Chapter 7

Generalized Stokes Problems

This chapter is devoted to maximal L_p-regularity of one-phase linear generalized Stokes problems on domains $\Omega \subset \mathbb{R}^n$ which are either $\mathbb{R}^n$, $\mathbb{R}^n_+$, or domains with compact boundary $\partial\Omega$ of class C^3, i.e., interior or exterior domains. Here we only consider the physically natural boundary conditions no-slip, pure slip, outflow, and free. As in Chapter 6, our approach is based on vector-valued Fourier multiplier theory, perturbation, and localization. It turns out that due to the divergence condition (and the pressure), the analysis for the half-space as well as the localization procedure are much more involved than in the previous chapter. Nevertheless, besides some extra compatibility condition which comes from the divergence condition, the main results will parallel those in Chapter 6.

7.1 The Generalized Stokes Problem on $\mathbb{R}^n$

1.1 Constant Coefficients

We consider the problem

$$\begin{aligned} \partial_t u(t,x) + \mathcal{A}(D)u(t,x) + \nabla\pi(t,x) &= f(t,x) \quad \text{in } \mathbb{R}^n, \\ \operatorname{div} u(t,x) &= g(t,x) \quad \text{in } \mathbb{R}^n, \\ u(0,x) &= u_0(x) \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{7.1}$$

Here $\mathcal{A}(D) = \sum_{k,l=1}^n a^{kl} D_k D_l$ denotes a differential operator with constant coefficient matrices a^{kl} acting on $\mathbb{C}^n$-valued functions. We assume that $\mathcal{A}(D)$ is *strongly elliptic.* As we have seen in the previous chapter, this implies that the problem

$$\begin{aligned} \partial_t u(t,x) + \mathcal{A}(D)u(t,x) &= f(t,x) \quad \text{in } \mathbb{R}^n, \\ u(0,x) &= u_0(x) \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{7.2}$$

has maximal $L_{p,\mu} - L_q$-regularity, $1 < p, q < \infty$, $\mu \in (1/p, 1]$. We want to show that the same assertion is valid for the generalized Stokes problem (7.1). More precisely, we have the following result.

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Theorem 7.1.1. *Let $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and assume that $\mathcal{A}(D)$ is strongly elliptic.*

Then (7.1) has maximal $L_{p,\mu}-L_q$-regularity in the following sense. There is a unique solution (u, π) of (7.1) with $u \in L_{1,loc}(\mathbb{R}_+; H^2_q(\mathbb{R}^n; \mathbb{C}^n))$ such that

$$\partial_t u_k, \partial_i \partial_j u_k \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n)), \quad \pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\mathbb{R}^n)),$$

if and only if the data (f, g, u_0) satisfy the subsequent conditions.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; \mathbb{C}^n))$;

(b) $\partial_t g \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_q(\mathbb{R}^n))$ *and* $\nabla g \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; \mathbb{C}^n))$;

(c) $u_0 \in B^{2(\mu-1/p)}_{qp}(\mathbb{R}^n; \mathbb{C}^n)$ *and* $\operatorname{div} u_0 = g(0)$ *in* $\mathcal{D}'(\mathbb{R}^n)$.

The solution (u, π) depends continuously on the data in the corresponding spaces.

Proof. Necessity follows easily by trace theory. To prove sufficiency of the conditions, note that by the open mapping theorem, the continuity assertion follows as soon as the solvability assertion is proved. So let data (f, g, u_0) be given which are subject to conditions (a), (b), and (c). We first solve the parabolic problem

$$\partial_t v + \mathcal{A}(D)v = f, \quad v(0) = u_0,$$

with maximal $L_{p,\mu} - L_q$-regularity, applying Theorem 6.1.8 and Theorem 4.4.4. Then $w = u - v$ must be a solution of the system

$$\partial_t w + \mathcal{A}(D)w + \nabla\pi = 0, \quad \operatorname{div} w = g_0, \quad w(0) = 0,$$

where $g_0 = g - \operatorname{div} v$ has the same regularity as g and trace 0 at time $t = 0$.

Suppose the pressure π is already known. Taking Fourier transform in the space variables and Laplace transform in the time variable we obtain the system

$$\begin{aligned} \lambda\hat{w} + \mathcal{A}(\xi)\hat{w} &= -i\xi\hat{\pi}, \\ i(\hat{w}|\xi) &= \hat{g}_0. \end{aligned} \tag{7.3}$$

Solving for $\hat{w}$ this yields

$$\hat{w} = -i(\lambda + \mathcal{A}(\xi))^{-1}\xi\hat{\pi},$$

and inserting this relation into the second equation of (7.3) we obtain

$$\hat{g}_0 = ((\lambda + \mathcal{A}(\xi))^{-1}\xi|\xi)\hat{\pi}.$$

Set $\eta = (\lambda + \mathcal{A}(\xi))^{-1}\xi$. Then $\eta \neq 0$ unless $\xi = 0$, and

$$\alpha(\lambda, \xi) := ((\lambda + \mathcal{A}(\xi))^{-1}\xi|\xi) = \overline{\lambda}|\eta|^2 + (\eta|\mathcal{A}(\xi)\eta).$$

Therefore, strong ellipticity of $\mathcal{A}(D)$ implies $\alpha(\lambda, \xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, $\operatorname{Re}\lambda \geq 0$, with $|\xi| + |\lambda| \neq 0$. We may now solve for $\hat{\pi}$ to the result

$$\hat{\pi}(\lambda, \xi) = \hat{g}_0(\lambda, \xi)/\alpha(\lambda, \xi),$$

and for $\hat{w}$ we get

$$\hat{w}(\lambda,\xi) = -i\,\frac{(\lambda+\mathcal{A}(\xi))^{-1}\xi}{\alpha(\lambda,\xi)}\hat{g}_0(\lambda,\xi).$$

Choose $v_0 \in L_{p,loc}(\mathbb{R}_+; H^2_q(\mathbb{R}^n;\mathbb{C}^n))$ such that

$$\partial_t v_{0k}, \partial_i\partial_j v_{0k} \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n)), \quad \operatorname{div} v_0 = g_0.$$

This is possible by assumption (b) on the function g. In fact, setting

$$g_1 = (-\Delta)^{-1/2}\partial_t g_0 + (-\Delta)^{1/2} g_0 \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n))$$

we obtain $g_0 = -\operatorname{div} R(\partial_t - \Delta)^{-1} g_1$, where R denotes the Riesz transform defined by the symbol $i\xi/|\xi|$, i.e., we may choose $v_0 = -R(\partial_t-\Delta)^{-1}g_1$. Therefore,

$$(\partial_t - \Delta)w = T_1(\partial_t-\Delta)v_0, \quad \nabla\pi = T_2(\partial_t-\Delta)v_0,$$

where T_j are defined by means of their Fourier-Laplace symbols

$$\hat{T}_1(\lambda,\xi) = \frac{(\lambda+\mathcal{A}(\xi))^{-1}\xi\otimes\xi}{\alpha(\lambda,\xi)}, \quad \hat{T}_2(\lambda,\xi) = -\frac{\xi\otimes\xi}{(\lambda+|\xi|^2)\alpha(\lambda,\xi)}.$$

Thus, to prove the theorem, it is enough to show that the operators T_j with symbols $\hat{T}_j(\lambda,\xi)$ are bounded in $L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;\mathbb{C}^n))$.

This in turn will follow by an application of the Kalton-Weis theorem and $\mathcal{R}$-boundedness of families of Fourier multipliers. By the scaling $\mu = \lambda/|\xi|^2$, $\zeta = \xi/|\xi|$, we may rewrite the symbols as

$$\hat{T}_1(\lambda,\xi) = \frac{(\mu+\mathcal{A}(\zeta))^{-1}\zeta\otimes\zeta}{\alpha(\mu,\zeta)}, \quad \hat{T}_2(\lambda,\xi) = -\frac{\zeta\otimes\zeta}{(1+\mu)\alpha(\mu,\zeta)}.$$

By strong ellipticity, we already know $\alpha(\mu,\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$, $|\zeta| = 1$, and $\operatorname{Re}\mu \geq 0$. As $|\mu| \to \infty$ we have $\mu\alpha(\mu,\zeta) \to 1$, while $\alpha(\mu,\zeta) \to \alpha(0,\zeta) = (\mathcal{A}(\zeta)^{-1}\zeta|\zeta) \neq 0$ as $\mu \to 0$. This shows that we may extend the range of $\mu \in \mathbb{C}$ to some sector Σ_ϕ, with $\phi > \pi/2$. Furthermore, by compactness, $|(1+\mu)\alpha(\mu,\zeta)| \geq \alpha_0 > 0$ for all such ζ and μ, where α_0 denotes a constant. This implies boundedness of the symbols $\hat{T}_j(\mu|\xi|^2,\xi)$, uniformly in ξ and μ. Furthermore, $\hat{T}_j(\mu|\xi|^2,\xi)$ are homogeneous in ξ of degree 0, and so $|\xi|^{|\beta|}D^\beta_\xi \hat{T}_j(\mu|\xi|^2,\xi)$ are also uniformly bounded in ξ and μ, for each multi-index $\beta \in \mathbb{N}_0^n$. The Lizorkin multiplier theorem, Theorem 4.3.9, then implies that these symbols are Fourier multipliers in $L_q(\mathbb{R}^n;E_j)$ w.r.t. ξ, which yields a holomorphic $\mathcal{R}$-bounded family of operators $\{T_j(\mu)\}_{\mu\in\Sigma_\phi} \subset \mathcal{B}(L_q(\mathbb{R}^n;E_j))$ for $j = 1,2$, where $E_1 = \mathbb{C}^n$, $E_2 = \mathbb{C}$. By canonical extension, it is also $\mathcal{R}$-bounded in $L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;E_j))$. Since the operator $L := \partial_t(-\Delta)^{-1}$ admits an $\mathcal{H}^\infty$-calculus in $L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;E_j))$ of angle $\pi/2$, the Kalton-Weis theorem, Theorem 4.5.6, implies boundedness of $T_j(L)$ in $L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;E_j))$. This completes the proof of Theorem 7.1.1. □

1.2 The Generalized Stokes Operator

Let $\mathcal{A}(D)$ be strongly elliptic as in the previous section and consider (7.1) with $(\operatorname{div} f, g, u_0) = 0$. Then, according to Theorem 7.1.1, Problem (7.1) admits a unique solution (u, π) with maximal $L_{p,\mu}-L_q$-regularity, which means

$$u \in L_{1,loc}(\mathbb{R}_+; H_q^2(\mathbb{R}^n; \mathbb{C}^n)), \quad \pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}_q^1(\mathbb{R}^n)),$$
$$\partial_t u_k, \partial_i \partial_j u_k \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n)),$$

whenever $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; \mathbb{C}^n))$.

Define the base space X_0 by means of

$$X_0 = L_{q,\sigma}(\mathbb{R}^n) := \{u \in L_q(\mathbb{R}^n; \mathbb{C}^n) : \operatorname{div} u = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n)\},$$

and let $P_H := I - R \otimes R$ denote the *Helmholtz projection* from $L_q(\mathbb{R}^n; \mathbb{C}^n)$ onto X_0, where R means the Riesz operator defined via its symbol $\tilde{R} = i\xi/|\xi|$, as before. The **generalized Stokes operator** A associated to $\mathcal{A}(D)$ is defined according to

$$(Au)(x) := [P_H \mathcal{A}(D)u](x), \quad x \in \mathbb{R}^n, \tag{7.4}$$

with domain

$$\mathsf{D}(A) := H_q^2(\mathbb{R}^n; \mathbb{C}^n) \cap L_{q,\sigma}(\mathbb{R}^n).$$

Then $u \in L_{1,loc}(\mathbb{R}_+; X_0)$ is the unique solution of the evolution equation

$$\dot{u} + Au = f, \quad t > 0, \quad u(0) = u_0, \tag{7.5}$$

in the base space X_0. It belongs to the maximal regularity class $\partial_t u, Au \in L_{p,\mu}(\mathbb{R}_+; X_0)$, i.e., (7.5) has maximal $L_{p,\mu}$-regularity. Then Theorem 4.4.4 and Proposition 3.5.2 imply that A is $\mathcal{R}$-sectorial with angle $\phi_A < \pi/2$. But even more is true.

Theorem 7.1.2. *Let $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and assume that $\mathcal{A}(D)$ is strongly elliptic. Let A be defined by (7.4) in $X_0 = L_{q,\sigma}(\mathbb{R}^n)$.*

Then $A \in \mathcal{H}^\infty(X_0)$ with $\mathcal{H}^\infty$-angle $\phi_A^\infty \le \phi_{\mathcal{A}}$, where

$$\phi_{\mathcal{A}} \le \max\{|\arg(\mathcal{A}(\xi)v|v)| : \xi \in \mathbb{R}^n, \ v \in \mathbb{C}^n\} < \pi/2.$$

In particular, $A \in \mathcal{RS}(X_0)$ with $\mathcal{R}$-angle $\phi_A^R \le \phi_{\mathcal{A}}$, and (7.5) has maximal $L_{p,\mu}-L_q$-regularity.

Proof. From the previous subsection we have for the resolvent $(\lambda + A)^{-1}$ of A the symbolic representation

$$\mathcal{F}(\lambda + A)^{-1}(\xi) = [I - (\lambda + \mathcal{A}(\xi))^{-1}\xi \otimes \xi/\alpha(\lambda, \xi)](\lambda + \mathcal{A}(\xi))^{-1}, \quad \xi \in \mathbb{R}^n,$$

where $\alpha(\lambda, \xi) = ((\lambda + \mathcal{A}(\xi))^{-1}\xi|\xi)$. We proceed as in the proof of Theorem 6.1.8. So let $h \in H_0(\Sigma_\phi)$ with $\phi > \phi_{\mathcal{A}}$ be given. Then the symbol of $h(A)$ reads

$$\mathcal{F}h(A)(\xi) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)\mathcal{F}(\lambda - A)^{-1}(\xi)\, d\lambda, \quad \xi \in \mathbb{R}^n,$$

where Γ denotes the contour $\Gamma = (\infty, 0]e^{i\theta} \cup [0,\infty)e^{-i\theta}$ with $\theta \in (\phi_{\mathcal{A}}, \phi)$. Employing the scaling $\xi = \rho\zeta$, $\rho = |\xi|$, and $\lambda = \mu\rho^2$, we obtain

$$\mathcal{F}h(A)(\xi) = \frac{1}{2\pi i}\int_{\Gamma} h(\rho^2\mu)\big(I - (\mu - \mathcal{A}(\zeta))^{-1}\zeta\otimes\zeta/\alpha(\mu,\zeta)\big)(\mu - \mathcal{A}(\zeta))^{-1}\,d\mu.$$

As $\mathsf{n}_0 = \cup_{|\zeta|=1}\mathsf{n}(\mathcal{A}(\zeta))$, where n denotes the numerical range, is compact and contained in $\Sigma_{\phi_{\mathcal{A}}}$, according to Cauchy's theorem, we may deform the contour within Σ_θ into a closed compact contour Γ_0 surrounding n_0 counter-clockwise to obtain the representation

$$\mathcal{F}h(A)(\xi) = \frac{1}{2\pi i}\int_{\Gamma_0} h(\rho^2\mu)\big(I - (\mu - \mathcal{A}(\zeta))^{-1}\zeta\otimes\zeta/\alpha(\mu,\zeta)\big)(\mu - \mathcal{A}(\zeta))^{-1}\,d\mu.$$

By compactness of Γ_0 and $\mathbb{S}^{n-1}$ this implies boundedness of the symbol $\mathcal{F}h(A)(\xi)$ in terms of $|h|_{H^\infty(\Sigma_\phi)}$. As in the proof of Theorem 6.1.8 we also obtain bounds for the derivatives $|\xi|^{|\alpha|}|D_\xi^\alpha\mathcal{F}h(A)(\xi)|$, hence by the classical Mikhlin multiplier theorem we obtain

$$|h(A)|_{\mathcal{B}(L_q)} \le C|h|_{H^\infty(\Sigma_\phi)}, \quad h \in H_0(\Sigma_\phi).$$

Therefore, the generalized Stokes operator A admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\phi_A^\infty \le \phi_{\mathcal{A}}$. □

We observe that for the trace spaces $X_{\gamma,\mu}$ of A we obtain

$$\begin{aligned}X_{\gamma,\mu} &:= (X_0, \mathsf{D}(A))_{\mu-1/p,p} = (L_q(\mathbb{R}^n;\mathbb{C}^n)\cap X_0, H_q^2(\mathbb{R}^n;\mathbb{C}^n)\cap X_0)_{\mu-1/p,p}\\ &= (L_q(\mathbb{R}^n;\mathbb{C}^n), H_q^2(\mathbb{R}^n;\mathbb{C}^n))_{\mu-1/p,p}\cap X_0 = B_{qp}^{2(\mu-1/p)}(\mathbb{R}^n;\mathbb{C}^n)\cap X_0,\end{aligned}$$

for $1 < p, q < \infty$ and $\mu \in (1/p, 1]$. For the fractional power spaces we have

$$\begin{aligned}\mathsf{D}(A^\alpha) &= (X_0, \mathsf{D}(A))_\alpha = (L_q(\mathbb{R}^n;\mathbb{C}^n)\cap X_0, H_q^2(\mathbb{R}^n;\mathbb{C}^n)\cap X_0)_\alpha\\ &= (L_q(\mathbb{R}^n;\mathbb{C}^n), H_q^2(\mathbb{R}^n;\mathbb{C}^n))_\alpha\cap X_0 = H_q^{2\alpha}(\mathbb{R}^n;\mathbb{C}^n)\cap X_0,\end{aligned}$$

for each $\alpha \in (0,1)$, as A admits an $\mathcal{H}^\infty$-calculus.

1.3 Variable Coefficients

(i) We can easily extend Theorem 7.1.1 to the case of variable coefficients with small deviation from constant ones. To see this, let $\mathcal{A}(x,D) = \mathcal{A}_0(D) + \mathcal{A}_1(x,D)$, where $\mathcal{A}_1(x,D) = \sum_{k,l} a_1^{kl}(x)D_kD_l$ with

$$\sup\{|a_1^{kl}(x)| : k,l = 1,\dots n,\ x \in \mathbb{R}^n\} \le \eta.$$

Let S denote the solution operator of the generalized Stokes problem (7.1) from Theorem 7.1.1 for $\mathcal{A}_0(D)$, and T that of the perturbed problem. Then we obtain the identity

$$T = S - SBT, \quad \text{where} \quad B = \begin{bmatrix}\mathcal{A}_1(x,D) & 0\\ 0 & 0\end{bmatrix}.$$

The norm of B as an operator from $H^2_q(\mathbb{R}^n;\mathbb{C}^n)$ into $L_q(\mathbb{R}^n;\mathbb{C}^n)$ is bounded by $C\eta$, where $C>0$ denotes a constant independent of η. Let $|S|$ stand for the norm of the solution operator from the data space to the maximal regularity space. If $|S|C\eta<1$, then a Neumann series argument shows that $T=(I+SB)^{-1}S$ in fact exists and is bounded as a map from the data space to the maximal regularity space as well. Let us state this as

Corollary 7.1.3. *The assertions of Theorem* 7.1.1 *remain valid in the case of variable coefficients $\mathcal{A}(x,D)=\mathcal{A}_0(D)+\mathcal{A}_1(x,D)$, provided the coefficients $a_1^{kl}(x)$ of $\mathcal{A}_1(D)$ are subject to*

$$\sup\{|a_1^{kl}(x)| : k,l=1,\dots n,\, x\in\mathbb{R}^n\}\le\eta,$$

for some sufficiently small $\eta>0$, which only depends on p,q,μ, $\max_{k,l}|a_0^{kl}|$, and the ellipticity constant of $\mathcal{A}_0(D)$.

(ii) Below we will need a certain decomposition of the solution operator. For this purpose observe that from the proof of Theorem 7.1.1 we have the representations

$$\hat{u}=[I-(\lambda+\mathcal{A}(\xi))^{-1}\xi\otimes\xi/\alpha(\lambda,\xi)](\lambda+\mathcal{A}(\xi))^{-1}(\hat{f}+\tilde{u}_0)-i\alpha^{-1}(\lambda+\mathcal{A}(\xi))^{-1}\xi\hat{g},$$

and

$$\hat{\pi}=-i\alpha^{-1}((\lambda+\mathcal{A}(\xi))^{-1}(\hat{f}+\tilde{u}_0)|\xi)+\hat{g}/\alpha.$$

Let us have a closer look at the term $1/\alpha(\lambda,\xi)$. We may write

$$\begin{aligned}\frac{1}{\alpha(\lambda,\xi)}&=(\mu+1)\frac{1}{(\mu+1)((\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}\\&=\mu+1+(\mu+1)[\frac{1}{(\mu+1)((\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}-1]\\&=\mu+1+\frac{(\mu+1)[((\mu+\mathcal{A}(\zeta))-(\mu+1)](\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}{(\mu+1)((\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}\\&=\mu+1+\frac{([\mathcal{A}(\zeta)-1](\mu+1)(\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}{(\mu+1)((\mu+\mathcal{A}(\zeta))^{-1}\zeta|\zeta)}\\&=\lambda/|\xi|^2+1+M_{22}(\lambda,\xi),\end{aligned}$$

where we used again the notation $\mu=\lambda/|\xi|^2$ and $\zeta=\xi/|\xi|$. As in the proof of Theorem 7.1.1, $\xi\mapsto M_{22}(\mu|\xi|^2,\xi)$ is homogeneous of degree 0 and bounded, uniformly in $\xi\in\mathbb{R}^n$ and $\lambda\in\Sigma_\phi$. The arguments given there apply again to the result that there is an $L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n;\mathbb{C}^n)))$-bounded operator S_{22} with symbol $\hat{S}_{22}=M_{22}$. In a similar way we decompose

$$-i\alpha^{-1}(\lambda+\mathcal{A}(\xi))^{-1}\xi=-i\xi/|\xi|^2+(\lambda+|\xi|^2)^{-1}|\xi|M_{21}(\lambda,\xi),$$

where M_{21} is the symbol of an $L_{p,\mu}-L_q$-bounded operator S_{21}, as well as

$$-i((\lambda+\mathcal{A}(\xi))^{-1}\cdot|\xi)/\alpha(\lambda,\xi)=-i(\xi/|\xi|^2|\cdot)+(\lambda+|\xi|^2)^{-1}|\xi|M_{12}(\lambda,\xi),$$

and M_{12} is the symbol of an $L_{p,\mu}-L_q$-bounded operator S_{12}. Last but not least, in the same way we obtain the decomposition

$$\begin{aligned}[I-(\lambda+\mathcal{A}(\xi))^{-1}\xi\otimes\xi/\alpha(\lambda,\xi)](\lambda+\mathcal{A}(\xi))^{-1} &= (\lambda+|\xi|^2)^{-1}(I-\xi\otimes\xi/(\lambda+|\xi|^2)) \\ &\quad + (\lambda+|\xi|^2)^{-2}|\xi|^2 M_{11}(\lambda,\xi),\end{aligned}$$

with M_{11} the symbol of an $L_{p,\mu}-L_q$-bounded operator S_{11}. Thus the solution operator S of the generalized Stokes problem splits as $S = S_0 + S_1$, where the symbols of S_j are given by

$$\hat{S}_0 = \begin{bmatrix} (\lambda+|\xi|^2)^{-1}(I-\xi\otimes\xi/(\lambda+|\xi|^2)) & -i\xi/|\xi|^2 \\ -i\xi^{\mathsf{T}}/|\xi|^2 & (\lambda+|\xi|^2)/|\xi|^2 \end{bmatrix}, \tag{7.6}$$

and

$$\hat{S}_1 = \begin{bmatrix} (\lambda+|\xi|^2)^{-2}|\xi|^2 M_{11}(\lambda,\xi) & (\lambda+|\xi|^2)^{-1}|\xi| M_{12}(\lambda,\xi) \\ (\lambda+|\xi|^2)^{-1}|\xi| M_{21}(\lambda,\xi) & M_{22}(\lambda,\xi) \end{bmatrix}. \tag{7.7}$$

It is interesting to note that S_0 is independent of the coefficients of $\mathcal{A}(D)$, in fact, it is the solution of the classical Stokes problem where $\mathcal{A}(D) = -\Delta$. The operator S_1 factors as

$$\hat{S}_1 = \begin{bmatrix} \frac{1}{\lambda+|\xi|^2} & 0 \\ 0 & \frac{1}{|\xi|} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \frac{|\xi|^2}{\lambda+|\xi|^2} & 0 \\ 0 & |\xi| \end{bmatrix}.$$

Here $M = [M_{ij}]$ is the symbol of an $L_{p,\mu}-L_q$-bounded operator matrix.

It is a remarkable fact that such a decomposition remains valid in the variable coefficient case of Corollary 7.1.3. This can be seen as follows. We have the Neumann series for T which reads

$$T = S + \sum_{n\geq 1}(SB)^n S = S_0 + S_1 + \sum_{n\geq 1}(SB)^n S.$$

By induction we obtain

$$(SB)^n = \begin{bmatrix} (S_{11}\mathcal{A}_1)^n & 0 \\ S_{21}\mathcal{A}_1(S_{11}\mathcal{A}_1)^{n-1} & 0 \end{bmatrix},$$

and

$$(SB)^n S = \begin{bmatrix} (S_{11}\mathcal{A}_1)^n S_{11} & (S_{11}\mathcal{A}_1)^n S_{12} \\ S_{21}\mathcal{A}_1(S_{11}\mathcal{A}_1)^{n-1}S_{11} & S_{21}\mathcal{A}_1(S_{11}\mathcal{A}_1)^{n-1}S_{12} \end{bmatrix}.$$

In symbolic notation, using the factorization of S this yields for the first entry

$$\begin{aligned}&(S_{11}\mathcal{A}_1)^n S_{11} \\ &= \frac{1}{\lambda+|\xi|^2}(1+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{11})(\mathcal{A}_1(D)S_{11})^{n-1}\mathcal{A}_1(\zeta)(1+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{11})\frac{|\xi|^2}{\lambda+|\xi|^2}.\end{aligned}$$

Similarly, for the second entry we get

$$
\begin{aligned}
&(S_{11}\mathcal{A}_1)^n S_{12}\\
&= \frac{1}{\lambda+|\xi|^2}(1+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{11})(\mathcal{A}_1(D)S_{11})^{n-1}\mathcal{A}_1(\zeta)(\frac{-i\xi}{|\xi|}+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{12})|\xi|.
\end{aligned}
$$

In the same way the third entry becomes

$$
\begin{aligned}
&S_{21}(\mathcal{A}_1 S_{11})^n\\
&= \frac{1}{|\xi|}(\frac{-i\xi}{|\xi|}+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{21})(\mathcal{A}_1(D)S_{11})^{n-1}\mathcal{A}_1(\zeta)(1+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{11})\frac{|\xi|^2}{\lambda+|\xi|^2},
\end{aligned}
$$

and finally the last entry is

$$
\begin{aligned}
&S_{21}\mathcal{A}_1(S_{11}\mathcal{A}_1)^{n-1}S_{12}\\
&= \frac{1}{|\xi|}(\frac{-i\xi}{|\xi|}+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{12})(\mathcal{A}_1(D)S_{11})^{n-1}\mathcal{A}_1(\zeta)(\frac{-i\xi}{|\xi|}+\frac{|\xi|^2}{\lambda+|\xi|^2}M_{12})|\xi|.
\end{aligned}
$$

This proves the assertion.

(iii) It is very useful to consider also the shifted Stokes problem

$$
\begin{aligned}
\partial_t u(t,x)+\omega u(t,x)+\mathcal{A}(D)u(t,x)+\nabla\pi(t,x) &= f(t,x) \quad \text{in } \mathbb{R}^n,\\
\operatorname{div} u(t,x) &= g(t,x) \quad \text{in } \mathbb{R}^n,\\
u(0,x) &= u_0(x) \quad \text{in } \mathbb{R}^n,
\end{aligned}
\tag{7.8}
$$

for $t>0$, where $\omega>0$ is fixed. One should note that the substitutions $u_\omega = e^{-\omega t}u$, $f_\omega = e^{-\omega t}f$, and $g_\omega = e^{-\omega t}g$ transform the system (7.1) into (7.8). The advantage lies in the fact that we also obtain estimates for the $L_{p,\mu}-L_q$-norm. In fact, we get the following estimates for the solution u of (7.8). Setting

$$
\mathbb{E}_{0\mu} = L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n;\mathbb{C}^n)), \quad \mathbb{G}_{1\mu} = H^1_{p,\mu}(\mathbb{R}_+;\dot H^{-1}_q(\mathbb{R}^n))\cap L_{p,\mu}(\mathbb{R}_+;H^1_q(\mathbb{R}^n)),
$$

and $X_{\gamma,\mu} = B^{2(\mu-1/p)}_{qp}(\mathbb{R}^n;\mathbb{C}^n)$, there is a constant $C>0$ such that

$$
\begin{aligned}
&\omega|u|_{\mathbb{E}_{0\mu}} + |\partial_t u|_{\mathbb{E}_{0\mu}} + |\nabla^2 u|_{\mathbb{E}_{0\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}}\\
&\qquad\le C\big(|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0,\mu}} + |g|_{\mathbb{G}_{1\mu}} + \omega|g|_{L_{p,\mu}(\dot H^{-1}_q)}\big),
\end{aligned}
\tag{7.9}
$$

for all $(f,g,u_0)\in\mathbb{E}_{0,\mu}\times\mathbb{G}_{1\mu}\times X_{\gamma,\mu}$ such that $\operatorname{div} u_0 = g(0)$ in $\mathcal{D}'(\mathbb{R}^n)$. Here the constant C depends only on p,q,μ and on the symbol $\mathcal{A}(\zeta)$. This result follows directly from the representation of the symbol of S, one only has to observe that the exponential shift replaces λ by $\lambda+\omega$.

(iv) At several places it will be convenient to reduce the full Stokes problem to a problem for the Stokes operator. This can be achieved as follows. We first solve the problem

$$\begin{aligned} \partial_t v + \omega v + \mathcal{A}(D)v + \nabla\pi &= f && \text{in } \mathbb{R}^n, \\ \operatorname{div} v &= g && \text{in } \mathbb{R}^n, \\ v(0) &= v_0 && \text{in } \mathbb{R}^n, \end{aligned} \tag{7.10}$$

for $t > 0$, with ω sufficiently large. Then $w = u - v$ must satisfy

$$\dot{w} + Aw = \omega v, \quad t > 0, \quad w(0) = 0.$$

This reduction will be useful in several situations.

1.4 Localization

Now we are in position for the general case, i.e., we consider the problem

$$\begin{aligned} \partial_t v + \omega v + \mathcal{A}(x, D)v + \nabla q &= f && \text{in } \mathbb{R}^n, \\ \operatorname{div} v &= g && \text{in } \mathbb{R}^n, \\ v(0) &= v_0 && \text{in } \mathbb{R}^n. \end{aligned} \tag{7.11}$$

As before the data (f, g, v_0) are given, and we assume that the differential operator $\mathcal{A}(x, D) = \sum_{k,l} a^{kl}(x) D_k D_l$ has coefficients $a^{kl} \in C_l(\mathbb{R}^n; \mathcal{B}(\mathbb{C}^n))$ and that $\mathcal{A}(x, D)$ is uniformly strongly elliptic, i.e.,

$$\operatorname{Re}(\mathcal{A}(x, \xi)v|v) \geq c_0|\xi|^2|v|^2, \quad \xi \in \mathbb{R}^n,\ v \in \mathbb{C}^n,\ x \in \mathbb{R}^n,$$

with some constant $c_0 > 0$. The parameter $\omega \geq 0$ will be chosen later. Observe that maximal regularity on finite intervals does not depend on ω.

First, we reduce the problem as above to the case $(f, u_0) = 0$, employing the results of Chapter 6. To solve the remaining problem we employ the method of localization. Choose a large ball $B(0, R)$ such that

$$\sup\{|a(x) - a(\infty)| : |x| \geq R\} \leq \eta.$$

Cover the ball $\bar{B}(0, R)$ by finitely many balls $B(x_k, r)$, $k = 1, \dots, N$, such that

$$\sup\{|a(x) - a(x_k)| : x \in B(x_k, r)\} \leq \eta.$$

Fix a C^∞-partition of unity ϕ_k which is subordinate to the covering $\bar{B}(0, R)^c \cup \cup_{k=1}^N B(x_k, r)$ of $\mathbb{R}^n$. The index $k = 0$ corresponds to the chart at infinity. Define local operators $\mathcal{A}_k(D) = \mathcal{A}(x, D)$ for each chart $B(x_k, r)$, $k = 1, \dots, N$, and $\mathcal{A}_0(D) = \mathcal{A}(x, D)$, extend these coefficients to all of $\mathbb{R}^n$, say by reflection at the boundary of the corresponding ball. Corollary 7.1.3 shows that each of these operators has maximal regularity, provided $\eta > 0$ is sufficiently small, but independent of k.

Suppose (v, q) is a solution of (7.11) (with $(f, v_0) = 0$). In the sequel we normalize the pressure by $\int_{B(0,2R)} q(t, x)\, dx = 0$. Setting $v_k = \phi_k v$, $q_k = \phi_k q$,

$g_k = \phi_k g$ we obtain the following problem for the functions v_k and q_k.

$$
\begin{aligned}
\partial_t v_k + \omega v_k + \mathcal{A}_k(D)v_k + \nabla q_k &= (\nabla\phi_k)q + [\mathcal{A}, \phi_k]v && \text{in } \mathbb{R}^n,\\
\operatorname{div} v_k &= g_k + (\nabla\phi_k|v) && \text{in } \mathbb{R}^n,\\
v_k(0) &= 0 && \text{in } \mathbb{R}^n,
\end{aligned}
\tag{7.12}
$$

where $[\mathcal{A}(x,D), \phi_k]v = \mathcal{A}(x,D)(\phi_k v) - \phi_k\mathcal{A}(x,D)v$ means the commutator of $\mathcal{A}(x,D)$ and ϕ_k. Denote the solution operator of the generalized Stokes problem for $\omega + \mathcal{A}_k$ by S^k. Then we have the representation

$$
\begin{bmatrix} v_k \\ q_k \end{bmatrix} = S^k \begin{bmatrix} (\nabla\phi_k)q + [\mathcal{A}, \phi_k]v \\ g_k + (\nabla\phi_k|v) \end{bmatrix}.
$$

Summing over all charts k we deduce

$$
\begin{bmatrix} v \\ q \end{bmatrix} = \sum_{k=0}^{N} \begin{bmatrix} v_k \\ q_k \end{bmatrix} = \sum_{k=0}^{N} S^k \begin{bmatrix} (\nabla\phi_k)q + [\mathcal{A}, \phi_k]v \\ g_k + (\nabla\phi_k|v) \end{bmatrix}.
$$

We decompose this representation of the solution as

$$
\begin{bmatrix} v \\ q \end{bmatrix} = \sum_{k=0}^{N} S^k \begin{bmatrix} 0 \\ g_k \end{bmatrix} + T \begin{bmatrix} q \\ v \end{bmatrix} + Rv,
$$

where

$$
T = \sum_{k=0}^{N} S^k \nabla\phi_k \quad \text{and} \quad R = \sum_{k=0}^{N} S^k \begin{bmatrix} [\mathcal{A}, \phi_k] \\ 0 \end{bmatrix}.
$$

We estimate T and R separately. For this purpose, we define the maximal regularity space

$$
\mathbb{E}_{1\mu} := [H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;\mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\mathbb{R}^n;\mathbb{C}^n))] \times L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\mathbb{R}^n)),
$$

and recall the definition of the spaces $\mathbb{E}_{0\mu}$ and $\mathbb{G}_{1\mu}$ from above. To begin with T, recall that each S^k splits into $S^k = S_0 + S_1^k$, with the same S_0 for each k, since the latter does not depend on the coefficients of $\mathcal{A}_k$. Hence

$$
T = \sum_{k=0}^{N} S^k\nabla\phi_k = \sum_{k=0}^{N} S_1^k\nabla\phi_k + S_0\nabla\sum_{k=0}^{N}\phi_k = \sum_{k=0}^{N} S_1^k\nabla\phi_k,
$$

since ϕ_k forms a partition of unity. Let us decompose T into its components, employing the factorization of S_1 obtained in Section 7.1.3. We have

$$\begin{aligned}
T_{11}q &= (\partial_t + \omega - \Delta)^{-1} \sum_k S_{11}^k(-\Delta)(\partial_t + \omega - \Delta)^{-1}(q\nabla\phi_k),\\
T_{21}q &= (-\Delta)^{-1/2} \sum_k S_{21}^k(-\Delta)(\partial_t + \omega - \Delta)^{-1}(q\nabla\phi_k),\\
T_{12}v &= (\partial_t + \omega - \Delta)^{-1} \sum_k S_{12}^k(-\Delta)^{1/2}(\nabla\phi_k|v),\\
T_{22}v &= (-\Delta)^{-1/2} \sum_k S_{22}^k(-\Delta)^{1/2}(\nabla\phi_k|v).
\end{aligned}$$

Since $\nabla\phi_k$ has compact support also for $k = 0$, we see that $(\nabla\phi_k)q$ belongs to $L_{p,\mu}(\mathbb{R}_+; H_q^1(\mathbb{R}^n))$, and

$$|(-\Delta)^{1/2}(q\nabla\phi_k)|_{\mathbb{E}_{0,\mu}} \le C|\nabla q|_{\mathbb{E}_{0,\mu}}$$

holds with some constant $C > 0$; recall the normalization of the pressure $\int_{B(0,2R)} q(t,x)\,dx = 0$, hence Poincaré's inequality is valid. Therefore,

$$|(-\Delta)(\partial_t + \omega - \Delta)^{-1}(q\nabla\phi_k)|_{\mathbb{E}_{0,\mu}} \le \frac{C}{\sqrt{\omega}}|\nabla q|_{\mathbb{E}_{0,\mu}}.$$

Similarly, there is a constant $C > 0$ such that

$$|(-\Delta)^{1/2}(\nabla\phi_k|v)|_{\mathbb{E}_{0\mu}} \le \frac{C}{\sqrt{\omega}}|(\partial_t + \omega - \Delta)v|_{\mathbb{E}_{0,\mu}}.$$

As a consequence, the operator T satisfies

$$\omega\left|T\begin{bmatrix} q \\ v \end{bmatrix}\right|_{\mathbb{E}_{0\mu}} + \left|T\begin{bmatrix} q \\ v \end{bmatrix}\right|_{\mathbb{E}_{1\mu}} \le \frac{C}{\sqrt{\omega}}\Big(\left|\begin{bmatrix} v \\ q \end{bmatrix}\right|_{\mathbb{E}_{1\mu}} + \omega\left|\begin{bmatrix} v \\ q \end{bmatrix}\right|_{\mathbb{E}_{0\mu}}\Big).$$

Next, R is given by

$$R\begin{bmatrix} q \\ v \end{bmatrix} = \sum_k S^k \begin{bmatrix} [\mathcal{A}, \phi_k]v \\ 0 \end{bmatrix}.$$

The commutator $[\mathcal{A}(x,D), \phi_k]$ is a differential operator of first order, hence

$$\omega\left|R\begin{bmatrix} q \\ v \end{bmatrix}\right|_{\mathbb{E}_{0\mu}} + \left|R\begin{bmatrix} q \\ v \end{bmatrix}\right|_{\mathbb{E}_{1\mu}} \le \frac{C}{\sqrt{\omega}}\Big(\left|\begin{bmatrix} v \\ q \end{bmatrix}\right|_{\mathbb{E}_{1\mu}} + \omega\left|\begin{bmatrix} v \\ q \end{bmatrix}\right|_{\mathbb{E}_{0\mu}}\Big).$$

The above arguments show that, choosing first $\eta > 0$ small and then $\omega > 0$ large enough, there is a constant $C > 0$ such that the estimate

$$\omega|v|_{\mathbb{E}_{0\mu}} + |v|_{\mathbb{E}_{1\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}} \le C\big(|g|_{\mathbb{G}_{1\mu}} + \omega|g|_{L_{p,\mu}(\dot{H}_q^{-1})}\big) \tag{7.13}$$

holds. Therefore, the operator L defined by the first two lines of the left-hand side of (7.11) is injective and has closed range, hence it is semi-Fredholm, for each set of coefficients which are continuous on $\mathbb{R}^n$, admit uniform limits as $|x| \to \infty$, and are uniformly strongly elliptic. Define the family $\mathcal{A}_\tau = \tau\mathcal{A} + (1-\tau)(-\Delta)$. By strong ellipticity, we then may conclude that for each $\tau \in [0,1]$, the corresponding operator L_τ is injective and has closed range. By the continuity of the Fredholm index, it must be constant, i.e., the index is zero for all $\tau \in [0,1]$ since L_0 is bijective by Theorem 7.1.1. This shows that $L = L_1$ is also surjective.

Summarizing, for the problem with variable coefficients

$$\begin{aligned} \partial_t v + \omega v + \mathcal{A}(x,D)v + \nabla\pi = f &\quad \text{in } \mathbb{R}^n, \\ \operatorname{div} v = g &\quad \text{in } \mathbb{R}^n, \\ v(0) = v_0 &\quad \text{in } \mathbb{R}^n, \end{aligned} \tag{7.14}$$

we have proved the following result.

Theorem 7.1.4. *Let $1 < p,q < \infty$, $\mu \in (1/p, 1]$, and assume that $\mathcal{A}(x,D)$ is a second-order differential operator with coefficients $a^{kl} \in C_l(\mathbb{R}^n; \mathcal{B}(\mathbb{C}^n))$ which is uniformly strongly elliptic.*

Then there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, (7.14) has maximal $L_{p,\mu}$–L_q-regularity in the following sense. There is a unique solution (u,π) of (7.14) with

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;\mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\mathbb{R}^n;\mathbb{C}^n)), \quad \pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\mathbb{R}^n)),$$

if and only if the data f, g, u_0 satisfy the subsequent conditions.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n;\mathbb{C}^n))$;

(b) $g \in H^1_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_q(\mathbb{R}^n)) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n))$;

(c) $u_0 \in B^{2(\mu-1/p)}_{qp}(\mathbb{R}^n;\mathbb{C}^n)$ *and* $\operatorname{div} u_0 = g(0)$ *in* $\mathcal{D}'(\mathbb{R}^n)$.

The solution (u,π) depends continuously on the data in the corresponding spaces. Moreover, the estimate (7.9) *is valid.*

We may now define *the generalized Stokes operator A* in the case of variable coefficients as in Section 1.2, to the result that $\omega + A \in \mathcal{MR}_p(X_0)$ for $\omega > \omega_0$. The lower bound for ω_0 is easily seen to be $s(-A)$, the spectral bound of $-A$.

7.2 Generalized Stokes Problems in a Half-Space

In this section we consider the generalized Stokes problem in $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$ with either one of the four boundary conditions explained below. Thus we consider the problem

$$\begin{aligned} (\partial_t + \omega)u + \mathcal{A}(D)u + \nabla\pi = f(t,x) &\quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} u = g(t,x) &\quad \text{in } \mathbb{R}^n_+, \\ u(0,x) = u_0(x) &\quad \text{in } \mathbb{R}^n_+, \end{aligned} \tag{7.15}$$

with $t > 0$. Here, as in Section 7.1, $\mathcal{A}(D) = \sum_{k,l=1}^n a^{kl} D_k D_l$ denotes a strongly elliptic differential operator with constant coefficients acting on $\mathbb{C}^n$-valued functions, $J = \mathbb{R}_+$, and $\omega \geq 0$.

In the sequel, $\mathcal{P}_\Sigma$ denotes the projection onto the tangent bundle of Σ; more precisely, $\mathcal{P}_\Sigma(p)$ means the orthogonal projection onto the tangent space $T_p\Sigma$. With $\nu = -e_n$, the n-th unit vector in $\mathbb{R}^n$, the boundary conditions are either
(i) no slip

$$u = h_0 \quad \text{on } \partial\mathbb{R}^n_+, \tag{7.16}$$

(ii) pure slip

$$(u|\nu) = h_{0\nu},\ \mathcal{P}_\Sigma \nu_k a^{kl} D_l u = h_\Sigma \quad \text{on } \partial\mathbb{R}^n_+, \tag{7.17}$$

(iii) outflow

$$\mathcal{P}_\Sigma u = h_{0\Sigma},\ (\nu_k a^{kl} D_l u|\nu) + i\pi = h_\nu \quad \text{on } \partial\mathbb{R}^n_+, \tag{7.18}$$

(iv) free

$$\nu_k a^{kl} D_l u\nu + i\pi\nu = h \quad \text{on } \partial\mathbb{R}^n_+. \tag{7.19}$$

Of course, appropriate compatibility conditions have to be satisfied. Assuming normal strong ellipticity, as in Section 6.2.5, it is easily verified that the parabolic problem without pressure and divergence condition satisfies the Lopatinskii-Shapiro condition for these boundary conditions, hence is well-posed and has maximal L_p-regularity for $1 < p < \infty$. The main result of this section states that these properties carry over to the generalized Stokes problem.

For this we need some notation. If $\Omega \subset \mathbb{R}^n$ is a C^1 domain, $\Sigma \subset \partial\Omega$ open, $1 < q < \infty$, we define

$$\dot{H}^1_q(\Omega) = \{w \in L_{1,loc}(\Omega) : \nabla w \in L_q(\Omega)\}.$$

By means of standard arguments in the theory of function spaces, $\dot{H}^1_q(\Omega)$ embeds into $H^1_q(\Omega \cap \mathbb{B}(0,R))$, for each $R > 0$. This shows that traces of functions in $\dot{H}^1_q(\Omega)$ are well defined, and that in this space localization is possible. In fact, if χ is $\mathcal{D}(\mathbb{R}^n)$, then by the Poincaré-Wirtinger inequality, $\chi u \in H^1_q(\Omega)$ for each $u \in \dot{H}^1_q(\Omega)$. In the case of $\Omega = \mathbb{R}^n$ it is true that

$$\dot{H}^1_q(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) :\ \nabla u \in L_q(\mathbb{R}^n)\}.$$

We next define

$$\dot{H}^1_{q,\Sigma}(\Omega) = \{w \in L_{1,loc}(\Omega) : \nabla w \in L_q(\Omega),\ w = 0 \text{ on } \Sigma\};$$

in particular, $\dot{H}^1_{q,\emptyset}(\Omega) = \dot{H}^1_q(\Omega)$. Then $\dot{H}^{-1}_{q,\Sigma}(\Omega)$ is defined as

$$\dot{H}^{-1}_{q,\Sigma}(\Omega) := [\dot{H}^1_{q',\partial\Omega\setminus\Sigma}(\Omega)]^*.$$

Especially,

$$\dot{H}_q^{-1}(\Omega) = \dot{H}_{q,\emptyset}^{-1}(\Omega), \quad {}_0\dot{H}_q^{-1}(\Omega) = \dot{H}_{q,\partial\Omega}^{-1}(\Omega).$$

Observe that $\dot{H}_q^{-1}(\Omega)$ consists solely of distributions in Ω, but ${}_0\dot{H}_q^{-1}(\Omega)$ does not have this property.

Assume that (7.15) admits a solution (u, π) in the regularity class

$$u \in H^1_{p,\mu}(J; L_q(\Omega))^n \cap L_{p,\mu}(J; H^2_q(\Omega))^n, \quad \pi \in L_{p,\mu}(J; \dot{H}^1_q(\Omega)).$$

By trace theory, the conditions for the right-hand side f and for the initial value u_0 are the same as in the previous section. They are collected in *condition* **(D)**

(a) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; \mathbb{C}^n))$, $u_0 \in B^{2\mu-2/p}_{qp}(\mathbb{R}^n_+; \mathbb{C}^n)$.

For g, trace theory yields

(b) $g \in H^1_{p,\mu}(\mathbb{R}_+; \dot{H}_q^{-1}(\mathbb{R}^n_+)) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+))$, $\operatorname{div} u_0 = g(0)$.

The boundary data must satisfy

(d0) for no-slip (Dirichlet) boundary conditions:
$h_0 \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; \mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\mathbb{R}^{n-1}; \mathbb{C}^n))$ and for $\mu > 3/2p$ in addition $h(0) = u_0$.

Similarly, we have

(ds) for pure slip boundary conditions:
$h_{0\nu} \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\mathbb{R}^{n-1}))$;
$h_\Sigma \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; \mathbb{C}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; \mathbb{C}^{n-1}))$ and $\mathcal{P}_\Sigma \nu_k a^{kl} D_l u_0 = h_\Sigma(0)$ for $\mu > 3/p$;

(do) for outflow boundary conditions:
$h_{0\Sigma} \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; \mathbb{C}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\mathbb{R}^{n-1}; \mathbb{C}^{n-1}))$;
$h_\nu \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}))$ and $\mathcal{P}_\Sigma u_0 = h_{0\Sigma}(0)$ for $\mu > 3/2p$;

(dn) for free (Neumann) boundary conditions:
$h \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; \mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; \mathbb{C}^n))$ and $\mathcal{P}_\Sigma \nu_k a^{kl} D_l u_0 = P_\Sigma h(0)$ for $\mu > 3/p$.

In case of outflow or Neumann conditions these are all requirements needed. In case of slip or Dirichlet conditions we have the additional property

(e) $(g, h_{0\nu}) \in H^1_{p,\mu}(\mathbb{R}_+; {}_0\dot{H}_q^{-1}(\mathbb{R}^n_+))$ and $h_{0\nu}(0) = (\nu|u_0)$.

Observe that the last condition is a compatibility condition which comes from the divergence equation, as the identity

$$\begin{aligned}
-\int_{\mathbb{R}^n_+} u\cdot\nabla\phi\, d(x,y) &= \int_{\mathbb{R}^n_+} \operatorname{div} u\phi\, d(x,y) - \int_{\mathbb{R}^{n-1}} u\cdot\nu\,\phi\, dx \\
&= \int_{\mathbb{R}^n_+} g\phi\, d(x,y) - \int_{\mathbb{R}^{n-1}} h_{0\nu}\phi\, dx =: \langle (g, h_{0\nu})|\phi\rangle
\end{aligned}$$

shows. Here $\phi \in \dot{H}^1_{q'}(\mathbb{R}^n_+)$.

After these preliminaries we can state the main result of this section.

Theorem 7.2.1. *Let* $1 < p, q < \infty$, $1 \geq \mu > 1/p$, $\mu \neq 3/2p, 3/p$, *and assume that* $\mathcal{A}(D) = \sum_{k,l=1}^n a^{kl} D_k D_l$ *is normally strongly elliptic. Then for each* $\omega > 0$, (7.15) *with boundary conditions* (7.16) *or* (7.17) *or* (7.18) *or* (7.19) *has maximal* $L_{p,\mu} - L_q$*-regularity in the following sense. There is a unique solution* (u, π) *of* (7.15) *in the class*

$$u \in H^1_{p,\mu}(J; L_q(\mathbb{R}^n_+; \mathbb{C}^n)) \cap L_{p,\mu}(J; H^2_q(\mathbb{R}^n_+; \mathbb{C}^n)), \quad \pi \in L_{p,\mu}(J; \dot{H}^1_q(\mathbb{R}^n_+)),$$

satisfying the corresponding boundary condition, and in addition

$$\pi \in F^{1/2-1/2q}_{pq,\mu}(J; L_q(\partial\mathbb{R}^n_+))$$

in case of outflow or Neumann boundary condition, if and only if the data (f, g, h, u_0) *satisfy the conditions* **(D)**. *The solution* u *depends continuously on the data in the corresponding spaces.*

The next subsections are devoted to the proof of this result.

2.1 Reductions

According to the discussion above, we only need to show the sufficiency part. Let data (f, g, u_0) and boundary data h with the corresponding regularity be given. Without loss of generality we may assume $(f, g, u_0) = 0$ and trace 0 of h at $t = 0$ in case it exists. This can be seen as follows. Firstly, extend the initial value to all of $\mathbb{R}^n$ in the class $B^{2\mu-2/p}_{qp}(\mathbb{R}^n)^n$, and extend f trivially to $f \in L_{p,\mu}(J; L_q(\mathbb{R}^n))^n$. Solving the parabolic initial-boundary value problem without pressure and divergence condition on all of $\mathbb{R}^n$ yields a function u_1 in the right regularity class. Then $u_2 := u - u_1$ and $\pi_2 := \pi$ should solve the problem with $(f, u_0) = 0$ and g replaced by $g_1 := g - \operatorname{div} u_1$, which belongs to the same regularity class but has trace 0 at $t = 0$. Extend g_1 evenly in x_n to all of $J \times \mathbb{R}^n$, and solve the full-space generalized Stokes problem (7.1) with $(f, u_0) = 0$ to obtain a pair (u_3, π_3) in the right regularity class. Then the pair (u_4, π_4) defined by $u_4 := u_2 - u_3$, $\pi_4 := \pi_2 - \pi_3$ should solve (7.15) with the boundary condition in question, where $(f, g, u_0) = 0$ and $h_4 = h - \mathcal{B}(D)(u_1 + u_3, \pi_3)$; here $\mathcal{B}(D)$ denotes the boundary operator under consideration. Note that the new boundary datum h belongs to the right regularity class and has trace 0 at $t = 0$ whenever it exists. The compatibility condition (e) becomes now

$$h_{0,\nu} \in {}_0H^1_{p,\mu}(J; \dot{W}_q^{-1/q}(\mathbb{R}^{n-1})).$$

So we have to solve the homogeneous problem (7.15) with one of the inhomogeneous boundary conditions. It is convenient to replace the spatial variables by (x, y), where $x \in \mathbb{R}^{n-1}$ and $y > 0$; recall that $\nu = -e_n$. Similarly we decompose $u = (v, w)$, with $v \in \mathbb{R}^{n-1}$ the tangential and $w \in \mathbb{R}$ the normal velocity.

2.2 Fourier-Laplace Transform

Taking Fourier transform in the tangential space directions, Laplace transform in t we obtain the parameter dependent ODE-problem

$$\begin{aligned}
(\lambda + \mathcal{A}_{11}(\xi + e_n D_y))\hat{v} + \mathcal{A}_{12}(\xi + e_n D_y)\hat{w} + i\xi\hat{\pi} &= 0, \qquad y > 0,\\
\mathcal{A}_{21}(\xi + e_n D_y)\hat{v} + (\lambda + \mathcal{A}_{22}(\xi + e_n D_y))\hat{w} + \partial_y\hat{\pi} &= 0, \qquad y > 0,\\
i\xi^{\mathsf{T}}\hat{v} + iD_y\hat{w} &= 0, \qquad y > 0,\\
\mathcal{B}_{11}(\xi + e_n D_y)\hat{v}(0) + \mathcal{B}_{12}(\xi + e_n D_y)\hat{w}(0) &= \hat{h}_v,\\
\mathcal{B}_{21}(\xi + e_n D_y)\hat{v}(0) + \mathcal{B}_{22}(\xi + e_n D_y)\hat{w}(0) + \mathcal{B}_{23}\hat{\pi}(0) &= \hat{h}_w,
\end{aligned} \tag{7.20}$$

where $\mathcal{B}$ is defined by one of the boundary conditions (7.16), (7.17), (7.18) or (7.19). The parameters ξ and λ satisfy $(\xi, \lambda) \in \mathbb{R}^n \times \Sigma_\phi$, for some $\phi > \pi/2$ and $\xi_n = 0$. Here and below we identify $\xi \in \mathbb{R}^{n-1}$ with $(\xi, 0) \in \mathbb{R}^n$. Introducing the vector

$$\mathsf{x} = [\hat{v}, \hat{w}, \partial_y\hat{v}, \partial_y\hat{w}, \hat{\pi}]^{\mathsf{T}},$$

we rewrite this problem as the first-order system

$$E\partial_y\mathsf{x} + A\mathsf{x} = 0, \quad y > 0, \quad B\mathsf{x}(0) = \hat{h}, \tag{7.21}$$

where the dependence on (λ, ξ) has been dropped. Here the $(2n + 1)$-dimensional square matrix E is defined as

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & A^0_{11} & A^0_{12} & 0\\ 0 & 0 & A^0_{21} & A^0_{22} & -1\\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and A by

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ -(\lambda + A^2_{11}) & -A^2_{12} & A^1_{11} & A^1_{12} & -i\xi\\ -A^2_{21} & -(\lambda + A^2_{22}) & A^1_{21} & A^1_{22} & 0\\ i\xi^{\mathsf{T}} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We used the abbreviations

$$A^2 = (a^{kl}\xi_k\xi_l), \quad A^1 = i(a^{kl}\nu_k\xi_l + a^{kl}\nu_l\xi_k), \quad A^0 = (a^{kl}\nu_k\nu_l)$$

recalling the summation convention. Observe that A^k are homogeneous in ξ of order k; in particular A^0 is constant and invertible by ellipticity. Also note that E does neither depend on λ nor on ξ. The boundary matrices B are

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

in case of Dirchlet conditions,

$$B = \begin{bmatrix} B_{11}^1 & B_{12}^1 & B_{11}^0 & B_{12}^0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

for slip conditions,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ B_{21}^1 & B_{22}^1 & B_{21}^0 & B_{22}^0 & -1 \end{bmatrix}$$

for outflow conditions, and

$$B = \begin{bmatrix} B_{11}^1 & B_{12}^1 & B_{11}^0 & B_{12}^0 & 0 \\ B_{21}^1 & B_{22}^1 & B_{21}^0 & B_{22}^0 & -1 \end{bmatrix}$$

in the case of Neumann conditions. Here B_{ij}^k are homogeneous of order k in ξ, and $B^0 = A^0$. Recall that the Lopatinskii-Shapiro condition means that system (7.21) admits at most one solution $\mathsf{x} \in C_0(\mathbb{R}_+; \mathbb{C}^{2n+1})$, for each $\hat{h} \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^{n-1}$, $\operatorname{Re}\lambda \geq 0$, $\xi \neq 0$. This follows from normal strong ellipticity as in Section 6.2.5, as the crucial identity (6.39) holds also in the Stokes cases for the four boundary conditions under consideration

2.3 The DAE-System
It is our purpose to derive a representation formula of the function x in terms of the given data $\hat{h}$, which is accessible to inversion of the Fourier and Laplace transform.

So, assume that $\mathsf{x} \in C_0(\mathbb{R}_+; \mathbb{C}^{2n+1})$ is a solution of (7.21). Taking Laplace transform $\mathcal{L}$ in y, this yields

$$(zE + A)\mathcal{L}\mathsf{x}(z) = E\mathsf{x}^0, \quad \operatorname{Re} z > 0, \quad B\mathsf{x}^0 = \hat{h},$$

where $\mathsf{x}^0 = \mathsf{x}(0)$ denotes the initial value of x. To obtain a representation of x we have to study the operator pencil $zE + A$. To this end note that E is not invertible but its kernel $\mathsf{N}(E)$ is one-dimensional, and $\mathsf{N}(E^2) = \mathsf{N}(E)$, hence $\mathsf{N}(E) \oplus \mathsf{R}(E) = \mathbb{C}^{2n+1}$. Therefore, (7.21) is a differential-algebraic system of index ≥ 1. This implies that the characteristic polynomial $p(z) = \det(zE + A)$ has at most order $2n$. Let us show that it is precisely of order $2n$, i.e., that the index is 1. This can be seen as follows. Expand $\det(zE+A)$ first w.r.t. the last column and the last row and then w.r.t. the second row. This yields up to a sign

$$p(z) = z^2 \det \begin{bmatrix} z & -1 \\ -(\lambda + A^2_{11}) & zA^0_{11} + A^1_{11} \end{bmatrix} + q(z),$$

where $q(z)$ is of order less than $2n$. Asymptotically this yields for large z

$$p(z) \sim z^2 \det \begin{bmatrix} z & 0 \\ 0 & zA^0_{11} \end{bmatrix} = z^{2n} \det A^0_{11},$$

and $\det A^0_{11} \neq 0$ by strong ellipticity. Therefore, $p(z)$ is of order $2n$. Ellipticity shows also that $p(z)$ has no zeros on the imaginary axis, for $\xi \neq 0$. Now consider the case $\xi = 0$. Then we see by the same procedure that $p(z)$ is in fact a function of z^2, i.e., if z_0 is a zero of p then $-z_0$ is one as well. Unfortunately, $z = 0$ is a solution in case $\xi = 0$, here the degeneracy of the Stokes problem shows up. We have to look at this zero more closely.

The eigenvalue problem for these small zeros $z(\xi)$ for small ξ (or large λ) becomes

$$(A(z,\xi) - \lambda) \begin{bmatrix} \mathsf{x}_1 \\ \mathsf{x}_2 \end{bmatrix} = \begin{bmatrix} i\xi \\ z \end{bmatrix}, \quad (i\xi|\mathsf{x}_1) + z\mathsf{x}_2 = 0,$$

where

$$A(z,\xi) = z^2 A^0 + zA^1(\xi) - A^2(\xi).$$

Since by $\lambda \neq 0$ we have invertibility of $A(z,\xi) - \lambda$, this implies the condition

$$\left(\begin{bmatrix} i\xi \\ z \end{bmatrix} \middle| (A(z,\xi) - \lambda)^{-1} \begin{bmatrix} i\xi \\ z \end{bmatrix} \right) = 0$$

for the small eigenvalues. Writing $(A(z,\xi) - \lambda)^{-1}$ as a Neumann series, this condition becomes

$$z^2 - |\xi|^2 + O((|\xi| + |z|)^4) = 0,$$

which shows that $z = \pm|\xi| + O(|\xi|^2)$ near $\xi = 0$. Therefore, the double zero $z(0) = 0$ for $\xi = 0$ splits into two simple real zeros which behave like $z_1^{\pm}(\xi) \sim \pm|\xi|$ near $\xi = 0$.

Varying now ξ we may conclude that $p(z)$ has exactly n roots with positive real parts, counting with multiplicity, for each $\xi \in \mathbb{R}^{n-1}$, $\operatorname{Re}\lambda > 0$, $\xi \neq 0$, since none of them can cross the imaginary axis by ellipticity.

We may now write

$$\mathcal{L}\mathsf{x}(z) = (zE + A)^{-1} E\mathsf{x}^0, \quad B\mathsf{x}^0 = \hat{h},$$

for the Laplace transform of x. The initial value x^0 thus must be chosen in such a way that $\mathcal{L}\mathsf{x}(z)$ has no poles in the right half-plane, and $B\mathsf{x}^0 = \hat{h}$ holds.

Define the projection P^+ by means of

$$P^+ = \frac{1}{2\pi i}\int_{\Gamma_+}(zE+A)^{-1}E\,dz,$$

where Γ_+ denotes a closed simple contour in the right half-plane surrounding the poles of $(zE+A)^{-1}$, i.e., the zeros of $p(z)$ in the right half-plane. Let z_k, $k=1,\dots,m^+$, denote the zeros of $p(z)$ in the right and for $k=-m^-,\dots,-1$ in the left half-plane. Set

$$P_k = \frac{1}{2\pi i}\int_{|z-z_k|=r}(zE+A)^{-1}E\,dz.$$

These operators are mutually disjoint projections and by Cauchy's theorem we have

$$P^+ = \sum_{k=1}^{m^+} P_k.$$

It can be seen e.g. by Cramer's rule that $(zE+A)^{-1}$ is a rational function which is bounded at ∞, hence admits a limit as $|z|\to\infty$. Therefore

$$z(zE+A)^{-1}E = I-(zE+A)^{-1}A$$

is bounded at ∞ as well and admits the limit

$$Q_0 = \lim_{z\to\infty} z(zE+A)^{-1}E,$$

which is a projection, too. We set $P_0 = I - Q_0$. Obviously, $Q_0\mathsf{x} = 0$ for each $\mathsf{x}\in\mathsf{N}(E)$, and on the other hand, we have

$$EQ_0 = \lim_{z\to\infty} zE(zE+A)^{-1}E = \lim_{z\to\infty}(E - A(zE+A)^{-1}E) = E.$$

This implies that P_0 projects onto the kernel of E. Moreover,

$$\sum_k P_k = P_0 + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{|z|=R}(zE+A)^{-1}E\,dz = P_0+Q_0 = I,$$

which also shows that $P_0P_k = P_kP_0 = 0$ for all $k\neq 0$. Linear algebra implies further that the dimension of the range of P_k is m_k, hence P^+ has dimension n. Since

$$\mathsf{x}^0 = \mathsf{x}(0) = \lim_{t\to 0+}\mathsf{x}(t) = \lim_{\mathbb{R}\ni z\to\infty} z\mathcal{L}\mathsf{x}(z) = \lim_{z\to\infty} z(zE+A)^{-1}E\mathsf{x}^0 = Q_0\mathsf{x}^0,$$

we must have $P_0\mathsf{x}^0 = 0$. It is not difficult to compute the projection P_0, it is given by

$$P_0\mathsf{x} = \frac{\mathsf{x}_4 + (i\xi|\mathsf{x}_1)}{\alpha_0}\begin{bmatrix} 0 \\ {A^0}^{-1}\begin{bmatrix} 0 \\ -1\end{bmatrix} \\ 1\end{bmatrix},$$

where

$$\alpha_0 := \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big| A^{0^{-1}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

is nonzero by ellipticity. Observe that

$$P_0 \mathsf{x}^0 = 0 \quad \Leftrightarrow \quad \mathsf{x}_4^0 + (i\xi | \mathsf{x}_1^0) = 0.$$

For later purposes we also compute the projection $P_1^{\pm}$ corresponding to the small eigenvalue $z_1^{\pm}(\xi) \sim \pm|\xi|$ for small ξ. The analysis of $z_1^{\pm}$ given above shows that an eigenvector is given by

$$e_1^{\pm} = [(A(z_1^{\pm}) - \lambda)^{-1} \begin{bmatrix} i\xi \\ z_1^{\pm} \end{bmatrix}, z_1^{\pm}(A(z_1^{\pm}) - \lambda)^{-1} \begin{bmatrix} i\xi \\ z_1^{\pm} \end{bmatrix}, 1]^{\mathsf{T}} \sim [\frac{1}{\lambda} \begin{bmatrix} -i\xi \\ \mp|\xi| \end{bmatrix}, 0, 1]^{\mathsf{T}}.$$

For a dual eigenvector we get similarly

$$e_1^{*\pm} = [(z_1^{\pm} A^0 + A^1)^{\mathsf{T}} (A(z_1^{\pm})^{\mathsf{T}} - \lambda)^{-1} \begin{bmatrix} i\xi \\ z_1^{\pm} \end{bmatrix}, (A(z_1^{\pm})^{\mathsf{T}} - \lambda)^{-1} \begin{bmatrix} i\xi \\ z_1^{\pm} \end{bmatrix}, -1]^{\mathsf{T}},$$

hence

$$e_1^{*\pm} \sim [0, \frac{1}{\lambda} \begin{bmatrix} -i\xi \\ \mp|\xi| \end{bmatrix}, -1]^{\mathsf{T}}.$$

The projections are then $P_1^{\pm}\mathsf{x} = \frac{(e_1^{*\pm}|E\mathsf{x})}{(e_1^{*\pm}|Ee_1^{\pm})} e_1^{\pm}$. Note that $(e_1^{*\pm}|Ee_1^{\pm}) \sim \pm 2|\xi|/\lambda$ for small ξ, and the asymptotics of $z_1^{\pm}$, $e_1^{\pm}$ and $e_1^{*\pm}$ do not depend on the coefficients a_{ij}^{kl}. Note also that

$$P_1^{+}\mathsf{x}^0 = 0 \quad \Leftrightarrow \quad (e_1^{*+}|E\mathsf{x}^0) = 0,$$

which asymptotically yields the condition

$$\mathsf{x}_5^0 - \frac{\lambda}{|\xi|}\mathsf{x}_2^0 \sim \left(\begin{bmatrix} i\xi/|\xi| \\ 1 \end{bmatrix} \Big| A^0 \begin{bmatrix} \mathsf{x}_3^0 \\ \mathsf{x}_4^0 \end{bmatrix} \right).$$

2.4 The Boundary Value Problem for the DAE-System

To determine the initial value x^0 we therefore have to solve the linear system

$$B\mathsf{x}^0 = \hat{h}, \quad P^{+}\mathsf{x}^0 = 0, \quad P_0\mathsf{x}^0 = 0. \tag{7.22}$$

The Lopatinskii-Shapiro condition is equivalent to the uniqueness of the solution x^0 of this system, for $\xi \neq 0$. To see that it is solvable for each $\hat{h} \in \mathbb{C}^n$, observe that the kernel N of $P^{+} + P_0$ has dimension n. $B : N \to \mathbb{C}^n$ is injective, hence the rank theorem implies that it is also surjective. Thus there is a linear operator $M_0(\lambda, \xi)$ such that $\mathsf{x}^0 = M_0(\lambda, \xi)\hat{h}$ gives the unique solution of (7.22). We have the explicit representation

$$\mathsf{x}^0 = (B^*B + (P^{+})^*P^{+} + P_0^*P_0)^{-1}B^*\hat{h},$$

which shows that $M_0(\lambda, \xi)$ is holomorphic as B, P_0, and P^+ have this property. By homogeneity, λ can even be taken from a sector Σ_ϕ for some $\phi > \pi/2$, but $\xi \neq 0$, in general.

Therefore, we have to look more closely at $\xi = 0$. Note that the projections $P_1^\pm$ are not holomorphic at $\xi = 0$. However, $P_1^0 := P_1^+ + P_1^-$ does have this property. A simple calculation shows that for $\xi = 0$ we have

$$P_1^0 \mathsf{x} = \mathsf{x}_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (\mathsf{x}_5 - A_{21}^0 \mathsf{x}_3 - A_{22}^0 \mathsf{x}_4) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, it is convenient to decompose $\mathsf{x}^0 = \mathsf{y}^0 + \alpha e_1^-$, with $\alpha \in \mathbb{C}$ and $P_1^- \mathsf{y}^0 = 0$. Setting $P = P_0 + P^+ + P_1^-$, we therefore have to solve the system

$$B\mathsf{y}^0 + \alpha B e_1^- = \hat{h}, \quad P\mathsf{y}^0 = 0.$$

From $P\mathsf{y}^0 = 0$ we obtain $\mathsf{y}_2^0 = 0$, $\mathsf{y}_4^0 = 0$ and $\mathsf{y}_5^0 = A_{21}^0 \mathsf{y}_3^0$. Solving the system $(zE + A)\mathsf{x} = E\mathsf{x}^0$, we obtain with $e_1^- = [0,0,0,0,1]^\mathsf{T}$ and $\mathsf{x}_2^0 = \mathsf{y}_2^0 = \mathsf{x}_4^0 = \mathsf{y}_4^0 = 0$ the relations $\mathsf{x}_2 = \mathsf{x}_4 = 0$ and

$$(z^2 A_{11}^0 - \lambda)\mathsf{x}_1 = A_{11}^0(\mathsf{x}_3^0 + z\mathsf{x}_1^0), \quad \mathsf{x}_3 = z\mathsf{x}_1 - \mathsf{x}_1^0, \quad \mathsf{x}_5 = A_{21}^0 \mathsf{x}_3 + \alpha/z,$$

since $\mathsf{x}_5^0 - A_{21}^0 \mathsf{x}_3^0 = \alpha + \mathsf{y}_5^0 - A_{21}^0 \mathsf{y}_3^0 = \alpha$. By strong ellipticity, A_{11}^0 is invertible and has spectrum in the open right half-plane. Hence we may compute further

$$\begin{aligned} \mathsf{x}_1(z) = &\frac{1}{2}(z + \sqrt{\lambda}(A_{11}^0)^{-1/2})^{-1}(\mathsf{y}_1^0 + (A_{11}^0)^{1/2}\mathsf{y}_3^0/\sqrt{\lambda}) \\ &+ \frac{1}{2}(z - \sqrt{\lambda}(A_{11}^0)^{-1/2})^{-1}(\mathsf{y}_1^0 - (A_{11}^0)^{1/2}\mathsf{y}_3^0/\sqrt{\lambda}). \end{aligned}$$

Now, $\mathsf{x}_1(z)$ must be holomorphic in the right half-plane, which means that necessarily we have $\mathsf{y}_3^0 = -\sqrt{\lambda}(A_{11}^0)^{-1/2}\mathsf{y}_1^0$. The boundary condition yields in the Dirichlet and outflow cases $\mathsf{x}_1^0 = \mathsf{y}_1^0 = \hat{h}_1$, and in the slip or Neumann case $\mathsf{x}_3^0 = \mathsf{y}_3^0 = (A_{11}^0)^{-1}\hat{h}_3$. Note that in the outflow and Neumann cases, $\alpha = -\hat{h}_4$ is uniquely determined, in contrast to the Dirichlet or slip case, where α is not unique. In fact, the function $\alpha(\lambda, \xi)$ is discontinuous at $\xi = 0$ for the latter, but holomorphic in the outflow and Neumann case.

Now, for $\xi \neq 0$ small, we may parameterize the kernel of P by a holomorphic map

$$\mathsf{y} \mapsto R(\lambda, \xi)\mathsf{y} := [\mathsf{y}, 0, -\sqrt{\lambda}(A_{11}^0)^{-1/2}\mathsf{y}, 0, -A_{21}^0\sqrt{\lambda}(A_{11}^0)^{-1/2}\mathsf{y}]^\mathsf{T} + R^1(\lambda, \xi)\mathsf{y},$$

where $R^1 = O(|\xi|)$ near $\xi = 0$, with $\mathsf{y} \in \mathbb{C}^{n-1}$. Then we have to solve the equation $BR\mathsf{y} + \alpha B e_1^- = \hat{h}$. For the outflow and Neumann cases it then follows that y

and α are uniquely determined and holomorphic near $\xi = 0$, hence $M_0(\lambda, \xi)$ is holomorphic also at $\xi = 0$.

However, in the other cases things are more involved. We begin with the Dirichlet case. Then the system becomes

$$\mathsf{y} - i\alpha\xi/\lambda = \hat{h}_1 + O(|\xi|)\mathsf{y} + O(|\xi|^2)\alpha, \quad \alpha|\xi|/\lambda = \hat{h}_2 + O(|\xi|)\mathsf{y} + O(|\xi|^2)\alpha,$$

hence

$$\alpha \sim \lambda\hat{h}_2/|\xi|, \quad \mathsf{y} \sim \hat{h}_1 + \frac{i\xi}{|\xi|}\hat{h}_2.$$

In the case of slip conditions we have similarly

$$\begin{aligned} &-\sqrt{\lambda}{A_{11}^0}^{1/2}\mathsf{y} - \alpha A_{11}^0 i\xi/\sqrt{\lambda} = \hat{h}_3 + O(|\xi|)\mathsf{y} + O(|\xi|^2)\alpha, \\ &\alpha|\xi|/\lambda = \hat{h}_2 + O(|\xi|)\mathsf{y} + O(|\xi|^2)\alpha, \end{aligned}$$

and so

$$\alpha \sim \lambda\hat{h}_2/|\xi|, \quad \mathsf{y} \sim -{A_{11}^0}^{1/2}\Big({A_{11}^0}^{-1}\hat{h}_3 + \frac{i\xi}{|\xi|}\hat{h}_2\Big)/\sqrt{\lambda}.$$

Thus there are holomorphic functions $M_{00}(\lambda, \xi)$ and $\alpha_0(\lambda, \xi)$ such that

$$M_0(\lambda, \xi)\hat{h} = M_{00}(\lambda, \xi)\hat{h} + \Big[\frac{\lambda}{|\xi|}\hat{h}_2 + (\alpha_0(\lambda, \xi)|\hat{h})\Big]e_1^-,$$

where $\hat{h}_2$ denotes the normal component of $\hat{u}$ at the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$.

2.5 Harmonic Analysis

We may now write the following representation of the solution $\mathsf{x}(y) = \mathsf{x}(y, \lambda, \xi)$ of (7.21).

$$\mathsf{x}(y, \lambda, \xi) = \frac{1}{2\pi i}\int_{\Gamma_-} e^{zy}(zE + A(\lambda, \xi))^{-1}EM_0(\lambda, \xi)\hat{h}(\lambda, \xi)\,dz, \tag{7.23}$$

where Γ_- denotes a closed simple contour in the open left half-plane surrounding the zeros of $p(z) = p(z, \lambda, \xi)$ in the left half-plane. Employing residue calculus this representation can be rewritten as

$$\mathsf{x}(y, \lambda, \xi) = \sum_{\mathrm{Re} z_k < 0} \mathrm{Res}_{z=z_k(\lambda,\xi)}[e^{zy}(zE + A(\lambda, \xi))^{-1}E]M_0(\lambda, \xi)\hat{h}(\lambda, \xi),$$

hence it is an exponential polynomial in y.

Note that the zeros z_k of $p(z) = p(z, \lambda, \xi)$ depend on ξ and λ, hence the integration path in (7.23) cannot be chosen independently of ξ and λ. To remove this defect a scaling argument will help. With $\rho = \sqrt{\lambda + |\xi|^2}$, the standard parabolic symbol, and $\sigma = \lambda/\rho^2$, $\zeta = \xi/\rho$, the pair (σ, ζ) belongs to a compact subset of $\mathbb{C}^n \setminus \{0\}$. Replace $\hat{\pi}(y)$ by $\hat{\pi}(\rho y)/\rho$, $\mathsf{x}(y)$ by $\mathsf{x}(\rho y)$, Neumann data $\hat{h}_k$ by $\hat{h}_k/\rho$, and leave Dirichlet data unchanged. Then homogeneity of $\mathcal{A}$ and $\mathcal{B}$ yield the modified representation formula

$$\mathsf{x}(y,\lambda,\xi) = \frac{1}{2\pi i}\int_{\Gamma_-} e^{\rho z y}(zE + A(\sigma,\zeta))^{-1} E M_0(\sigma,\zeta)\hat{h}(\lambda,\xi)\,dz. \tag{7.24}$$

Since the poles of $(zE + A(\sigma,\zeta))^{-1}$ stay in a compact set in the left half-plane, we may now choose the contour Γ_- independently of (σ,ζ). This argument parallels the scaling employed in Section 6.2 for the parabolic case.

Observe that the scaling of h induces

$$h \in {}_0\mathbb{F}_{1\mu} := {}_0F^{1-1/2q}_{pq,\mu}(J; L_q(\mathbb{R}^{n-1};\mathbb{C}^n)) \cap L_{p,\mu}(J; B^{2-1/q}_{qq}(\mathbb{R}^{n-1};\mathbb{C}^n)),$$

which is independent of the choice of the boundary conditions. Let

$$L := (\partial_t + \omega - \Delta_x)^{1/2}, \quad \mathsf{D}(L) = {}_0H^{1/2}_{p,\mu}(J; L_q(\mathbb{R}^{n-1};\mathbb{C}^n)) \cap L_{p,\mu}(J; H^1_q(\mathbb{R}^{n-1};\mathbb{C}^n)).$$

Then by Lemma 6.2.4 with $m=1$, $h \in Y$ implies $\hat{v}(y) := L^2 e^{-L\cdot} h \in \mathbb{E}_{0\mu}$. The symbol of L is $\sqrt{\lambda + |\xi|^2}$ which is precisely ρ. By means of the identity

$$\hat{h} = \int_0^\infty 2\rho e^{-2\rho\bar{y}}\hat{h}\,d\bar{y} = \frac{2}{\rho}\int_0^\infty e^{-\rho\bar{y}}\hat{v}(\bar{y})\,d\bar{y},$$

we may rewrite the representation of $\mathsf{x}(y)$ in the following way.

$$\mathsf{x}(y,\lambda,\xi) = \operatorname{diag}\left[\frac{1}{\rho^2},\frac{1}{\rho^2},\frac{1}{\rho^2},\frac{1}{\rho^2},\frac{1}{\rho|\xi|}\right]\int_0^\infty \hat{k}(y,\bar{y},\lambda,\xi)\hat{v}(\bar{y},\lambda,\xi)\,d\bar{y}, \tag{7.25}$$

where the Fourier-Laplace transform of k is given by

$$\hat{k}(y,\bar{y},\lambda,\xi) = \frac{1}{i\pi}\int_{\Gamma_-} e^{\rho(yz-\bar{y})} D(\rho,|\xi|)(zE + A(\sigma,\zeta))^{-1} E M_0(\sigma,\zeta)\,dz, \tag{7.26}$$

where $D(\rho,|\xi|) = \operatorname{diag}[\rho,\rho,\rho,\rho,|\xi|]$.

It remains to be shown that the integral operator $K(\lambda)$ with operator-valued kernel $k(y,\bar{y},\lambda,D_x)$ is $\mathcal{R}$-bounded from $L_q(\mathbb{R}_+; L_q(\mathbb{R}^{n-1};\mathbb{C}^n))$ to $L_q(\mathbb{R}_+; L_q(\mathbb{R}^{n-1};\mathbb{C}^{2n+1})$, where the symbol of $K(y,\bar{y},\lambda,D_x)$ is $\hat{k}(y,\bar{y},\lambda,\xi)$ from (7.26). This will imply that u belongs to the maximal regularity space, and the remaining regularity statements concerning the pressure π follow from the equations.

2.6 Large Frequencies

However, due to the presence of the small eigenvalues $z_1^\pm(\xi)$ introduced above, there are difficulties at $\zeta = 0$. We have to deal with the cases $|\zeta| \le \eta$ and $|\zeta| > \eta$ for some small $\eta > 0$ separately. For this purpose we introduce a cut-off function

$\chi(|\zeta|^2)$, where χ belongs to C^∞, is 1 in $B(0,\eta)$, 0 outside of $B(0,2\eta)$ and between 0 and 1 elsewhere. Then we may decompose $\hat{k}(y,\bar{y},\lambda,\xi)$ as $\hat{k} = \hat{k}_S + \hat{k}_R$, where

$$\hat{k}_R(y,\bar{y},\lambda,\xi) = \frac{1}{2\pi i}\int_{\Gamma_-}(1-\chi(\zeta))D(\rho,|\xi|)e^{\rho(zy-\bar{y})}(zE+A(\sigma,\zeta))^{-1}EM_0(\sigma,\zeta)\,dz. \tag{7.27}$$

Let us first deal with $\hat{k}_R$ and invert the Fourier transform via Mikhlin's theorem. Since Γ_- is compact and contained in the open left half-plane, for $|\zeta| > \eta$, (σ,ζ) runs through a compact subset of $\mathbb{C}^n$, and

$$\mathrm{Re}\,\rho \le |\rho| \le c_\phi \mathrm{Re}\,\rho,$$

we obtain

$$|\hat{k}_R(y,\bar{y},\lambda,\xi)| \le C|\rho|e^{-c|\rho|(y+\bar{y})} \le \frac{C}{y+\bar{y}}, \quad y,\bar{y} > 0,$$

where $C, c > 0$ are independent of $y, \bar{y}$, λ and ξ. This is already sufficient in case $p = 2$, by Plancherel's theorem. For the case of general $p \in (1,\infty)$, note first that

$$|\xi||\frac{1}{\rho}\partial_{\xi_k}\rho| = |\xi||\xi_k/\rho^2| \le |\xi|^2/\rho^2 \le 1,$$

and similarly we have by induction $|\xi|^{|\alpha|}|D_\xi^\alpha \rho| \le M_\alpha$, for each multiindex $\alpha \in \mathbb{N}_0^{n-1}$. Next,

$$|\xi||\partial_{\xi_k}\zeta_j| = |\xi||\delta_{kj}/\rho - \zeta_j\partial_{\xi_k}\rho/\rho^2| \le M_1,$$

and similarly for higher derivatives, by induction. The relation $\sigma = 1 - |\xi|^2/\rho^2$ shows that also $|\xi|^{|\alpha|}|D_\xi^\alpha\sigma|$ is uniformly bounded for each α. Next

$$|\xi||\partial_{\xi_k}e^{\rho(yz-\bar{y})}| \le |\xi||\partial_{\xi_k}\rho/\rho^2||\rho^2(yz-\bar{y})e^{\rho(yz-\bar{y})}| \le C|\rho|e^{-c|\rho|(y+\bar{y})} \le \frac{C}{y+\bar{y}},$$

and similarly by induction also for all higher derivatives. Therefore we may conclude that

$$|\xi|^{|\alpha|}|D_\xi^\alpha \hat{k}_R(y,\bar{y},\lambda,\xi)| \le \frac{M_\alpha}{y+\bar{y}}, \quad y,\bar{y} > 0,$$

for each multi-index α, where M_α is independent of $y, \bar{y}$, and of λ and ξ.

2.7 Small Frequencies

Now we deal with the other part of $\hat{k}$. Since we have enough information about the small eigenvalue $z_1(\xi)$ we may use residue calculus to decompose $\hat{k}_S = \hat{k}_{S0} + \hat{k}_{S1}$, where

$$\hat{k}_{S1}(y,\bar{y},\lambda,\xi) = \frac{1}{i\pi}\int_{\Gamma_-}\chi(\zeta)e^{\rho(yz-\bar{y})}D(\rho,|\xi|)(zE+A(\sigma,\zeta))^{-1}E(I-P_1^-)M_0(\sigma,\zeta)dz,$$

with a fixed contour Γ_- contained in the open left half-plane. The part $\hat{k}_{S1}$ can then be treated as above.

The essential part is $\hat{k}_{S0}$, which is given by

$$\hat{k}_{S0}(y,\bar{y},\lambda,\xi) = \chi(\zeta)e^{\rho(z_1^-(\sigma,\zeta)y-\bar{y})}D(\rho,|\xi|)P_1^-(\sigma,\zeta)M_0(\sigma,\zeta).$$

Using the decomposition $x^0 = y^0 + \alpha e_1^-$ as above, this yields

$$\hat{k}_{S0}(y,\bar{y},\lambda,\xi) = \chi(\zeta)|\xi|e^{\rho(z_1^-(\sigma,\zeta)y-\bar{y})}D(\rho/|\xi|,1)e_1^-(\lambda,\xi)\otimes\alpha(\lambda,\xi).$$

In the outflow and Neumann cases, α is holomorphic and

$$D(\rho/|\xi|,1)e_1^-(\lambda,\xi) = [0,0,-i\xi^{\mathsf{T}}\rho/\lambda,-|\xi|\rho/\lambda,1]^{\mathsf{T}}$$

is bounded and satisfies the Mikhlin condition. Since $z_1^- \sim -|\xi|$ we obtain as above an estimate of the form

$$|\xi|^{|\alpha|}|D_\xi^\alpha\hat{k}_{S0}(y,\bar{y},\lambda,\xi)| \le \frac{M_\alpha}{y+\bar{y}},$$

where M_α is independent of $y,\bar{y}$, ξ and λ.

The argument is more involved in the case of Dirichlet or slip conditions. It is here where the extra time regularity of the normal velocity h_2 comes in. As shown above, α decomposes as

$$\alpha(\lambda,\xi) = \alpha_0(\lambda,\xi) + \frac{\lambda}{|\xi|}\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\alpha_0(\lambda,\xi)$ is holomorphic. Since the term containing α_0 can be treated as before, we concentrate on the extra term. This yields the kernel k_{S00}, defined by

$$\hat{k}_{S00}(y,\bar{y},\lambda,\xi) = \chi(\zeta)|\xi|e^{\rho(z_1^-(\sigma,\zeta)y-\bar{y})}D(\rho/|\xi|,1)e_1^-(\lambda,\xi)\frac{\lambda}{|\xi|}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since by assumption $\hat{h}_2$ is the Fourier-Laplace transform of a function of class ${}_0H^1_{p,\mu}(\mathbb{R}_+;\dot{W}_q^{-1/q}(\mathbb{R}^{n-1}))$, we see that $\lambda\hat{h}_2/|\xi|$ is the Fourier-Laplace transform of a function in $L_{p,\mu}(\mathbb{R}_+;\dot{W}_q^{1-1/q}(\mathbb{R}^{n-1}))$. Thus we obtain $g_0 \in L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n_+))$ such that

$$\hat{g}_0(\bar{y},\lambda,\xi) = |\xi|e^{-|\xi|\bar{y}}\lambda\hat{h}_2(\lambda,\xi)/|\xi|.$$

Writing

$$(\lambda/|\xi|)\hat{h}_2 = 2\int_0^\infty |\xi|e^{-2|\xi|\bar{y}}\lambda\hat{h}_2/|\xi|\,d\bar{y} = 2\int_0^\infty e^{-|\xi|\bar{y}}g_0(\bar{y})\,d\bar{y},$$

we have

$$|\xi|e^{\rho z_1^- y}D(\rho/\xi,1)e_1^-\lambda\hat{h}_2/|\xi| = \int_0^\infty |\xi|e^{\rho z_1^- y-|\xi|\bar{y}}D(\rho/\xi,1)e_1^-\hat{g}_0(\bar{y},\lambda,\xi)\,d\bar{y}$$

and the kernel of this representation can be estimated as before.

2.8 End of the Proof

Summarizing, we have obtained kernels $k(y,\bar{y},\lambda,\xi) \in \mathcal{B}(\mathbb{C}^n)$ such that the family $\{k(y,\bar{y},\lambda,\xi) : \xi \in \mathbb{R}^{n-1}, y,\bar{y} > 0, \lambda \in \Sigma_\phi\}$ satisfies the uniform Mikhlin condition

$$|\xi|^{|\alpha|}|D_\xi^\alpha \hat{k}(y,\bar{y},\lambda,\xi)| \leq \frac{M_\alpha}{y+\bar{y}}, \quad y,\bar{y} > 0,\ \xi \in \mathbb{R}^{n-1},\ \lambda \in \Sigma_\phi.$$

The Lizorkin Fourier multiplier theorem, Theorem 4.3.9, implies that the family of operators

$$\{(y+\bar{y})k(y,\bar{y},\lambda,D_x) :\ y,\bar{y} > 0,\ \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_q(\mathbb{R}^{n-1};\mathbb{C}^n); L_q(\mathbb{R}^{n-1};\mathbb{C}^{2n+1}))$$

is $\mathcal{R}$-bounded. As the Hilbert transform with kernel $k_0(y,\bar{y}) = 1/(y+\bar{y})$ is bounded on $L_q(\mathbb{R}_+)$, Proposition 4.1.5 shows that the family of integral operators $\{K(\lambda) : \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_q(\mathbb{R}^n_+;\mathbb{C}^n); L_q(\mathbb{R}^n_+;\mathbb{C}^{2n+1}))$ with kernels $k(\lambda,y,\bar{y})$ is also $\mathcal{R}$-bounded, hence by canonical extension also in $\mathcal{B}(L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n_+;\mathbb{C}^n)), L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n_+;\mathbb{C}^{2n+1})))$. In addition, this operator family is holomorphic on Σ_ϕ, and as $L_q(\mathbb{R}^n_+)$ is of class $\mathcal{HT}$, the Kalton-Weis theorem, Theorem 4.5.6, implies that $K(\partial_t+\omega)$ is bounded in $\mathbb{E}_{0\mu}$. This completes the proof of Theorem 7.2.1.

2.9 Estimates for the Solution

As in the whole space case it is useful to have estimates for the solution in terms of the data which are uniform in the parameter $\omega \geq \omega_0 > 0$. These follow directly from the proof of Theorem 7.2.1 but are more elaborate than those for the case $\Omega = \mathbb{R}^n$, as they depend on the boundary conditions in question. For this purpose we fix some function spaces as follows.

$$\begin{aligned}
&\mathbb{E}_{0\mu} := L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n_+)^n), \qquad \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n_+)^n) \cap L_{p,\mu}(\mathbb{R}_+;H^2_q(\mathbb{R}^n_+)^n),\\
&\mathbb{G}_{0\mu} := L_{p,\mu}(\mathbb{R}_+;\dot{H}^{-1}_q(\mathbb{R}^n_+)), \quad \mathbb{G}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+;\dot{H}^{-1}_q(\mathbb{R}^n_+)) \cap L_{p,\mu}(\mathbb{R}_+;H^1_q(\mathbb{R}^n_+)),\\
&\mathbb{G}^0_\mu := L_{p,\mu}(\mathbb{R}_+;{}_0\dot{H}^{-1}_q(\mathbb{R}^n_+)), \quad \mathbb{G}^1_\mu := H^1_{p,\mu}(\mathbb{R}_+;{}_0\dot{H}^{-1}_q(\mathbb{R}^n_+)),\\
&\mathbb{F}_{0\mu} := F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+;B^{1-1/q}_{qq}(\mathbb{R}^{n-1})),\\
&\mathbb{F}_{1\mu} := F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^{n-1})) \cap L_{p,\mu}(\mathbb{R}_+;B^{2-1/q}_{qq}(\mathbb{R}^{n-1})),
\end{aligned}$$

and $X_{\gamma,\mu} = B^{2(\mu-1/p)}_{qp}(\mathbb{R}^n_+;\mathbb{C}^n)$. The estimates read as follows. For each $\omega_0 > 0$ there is a constant $C > 0$ such that for all $\omega \geq \omega_0$ and all data subject to the corresponding compatibility conditions, the solution (u,π) satisfies

(i) no-slip

$$\begin{aligned}
\omega|u|_{\mathbb{E}_{0\mu}} + |u|_{\mathbb{E}_{1\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}} \leq C\{&|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + (|g|_{\mathbb{G}_{1\mu}} + \omega|g|_{\mathbb{G}_{0\mu}}) \\
&+ (|h_0|_{\mathbb{F}^n_{1\mu}} + \omega|e^{-L_\omega y}h_0|_{\mathbb{E}_{0\mu}}) + (|(g,h_{0\nu})|_{\mathbb{G}^1_\mu} + \omega|(g,h_{0\nu})|_{\mathbb{G}^0_\mu})\}.
\end{aligned} \tag{7.28}$$

(ii) pure slip

$$\begin{aligned}\omega|u|_{\mathbb{E}_{0\mu}} + |u|_{\mathbb{E}_{1\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}} \le C\{&|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + (|g|_{\mathbb{G}_{1\mu}} + \omega|g|_{\mathbb{G}_{0\mu}}) \\ &+ (|h_{0\nu}|_{\mathbb{F}_{1\mu}} + \omega|e^{-L_\omega y}h_{0\nu}|_{\mathbb{E}_{0\mu}}) + (|h_\Sigma|_{\mathbb{F}^n_{0\mu}} + \omega^{1/2}|e^{-L_\omega y}h_\Sigma|_{\mathbb{E}_{0\mu}}) \\ &+ (|(g,h_{0\nu})|_{\mathbb{G}^1_\mu} + \omega|(g,h_{0\nu})|_{\mathbb{G}^0_\mu})\}.\end{aligned} \tag{7.29}$$

(iii) outflow

$$\begin{aligned}\omega|u|_{\mathbb{E}_{0\mu}} + |u|_{\mathbb{E}_{1\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}} \le C\{&|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + (|g|_{\mathbb{G}_{1\mu}} + \omega|g|_{\mathbb{G}_{0\mu}}) \\ &+ (|h_\nu|_{\mathbb{F}_{0\mu}} + \omega^{1/2}|e^{-L_\omega y}h_\nu|_{\mathbb{E}_{0\mu}}) + (|h_{0\Sigma}|_{\mathbb{F}^n_{1\mu}} + \omega|e^{-L_\omega y}h_{0\Sigma}|_{\mathbb{E}_{0\mu}})\}\end{aligned} \tag{7.30}$$

(iv) free

$$\begin{aligned}\omega|u|_{\mathbb{E}_{0\mu}} + |u|_{\mathbb{E}_{1\mu}} + |\nabla\pi|_{\mathbb{E}_{0\mu}} \le C\{&|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + (|g|_{\mathbb{G}_{1\mu}} + \omega|g|_{\mathbb{G}_{0\mu}}) \\ &+ (|h|_{\mathbb{F}^n_{0\mu}} + \omega^{1/2}|e^{-L_\omega y}h|_{\mathbb{E}_{0\mu}})\}\end{aligned} \tag{7.31}$$

We recall that $L_\omega = (\partial_t + \omega - \Delta)^{-1/2}$. As in the previous chapter, we may estimate

$$|e^{-L_\omega y}h|_{\mathbb{E}_{0\mu}} \le \omega^{-1/2q}|h|_{L_{p,\mu}(L_q)},$$

which has the advantage that only norms of the boundary data are involved, but slightly loosing sharpness. For perturbations of highest order we have to use the sharp estimates, but for localization the weaker version is sufficient.

7.3 General Domains

In this section we state and prove the main result of this chapter, which is maximal $L_{p,\mu} - L_q$-regularity of the generalized Stokes problem on interior and exterior domains. To state the result, let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary $\Sigma := \partial\Omega$ of class C^{3-}, and assume that the coefficients a^{kl} of the normally strongly elliptic differential operator $\mathcal{A}(x,D) = \sum_{k,l=1}^n D_k a^{kl}(x) D_l$ belong to $C^{1-}(\bar\Omega; \mathcal{B}(\mathbb{C}^n))$. Consider the Stokes problem

$$\begin{aligned}(\partial_t + \omega)u + \mathcal{A}(x,D)u + \nabla\pi &= f(t,x) \quad \text{in } \Omega,\\ \operatorname{div} u &= g(t,x) \quad \text{in } \Omega,\\ u(0,x) &= u_0(x) \quad \text{in } \Omega,\end{aligned} \tag{7.32}$$

for $t > 0$, with the following types of natural boundary conditions
(i) no-slip

$$u = h_0 \quad \text{on } \Sigma_d;$$

(ii) pure slip

$$u \cdot \nu = h_{0\nu}, \quad \mathcal{P}_\Sigma \nu_k a^{kl}(x) D_l u = h_\Sigma \quad \text{on } \Sigma_s;$$

(iii) outflow

$$\mathcal{P}_\Sigma u = h_{0\Sigma}, \quad (\nu_k a^{kl}(x) D_l u|\nu) + i\pi = h_\nu \quad \text{on } \Sigma_o;$$

(iv) free

$$\nu_k a^{kl}(x) D_l u + i\pi\nu = h \quad \text{on } \Sigma_n.$$

Here we assume that Σ decomposes disjointly into four parts, i.e.,

$$\Sigma = \Sigma_d \cup \Sigma_s \cup \Sigma_o \cup \Sigma_n,$$

where each set Σ_j is open and closed in Σ. Note that up to three of these sets may be empty. As before, $\mathcal{P}_\Sigma$ denotes the orthogonal projection onto the tangent bundle of Σ. By trace theory, the necessary conditions for solvability of this problems are the following conditions $(\mathbf{D_\Omega})$.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\Omega; \mathbb{C}^n))$, $u_0 \in B^{2\mu-2/p}_{qp}(\Omega; \mathbb{C}^n)$.

(b) $g \in H^1_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_q(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega))$, $\operatorname{div} u_0 = g(0)$.

(d0) for no-slip (Dirichlet) boundary conditions:
$h_0 \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_d; \mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\Sigma_d; \mathbb{C}^n))$ and
for $\mu > 3/2p$ in addition $h_0(0) = u_0$ on Σ_d.

(ds) for pure slip boundary conditions:
$h_{0\nu} \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_s)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\Sigma_s))$;
$h_\Sigma \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_s; T\Sigma) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\Sigma_s; T\Sigma))$ and
$\mathcal{P}_\Sigma \nu_k a^{kl} D_l u_0 = h_\Sigma(0)$ for $\mu > 3/p$;

(do) for outflow boundary conditions:
$h_{0\Sigma} \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_o; T\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2-1/q}_{qq}(\Sigma_0; T\Sigma))$;
$h_\nu \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_o)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\Sigma_o))$ and
$\mathcal{P}_\Sigma u_0 = h_{0\Sigma}(0)$ for $\mu > 3/2p$;

(dn) for free (Neumann) boundary conditions:
$h \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma_n; \mathbb{C}^n)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\Sigma_n; \mathbb{C}^n))$ and
$\mathcal{P}_\Sigma \nu_k a^{kl} D_l u_0 = P_\Sigma h(0)$ for $\mu > 3/p$.

In addition,

(e) $(g, h_{0\nu}) \in H^1_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_{q,\Sigma_d \cup \Sigma_s}(\Omega))$ and $h_{0\nu}(0) = (\nu|u_0)$ on $\Sigma_d \cup \Sigma_s$.

After these preliminaries we can state the main result of this section.

Theorem 7.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary $\Sigma := \partial\Omega$ of class C^{3-}, $1 < p, q < \infty$, $1 \geq \mu > 1/p$, $\mu \neq 3/2p, 3/p$, and assume that $\mathcal{A}(x, D) = \sum_{k,l=1}^n D_k a^{kl}(x) D_l$ is uniformly normally strongly elliptic with coefficients*

$$a^{kl} \in C^{1-}(\bar{\Omega}; \mathcal{B}(\mathbb{C}^n))) \cap C_l(\bar{\Omega}; \mathcal{B}(\mathbb{C}^n)).$$

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, (7.32) with the boundary conditions explained above has maximal $L_{p,\mu} - L_q$-regularity in the following sense. There is a unique solution (u, π) of (7.32) in the class

$$u \in H^1_{p,\mu}(J; L_q(\Omega; \mathbb{C}^n)) \cap L_{p,\mu}(J; H^2_q(\Omega; \mathbb{C}^n)), \quad \pi \in L_{p,\mu}(J; \dot{H}^1_q(\Omega)),$$

satisfying the corresponding boundary condition, and in addition with

$$\pi \in F^{1/2-1/2q}_{pq,\mu}(J; L_q(\Sigma_o \cup \Sigma_n)),$$

if and only if the data (f, g, h_j, u_0) satisfy the conditions $(\mathbf{D_\Omega})$. The solution u depends continuously on the data in the corresponding spaces.

Observe that the pressure π is unique for $\Sigma_o \cup \Sigma_n \neq \emptyset$, but otherwise only unique up to a constant.

By means of this result we can introduce the *generalized Stokes operator* for the four natural boundary conditions. For this, we employ the *Helmholtz-Weyl projection* on $L_q(\Omega; \mathbb{C}^n)$ w.r.t. the given decomposition of Σ, cf. Corollary 7.4.4 below. It is defined in the following way. Given $f \in L_q(\Omega; \mathbb{C}^n)$, solve the following weak mixed Dirichlet-Neumann problem according to Theorem 7.4.3.

$$\begin{aligned} \Delta\phi &= \operatorname{div} f && \text{in } \Omega, \\ \partial_\nu \phi &= f \cdot \nu && \text{on } \Sigma_d \cup \Sigma_s, \\ \phi &= 0 && \text{on } \Sigma_o \cup \Sigma_n, \end{aligned} \tag{7.33}$$

and set $P_{HW} f = f - \nabla\phi$. This is a bounded projection in $L_q(\Omega; \mathbb{C}^n)$ along the gradients onto $X_0 := \{u \in L_q(\Omega; \mathbb{C}^n) : \nabla^* u = 0\}$, where

$$\nabla : \dot{H}^1_{q', \Sigma_o \cup \Sigma_n} \to L_{q'}(\Omega; \mathbb{C}^n).$$

Thus $X_0 = \mathsf{N}(\nabla^*)$, which formally reads

$$X_0 = \{u \in L_q(\Omega; \mathbb{C}^n); \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot \nu = 0 \text{ on } \Sigma_d \cup \Sigma_s\}.$$

Then we define

$$Au := P_{HW} \mathcal{A}(x, D) u, \quad u \in \mathsf{D}(A),$$

with

$$\mathsf{D}(A) = \{u \in H^2_q(\Omega; \mathbb{C}^n) \cap X_0 : \mathcal{P}_\Sigma u = 0 \text{ on } \Sigma_d \cup \Sigma_o,\ \mathcal{P}_\Sigma \nu_k a^{kl} D_l u = 0 \text{ on } \Sigma_s \cup \Sigma_n\}.$$

Problem (7.32) with trivial data except for f and u_0 is equivalent to the abstract evolution equation

$$\dot{u} + \omega u + Au = f, \quad t > 0, \quad u(0) = u_0. \tag{7.34}$$

In fact, one implication is obvious. To obtain the reverse one, we have to recover the pressure π from the weak mixed Dirichlet-Neumann problem

$$\begin{aligned} \Delta\pi &= \operatorname{div}\big(f - \partial_t u - \omega u - \mathcal{A}(x,D)u\big) && \text{in } \Omega, \\ \partial_\nu \pi &= \big(f - \partial_t u - \omega u - \mathcal{A}(x,D)u\big)\cdot\nu && \text{on } \Sigma_d \cup \Sigma_s, \\ \pi &= (\nu \cdot a\nabla u|\nu) && \text{on } \Sigma_o \cup \Sigma_n. \end{aligned} \tag{7.35}$$

By Theorem 7.4.3 this problem admits a unique solution $\pi \in \dot{H}^1_q(\Omega)$. By Theorem 7.3.1 it follows that (7.34) has the property of maximal L_p-regularity, hence the generalized Stokes operators A is the negative generator of an analytic C_0-semigroup in X_0. More precisely we have

Theorem 7.3.2. *Let $\Omega \subset \mathbb{R}^n$ a domain with compact boundary $\Sigma := \partial\Omega$ of class C^{3-}, $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and assume that $\mathcal{A}(x,D)$ is uniformly normally strongly elliptic with coefficients in the class*

$$a^{kl} \in C_b^{1-}(\bar{\Omega}; \mathcal{B}(\mathbb{C}^n))) \cap C_l(\bar{\Omega}; \mathcal{B}(\mathbb{C}^n)),$$

and let the Stokes operator A be defined as above in X_0.

Then (7.34) has maximal $L_{p,\mu}$–L_q-regularity; hence $\omega + A \in \mathcal{MR}_p(X_0)$, for any $\omega > \omega_0 := \mathsf{s}(-A)$.

Consequently the minimal ω_0 in Theorem 7.3.1 is the spectral bound $\mathsf{s}(-A)$. The next subsections are devoted to the proof of Theorem 7.3.1.

3.1 Half-Space: Variable Coefficients

We can easily extend Theorem 7.2.1 to the case of variable coefficients with small deviation from constant ones. To see this, let $\mathcal{A}(x,D) = \mathcal{A}_0(D) + \mathcal{A}_1(x,D)$, where $a_1^{kl} \in C_b^{1-}(\mathbb{R}^n_+; \mathcal{B}(\mathbb{C}^n))$ and

$$\sup\{|a_1^{kl}| : k,l = 1,\dots n, x \in \mathbb{R}^n\} \le \eta.$$

Let S denote the solution operator of the generalized Stokes problem (7.15) from Theorem 7.2.1 for $\mathcal{A}_0(D)$ with one of the boundary conditions under consideration, and let T be that of the perturbed problem. Then we obtain the identity

$$T = S + SBT, \quad \text{where} \quad B = \begin{bmatrix} -\mathcal{A}_1(x,D) & 0 \\ 0 & 0 \\ -\mathcal{B}_1(x,D) & 0 \end{bmatrix}.$$

Here $\mathcal{B}_1$ has the obvious meaning of the corresponding boundary operator generated by the perturbation $\mathcal{A}_1$. The norm of the first component of B as an operator from the maximal regularity space $\mathbb{E}_{1\mu}$ into $\mathbb{E}_{0\mu}$ is bounded by $C\eta$, where $C > 0$ denotes a constant independent of η, and the norm of its third component in the boundary space $\mathbb{F}_{0\mu}$ is estimated as in Section 6.2 to the result

$$|\mathcal{B}_1(\cdot, D)u|_{\mathbb{F}_{0\mu}} \leq \eta|u|_{\mathbb{E}_{1\mu}} + C|a_1|_{C_b^{1-}}|u|_{\mathbb{E}_{1\mu}}^{\gamma}|u|_{\mathbb{E}_{0\mu}}^{1-\gamma},$$

for some $\gamma \in (0, 1]$.

Therefore, as in Section 6.2, a Neumann series argument shows that $T = (I - SB)^{-1}S$ in fact exists, is bounded as a map from the data space to the maximal regularity space as well, and the estimates from Section 7.2.9 remain valid. Let us state this as

Corollary 7.3.3. *The assertions of Theorem 7.2.1 as well as the estimates* (7.28), (7.29), (7.30), (7.31) *remain valid in the case of variable coefficients*

$$\mathcal{A}(x, D) = \mathcal{A}_0(D) + \mathcal{A}_1(x, D),$$

provided

$$a_1^{kl} \in C_b^{1-}(\mathbb{R}^n_+; \mathcal{B}(\mathbb{C}^n)) \quad \textit{and} \quad \sup\{|a_1^{kl}(x)| : k, l = 1, \dots n, x \in \mathbb{R}^n\} \leq \eta,$$

uniformly for $0 < \eta \leq \eta_0$.

3.2 Bent Half-Spaces

In contrast to the parabolic case, we only are able to consider bent half-spaces which are tangentially close to a planar boundary. This comes from the fact that the Stokes-problem has no invariance properties except for the trivial ones, i.e., translation and rotation. As before, replacing the variable $x \in \mathbb{R}^n_+$ by (x, y), the bent half-space is defined by the mapping

$$\Phi(x, y) = [x, y + \phi(x)]^{\mathsf{T}}, \quad x \in \mathbb{R}^{n-1},\ y \geq 0.$$

Then $\Omega = \Phi(\mathbb{R}^n_+)$ and $\Gamma := \partial\Omega = \Phi(\mathbb{R}^{n-1} \times \{0\}) = \Phi(\Sigma)$, where $\Sigma = \mathbb{R}^{n-1} \times \{0\}$. For the normal of Γ we obtain

$$\nu_\Gamma(x, \phi(x)) = \beta(x)[\nabla\phi(x), -1]^{\mathsf{T}}, \quad \beta(x) = (1 + |\nabla\phi(x)|^2)^{-1/2},\ x \in \mathbb{R}^{n-1}.$$

We employ the transformation to the domain $\mathbb{R}^{n-1}$ by means of

$$u(\Phi(x, y)) = \bar{u}(x, y), \quad \pi(\Phi(x, y)) = \bar{\pi}(x, y), \quad x \in \mathbb{R}^{n-1},\ y \geq 0.$$

This implies the relations

$$\nabla\pi \circ \Phi(x, y) = (M\nabla)\bar{\pi}, \quad \nabla u \circ \Phi(x, y) = (M\nabla)\bar{u},$$

where

$$M(x, y) = (\partial\Phi)^{-1}(x, y) = \begin{bmatrix} I & -\nabla\phi \\ 0 & 1 \end{bmatrix} = M(x).$$

Similarly,

$$\operatorname{div} u \circ \Phi(x,y) = \operatorname{tr}(M(x)\nabla\bar{u}(x,y)).$$

In more explicit form, these identities read

$$\nabla\pi \circ \Phi = \nabla\bar{\pi} - \nabla\phi\partial_y\bar{\pi}, \quad \operatorname{div} u \circ \Phi = \operatorname{div}\bar{u} - \nabla\phi \cdot \partial_y\bar{u}.$$

Using these transformation laws, the problem on a bent half-space transforms to a problem on a half-space, which reads as follows, dropping the bars.

$$\begin{aligned} (\partial_t + \omega)u + \mathcal{A}^\Phi(D)u + \nabla\pi &= f + \mathcal{A}_1(D)u + B_1\pi && \text{in } \mathbb{R}^n_+, \\ \operatorname{div} u &= g + B_2 u && \text{in } \mathbb{R}^n_+, \\ u(0) &= u_0 && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{7.36}$$

for $t > 0$. Here $\mathcal{A}^\Phi$ is defined by its coefficients $a_\Phi = \partial\Phi^{-1}(a \circ \Phi)\partial\Phi^{-\mathsf{T}}$, and $\mathcal{A}_1$ is lower order, but contains second-order derivatives of ϕ. The natural boundary conditions are perturbed in the following way.
(i) no-slip

$$u = h_0 \quad \text{on } \Sigma_d;$$

(ii) pure slip

$$u \cdot \nu_\Sigma = h_{0\nu}/\beta + B_3 u, \quad \mathcal{P}_\Sigma \nu_\Sigma a_\Phi(x) Du = \mathcal{P}_\Sigma h_\Sigma + B_4 u \quad \text{on } \Sigma_s;$$

(iii) outflow

$$\mathcal{P}_\Sigma u = \mathcal{P}_\Sigma h_{0\Sigma} + B_5 u, \quad (\nu_\Sigma a_\Phi(x) Du | \nu_\Sigma) + i\pi = h_\nu + B_6 u \quad \text{on } \Sigma_o;$$

(iv) free

$$\mathcal{P}_\Sigma \nu_\Sigma a_\Phi(x) Du = \mathcal{P}_\Sigma h + B_4 u, \quad (\nu_\Sigma a_\Phi(x) Du | \nu_\Sigma) + i\pi = h_\nu + B_6 u \quad \text{on } \Sigma_n.$$

Here the perturbation operators are defined as follows.

$$\begin{aligned} B_1\phi &= \nabla\phi\partial_y\pi, & B_2 u &= \nabla\phi \cdot \partial_y u, \\ B_3 u &= u \cdot (\nu_\Sigma - \nu_\Gamma/\beta), & B_4 u &= \mathcal{P}_\Sigma(\mathcal{P}_\Sigma - \mathcal{P}_\Gamma)\nu_\Sigma a_\Phi \nabla u, \\ B_5 u &= \mathcal{P}_\Sigma(\mathcal{P}_\Sigma - \mathcal{P}_\Gamma)u, & B_6 u &= \nu_\Sigma a_\Phi \nabla u(\nu_\Sigma - \nu_\Gamma). \end{aligned}$$

Observe that

$$\begin{aligned} \nu_\Sigma - \nu_\Gamma &= [-\beta\nabla\phi, |\nabla\phi|^2/(1+\beta)]^\mathsf{T}, \\ \mathcal{P}_\Sigma - \mathcal{P}_\Gamma &= \nu_\Gamma \otimes \nu_\Gamma - \nu_\Sigma \otimes \nu_\Sigma. \end{aligned}$$

Both are analytic in $\nabla\phi$ and of order $\nabla\phi$ if the latter is close to zero, hence all perturbation operators B_j are of order $\nabla\phi$.

This is a perturbation of the half-space problem. The estimates for the right-hand sides are the same as in Section 6, they are small if $|\nabla\phi|_{L_\infty}$ is small. The exception is that we need to consider B_2u in $L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+))$, as well as the pair (B_2u, B_3u) in $H^1_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_q(\mathbb{R}^n_+))$. We easily obtain

$$\begin{aligned}|B_2u|_{L_{p,\mu}(H^1_q)} + \omega|B_2u|_{L_{p,\mu}(\dot{H}^{-1}_q)} &\leq |\nabla\phi|_{L_\infty}|u|_{L_{p,\mu}(H^2_q)} + |\nabla^2\phi|_{L_\infty}|u|_{L_{p,\mu}(H^1_q)}\\ &\leq \big(|\nabla\phi|_{L_\infty} + \eta + \frac{C_\eta}{\omega^{1/2}}\big)(|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}\big).\end{aligned}$$

Further, as

$$\int_{\mathbb{R}^n_+} B_2u\psi d(x,y) - \int_{\mathbb{R}^{n-1}} B_3u\psi dx = -\int_{\mathbb{R}^n_+} u\cdot\nabla\phi\partial_y\psi\, d(x,y),$$

it is also clear that

$$|(B_2u, B_3u)|_{H^1_{p,\mu}(\dot{H}^{-1}_q)} + \omega|(B_2u, B_3u)|_{L_{p,\mu}(\dot{H}^{-1}_q)} \leq |\nabla\phi|_{L_\infty}[|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}].$$

Therefore, by perturbation, the half-space result Theorem 7.2.1 is also true in bent half-spaces, provided $\phi \in C^{3-}_b(\mathbb{R}^{n-1})$ and $|\nabla\phi|_{L_\infty}$ is small enough.

Corollary 7.3.4. *The assertions of Theorem* 7.2.1 *as well as the estimates* (7.28), (7.29), (7.30), (7.31) *remain valid in the case of variable coefficients*

$$\mathcal{A}(x, D) = \mathcal{A}_0(D) + \mathcal{A}_1(x, D)$$

in bent half-spaces provided

$$a^{kl}_1 \in C^{1-}_b(\mathbb{R}^n_+; \mathcal{B}(\mathbb{C}^n)) \quad \textit{and} \quad \sup\{|a^{kl}_1(x)| : k, l = 1, \dots n, x \in \mathbb{R}^n_+\} \leq \eta,$$

and

$$\phi \in C^{3-}_b(\mathbb{R}^{n-1}) \quad \textit{and} \quad |\nabla\phi|_{L_\infty} \leq \eta,$$

uniformly for $0 < \eta \leq \eta_0$.

3.3 Pressure Regularity

The pressure π has in general no time regularity. But in special situations we do have regularity in time.

Proposition 7.3.5. *In the situation of Theorem 7.3.1, assume further*

$$\begin{aligned}u_0 = 0, \quad g = 0, \quad \operatorname{div} f = 0 \quad &\text{in } \Omega,\\ h_{0\nu} = 0, \quad f\cdot\nu = 0 \quad &\text{on } \Sigma_0 \cup \Sigma_s.\end{aligned}$$

Then

(i) *If* Ω *is bounded,* $P_0\pi \in H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\Omega))$*, for* $\alpha \in (0, 1/2 - 1/2q)$*, and for any fixed* $s > 1/q$

$$|P_0\pi|_{L_{p,\mu}(L_q)} \leq C\Big(|h_\nu|_{L_{p,\mu}(L_q(\Sigma))} + |u|_{L_{p,\mu}(H^{1+s}_q(\Omega))}\Big),$$

where $P_0 = I$ in case $\Sigma_o \cup \Sigma_n \neq \emptyset$, and $P_0\pi$ denotes the mean zero part of π otherwise.

(ii) *If Ω is unbounded, with $\Omega_R = \Omega \cap B(0,R)$, R large, then $P_{0R}\pi \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\Omega_R)$ for $\alpha < 1/2 - 1/2q$, and for $s > 1/q$*

$$|P_{0R}\pi|_{L_{p\mu}(L_q(\Omega_R)} \le C_R\Big(|h_\nu|_{L_{p,\mu}(L_q(\Sigma))} + |u|_{L_{p,\mu}(H^{1+s}_q(\Omega))}\Big),$$

where $P_{0R} = I$ in case $\Sigma_o \cup \Sigma_n \neq \emptyset$, and $P_{0R}\pi$ denotes the mean zero part of π w.r.t. Ω_R otherwise.

Proof. (i) First we assume that Ω is bounded. In case $\Sigma_o \cup \Sigma_n = \emptyset$ we normalize the pressure by zero mean value. Fix any $\phi \in L_{q'}(\Omega)$ with mean zero and solve the elliptic problem

$$\begin{aligned} \Delta\psi = \phi &\quad \text{in } \Omega,\\ \partial_\nu\psi = 0 &\quad \text{on } \Sigma_d \cup \Sigma_s,\\ \psi = 0 &\quad \text{on } \Sigma_o \cup \Sigma_n, \end{aligned}$$

to obtain a unique solution $\psi \in H^2_q(\Omega)$ with mean zero, according to Corollary 7.4.5. Then we obtain with two integrations by parts

$$\begin{aligned} (\pi|\phi)_\Omega = (\pi|\Delta\psi)_\Omega &= (\pi|\partial_\nu\psi)_\Sigma - (\nabla\pi|\nabla\psi)_\Omega\\ &= (\pi|\partial_\nu\psi)_\Sigma + (\partial_t u + \omega u - f|\nabla\psi)_\Omega - (\partial_k a^k l\partial_l u|\nabla\psi)_\Omega\\ &= (\pi|\partial_\nu\psi)_{\Sigma_o\cup\Sigma_n} + (a^{kl}\partial_l u|\nabla\partial_k\psi)_\Omega - (\nu_k a^{kl}\partial_l u|\nabla\psi)_\Sigma\\ &= (h_\nu|\partial_\nu\psi)_{\Sigma_o\cup\Sigma_n} + (a^{kl}\partial_l u|\nabla\partial_k\psi)_\Omega - (\nu_k a^{kl}\partial_l u|\nabla_\Sigma\psi)_\Sigma \end{aligned}$$

as $(f\cdot\nu, \operatorname{div} f, g, h_{0,\nu}) = 0$. As $u_0 = 0$ we may apply the fractional time derivative ∂_t^α to the result

$$(\partial_t^\alpha\pi|\phi)_\Omega = (\partial_t^\alpha\pi|\partial_\nu\psi)_{\Sigma_o\cup\Sigma_n} + (a^{kl}\partial_l\partial_t^\alpha u|\nabla\partial_k\psi)_\Omega - (\nu_k a^{kl}\partial_l\partial_t^\alpha u|\nabla\psi)_\Sigma,$$

which shows that $\pi \in H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\Omega))$ provided $0 < \alpha < 1/2 - 1/2q$. This also implies the claimed estimate.

(ii) If Ω is an exterior domain, we choose any ball $B(0,R) \subset \mathbb{R}^n$ such that $\Sigma \subset B(0,R)$, and let $\Omega_R = \Omega \cap B(0,R)$. Take any function $\phi \in L_{q'}(\Omega_R)$, with mean value 0 in case $\Sigma_0 \cup \Sigma_n = \emptyset$. Then $\phi \in \dot H^{-1}_{q,\Sigma_d\cap\Sigma_s}(\Omega)$, by Poincaré's inequality. This implies by Theorem 7.4.3 that there is a solution ψ of the elliptic problem

$$\begin{aligned} \Delta\psi = \phi &\quad \text{in } \Omega,\\ \partial_\nu\psi = 0 &\quad \text{on } \Sigma_d \cup \Sigma_s,\\ \psi = 0 &\quad \text{on } \Sigma_o \cup \Sigma_n, \end{aligned}$$

where ϕ is extended trivially to all of Ω. ψ is unique in case $\Sigma_o \cup \Sigma_n \neq \emptyset$, but $\nabla\psi \in H^1_{q'}(\Omega)$ is always unique, and there is a constant $C > 0$ such that

$$|\nabla\psi|_{H^1_{q'}(\Omega)} \le C|\phi|_{L_{q'}(\Omega_R)}.$$

Now we can perform the same computation as in (i), to the result

$$(\pi|\phi)_{\Omega_R} = (h_\nu|\partial_\nu\psi)_{\Sigma_o\cup\Sigma_n} + (a^{kl}\partial_l u|\nabla\partial_k\psi)_\Omega - (\nu_k a^{kl}\partial_l u|\nabla_\Sigma\psi)_\Sigma.$$

This implies $\pi \in {}_0H^\alpha_{p\mu}(\mathbb{R}_+; L_q(\Omega_R))$ for each R sufficiently large, and also the asserted estimate. □

To be able to apply Proposition 7.3.5, it is convenient to reduce the case of general data to such data for which the assumptions of Proposition 7.3.5 are valid. This will be achieved in two steps. First we extend u_0 to some globally defined $u_0 \in B^{2(\mu-1/p)}_{pq}(\mathbb{R}^n;\mathbb{C})^n$ and solve the whole space problem

$$\partial_t u_1 + \omega u_1 + \mathcal{A}(x,D)u_1 = f, \quad t > 0, \quad u_1(0) = u_0.$$

This removes the initial condition and trivializes the compatibility conditions at $t = 0$, while the regularity of the data remains unchanged. So we may assume $u_0 = 0$. In the second step we remove g and $h_{0\nu}$, as well as the compatibility condition (e). For this purpose, by Corollary 7.4.5 we solve the elliptic problem

$$\begin{aligned} \Delta\phi &= g && \text{in } \Omega,\\ \partial_\nu\phi &= h_{0\nu} && \text{on } \Sigma_d\cup\Sigma_s,\\ \phi &= 0 && \text{on } \Sigma_o\cup\Sigma_n. \end{aligned}$$

Then we set $u_2 = u - \nabla\phi$ and $\pi_2 = \pi + (\partial_t+\omega)\phi + \psi$, where, using Theorem 7.4.3, ψ solves the problem

$$\begin{aligned} \Delta\psi &= \operatorname{div}\big(\mathcal{A}(x,D)\nabla\phi\big) && \text{in } \Omega,\\ \partial_\nu\psi &= \nu\cdot\big(\mathcal{A}(x,D)\nabla\phi\big) && \text{on } \Sigma_d\cup\Sigma_s,\\ \psi &= 0 && \text{on } \Sigma_o\cup\Sigma_n. \end{aligned}$$

Then (u_2,π_2) satisfies (7.32) with the boundary conditions in question, with data subject to

$$(f\cdot\nu, \operatorname{div} f, g, h_{0\nu}, u_0) = 0,$$

hence π_2 has the time regularity asserted in Proposition 7.3.5. So the only remaining data are

(i) $f \in L_{p,\mu}(\mathbb{R}_+; X_0)$;

(ii) $h_{0\Sigma} \in {}_0F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma; T\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-1/q}_q(\Sigma; T\Sigma))$;

(iii) $h \in {}_0F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma;\mathbb{R}^n)) \cap L_{p,\mu}(\mathbb{R}_+; W^{1-1/q}_q(\Sigma;\mathbb{R}^n))$.

Here we have set $h_{0\Sigma} = 0$ on $\Sigma_o\cup\Sigma_n$ and $h = 0$ on $\Sigma_d\cup\Sigma_s$, for convenience. We remark, that in case $\mathcal{A} = -\Delta$, we can even achieve $f = 0$. Indeed, as ∇ commutes with $\mathcal{A} = -\Delta$ we may choose $\pi_2 = \pi + (\partial_t+\omega)\phi - \Delta\phi$.

3.4 Localization

Here we employ the notation of Sections 6.2.4 and 6.3.3, to introduce the charts and the local operators $\mathcal{A}^k$. If $\Omega \subset \mathbb{R}^n$ is unbounded, i.e., an exterior domain, we choose a large ball $B(0,R) \supset \partial\Omega$ and define $U_0 = \mathbb{R}^n \setminus \bar{B}(0,R)$; otherwise U_0 is void. We cover the compact set $\Sigma := \partial\Omega \subset \mathbb{R}^n$ by balls $B(x_k, r/2)$ with $x_k \in \partial\Omega$, $k = 1, \dots, N_1$, such that each part $\partial\Omega \cap B(x_k, 2r)$ of the boundary Σ can be parameterized by a function $\rho_k \in C^{3-}$ as a graph over the tangent space $T_{x_k}\Sigma$. We extend this function ρ_k to a global function by a cut-off procedure, and denote the resulting bent half-space by $\mathbb{H}_k$. This is possible by the regularity assumption $\Sigma \in C^{3-}$ as well as by compactness of Σ. Define $U_k = B(x_k, r) \cap \Omega$, $k = 1, \dots, N_1$. We cover the compact set $\bar{\Omega} \setminus \cup_{k=0}^{N_1} U_k$ by finitely many balls $B(x_k, r/2)$, $k = N_1 + 1, \dots, N_2$, and set $U_k = B(x_k, r)$. Then $\{U_k\}_{k=0}^{N_2}$ is a finite open covering of $\bar{\Omega}$. Fix a C^∞-partition of unity $\{\varphi_k\}_{k=1}^{N_2}$ subordinate to this open covering of $\bar{\Omega}$, and let χ_k denote C^∞-functions with $\chi_k = 1$ on $\operatorname{supp}\varphi_k$, $\operatorname{supp}\chi_k \subset U_k$.

We assume in the sequel that the operator $\mathcal{A}(x_0, D)$ is strongly elliptic, for each $x_0 \in \bar{\Omega} \cup \{\infty\}$, and normally strongly elliptic for each $x_0 \in \Sigma$. Then the maximal regularity constants for the problems with frozen coefficients will be uniform in $x_0 \in \bar{\Omega} \cup \{\infty\}$, by continuity and compactness, hence η_0 in Corollaries 7.3.3 and 7.3.4 will be uniform in x_0 as well. Now we fix any $\eta \in (0, \eta_0]$, and choose the radius of the chart $r > 0$ so small that the assumptions of these corollaries are met, and each chart only intersects one of the boundary parts Σ_j. According to the previous subsection, we may also assume

$$(\operatorname{div} f, g, u_0) = 0 \quad \text{in } \Omega, \quad h_{0,\nu} = f \cdot \nu = 0 \quad \text{on } \Gamma_d \cup \Gamma_s, \quad h_\nu = 0 \quad \text{on } \Gamma_o \cup \Gamma_n.$$

Therefore Proposition 7.3.5 is available.

To define local operators $\mathcal{A}^k(x, D)$ and $\mathcal{B}_j^k(x, D)$ we proceed as follows. For the interior charts $k = 0$, $k = N_1 + 1, \dots, N_2$, we define the coefficients of $\mathcal{A}^k(x, D)$ by reflection of the coefficients at the boundary of U_k. This is the same trick as in Section 6.1.4. For the boundary charts $k = 1, \dots, N_1$ we first transform the coefficients of $\mathcal{A}(x, D)$ and $\mathcal{B}_j(x, D)$ in U_k to a half-space, extend them as in Section 6.2.4, and then transform them back to the bent half-space $\mathbb{H}_k$. Having defined the local differential operators, we may proceed as in Section 6.2.4, introducing local problems for the functions $u^k = \varphi_k u$, which for the interior charts $k = 0$, and $k = N_1 + 1, \dots, N_2$ are problems on $\mathbb{R}^n$, and for the boundary charts $k = 1, \dots, N_1$ are problems on the bent half-spaces $\mathbb{H}_k$ with boundary $\partial\mathbb{H}_k$. This yields the following problems. For $k = 0$ and $k = N_1 + 1, \dots, N_2$ we have the whole space problems

$$\begin{aligned} \partial_t u^k + \omega u^k + \mathcal{A}^k(x, D) u^k + \nabla \pi^k &= f_k + F_k(u, \pi) && \text{in } \mathbb{R}^n, \\ \operatorname{div} u^k &= u \cdot \nabla \varphi_k && \text{in } \mathbb{R}^n, \\ u^k(0) &= 0 && \text{in } \mathbb{R}^n, \end{aligned}$$

for $t > 0$, where $f_k = f\varphi_k$ and $F_k(u, \pi) = [\mathcal{A}(x, D), \varphi_k]u + \pi \nabla \varphi_k$. For the boundary charts $k = 1, \dots, N_1$ we have the problems

$$\begin{aligned} \partial_t u^k + \omega u^k + \mathcal{A}^k(x,D)u^k + \nabla\pi^k &= f_k + F_k(u,\pi) && \text{in } \mathbb{H}_k, \\ \operatorname{div} u^k &= \nabla\varphi_k \cdot u && \text{in } \mathbb{H}_k, \\ u^k(0) &= 0 && \text{in } \mathbb{H}_k, \end{aligned}$$

for $t > 0$, together with the following boundary conditions

$$\begin{aligned} \mathcal{P}_{\partial\mathbb{H}_k} u^k &= h^k_{0\Sigma} && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_d \cup \Sigma_0) \neq \emptyset; \\ (u^k|\nu) &= 0 && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_d \cup \Sigma_s) \neq \emptyset; \\ \mathcal{P}_\Sigma \nu a : \nabla u^k &= h^k_\Sigma + H_{\Sigma k}(u) && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_s \cup \Sigma_n) \neq \emptyset; \\ -\nu a : \nabla u^k \nu + \pi^k &= H_{\nu k}(u) && \text{on } \partial\mathbb{H}_k && \text{if } U_k \cap (\Sigma_o \cup \Sigma_n) \neq \emptyset. \end{aligned}$$

Here $h^k_{0\Sigma} = h_{0\Sigma}\varphi_k$, $h^k_\Sigma = h_\Sigma\varphi_k$, $H_{\Sigma k}u = \mathcal{P}_\Sigma \nu a \nabla\varphi_k u$, and $H_{\nu k}(u) = -\nu a \nabla\varphi_k u \nu$. In short-hand notation we may write this problem as

$$L_k z_k = g_k + [L, \varphi_k]z,$$

where $z = (u,\pi)$, $z_k = \varphi_k z$, $g_k = \varphi_k(f, 0, h)$, and the notations L and L_k are obvious.

Unfortunately, the commutator $[L, \phi_k]$ in this case is not lower order, so we cannot continue as in Section 6.2.2 and some additional arguments are needed. It turns out that all perturbation terms on the right-hand sides of these equations are lower order, hence can be estimated as in Section 6.2.2, except for $\nabla\varphi_k \cdot u$ in the divergence equation. In fact, as in Section 6.2.2 we have

$$|[\mathcal{A}, \varphi_k]u|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} \le C\omega^{-1/2}\big(\omega|u|_{\mathbb{E}_{0\mu}(\Omega)} + |u|_{\mathbb{E}_{1\mu}(\Omega)}\big), \tag{7.37}$$

as well as

$$|H_k|_{\mathbb{F}_{0\mu}(\partial\mathbb{H}_k)} + \omega^{1/2}|H_k|_{L_{p,\mu}(L_q(\partial\mathbb{H}_k))} \le C\omega^{-1/2}\big(\omega|u|_{\mathbb{E}_{0\mu}(\Omega)} + |u|_{\mathbb{E}_{1\mu}(\Omega)}\big). \tag{7.38}$$

Further, by Proposition 7.3.5,

$$|\pi\nabla\varphi_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} \le C\omega^{-\gamma}\big(\omega|u|_{\mathbb{E}_{0\mu}(\Omega)} + |u|_{\mathbb{E}_{1\mu}(\Omega)}\big), \tag{7.39}$$

for some $\gamma > 0$, here the additional pressure regularity comes in.

Next we remove the inhomogeneous part $\varphi_k[f, 0, h]$ by solving the corresponding bent half-space problems to obtain $z^0_k = (u^0_k, \pi^0_k)$ in the right regularity classes.

To remove the inhomogeneity $u \cdot \nabla\varphi_k$ in the divergence equation, we decompose $u^k = u^0_k + \tilde{u}_k + \nabla\phi_k$, where ϕ_k solves the elliptic problem

$$\begin{aligned} \Delta\phi_k = u \cdot \nabla\varphi_k &= \operatorname{div}(u\varphi_k) && \text{in } \mathbb{H}_k, \\ \partial_\nu\phi_k &= 0 && \text{on } \partial\mathbb{H}_k, \quad \text{if } U_k \cap (\Sigma_d \cup \Sigma_s) \neq \emptyset, \\ \phi_k &= 0 && \text{on } \partial\mathbb{H}_k, \quad \text{if } U_k \cap (\Sigma_o \cup \Sigma_n) \neq \emptyset, \end{aligned} \tag{7.40}$$

where $\mathbb{H}_k = \mathbb{R}^n$ for $k = 0, N_1+1, \dots, N_2$. By Corollary 7.4.2, this problem admits a solution ϕ_k such that $\nabla\phi_k$ is unique, with regularity

$$\nabla\phi_k \in {}_0H^1_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{H}_k)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\mathbb{H}_k)).$$

Moreover, we have the estimates

$$\begin{aligned} |\nabla\phi_k|_{L_{p,\mu}(H^1_q(\mathbb{H}_k))} &\le C|u|_{\mathbb{E}_{0\mu}(\Omega)}, \\ |\nabla\phi_k|_{\mathbb{E}_{1\mu}(\mathbb{H}_k)} + |\nabla^2\phi_k|_{\mathbb{E}_{1\mu}(\mathbb{H}_k)} &\le C|u|_{\mathbb{E}_{1\mu}(\Omega)}, \qquad (7.41)\\ |\nabla\phi_k|_{H^{1/2}_{p,\mu}(L_q(\mathbb{H}_k))} + |\nabla\phi_k|_{L_{p,\mu}(H^2_q(\mathbb{H}_k))} &\le C\omega^{-1/2}\big(\omega|u|_{\mathbb{E}_{0\mu}(\Omega)} + |u|_{\mathbb{E}_{1\mu}(\Omega)}\big). \end{aligned}$$

Next we employ the Helmholtz projection in case $U_k \cap (\Sigma_d \cup \Sigma_s) \neq \emptyset$ resp. the Weyl projection in case $U_k \cap (\Sigma_o \cup \Sigma_n) \neq \emptyset$, denoted by P_k, to decompose

$$\tilde{F}_k(u,\pi) := F_k(u,\pi) - \mathcal{A}^k\nabla\phi_k = \nabla\psi_k + P_k\tilde{F}_k(u,\pi).$$

Introducing a new pressure $\tilde{\pi}_k$ by means of

$$\tilde{\pi}_k = \pi^k + (\partial_t + \omega)\phi_k - \psi_k - \pi^0_k,$$

we arrive at the modified problems

$$\begin{aligned} \partial_t\tilde{u}_k + \omega\tilde{u}_k + \mathcal{A}^k(x,D)\tilde{u}_k + \nabla\tilde{\pi}_k &= P_k\tilde{F}_k(u,\pi) && \text{in } \mathbb{H}_k, \\ \operatorname{div}\tilde{u}_k &= 0 && \text{in } \mathbb{H}_k, \\ \tilde{u}_k(0) &= 0 && \text{in } \mathbb{H}_k. \end{aligned}$$

For the boundary charts $k = 1, \dots, N_1$ these problems are complemented by the boundary conditions

$$\begin{aligned} \mathcal{P}_{\partial\mathbb{H}_k}\tilde{u} &= -\nabla_\Sigma\phi_k && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_d \cup \Sigma_o) \neq \emptyset; \\ (\tilde{u}_k|\nu) &= 0 && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_d \cup \Sigma_s) \neq \emptyset; \\ \mathcal{P}_{\partial\mathbb{H}_k}\nu a\nabla\tilde{u}_k &= \tilde{H}_{\Sigma k}(u) && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_s \cup \Sigma_n) \neq \emptyset; \\ -\nu a\nabla\tilde{u}_k\nu + \pi_k &= \tilde{H}_{\nu k}(u) && \text{on } \partial\mathbb{H}_k, && \text{if } U_k \cap (\Sigma_o \cup \Sigma_n) \neq \emptyset. \end{aligned}$$

Here $\tilde{H}_{\Sigma k}(u) = H_{\Sigma k}(u) - \mathcal{P}_\Sigma\nu a_k\nabla^2\phi_k$, and $\tilde{H}_{\nu k}(u) = H_{\nu k}(u) + \nu a_k\nabla^2\phi_k\nu$. Note that $\tilde{F}_k$, $P_k\tilde{F}_k$ and $\tilde{H}_k$ are subject to the same estimates as F_k and H_k, with probably larger constants C, thanks to (7.41).

Next, we introduce the operators

$$T_k z = (\nabla\phi_k, (\partial_t + \omega)\phi_k - \psi_k).$$

With this notation we can rewrite the localized solution as

$$z_k = z^0_k + \tilde{z}_k + T_k z,$$

where $\tilde{z}_k$ solves the problem

$$L_k \tilde{z}_k = G_k z,$$

with

$$\begin{aligned} G_k z &= [L, \varphi_k] z - L_k T_k z \\ &= [P_k([\mathcal{A}, \varphi_k] u + \pi \nabla \varphi_k - \mathcal{A}^k \nabla \phi_k, 0, [\mathcal{B}, \varphi_k] u - \mathcal{B}^k \nabla \phi_k]^\mathsf{T}, \end{aligned}$$

where $\mathcal{B}^k$ denotes the appropriate boundary operator. More precisely, $[\varphi_k, \mathcal{B}]u = 0$ if $U_k \cap (\Sigma_o \cup \Sigma_n) = \emptyset$ and $[\varphi_k, \mathcal{B}]u = \nu a \nabla^2 \varphi_k u$, otherwise.

It is useful to introduce norms for the solutions and for the data which depend on ω. We set

$$\|z_k\| = \omega |u_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} + |u_k|_{\mathbb{E}_{1\mu}(\mathbb{H}_k)} + |\nabla \pi_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)},$$

and similarly we define $\|z\|$ on Ω. For the data we set

$$\begin{aligned} \|g_k\| = |f_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} &+ \omega^{1-1/2q} |h_0^k|_{L_{p,\mu}(L_q(\partial \mathbb{H}_k))} + |h_0^k|_{\mathbb{F}_{1\mu}(\partial \mathbb{H}_k)} \\ &+ \omega^{1/2-1/2q} |h^k|_{L_{p,\mu}(L_q(\partial \mathbb{H}_k))} + |h^k|_{\mathbb{F}_{0\mu}(\partial \mathbb{H}_k)}, \end{aligned}$$

and similarly for g on Ω. Then we obtain by maximal regularity on a bent half-space

$$\|z_k^0\| \le C \|g_k\| \le C \|g\|, \quad \|\tilde{z}_k\| \le C \omega^{-\gamma} \|z\|,$$

with a constant $C > 0$ independent of ω and k. Here we employed estimates (7.37), (7.38), (7.39), and (7.41).

To estimate $T_k z$, we employ again (7.37), (7.38), (7.39), and (7.41) to obtain

$$|\nabla \phi_k|_{L_{p,\mu}(H_q^1(\mathbb{H}_k))} + |\nabla \psi_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} \le C \omega^{-\gamma} \|z\|.$$

Finally, it remains to estimate $(\partial_t + \omega) \nabla \phi_k$. For this purpose, we employ the identity

$$(\partial_t + \omega) \phi_k = \tilde{\pi}_k - \pi_k + \psi_k - \pi_k^0.$$

Applying Poincaré's inequality to π_k^0 and ψ_k, and Proposition 7.3.5 to π and $\tilde{\pi}_k$, we obtain

$$\begin{aligned} |(\partial_t + \omega) \phi_k|_{L_{p,\mu}(L_q(U_k))} &\le |\tilde{\pi}_k|_{L_{p,\mu}(L_q(U_k))} + |\pi_k|_{L_{p,\mu}(L_q(U_k))} + |\pi_k^0 + \psi_k|_{L_{p,\mu}(L_q(U_k))} \\ &\le |\tilde{\pi}_k|_{\mathbb{E}_0(\mathbb{H}_k \cap B(0,R))} + |\pi_k|_{\mathbb{E}_{0\mu}(\Omega \cap B(0,R))} \\ &\quad + C(|\nabla \pi_k^0|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)} + |\nabla \psi_k|_{\mathbb{E}_{0\mu}(\mathbb{H}_k)}) \\ &\le C\Big(\|g\| + \omega^{-\gamma} \|z\|\Big). \end{aligned}$$

By interpolation with (7.41) this yields

$$|(\partial_t + \omega) \nabla \phi_k|_{\mathbb{E}_{0\mu}(U_k)} \le C \|g\| + C \omega^{-\gamma/2} \|z\|.$$

Summing over k yields the a priori estimate for $z = \sum_k \chi_k z_k$, which reads

$$\|z\| \leq \sum_k \|\chi_k z_k\| \leq C\|g\| + C\omega^{-\gamma}\|z\|,$$

for some $\gamma > 0$, and a constant $C > 0$ which is independent of ω. Choosing $\omega > 2C$ this implies

$$\|z\| \leq 2C\|g\|.$$

Therefore, the operator L on Ω is injective and has closed range. We even can write down a left inverse S as follows. From the identity

$$\begin{aligned} z = \sum_k \chi_k z_k &= \sum_k \chi_k (z_k^0 + \tilde{z}_k + T_k z) \\ &= \sum_k \chi_k L_k^{-1}\varphi_k g + \sum_k \chi_k (L_k^{-1}G_k + T_k) z \\ &= \sum_k \chi_k L_k^{-1}\varphi_k g + G^L z, \end{aligned}$$

we obtain

$$z = Sg := (I - G^L)^{-1}\big(\sum_k \chi_k L_k^{-1}\varphi_k\big) g,$$

as $\|G^L\| < 1$ for ω large.

So it remains to prove surjectivity of L. For this purpose, we assume $f = 0$ for the moment. Set $z = Sg$ as just defined, i.e.,

$$\begin{aligned} z &= \sum_k \chi_k L_k^{-1}\varphi_k g + \sum_k \chi_k (L_k^{-1}G_k + T_k) z \\ &= \sum_k \chi_k L_k^{-1}\varphi g + G^L z, \end{aligned}$$

and apply L, to the result

$$\begin{aligned} L(z - G^L z) &= \sum_k \chi_k L_k L_k^{-1}\varphi_k g + \sum_k [L, \chi_k] L_k^{-1}\varphi_k g \\ &= g + \sum_k \tilde{G}_k L_k^{-1}\varphi_k g + L \sum_k \tilde{T}_k \varphi_k g, \end{aligned}$$

where $\tilde{G}_k = [L, \chi_k] - L\tilde{T}_k$ and $\tilde{T}_k$ is defined in the same way as T_k, replacing φ_k by χ_k. This implies

$$L(z - G^L z - \sum_k \tilde{T}_k \varphi_k g) = g + \sum_k \tilde{G}_k L_k^{-1}\varphi_k g = (I + G^R) g.$$

To conclude the argument, we only have to show that the operator G^R in the data space has norm smaller than 1, as this implies surjectivity of L, and then

$$(S - G^L S - \sum_k \tilde{T}_k \varphi_k)(I + G^R)^{-1}$$

is a right inverse of L. Now $\tilde{G}_k$ can be estimated in the same way as G_k, as $f = 0$, hence we have surjectivity in this case.

To deal with general f, we employ a homotopy argument. Replacing $\mathcal{A}$ by $\tau\mathcal{A} - (1-\tau)\Delta$, we see that the corresponding operators L^τ are injective and have closed ranges for all $\tau \in [0,1]$, as these operators are uniformly normally strongly elliptc, uniformly w.r.t. τ. Therefore the Fredholm index of L^τ is constant, and this shows that L^1 is surjective if and only if L^0 is surjective. For $\tau = 0$ we have the classical case $\mathcal{A} = -\Delta$, and as we have noted above, we may then assume $f = 0$. This completes the proof of Theorem 7.3.1.

7.4 Boundary Value Problems for the Laplacian

Here we state and prove some results for the Laplace equation which have been employed in Section 7.3.

4.1 Whole Space
We begin with the case $\Omega = \mathbb{R}^n$. By the very definition of the homogeneous Bessel potential spaces $\dot{H}^s_q(\mathbb{R}^n)$, namely

$$\dot{H}^s_q(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}|\xi|^s \mathcal{F}u \in L_q(\mathbb{R}^n)\},$$

where $1 < q < \infty$ and $s \in \mathbb{R}$, it is clear that Δ is an isomorphism between the spaces $\dot{H}^{s+2}_q(\mathbb{R}^n)$ and $\dot{H}^s_q(\mathbb{R}^n)$.

4.2 Half Space
The half-space case $\Omega = \mathbb{R}^n_+$ is a little more involved.

(i) We first consider the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n_+, \qquad u = h \quad \text{on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}.$$

Defining the *Poisson semigroup* $P(y)$ by means of

$$P(y)h = \mathcal{F}^{-1} e^{-y|\xi|} \mathcal{F}h,$$

$u = P(y)h$ is the unique solution of the Dirichlet problem. This shows that $u \in \dot{H}^k_q(\mathbb{R}^n_+)$ if and only if $h \in \dot{W}^{k-1/q}_q(\mathbb{R}^{n-1})$, for all $q \in (1,\infty)$ and $k \geq 0$.

(ii) In the next step we consider the Neumann problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n_+, \qquad -\partial_y u = g \quad \text{on } \partial\mathbb{R}^n_+.$$

Denoting the generator of the Poisson semigroup by $\dot{D}$, the unique solution of the Neumann problem is given by $u = P(y)\dot{D}^{-1}g$. As $\dot{D}$ has symbol $|\xi|$, it is clear that $\dot{D}$ is an isomorphism from $\dot{H}^{s+1}_q(\mathbb{R}^{n-1})$ to $\dot{H}^s_q(\mathbb{R}^{n-1})$, for all $q \in (1,\infty)$, $s \in \mathbb{R}$. Therefore the solution u of the Neumann problem belongs to the class $\dot{H}^k_q(\mathbb{R}^n_+)$ if and only if $g \in \dot{W}^{k-1-1/q}_q(\mathbb{R}^{n-1})$, for all $q \in (1,\infty)$, $k \geq 0$.

(iii) Now we consider the inhomogeneous Dirichlet problem

$$-\Delta u = f \quad \text{in } \mathbb{R}^n_+, \quad u = 0 \quad \text{on } \partial\mathbb{R}^n_+.$$

The unique solution of this problem is given by

$$u = G_D f := \frac{\dot{D}^{-1}}{2} \int_0^\infty \big(P(|y-s|) - P(y+s)\big) f(s)\, ds.$$

This representation shows $u \in \dot{H}^2_q(\mathbb{R}^n_+)$ if and only if $f \in L_q(\mathbb{R}^n_+)$.

(iv) Similarly, the solution of the inhomogeneous Neumann problem

$$-\Delta u = f \quad \text{in } \mathbb{R}^n_+, \quad \partial_y u = 0 \quad \text{on } \partial\mathbb{R}^n_+.$$

is given by

$$u = G_N f := \frac{\dot{D}^{-1}}{2} \int_0^\infty \big(P(|y-s|) + P(y+s)\big) f(s)\, ds.$$

This representation shows $u \in \dot{H}^2_q(\mathbb{R}^n_+)$ if and only if $f \in L_q(\mathbb{R}^n_+)$.

(v) Higher order regularity.
If $f \in \dot{H}^1_q(\mathbb{R}^n_+)$ then differentiating the equations (or the solution formulas) first tangentially we obtain $\nabla_x u \in \dot{H}^2_q(\mathbb{R}^n_+)$, and then normally, we find $u \in \dot{H}^3_q(\mathbb{R}^n_+)$. In the Dirichlet case we also use **(i)** with $g = f|_{\mathbb{R}^{n-1}} \in \dot{W}^{1-1/q}_q(\mathbb{R}^{n-1})$.

(vi) Weak solutions.
Finally, we consider the weak Dirichlet problem

$$\Delta u = \operatorname{div} f \quad \text{in } \mathbb{R}^n_+, \quad u = 0 \quad \text{on } \partial\mathbb{R}^n_+,$$

where $f = [f_x, f_y]^\mathsf{T} \in L_q(\mathbb{R}^n_+; \mathbb{C}^n)$. In this case the solution u is given by

$$u = \nabla_x \cdot G_D f_x + \partial_y G_N f_y,$$

hence $u \in \dot{H}^1_q(\mathbb{R}^n_+)$. Similarly, for the weak Neumann problem

$$\Delta u = \operatorname{div} f \quad \text{in } \mathbb{R}^n_+, \quad \partial_y u = f_y \quad \text{on } \partial\mathbb{R}^n_+,$$

we have

$$u = \nabla_x \cdot G_N f_x + \partial_y G_D f_y,$$

and so also in this case $u \in \dot{H}^1_q(\mathbb{R}^n_+)$.

4.3 Bent Half Spaces
In the next step we extend the results from the previous subsection to the case of certain bent half-spaces.

(a) Coordinate Transformations.
Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary of class C^1, such that $\partial\Omega =: \Sigma$ decomposes disjointly as $\Sigma = \Sigma_0 \cup \Sigma_1$ with Σ_j open and closed in Σ. Suppose $\Phi : \overline{\Omega} \to \mathbb{R}^n$ is bijective, of class C^1 such that

$$0 < c \leq |\det \partial\Phi(x)| \leq 1/c, \quad x \in \overline{\Omega},$$

and assume $\Phi(\Sigma) = \partial\Phi(\Omega)$. We set $\Omega^\Phi = \Phi(\Omega)$ and $\Sigma_j^\Phi = \Phi(\Sigma_j)$, $j = 0, 1$.

Consider the weak Dirichlet-Neumann problem

$$\begin{aligned} (\nabla u|\nabla v)_{\Omega^\Phi} &= (f|\nabla v)_{\Omega^\Phi}, \quad v \in \dot{H}^1_{q',\Sigma_0^\Phi}(\Omega^\Phi), \\ u &= h \quad \text{on } \Sigma_0^\Phi. \end{aligned} \tag{7.42}$$

By means of the transformation Φ, this problem can be reformulated as a weak problem on Ω in the following way. By means of the pull backs

$$\bar{u}(x) = u(\Phi(x)),\ \bar{v}(x) = v(\Phi(x)),\ \bar{h}(x) = h(\Phi(x)),$$

and with

$$\nabla_x \bar{u}(x) = \nabla_x u(\Phi(x)) = \partial\Phi(x)^{\mathsf{T}} \nabla_y u \circ \Phi(x),$$

the transformation rule yields for a weak solution u on Ω^Φ

$$\begin{aligned} 0 = (\nabla u - f|\nabla v)_{\Omega^\Phi} &= \int_{\Phi(\Omega)} \big(\nabla_y u(y) - f(y)\big) \cdot \nabla_y v(y)\, dy \\ &= \int_\Omega \big(\nabla_y u(\Phi(x)) - f(\Phi(x))\big) \cdot \nabla_y v(\Phi(x)) |\det \partial\Phi(x)|\, dx \\ &= \int_\Omega \big((|\det \partial\Phi(x)|\partial\Phi(x)^{-1}\partial\Phi(x)^{-\mathsf{T}})\nabla_x \bar{u}(x) - \bar{f}(x)\big) \cdot \nabla_x \bar{v}(x)\, dx, \end{aligned}$$

where

$$\bar{f}(x) = |\det \partial\Phi(x)|\partial\Phi(x)^{-\mathsf{T}} f(\Phi(x)), \quad x \in \Omega.$$

This shows that Problem (7.42) becomes

$$\begin{aligned} 0 &= (\mathcal{A}\nabla\bar{u} - \bar{f}|\nabla\bar{v}), \quad \bar{v} \in \dot{H}^1_{q',\Sigma_0}(\Omega), \\ \bar{u} &= \bar{h} \quad \text{on } \Sigma_0. \end{aligned} \tag{7.43}$$

Here the coefficient matrix $\mathcal{A}(x)$ is defined by

$$\mathcal{A}(x) = |\det \partial\Phi(x)|\partial\Phi(x)^{-1}\partial\Phi(x)^{-\mathsf{T}},$$

hence $\mathcal{A}$ is continuous and bounded.

Note that by the assumptions on Φ, the map T_Φ defined by $T_\phi u := \bar{u}$ is an isomorphism from $L_q(\Omega^\Phi)$ to $L_q(\Omega)$ and from $\dot{H}^1_{q,\Sigma_0^\Phi}(\Omega^\Phi)$ to $\dot{H}^1_{q,\Sigma_0}(\Omega)$, hence by interpolation also from $\dot{H}^s_{q,\Sigma_0^\Phi}(\Omega^\Phi)$ to $\dot{H}^s_{q,\Sigma_0}(\Omega)$, $s \in [0, 1]$, and from $H^s_{q,\Sigma_0^\Phi}(\Omega^\Phi)$

to $H^s_{q,\Sigma_0}(\Omega)$, $s \in [0,1]$. As T_Φ respects boundary traces by assumption, we also see that $h \in \dot{W}_q^{1-1/q}(\Sigma_0^\Phi)$ if and only if $\bar{h} \in \dot{W}_q^{1-1/q}(\Sigma_0)$. Finally, we have $f \in L_q(\Omega^\Phi;\mathbb{R}^n)$ if and only if $\bar{f} \in L_q(\Omega;\mathbb{R}^n)$.

These arguments show that (7.42) is well-posed in Ω^Φ if and only if (7.43) is well-posed in Ω.

(b) Perturbed Half-Spaces

Now we consider the special case where $\Omega = \mathbb{R}^n_+$ and $\Phi(x,y) = [x, y+h(x)]^\mathsf{T}$ with $x \in \mathbb{R}^{n-1}$ and $y > 0$, as well as $h \in C^1_b(\mathbb{R}^{n-1})$. This means that Ω^Φ is a bent half-space. Easy computations show $\det \partial\Phi(x,y) = 1$, as well as

$$\mathcal{A}(x,y) = \partial\Phi(x,y)^{-1}\partial\Phi(x,y)^\mathsf{T} = \begin{bmatrix} I & -\nabla_x h(x) \\ -\nabla_x h(x)^\mathsf{T} & 1+|\nabla_x h|_2^2 \end{bmatrix},$$

hence $\mathcal{A}(x,y) = I - \mathcal{B}(x)$, where $|\mathcal{B}(x)| \le C|\nabla_x h|_\infty$. So, dropping the bars, the transformed problem can be rewritten as the problem

$$\begin{aligned} (\nabla u|\nabla v)_{\mathbb{R}^n_+} &= (f|\nabla v)_{\mathbb{R}^n_+} + (\mathcal{B}\nabla u|\nabla v)_{\mathbb{R}^n_+}, \quad v \in {}_0\dot{H}^1_{q'}(\mathbb{R}^n_+), \\ u &= h \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{7.44}$$

in the Dirichlet case, i.e., $\Sigma_1 = \emptyset$, and

$$(\nabla u|\nabla v)_{\mathbb{R}^n_+} = (f|\nabla v)_{\mathbb{R}^n_+} + (\mathcal{B}\nabla u|\nabla v)_{\mathbb{R}^n_+}, \quad v \in \dot{H}^1_{q'}(\mathbb{R}^n_+), \tag{7.45}$$

in the Neumann case, i.e., $\Sigma_0 = \emptyset$. These are perturbations of the half-space problems in Section 7.4.2, provided $|\nabla_x h|_\infty$ is small.

More precisely, let $L_D : L_q(\mathbb{R}^n_+;\mathbb{R}^n) \times \dot{W}_q^{1-1/q}(\mathbb{R}^{n-1}) \to \dot{H}^1_q(\mathbb{R}^n_+)$ denote the bounded solution map from Section 7.4.2 for the Dirichlet problem and $L_N : L_q(\mathbb{R}^n_+;\mathbb{R}^n) \to \dot{H}^1_q(\mathbb{R}^n_+)$ that for the Neumann problem in the half-space. Then the perturbed problems can rewritten abstractly as

$$u = L_D(f,h) + L_D(\mathcal{B}\nabla u, 0), \quad u = L_N f + L_N \mathcal{B}\nabla u,$$

respectively. Thus by a Neumann series argument, there is a number $\eta_0 > 0$ such that whenever $|\nabla_x h|_\infty \le \eta_0$, then the perturbed equations are also uniquely solvable.

Note that this number $\eta_0 > 0$ is universal for the Laplacian, it only depends on q. Bent half-spaces will be called *perturbed half-spaces* if the corresponding height function h is subject to $|\nabla_x h|_\infty \le \eta_0$. If in addition the support of h is compact, then we use the term *compactly perturbed half-space.*

Let us summarize.

Theorem 7.4.1. *Let $\Omega = \mathbb{H}$ denote a perturbed half-space, and $q \in (1,\infty)$. Then*

(i) Neumann problem

For each $f \in L_q(\mathbb{H})$ there is a unique solution of

$$(\nabla u|\nabla v)_{\mathbb{H}} = (f|\nabla v)_{\mathbb{H}}, \quad v \in \dot{H}^1_{q'}(\mathbb{H}). \tag{7.46}$$

There is a constant $c > 0$ such that

$$c|\nabla u|_q \leq |f|_q, \quad f \in L_q(\mathbb{H}),$$

and

$$c|\nabla u|_q \leq \sup\{|(\nabla u|\nabla v)_{\mathbb{H}} : v \in \dot{H}^1_{q'}(\mathbb{H}),\ |\nabla v|_{q'} \leq 1\}. \tag{7.47}$$

(ii) Dirichlet problem
For each $f \in L_q(\mathbb{H})$ and $h \in \dot{W}^{1-1/q}_q(\partial\mathbb{H})$, there is a unique solution of

$$(\nabla u|\nabla v)_{\mathbb{H}} = (f|\nabla v)_{\mathbb{H}}, \quad v \in {}_0\dot{H}^1_{q'}(\mathbb{H}), \quad u = h \text{ on } \partial\mathbb{H}. \tag{7.48}$$

There is a constant $c > 0$ such that

$$c|\nabla u|_q \leq |f|_q + |h|_{\dot{W}^{1-1/q}_q}, \quad f \in L_q(\mathbb{H}),\ h \in \dot{W}^{1-1/q}_q(\partial\mathbb{H}).$$

Furthermore, in case $h = 0$,

$$c|\nabla u|_q \leq \sup\{|(\nabla u|\nabla v)_{\mathbb{H}} : v \in {}_0\dot{H}^1_{q'}(\mathbb{H}),\ |\nabla v|_{q'} \leq 1\}. \tag{7.49}$$

For the proof of the variational inequalities note that (7.46) is equivalent to $\nabla^*_{q'}\nabla_q u = \nabla^*_{q'} f$, and the right-hand side of (7.47) is precisely the norm of this quantity in ${}_0H^{-1}_q(\mathbb{H})$. A similar argument is valid for the Dirichlet problem, provided $h = 0$.

Concerning higher regularity, the results for perturbed half-spaces are not as precise as those for the half-space case, as lower order terms occur. However, the assertions in the next corollary follow from the corresponding half-space results, again by Neumann series arguments.

Corollary 7.4.2. *Let $\Omega = \mathbb{H}$ denote a perturbed half-space, $q \in (1,\infty)$, $s \in \{0,1\}$, and $h \in C^{(2+s)-}_b(\mathbb{R}^{n-1})$.*

(i) Neumann problem
If $f \in H^s_q(\mathbb{H})$, $g \in W^{1+s-1/q}_q(\partial\mathbb{H})$ such that $(f,g) \in {}_0\dot{H}^{-1}_q(\mathbb{H}))$, then the problem

$$\Delta u = f \quad \text{in } \mathbb{H}, \quad \partial_\nu u = g \quad \text{on } \partial\mathbb{H}$$

has a unique solution u such that $\nabla u \in H^{1+s}_q(\mathbb{H})$.

(ii) Dirichlet problem
If $f \in H^s_q(\mathbb{H})$, $h \in W^{2+s-1/q}_q(\partial\mathbb{H})$ such that $f \in \dot{H}^{-1}_q(\mathbb{H}))$, then the problem

$$\Delta u = f \quad \text{in } \mathbb{H}, \quad u = h \quad \text{on } \partial\mathbb{H}$$

has a unique solution u such that $\nabla u \in H^{1+s}_q(\mathbb{H})$.

4.4 General Domains

Now we are ready to consider domains with compact boundary, which means domains which are either bounded or exterior.

Theorem 7.4.3. *Suppose that Ω is domain in $\mathbb{R}^n$ with compact boundary $\partial\Omega := \Sigma$ of class C^1, and suppose that Σ decomposes disjointly into $\Sigma = \Sigma_0 \cup \Sigma_1$, where Σ_j are open and closed in Σ. Let $f \in L_q(\Omega)$, $h \in W_q^{1-1/q}(\Sigma_0)$, with $q \in (1, \infty)$. Then the problem*

$$\begin{aligned} (\nabla u|\nabla v)_\Omega &= (f|\nabla v)_\Omega, \quad v \in \dot{H}^1_{q',\Sigma_0}(\Omega), \\ u &= h \qquad\qquad\quad \text{on } \Sigma_0, \end{aligned} \tag{7.50}$$

admits a unique solution $u \in \dot{H}^1_q(\Omega)$. There is a constant $C > 0$ such that

$$|\nabla u|_{L_q} \le C\big(|f|_{L_q} + |h|_{W_q^{1-1/q}}\big) \tag{7.51}$$

holds for all $f \in L_q(\Omega)$ and $h \in W_q^{1-1/q}(\Sigma_0)$.

Recall $\dot{H}^1_{q,\emptyset}(\Omega) = \dot{H}^1_q(\Omega)/constants$, hence uniqueness in $\dot{H}^1_{q,\Sigma_0}(\Omega)$ means uniqueness up to a constant in case $\Sigma_0 = \emptyset$, and even uniqueness otherwise. If $\Sigma_0 = \emptyset$, we normalize the solution by mean value zero if Ω is bounded, and by mean zero on $\Omega \cap B(0, R)$, for some large fixed ball $B(0, R)$ which contains Σ.

Proof. The proof consists of several steps. The first step concerns uniqueness.

(a) Uniqueness

Suppose

$$(\nabla u|\nabla v)_\Omega = 0, \quad v \in \dot{H}^1_{q',\Sigma_0}(\Omega), \quad u = 0 \quad \text{on } \Sigma_0.$$

We show that this implies $u = 0$ in $\dot{H}^1_{q,\Sigma_0}(\Omega)$. For this purpose, we prove two assertions, namely

(i) For each $x_0 \in \overline{\Omega}$ there is a ball $B(x_0, r)$ such that $\nabla u \in L_2(B(x_0, r))$.

(ii) There is a ball $B(0, r) \supset \Sigma$, such that $\nabla u \in L_2(\mathbb{R}^n \setminus B(0, r))$.

Here (ii) is void in case Ω is bounded.

Assuming (i) and (ii), by compactness we obtain $\nabla u \in L_2(\Omega)$ and so we may use $v = u$ as a test function to obtain $|\nabla u|_2^2 = 0$, which yields the assertion.

(i) If $q \ge 2$ this is obvious, as $L_q(B(x_0, r)) \subset L_2(B(x_0, r))$, for each $r > 0$. So let $q \in (1, 2)$. Set $q_0 = q$ and define inductively q_j by

$$\frac{1}{q_j} = \frac{1}{q_{j-1}} - \frac{1}{n} = \frac{1}{q} - \frac{j}{n};$$

clearly $q_k \ge 2$ if $k \ge n(2 - q)/2q$. Choose a radius $r_0 > 0$ small enough so that $B(x_0, r_0) \subset \Omega$ in case $x_0 \in \Omega$ – then we set $\mathbb{H}_{x_0} = \mathbb{R}^n$ –, and if $x_0 \in \partial\Omega$, such that

$\Omega \cap B(x_0, r_0)$ is part of the boundary of a perturbed half-space $\mathbb{H}_{x_0}$. Below we will be using the inequalities (7.47) and (7.49) for perurbed half-spaces as well as for the whole space.

Next we choose cut-off functions χ_j with $\operatorname{supp}\chi_j \subset B(x_0, r_j)$, $\chi_j = 1$ on $B(x_0, r_{j+1})$. We proceed by induction. By assumption we know $\nabla u \in L_{q_0}(B(x_0, r_0))$. Assume $\nabla u \in L_{q_j}(B(x_0, r_0))$, and consider $\nabla(\chi_j u)$. We have

$$|\nabla(\chi_j u)|_{q_{j+1}} \leq c \sup\{(\nabla(\chi_j u)|\nabla v)_{\mathbb{H}_{x_0}} : |\nabla v|_{q'_{j+1}} \leq 1\},$$

where we may normalize v by mean value zero on $B(x_0, r_j)$, in case $x_0 \in \Omega \cup \Sigma_1$. hence with

$$\begin{aligned}(\nabla(\chi_j u)|\nabla v)_{\mathbb{H}_{x_0}} &= (\nabla u|\nabla(\chi_j v))_{\mathbb{H}_{x_0}} - (\nabla u|v\nabla\chi_j)_{\mathbb{H}_{x_0}} + (u\nabla\chi_j|\nabla v)_{\mathbb{H}_{x_0}}\\ &= -(\nabla u|v\nabla\chi_j)_{\mathbb{H}_{x_0}} + (u\nabla\chi_j|\nabla v)_{\mathbb{H}_{x_0}},\end{aligned}$$

by assumption, as $\chi_j v$ belongs to ${}_0\dot{H}^1_{q'}(\mathbb{H}_{x_0})$ if $x_0 \in \Sigma_0$, and to $\dot{H}^1_{q'}(\mathbb{H}_{x_0})$ otherwise. Since $\nabla\chi_j$ has support in $\bar{B}(x_0, r_j) \setminus B(x_0, r_{j+1})$, we obtain

$$|(\nabla u|v\nabla\chi_j)_{\mathbb{H}_{x_0}}| \leq C|\nabla u|_{L_{q_j}(B(x_0,r_j))}|v|_{L_{q'}(B(x_0,r_j))},$$

and also

$$|(u\nabla\chi_j|\nabla v)_{\mathbb{H}_{x_0}}| \leq C|u|_{L_{q_j}(B(x_0,r_j))}|\nabla v|_{L_{q'}(B(x_0,r_j))}.$$

Consequently, by Poincaré's inequlity we have

$$\begin{aligned}|\nabla(\chi_j u)|_{q_{j+1}} &\leq C|u|_{H^1_{q_j}(B(x_0,r_j))}|v|_{H^1_{q'_j}(B(x_0,r_j))}\\ &\leq C|u|_{H^1_{q_j}(B(x_0,r_j))}|\nabla v|_{L_{q'_j}(B(x_0,r_j))}\\ &\leq C|u|_{H^1_{q_j}(B(x_0,r_j))}|\nabla v|_{L_{q'_j}(\mathbb{H}_{x_0}))} \leq C|u|_{H^1_{q_j}(B(x_0,r_j))},\end{aligned}$$

and as $\chi_j = 1$ on $B(x_0, r_{j+1})$ this yields

$$|\nabla u|_{L_{q_{j+1}}(B(x_0,r_{j+1}))} \leq C|u|_{H^1_{q_j}(B(x_0,r_j))}.$$

This proves **(i)**.

(ii) We have to distinguish the cases $q \geq 2$ and $1 < q < 2$. If $q \geq 2$, choose a ball $B(0, r_0)$ such that $\Sigma \subset B(0, r_0 - 1)$, and fix a cut-off function χ_0 which equals 0 in $B(0, r_0 - 1)$ and equals one outside the ball $B(0, r_0)$. Then we have

$$c|\nabla(\chi_0 u)|_{L_2(\mathbb{R}^n)} \leq \sup\{(\nabla(\chi_0 u)|\nabla v)_{\mathbb{R}^n} : |\nabla v|_{L_2(\mathbb{R}^n)} \leq 1\}.$$

As above

$$(\nabla(\chi_0 u)|\nabla v)_{\mathbb{R}^n} = -(\nabla u|v\nabla\chi_0)_{\mathbb{R}^n} + (u\nabla\chi_0|\nabla v)_{\mathbb{R}^n},$$

hence

$$\begin{aligned}|(\nabla(\chi_0 u)|\nabla v)_{\mathbb{R}^n}| &\leq C|u|_{H_2^1(A_0)}|v|_{H_2^1(A_0)}\\ &\leq C|u|_{H_q^1(A_0)}|v|_{H_2^1(A_0)},\end{aligned}$$

where $A_0 = B(0, r_0) \setminus B(0, r_0 - 1)$. As we may normalize v by mean value zero over A_0, and $\chi_0 = 1$ on $\mathbb{R}^n \setminus B(0, r_0)$ this shows $\nabla u \in L_2(\mathbb{R}^n \setminus B(0, r_0))$.

On the other hand, if $1 < q < 2$ then we set $r_j = jr_0$, and choose cut-offs such that $\operatorname{supp} \chi_j \subset \mathbb{R}^n \setminus B(0, r_j)$, and $\chi_j = 1$ on $\mathbb{R}^n \setminus B(0, r_{j+1})$. Then by

$$c|\nabla(\chi_j u)|_{q_{j+1}} \leq \sup\{(\nabla(\chi_j u)|\nabla v)_{\mathbb{R}^n} : |\nabla v|_{q'_{j+1}} \leq 1\},$$

we obtain as before

$$\begin{aligned}|(\nabla(\chi_j u)|\nabla v)_{\mathbb{R}^n}| &\leq C|u|_{H_2^1(A_j)}|v|_{H_2^1(A_j)}\\ &\leq C|u|_{H_q^1(A_j)}|v|_{H_2^1(A_j)},\end{aligned}$$

and so the same argument as in **(i)** implies $\nabla u \in L_2(\mathbb{R}^n \setminus B(0, r_k))$, by induction.

As a consequence, we obtain $u \in L_2(\mathbb{R}^n \setminus B(0, r))$ for some $r > 0$.

(b) Lower Bound

(i) Suppose that the inequality (with $h = 0$)

$$c|\nabla u|_q \leq \sup\{|(\nabla u|\nabla v)_\Omega| : |\nabla v|_{q'} \leq 1\}$$

does not hold. Then there is a sequence $(u_k) \subset \dot{H}^1_{q,\Sigma_0}(\Omega)$ with $|\nabla u_k|_q = 1$ such that

$$\varepsilon_k := \sup\{|(\nabla u_k|\nabla v)_\Omega| : |\nabla v|_{q'} \leq 1\} \to 0 \quad \text{as } k \to \infty.$$

Since $L_q(\Omega)$ is reflexive, there is a subsequence (w.l.o.g. the whole sequence) such that $\nabla u_k \rightharpoonup \nabla u$ in $L_q(\Omega)$. This implies with $\varepsilon_k \to 0$

$$(\nabla u_k|\nabla v)_\Omega \to (\nabla u|\nabla v)_\Omega = 0, \quad \text{for all } v \in \dot{H}^1_{q',\Sigma_0}(\Omega).$$

Then **(a)** implies $u = 0$.

(ii) Next we localize as e.g. in Section 6.3.3; below we use the notation from there. Then by the previous subsection we know

$$c|\nabla(\varphi_j u_k)|_q \leq \sup\{|(\nabla(\varphi_j u_k)|\nabla v)_{\mathbb{H}_j}| : |\nabla v|_{q'} \leq 1\} =: d_{kj}$$

on each perturbed half-space or whole space $\mathbb{H}_j$, $j = 0, \ldots, N$. We want to prove $d_{kj} \to 0$ as $k \to \infty$, for each j. If this is true, then

$$|\nabla u_k|_q = |\sum_{j=0}^N \nabla(\varphi_j u_k)|_q \leq \sum_{j=0}^N |\nabla(\varphi_j u_k)|_q \leq C \sum_{j=0}^N d_{kj} \to 0$$

as $k \to \infty$, a contradiction as $|\nabla u_k|_q = 1$ by assumption.

(iii) For a fixed $j \in \{0, \dots, N\}$ choose $v_{kj} \in \dot{H}^1_{q'}(\mathbb{H}_j)$ normalized by $|\nabla v_{kj}|_{q'} = 1$, and by mean value zero over U_j in case $U_j \cap \Sigma_0 = \emptyset$, such that

$$d_{kj} \le \frac{1}{k} + (\nabla(\varphi_j u_k)|\nabla v_{kj})_{\mathbb{H}_j}.$$

We have

$$(\nabla(\varphi_j u_k)|\nabla v_{kj})_{\mathbb{H}_j} = (\nabla u_k|\nabla(\varphi_j v_{kj})_{\mathbb{H}_j} - (\nabla u_k|\nabla\varphi_j v_{kj})_{\mathbb{H}_j} + (u_k \nabla\varphi_j|v_{kj})_{\mathbb{H}_j},$$

hence

$$d_{kj} \le \frac{1}{k} + \varepsilon_k|\nabla(\varphi_j v_{kj})|_{q'} + |(\nabla u_k|\nabla\varphi_j v_{kj})_{\mathbb{H}_j}| + |(u_k\nabla\varphi_j|\nabla v_{kj})_{\mathbb{H}_j}|.$$

Clearly the first two terms on the right-hand side of this inequality converge to zero as $k \to \infty$. The third term tends to zero, as $\nabla u_k \rightharpoonup 0$ in $L_q(\Omega)$ and by Poincaré's inequality and compact embedding, the set $\{\nabla\varphi_j v_{kj}\}_{k\ge 0}$ is relatively compact in $L_{q'}(\Omega)$. Finally, the last term converges also to zero, as $u_k\nabla\varphi_j \to 0$ as $k \to \infty$ by compact embedding, and ∇v_{kj} is bounded in $L_{q'}$, by construction.

(c) The Isomorphism
Let

$$\nabla_q : \dot{H}^1_{q,\Sigma_0}(\Omega) \to L_q(\Omega)$$

be defined by $(\nabla_q u)(x) = (\nabla u)(x)$, $x \in \Omega$. This operator is bounded, linear, injective, and has closed range. Therefore its dual

$$\nabla_q^* : L_{q'}(\Omega) \to [\dot{H}^1_{q,\Sigma_0}(\Omega)]^* = \dot{H}^{-1}_{q',\Sigma_1}(\Omega)$$

is linear, bounded, and surjective. Define

$$A_q : \dot{H}^1_{q,\Sigma_0}(\Omega) \to \dot{H}^{-1}_{q,\Sigma_1}(\Omega)$$

by means of $A_q u := \nabla^*_{q'}\nabla_q$; then A_q is bounded linear, and $A_q^* = A_{q'}$. We have

$$A_q u = f \quad \Leftrightarrow \quad (\nabla u|\nabla v)_\Omega = (f|\nabla v)_\Omega \ \text{ for all } v \in \dot{H}^1_{q',\Sigma_0}(\Omega), \quad u = 0 \text{ on } \Sigma_0.$$

By **(a)** we see that A_q is injective, for $q \in (1,\infty)$, and **(b)** implies that A_q has closed range. Therefore, as $A_q^* = A_{q'}$ is also injective, it is bijective, i.e., A_q is an isomorphism for each $q \in (1,\infty)$.

(d) Inhomogeneous Dirichlet Data
Finally we consider the case $f = 0$ but $h \ne 0$. For this purpose we first solve

$$u_0 - \Delta u_0 = 0 \text{ in } \Omega, \quad \partial_\nu u_0 = 0 \text{ on } \Sigma_1, \quad u_0 = h \text{ on } \Sigma_0.$$

Section 6.3.6 yields a unique $u_0 \in H^1_q(\Omega)$. Then $u_1 = u - u_0$ must solve

$$A_q u_1 = \Delta u_0 \in \dot{H}^{-1}_{q,\Sigma_1}(\Omega),$$

which by **(c)** admits a unique solution $u_1 \in \dot{H}^1_{q,\Sigma_0}(\Omega)$. This completes the proof. $\square$

As a first consequence we obtain the **Helmholtz-Weyl projection**.

Corollary 7.4.4. *Let $1 < q < \infty$, Ω be either the whole space $\mathbb{R}^n$, or a perturbed half-space, or a domain with compact C^1-boundary $\partial\Omega =: \Sigma$. Suppose that $\Sigma = \Sigma_0 \cup \Sigma_1$ with disjoint parts Σ_j which are open and closed in Σ.*
Then given $f \in L_q(\Omega; \mathbb{C}^n)$, there are unique functions $\phi \in \dot{H}^1_{q,\Sigma_0}(\Omega)$ and $w \in \mathsf{N}(\nabla^*_{q'})$ such that

$$f = \nabla\phi + w,$$

and there is a constant such that

$$|w|_{L_q} \le C|f|_{L_q}, \quad \textit{for all } f \in L_q(\Omega).$$

The bounded linear operator $P_{HW} \in \mathcal{B}(L_q(\Omega))$ defined by $P_{HW} f := w$ is a projection, called the Helmholtz-Weyl *projection associated to the decomposition $\Sigma = \Sigma_0 \cup \Sigma_1$ of the boundary $\Sigma = \partial\Omega$ of Ω.*

This result follows by solving the problem $A_q\phi = \nabla^*_{q'} f$ according to Theorem 7.4.3. Then obviously $w = f - \nabla\phi \in \mathsf{N}(\nabla^*_{q'})$.

The final result concerns higher regularity.

Corollary 7.4.5. *Suppose that Ω is a domain in $\mathbb{R}^n$ with compact boundary $\partial\Omega := \Sigma$ of class $C^{(2+s)-}$, $s = 0, 1$, and suppose that Σ decomposes disjointly into $\Sigma = \Sigma_0 \cup \Sigma_1$, where Σ_j are open and closed in Σ. Let $f \in H^s_q(\Omega)$, $g \in W^{1+s-1/q}_q(\Sigma_1)$, $h \in W^{2+s-1/q}_q(\Sigma_0)$, and assume $(f, g) \in \dot{H}^{-1}_{q,\Sigma_1}(\Omega)$.*

Then the problem

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ \partial_\nu u &= g \quad \text{on } \Sigma_1, \\ u &= h \quad \text{on } \Sigma_0, \end{aligned} \tag{7.52}$$

admits a unique solution u with $\nabla u \in H^{1+s}_q(\Omega)$. There is a constant $C > 0$ such that

$$|\nabla u|_{H^{1+s}_q} \le C\big(|(f,g)|_{\dot{H}^{-1}_{q,\Sigma_1}} + |f|_{H^s_q} + |g|_{W^{1+s-1/q}_q} + |h|_{W^{2+s-1/q}_q}\big) \tag{7.53}$$

holds for all $(f, g, h) \in H^s_q(\Omega) \times W^{1+s-1/q}_q(\Sigma_1) \times W^{2+s-1/q}_q(\Sigma_0)$, $s = 0, 1$.

Proof. First we may reduce to the case $(g, h) = 0$, solving the problem

$$\begin{aligned} u_0 - \Delta u_0 &= 0 \quad \text{in } \Omega, \\ \partial_\nu u_0 &= g \quad \text{on } \Sigma_1, \\ u_0 &= h \quad \text{on } \Sigma_0, \end{aligned}$$

as in **(d)** above.

Let $\mathbb{H}_j$ and φ_j, $j = 0, \ldots, N$, be as above. Let $v \in \dot{H}^1_{q'}(\mathbb{H}_j)$ if $x_j \in \Omega \cup \Sigma_1$, and $v \in {}_0\dot{H}^1_{q'}(\mathbb{H}_j)$ otherwise. Then we have

$$\begin{aligned} (\nabla(\varphi_j u)|\nabla v)_{\mathbb{H}_j} &= (\nabla u|\nabla(\varphi_j v))_{\mathbb{H}_j} - (\nabla u \nabla\varphi_j|v)_{\mathbb{H}_j} + (u\nabla\varphi_j|\nabla v)_{\mathbb{H}_j} \\ &= (\nabla u|\nabla(\varphi_j v)))_{\mathbb{H}_j} - (\nabla u \nabla\varphi_j|v)_{\mathbb{H}_j} - (\mathrm{div}(u\nabla\varphi_j)|v)_{\mathbb{H}_j} \\ &= -(f\varphi_j + 2\nabla u\nabla\varphi_j + u\Delta\varphi_j|v)_{\mathbb{H}_j} = -(f_j|v)_{\mathbb{H}_j}, \end{aligned}$$

with $f_j := f\varphi_j + 2\nabla u\nabla\varphi_j + u\Delta\varphi_j \in L_q(\mathbb{H}_j)$. This shows that $\varphi_j u$ is the weak solution in $\mathbb{H}_j$ with right-hand side $f_j \in L_q(\mathbb{H}_j)$. The results in Section 7.4.3 show that $\nabla(\varphi_j u) \in H^{1+s}_q(\mathbb{H}_j)$, hence summing over j we obtain the assertion. □

Remark. In all of this section we restricted our analysis to the Laplacian. However, Δ can be replaced by any uniformly strongly elliptic operator $\mathrm{div}(\mathcal{A}(x)\nabla)$ with coefficients $\mathcal{A} \in C_l(\overline{\Omega}; \mathbb{R}^{n\times n})$ for weak solutions, and additionally $\mathcal{A} \in W^{1+s}_\infty(\Omega; \mathbb{R}^{n\times n})$ for higher regularity. This extension is straightforward, and its implementation is left for the curious reader as well as to researchers who are in need of such results.