Chapter 9

Local Well-Posedness and Regularity

In this chapter we study local well-posedness and regularity of the solutions of Problems **(P1)**$\sim$**(P6)**. Here we employ without further comments the notations introduced in Chapters 1 and 2, in particular those in connection with Conditions **(H1)**$\sim$**(H6)** from Chapter 1, the Hanzawa transform, and the transformed problems on the fixed domain $\Omega \setminus \Sigma$ in Section 1.3. In the first section of this chapter we reformulate Problems **(P1)**$\sim$**(P6)** in a way which is amenable to a joint analysis, which will be based on maximal L_p-regularity as well as on the contraction mapping principle in Section 9.2, and on the implicit function theorem for dependence on the data in Section 9.3. For regularity we employ in Section 9.4 the so-called parameter trick, which is also based on maximal L_p-regularity and the implicit function theorem. This way we can show that the solutions obtained in Section 9.2 are in fact classical solutions. The proofs for the technical results on the nonlinearities are postponed to the last section of this chapter.

9.1 Reformulation on the Fixed Domain

The main goal of this section is the reformulation of the transformed problems **(P1)**$\sim$**(P6)** in abstract form $Lz = N(z)$. We call L the *principal linearization.* The mapping N collects all nonlinear and lower order terms. We have to set up function spaces such that L has the property of maximal regularity, and N is Lipschitz continuous. Furthermore, we use the decomposition $z = \bar{z} + \tilde{z}$, where $\bar{z}$ resolves the compatibility conditions and satisfies the initial condition, and $\tilde{z}$ has vanishing trace at time $t = 0$. This has to be done separately for each problem in question. We begin with the simplest one.

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1.1 The Stefan Problem with Surface Tension

In the sequel we assume Condition **(H1)**, and the compatibility condition

$$(\mathbf{C1}) \quad \varrho[\![\psi(\theta_0)]\!] + \sigma H_{\Gamma_0} = 0 \ \text{ on } \Gamma_0, \quad [\![d(\theta_0)\partial_\nu\theta_0]\!] \in W_p^{2-6/p}(\Gamma_0).$$

The transformed problem **(P1)** reads as follows (w.l.o.g. $\varrho = 1$).

$$\begin{aligned} \partial_t\theta + \mathcal{A}_\theta(\theta, h) : \nabla^2\theta &= F_\theta(\theta, h) && \text{in } \Omega\setminus\Sigma,\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ [\![\theta]\!] = 0, \quad \varphi(\theta) + \sigma H_\Gamma(h) &= 0 && \text{on } \Sigma,\\ \partial_t h + [\![\mathcal{B}_\theta(\theta, h)\nabla\theta]\!] &= 0 && \text{on } \Sigma,\\ h(0) = h_0 \text{ on } \Sigma, \quad \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{9.1}$$

Recall that $\varphi(\theta) = [\![\psi(\theta)]\!]$, where ψ denotes the free energy, and $l(\theta) = \theta\varphi'(\theta)$ is the latent heat. We assume here that $l(\theta_0) \neq 0$. The map F_θ collects lower order terms and we have

$$\begin{aligned} \kappa(\theta)F_\theta(\theta, h) = \kappa(\theta)m_0(h)\partial_t h \circ \Pi_\Sigma(\nu_\Sigma\cdot\nabla\theta) + d'(\theta)|(1 - M_1(h))\nabla\theta|^2\\ - d(\theta)\big[(I - M_1(h))\nabla\big]M_1(h)\big)\cdot\nabla\theta. \end{aligned}$$

Note that the time derivative of h appears in $\mathbb{F}_\theta$. On the other hand, the curvature operator according to Section 2.2.5 is given by

$$\begin{aligned} H_\Gamma(h) = \beta(h)\{\mathrm{tr}[M_0(h)(L_\Sigma + \nabla_\Sigma(M_0(h)\nabla_\Sigma h))]\\ - \beta^2(h)(M_0^2(h)\nabla_\Sigma h|\nabla_\Sigma\big(M_0(h)\nabla_\Sigma(h)\big)M_0(h)\nabla_\Sigma h)\}. \end{aligned}$$

Finally, $\mathcal{A}_\theta(\theta, h)$ and $\mathcal{B}_\theta(\theta, h)$ are defined by

$$\begin{aligned} \mathcal{A}_\theta(\theta, h) &= -\big(d(\theta)/\kappa(\theta)\big)(I - M_1(h)^{\mathsf{T}})(I - M_1(h)),\\ \mathcal{B}_\theta(\theta, h) &= -\big(d(\theta)/l(\theta)\big)(1 - M_1(h)^{\mathsf{T}})(\nu_\Sigma - M_0(h)\nabla_\Sigma h). \end{aligned}$$

To formulate the problem abstractly, let $J = (0, a)$ where $a > 0$ will be fixed later. We first set

$$\begin{aligned} \mathbb{E}_{\theta,\mu}(J) &= H^1_{p,\mu}(J; L_p(\Omega)) \cap L_{p,\mu}(J; H^2_p(\Omega\setminus\Sigma)),\\ \mathbb{E}^\theta_{h,\mu}(J) &= W^{3/2-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(J; H^2_p(\Sigma)) \cap L_{p,\mu}(J; W_p^{4-1/p}(\Sigma)), \end{aligned}$$

and define the solution space for $z = (\theta, h)$ by

$$\mathbb{E}^1_\mu(a) = \{z = (\theta, h) \in \mathbb{E}_{\theta,\mu}(J) \times \mathbb{E}^\theta_{h,\mu}(J) : [\![\theta]\!] = 0 \text{ on } \Sigma,\ \partial_\nu\theta = 0 \text{ on } \partial\Omega\}.$$

The space of data for $(f_\theta, g_\theta, f_h)$ will be

$$\mathbb{F}^1_\mu(a) = \mathbb{F}_{\theta,\mu}(J) \times \mathbb{F}^\theta_{h,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J),$$

with

$$\begin{aligned}
\mathbb{F}_{\theta,\mu}(J) &= L_{p,\mu}(J; L_p(\Omega)),\\
\mathbb{F}^{\theta}_{h,\mu}(J) &= W^{1-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap L_{p,\mu}(J; W^{2-1/p}_p(\Sigma)),\\
\mathbb{F}^{u}_{h,\mu}(J) &= W^{1/2-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Sigma)).
\end{aligned}$$

Recall from Subsection 3.4.6 that the time trace space $X^1_{\gamma,\mu}$ of $\mathbb{E}^1_\mu(a)$ is given by

$$X^1_{\gamma,\mu} = \{(\theta, h) \in W^{2\mu-2/p}_p(\Omega\setminus\Sigma) \times W^{2+2\mu-3/p}_p(\Sigma) : [\![\theta]\!] = 0 \text{ on } \Sigma,\ \partial_\nu\theta = 0 \text{ on } \partial\Omega\}.$$

We observe that

$$X^1_{\gamma,\mu} \hookrightarrow C^1_{ub}(\Omega\setminus\Sigma) \times C^3(\Sigma), \quad \text{provided } 1 \ge \mu > \frac{1}{2} + \frac{n+2}{2p}. \tag{9.2}$$

We will use this restriction in the sequel, although it would be enough to require

$$X^1_{\gamma,\mu} \hookrightarrow C_{ub}(\Omega\setminus\Sigma) \times C^2(\Sigma), \qquad \text{valid for } 1 \ge \mu > \frac{n+2}{2p}.$$

However, this would involve more technical efforts, and we refrain from carrying this out here. Observe that the last restriction cannot be relaxed, since we definitely need continuity of temperature and of curvature; the interfaces ought to be of class C^2.

Unfortunately, $(\theta_0, h_0) \in X^1_{\gamma,\mu}$ do not have enough regularity for the space $\mathbb{F}^{\theta}_{h,\mu}(J)$, as $\varphi'(\theta_0)$ fails to be a pointwise multiplier for this space. For this reason we cannot freeze coefficients in the stationary interface equation. Therefore, we extend the initial value θ_0 to some function $\bar\theta$ in $\mathbb{E}_{\theta,\mu}(\mathbb{R}_+)$, for instance by solving the problem

$$\begin{aligned}
\partial_t\bar\theta - \Delta\bar\theta &= 0 && \text{in } \Omega,\\
\partial_\nu\bar\theta &= 0 && \text{on } \partial\Omega,\\
\bar\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}$$

Similarly, we extend h_0 and $h_1 := -[\![\mathcal{B}(\theta_0, h_0)\nabla\theta_0]\!]$ as in Section 6.6.2 to a function $\bar h \in \mathbb{E}^{\theta}_{h,\mu}(\mathbb{R}_+)$ such that $\bar h(0) = h_0$ and $\partial_t \bar h(0) = h_1$. Further we set $\tilde\theta = \theta - \bar\theta$ and $\tilde h = h - \bar h$. This way, we have trivialized the initial conditions and at the same time resolved the compatibility conditions. Writing

$$\varphi(\theta) = \varphi(\bar\theta) + \varphi'(\bar\theta)\tilde\theta + r_\theta(\tilde\theta, \bar\theta)$$

and

$$H_\Gamma(h) = H_\Gamma(\bar h) + H'_\Gamma(h_0)\tilde h + r_h(\tilde h, \bar h)$$

we may replace the stationary interface condition by

$$\varphi'(\bar\theta)\tilde\theta + \sigma H'_\Gamma(h_0)\tilde h = \bar g_\theta - r_\theta(\tilde\theta, \bar\theta) - \sigma r_h(\tilde h, \bar h)$$

where

$$\bar{g}_\theta = -\big(\varphi(\bar{\theta}) + \sigma H_\Gamma(\bar{h})\big) \in {}_0\mathbb{F}^\theta_{h,\mu}(\mathbb{R}_+)$$

by the compatibility condition **(C1)**. Now we can rewrite the problem abstractly as

$$L_1\tilde{z} = N_1(\tilde{z},\bar{z}), \quad \tilde{z}(0) = 0, \tag{9.3}$$

with $N_1 : {}_0\mathbb{E}^1_\mu(a)\times\mathbb{E}^1_\mu(\infty) \to {}_0\mathbb{F}^1_\mu(a)$, and $L_1 : \mathbb{E}^1_\mu(a) \to \mathbb{F}^1_\mu(a)$ linear and bounded, given by

$$L_1\tilde{z} = \begin{bmatrix} \partial_t\tilde{\theta} + \mathcal{A}_\theta(\theta_0,h_0) : \nabla^2\tilde{\theta} \\ \varphi'(\bar{\theta})\tilde{\theta} - \sigma\,\mathcal{C}_\Sigma(h_0)\tilde{h} \\ \partial_t\tilde{h} + [\![\mathcal{B}_\theta(\theta_0,h_0)\nabla\tilde{\vartheta}]\!] \end{bmatrix},$$

where $\mathcal{C}_\Sigma(h_0)$ denotes the principal part of the curvature operator $-H'_\Gamma(h_0)$. The operator L_1 has maximal L_p-regularity by Section 6.6.

The nonlinearity N_1 is given by

$$N_1(\tilde{z},\bar{z}) = \begin{bmatrix} F_\theta(\theta,h) - \partial_t\bar{\theta} - \mathcal{A}_\theta(\theta,h) : \nabla^2\bar{\theta} + (\mathcal{A}_\theta(\theta_0,h_0) - \mathcal{A}_\theta(\theta,h)) : \nabla^2\tilde{\theta} \\ \bar{g}_\theta + r_\theta(\tilde{\theta},\bar{\theta}) + \sigma r_h(\tilde{h},\bar{h}) - \sigma(\mathcal{C}_\Sigma(h_0) + H'_\Gamma(\bar{h}))\tilde{h} \\ [\![(\mathcal{B}_\theta(\theta_0,h_0) - \mathcal{B}_\theta(\theta,h))\nabla\tilde{\theta} - \mathcal{B}_\theta(\theta,h)\nabla\bar{\theta}]\!] - \partial_t\bar{h} \end{bmatrix}.$$

Observe that

$$N_1(0,\bar{z}) = \begin{bmatrix} F_\theta(\bar{\theta},\bar{h}) - \partial_t\bar{\theta} - \mathcal{A}_\theta(\bar{\theta},\bar{h}) : \nabla^2\bar{\theta} \\ \bar{g}_\theta \\ -\partial_t\bar{h} - [\![\mathcal{B}_\theta(\bar{\theta},h)\nabla\bar{\theta}]\!] \end{bmatrix}$$

satisfies $|N_1(0,\bar{z})|_{\mathbb{F}_\mu(a)} \to 0$ as $a \to 0$.

1.2 The Two-Phase Navier-Stokes Problem with Surface Tension

In the sequel we assume Condition **(H2)** and the compatibility condition

$$(\textbf{C2}) \quad \operatorname{div} u_0 = 0 \ \text{ in } \Omega\setminus\Gamma_0, \quad [\![d(\theta_0)\partial_\nu\theta_0]\!],\ \mathcal{P}_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_{\Gamma_0}]\!] = 0 \ \text{ on } \Gamma_0.$$

The transformed problem **(P2)** reads as follows.

$$\begin{aligned} \partial_t u + \mathcal{A}_u(\theta,h) : \nabla^2 u + (I - M_1(h))\nabla\pi/\varrho &= F_u(u,\theta,h) && \text{in } \Omega\setminus\Sigma, \\ (I - M_1(h))\nabla\cdot u &= 0 && \text{in } \Omega\setminus\Sigma, \\ \partial_t\theta + \mathcal{A}_\theta(\theta,h) : \nabla^2\theta &= F_\theta(\theta,h) && \text{in } \Omega\setminus\Sigma, \\ u, \partial_\nu\theta &= 0 && \text{on } \partial\Omega, \\ [\![u]\!], [\![\theta]\!], [\![\mathcal{B}_\theta(\theta,h)\nabla\theta]\!] &= 0 && \text{on } \Sigma, \\ -[\![S(u,\theta,h)]\!]\nu_\Gamma + \big([\![\pi]\!] - \sigma H_\Gamma(h)\big)\nu_\Gamma &= 0 && \text{on } \Sigma, \\ \partial_t h - (u|\nu_\Sigma - M_0(h)\nabla_\Sigma h) &= 0 && \text{on } \Sigma, \\ h(0) = h_0 \text{ on } \Sigma, \quad u(0) = u_0,\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{9.4}$$

Note that here we used the abbreviations

$$\begin{aligned}\mathcal{A}_u(\theta,h) &= (\mu(\theta)/\varrho)(I-M_1(h)^{\mathsf T})(I-M_1(h)),\\ \mathcal{A}_\theta(\theta,h) &= (d(\theta)/\varrho\kappa(\theta))(I-M_1(h)^{\mathsf T})(I-M_1(h)),\\ \mathcal{B}_\theta(\theta,h) &= d(\theta)(1-M_1(h)^{\mathsf T})(\nu_\Sigma - M_0(h)\nabla_\Sigma h).\end{aligned}$$

The nonlinearities F_u and F_θ collect all lower order terms, i.e.,

$$\begin{aligned}\varrho F_u =& -\varrho u\cdot(I-M_1(h))\nabla u + \varrho m_0(h)\partial_t h\circ \Pi_\Sigma(\nu_\Sigma\cdot\nabla\theta)\\ &+\mu'(\theta)(I-M_1(h))\nabla\theta\cdot D(u,h)\\ &+\mu(\theta)\big((I-M_1(h))\nabla\cdot M_1(h)\nabla u + [\nabla u]^{\mathsf T} : (I-M_1(h))\nabla M_1(h)\\ &-(I-M_1(h))\nabla\otimes M_1(h):\nabla u\big),\end{aligned}$$

and

$$\begin{aligned}\varrho\kappa(\theta)F_\theta =& \varrho\kappa(\theta)m_0(h)\partial_t h\circ\Pi_\Sigma(\nu_\Sigma\cdot\nabla\theta) - \varrho\kappa(\theta)u\cdot(I-M_1(h))\nabla\theta\\ &+ d'(\theta)|(1-M_1(h))\nabla\theta|^2 - d(\theta)\big[(I-M_1(h))\nabla]M_1(h)\big)\cdot\nabla\theta + 2\mu(\theta)|D|^2.\end{aligned}$$

Note that $F_\theta(0,\theta,h)$ coincides with F_θ from the previous subsection. Furthermore, recall that

$$S = S(u,\theta,h) = 2\mu(\theta)D(u,h),\quad 2D(u,h) = (I-M_1(h))\nabla u + [\nabla u]^{\mathsf T}(I-M_1(h))^{\mathsf T}.$$

To obtain the abstract formulation of the problem, we choose as the system variable $z=(u,\theta,h,\pi,q)$, where $q=[\![\pi]\!]$ is a dummy variable which we introduce for convenience. The regularity space for z is

$$\begin{aligned}z\in\mathbb{E}^2_\mu(a) := \{z\in \mathbb{E}_{u,\mu}(J)\times\mathbb{E}_{\theta,\mu}(J)\times\mathbb{E}^u_{h,\mu}(J)\times\mathbb{E}_{\pi,\mu}(J)\times\mathbb{E}_{q,\mu}(J):\ [\![\pi]\!]=q,\\ [\![\theta]\!],[\![u]\!]=0 \text{ on } \Sigma,\ u,\partial_\nu\theta = 0 \text{ on } \partial\Omega\},\end{aligned}$$

where

$$\mathbb{E}_{u,\mu}(J)=\mathbb{E}_{\theta,\mu}(J)^n,\quad \mathbb{E}_{\pi,\mu}(J) = L_{p,\mu}(J;\dot H^1_p(\Omega\setminus\Sigma)),\quad \mathbb{E}_{q,\mu}(J)=\mathbb{F}^u_{h,\mu}(J).$$

Here we set

$$\mathbb{E}^u_{h,\mu}(J) = W^{2-1/2p}_{p,\mu}(J;L_p(\Sigma))\cap H^1_{p,\mu}(J;W^{2-1/p}_p(\Sigma))\cap L_{p,\mu}(J;W^{3-1/p}_p(\Sigma)),$$

which differs from the space for h in the previous subsection. Note that, according to Section 8.2, the time-trace space of (u,θ,h) in this case reads

$$\begin{aligned}X^2_{\gamma,\mu} = \{(u,\theta,h)\in W^{2\mu-2/p}_p(\Omega\setminus\Sigma)^{n+1}\times W^{2+\mu-2/p}_p(\Sigma) : [\![u]\!],[\![\theta]\!]=0 \text{ on } \Sigma,\\ u,\partial_\nu\theta = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

The data space $\mathbb{F}^2_\mu(a)$ is given by

$$\mathbb{F}^2_\mu(a) = \mathbb{F}_{u,\mu}(J) \times \mathbb{F}^2_{\pi,\mu}(J) \times \mathbb{F}_{\theta,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J)^{n+1} \times \mathbb{F}^\theta_{h,\mu}(J),$$

with

$$\mathbb{F}_{u,\mu}(J) = \mathbb{F}_{\theta,\mu}(J)^n, \quad \mathbb{F}^2_{\pi,\mu}(J) = H^1_{p,\mu}(J; {}_0\dot{H}^{-1}_p(\Omega)) \cap L_{p,\mu}(J; H^1_p(\Omega \setminus \Sigma)),$$

Next we define suitable extensions of $z_0 \in X^2_{\gamma,\mu}$ in the following way. We solve the diffusion problem

$$\begin{aligned} \partial_t \bar{u} - \Delta \bar{u} &= 0 && \text{in } \Omega, \\ \bar{u} &= 0 && \text{on } \partial\Omega, \\ \bar{u}(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

to obtain a function $\bar{u} \in \mathbb{E}_{u,\mu}(\mathbb{R}_+)$. Also we define $\bar{\theta} \in \mathbb{E}_{\theta,\mu}(\mathbb{R}_+)$ as in the previous subsection. Next we extend the initial values h_0 and

$$h_1 := u_0 \cdot (\nu_\Sigma - M_0(h_0)\nabla_\Sigma h_0) \in W^{2\mu-3/p}_p(\Sigma)$$

as in Section 8.2.2 to obtain a function $\bar{h} \in \mathbb{E}^u_{h,\mu}(\mathbb{R}_+)$ with initial values $\bar{h}(0) = h_0$ and $\partial_t \bar{h}(0) = h_1$. Finally, we extend the pressure jump q_0 defined by

$$q_0 := \sigma H_\Gamma(h_0) + ([\![S(u_0, \theta_0, h_0]\!]\nu_\Gamma(h_0)|\nu_\Gamma(h_0)) \in W^{2\mu-1-3/p}_p(\Sigma)$$

by means of

$$\bar{q} = e^{-(I-\Delta_\Sigma)t} q_0 \in \mathbb{E}_{q,\mu}(\mathbb{R}_+),$$

and define $\bar{\pi} \in \mathbb{E}_{\pi,\mu}(\mathbb{R}_+)$ as the solution of the elliptic transmission problem

$$\begin{aligned} \Delta \bar{\pi} &= 0 && \text{in} && \Omega \setminus \Sigma, \\ \partial_\nu \bar{\pi} &= 0 && \text{on} && \partial\Omega, \\ [\![\partial_\nu \bar{\pi}]\!] = 0, \quad [\![\bar{\pi}]\!] &= \bar{q} && \text{on} && \Sigma, \end{aligned}$$

see Proposition 8.6.2, We denote the projection onto mean value zero by P_0. Then with $\bar{z} = (\bar{u}, \bar{\theta}, \bar{h}, \bar{\pi}, \bar{q})$, we decompose as in the previous section $z = \bar{z} + \tilde{z}$, and obtain the abstract equation

$$L_2 \tilde{z} = N_2(\tilde{z}, \bar{z}), \quad \tilde{z}(0) = 0,$$

with $L_2 : \mathbb{E}^2_\mu(a) \to \mathbb{F}^2_\mu(a)$ linear and bounded, $N_2 : {}_0\mathbb{E}^2_\mu(a) \times \mathbb{E}^2_\mu(\infty) \to {}_0\mathbb{F}^2_\mu(a)$. Here L_2 is given by

$$L_2 \tilde{z} = \left[\begin{array}{l} \partial_t \tilde{u} + \mathcal{A}_u(\theta_0, h_0) : \nabla^2 \tilde{u} + (1 - M_1(h_0))\nabla \tilde{\pi}/\varrho \\ (I - P_0 M_1(h_0))\nabla \cdot \tilde{u}) \\ \partial_t \tilde{\theta} + \mathcal{A}_\theta(\theta_0, h_0) : \nabla^2 \tilde{\theta} \\ -[\![S(\tilde{u}, \theta_0, h_0)]\!]\nu_\Sigma + (\tilde{q} + \sigma \mathcal{C}_\Sigma(h_0) : \nabla^2_\Sigma \tilde{h})\nu_\Sigma \\ [\![\mathcal{B}_\theta(\theta_0, h_0)\nabla \tilde{\theta}]\!] \\ \partial_t \tilde{h} - \tilde{u} \cdot (\nu_\Sigma - M_0(h_0)\nabla_\Sigma \bar{h}) + \bar{u} \cdot M_0(h_0)\nabla_\Sigma \tilde{h} \end{array} \right].$$

Note that the temperature decouples completely from the problem for (u, π, h), it has maximal L_p-regularity by Section 6.5. The remaining problem for (u, π, h) has been analyzed in Chapter 8 for the case $h_0 = 0$. There, maximal L_p-regularity has been shown for $(h_0, h_1) = 0$ which, by perturbation, extends to nontrivial h_0 with small norm in $C^1(\Sigma)$, and also to arbitrary h_1 provided the time interval $J = (0, a)$ is small. Observe that in the part for h we cannot replace $\bar{h}$ by h_0 everywhere, as h_0 does not have enough regularity.

The nonlinearity N_2 reads

$$N_2 = \begin{bmatrix} F_u(u,\theta,h) - \partial_t\bar{u} - (I - M_1(h))\nabla\bar{\pi}/\varrho + (M_1(h) - M_1(h_0))\nabla\tilde{\pi} \\ \quad +(\mathcal{A}_u(\theta_0,h_0) - \mathcal{A}_u(\theta,h)) : \nabla^2\tilde{u} - \mathcal{A}_u(\theta,h) : \nabla^2\bar{u} \\ P_0(M_1(h) - M_1(h_0))\nabla\cdot\tilde{u} + P_0(M_1(h) - I)\nabla\cdot\bar{u} \\ F_\theta(u,\theta,h) - \partial_t\bar{\theta} - (\mathcal{A}_\theta(\theta_0,h_0) - \mathcal{A}_\theta(\theta,h)) : \nabla^2\tilde{\theta} - \mathcal{A}(\theta,h) : \nabla^2\bar{\theta} \\ \tilde{T}_0 M_0(h)\nabla_\Sigma h + \big([\![S(u,\theta,h) - S(\tilde{u},\theta_0,h_0) - \bar{\pi}]\!] \\ \quad +\sigma(H_\Gamma(\bar{h}) + H'_\Gamma(\bar{h}) - H'_\Gamma(h_0))\tilde{h} + r_h(\bar{h},\tilde{h})\big)\nu_\Gamma(h)/\beta \\ [\![(\mathcal{B}_\theta(\theta_0,h_0) - \mathcal{B}_\theta(\theta,h))\nabla\tilde{\theta} - \mathcal{B}(\theta,h)\nabla\bar{\theta} \\ -\partial_t\bar{h} + \bar{u}\cdot(\nu_\Sigma - M_0(h)\nabla_\Sigma\bar{h}) + \bar{u}\cdot(M_0(h_0) - M_0(h))\nabla_\Sigma\tilde{h} \\ \quad +\tilde{u}\cdot((M_0(h_0) - M_0(h))\nabla_\Sigma\bar{h} - M_0(h)\nabla_\Sigma\tilde{h}) \end{bmatrix};$$

here we employed the abbreviation

$$\tilde{T}_0 = -[\![S(\tilde{u},\theta_0,h_0) - \tilde{\pi}]\!] - \sigma H'_\Gamma(h_0)\tilde{h}.$$

Note that

$$N_2(0,\bar{z}) = \begin{bmatrix} F_u(\bar{u},\bar{\theta},\bar{h}) - \partial_t\bar{u} - (I - M_1(\bar{h}))\nabla\bar{\pi}/\rho - \mathcal{A}_u(\bar{\theta},\bar{h}) : \nabla^2\bar{u} \\ P_0((M_1(\bar{h}) - I)\nabla\cdot\bar{u}) \\ F_\theta(\bar{u},\bar{\theta},\bar{h}) - \partial_t\bar{\theta} - \mathcal{A}(\bar{\theta},\bar{h}) : \nabla^2\bar{\theta} \\ \big([\![S(\bar{u},\bar{\theta},\bar{h}) - \bar{\pi}]\!] + \sigma H_\Gamma(\bar{h})\nu_\Gamma(\bar{h})/\beta \\ -[\![\mathcal{B}(\bar{\theta},\bar{h})\nabla\bar{\theta} \\ -\partial_t\bar{h} + \bar{u}\cdot(\nu_\Sigma - M_0(\bar{h})\nabla_\Sigma\bar{h}) \end{bmatrix}.$$

Then we see that $|N_2(0,\bar{z})|_{\mathbb{F}_\mu(a)} \to 0$ as $a \to 0$.

1.3 Phase Transitions: Equal Densities

In the sequel we assume Condition **(H3)** and the compatibility condition

$$\begin{aligned} &(\mathbf{C3}) \quad \varrho[\![\psi(\theta_0)]\!] + \sigma H_{\Gamma_0} = 0 \ \text{ on } \Gamma_0, \quad [\![d(\theta_0)\partial_\nu\theta_0]\!] \in W_p^{2-6/p}(\Gamma_0), \\ &\qquad\quad \operatorname{div} u_0 = 0 \ \text{ in } \Omega\setminus\Gamma_0, \qquad\qquad \mathcal{P}_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_{\Gamma_0}]\!] = 0 \ \text{ on } \Gamma_0. \end{aligned}$$

Here we have $\varrho_1 = \varrho_2 = 1$ w.l.o.g. and we may express the phase flux j_Σ by

$$j_\Sigma = [\![\mathcal{B}_\theta(\theta,h)\nabla\theta]\!],$$

insert it into the V_Γ-equation, and the Gibbs-Thomson relation into interface stress balance to the result

$$
\begin{aligned}
\partial_t u + \mathcal{A}_u(\theta,h) : \nabla^2 u + (I - M_1(h))\nabla\pi &= F_u(u,\theta,h) && \text{in } \Omega\setminus\Sigma,\\
(I - M_1(h))\nabla\cdot u &= 0 && \text{in } \Omega\setminus\Sigma,\\
\partial_t\theta + \mathcal{A}_\theta(\theta,h) : \nabla^2\theta &= F_\theta(\theta,h) && \text{in } \Omega\setminus\Sigma,\\
u, \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\
[\![u]\!], [\![\theta]\!] &= 0 && \text{on } \Sigma,\\
-[\![S(u,\theta,h)]\!]\nu_\Gamma + \big([\![\pi]\!] + \varphi(\theta)\big)\nu_\Gamma &= 0 && \text{on } \Sigma,\\
\varphi(\theta) + \sigma H_\Gamma(h) &= 0 && \text{on } \Sigma,\\
\partial_t h - (u|\nu_\Sigma - M_0(h)\nabla_\Sigma h) + [\![\mathcal{B}_\theta(\theta,h)\nabla\theta]\!] &= 0 && \text{on } \Sigma,\\
h(0) = h_0 \text{ on } \Sigma, \quad u(0) = u_0,\ \theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}
$$

Here $(\mathcal{A}_u, \mathcal{A}_\theta, \mathcal{B}_\theta)$ are as before. The extensions $(\bar{u}, \bar{\theta})$ are as in the previous subsection, whereas the extension $\bar{h}$ is that from Section 9.1.1. As a result we obtain again a problem of the form

$$L_3\tilde{z} = N_3(\tilde{z}, \bar{z}), \quad \tilde{z} = 0,$$

where L_3 is defined by

$$
L_3\tilde{z} = \begin{bmatrix}
\partial_t\tilde{u} + \mathcal{A}_u(\theta_0,h_0) : \nabla^2\tilde{u} + (1 - M_1(h_0))\nabla\tilde{\pi}\\
(I - P_0M_1(h_0))\nabla\cdot\tilde{u})\\
\partial_t\tilde{\theta} + \mathcal{A}_\theta(\theta_0,h_0) : \nabla^2\tilde{\theta}\\
-[\![S(\tilde{u},\theta_0,h_0)]\!]\nu_\Sigma + [\![\tilde{\pi}]\!]\nu_\Sigma\\
\varphi'(\bar{\theta})\tilde{\theta} - \sigma\mathcal{C}_\Sigma(h_0)\tilde{h}\\
\partial_t\tilde{h} + [\![\mathcal{B}_\theta(\theta_0,h_0)\nabla\tilde{\theta}]\!]
\end{bmatrix}.
$$

Note that the term $\varphi(\theta)$ in the stress balance on the interface as well the term $u\cdot\nu_\Gamma$ in the equation for h are lower order and can be subsumed in N_3. Here we define with $z = (u,\theta,h,\pi,q)$ the regularity space as

$$
\begin{aligned}
z \in \mathbb{E}^3_\mu(a) := \{z \in \mathbb{E}_{u,\mu}(J) \times \mathbb{E}_{\theta,\mu}(J) \times \mathbb{E}^\theta_{h,\mu}(J) \times \mathbb{E}_{\pi,\mu}(J) \times \mathbb{E}_{q,\mu}(J) :\\
[\![\theta]\!], [\![u]\!] = 0,\ [\![\pi]\!] = q \text{ on } \Sigma,\ u, \partial_\nu\theta = 0 \text{ on } \partial\Omega\},
\end{aligned}
$$

and the space of data by

$$\mathbb{F}^3_\mu(a) = \mathbb{F}_{u,\mu}(J) \times \mathbb{F}^2_{\pi,\mu}(J) \times \mathbb{F}_{\theta,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J) \times \mathbb{F}^\theta_{h,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J).$$

Observe that up to lower order terms, the problems for (u,π) and (θ,h) decouple. Therefore, for (u,π) we have at the linear level a two-phase Stokes problem on a fixed domain, and for (θ,h) we have the same principal part as in Section 9.1.1. By the previous subsections, L_3 has maximal regularity and $L_3 : {}_0\mathbb{E}_\mu(a) \to {}_0\mathbb{F}_\mu(a)$ is an isomorphism with $|L_3^{-1}|$ uniformly bounded for $a \in (0,1]$. The nonlinearity

N_3 is similar to N_2 and N_1. In particular, we have again $|N_3(0,\bar{z})|_{\mathbb{F}_\mu(a)} \to 0$ as $a \to 0$.

1.4 Phase Transitions: Different Densities
In the sequel we assume Condition **(H4)** and the compatibility condition

$$\begin{aligned}
&(\mathbf{C4})\quad \operatorname{div} u_0 = 0 \ \text{ in } \Omega\setminus\Gamma_0, \quad \mathcal{P}_{\Gamma_0}[\![u_0]\!] = 0,\\
&\qquad\quad \mathcal{P}_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_{\Gamma_0}]\!],\ l(\theta_0)[\![u_0\cdot\nu_{\Gamma_0}]\!] + [\![1/\varrho]\!][\![d(\theta_0)\partial_\nu\theta_0]\!] = 0 \ \text{ on } \Gamma_0,
\end{aligned}$$

As shown in Chapter 1, with $[\![\varrho]\!] \neq 0$, we may eliminate j_Σ to obtain

$$j_\Sigma(u,h) = [\![u\cdot\nu_\Sigma]\!]/\beta(h)[\![1/\varrho]\!], \quad V_\Gamma = \beta(h)\partial_t h = [\![\varrho u\cdot\nu_\Gamma]\!]/[\![\varrho]\!].$$

Then the transformed problem **(P4)** becomes

$$\begin{aligned}
\partial_t u + \mathcal{A}_u(\theta,h):\nabla^2 u + (I - M_1(h))\nabla\pi/\varrho = F_u(u,\theta,h) &\quad \text{in } \Omega\setminus\Sigma,\\
(I - M_1(h))\nabla\cdot u = 0 &\quad \text{in } \Omega\setminus\Sigma,\\
\partial_t\theta + \mathcal{A}_\theta(\theta,h):\nabla^2\theta = F_\theta(\theta,h) &\quad \text{in } \Omega\setminus\Sigma,\\
u, \partial_\nu\theta = 0 &\quad \text{on } \partial\Omega,\\
[\![\theta\eta(\theta)]\!]j_\Sigma - [\![d(\theta)\nu_\Gamma\cdot(I - M_1(h))\nabla\theta]\!] = 0 &\quad \text{on } \Sigma,\\
\mathcal{P}_\Gamma[\![u]\!], [\![\theta]\!] = 0 &\quad \text{on } \Sigma,\\
-[\![S(u,\theta,h)]\!]\nu_\Gamma + \big([\![\pi]\!] + [\![1/\varrho]\!]j_\Sigma^2 - \sigma H_\Gamma(h)\big)\nu_\Gamma = 0 &\quad \text{on } \Sigma,\\
\varphi(\theta) + [\![1/2\varrho^2]\!]j_\Sigma^2 - [\![S(u,\theta,h)\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] + [\![\pi/\varrho]\!] = 0 &\quad \text{on } \Sigma,\\
\partial_t h - [\![(\varrho u|\nu_\Sigma - M_0(h)\nabla_\Sigma h)]\!]/[\![\varrho]\!] = 0 &\quad \text{on } \Sigma,\\
h(0) = h_0 \text{ on } \Sigma, \quad u(0) = u_0,\ \theta(0) = \theta_0 &\quad \text{in } \Omega.
\end{aligned}\tag{9.5}$$

Here the heat problem is only weakly coupled to the system for (u,π,h). However, the system for (u,π,h) leads to the asymmetric Stokes problem, which differs from the one considered above. The regularity of h is the same as in Section 9.1.2; the problem is *velocity dominated*. We proceed as before, extending the initial values $(u_0,\theta_0,h_0) \in X^4_{\gamma,\mu}$ as in Section 9.1.2 to obtain $(\bar{u},\bar{\theta},\bar{h})$. Furthermore, we solve the Gibbs-Thomson relation combined with the normal component of the stress transmission condition on the interface at time $t = 0$, to obtain unique initial values q_{j0} for the pressures π_j on the interface. We extend these by defining

$$\bar{q}_j = e^{-(1-\Delta_\Sigma)t}q_{j0}, \quad t > 0, \ \ j = 1,2,$$

and then solve the two elliptic problems

$$\begin{aligned}
\Delta\bar{\pi}_2 = 0 &\quad \text{in } \Omega_2,\\
\partial_\nu\bar{\pi}_2 = 0 &\quad \text{on } \partial\Omega,\\
\bar{\pi}_2 = \bar{q}_2 &\quad \text{on } \Sigma,
\end{aligned}$$

and

$$\begin{aligned}\Delta \bar{\pi}_1 &= 0 \quad \text{in } \Omega_1,\\ \bar{\pi}_1 &= \bar{q}_1 \quad \text{on } \Sigma.\end{aligned}$$

From the above construction it is evident that $\bar{z} \in \mathbb{E}^4_\mu(\infty)$ trivializes the initial conditions and resolves the compatibilities. The relevant variables are here $z = (u, \theta, h, \pi, q_1, q_2)$, where q_j denote the surface pressures on Σ, and the solution space $z \in \mathbb{E}^4_\mu(a)$ is

$$\begin{aligned}\mathbb{E}^4_\mu(a) :=\{z \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\theta,\mu}(J) \times \mathbb{E}^u_{h,\mu}(J) \times \mathbb{E}_{\pi,\mu}(J) \times \mathbb{E}_{q,\mu}(J) \times \mathbb{E}_{q,\mu}(J) :\\ [\![\theta]\!] = 0, \;\; \pi_j = q_j \text{ on } \Sigma, \;\; u, \partial_\nu \theta = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

The image space in this case will be

$$\mathbb{F}^4_\mu(a) := \mathbb{F}_{u,\mu}(J) \times \mathbb{F}^4_{\pi,\mu}(J) \times \mathbb{F}_{\theta,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J) \times \mathcal{P}_\Sigma \mathbb{F}^\theta_{h,\mu}(J)^n \times \mathbb{F}^u_{h,\mu}(J)^{n+1} \times \mathbb{F}^\theta_{h,\mu}(J),$$

with

$$\mathbb{F}^4_{\pi,\mu}(J) = H^1_{p,\mu}(J; H^{-1}_{p,\partial\Omega}(\Omega \setminus \Sigma)) \cap L_{p,\mu}(J; H^1_p(\Omega \setminus \Sigma)).$$

Compared to the previous cases, the equation for h is different from that in Section 9.1.2, but it has a similar structure and hence needs no additional comments. On the other hand, the transmission condition $[\![u]\!] = 0$ is replaced by $\mathcal{P}_\Gamma [\![u]\!] = 0$, which by application of $\mathcal{P}_\Sigma$ leads to the decomposition

$$\mathcal{P}_\Sigma [\![u]\!] + \beta(h) M_0 \nabla_\Sigma h [\![\nu_\Gamma(h) \cdot u]\!] = 0.$$

This equation is linearized in the same way as the equation for h. Furthermore, note that the terms $\varphi(\theta)$ and $[\![1/2\varrho^2]\!] j_\Sigma^2$ in the Gibbs-Thomson law are lower order. The remaining part is linearized in the same way as the stress boundary condition.

As a result we obtain again a problem of the form

$$L_4 \tilde{z} = N_4(\tilde{z}, \bar{z}), \quad \tilde{z} = 0,$$

where L_4 is defined by

$$L_4 \tilde{z} = \begin{bmatrix} \partial_t \tilde{u} + \mathcal{A}_u(\theta_0, h_0) \nabla^2 \tilde{u} + (1 - M_1(h_0)) \nabla \tilde{\pi} \\ (I - M_1(h_0)) \nabla \cdot \tilde{u}) \\ \partial_t \tilde{\theta} + \mathcal{A}_\theta(\theta_0, h_0) : \nabla^2 \tilde{\theta} \\ [\![\mathcal{B}_\theta(\theta_0, h_0) \nabla \tilde{\theta}]\!] \\ \mathcal{P}_\Sigma [\![\tilde{u}]\!] + [\![\tilde{u} \cdot \nu_\Sigma]\!] M_0(h_0) \nabla_\Sigma \bar{h} + [\![\bar{u} \cdot \nu_\Sigma]\!] M_0(h_0) \nabla_\Sigma \tilde{h} \\ -[\![S(\tilde{u}, \theta_0, h_0)]\!] \nu_\Sigma + ([\![\tilde{\pi}]\!] + \sigma \mathcal{C}_\Sigma(h_0) \tilde{h}) \nu_\Sigma \\ -[\![S(\tilde{u}, \theta_0, h_0) \nu_\Sigma \cdot \nu_\Sigma / \varrho]\!] + [\![\tilde{\pi}/\varrho]\!] \\ \partial_t \tilde{h} - ([\![\varrho \tilde{u} \cdot (\nu_\Sigma - M_0(h_0) \nabla_\Sigma \bar{h})]\!] - [\![\varrho \bar{u} \cdot M_0(h_0) \nabla_\Sigma \tilde{h}]\!]) / [\![\varrho]\!] \end{bmatrix}.$$

On the linear level we have an asymmetric Stokes problem for (u, π, h) and a transmission problem for θ. Maximal L_p-regularity of the transmission problem follows from Section 6.5, and the asymmetric Stokes problem has been studied in Chapter 8. As shown there, it has maximal L_p-regularity in case $(h_0, h_1) = 0$. By perturbation, this extends to nontrivial h_0 which are small in $C^1(\Sigma)$, as well as to arbitrary h_1 provided the interval $J = (0, a)$ is small.

1.5 Phase Transitions and Marangoni Forces: Different Densities
In the sequel we assume Condition **(H6)** and the compatibility condition

$$\textbf{(C6)} \quad \operatorname{div} u_0 = 0 \ \text{ in } \Omega \setminus \Gamma_0, \quad \mathcal{P}_{\Gamma_0}[\![u_0]\!] = 0,$$
$$2\mathcal{P}_{\Gamma_0}[\![\mu(\theta_0)D(u_0)\nu_{\Gamma_0}]\!] + \sigma'(\theta_0)\nabla_{\Gamma_0}\theta_0 = 0 \ \text{ on } \Gamma_0.$$

We eliminate j_Σ as before and obtain the transformed problem **(P6)**

$$\begin{aligned}
\partial_t u + \mathcal{A}_u(\theta,h) : \nabla^2 u + (I - M_1(h))\nabla\pi/\varrho &= F_u(u,\theta,h) && \text{in } \Omega\setminus\Sigma,\\
(I - M_1(h))\nabla\cdot u &= 0 && \text{in } \Omega\setminus\Sigma,\\
\partial_t\theta + \mathcal{A}_\theta(\theta,h) : \nabla^2\theta &= F_\theta(\theta,h) && \text{in } \Omega\setminus\Sigma,\\
u, \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\
\partial_t\theta_\Sigma + \mathcal{A}_{\theta_\Sigma}(\theta_\Sigma,h) : \nabla_\Sigma^2\theta_\Sigma &= F_{\theta_\Sigma}(u,\theta,h,\theta_\Sigma) && \text{on } \Sigma,\\
\theta = \theta_\Sigma, \ \ \mathcal{P}_\Gamma[\![u]\!], [\![\theta]\!] &= 0 && \text{on } \Sigma,\\
-[\![S(u,\theta,h)]\!]\nu_\Gamma + \big([\![\pi]\!] + [\![1/\varrho]\!]j_\Sigma^2 - \sigma(\theta_\Sigma)H_\Gamma(h)\big)\nu_\Gamma &= \sigma'(\theta_\Sigma)\nabla_\Gamma\theta_\Gamma && \text{on } \Sigma,\\
\varphi(\theta) + [\![1/2\varrho^2]\!]j_\Sigma^2 - [\![S(u,\theta,h)\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] + [\![\pi/\varrho]\!] &= 0 && \text{on } \Sigma,\\
\partial_t h + [\![\varrho u\cdot(\nu_\Sigma - M_0(h)\nabla_\Sigma h)]\!]/[\![\varrho]\!] &= 0 && \text{on } \Sigma,\\
h(0) = h_0 \text{ on } \Sigma, \ \ u(0) = u_0, \ \ \theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned} \tag{9.6}$$

The differential operators $(\mathcal{A}_u, \mathcal{A}_\theta, \mathcal{B}_\theta)$ are defined as previously, and with Section 2.2, $\mathcal{A}_{\theta_\Sigma}$ is given by

$$\mathcal{A}_{\theta_\Sigma} : \nabla_\Sigma^2 = -(d_\Gamma(\theta_\Sigma)/\kappa_\Gamma(\theta_\Sigma))M_0(h)P_\Gamma(h)M_0(h) : \nabla_\Sigma^2.$$

Here we employed the relation

$$\frac{D}{Dt}\theta_\Sigma = \partial_t\theta_\Sigma + (I - M_1^{\mathsf{T}}(h))u_\Sigma\cdot\nabla_\Sigma\theta_\Sigma,$$

taken from Section 1.3.2. $F_{\theta_\Sigma} = F_{\theta_\Sigma}(u,\theta,\theta_\Sigma,h)$ is defined by

$$\begin{aligned}
&\kappa_\Gamma(\theta_\Sigma)F_{\theta_\Sigma}\\
&= M_0(h)P_\Gamma(h)\nabla_\Sigma\cdot(d_\Gamma(\theta_\Sigma)P_\Gamma(h)M_0(h))\nabla_\Sigma\theta_\Sigma - \kappa_\Gamma(\theta_\Sigma)(I - M_1^{\mathsf{T}}(h))u_\Sigma\cdot\nabla_\Sigma\theta_\Sigma\\
&\quad + \theta_\Sigma\sigma'(\theta_\Sigma)(P_\Gamma(h)M_0(h)\nabla_\Sigma\cdot\mathcal{P}_\Sigma u - H_\Gamma(h)V_\Gamma\nu_\Gamma) - [\![\theta\eta(\theta)]\!]j_\Sigma - [\![\mathcal{B}_\theta(\theta,h)\nabla\theta]\!],
\end{aligned}$$

it collects all lower order terms. Recall that

$$j_\Sigma = [\![u\cdot\nu_\Gamma]\!]/[\![1/\varrho]\!], \quad V_\Gamma = [\![\varrho u\cdot\nu_\Gamma]\!]/[\![\varrho]\!].$$

We extend (u_0, h_0) as in Section 9.1.4, but we have to be more careful with θ_0 due to the dynamic equation for θ_Σ on Σ. We first extend $\theta_{\Sigma 0} = \theta_0|_\Sigma$ on Σ by $\bar\theta_\Sigma = e^{-(1-\Delta_\Sigma)t}\theta_{\Sigma 0}$ and then solve the two one-phase parabolic problems

$$\begin{aligned}
\partial_t\bar\theta - \Delta\bar\theta &= 0 && \text{in } \Omega\setminus\Sigma,\\
\partial_\nu\bar\theta &= 0 && \text{on } \partial\Omega,\\
\bar\theta &= \bar\theta_\Sigma && \text{on } \Sigma,\\
\bar\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}$$

Observe that the heat equation on Σ decouples to highest order from the remaining equations, and the heat problem in $\Omega\setminus\Sigma$ decouples from the system for (u,π,h). The latter is as in the previous subsection governed by an asymmetric Stokes problem. The solution space for $z=(u,\theta,\theta_\Sigma,h,\pi,q_1,q_2)$ is here defined by

$$\begin{aligned}\mathbb{E}^6_\mu(a) :=\{z\in \mathbb{E}_{u,\mu}\times\mathbb{E}_{\theta,\mu}(J)\times\mathbb{E}_{\theta_\Sigma,\mu}(J)\times\mathbb{E}^u_{h,\mu}(J)\times\mathbb{E}_{\pi,\mu}(J)\times\mathbb{F}^u_{u,\mu}(J)^2 :\\ \mathcal{P}_\Sigma[\![u]\!],[\![\theta]\!]=0,\ \theta=\theta_\Sigma,\ \pi_{|\partial\Omega_j}=q_j \text{ on } \Sigma,\ \ u,\partial_\nu\theta=0 \text{ on } \partial\Omega\},\end{aligned}$$

with

$$\mathbb{E}_{\theta_\Sigma,\mu}(J)=H^1_{p,\mu}(J;W^{-1/p}_p(\Sigma))\cap L_{p,\mu}(J;W^{2-1/p}_p(\Sigma)).$$

For the space of data we may take here

$$\mathbb{F}^6_\mu(a)=\mathbb{F}_{u,\mu}(J)\times\mathbb{F}^4_{\pi,\mu}(J)\times\mathbb{F}_{\theta,\mu}(J)\times\mathbb{F}_{\theta_\Sigma,\mu}(J)\times\mathcal{P}_\Sigma\mathbb{F}^\theta_{h,\mu}(J)^n\times\mathbb{F}^u_{h,\mu}(J)^{n+1}\times\mathbb{F}^\theta_{h,\mu}(J),$$

where

$$\mathbb{F}_{\theta_\Sigma,\mu}(J)=L_{p,\mu}(J;W^{-1/p}_p(\Sigma)).$$

This way we obtain the abstract form

$$L_6\tilde z=N_6(\tilde z,\bar z),\quad \tilde z=0,$$

with $N_6: {}_0\mathbb{E}^6_\mu(a)\times\mathbb{E}^6_\mu(\infty)\to {}_0\mathbb{F}^6_\mu(a)$ and $L_6:\mathbb{E}^6_\mu(a)\to\mathbb{F}^6_\mu(a)$ linear and bounded. More precisely, L_6 is defined by

$$L_6\tilde z=\begin{bmatrix}\partial_t\tilde u+\mathcal{A}_u(\theta_0,h_0)\nabla^2\tilde u+(1-M_1(h_0))\nabla\tilde\pi\\ (I-M_1(h_0))\nabla\cdot\tilde u)\\ \partial_t\tilde\theta+\mathcal{A}_\theta(\theta_0,h_0):\nabla^2\tilde\theta\\ \partial_t\tilde\theta_\Sigma+\mathcal{A}_{\theta_\Sigma}(\theta_0,h_0):\nabla^2_\Sigma\tilde\theta_\Sigma\\ \mathcal{P}_\Sigma[\![\tilde u]\!]+[\![\tilde u\cdot\nu_\Sigma]\!]M_0(h_0)\nabla_\Sigma\bar h+[\![\bar u\cdot\nu_\Sigma]\!]M_0(h_0)\nabla_\Sigma\tilde h\\ -[\![S(\tilde u,\theta_0,h_0)]\!]\nu_\Sigma+([\![\tilde\pi]\!]+\sigma(\theta_0)\mathcal{C}_\Sigma(h_0))\nu_\Sigma-\sigma'(\theta_0)\nabla_\Sigma\tilde\theta_\Sigma\\ -[\![S(\tilde u,\theta_0,h_0)\nu_\Sigma\cdot\nu_\Sigma/\varrho]\!]+[\![\tilde\pi/\varrho]\!]\\ \partial_t\tilde h-([\![\varrho\tilde u\nu_\Sigma]\!]-[\![\varrho\tilde u\cdot M_0(h_0)\nabla_\Sigma\bar h]\!]-[\![\varrho\bar u\cdot M_0(h_0)\nabla_\Sigma\tilde h]\!])/[\![\varrho]\!]\end{bmatrix}.$$

As the operator for θ_Σ has maximal L_p-regularity by Section 6.3, that for θ has this property by Section 6.3, and the remaining asymmetric Stokes operator does so as we have seen in the previous subsection, we conclude that L_6 has maximal regularity, which shows that $L_6: {}_0\mathbb{E}^6_\mu(a)\to {}_0\mathbb{F}^6_\mu(a)$ is an isomorphism, with uniform bounds in $a\in(0,1]$.

1.6 Phase Transitions and Marangoni Forces: Equal Densities
In the sequel we assume Condition **(H5)** and the compatibility condition

$$\begin{aligned}\textbf{(C5)}\quad &[\![\psi(\theta_0)]\!]+\sigma(\theta_0)H_{\Gamma_0}=0 \ \text{ on } \Gamma_0,\quad \operatorname{div}u_0=0\ \text{ in } \Omega\setminus\Gamma_0,\\ &\mathcal{P}_{\Gamma_0}[\![2\mu(\theta_0)D(u_0)\nu_{\Gamma_0}]\!]+\sigma'(\theta_0)\nabla_{\Gamma_0}\theta_0=0\ \text{ on } \Gamma_0.\end{aligned}$$

Here we have once more $\varrho_1=\varrho_2=1$ w.l.o.g, and we solve for j_Γ according to $j_\Gamma=u\cdot\nu_\Gamma-V_\Gamma$, and insert this into the interface energy balance.

The case where undercooling is present is the simpler one, as both equations on the interface are dynamic equations, however it can be used as a guide. In particular, the Gibbs-Thomson identity

$$\gamma(\theta_\Gamma)V_\Gamma - \sigma(\theta_\Gamma)H_\Gamma(h) = \varphi(\theta_\Gamma)$$

can be understood as a *mean curvature flow* for the evolution of the surface, modified by physics.

If there is no undercooling, there is a hidden mean curvature flow which, however, is more complex. For the derivation, it is convenient to eliminate the time derivative of θ_Γ from the energy balance on the interface. In fact, differentiating the Gibbs-Thomson law w.r.t. time t and with $\lambda(s) = \varphi(s)/\sigma(s)$ we obtain

$$\lambda'(\theta_\Gamma)\frac{D_n}{Dt}\theta_\Gamma + H'_\Gamma(h)V_\Gamma = 0 \quad \text{on } \Gamma(t),$$

hence substitution into surface energy balance yields with

$$T_\Gamma(\theta_\Gamma) := \omega_\Gamma(\theta_\Gamma) - H'_\Gamma(h), \quad \omega_\Gamma(\theta_\Gamma) := \frac{\lambda'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}(l(\theta_\Gamma) - l_\Gamma(\theta_\Gamma)\lambda(\theta_\Gamma)) \tag{9.7}$$

the relation

$$\begin{aligned} T_\Gamma(\theta_\Gamma)V_\Gamma = \frac{\lambda'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\big\{&\mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma) - \kappa_\Gamma u_\Gamma\nabla_\Gamma\theta_\Gamma + [\![d(\theta)\partial_\nu\theta]\!] \\ &+ l_\Gamma \mathrm{div}_\Gamma u + l_0(\theta)u\cdot\nu_\Gamma\big\}. \end{aligned} \tag{9.8}$$

As V_Γ should be determined only by the state of the system and should not depend on time derivatives of other variables, this indicates that the problem without undercooling is not well-posed if the operator $T_\Gamma(\theta_\Gamma)$ is not invertible in $L_2(\Gamma)$, as V_Γ might not be well-defined. On the other hand, if $T_\Gamma(\theta_\Gamma)$ is invertible, then

$$\begin{aligned} V_\Gamma = T_\Gamma^{-1}\frac{\lambda'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\big\{&\mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma) - \kappa_\Gamma u_\Gamma\nabla_\Gamma\theta_\Gamma + [\![d(\theta)\partial_\nu\theta]\!] \\ &+ l_\Gamma \mathrm{div}_\Gamma u + l_0(\theta)u\cdot\nu_\Gamma\big\}. \end{aligned} \tag{9.9}$$

uniquely determines the interfacial velocity V_Γ, gaining two derivatives in space, and showing that all terms on the right-hand side of surface energy balance are of lower order. Note that

$$\omega_\Gamma(s) = s\sigma(s)[\lambda'(s)]^2/\kappa_\Gamma(s) \geq 0 \quad \text{in } (0,\theta_c), \tag{9.10}$$

and $\omega_\Gamma(s) = 0$ if and only if $\lambda'(s) = 0$. The well-posedness condition appears to be more complex, compared to the case $\kappa_\Gamma \equiv 0$.

Going one step further, taking the surface gradient of the Gibbs-Thomson relation yields the identity

$$\kappa_\Gamma(\theta_\Gamma)V_\Gamma - d_\Gamma(\theta_\Gamma)H_\Gamma = \kappa_\Gamma(\theta_\Gamma)\{f_\Gamma(\theta_\Gamma) + F_\Gamma(u,\theta,\theta_\Gamma)\}, \tag{9.11}$$

as will be shown below. Here the function f_Γ is the antiderivative of $\lambda(d_\Gamma/\kappa_\Gamma)'$ vanishing at $s=\theta_m$, and F_Γ is nonlocal in space and of lower order. So also in the case where undercooling is absent we obtain a *mean curvature flow*, modified by physics.

To derive (9.11), note that

$$\begin{aligned}
&\frac{\lambda'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma)\\
&\qquad=\frac{1}{\kappa_\Gamma(\theta_\Gamma)}\mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\lambda(\theta_\Gamma))-\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\lambda''(\theta_\Gamma)|\nabla_\Gamma\theta_\Gamma|^2\\
&\qquad=\mathrm{div}_\Gamma\Big(\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\nabla_\Gamma\lambda(\theta_\Gamma)\Big)-\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\{\lambda''(\theta_\Gamma)-\lambda'(\theta_\Gamma)\frac{\kappa_\Gamma'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\}|\nabla_\Gamma\theta_\Gamma|^2\\
&\qquad=\Delta_\Gamma g_\Gamma(\theta_\Gamma)-\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\{\lambda''(\theta_\Gamma)-\lambda'(\theta_\Gamma)\frac{\kappa_\Gamma'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\}|\nabla_\Gamma\theta_\Gamma|^2,
\end{aligned}$$

where g_Γ denotes the antiderivative of $d_\Gamma\lambda'/\kappa_\Gamma$ with $g_\Gamma(\theta_m)=0$. We note that by a partial integration

$$g_\Gamma(s)=\lambda(s)\frac{d_\Gamma(s)}{\kappa_\Gamma(s)}-\int_{\theta_m}^s\lambda(\tau)\Big(\frac{d_\Gamma}{\kappa_\Gamma}\Big)'(\tau)\,d\tau=:\lambda(s)\frac{d_\Gamma(s)}{\kappa_\Gamma(s)}-f_\Gamma(s).$$

Now employing $\lambda(\theta_\Gamma)=-H_\Gamma$, (9.8) leads to the identity

$$\begin{aligned}
&T_\Gamma(\theta_\Gamma)\{V_\Gamma-\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}H_\Gamma-f_\Gamma(\theta_\Gamma)\}\\
&=\frac{\lambda'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\{[\![d(\theta)\partial_\nu\theta]\!]-\kappa_\Gamma u_\Gamma\nabla_\Gamma\theta_\Gamma+l_\Gamma\mathrm{div}_\Gamma u+l_0(\theta)u\cdot\nu_\Gamma\}\\
&-\frac{d_\Gamma(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\{\lambda''(\theta_\Gamma)-\lambda'(\theta_\Gamma)\frac{\kappa_\Gamma'(\theta_\Gamma)}{\kappa_\Gamma(\theta_\Gamma)}\Big\}|\nabla_\Gamma\theta_\Gamma|^2+\{\omega_\Gamma(\theta_\Gamma)-\mathrm{tr}L_\Gamma^2\}g_\Gamma(\theta_\Gamma),
\end{aligned}$$

hence applying the inverse of $T_\Gamma(\theta_\Gamma)$ we arrive at (9.11), with

$$\begin{aligned}
F_\Gamma(u,\theta,\theta_\Gamma)=[\kappa_\Gamma(\theta_\Gamma)T_\Gamma(\theta_\Gamma)]^{-1}\big(&\lambda'(\theta_\Gamma)\{[\![d(\theta)\partial_\nu\theta]\!]-\kappa_\Gamma u_\Gamma\nabla_\Gamma\theta_\Gamma+l_\Gamma\mathrm{div}_\Gamma u+l_0(\theta)u\cdot\nu_\Gamma\}\\
&-d_\Gamma(\theta_\Gamma)\{(\lambda''(\theta_\Gamma)-\lambda'(\theta_\Gamma)\kappa_\Gamma'(\theta_\Gamma)/\kappa_\Gamma(\theta_\Gamma)\}|\nabla_\Gamma\theta_\Gamma|^2\\
&+\kappa_\Gamma(\theta_\Gamma)\{\omega_\Gamma(\theta_\Gamma)-\mathrm{tr}L_\Gamma^2\}g_\Gamma(\theta_\Gamma)\big).
\end{aligned}$$

In the sequel we replace the Gibbs-Thomson law by the dynamic equation (9.11) plus the compatibility condition $\varphi(\theta_{\Gamma0})+\sigma(\theta_{\Gamma0})H_{\Gamma_0}=0$ at time $t=0$.

Now we perform the Hanzawa transform to obtain a problem on Ω with fixed interface Σ. This yields the following problem.

$$
\begin{aligned}
\partial_t u + \mathcal{A}_u(\theta,h) : \nabla^2 u + (I - M_1(h))\nabla\pi/\varrho &= F_u(u,\theta,h) && \text{in } \Omega\setminus\Sigma,\\
(I - M_1(h))\nabla\cdot u &= 0 && \text{in } \Omega\setminus\Sigma,\\
\partial_t\theta + \mathcal{A}_\theta(\theta,h) : \nabla^2\theta &= F_\theta(\theta,h) && \text{in } \Omega\setminus\Sigma,\\
u, \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\
\partial_t\theta_\Sigma + \mathcal{A}_{\theta_\Sigma}(\theta_\Sigma,h) : \nabla^2_\Sigma\theta_\Sigma &= F_{\theta_\Sigma}(u,\theta,\theta_\Sigma,h) && \text{on } \Sigma,\\
\theta = \theta_\Sigma,\ \ [\![u]\!], [\![\theta]\!] &= 0 && \text{on } \Sigma,\\
-[\![S(u,\theta,h)]\!]\nu_\Gamma + \big([\![\pi]\!] - \sigma(\theta_\Sigma)H_\Gamma(h)\big)\nu_\Gamma &= \sigma'(\theta_\Sigma)\nabla_\Sigma\theta_\Sigma && \text{on } \Sigma,\\
\kappa_\Gamma(\theta_\Sigma)V_\Gamma - d_\Gamma H_\Gamma(h) - \kappa_\Gamma(\theta_\Sigma)f(\theta_\Sigma) &= \kappa_\Gamma(\theta_\Sigma)F_\Gamma(\theta,\theta_\Sigma,h)) && \text{on } \Sigma,\\
h(0) = h_0 \text{ on } \Sigma,\quad u(0) = u_0,\ \theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned}
\tag{9.12}
$$

The abstract setting of this problem differs from the previous cases. As variables we choose $z = (u,\theta,\theta_\Sigma,h,\pi,q)$ in the regularity space

$$
\begin{aligned}
\mathbb{E}^5_\mu(a) = &\{z \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\theta,\mu}(J) \times \times\mathbb{E}_{\theta_\Sigma,\mu}(J) \times \mathbb{E}^5_{h,\mu}(J) \times \mathbb{E}_{\pi,\mu}(J) \times \mathbb{E}_{q,\mu}(J) :\\
&[\![u]\!], [\![\theta]\!] = 0,\ \ \theta = \theta_\Sigma,\ [\![\pi]\!]_{|\Sigma} = q \text{ on } \Sigma,\ \ u, \partial_\nu\theta = 0 \text{ on } \partial\Omega\},
\end{aligned}
$$

with

$$\mathbb{E}^5_{h,\mu}(J) = H^1_{p,\mu}(J; W^{1-1/p}_p(\Sigma)) \cap L_{p,\mu}(J; W^{3-1/p}_p(\Sigma)).$$

For the space of data we may take here

$$\mathbb{F}^5_\mu(a) = \mathbb{F}_{u,\mu}(J) \times \mathbb{F}^2_{\pi,\mu}(J) \times \mathbb{F}_{\theta,\mu}(J) \times \mathbb{F}_{\theta_\Sigma,\mu}(J) \times \mathbb{F}^u_{h,\mu}(J)^n \times \mathbb{F}^5_{h,\mu}(J),$$

where

$$\mathbb{F}^5_{h,\mu}(J) = L_{p,\mu}(J; W^{1-1/p}_p(\Sigma)).$$

This way we obtain the abstract form of the problem

$$L_5\tilde{z} = N_5(\tilde{z},\bar{z}),\quad \tilde{z}(0) = 0,$$

with $N_5 : {}_0\mathbb{E}^5_\mu(a) \times \mathbb{E}^5_\mu(\infty) \to {}_0\mathbb{F}^5_\mu(a)$ and $L_5 : \mathbb{E}^5_\mu(a) \to \mathbb{F}^5_\mu(a)$ linear and bounded.

More precisely, L_5 is defined by

$$
L_5\tilde{z} = \begin{bmatrix}
\partial_t\tilde{u} + \mathcal{A}_u(\theta_0,h_0)\nabla^2\tilde{u} + (1 - M_1(h_0))\nabla\tilde{\pi}\\
(I - P_0M_1(h_0))\nabla\cdot\tilde{u})\\
\partial_t\tilde{\theta} + \mathcal{A}_\theta(\theta_0,h_0) : \nabla^2\tilde{\theta}\\
\partial_t\tilde{\theta}_\Sigma + \mathcal{A}_{\theta_\Sigma}(\theta_{\Sigma 0},h_0) : \nabla^2_\Sigma\tilde{\theta}_\Sigma\\
-[\![S(\tilde{u},\theta_0,h_0)]\!]\nu_\Sigma + \tilde{q}\nu_\Sigma + \sigma(\theta_{\Sigma 0})\mathcal{C}_\Sigma\tilde{h} - \sigma'(\theta_{\Sigma 0})\nabla_\Sigma\tilde{\theta}_\Sigma\\
\partial_t\tilde{h} + c_0(\theta_{\Sigma 0},h_0)\mathcal{C}_\Sigma\tilde{h} + c_1(\theta_{\Sigma 0})\tilde{\theta}_\Sigma
\end{bmatrix}.
$$

Here we have set

$$c_0(\theta_{\Sigma 0},h_0) = d_\Gamma(\theta_{\Sigma 0})/\kappa_\Gamma(\theta_{\Sigma 0}),\quad c_1(\theta_{\Sigma 0}) = -\lambda(\theta_{\Sigma 0})(d_\Gamma/\kappa_\Gamma)'(\theta_{\Sigma 0}).$$

The operator L_5 also has maximal regularity, as it has triangular structure, and each diagonal entry has maximal L_p-regularity.

9.2 The Fixed Point Argument

In the previous section we have seen that on the fixed domain all six problems can be reformulated as the abstract problem

$$L\tilde{z} = N(\tilde{z}, \bar{z}), \quad \tilde{z}(0) = 0, \tag{9.13}$$

where $L : \mathbb{E}_\mu(a) \to \mathbb{F}_\mu(a)$ is bounded linear, and $N : {}_0\mathbb{E}_\mu(a) \times \mathbb{E}_\mu(\infty) \to {}_0\mathbb{F}_\mu(a)$ is nonlinear. Of course, the specific spaces and operators differ from problem to problem, but they all share the following properties.

(MR) For each $a \in (0,1]$, the operator $L : {}_0\mathbb{E}_\mu(a) \to {}_0\mathbb{F}_\mu(a)$ is an isomorphism, and the norm of L^{-1} is bounded by some constant M independent of $a \in (0,1]$.

(NL) For each $a \in (0,1]$, the nonlinearity N is of class C^1. Moreover,

(i) $|N(0,\bar{z})|_{\mathbb{F}_\mu(a)} \to 0$ as $a \to 0$, for each fixed $\bar{z} \in \mathbb{E}_\mu(\infty)$;

(ii) $|D_1 N(0,\bar{z})|_{\mathcal{B}({}_0\mathbb{E}_\mu(a), {}_0\mathbb{F}_\mu(a))} \to 0$ as $a \to 0$, for each fixed $\bar{z} \in \mathbb{E}_\mu(\infty)$.

Condition **(NL)** will be verified in Section 9.5. It implies that for a given $\bar{z} \in \mathbb{E}_\mu(\infty)$,

$$\eta(a,r) := \sup\{|D_1 N(\tilde{z},\bar{z})|_{\mathcal{B}({}_0\mathbb{E}_\mu(a), {}_0\mathbb{F}_\mu(a))} : |\tilde{z}|_{\mathbb{E}_\mu(a)} \le r\}$$

satisfies $\eta(a,r) \to 0$ as $a, r \to 0$. This in turn implies

$$|N(\tilde{z}_1,\bar{z}) - N(\tilde{z}_2,\bar{z})|_{\mathbb{F}_\mu(a)} \le \eta(a,r)|\tilde{z}_1 - \tilde{z}_2|_{\mathbb{E}_\mu(a)}, \quad |\tilde{z}_j|_{\mathbb{E}_\mu(a)} \le r,$$

and

$$|N(\tilde{z},\bar{z})|_{\mathbb{F}_\mu(a)} \le |N(0,\bar{z})|_{\mathbb{F}_\mu(a)} + \eta(a,r)r, \quad |\tilde{z}|_{\mathbb{E}_\mu(a)} \le r.$$

As $|L^{-1}|_{\mathcal{B}({}_0\mathbb{E}_\mu(a), {}_0\mathbb{F}_\mu(a))}$ is uniformly bounded for $a \in (0,1]$, say by C, we see that $T(\tilde{z}) = L^{-1}N(\tilde{z},\bar{z})$ will be a contracting self-map on the ball $\bar{B}_{{}_0\mathbb{E}_\mu(a)}(0,r)$, by choosing a, r small enough. The contraction mapping principle then yields a unique fixed point $\tilde{z}_\odot \in \bar{B}_{{}_0\mathbb{E}_\mu(a)}(0,r)$, which means that (9.13) admits the unique solution $\tilde{z}_\odot$. This completes the proof of local existence and uniqueness for the six Problems **(P1)**$\sim$**(P6)**. This way we have proved

Theorem 9.2.1. *Let $p > n+2$, $1 \ge \mu > \frac{1}{2} + \frac{n+2}{2p}$, and suppose the following conditions are satisfied.*

(i) Regularity: *Condition* **(Hj)** *holds for Problem* **(Pj)**.

(ii) Well-Posedeness: $\theta_0 > 0$*;* $l(\theta_0) \neq 0$ *for Problems* **(P1), (P3)**,

$0 < \theta_0 < \theta_c$ *for Problems* **(P5), (P6)**,

$T_{\Gamma_0}(\theta_0)$ *is invertible in* $L_2(\Gamma_0)$ *for Problem* **(P5)**.

(iii) Compatibilities: *Condition* (**Cj**) *holds for Problem* **(Pj)**.

Then each Problem **(Pj)**, $j = 1, \ldots, 6$, *is locally uniquely solvable in the sense that for any initial value* $z_0 \in X^j_{\gamma,\mu}$, *there is* $a = a(z_0) > 0$ *such that the transformed problems admit a unique solution* $z \in \mathbb{E}^j_\mu(a)$.

9.3 Dependence on the Data

To study the dependence of the solution of (9.13) on the initial data, we will employ the implicit function theorem. For this purpose note that the map $E : X_{\gamma,\mu} \to \mathbb{E}_\mu(\infty)$ defined by $Ez_0 = \bar{z}$ is linear and bounded, hence real analytic. We rewrite problem (9.13) as

$$G(\tilde{z}, z_1) := L(z_1, Ez_1)\tilde{z} - N(\tilde{z}, Ez_1) = 0,$$

where $L(z_1, Ez_1)$ indicates the dependence of L on the initial value z_1 and, where applicable, on the pertinent extensions $\bar{z}_1 = Ez_1$ subsumed in the definition of L_j. Here

$$G : {}_0\mathbb{E}_\mu(a) \times B_{X_{\gamma,\mu}}(z_0, r) \to {}_0\mathbb{F}_\mu(a)$$

is at least of class C^1. We have $G(\tilde{z}_\odot, z_0) = 0$, and the Fréchet-derivative $D_1 G(\tilde{z}_\odot, z_0) \in \mathcal{B}({}_0\mathbb{E}_\mu(a), {}_0\mathbb{F}_\mu(a))$ is invertible, as we have seen in the previous section. Therefore, there is a radius $\delta > 0$ and a C^1-map $\tilde{z} : B_{X_{\gamma,\mu}}(z_0, \delta) \to {}_0\mathbb{E}_\mu(a)$ such that

$$\tilde{z}(z_0) = \tilde{z}_\odot \ \text{ and } \ G(\tilde{z}(z_1), z_1) = 0 \ \text{ for all } \ z_1 \in B_{X_{\gamma,\mu}}(z_0, \delta).$$

Moreover, by uniqueness there are no other solutions close to $\tilde{z}_\odot$, and so by causality there are no other solutions, at all.

Further, if G is of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, then $\tilde{z}$ has the same regularity; here ω means real analytic. We observe that L, N, and hence G, are of class C^k provided

$$\psi, \sigma \in C^{k+2}(0, \infty) \quad \text{ and } \quad d, d_\Gamma, \mu \in C^{k+1}(0, \infty).$$

Note that the maps $h \mapsto (m_0(h), M_0(h), M_1(h), \beta(h))$ are real analytic. This implies the following result.

Theorem 9.3.1. *In addition to the assumptions of Theorem* 9.2.1 *assume that*

$$\psi, \sigma \in C^{k+2}(0, \infty) \quad \textit{and} \quad d, d_\Gamma, \mu \in C^{k+1}(0, \infty),$$

for some $k \in \mathbb{N} \cup \{\infty, \omega\}$.

Then the solution map is of class C^k *from the data space* $X^j_{\gamma,\mu}$ *into the solution space* $\mathbb{E}^j_\mu(a)$*, for each* $j = 1, \ldots, 6$.

9.4 Regularity: The Parameter Trick

In Section 5.3 we used a scaling argument for time t to extract more time regularity from the regularity properties of the nonlinearity $A(u)u - F(u)$ for the solution of the quasilinear parabolic evolution equation

$$\dot{u} + A(u)u = F(u), \quad t \in J, \ \ u(0) = u_0.$$

In this section we extend this method to obtain regularity of z in the 6 problems studied above. The implicit function theorem as well as maximal L_p-regularity will again be the main tools.

9.4.1 Interior Regularity

Let $G : \mathbb{E}_\mu(a) \to \mathbb{F}_\mu(a)$ denote the functions G^j from the previous section, where we now fix the initial values and suppress them in our notation, with the corresponding function spaces $\mathbb{E}^j_\mu(a)$ and $\mathbb{F}^j_\mu(a)$. We assume that G is in the class C^k, with $k \in \mathbb{N} \cup \{\infty, \omega\}$, where, as before, ω means real analytic. We want to show

$$(u, \theta), \partial_i(u, \theta) \in C^k((0, a) \times (\Omega \setminus \Sigma))^{n+1}, \quad i = 1, \ldots, n.$$

This then implies also pressure regularity $\pi, \partial_i \pi \in C^{k-1}((0, a) \times (\Omega \setminus \Sigma))$, for all $i = 1, \ldots, n$, by the equation for u.

For this purpose we fix $(t_0, x_0) \in (0, a) \times (\Omega \setminus \Sigma)$. Recall that regularity is a local property, so we need only to show regularity of (u, θ) in $(t_0 - r, t_0 + r) \times B(x_0, r)$ where $r > 0$ is small enough. We fix $R > 0$ such that $3R < t_0 < a - 3R$, and $B(x_0, 3R) \subset \Omega \setminus \Sigma$. Further we may let $a \leq a_0$ by causality; otherwise we shift the time interval in question, and repeat the argument finitely many times. Furthermore, we assume that $B(x_0, 3R)$ does not intersect the tubular neighbourhood of width $3a_\Sigma$ around Σ; we comment on this assumption later.

Next we choose standard C^∞-cut-off functions χ_{t_0} and χ_{x_0}, which are 1 for $|t - t_0| < R$, resp. $|x - x_0| < R$, and 0 for $|t - t_0| > 2R$, resp. $|x - x_0| > 2R$, between 0 and 1 elsewhere.

We introduce a coordinate transform $\tau_{(\lambda,\xi)}$ by means of

$$\tau_{\lambda,\xi}(t, x) = (t + \lambda \chi_{t_0}(t), x + t\xi \chi_{x_0}(x)), \quad (t, x) \in (0, a) \times \Omega).$$

It is easy to see that $\tau_{(\lambda,\xi)} : (0, a) \times \Omega$ is a diffeomorphism of class C^∞, so that the map

$$\tau : (\lambda, \xi) \mapsto \tau_{\lambda,\xi}, \quad (-r, r) \times B_{\mathbb{R}^n}(0, r) \to \mathrm{Diff}^\infty((0, a) \times \Omega)$$

is well-defined, provided r is sufficiently small. Observe that $\tau_{0,0} = id$, and that $\tau_{(\lambda,\xi)} = id$ outside the cube $(-2R, 2R) \times B_{\mathbb{R}^n}(0, 2R)$.

In the next step we introduce the lifted coordinate transforms $T_{\lambda,\xi}$ by

$$T_{\lambda,\xi} z(t, x) = z(\tau_{\lambda,\xi}(t, x)) = z(t + \lambda \chi_{t_0}(t), x + t\xi \chi_{x_0}(x)), \quad t \in (0, a), \ \ x \in \Omega,$$

where $(\lambda, \xi) \in (-r, r) \times B_{\mathbb{R}^n}(0, r)$. It is not difficult to show that $T_{\lambda,\xi}$ is an isomorphism in $\mathbb{E}_\mu(a)$ as well as in $\mathbb{F}_\mu(a)$; one only needs to recall the transformation rules from Section 6.3. Note that $T_{\lambda,\xi}$ is leaving *the initial values unchanged.* This property is very important, as it will show that the obtained regularity does not depend on the regularity of the initial value z_0.

By the transformation rules from Section 6.3, we obtain the relations

$$T_{\lambda,\xi} \nabla z = \nabla z \circ \tau_{\lambda,\xi} = (I - m_1(\lambda, \xi)) \nabla T_{\lambda,\xi} z,$$

and

$$T_{\lambda,\xi}\partial_t z = \partial_t z \circ \tau_{\lambda,\xi} = (1+\lambda\chi'_{t_0})^{-1}[\partial_t T_{\lambda,\xi} z - m_0(\lambda,\xi)(\xi|\nabla)T_{\lambda,\xi} z],$$

where

$$m_0(\lambda,\xi) = \frac{\chi_{x_0}}{1+t(\xi|\nabla\chi_{x_0})}, \quad m_1(\lambda,\xi) = \frac{t\nabla\chi_{x_0}\otimes\xi}{1+t(\xi|\nabla\chi_{x_0})}.$$

Note that m_0, m_1 are real analytic in (λ,ξ) and of class C^∞ in (t,x).

Given the solution $z_\odot$ of $G(z_\odot)=0$ from the previous section, we see that

$$0 = T_{\lambda,\xi}G(z_\odot) = T_{\lambda,\xi}G(T^{-1}_{\lambda,\xi}T_{\lambda,\xi}z_\odot),$$

hence with

$$\mathsf{G}(\lambda,\xi,\bar z) = T_{\lambda,\xi}G(T^{-1}_{\lambda,\xi}\bar z)$$

and setting $\bar z_\odot = T_{\lambda,\xi}z_\odot = z_\odot\circ\tau_{\lambda,\xi}$, it is obvious that $\bar z_\odot$ satisfies the equation

$$\mathsf{G}(\lambda,\xi,\bar z_\odot) = 0.$$

So it is natural to employ the implicit function theorem to solve for $\bar z_\odot$. As we are interested in the regularity of solutions for $t>0$, we may, and we will, assume that the fixed initial value z_0 is in the regularity space X_1. We then consider

$$\mathsf{G} : (-r,r)\times B_{\mathbb{R}^n}(0,r)\times \mathbb{E}_1^{z_0}(a) \to {}_0\mathbb{F}_1(a),$$

where $\mathbb{E}_1^{z_0}(a)$ denotes the affine linear subspace of $\mathbb{E}_1(a)$ with fixed initial values $u(0)=u_0$, $\theta(0)=\theta_0$, $h(0)=h_0$, $\partial_t h(0)=h_1$, where these data are subject to the appropriate compatibility conditions. Employing the transformation rules for ∇ and ∂_t from above, as in the previous subsection it follows from Section 9.5 that G is of class C^k, $k\in\mathbb{N}\cup\{\infty,\omega\}$, whenever

$$\psi,\sigma\in C^{k+2}(0,\infty) \quad \text{and} \quad d, d_\Gamma, \mu \in C^{k+1}(0,\infty).$$

Furthermore, we have $\mathsf{G}(0,0,z_\odot)=0$ and

$$D_z\mathsf{G}(0,0,z_\odot) = D_z G(z_\odot) : {}_0\mathbb{E}_1(a)\to {}_0\mathbb{F}_1(a)$$

is invertible, by maximal regularity, as known from Section 9.2. Hence by the implicit function theorem, there is a neighbourhood $(-\delta,\delta)\times B_{\mathbb{R}^n}(0,\delta)$ of $(0,0)$ and a map

$$\Phi : (-\delta,\delta)\times B_{\mathbb{R}^n}(0,\delta)\to\mathbb{E}_1(a),$$

of class C^k with $\Phi(0,0)=z_\odot$ such that $\mathsf{G}(\lambda,\xi,\Phi(\lambda,\xi))=0$. By uniqueness, this implies $\Phi(\lambda,\xi)=z_\odot\circ\tau_{\lambda,\xi}$. As a consequence, the projection-embedding

$$\mathbb{E}_1(a)\to C^\alpha((0,a);C^{1+\alpha}(\Omega\setminus\Sigma))^{n+1}, \quad z\mapsto(u,\theta),$$

with $\alpha \in (0,1)$ sufficiently small, shows that

$$(\lambda, \xi) \mapsto (u, \theta)(t + \lambda \chi_{t_0}(t), x + t\xi \chi_{x_0}(x))$$

is of class C^k. But this implies that this map is even $C^{k+\alpha}$ with image space $C((0,a); C^1(\Omega \setminus \Sigma))^{n+1}$; this is a *transfer of regularity* induced by the definition of τ. Setting $t = t_0$ and $x = x_0$ this shows that the function

$$(\lambda, \xi) \mapsto (u, \theta)(t_0 + \lambda, x_0 + t_0 \xi)$$

is of class $C^{k+\alpha}$ near (t_0, x_0). Repeating the same argument with $\nabla_x(u, \theta)$ we see in the same way that $\nabla_x(u, \theta) \in C^{k+\alpha}$ near (t_0, x_0).

If x_0 does belong to the tubular neighbourhood of Σ, we re-parameterize near t_0 in such a way that x_0 does not belong to the new tubular neighbourhood, and proceed as before. This yields the following result on interior regularity.

Theorem 9.4.1. *Let the assumptions of Theorems* 9.2.1 *and Theorem* 9.3.1 *be valid, for some* $k \in \mathbb{N} \cup \{\infty, \omega\}$.

Then there is $\alpha \in (0,1)$ *such that in all 6 problems we have*

$$(u, \theta), \partial_i(u, \theta) \in C^{k+\alpha}((0,a) \times (\Omega \setminus \Sigma))^{n+1},$$

and

$$\pi, \partial_i \pi \in C^{k-1+\alpha}((0,a) \times (\Omega \setminus \Sigma)),$$

where $i = 1, \dots, n$. *In particular, in each problem we have classical solutions in the interior, even for* $k = 1$.

9.4.2 Regularity on the Interface

By means of the parameter trick it is also possible to prove regularity in time and tangential directions on the interface. However, here the construction of the map $\tau_{\lambda,\xi}$ is more involved, but also quite natural. We fix a point $(t_0, x_0) \in (0,a) \times \Sigma$ and choose a parameterization $\varphi : B_{\mathbb{R}^{n-1}}(0, 3R) \to \mathbb{R}^n$ for Σ near x_0; as Σ is real analytic we may choose φ real analytic. Here the chosen optimal smoothness of the reference Σ pays off! Next we extend φ by means of

$$\phi(p,q) = \varphi(p) + q\nu_\Sigma(\varphi(p)), \quad (p,q) \in B_{\mathbb{R}^{n-1}}(0,3R) \times (-3a_\Sigma, 3a_\Sigma),$$

to a neighbourhood of $x_0 \in \Sigma$, with $3a_\Sigma$ the with of the tubular neighbourhood of Σ as chosen in Section 2.3. This map is again real analytic and it is a diffeomorphism onto its image if $R > 0$ is small enough. Observe that $\Pi_\Sigma \phi(p,q) = \varphi(p)$. Then we define the truncated shift

$$\tau_\xi(p,q) = (p + \xi \chi_0(p) \zeta_0(q), q), \quad (p,q) \in B_{\mathbb{R}^{n-1}}(0,3R) \times (-3a_\Sigma, 3a_\Sigma).$$

Here, χ_0 is a smooth cut-off function on $\mathbb{R}^{n-1}$ which is one for $|p| \le R$ and zero for $|p| > 2R$, while ζ_0 is a smooth cut-off function on $\mathbb{R}$ which is one for $|q| \le 2a_\Sigma$

and 0 for $|q| \geq 5a_\Sigma/2$. Note that here $\xi \in B_{\mathbb{R}^{n-1}}(0,r)$ acts only tangentially. Then we set

$$\tau_{\lambda,\xi}(t,x) = (t + \lambda\chi_{t_0}(t), \phi(\tau_{t\xi}(\phi^{-1}(x)))) = (\tau^1_{\lambda,\xi}, \tau^2_{\lambda,\xi}), \quad (t,x) \in (0,a) \times U,$$

and

$$\tau_{\lambda,\xi}(t,x) = (t + \lambda\chi_{t_0}(t), x), \quad (t,x) \in (0,a) \times (\Omega \setminus U).$$

Here $U := \phi\big(B_{\mathbb{R}^{n-1}}(0,3R) \times (-3a_\Sigma, 3a_\Sigma)\big)$ is an open tubular neighbourhood of $x_0 \in \Sigma$. Observe that $\tau_{\lambda,\xi}$ commutes with Π_Σ on $U_{2a_\Sigma} = \phi\big(B_{\mathbb{R}^{n-1}}(0,3R) \times (-2a_\Sigma, 2a_\Sigma)\big)$, which implies

$$\begin{aligned} h \circ (id, \Pi_\Sigma) \circ \tau_{\lambda,\xi}(t,x) &= h(\tau^1_{\lambda,\xi}(t), \Pi_\Sigma \tau^2_{\lambda,\xi}(t,x)) \\ &= h(\tau_{\lambda,\xi}(t, \Pi_\Sigma x)) = h \circ \tau_{\lambda,\xi} \circ (id, \Pi_\Sigma)(t,x), \end{aligned}$$

for each $(t,x) \in (0,a) \times U_{2a_\Sigma}$. Recalling the definition of Ξ_h from Section 1.3, we then have

$$\big([\chi \circ (d_\Sigma/a_\Sigma)]\,[h \circ (id, \Pi_\Sigma)]\big) \circ \tau_{\lambda,\xi} = [\chi \circ (d_\Sigma/a_\Sigma)]\,[h \circ \tau_{\lambda,\xi} \circ (id, \Pi_\Sigma)]$$

on $(0,a) \times U$, as $d_\Sigma \circ \tau_{\lambda,\xi} = d_\Sigma$ on U, and $\chi \circ (d_\Sigma/a_\Sigma) = 0$ on $U \setminus U_{2a_\Sigma}$. If we choose $r > 0$ sufficiently small, then

$$\tau : (\lambda, \xi) \mapsto \tau_{\lambda,\xi}, \quad (-r,r) \times B_{\mathbb{R}^n}(0,r) \to \mathrm{Diff}^\infty((0,a) \times \Omega),$$

as in the simpler previous case of Section 9.4.1. Note that we do not shift in the vertical direction, as this would distort the interface Σ which needs to be kept fixed.

Then as before, we lift the coordinate transform $\tau_{\lambda,\xi}$ to an operator $T_{\lambda,\xi}$, which is a linear and bounded isomorphism in the spaces $\mathbb{E}_1(a)$ of solutions as well as in the space of date $\mathbb{F}_1(a)$. The function G is then defined as in the previous section, and we see that G is of class C^k, provided G has this property.

Hence again by the implicit function theorem, there is a ball $(-\delta,\delta) \times B_{\mathbb{R}^{n-1}}(0,\delta)$ and a map

$$\Phi : (-\delta,\delta) \times B_{\mathbb{R}^{n-1}}(0,\delta) \to \mathbb{E}_1^{z_0}(a),$$

of class C^k with $\Phi(0,0) = z_\odot$ such that $\mathsf{G}(\lambda,\xi,\Phi(\lambda,\xi)) = 0$. By uniqueness, we have again $\Phi(\lambda,\xi) = z_\odot \circ \tau_{\lambda,\xi}$.

Now we extract the height function h on the interface to obtain $(\lambda,\xi) \mapsto h \circ \tau_{(\lambda,\xi)} \in \mathbb{E}^j_{h,1}(a)$, where with $J = (0,a)$, and for some $\alpha \in (0,1)$ small,

$$\begin{aligned} \mathbb{E}^\theta_{h,1}(a) &= W^{3/2-1/2p}_p(J; L_p(\Sigma)) \cap W^{1-1/2p}_p(J; H^2_p(\Sigma)) \cap L_p(J; W^{4-1/p}_p(\Sigma)) \\ &\hookrightarrow C^\alpha((0,a); C^{3+\alpha}(\Sigma)) \cap C^{1+\alpha}((0,a); C^{1+\alpha}(\Sigma)), \end{aligned}$$

for Problems **(Pj)**, $j = 1, 3, 5$. where we need to assume $p > n+5$ for the embedding into $C^{1+\alpha}((0,a); C^{1+\alpha}(\Sigma))$. Moreover,

$$\begin{aligned}\mathbb{E}^u_{h,1}(a) &= W_p^{2-1/2p}(J; L_p(\Sigma)) \cap H_p^1(J; W_p^{2-1/p}(\Sigma)) \cap L_p(J; W_p^{3-1/p}(\Sigma) \\ &\hookrightarrow C^\alpha((0,a); C^{2+\alpha}(\Sigma)) \cap C^{1+\alpha}((0,a); C^{1+\alpha}(\Sigma)),\end{aligned}$$

for Problems **(Pj)**, $j = 2, 4, 6$. Employing exchange of regularity as before, point evaluation implies

$$h \in C^{k+1+\alpha}((0,a) \times \Sigma), \nabla^i_\Sigma h \in C^{k+\alpha}((0,a) \times \Sigma)^{i \times n},$$

$i \leq 3$ for **(P1), (P3), (P5)**, with $p > n+5$, $i \leq 2$ for **(P2), (P4), (P6), (P5)** with $p > n+2$.

In the same way, we obtain regularity of the boundary pressures $q = [\![\pi]\!]$ in Problems **(P2), (P3), (P5)** and also of the on-sided pressures π_1, π_2 in Problems **(P4), (P6)**. In fact, $[\![\pi]\!], \pi_1, \pi_2 \in \mathbb{F}^u_{h,1}(a)$ yields

$$[\![\pi]\!], \pi_1, \pi_2 \in C^{k+\alpha}((0,a) \times \Sigma).$$

As in all problems we have for the surface temperature $\theta_\Sigma \in \mathbb{F}^\theta_{h,1}(a)$, this technique yields

$$(\theta_\Sigma, \nabla_\Sigma \theta_\Sigma) \in C^{k+\alpha}((0,a) \times \Sigma)^{n+1}.$$

Similarly, (the one-sided) traces u_i of u at the interface satisfy

$$(u_i, \nabla_\Sigma u_i) \in C^{k+\alpha}((0,a) \times \Sigma)^{n \times (n+1)}.$$

This shows that all equations in each Problem **(Pj)** are satisfied pointwise, i.e., the solutions obtained in Section 9.2 are all classical.

We summarize these results in

Theorem 9.4.2. *Let the assumptions of Theorems* 9.2.1 *and Theorem* 9.3.1 *be valid for some* $k \in \mathbb{N} \cup \{\infty, \omega\}$.

Then there is $\alpha \in (0,1)$ *such that in all 6 problems we have*

$$h \in C^{1+k+\alpha}((0,a) \times \Sigma), \quad \nabla^i_\Sigma h \in C^{k+\alpha}((0,a) \times \Sigma)^{i \times n}$$

$i \leq 3$ *for* **(P1), (P3), (P5)** *with* $p > n+5$, *and* $i \leq 2$ *for* **(P2), (P4), (P6)** *with* $p > n+2$. *Furthermore,*

$$[\![\pi]\!], \pi_1, \pi_2, \theta_\Sigma \in C^{k+\alpha}((0,a) \times \Sigma),$$

and

$$u, \nabla_\Sigma \theta \in C^{k+\alpha}((0,a) \times \Sigma)^n, \quad \nabla_\Sigma u \in C^{k+\alpha}((0,a) \times \Sigma)^{n \times n}.$$

In particular, in each problem the solutions are classical also on the interface, even for $k = 1$.

Observe that in case $k = \omega$, which means that all coefficient functions are real analytic, then h will be so jointly in time and space, hence the interfaces $\Gamma(t)$ become real analytic, instantaneously. This shows the strong regularizing effect which is inherent in quasilinear parabolic problems.

9.5 Estimates for the Nonlinearities

The basis of all considerations below are the following embeddings which are due to the restriction $1 \geq \mu > \frac{1}{2} + \frac{n+2}{2p}$.

$$\begin{aligned} \mathbb{E}_{u,\mu}(a) \times \mathbb{E}_{\theta,\mu}(a) &\hookrightarrow C^{1/2}([0,a]; C_{ub}(\Omega \setminus \Sigma))^{n+1}, \\ \mathbb{E}_{u,\mu}(a) \times \mathbb{E}_{\theta,\mu}(a) &\hookrightarrow C([0,a]; C^1_{ub}(\Omega \setminus \Sigma))^{n+1}, \\ \mathbb{E}^{\theta}_{h,\mu}(a) &\hookrightarrow C^{1-}([0,a]; C(\Sigma)) \cap C([0,a]; C^3(\Sigma)), \\ \mathbb{E}^{u}_{h,\mu}(a) &\hookrightarrow C^{1-}([0,a]; C^1(\Sigma)) \cap C([0,a]; C^2(\Sigma))). \end{aligned} \tag{9.14}$$

In general, the embedding constants will blow up as $a \to 0$, however, they do not depend on a, provided we restrict to time trace 0. This can be seen by the following simple extension argument. If a function v is defined on $[0,a]$, say for $a \leq 1$, and has time trace 0, we may extend it by

$$Ev(t) = \begin{cases} v(t), & 0 \leq t \leq a, \\ v(2a-t), & a \leq t \leq 2a, \\ 0, & 2a \leq t \leq 2. \end{cases}$$

Then $\sup_{0\leq t\leq a} |v(t)| \leq \sup_{0\leq t\leq 2} |Ev(t)|$ can be estimated by the relevant embedding for the fixed interval $[0,2]$. This simple observation is very important, and besides of the compatibilities this is another reason to reduce all problems to the case of vanishing time traces at $t = 0$.

5.1. The Nonlinearities in $\mathbb{F}_{\theta,\mu}$ and $\mathbb{F}_{u,\mu}$

(a) The nonlinearities F_θ and the components of F_u live in $L_{p,\mu}((0,a); L_p(\Omega))$. They consist of sums and products of $\nabla\theta$, u, ∇u, as well as of $d(\theta)$, $d'(\theta)$, $\mu(\theta)$, $\mu'(\theta)$, $1/\kappa(\theta)$, $M_1(h)$, $\nabla M_1(h)$, and $m_0(h)\partial_t h \circ \Pi_\Sigma$. As the functions μ and d are C^2 and $\kappa \in C^1$, the maps $\theta \mapsto \mu(\theta), \mu'(\theta), d(\theta), d'(\theta), \kappa(\theta)$ are of class C^1 from $C([0,a] \times \bar{\Omega})$ into itself, hence by the embeddings (9.14) it follows easily that F_θ and F_u are of class C^1, for all six problems under consideration.

Moreover, $F_\theta(z), F_u(z)$ belong to $L_\infty((0,a) \times \Omega)$, for each $z \in \mathbb{E}^j_\mu$, hence we obtain estimates of the form

$$|F_k(z)|_{\mathbb{F}_{k,\mu}(a)} \leq |F_k(z)|_\infty |\Omega| [\int_0^a t^{p(1-\mu)} dt]^{1/p} \leq C(|\bar{z}|_{\mathbb{E}_{k,\mu}} + R)^m a^{1-\mu+1/p},$$

with some constants $m \in \mathbb{N}$, $C > 0$, for all $\tilde{z} \in \bar{B}_{\mathbb{E}^j_\mu}(0,R)$, where $z = \tilde{z} + \bar{z}$. Therefore, these terms become small by choosing the time interval $J = (0,a)$ small. The same argument also applies to their Fréchet derivatives DF_k.

(b) On the other hand, there appear terms of highest order in the θ- and u-components of N_j; however these are only linear in the highest order derivative. For instance, we have the terms $F_1(\tilde{z},\bar{z}) = (\mathcal{A}_\theta(\theta,h) - \mathcal{A}_\theta(\theta_0,h_0) : \nabla^2\tilde{\theta}$ and $F_2(\tilde{z},\bar{z}) = \partial_t\bar{\theta} + \mathcal{A}_\theta(\theta,h) : \nabla^2\bar{\theta}$ in the θ-component of N_j, and similar terms in the u-components. For the analysis of such terms we first observe that bilinear mappings

$$b : L_\infty((0,a)\times\Omega)\times\mathbb{F}_{\theta,\mu}(J)\to\mathbb{F}_{\theta,\mu}(J),\quad (m,f)\mapsto mf,$$

are bounded, since $|b(m,f)|_{\mathbb{F}_{\theta,\mu}(J)}\le|m|_\infty|f|_{\mathbb{F}_{\theta,\mu}(J)}$; hence this map is real analytic. Therefore, composite mappings like

$$(\bar z,\tilde z)\mapsto(\mathcal{A}_\theta(\theta,h),\nabla^2\bar\theta)\mapsto\mathcal{A}_\theta(\theta,h):\nabla^2\bar\theta$$

are as smooth as the coefficients d, κ, in particular of class C^k if d,κ are C^k. The Fréchet derivatives are given by

$$D_1F_1(0,\bar z)=(\mathcal{A}_\theta(\bar\theta,\bar h)-\mathcal{A}_\theta(\theta_0,h_0)):\nabla^2,$$

and

$$D_1F_2(0,\bar z)\tilde z=[\partial_\theta\mathcal{A}_\theta(\bar\theta,\bar h)\tilde\theta+\partial_h\mathcal{A}_\theta(\bar\theta,\bar h)\tilde h]:\nabla^2\bar\theta.$$

Therefore, we obtain

$$|D_1F_1(0,\bar z)\tilde z|_{\mathbb{F}_{\theta,\mu}(J)}\le|(\mathcal{A}_\theta(\bar\theta,\bar h)-\mathcal{A}_\theta(\theta_0,h_0)|_\infty|\nabla^2\tilde\theta|_{\mathbb{F}^{n\times n}_{\theta,\mu}(J)}\le\eta|z|_{\mathbb{E}^j_\mu(a)},$$

provided a is sufficiently small, depending only on the fixed function $\bar z$ which is continuous.

Similarly, we have

$$|D_1F_2(0,\bar z)\tilde z|_{\mathbb{F}_{\theta,\mu}(J)}\le C(|\tilde\theta|_\infty+|\tilde h|_\infty)|\nabla^2\bar\theta|_{\mathbb{F}^{n\times n}_{\theta,\mu}(J)},$$

where C dos not depend on $\tilde z$. By the embeddings (9.14) and trace 0 for $\tilde z$ we obtain further

$$|D_1F_2(0,\bar z)\tilde z|_{\mathbb{F}_{\theta,\mu}(J)}\le C|\tilde z|_{\mathbb{E}^j_\mu(a)}|\nabla^2\bar\theta|_{\mathbb{F}_{\theta,\mu}(J)}\le\eta|\tilde z|_{\mathbb{E}^j_\mu(a)},$$

whenever a is chosen small enough, depending only on $\bar z$, but not on $\tilde z$.

This proves Condition **(NL)** for the θ-part of N_j, and similarly it also holds for the u-part of N_j.

5.2. The Nonlinearity in $\mathbb{F}^j_{\pi,\mu}$

The corresponding term appearing in N_j, $j=4,6$, reads

$$F(\tilde z,\bar z)=(M_1(h)-I)\nabla\cdot\bar u+(M_1(h)-M_1(h_0))\nabla\cdot\tilde u=F_1+F_2,$$

and for $j=2,3,5$ we apply the projection P_0 onto mean value zero. Note that F_i, $i=1,2$, are linear in the terms of highest order, namely ∇u. We consider first

(a) $L_{p,\mu}(J;H^1_p(\Omega\setminus\Sigma)$

The coefficients depend on h and $\nabla_\Sigma h$, hence belong to $C([0,a];C^1(\bar\Omega))$, and vanish outside a tubular neighbourhood of Σ. Therefore, we may use here the bilinear map

$$C([0,a];C^1(\bar{\Omega}))\times L_{p,\mu}(J;H^1_p(\Omega\setminus\Sigma))\to L_{p,\mu}(J;H^1_p(\Omega\setminus\Sigma)),\quad (m,u)\mapsto mu,$$

which is easily seen to be bounded. Therefore,

$$F:{}_0\mathbb{E}_\mu(a)\times\mathbb{E}_\mu(\infty)\to L_{p,\mu}((0,a);H^1_p(\Omega\setminus\Sigma))$$

belongs to the class C^k. Moreover, we have $F(0,\bar{z})=(M_1(\bar{h})-I)\nabla\cdot\bar{u}$, and

$$D_1F_1(0,\bar{z})\tilde{z}=M_1'(\bar{h})\tilde{h}\nabla\cdot\bar{u},\quad D_1F_2(0,\bar{z})\tilde{z}=(M_1(\bar{h})-M_1(h_0))\nabla\cdot\tilde{u}.$$

This implies

$$|F(0,\bar{z})|_{L_{p,\mu}(J;H^1_p)}\le|M_1(\bar{h})-I|_{C(J;C^1_b)}|\nabla\bar{u}|_{L_{p,\mu}(J;H^1_p)}\to 0,$$

as $a\to 0$. Similarly

$$|D_1F_2(0,\bar{z})\tilde{z}|_{L_{p,\mu}(J;H^1_p)}\le|M_1(\bar{h})-M_1(h_0))|_{C(J;C^1_b)}|\tilde{u}|_{L_{p,\mu}(J;H^2_p)}\le\eta|\tilde{z}|_{\mathbb{E}_\mu(a)},$$

provided $a>0$ is small enough. Moreover, we also have

$$|D_1F_1(0,\bar{z})\tilde{z}|_{L_{p,\mu}(J;H^1_p)}=|M_1'(\bar{h})|_{C(J;C^1)}|\tilde{z}|_{C(J;C^2)}|\nabla\bar{u}|_{L_{p,\mu}(H^1_p)}\le\eta|\tilde{z}|_{\mathbb{E}_\mu(a)},$$

if $a>0$ is small enough, as $\bar{u}$ is a fixed function, and the embedding

$${}_0\mathbb{E}^u_{h,\mu}(J)\hookrightarrow C(J;C^2(\Sigma))$$

is uniform in a.

As P_0 is bounded linear, the same assertions hold for P_0F.

(b) $H^1_{p,\mu}(J;{}_0\dot{H}^{-1}_p(\Omega))$
This space is needed for Problems **(P2), (P3), (P5)**. Here we observe that for given $\phi\in\dot{H}^1_{p'}(\Omega)$ we have

$$\int_\Omega(P_0F_j)\phi\,dx=\int_\Omega P_0F_jP_0\phi\,dx=\int_\Omega F_jP_0\phi\,dx,$$

hence

$$\begin{aligned}\int_\Omega P_0F_1\phi\,dx&=\int_\Omega(M_1(h)-I)\nabla\cdot\bar{u}P_0\phi\,dx\\&=\int_\Omega\bar{u}\cdot[(I-M_1(h))\nabla\phi-(\operatorname{div}M_1(h)^{\mathsf{T}})P_0\phi]\,dx,\end{aligned}$$

and similarly

$$\int_\Omega P_0 F_2 \phi \, dx = \int_\Omega (M_1(h) - M_1(h_0)) \nabla \cdot \tilde{u} P_0 \phi \, dx$$
$$= \int_\Omega \tilde{u} \cdot [(M_1(h_0) - M_1(h)) \nabla \phi + (\mathrm{div}(M_1(h_0) - M_1(h))^\mathsf{T}) P_0 \phi] \, dx.$$

Now we may differentiate in time, apply Hölder's inequality and Poincaré's inequality to see as in 5.1 above that Condition **(NL)** holds for this nonlinearity.

(c) $H^1_{p,\mu}(J; H^{-1}_{p,\partial\Omega}(\Omega \setminus \Sigma))$. Here the same arguments as in **(b)** are valid, as in this case ϕ vanishes on Σ, and so the projection P_0 is not needed.

5.3 Analysis in Fractional Sobolev Spaces

Before we continue, note that $\mathbb{F}^u_{h,\mu}(a)$ as well as $\mathbb{F}^\theta_{h,\mu}(a)$ are Banach algebras, due to the restriction $1 \geq \mu > \frac{1}{2} + \frac{n+2}{2p}$. In fact, this follows easily from the embeddings

$$\mathbb{F}^u_{h,\mu}(a) \hookrightarrow C([0,a]; C(\Sigma)),$$
$$\mathbb{F}^\theta_{h,\mu}(a) \hookrightarrow C([0,a]; C^1(\Sigma)). \tag{9.15}$$

As above, the embedding constants do not depend on a, provided we restrict to functions with time-trace 0 at $t = 0$. Recall that a norm for $W^s_p(\Sigma)$, $s \in (0,1)$, is given by

$$|v|_{W^s_p(\Sigma)} = |v|_{L_p} + [\int_\Sigma \int_\Sigma \frac{|v(x) - v(y)|^p}{|x-y|^{sp+n-1}} d\Sigma(x) d\Sigma(y)]^{1/p}.$$

There are several well-known fundamental estimates in fractional Sobolev spaces, which we want to recall here.

(i) The first one, which we already used before, concerns products and reads as

$$|mw|_{W^s_p} \leq |m|_\infty |w|_{W^s_p} + |w|_\infty |m|_{W^s_p},$$

valid for all functions $m, w \in W^s_p \cap L_\infty$, $s \in (0,1)$. In case $W^s_p(\Sigma) \hookrightarrow C(\Sigma)$ and $m \in C^1(\Sigma)$ it simplifies to

$$|mw|_{W^s_p} \leq C |m|_{C^1} |w|_{W^s_p}.$$

This estimate can easily be extended to the space $\mathbb{F}^u_{h,\mu}(J)$ with $1 \geq \mu > 1/2 + (n+2)/2p$. If

$$m \in \mathbb{G}_\theta(J) := C^{1/2}(J; C(\Sigma)) \cap C(J; C^1(\Sigma)),$$

$w \in \mathbb{F}^u_{h,\mu}(J)$, we have

$$|mw|_{\mathbb{F}^u_{h,\mu}(J)} \leq C |m|_{\mathbb{G}_\theta(J)} |w|_{\mathbb{F}^u_{h,\mu}(J)}.$$

However, we emphasize that the constant C will depend on the length of the interval a, unless w has trace 0 at $t = 0$.

(ii) In the sequel, we will need the following little trick. Let $m \in \mathbb{G}_\theta(J)$, $v \in {}_0\mathbb{F}^\theta_{h,\mu}(J)$, $w \in \mathbb{F}^u_{h,\mu}(\mathbb{R}_+)$ and suppose the trace of w vanishes at time $t = 0$. Then with $s = 1 - 1/p$

$$|mvw|_{\mathbb{F}^u_{h,\mu}(J)} \leq C|m|_{\mathbb{G}_\theta(J)}|vw|_{\mathbb{F}^u_{h,\mu}(J)} \leq C|m|_{\mathbb{G}_\theta(J)}|v|_{W^{s/2}_{p,\mu}(J;W^s_p(\Sigma))}|w|_{\mathbb{F}^u_{h,\mu}(\mathbb{R}_+)},$$

with a constant C independent of a. On the other hand,

$${}_0\mathbb{F}^\theta_{h,\mu}(J) \hookrightarrow {}_0W^{1/2}_{p,\mu}(J;W^s_p(\Sigma)) \hookrightarrow {}_0W^{s/2}_{p,\mu}(J;W^s_p(\Sigma))$$

with uniform embedding constant, and with

$$|v|_{W^{s/2}_{p,\mu}(J;W^s_p(\Sigma))} \leq ca^{1/2p}|v|_{W^{1/2}_{p,\mu}(J;W^s_p(\Sigma))}$$

this yields

$$|mvw|_{\mathbb{F}^u_{h,\mu}(J)} \leq a^{1/2p}C|m|_{\mathbb{G}_\theta(J)}|v|_{\mathbb{F}^\theta_{h,\mu}(J)}|w|_{\mathbb{F}^u_{h,\mu}(\mathbb{R}_+)}.$$

(iii) In a similar, but more elaborate way we also obtain the estimate

$$|bw|_{\mathbb{F}^\theta_{h,\mu}(J)} \leq C|b|_{\mathbb{G}_h(J)}|w|_{\mathbb{F}^\theta_{h,\mu}(\mathbb{R}_+)},$$

with a constant independent of a, provided

$$b \in \mathbb{G}_h(h) := W^s_{p,\mu}((0,a);C(\Sigma)) \cap C([0,a];W^{2s}_p(\Sigma)), \quad s = 1 - 1/2p,$$

has vanishing time trace and $w \in \mathbb{F}^\theta_{h,\mu}(\mathbb{R}_+)$. Of course, ${}_0\mathbb{F}^\theta_{h,\mu}(J)$ is also a Banach algebra, as ${}_0\mathbb{F}^\theta_{h,\mu}(J) \hookrightarrow C([0,a];C^1(\Sigma))$.

Next we consider substitution operators in W^s_p of the form $\phi(v)$ with $\phi \in C^2$.

(iv) Based on the identity

$$\begin{aligned}
&[\phi(v(x)) - \phi(w(x))] - [\phi(v(y)) - \phi(w(y))]\\
&= \int_0^1\int_0^1 \frac{d}{dt}\frac{d}{ds}\phi(s[tv(x) + (1-t)w(x)] + (1-s)[tv(y) + (1-t)w(y)])\,dsdt\\
&= \int_0^1\int_0^1 \phi'(\xi(t,s))([v(x) - w(x)] - [v(y) - w(y)])\,dsdt\\
&\quad + \int_0^1\int_0^1 \phi''(\xi(t,s))([tv(x) + (1-t)w(x)] - [tv(y) + (1-t)w(y)])\cdot\\
&\quad \cdot (s[v(x) - w(x)] + (1-s)[v(y) - w(y)])\,dtds
\end{aligned}$$

we obtain

$$\begin{aligned}
&|[\phi(v(x)) - \phi(w(x))] - [\phi(v(y) - \phi(w(y))]| \leq |\phi'|_\infty|(v(x) - w(x)) - (v(y) - w(y))|\\
&+ |\phi''|_\infty\{|(v(x) - w(x)) - (v(y) - w(y))| + |w(x) - w(y)|\}|v - w|_\infty
\end{aligned}$$

This implies

$$|\phi(v) - \phi(w)|_{W_p^s} \leq |\phi|_{C_b^2}\big[|v-w|_{W_p^s}(1+|v-w|_\infty) + |v-w|_\infty |w|_{W_p^s}\big].$$

This estimate implies that the substitution operator $v \mapsto \phi(v)$ is locally Lipschitz in $W_p^s \cap L_\infty$.

(v) We have

$$l(r,h) := \phi(r+h) - \phi(r) - \phi'(r)h = \int_0^1 (\phi'(r+sh) - \phi'(r))\, dsh,$$

hence with $\delta h = h - \bar{h}$, $\delta r = r - \bar{r}$, $\delta l = l(r,h) - l(\bar{r},\bar{h})$

$$\begin{aligned}
\delta l = \int_0^1 \frac{d}{dt}\Big(\int_0^1 [\phi'(t(r+sh) + (1-t)(\bar{r}+s\bar{h}))& \\
- \phi'(tr + (1-t)\bar{r}]ds(th + (1-t)\bar{h})\Big)\, dt& \\
= \int_0^1\int_0^1 [\phi'(t(r+sh) + (1-t)(\bar{r}+s\bar{h})) - \phi'(tr+(1-t)\bar{r})]\, dsdt\, \delta h& \\
+ \int_0^1\int_0^1 [[\phi''(t(r+sh) + (1-t)(\bar{r}+s\bar{h}))& \\
- \phi''(tr+(1-t)\bar{r})]\delta r(\bar{h} + t\delta h)\, dsdt& \\
+ \int_0^1\int_0^1 \phi''(t(r+sh) + (1-t)(\bar{r}+s\bar{h}))s\delta h(\bar{h}+t\delta h)\, dsdt.&
\end{aligned}$$

This implies by continuity of ϕ' and ϕ''

$$|\delta l| \leq \varepsilon|\delta h| + \varepsilon|\delta r|\max\{|h|,|\bar{h}|\} + |\phi''|_\infty|\delta h|\max\{|\bar{h}|,|h|\},$$

provided $|h|, |\bar{h}|$ are small enough. Setting $r = w(x)$, $\bar{r} = w(y)$, $h = h(x)$, $\bar{h} = h(y)$, we obtain

$$\begin{aligned}
&|[\phi(w(x)+h(x)) - \phi(w(x)) - \phi'(w(x))h(x)] \\
&\qquad - [\phi(w(y)+h(y)) - \phi(w(y)) - \phi'(w(y))h(y)]| \\
&\leq \varepsilon|h(x)-h(y)| + \varepsilon|w(x)-w(y)||h|_\infty + |\phi''|_\infty|h|_\infty|h(x)-h(y)|.
\end{aligned}$$

From this estimate the Fréchet-differentiability of the substitution operator $\Phi : v \mapsto \phi(v)$ in $W_p^s \cap L_\infty$ follows, as soon as $\phi \in C^2$. The derivative is given by

$$(\Phi'(w)h)(x) = \phi'(w(x)h(x), \quad x \in \Sigma, \quad w, h \in W_p^s \cap L_\infty,$$

and so Φ is of class C^1. By induction we easily get $\Phi \in C^k$ if $\phi \in C^{k+1}$, for all $k \in \mathbb{N} \cup \{\infty\}$, and also $\Phi \in C^\omega$ in case $\phi \in C^\omega$, estimating the remainders in the Taylor expansions.

(vi) Let again $s \in (0,1)$, and consider a substitution operator $\Phi : v \mapsto \phi(v)$ in $W_p^{1+s}(\Sigma) \cap W_\infty^1(\Sigma)$. Here the main estimate concerns the derivative of $\phi(v)$, i.e., $\phi'(v)v'$. This case is simpler, as v has more regularity and so $\phi'(v)$ has so as well. By the results of the previous paragraphs it implies that $\Phi \in C^k$, provided $\phi \in C^{k+2}$, for all $k \in \mathbb{N} \cup \{\infty, \omega\}$.

5.4. The Nonlinearities in $\mathbb{F}_{\theta_\Sigma,\mu}$
Here we may argue for the lower order nonlinearities F_{θ_Σ} as in the previous subsection in $L_p(\Sigma)$ and then use the embedding $L_p(\Sigma) \hookrightarrow W_p^{-1/p}(\Sigma)$.

For the highest order terms recall the definition of the norm in $W_p^{-s}(\Sigma)$.

$$|v|_{W_p^{-s}(\Sigma)} = \sup\{\int_\Sigma v\varphi\, d\Sigma : \varphi \in W_{p'}^s(\Sigma),\ |\varphi|_{W_{p'}^s(\Sigma)} \le 1\}.$$

This implies the estimate

$$|(mv|\varphi)| = |(v|m\varphi)| \le |v|_{W_p^{-s}(\Sigma)}|m\varphi|_{W_{p'}^s(\Sigma)} \le C|m|_{C^1(\Sigma)}|v|_{W_p^{-s}(\Sigma)}|\varphi|_{W_{p'}^s(\Sigma)},$$

which yields

$$|mv|_{W_p^{-s}(\Sigma)} \le C|m|_{C^1(\Sigma)}|v|_{W_p^{-s}(\Sigma)}, \quad |mv|_{\mathbb{F}_{\theta_\Sigma,\mu}} \le C|m|_{C(J;C^1(\Sigma))}|v|_{\mathbb{F}_{\theta_\Sigma,\mu}}.$$

The highest order terms are

$$F_1(\tilde{z},\bar{z}) = (\mathcal{A}_{\theta_\Sigma}(\theta_\Sigma,h) - \mathcal{A}_{\theta_\Sigma}(\theta_{\Sigma 0},h_0)) : \nabla_\Sigma^2\tilde{\theta}_\Sigma$$

and $F_2(\tilde{z},\bar{z}) = \partial_t\bar{\theta}_\Sigma + \mathcal{A}_{\theta_\Sigma}(\theta_\Sigma,h) : \nabla_\Sigma^2\bar{\theta}_\Sigma$. As in the previous subsection these are linear in the highest derivative, fortunately.

Here the bilinear map $(m,g) \mapsto mg$ is bounded from $C([0,a];C^1(\Sigma)) \times \mathbb{F}_{\theta_\Sigma,\mu}$ to $\mathbb{F}_{\theta_\Sigma,\mu}$, hence it is real analytic, and so the composition maps

$$(\tilde{z},\bar{z}) \mapsto (\mathcal{A}_{\theta_\Sigma}(\theta_\Sigma,h), \nabla_\Sigma^2\tilde{\theta}_\Sigma, \nabla_\Sigma^2\bar{\theta}_\Sigma) \mapsto F_j(\tilde{z},\bar{z})$$

are of class C^k, provided the coefficient functions d_Σ, κ_Σ are of class C^{k+1}. Then we may estimate similarly as in Section 9.5.1

$$\begin{aligned}|D_1F_1(0,\bar{z})\tilde{z}|_{\mathbb{F}_{\theta_\Sigma,\mu}(J)} &\le |(\mathcal{A}_{\theta_\Sigma}(\bar{\theta}_\Sigma,\bar{h}) - \mathcal{A}_{\theta_\Sigma}(\theta_{\Sigma 0},h_0)|_{C([0,a];C^1(\Sigma))}\\ &\quad \cdot |\nabla_\Sigma^2\tilde{\theta}_\Sigma|_{\mathbb{F}_{\theta_\Sigma,\mu}(J)} \le \eta|z|_{\mathbb{E}_\mu^j(a)},\end{aligned}$$

and

$$|D_1F_2(0,\bar{z})\tilde{z}|_{\mathbb{F}_{\theta_\Sigma,\mu}(J)} \le C|\tilde{z}|_{\mathbb{E}_\mu^j(a)}|\nabla_\Sigma^2\bar{\theta}_\Sigma|_{\mathbb{F}_{\theta_\Sigma,\mu}(J)} \le \eta|\tilde{z}|_{\mathbb{E}_\mu^j(a)},$$

provided a is sufficiently small, depending only on the fixed function $\bar{z}$. This shows Condition **(NL)** for the θ_Σ-components of N_5 and N_6.

5.5. The Nonlinearities in $\mathbb{F}_{h,\mu}^u$
There are only few lower order terms appearing in this boundary space. These

are $u\cdot\nu_\Gamma$ in the h-component of N_3, N_5, $[\![\theta\eta(\theta)]\!]j_\Sigma$ in N_4, and $[\![\psi(\theta)]\!]$, $[\![1/\varrho]\!]j_\Sigma^2\nu_\Gamma$, $[\![1/2\varrho^2]\!]j_\Sigma^2$ in N_4, N_6. These terms can be handled in the same way as the lower order terms in Sections 9.5.1 and 9.5.4. We now study the highest order terms in the same way as above.

(a) $[\![\mathcal{B}_\theta(\theta,h)\nabla\theta]\!]$
We set $F_1(\tilde z,\bar z)=[\![(\mathcal{B}_\theta(\theta,h)-\mathcal{B}_\theta(\theta_0,h_0))\nabla\tilde\theta]\!]$ and $F_2(\tilde z,\bar z)=\partial_t\bar\theta+[\![\mathcal{B}_\theta(\theta,h)\nabla\bar\theta]\!]$. Since $\theta\in\mathbb{G}_\theta(J)=C^{1/2}([0,a];C(\Sigma))\cap C([0,a];C^1(\Sigma))$ we may employ here the bilinear map $(m,g)\mapsto mg$ from $\mathbb{G}_\theta(J)\times\mathbb{F}^u_{h,\mu}(J)$ to $\mathbb{F}^u_{h,\mu}(J)$ which is bounded, to see as before that F_k are of class C^k provided d,l are of class C^{k+1}. For their Fréchet derivatives, by Section 9.5.3**(i)**,**(ii)**, we have the estimates

$$|D_1F_1(0,\bar z)\tilde z|_{\mathbb{F}^u_{h,\mu}(J)}\le C|\mathcal{B}_\theta(\bar\theta,\bar h)-\mathcal{B}_\theta(\theta_0,h_0))|_{\mathbb{G}_\theta(J)}|\nabla\tilde\theta|_{\mathbb{F}^u_{h,\mu}(J)}\le\eta|\tilde z|_{\mathbb{E}_\mu(a)},$$

and

$$\begin{aligned}|D_2F_2(0,\bar z)\tilde z|_{\mathbb{F}^u_{h,\mu}(J)}\le a^{1/2p}C\{&|\partial_\theta\mathcal{B}_\theta(\bar\theta,\bar h)|_{\mathbb{G}_\theta(J)}|\tilde\theta|_{\mathbb{E}_{\theta,\mu}(J)}\\&+|\partial_h\mathcal{B}_\theta(\bar\theta,\bar h)|_{\mathbb{G}_\theta(J)}|\tilde h|_{\mathbb{E}^k_{h,\mu}(J)}\}|\nabla\bar\theta|_{\mathbb{F}^u_{h,\mu}(\mathbb{R}_+)}\le\eta|\tilde z|_{\mathbb{E}_\mu(a)},\end{aligned}$$

provided a is chosen small enough, independently of $\tilde z$, as ${}_0\mathbb{E}_{\theta,\mu}(J)$ embeds into $\mathbb{G}_\theta(J)$ with uniform embedding constant. This shows Condition **(NL)** for this nonlinearity.

(b) $\sigma'(\theta_\Sigma)\nabla_\Sigma\theta_\Sigma$
This term can be handled in the same way. We employ the technique from **(a)** to the functions

$$F_1(\tilde z,\bar z)=(\sigma'(\theta_\Sigma)-\sigma'(\theta_{\Sigma 0})\nabla_\Sigma\tilde\theta_\Sigma,\quad F_2(\tilde z,\bar z)=\sigma'(\theta_\Sigma)\nabla_\Sigma\bar\theta_\Sigma.$$

As a result we obtain that this term is of class C^k, provided $\sigma\in C^{k+2}$, and so Condition **(NL)** is valid.

(c) $S(u,\theta,h)\nu_\Gamma(h)$
We rewrite this term as $\mathcal{B}_u(\theta,h)\nabla u$, where $\mathcal{B}_u$ is a tensor of degree 3 which depends only on $\theta,h,\nabla_\Sigma h$, hence is of lower order. Here we define

$$F_1(\tilde z,\bar z)=(\mathcal{B}_u(\theta,h)-\mathcal{B}_u(\theta_0,h_0))\nabla\tilde u,\quad F_2(\tilde z,\bar z)=\mathcal{B}_u(\theta,h)\nabla\bar u.$$

Then we have the same structure as in **(a)** and so the same argument as there proves **(NL)** for the jump of the normal stress. A similar argument can be employed for $[\![S(u,\theta,h)\nu_\Gamma(h)\cdot\nu_\Gamma(h)/\varrho]\!]$.

(d) $H_\Gamma(h)$
According to Section 2.2.5, the curvature reads as

$$H_\Gamma(h)=\mathcal{C}_0(h):\nabla_\Sigma^2h+\mathcal{C}_1(h),$$

where $\mathcal{C}_j(h)$ depend only on h and $\nabla_\Sigma h$, and hence are of lower order. Therefore, $H_\Gamma(h)$ fortunately has a quasilinear structure. Note that

$$\mathcal{C}_\Sigma(h) = -\mathcal{C}_0(h) : \nabla_\Sigma^2. \tag{9.16}$$

In the following we concentrate on the first term $\mathcal{C}_0(h) : \nabla_\Sigma^2 h$. Here

$$\mathcal{C}_0(h) = \beta(h)(M_0^2(h) - \beta^2(h)M_0^2(h)\nabla_\Sigma h \otimes M_0^2(h)\nabla_\Sigma h)$$

is real analytic in h and $\nabla_\Sigma h$. The highest order contribution of the term

$$H_\Gamma(h) - H_\Gamma(\bar{h}) - H'(h_0)\tilde{h}$$

to N_j in the normal stress condition on Σ is given by $F(\tilde{h},\bar{h}) = F_1(\tilde{h},\bar{h}) + F_2(\tilde{h},\bar{h})$, where

$$F_1(\tilde{h},\bar{h}) = (\mathcal{C}_0(h) - \mathcal{C}_0(h_0)) : \nabla_\Sigma^2\tilde{h}, \quad F_2(\tilde{h},\bar{h}) = (\mathcal{C}_0(h) - \mathcal{C}_0(\bar{h})) : \nabla_\Sigma^2\bar{h},$$

and so $F_i(0,\bar{h}) = 0$, and

$$D_1F_1(0,\bar{h})\tilde{h} = (\mathcal{C}_0(\bar{h}) - \mathcal{C}_0(h_0)) : \nabla_\Sigma^2\tilde{h}, \quad D_1F_2(0,\bar{h})\tilde{h} = \mathcal{C}_0(\bar{h})\tilde{h} : \nabla_\Sigma^2\bar{h}.$$

As in any of the 6 problems,

$$\nabla_\Sigma h \in W_{p,\mu}^{1-1/2p}(J; W_p^{1-1/p}(\Sigma)) \cap L_{p,\mu}(J; W_p^{2-1/p}(\Sigma) \hookrightarrow \mathbb{F}_{h,\mu}^u(J),$$

and we may estimate as in **(a)** to see that

$$|D_1F(0,\bar{h})\tilde{h}|_{\mathbb{F}_{h,\mu}^u(a)} \le \eta|\tilde{z}|_{\mathbb{E}_\mu(a)},$$

if a is small, hence Condition **(NL)** holds also for this nonlinearity.

5.6. The Nonlinearities in $\mathbb{F}_{h,\mu}^\theta$

(a) First we focus on the term $u \cdot \nu_\Gamma/\beta$ from the equation for h in Problem **(P2)**. The terms $[\![\varrho u \cdot \nu_\Gamma/\beta]\!]$ and $\mathcal{P}_\Gamma[\![u]\!] = [\![u]\!] - [\![u \cdot \nu_\Gamma]\!]\nu_\Gamma$ appearing in **(P4)** and **(P6)** can be estimated in the same way.

The corresponding term in N_2 looks like $F = F_1 + F_2$, with

$$F_1(\tilde{z},\bar{z}) = \bar{u} \cdot (M_0(h_0) - M_0(h))\nabla_\Sigma\tilde{h} + \tilde{u} \cdot (M_0(h_0) - M_0(h))\nabla_\Sigma\bar{h} - \tilde{u} \cdot M_0(h)\nabla_\Sigma\tilde{h},$$

and

$$F_2(\tilde{z},\bar{z}) = \bar{u} \cdot (\nu_\Sigma - M_0(h_0)\nabla_\Sigma\bar{h}).$$

Since $\mathbb{F}_{h,\mu}^\theta(a)$ is a multiplication algebra and M_0 is real analytic, it follows easily that F is also real analytic. To verify **(NL) (ii)** for F_1, it is sufficient to show that triple products of the form bvw become small if a is small, where $b \in \mathbb{G}_h(J)$ and $w \in \mathbb{F}_{h,\mu}^\theta(J)$ have zero trace, and $v \in \mathbb{F}_{h,\mu}^\theta(\mathbb{R}_+)$. Here $b = M_0(h_0) - M_0(h)$, and

$v = \bar{u}$, $w = \nabla_\Sigma \tilde{h}$, or bar and tilde in the latter ones interchanged. To do so we first use the Banach algebra property to obtain

$$|bvw|_{\mathbb{F}^\theta_{h,\mu}(J)} \le C|bv|_{\mathbb{F}^\theta_{h,\mu}(J)}|w|_{\mathbb{F}^\theta_{h,\mu}(J)},$$

with a constant C independent of a, as bv and w have both trace zero. Then we apply Section 9.5.3**(iii)** to obtain

$$|bv|_{\mathbb{F}^\theta_{h,\mu}(J)} \le C|b|_{\mathbb{G}_h(J)}|v|_{\mathbb{F}^\theta_{h,\mu}(\mathbb{R}_+)}.$$

As $|b|_{\mathbb{G}_h(J)} \to 0$ as $a \to$, the claim follows for F_1.

Further, we have

$$D_1F_2(0,\bar{z})\tilde{z} = -\bar{u}\cdot M_0'(\bar{h})\tilde{h}\nabla_\Sigma \bar{h},$$

hence we obtain by 5.3(i),(iii), as ${}_0\mathbb{E}^u_{h,\mu}(J) \hookrightarrow \mathbb{G}_h(J)$,

$$\begin{aligned}
|\tilde{h}M_0'(\bar{h})\nabla_\Sigma\bar{h}\cdot\bar{u}|_{\mathbb{F}^\theta_{h,\mu}(J)} &\le C|\tilde{h}M_0'(\bar{h})\nabla_\Sigma\bar{h}|_{\mathbb{F}^\theta_{h,\mu}(J)}|\bar{u}|_{\mathbb{E}^2_\mu(\mathbb{R}_+)}\\
&\le C|\tilde{h}|_{\mathbb{G}_h(J)}|M_0'(\bar{h})\nabla_\Sigma\bar{h}|_{\mathbb{F}^\theta_{h,\mu}(\mathbb{R}_+)}|\bar{u}|_{\mathbb{E}^2_\mu(\mathbb{R}_+)}\\
&\le C|\tilde{h}|_{\mathbb{G}_h(J)}|\bar{h}|_{\mathbb{E}^2_\mu(\mathbb{R}_+)}|\bar{u}|_{\mathbb{E}^2_\mu(\mathbb{R}_+)}.
\end{aligned}$$

In the last step we used fact that $M_0'(\bar{h})$ is a multiplier for $\mathbb{F}_{h,\mu}(\mathbb{R}_+)$. Finally, there is some $\alpha > 0$ such that

$${}_0\mathbb{E}^u_{h,\mu}(J) \hookrightarrow C^{1+\alpha}([0,a];C(\Sigma)) \cap C^\alpha([0,a];C^2(\Sigma)) =: \mathbb{G}^\alpha_h(J),$$

therefore

$$|\tilde{h}|_{\mathbb{G}_h(J)} \le a^\alpha|\tilde{h}|_{\mathbb{G}^\alpha_h(J)} \le Ca^\alpha|\tilde{h}|_{\mathbb{E}^2_{h,\mu}}.$$

This shows that F_2 is also subject to **(NL) (ii)**.

(b) $\varphi(\theta)$

We consider the term $\varphi(\theta)$ appearing in the Gibbs-Thomson condition in Problems **(P1)** and **(P3)**. The corresponding term in N_j, $j = 1,3$, is given by

$$F(\tilde{z},\bar{z}) = r_\theta(\tilde{\theta},\bar{\theta}) = \varphi(\theta) - \varphi(\bar{\theta}) - \varphi'(\bar{\theta})\tilde{\theta}.$$

From Section 9.5.3**(v)**,**(vi)** we see that F is of class C^k provided φ belongs to C^{k+2}, i.e., if $\psi \in C^{k+2}$. Further we obtain $D_1F(0,\bar{z})\tilde{z} = 0$, hence **(NL) (ii)** is satisfied trivially.

(c) $H_\Gamma(h)$

Employing the same decomposition of the relevant nonlinearity F as in Section 9.5.5**(d)**, we may argue as in **(a)** above to obtain

$$|F(0,\bar{z})|_{\mathbb{F}^\theta_{h,\mu}(J)} + |D_1F(0,\bar{z})|_{\mathcal{B}({}_0\mathbb{E}_\mu(a);\mathbb{F}_\mu(a))} \to 0,$$

as $a \to 0$, as the function $\bar{z}$ is fixed.