of degrees between those gaped assertions. Since, however, even single piling problems are clarified only recently, as we have mentioned above, in spite of a long history of monotone matrix functions and convex matrix functions, little is known for the double piling structure except the result by Mathias ([22]), which asserts that a $2n$-monotone function in the positive half-line $[0,\infty)$ becomes n-concave.

Now in order to make our investigations more transparently we mainly concentrate our discussions to the relationships between (1), (2) and (5). In fact, we need not say anything about (4) when $n = 1$, and for the reason choosing (2) instead of (3) we just borrow the witty expression in [10], "correctness must bow to applicability". Before going into our discussions, we state each assertion in a precise way but skipping the condition of the spectrum of a matrix A. Namely, in the interval $[0,\alpha)$ we consider the following assertions.

(i) $f(0) \leq 0$, and f is n-convex.

(ii) For each positive semidefinite element A with its spectrum in $[0,\alpha)$ and a contraction C in M_n, we have

$$f(C^*AC) \leq C^*f(A)C.$$

(iii) The function $g(t) = \frac{f(t)}{t}$ is n-monotone in the interval $(0,\alpha)$.

We shall show then the equivalency of the assertions (ii) and (iii). Hence the problem is reduced to the relationship between (i) and (iii) (or (ii)). Namely, we have the following

Theorem 3.1 ([26]). *Let* $n \in \mathbf{N}$.

1. *The assertions* $(\mathrm{ii})_n$ *and* $(\mathrm{iii})_n$ *are equivalent,*
2. *The assertion* $(\mathrm{i})_n$ *implies the assertion* $(\mathrm{iii})_{n-1}$.

When f is a convex function, $-f$ is a concave function. Hence we have the following.

Theorem 3.2 ([17]). *Let* $f\colon [0,\alpha) \to \mathbf{R}$ $(0 < \alpha \leq \infty)$ *be a continuous function. Consider the following three assertions:*

(i) $f(0) \geq 0$, *and* f *is* n*-concave,*

(ii) *For each positive semidefinite element* A *with its spectrum in* $[0,\alpha)$ *and a contraction* C *in* M_n, *we have*

$$f(C^*AC) \geq C^*f(A)C.$$

(iii) *The function* $g(t) = \frac{t}{f(t)}$ *is* n*-monotone in the interval* $(0,\alpha)$.

Then we have for each $n \in \mathbf{N}$

1. *The assertions* $(\mathrm{ii})_n$ *and* $(\mathrm{iii})_n$ *are equivalent,*
2. *The assertion* $(\mathrm{i})_n$ *implies the assertion* $(\mathrm{iii})_{n-1}$.

4. Generalized Powers–Størmer inequality

One of the most basic tasks in quantum statistics is the discrimination of two different quantum states. In the quantum hypothesis testing problem, one has to decide between two states of a system. The state ρ_0 is the null hypothesis and ρ_1 is alternative hypothesis.

The problem is to decide which hypothesis is true. The decision is performed by a two-valued measurement $\{T, I-T\}$, where $0 \leq T \leq I$ is an observable. T corresponds to the acceptance of ρ_0 and $I-T$ corresponds to the acceptance of ρ_1. T is called a test.

The total error $\operatorname{Err}(T)$ of T is

$$\begin{aligned}\operatorname{Err}(T) &= \frac{1}{2}\operatorname{Tr}(\rho_0(I-T)) + \frac{1}{2}\operatorname{Tr}(\rho_1 T) \\ &= \frac{1}{2}\left\{1 - \operatorname{Tr}(T(\rho_0-\rho_1))\right\}.\end{aligned}$$

Then asymptotic error exponent for ρ_0 and ρ_1 is

$$\lim_{n\to\infty}\frac{1}{n}\log \operatorname{Err}_n(T_{(n)}),$$

where for all $n \in \mathbf{N}$ $T_{(n)}$ is a $d^n \times d^n$ quantum multiple test, and

$$\operatorname{Err}_n(T_{(n)}) := \frac{1}{2}\left\{1 - \operatorname{Tr}(T_{(n)}(\rho_0^{\otimes n} - \rho_1^{\otimes n}))\right\}.$$

If the limit $\lim_{n\to\infty}\frac{1}{n}\log \operatorname{Err}_n(T_{(n)})$ exists, we refer to it as *the asymptotic error exponent.*

The lower band and upper bounds for the asymptotic error exponent are given by he following.

Theorem 4.1 ([1], [23]). *Let $\{\rho_0, \rho_1\}$ be hypothetic states on $\mathbf{C}^d$, $T_{(n)}$ be quantum multiple test, and $Q_{(n)}$ be a support projections on $(\rho_0^{\otimes n} - \rho_1^{\otimes n})$. Then one has*

(i) (*M. Nussbaum and A. Szkola*)

$$\liminf_{n\to\infty}\frac{1}{n}\log \operatorname{Err}_n(T_{(n)}) \geq \inf\{\log \operatorname{Tr}(\rho_0^{1-s}\rho_1^{s}) \mid 0 \leq s \leq 1\}.$$

(ii) (*K.M.R. Audenaert, et al.*)

$$\limsup_{n\to\infty}\frac{1}{n}\log \operatorname{Err}_n(Q_{(n)}) \leq \inf\{\log \operatorname{Tr}(\rho_0^{1-s}\rho_1^{s}) \mid 0 \leq s \leq 1\}.$$

In the proof of the previous Theorem 4.1(ii) the following inequality played a key role.

Theorem 4.2 ([1]). *For any positive matrices A and B on $\mathbf{C}^d$ we have*

$$\frac{1}{2}(\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A-B|) \leq \operatorname{Tr}(A^{1-s}B^s) \ (s \in [0,1]).$$

If we consider a function $f(t) = t^{1-s}$ and $g(t) = t^s = \dfrac{t}{f(t)}$, then both functions f and g are operator monotone. The inequality, then, can be reformed by

$$\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A-B| \leq 2\operatorname{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}).$$

Theorem 4.3 ([16], [17]). *Let f be a $2n$-monotone function (or $(n+1)$-concave function) on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$*

$$\mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A-B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = \frac{t}{f(t)}$.

We give a sketch of the proof.

Let A, B be positive matrices and, let

$$A - B = (A-B)_+ - (A-B)_- = P - Q$$

and $|A-B| = P + Q$. We may, then, show that

$$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \leq \mathrm{Tr}(P)$$

holds as follows:

$$\begin{aligned}
&\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(f(A)^{\frac{1}{2}} g(A) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad \leq \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B+P) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad \leq \mathrm{Tr}(f(B+P)^{\frac{1}{2}} (g(B+P) - g(B)) f(B+P)^{\frac{1}{2}}) \\
&\quad \leq \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B+P) f(B+P)^{\frac{1}{2}}) - \mathrm{Tr}(f(B)^{\frac{1}{2}} g(B) f(B)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}(P).
\end{aligned}$$

In particular we have

Corollary 4.4. *Let f be an operator monotone function on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$*

$$\mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A-B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = \frac{t}{f(t)}$.

Since any C^*-algebra can be realized as a closed selfadjoint $*$-algebra of $B(H)$ for some Hilbert space H. We can generalize Corollary 4.4 in the framework of C^*-algebras.

Theorem 4.5. *Let τ be a tracial functional on a C^*-algebra $\mathcal{A}$, f be a strictly positive, operator monotone function on $[0,\infty)$. Then for any pair of positive elements $A, B \in \mathcal{A}$*

$$\tau(A) + \tau(B) - \tau(|A-B|) \leq 2\tau(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}), \tag{4.1}$$

where $g(t) = tf(t)^{-1}$.

Proof. Let π be the universal representation of $\mathcal{A}$ and $\hat{\tau}$ be a positive linear functional on $\pi(\mathcal{A})$ by $\hat{\tau}(\pi(A)) = \tau(A)$ for $A \in \mathcal{A}$. Then $\hat{\tau}$ has the trace property. Since

g is operator monotone on $(0,\infty)$ by [10, Corollary 6], through the same steps in the proof of Theorem 4.3 we have that for any positive operators A and B in $\mathcal{A}$

$$\hat{\tau}(\pi(A)) + \hat{\tau}(\pi(B)) - \hat{\tau}(\pi(|A-B|)) \leq 2\hat{\tau}(f(\pi(A))^{\frac{1}{2}} g(\pi(B)) f(\pi(A))^{\frac{1}{2}}),$$

that is,

$$\tau(A) + \tau(B) - \tau(|A-B|) \leq 2\tau(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}).$$

□

5. Characterization of the trace property

In this section we shall show that the generalized Powers–Størmer inequality in the previous section guarantees the trace property for a positive linear functional on operator algebras.

Lemma 5.1 ([16]). *Let φ be a positive linear functional on M_n and f be a continuous function on $[0,\infty)$ such that $f(0) = 0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{5.1}$$

holds true for all $A, B \in M_n^+$, then φ should be a positive scalar multiple of the canonical trace Tr *on M_n, where*

$$g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}.$$

By analogy with a number of other similar cases (see [7] or [32]), the proof for the trace property of a positive normal functional satisfying the inequality (5.1) on a von Neumann algebra can be reduced to the case of the algebra M_2 of all matrices of order 2×2.

Theorem 5.2 ([16]). *Let φ be a positive normal linear functional on a von Neumann algebra $\mathcal{M}$ and f be a continuous function on $[0,\infty)$ such that $f(0) = 0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A) + \varphi(B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{5.2}$$

holds true for any pair $A, B \in \mathcal{M}^+$, then φ is a trace, where

$$g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}.$$

Proof. By [19, Proposition 8.1.1] we have only to show that $\varphi(P_1) = \varphi(P_2)$ for any pair of nonzero equivalent projections P_1 and P_2. Moreover, we may assume that P_1 and P_2 are mutually orthogonal. Indeed, considering mutually orthogonal equivalent projections $P_1' = P_1 \vee P_2 - P_1$ and $P_2' = P_1 - P_1 \wedge P_2$ we can show that

$$\varphi(P_1) = \frac{\varphi(P_1 \wedge P_2) + \varphi(P_1 \vee P_2)}{2}.$$

By symmetry we have $\varphi(P_1) = \varphi(P_2)$.

Hence we assume that P_1 and P_2 are nonzero mutually orthogonal equivalent projections in $\mathcal{M}$. Note that $(P_1+P_2)\mathcal{M}(P_1+P_2)$ is isomorphic to M_2. Then the inequality (5.2) still holds true for the operators in $\mathcal{N}$ and for the restriction of the functional φ to $\mathcal{N}$. According to Lemma 5.1, this restriction is a tracial functional on $\mathcal{N}$, and hence $\varphi(P_1) = \varphi(P_2)$. □

Corollary 5.3. *Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$ and f be a continuous function on $[0,\infty)$ such that $f(0)=0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A) + \varphi(B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{5.3}$$

holds true for any pair $A, B \in \mathcal{A}^+$, then φ is a tracial functional, where

$$g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t=0) \end{cases} .$$

The following is inspired by [30, Theorem 2.2].

Proposition 5.4 ([17]). *Let $n \in \mathbb{N}$ $(n \geq 2)$, and φ a positive linear functional on M_n. Let f be a strictly positive, continuous function on $(0,\infty)$. Assume that the function g on $(0,\infty)$ defined by $g(t) = \frac{t}{f(t)}$, is differentiable and strictly increasing on $(0,\infty)$. Suppose that*

$$\varphi(A) \leq \varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{5.4}$$

for any positive invertible $A, B \in M_n$ such that $0 < A \leq B$.

Then φ has the trace property if g satisfies the condition:

$$\inf_{\lambda > \mu} \frac{\sqrt{g'(\lambda) g'(\mu)}}{\frac{g(\lambda)-g(\mu)}{\lambda-\mu}} = 0. \tag{5.5}$$

6. Characterization of operator monotonicity

In this section, following the idea from [4] we give a characterization of operator monotonicity of matrix functions by the generalized Powers–Størmer type inequality. The following lemma is obvious.

Lemma 6.1. *Let $A = (a_{ij}), B = (b_{ij})$ be positive invertible in M_n and S a non-finite rank density operator on an infinite-dimensional, separable Hilbert space H. Suppose that $a_{11} > b_{11}$. Then there exist an orthogonal system $\{\xi_i\}_{i=1}^\infty \subset H$ and $\{\lambda_i\}_{i=1}^\infty \subset [0,1)$ such that $\sum_{i=1}^\infty \lambda_i = 1$, $S\xi_i = \lambda_i \xi_i$, and $\sum_{i=1}^n a_{ii}\lambda_i > \sum_{i=1}^n b_{ii}\lambda_i$.*

Theorem 6.2 ([17]). *Let H be an infinite-dimensional, separable Hilbert space and φ a normal state on $B(H)$ such that its corresponding density operator S_φ is not finite rank. Let f be a strictly positive, continuous function on $(0,\infty)$, and g be a function on $(0,\infty)$ defined by $g(t) = \frac{t}{f(t)}$. Suppose that*

$$\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{6.1}$$

for any positive invertible $A, B \in B(H)$. Then both of functions f and g on $(0, \infty)$ are operator monotone.

If $f(t) = \lambda t$ for some $\lambda > 0$, then g is constant on $(0, \infty)$. In this case, the inequality (6.1) automatically holds. When the range of the density operator S_φ, however, is proper subspace in a Hilbert space H, the inequality (6.1) does not hold for non-invertible positive operators.

Proposition 6.3 ([17]). *Let H be a separable Hilbert space and φ be a normal state on $B(H)$. Let f be a strictly positive, continuous function on $[0, \infty)$ with $f(0) = 0$, g a function on $(0, \infty)$ defined by $g(t) = \frac{t}{f(t)}$ on $(0, \infty)$ and $g(0) = 0$. Suppose that the range of the density operator S_φ of φ is a proper subspace of H. Then there exist positive non-invertible operators A and B which do not satisfy the inequality*

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}). \tag{6.2}$$

Proof. We shall give a sketch of the proof.

Let $\{\xi_i\}_{i \in \mathbf{N}}$ be an orthogonal system and $\{\lambda_i\} \subset [0, 1)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $S_\varphi \xi_i = \lambda_i \xi$ and $\sum_{i=1}^\infty \lambda_i = 1$.

Since $S_\varphi(H) \subsetneq H$, we take ξ_{i_0} such that $S_\varphi(\xi_{i_0}) = 0$. For $\delta, \varepsilon > 0$ such that $\delta > \varepsilon$ we set

$$A = \varepsilon |\xi_1\rangle\langle\xi_1| + \sqrt{\varepsilon(\delta - \varepsilon)}(|\xi_1\rangle\langle\xi_{i_0}| + |\xi_{i_0}\rangle\langle\xi_1|) + (\delta - \varepsilon)|\xi_{i_0}\rangle\langle\xi_{i_0}|$$

and

$$B = \varepsilon |\xi_1\rangle\langle\xi_1| - \sqrt{\varepsilon(\delta - \varepsilon)}(|\xi_1\rangle\langle\xi_{i_0}| + |\xi_{i_0}\rangle\langle\xi_1|) + (\delta - \varepsilon)|\xi_{i_0}\rangle\langle\xi_{i_0}|.$$

We have then

$$\begin{aligned}
\varphi(A + B) &= 2\,\mathrm{Tr}(S_\varphi(\varepsilon|\xi_1\rangle\langle\xi_1| + (\delta - \varepsilon)|\xi_{i_0}\rangle\langle\xi_{i_0}|)) \\
&= 2\,\mathrm{Tr}(\lambda_1 \varepsilon |\xi_1\rangle\langle\xi_1|) = 2\lambda_1\varepsilon \\
\varphi(|A - B|) &= 2\,\mathrm{Tr}(S_\varphi \sqrt{\varepsilon(\delta - \varepsilon)}(|\xi_1\rangle\langle\xi_1| + |\xi_{i_0}\rangle\langle\xi_{i_0}|)) \\
&= 2\lambda_1\sqrt{\varepsilon(\delta - \varepsilon)} \\
\varphi(f(A)^{1/2} g(B) f(A)^{1/2}) &= \varepsilon\lambda_1 \frac{(\delta - 2\varepsilon)^2}{\delta^2}.
\end{aligned}$$

Therefore, if positive operators A and B satisfy the inequality (6.2), we have

$$\varepsilon - \sqrt{\varepsilon(\delta - \varepsilon)} \leq \varepsilon \frac{(\delta - 2\varepsilon)^2}{\delta^2}.$$

But we have a contradiction if we take $\delta = \dfrac{4\varepsilon}{3}$. □

The following problem is plausible.

Problem 6.4. Let f and g be the functions. Suppose that for any n and any positive matrices $A, B \in M_n$

$$\mathrm{Tr}(A + B) - \mathrm{Tr}(|A - B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}).$$

Is the function f operator monotone?

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