Chapter 6

Elliptic and Parabolic Problems

In this chapter we prove maximal L_p-regularity for various linear parabolic and elliptic problems. These results will be crucial for our study of quasilinear parabolic problems, including those introduced in Chapter 1. The proofs are based on the vector-valued Fourier multiplier theorems and $\mathcal{H}^\infty$-calculi developed in Chapter 4, as well as on arguments involving perturbations, domain transformations, and localizations.

6.1 Elliptic and Parabolic Problems on $\mathbb{R}^n$

We begin with the constant coefficient case.

1.1 Kernel Estimates
Let $\mathcal{A}(\xi)$ denote a $\mathcal{B}(E)$-valued polynomial on $\mathbb{R}^n$ which is homogeneous of degree $m \in \mathbb{N}$, i.e.,

$$\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n,$$

where we use multi-index notation, and $a_\alpha \in \mathcal{B}(E)$, E a Banach space. We want to consider the vector-valued partial differential equation

$$\lambda u(x) + \mathcal{A}(D)u(x) = f(x), \quad x \in \mathbb{R}^n, \tag{6.1}$$

where the function f is given, $\lambda \in \mathbb{C}$, and $D = -i(\partial_1, \ldots, \partial_n)$. The purpose of this subsection is the derivation of a kernel representation for the solution $u(x)$ of the form

$$u(x) = \int_{\mathbb{R}^n} \gamma_\lambda(x - x')f(x')\,dx', \quad x \in \mathbb{R}^n, \tag{6.2}$$

as well as estimates for the kernel γ_λ.

Homogeneity of $\mathcal{A}$ of degree m implies that γ_λ must be of the form

$$\gamma_\lambda(x) = |\lambda|^{\frac{n}{m}-1}\gamma_\theta(|\lambda|^{1/m}x), \quad x \in \mathbb{R}^n,\ \arg(\lambda) = \theta,\ \lambda \neq 0. \tag{6.3}$$

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Here γ_θ denotes the fundamental solution of (6.1), i.e., it satisfies the equation

$$e^{i\theta}\gamma_\theta + \mathcal{A}(D)\gamma_\theta = \delta_0$$

in the sense of distributions.

In fact, a formal argument, which will become precise later, is as follows. Taking Fourier transforms we obtain for the solution of (6.1) the expression

$$\mathcal{F}u(\xi) = (\lambda + \mathcal{A}(\xi))^{-1}\mathcal{F}f(\xi), \quad \xi \in \mathbb{R}^n.$$

Taking inverse transforms this yields

$$u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} (\lambda + \mathcal{A}(\xi))^{-1}\mathcal{F}f(\xi)e^{ix\cdot\xi}\, d\xi.$$

By the convolution theorem we get

$$\gamma_\lambda(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} (\lambda + \mathcal{A}(\xi))^{-1}e^{ix\cdot\xi}\, d\xi,$$

which after the scaling $\xi = |\lambda|^{1/m}\xi'$ leads to the representation (6.3) with

$$\gamma_\theta(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} (e^{i\theta} + \mathcal{A}(\xi'))^{-1}e^{ix\cdot\xi'}\, d\xi', \tag{6.4}$$

where $\theta = \arg(\lambda)$.

For all this to make sense we surely must know that $\lambda + \mathcal{A}(\xi)$ is invertible for all $\xi \in \mathbb{R}^n$ and for all λ in question. This naturally leads to the basic assumption we make here, namely that of parameter-ellipticity.

Definition 6.1.1. *The $\mathcal{B}(E)$-valued polynomial $\mathcal{A}(\xi)$ is called* **parameter-elliptic** *if there is an angle $\phi \in [0, \pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ of $\mathcal{A}(\xi)$ satisfies*

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \quad \textit{for all } \xi \in \mathbb{R}^n,\ |\xi| = 1. \tag{6.5}$$

We call

$$\phi_{\mathcal{A}} := \inf\{\phi :\ (6.5)\ \textit{holds}\} = \sup_{|\xi|=1} |\arg\sigma(\mathcal{A}(\xi))|$$

angle of ellipticity *of $\mathcal{A}$. $\mathcal{A}(\xi)$ is called* **normally elliptic** *if it is parameter-elliptic with angle $\phi_{\mathcal{A}} < \pi/2$. We then call the differential operator $\mathcal{A}(D)$ parameter-elliptic resp. normally elliptic as well.*

Some remarks are in order.

Remark 6.1.2. (i) It is easy to see that parameter-ellipticity as well as $\phi_{\mathcal{A}}$ are invariant under orthogonal transformations, but even more is true. Consider a coordinate transformation of the form $Tu(x) = u(Qx)$ where $Q \in \mathbb{R}^{n\times n}$ is invertible. Then the transformed differential operator will be

$$\mathcal{A}_Q(D) := T^{-1}\mathcal{A}(D)T = \mathcal{A}(Q^{\mathsf{T}}D).$$

Hence with $\mathcal{A}(\xi)$ also $\mathcal{A}_Q(\xi) = \mathcal{A}(Q^{\mathsf{T}}\xi)$ is parameter-elliptic, and $\phi_{\mathcal{A}_Q} = \phi_{\mathcal{A}}$.

(ii) Note that m is necessarily even in case $\phi_{\mathcal{A}} < \pi/2$. Indeed,

$$\mathcal{A}(-\xi) = -\mathcal{A}(\xi), \quad \xi \in \mathbb{R}^n,$$

in case m is odd, and hence

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \cap -\Sigma_\phi = \emptyset, \quad |\xi| = 1,$$

which is impossible.

(iii) On the other hand, there are parameter-elliptic operators of odd order, e.g. for $n = 1$, $m = 1$, $\mathcal{A}(D) = iD$ is parameter-elliptic with $\phi_{\mathcal{A}} = \pi/2$.

(iv) Recall that the symbol $\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ is called **elliptic** if $0 \notin \sigma(\mathcal{A}(\xi))$ for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$. Obviously, each parameter-elliptic symbol is also elliptic, but not conversely. A famous counterexample is the Cauchy-Riemann operator $\mathcal{A}(\xi) = \xi_1 + i\xi_2$ with $n = 2$, $E = \mathbb{C}$; for this operator we have $\cup_{|\xi|=1}\sigma(\mathcal{A}(\xi)) = \mathbb{S}^1$, the unit sphere in $\mathbb{C}$.

If E is a Hilbert space, there is another notion of ellipticity.

Definition 6.1.3. *The $\mathcal{B}(E)$-valued polynomial $\mathcal{A}(\xi)$ is called* **strongly elliptic** *if there is a constant $c > 0$ such that*

$$\operatorname{Re}(\mathcal{A}(\xi)v|v)_E \geq c|\xi|^m |v|_E^2, \quad \xi \in \mathbb{R}^n,\ v \in E.$$

The largest such c will be called the **ellipticity constant** *$c_{\mathcal{A}}$ of $\mathcal{A}(D)$. The differential operator $\mathcal{A}(D)$ is then also called strongly elliptic.*

Also for this notion of ellipticity some remarks are in order.

Remark 6.1.4. **(i)** Observe that also strong ellipticity as well as $c_{\mathcal{A}}$ are invariant under orthogonal transformations. More generally, strong ellipticity is invariant also under general coordinate transformations, but the constant $c_{\mathcal{A}}$ does not have this property.

(ii) To understand the condition of strong ellipticity, recall that the **numerical range** $\mathsf{n}(B)$ of an operator $B \in \mathcal{B}(E)$ is defined by

$$\mathsf{n}(B) := \overline{\{z \in \mathbb{C} : z = (Bv|v)_E \text{ for some } v \in E,\ |v|_E = 1\}}.$$

It is easy to see that $\sigma(B) \subset \mathsf{n}(B)$, and that $\mathsf{n}(B) \subset \bar{B}_{\mathbb{C}}(0, |B|)$ holds. Therefore, $\mathcal{A}$ is strongly elliptic if the numerical range of $\mathcal{A}(\xi)$ is contained in the half-space $\operatorname{Re} z \geq c > 0$ for each $\xi \in \mathbb{R}^n$, $|\xi| = 1$. Consequently, if $\mathcal{A}$ is strongly elliptic then

$$\sigma(\mathcal{A}(\xi)) \subset \mathsf{n}(\mathcal{A}(\xi)) \subset \Sigma_\phi, \quad \xi \in \mathbb{R}^n,\ |\xi| = 1.$$

In particular, every strongly elliptic polynomial $\mathcal{A}$ is parameter-elliptic with

$$\phi_{\mathcal{A}} \le \sup\{|\arg(\mathcal{A}(\xi)v|v)_E| : v \in E,\ |v|_E = 1,\ \xi \in \mathbb{R}^n,\ |\xi| = 1\} < \pi/2,$$

hence even normally elliptic.

(iii) The class of strongly elliptic differential operators contains some of the most common elliptic operators arising in applications.

Now assume that $\mathcal{A}$ is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}}$ and let $\phi > \phi_{\mathcal{A}}$. We are going to justify the formal procedure from above for $|\theta| \le \pi - \phi$. For this purpose we consider the Fourier integral

$$\gamma_\theta^\varepsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (e^{i\theta} + \mathcal{A}(\xi'))^{-1} e^{ix\cdot\xi'} e^{-\varepsilon|\xi'|}\, d\xi', \tag{6.6}$$

with $\varepsilon > 0$ fixed. Note that this integral is absolutely convergent due to the additional exponential factor, in contrast to (6.4). For the moment we restrict attention to the case $n \ge 3$. We will comment at the end of this section on $n = 1, 2$. Fix $x \in \mathbb{R}^n$, $x \ne 0$, and choose a rotation Q such that $Qx = re_1$, where $r = |x|$ and e_1 means the first unit vector in $\mathbb{R}^n$. By means of the variable transformation

$$Q\xi' = (\eta, s\zeta), \quad \eta \in \mathbb{R},\ s > 0,\ \zeta \in \mathbb{S}^{n-2},$$

where $\mathbb{S}^k$ denotes the k-dimensional unit sphere, we obtain the following representation of $\gamma_\theta^\varepsilon$.

$$\gamma_\theta^\varepsilon(x) = \frac{1}{(2\pi)^n} \int_0^\infty s^{n-2} \int_{\mathbb{S}^{n-2}} \int_{\mathbb{R}} (e^{i\theta} + \mathcal{A}(Q^{\mathsf{T}}(\eta, s\zeta)))^{-1} e^{i|x|\eta} e^{-\varepsilon(\eta^2+s^2)^{1/2}}\, d\eta d\zeta ds.$$

Next we employ the scaling $\eta = (1+s)z$ for η and observe that by homogeneity of $\mathcal{A}$ we have

$$\begin{aligned}\mathcal{A}(Q^{\mathsf{T}}(\eta, s\zeta)) &= \sum_{k=0}^{m} \eta^{m-k} s^k \sum_{|\beta|=k} b_\beta \zeta^\beta \\ &= (1+s)^m \sum_{k=0}^{m} z^{m-k} \Big(1 - \frac{1}{1+s}\Big)^k b_k(\zeta) \\ &= (1+s)^m P(z, \zeta, 1/(1+s)),\end{aligned}$$

for some $b_\beta \in \mathcal{B}(E)$, $b_k(\zeta) = \sum_{|\beta|=k} b_\beta \zeta^\beta$. Then we set

$$H(z, \zeta, \sigma, \theta) = (2\pi)^{-n}(e^{i\theta}\sigma^m + P(z, \zeta, \sigma)),$$

and finally obtain the representation

$$\gamma_\theta^\varepsilon(x) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-1}} \Big[\int_{\mathbb{S}^{n-2}} h_\varepsilon(s, \zeta, \theta, r) d\zeta\Big]\, ds, \tag{6.7}$$

with

$$h_\varepsilon(s,\zeta,\theta,r) = \int_{\mathbb{R}} H(z,\zeta,1/(1+s),\theta)^{-1} e^{ir(1+s)z} e^{-\varepsilon(1+s)[z^2+(s/(1+s))^2]^{1/2}}\, dz.$$

The function $H(z,\zeta,\sigma,\theta)$ is a $\mathcal{B}(E)$-valued polynomial in z, with coefficients depending continuously on $p = (\zeta,\sigma,\theta) \in P := \mathbb{S}^{n-2} \times [0,1] \times [-\pi+\phi, \pi-\phi]$, a compact set.

By parameter-ellipticity, the set of $z \in \mathbb{C}$ such that $H(z,p)$ is not invertible is compact and does not contain real values. This set is upper-semicontinuous in p, hence the set of singularities of $H(\cdot,p)^{-1}$ is a compact set not intersecting the real line, uniformly for $p \in P$. Since H^{-1} is holomorphic in z we may therefore deform the path of integration to a contour Γ of the form

$$\Gamma := \{z = t + i\kappa(1+|t|) : \ t \in \mathbb{R}\},$$

where $\kappa > 0$ is small and independent of $p \in P$. Then we obtain by Cauchy's theorem

$$h_\varepsilon(s,\zeta,\theta,r) = \int_{\Gamma} H(z,\zeta,1/(1+s),\theta)^{-1} e^{ir(1+s)z} e^{-\varepsilon(1+s)[z^2+(s/(1+s))^2]^{1/2}}\, dz.$$

Since H^{-1} is bounded on Γ, and

$$|e^{ir(1+s)z}| = e^{-\kappa r(1+s)(1+|t|)},$$

the integral defining h_ε is absolutely convergent and

$$|h_\varepsilon(s,\zeta,\theta,r)| \le Ce^{-\kappa r(1+s)}/[r(1+s)],$$

independently of $\varepsilon > 0$. Hence we may pass to the limit $\varepsilon \to 0$ to the result

$$\gamma_\theta(x) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-1}} \Big[\int_{\mathbb{S}^{n-2}} h(s,\zeta,\theta,r)d\zeta\Big]\, ds \tag{6.8}$$

with

$$h(s,\zeta,\theta,r) = \int_{\Gamma} H(z,\zeta,1/(1+s),\theta)^{-1} e^{ir(1+s)z}\, dz.$$

Contracting the contour Γ in the set $\{\mathrm{Im}\, z > \kappa\} \subset \mathbb{C}$ into a smooth Jordan curve Γ_0 surrounding the singularities of H^{-1} in the upper half-plane, we finally get the following representation for h.

$$h(s,\zeta,\theta,r) = \int_{\Gamma_0} H(z,\zeta,1/(1+s),\theta)^{-1} e^{ir(1+s)z}\, dz. \tag{6.9}$$

This implies the estimate

$$|h(s,\zeta,\theta,r)| \le Ce^{-\kappa(1+s)r}, \quad s > 0, \ \zeta \in \mathbb{S}^{n-2}, \ |\theta| \le \pi - \phi \tag{6.10}$$

for h. We summarize these considerations in

Theorem 6.1.5. *Let $n, m \in \mathbb{N}$, E a Banach space, $a_\alpha \in \mathcal{B}(E)$, and suppose*

$$\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n,$$

is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}} < \pi$. Then for each $\phi > \phi_{\mathcal{A}}$ there is a constant C_ϕ such that the solution $\gamma_\theta(x)$ of

$$e^{i\theta} u + \mathcal{A}(D)u = \delta_0$$

satisfies the estimate

$$|\gamma_\theta(x)| \leq C_\phi p_0(|x|), \quad x \in \mathbb{R}^n,\ x \neq 0,\ |\theta| \leq \pi - \phi, \tag{6.11}$$

where p_0 is given by

$$p_0(r) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-1}} e^{-\kappa r(1+s)}\, ds,$$

for some $\kappa > 0$. The function $p_0 : (0,\infty) \to (0,\infty)$ is completely monotone, and satisfies

$$\int_0^\infty r^{n+\rho-1} p_0(r)\, dr < \infty \quad \textit{if and only if} \quad \rho > -m.$$

Note that we can estimate p_0 further by

$$p_0(r) \leq \begin{cases} ce^{-\kappa r} & \text{if} \quad n < m; \\ ce^{-\kappa r} \log(2 + 1/r) & \text{if} \quad n = m; \\ \frac{c}{r^{n-m}} e^{-\kappa r} & \text{if} \quad n > m. \end{cases}$$

Together with (6.8) and (6.3), Theorem 6.1.5 leads to a Poisson estimate for the kernel γ_λ from (6.2), i.e., for each $\phi > \phi_{\mathcal{A}}$ there is a constant $C_\phi > 0$ such that

$$|\gamma_\lambda(x)| \leq C_\phi |\lambda|^{\frac{n}{m}-1} p_0(|\lambda|^{1/m}|x|), \quad x \in \mathbb{R}^n,\ |\arg \lambda| \leq \pi - \phi. \tag{6.12}$$

However, even more is true. Applying the differential operator D^β to (6.6) and employing the same arguments as above we obtain

Corollary 6.1.6. *In the situation of Theorem* 6.1.5 *for each $k \in \mathbb{N}_0$, we have in addition*

$$|D^\beta \gamma_\theta(x)| \leq C_{\phi,k} p_k(|x|), \quad x \in \mathbb{R}^n,\ x \neq 0,\ |\theta| \leq \pi - \phi,\ |\beta| = k,$$

where p_k is given by

$$p_k(r) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-k-1}} e^{-\kappa r(1+s)}\, ds,$$

for some $\kappa > 0$.

Observe that this corollary implies the estimate

$$|D^\beta \gamma_\lambda(x)| \le C_{\phi,k}|\lambda|^{\frac{n+k}{m}-1} p_k(|\lambda|^{1/m}|x|), \quad x \in \mathbb{R}^n,\ |\arg\lambda| \le \pi - \phi, \tag{6.13}$$

for the derivatives of the fundamental solution γ_λ, with $k = |\beta|$. This yields $D^\beta\gamma_\lambda \in L_1(\mathbb{R}^n; \mathcal{B}(E))$ if $|\beta| < m$.

Concluding, some remarks concerning the cases $n = 1, 2$ have to be made. For $n = 1$, instead of the rotation Q we may use reflection; all above arguments remain valid for this case, simply dropping the integrals over s and ζ. In that case the functions p_k should be replaced by

$$p_k(r) = \int_0^\infty \frac{1}{(1+s)^{m-k}} e^{-\kappa r(1+s)}\, ds.$$

For $n = 2$ the arguments are also valid if we interpret $\mathbb{S}^0$ as the set consisting of the two points ± 1. Therefore the above results are valid for all dimensions $n \in \mathbb{N}$.

1.2 L_q-Realizations of Elliptic Differential Operators

Next we consider the L_q-realizations of the differential operator $\mathcal{A}(D)$.

Theorem 6.1.7. *Let $n, m \in \mathbb{N}$, E a Banach space, $a_\alpha \in \mathcal{B}(E)$, $1 < q < \infty$, and suppose $\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}} < \pi$. Define the L_q-realization A of $\mathcal{A}$ in $X_0 = L_q(\mathbb{R}^n; E)$ by means of $A = \overline{A_0}$, where*

$$[A_0 u](x) = \mathcal{A}(D)u(x), \quad x \in \mathbb{R}^n, \quad u \in \mathsf{D}(A_0) := H^m_q(\mathbb{R}^n; E).$$

Then A is sectorial with spectral angle $\phi_A \le \phi_{\mathcal{A}}$, and

$$H^m_q(\mathbb{R}^n; E) \subset \mathsf{D}(A) \subset H^{m-1}_q(\mathbb{R}^n; E).$$

Proof. Obviously, A has dense domain. If $f \in L_q(\mathbb{R}^n; E)$, choose a sequence $f_k \in \mathcal{D}(\mathbb{R}^n; E)$ such that $f_k \to f$ in $L_q(\mathbb{R}^n; E)$. For $\lambda \in \Sigma_{\pi-\phi}$, $\phi > \phi_{\mathcal{A}}$, we obtain $u_k = \gamma_\lambda * f_k \in H^m_q(\mathbb{R}^n; E)$ as well as $\lambda u_k + \mathcal{A}(D)u_k = f_k$, by uniqueness of the Fourier transform. Since $u_k \to u = \gamma_\lambda * f$ in $L_q(\mathbb{R}^n; E)$ as $k \to \infty$, we see that $u \in \mathsf{D}(A)$ and $\lambda u + Au = f$. This shows that $\lambda + A$ is invertible for each $\lambda \in \Sigma_\phi$ and $(\lambda + A)^{-1} f = \gamma_\lambda * f$. Thus by Corollary 6.1.6 we obtain the inclusions

$$H^m_q(\mathbb{R}^n; E) = \mathsf{D}(A_0) \subset \mathsf{D}(A) \subset H^{m-1}_q(\mathbb{R}^n; E),$$

and Theorem 6.1.5 yields $-\Sigma_{\pi-\phi} \subset \rho(A)$, as well as

$$|\lambda(\lambda + A)^{-1}|_{\mathcal{B}(L_q(\mathbb{R}^n;E))} \le M_{\pi-\phi}, \tag{6.14}$$

for each $\phi > \phi_{\mathcal{A}}$.

For $f \in \mathcal{D}(\mathbb{R}^n; E)$, supp $f \subset B(0, R)$, we have by Theorem 6.1.5

$$|\lambda\gamma_\lambda * f(x)| \le \int_{\mathbb{R}^n} p_0(|y|)|f(x - y/|\lambda|^{1/m})|\, dy \to 0$$

as $\lambda \to 0$, uniformly for bounded x. On the other hand, for $|x| \geq 2R$ we have $|x-y| \geq |x| - |y| \geq |y|$. Since p_0 is non-increasing this yields

$$\begin{aligned}|\lambda\gamma_\lambda * f(x)| &\leq |f|_\infty \int_{B_R(0)} |\lambda|^{n/m} p(|\lambda|^{1/m}|x-y|)\,dy\\ &\leq |f|_\infty \int_{B_R(0)} |\lambda|^{n/m} p(|\lambda|^{1/m}|y|)\,dy\\ &= |f|_\infty \int_0^{|\lambda|^{1/m}R} p(r)\,dr \to 0\end{aligned}$$

as $\lambda \to 0$. This implies $|\lambda(\lambda + A)^{-1} f|_\infty \to 0$ for $\lambda \to 0$, for each $f \in \mathcal{D}(\mathbb{R}^n; E)$, but then by interpolation

$$|\lambda\gamma_\lambda * f|_q \leq |\lambda\gamma_\lambda * f|_1^{1/q} |\lambda\gamma_\lambda * f|_\infty^{1/q'} \to 0.$$

Therefore, $A(\lambda + A)^{-1} \to I$ strongly as $\lambda \to 0$, i.e., $\mathsf{R}(A)$ is dense in $L_q(\mathbb{R}^n; E)$ and $\mathsf{N}(A) = 0$, for each $1 < q < \infty$. Thus A is sectorial and $\phi_A \leq \phi_{\mathcal{A}}$. □

One can show that we even have

$$H_q^m(\mathbb{R}^n; E) \hookrightarrow \mathsf{D}(A) \hookrightarrow H_q^s(\mathbb{R}^n; E), \quad \text{for each } s < m.$$

Nevertheless, we cannot prove the elliptic maximal L_q-regularity

$$\mathsf{D}(A) = H_q^m(\mathbb{R}^n; E)$$

unless more is known on the geometry of E. Here harmonic analysis comes into play.

1.3 $\mathcal{H}^\infty$-Calculus for Elliptic Operators

If E is a Banach space of class $\mathcal{HT}$, for differential operators with parameter-elliptic symbols the following result is valid.

Theorem 6.1.8. *Let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$, and $1 < q < \infty$. Suppose $\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ with $a_\alpha \in \mathcal{B}(E)$ is a homogeneous differential operator of order m which is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}}$. Let A denote its realization in $X_0 = L_q(\mathbb{R}^n; E)$ with domain $\mathsf{D}(A) = H_q^m(\mathbb{R}^n; E)$.*

Then $A \in \mathcal{H}^\infty(X_0)$ with $\mathcal{H}^\infty$-angle $\phi_A^\infty \leq \phi_{\mathcal{A}}$. In particular, A is $\mathcal{R}$-sectorial with $\phi_A^R \leq \phi_{\mathcal{A}}$.

Proof. (i) Observe first that the symbol $\mathcal{A}(\xi)$ is homogeneous of degree m, i.e., $\mathcal{A}(\xi) = \rho^m \mathcal{A}(\zeta)$, $\rho = |\xi|$. Parameter-ellipticity implies that $\mathcal{A}(\zeta)$ is invertible for each $|\zeta| = 1$ and $|\mathcal{A}(\zeta)^{-1}| \leq M_0$, where M_0 is independent of ζ, by compactness of the set $|\zeta| = 1$; this implies in particular $|\mathcal{A}(\xi)^{-1}| \leq M_0\rho^{-m}$. Hence $\xi^\alpha \mathcal{A}(\xi)^{-1} = \zeta^\alpha \mathcal{A}(\zeta)^{-1}$ is bounded for each $|\alpha| = m$. But since $\mathcal{A}(\zeta)$ is holomorphic, $\zeta^\alpha \mathcal{A}(\zeta)^{-1}$ is so as well, and since $\mathbb{S}^{n-1}$ is compact, $\{\xi^\alpha \mathcal{A}(\xi)^{-1} : \xi \in \mathbb{R}^n \setminus \{0\}\}$ is $\mathcal{R}$-bounded by Proposition 4.1.12. The same holds true for $\{|\xi|^k D^\beta[\xi^\alpha \mathcal{A}(\xi)^{-1}] : \xi \in \mathbb{R}^n \setminus \{0\}\}$, $|\beta| = k \in \mathbb{N}$, as a simple calculation shows. The vector-valued Mikhlin theorem, Theorem 4.3.11, then implies that there is a constant $C > 0$ such that

$$C^{-1}|D^\alpha u|_{X_0} \leq |\mathcal{A}(D)u|_{X_0}, \quad \text{for all } u \in H^m_q(\mathbb{R}^n; E),\ |\alpha| = m,$$

holds. In particular, we have $\mathsf{D}(A) = H^m_q(\mathbb{R}^n; E)$, and by (6.14) A is sectorial with spectral angle $\phi_A \leq \phi_{\mathcal{A}}$.

(ii) To show that A admits an $\mathcal{H}^\infty$-calculus such that the $\mathcal{H}^\infty$-angle satisfies $\phi_A^\infty \leq \phi_{\mathcal{A}}$, let $\phi > \phi_{\mathcal{A}}$ be fixed and choose a function $h \in H_0(\Sigma_\phi)$. Let Γ denote the contour $\Gamma = (\infty, 0]e^{i\theta} \cup (0, \infty)e^{-i\theta}$, where $\phi_{\mathcal{A}} < \theta < \phi$. Then $h(A)$ is well defined as the Dunford integral

$$h(A) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - A)^{-1}\, d\lambda.$$

For $u \in \mathcal{D}(\mathbb{R}^n; E)$, we may take Fourier transforms, to the result

$$\begin{aligned}\mathcal{F}[h(A)u](\xi) &= \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - \mathcal{A}(\xi))^{-1}\mathcal{F}u(\xi)\, d\lambda \\ &= h(\mathcal{A}(\xi))\mathcal{F}u(\xi),\end{aligned}$$

hence the symbol of $h(A)$ is given by $h(\mathcal{A}(\xi))$. Therefore, it is enough to show that this symbol is a Fourier multiplier for $L_q(\mathbb{R}^n; E)$, with norm $\leq C|h|_{H^\infty(\Sigma_\phi)}$. This will be done employing the vector-valued Mikhlin theorem another time.

By means of the rescalings $\xi = \rho\zeta$ and $\mu = \lambda\rho^{-m}$ we obtain the representation

$$h(\mathcal{A}(\xi)) = \frac{1}{2\pi i}\int_\Gamma h(\rho^m\mu)(\mu - \mathcal{A}(\zeta))^{-1}\, d\mu.$$

Since $\sigma_0 = \cup_{|\zeta|=1}\sigma(\mathcal{A}(\zeta))$ is compact and contained in Σ_{ϕ_0}, we may deform the contour Γ within Σ_θ into a compact simple smooth closed path Γ_0 surrounding σ_0 counter-clockwise, and by Cauchy's theorem

$$h(\mathcal{A}(\xi)) = \frac{1}{2\pi i}\int_{\Gamma_0} h(\rho^m\mu)(\mu - \mathcal{A}(\zeta))^{-1}\, d\mu.$$

By compactness of Γ_0 and of $\mathbb{S}^{n-1}$, in virtue of Proposition 4.1.12, $(\mu - \mathcal{A}(\zeta))^{-1}$ is $\mathcal{R}$-bounded on $\Gamma_0 \times \mathbb{S}^{n-1}$, hence this representation of $h(\mathcal{A}(\xi))$ yields

$$\mathcal{R}\{h(\mathcal{A}(\xi)) : \xi \in \mathbb{R}^n\} \leq (2\pi)^{-1}|h|_{H^\infty(\Sigma_\phi)} l(\Gamma_0)\mathcal{R}\{(\mu - \mathcal{A}(\zeta))^{-1} : \mu \in \Gamma_0, \zeta \in \mathbb{S}^{n-1}\}$$

where $l(\Gamma_0)$ denotes the length of Γ_0. Thus the symbol of $h(A)$ is $\mathcal{R}$-bounded.

To obtain appropriate bounds for the derivatives of $h(\mathcal{A}(\xi))$, observe the relation

$$D_\xi = -i\zeta\frac{\partial}{\partial\rho} + \frac{1}{\rho}(I - \zeta\otimes\zeta)D_\zeta.$$

With $G_0(\mu,\zeta) = (2\pi i)^{-1}(\mu - \mathcal{A}(\zeta))^{-1}$ we have

$$h(\mathcal{A}(\xi)) = \int_{\Gamma_0} h(\rho^m\mu)G_0(\mu,\zeta)\,d\mu,$$

hence differentiating this expression inductively we get

$$\rho^{|\alpha|}D_\xi^\alpha h(\mathcal{A}(\xi)) = \sum_{k=0}^{|\alpha|}\int_{\Gamma_0}(\rho^m\mu)^k h^{(k)}(\rho^m\mu)G_{\alpha,k}(\mu,\zeta)\,d\mu,$$

where the functions $G_{\alpha,k}(\mu,\zeta)$ are analytic in a neighbourhood of $\Gamma_0 \times \mathbb{S}^{n-1}$. Therefore we obtain

$$\mathcal{R}\{|\xi|^{|\alpha|}D_\xi^\alpha h(\mathcal{A}(\xi)) : \xi \in \mathbb{R}^n\} \le \sum_{k=0}^{|\alpha|} C_{\alpha,k}\sup_{z\in\Sigma_\theta}|z^k h^{(k)}(z)|.$$

Finally, by the Cauchy estimates we have $\sup_{z\in\Sigma_\theta}|z^k h^{(k)}(z)| \le c_k|h|_{H^\infty(\Sigma_\phi)}$, and so for each $\alpha \in \mathbb{N}_0^n$ there is a constant C_α such that

$$\mathcal{R}\{|\xi|^{|\alpha|}D_\xi^\alpha h(\mathcal{A}(\xi)) :\ \xi \in \mathbb{R}^n, \xi \neq 0\} \le C_\alpha|h|_{H^\infty(\Sigma_\phi)}$$

is satisfied. C_α is independent of h, it depends only on $\mathcal{A}(\xi)$, on the contour Γ_0, and on ϕ. By Theorem 4.3.11 we therefore obtain $|h(A)|_{\mathcal{B}(L_q(\mathbb{R}^n;E))} \le M_\phi|h|_{H^\infty(\Sigma_\phi)}$, which implies the assertion. □

In the situation of the last theorem, since $A \in \mathcal{H}^\infty(X_0)$ we have, by Theorem 3.3.7,

$$\mathsf{D}(A^\theta) = (L_q(\mathbb{R}^n;E), \mathsf{D}(A))_\theta = H_q^{m\theta}(\mathbb{R}^n;E)$$

for each $\theta \in [0,1]$, hence $D^\beta A^{-k/m}$ is bounded for each $|\beta| = k \le m$. On the other hand, for each $\nu \in (0,1)$ we have the representation

$$\lambda^{1-\nu}A^\nu(\lambda + A)^{-1} = \int_{-\infty}^{\infty}\frac{\lambda^{is}}{2\sin\pi(\nu+is)}A^{-is}\,ds, \quad \lambda \in \Sigma_{\pi-\phi}, \quad \phi > \phi_{\mathcal{A}}. \quad (6.15)$$

Convexity of $\mathcal{R}$-bounds and the contraction principle then show that the sets

$$\{\lambda^{1-\nu}A^\nu(\lambda + A)^{-1} :\ \lambda \in \Sigma_{\pi-\phi}\}$$

are $\mathcal{R}$-bounded. As a consequence we obtain

Corollary 6.1.9. *Let the assumptions of Theorem* 6.1.7 *be satisfied, and let* $\alpha \in (0,1)$, $q \in (1,\infty)$, $r \in [1,\infty]$. *Then*

(i) *The set*

$$\{\lambda^{1-k/m}D^\beta(\lambda + A)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\beta| = k \le m\}$$

is $\mathcal{R}$*-bounded in* $X_0 = L_q(\mathbb{R}^n;E)$*;*

(ii) $\mathsf{D}(A^\alpha) = (X_0, \mathsf{D}(A))_\alpha = H_q^{\alpha m}(\mathbb{R}^n; E)$;

(iii) $D_A(\alpha, r) = (X_0, \mathsf{D}(A))_{\alpha,r} = B_{qr}^{\alpha m}(\mathbb{R}^n; E)$.

1.4 Elliptic Operators with Variable Coefficients

Let E be a Banach space of class $\mathcal{HT}$, and consider the differential operator with variable $\mathcal{B}(E)$-valued coefficients

$$[Au](x) = \mathcal{A}(x, D)u(x), \quad x \in \mathbb{R}^n,\ u \in \mathsf{D}(A) = H_p^m(\mathbb{R}^n; E), \tag{6.16}$$

where

$$\mathcal{A}(x, D) = \sum_{\alpha| \leq m} a_\alpha(x) D^\alpha. \tag{6.17}$$

By means of the results on homogeneous parameter-elliptic operators with constant coefficients from the previous sections, perturbation and localization, we will prove the following result.

Theorem 6.1.10. *Let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$, and $1 < q < \infty$. Suppose $\mathcal{A}(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with $a_\alpha(x) \in \mathcal{B}(E)$ is a differential operator of order m with variable coefficients. Assume the following* **Condition (ra)***:*

(ra1) *$a_\alpha \in C_l(\mathbb{R}^n; \mathcal{B}(E))$ for each $|\alpha| = m$;*

(ra2) *$\mathcal{A}_\#(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is parameter-elliptic with angle of ellipticity $\leq \phi_\mathcal{A}$, for each $x \in \mathbb{R}^n \cup \{\infty\}$;*

(ra3) *$a_\alpha \in [L_{r_k} + L_\infty](\mathbb{R}^n; \mathcal{B}(E))$ for each $|\alpha| = k < m$, with $r_k \geq q$ and $m - k > n/r_k$.*

Let A denote the realization of $\mathcal{A}(x, D)$ in the base space $X_0 = L_q(\mathbb{R}^n; E)$ with domain $\mathsf{D}(A) = H_q^m(\mathbb{R}^n; E)$.

Then for each $\phi > \phi_\mathcal{A}$ there is $\mu_\phi \geq 0$ such that $\mu_\phi + A$ is $\mathcal{R}$-sectorial with $\phi^R_{\mu_\phi + A} \leq \phi$.

Proof. (a) Freeze the coefficients a_α, $|\alpha| = m$, at an arbitrary $x_0 \in \mathbb{R}^n \cup \{\infty\}$ and consider the homogeneous differential operator with constant coefficients $\mathcal{A}_\#(x_0, D)$; let A_0 denote its L_q-realization. Then we know from Theorem 6.1.8 that $\mathsf{D}(A_0) = H_q^m(\mathbb{R}^n; E)$ and that A_0 is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^R_{A_0} \leq \phi_\mathcal{A}$. By assumption **(ra1)** the coefficients a_α belong to a compact subset of $\mathcal{B}(E)$. By Corollary 6.1.9(i) and the perturbation results from Section 4.4, we see that the $\mathcal{R}$-bounds of $\lambda^{1-|\beta|/m} D^\beta (\lambda + A_0)^{-1}$ are upper semi-continuous in the coefficients, where $\lambda \in \Sigma_{\pi - \phi}$ for $\phi > \phi_\mathcal{A}$ fixed. Therefore, they are uniform in $x_0 \in \mathbb{R}^n \cup \{\infty\}$.

Applying the perturbation argument from Section 4.4 another time, we see that there is a number $\eta > 0$ independent of x_0 such that the L_q-realization $A_0 + A_1$ of $\mathcal{A}_0(D) + \mathcal{A}_1(x, D)$ is again $\mathcal{R}$-sectorial, whenever

$$\mathcal{A}_1(x, D) = \sum_{|\alpha|=m} a^1_\alpha(x) D^\alpha$$

has L_∞-coefficients subject to $|a^1_\alpha(x)|_{\mathcal{B}(E)} \le \eta$, uniformly in x, for each $|\alpha| = m$. The corresponding $\mathcal{R}$-bounds are also uniform in x_0, and the domain of $A_0 + A_1$ equals $H^m_q(\mathbb{R}^n; E)$.

(b) Here we assume $a_\alpha \in L_\infty$ for $|\alpha| < m$ and Condition **(ra1)**. Choose a large ball $B(0, r_0)$ such that

$$|a_\alpha(x) - a_\alpha(\infty)|_{\mathcal{B}(E)} \le \eta, \quad \text{for all } |x| \ge r_0,\ |\alpha| = m,$$

and set $U_0 = \mathbb{R}^n \setminus \bar{B}(0, r_0)$. Cover $\bar{B}(0, r_0)$ by finitely many balls $U_j = B(x_j, r_j)$ such that

$$|a_\alpha(x) - a_\alpha(x_j)|_{\mathcal{B}(E)} \le \eta, \quad \text{for all } |x - x_j| \le r_j,\ |\alpha| = m,\ j = 1, \ldots, N.$$

Define coefficients of local operators A_j e.g. by reflection, i.e.,

$$a^0_\alpha(x) = \begin{cases} a_\alpha(x), & x \notin \bar{B}(0, r_0) \\ a_\alpha\big(r_0^2 \frac{x}{|x|^2}\big), & x \in \bar{B}(0, r_0) \end{cases}$$

and

$$a^j_\alpha(x) = \begin{cases} a_\alpha(x) & x \in \bar{B}(x_j, r_j) \\ a_\alpha\big(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}\big), & x \notin \bar{B}(x_j, r_j) \end{cases}$$

for each $j = 1, \ldots, N$. Then $|a^j_\alpha(x) - a_\alpha(x_j)|_{\mathcal{B}(E)} \le \eta$, for each $x \in \mathbb{R}^n$ and $j = 0, \ldots, N$, hence by step (a) above the L_q-realizations A_j of

$$\mathcal{A}_j(x, D) = \sum_{|\alpha| = m} a^j_\alpha(x) D^\alpha$$

are $\mathcal{R}$-sectorial and the $\mathcal{R}$-bounds of the sets

$$\{\lambda^{1-k/m} D^\beta (\lambda + A_j)^{-1} :\ \lambda \in \Sigma_{\pi - \phi},\ |\beta| = k \le m\}$$

are finite. Next we choose a partition of unity $\varphi_j \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \varphi_j(x) \le 1$ and $\operatorname{supp} \varphi_j \subset U_j$. We may also choose $\psi_j \in \mathcal{D}(\mathbb{R}^n)$ such that $\operatorname{supp} \psi_j \subset U_j$ and $\psi_j = 1$ on $\operatorname{supp} \varphi_j$. Set $\mathcal{B}(x, D) = \sum_{|\alpha| < m} a_\alpha(x) D^\alpha$. We then obtain a representation of $(\lambda + A)^{-1}$ as follows.

$$\lambda u + Au = f \quad \text{iff} \quad \lambda u + \mathcal{A}_\#(x, D) u = f - \mathcal{B}(x, D) u.$$

Multiply by φ_j to obtain

$$\lambda(\varphi_j u) + \mathcal{A}_\#(x, D)(\varphi_j u) = \varphi_j f + [\mathcal{A}_\#(x, D), \varphi_j] u - \varphi_j \mathcal{B}(x, D) u.$$

Noting that $\mathcal{A}_\#(x, D)(\varphi_j u) = A_j(\varphi u)$, we employ the resolvent of A_j to the result

$$\varphi_j u = (\lambda + A_j)^{-1}(\varphi_j f) + (\lambda + A_j)^{-1}\{[\mathcal{A}_\#(x, D), \varphi_j] u - \varphi_j \mathcal{B}(x, D) u\}.$$

Observing $\psi_j = 1$ on $\operatorname{supp}\varphi_j$, multiplying with ψ_j and summing over j we finally get

$$u = \sum_j \psi_j(\lambda + A_j)^{-1}\varphi_j f + \sum_j \psi_j(\lambda + A_j)^{-1}\mathcal{C}_j(x, D)u, \tag{6.18}$$

where the differential operators

$$\mathcal{C}_j(x, D) := [\mathcal{A}_\#(x, D), \varphi_j] - \varphi_j\mathcal{B}(x, D) = \sum_{|\beta|<m} c^j_\beta(x)D^\beta$$

are in fact operators of order $\leq m - 1$. Hence for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|\mathcal{C}_j(x, D)v|_q \leq \varepsilon|D^m v|_q + C_\varepsilon|v|_q, \quad \text{for all } v \in \mathsf{D}(A),\ j = 0, \ldots, N.$$

By a Neumann series argument, (6.18) implies existence of a left inverse S_λ, which is given by

$$S_\lambda f = (I - \sum_j \psi_j(\lambda + A_j)^{-1}C_j(x, D))^{-1} \sum_j \psi_j(\lambda + A_j)^{-1}\varphi_j f,$$

for $\lambda \in \Sigma_{\pi-\phi}$, $|\lambda| \geq \lambda_0$ for some sufficiently large λ_0, as well as

$$|\lambda S_\lambda f|_q + |D^m S_\lambda f|_q \leq C|f|_q, \quad \lambda \in \Sigma_{\pi-\phi}, \quad |\lambda| \geq \lambda_0.$$

This shows that $\mu + A$ is sectorial for $\mu \geq \lambda_0$, and $\phi_{\mu+A} \leq \phi$, provided $\lambda + A$ is surjective, i.e., there is also a right inverse.

To show the latter we apply $\lambda + \mathcal{A}_\#(x, D)$ to $u = S_\lambda f$ which yields

$$\begin{aligned}(\lambda + \mathcal{A}_\#(D))S_\lambda &= \sum_j(\lambda + \mathcal{A}_\#(D))\psi_j(\lambda + A_j)^{-1}(\varphi_j + \mathcal{C}_j(x, D)S_\lambda) \\ &= \sum_j \psi_j\{\varphi_j + \mathcal{C}_j(x, D)S_\lambda\} + \sum_j[\mathcal{A}_\#(x, D), \psi_j](\lambda + A_j)^{-1}\{\varphi_j + \mathcal{C}_j(x, D)S_\lambda\}.\end{aligned}$$

Since $\psi_j = 1$ on $\operatorname{supp}\varphi_j$ and $\sum_j \varphi_j = 1$, as well as $\sum_j[\mathcal{A}_\#(x, D), \varphi_j] = 0$, we obtain

$$\sum_j \psi_j\{\varphi_j + \mathcal{C}_j(x, D)S_\lambda\} = \sum_j\{\varphi_j + \mathcal{C}_j(x, D)S_\lambda\} = I - \mathcal{B}(x, D)S_\lambda.$$

This yields the following identity

$$(\lambda + \mathcal{A}(x, D))S_\lambda = I + \sum_j[\mathcal{A}_\#(x, D), \psi_j](\lambda + A_j)^{-1}\{\varphi_j + \mathcal{C}_j(x, D)S_\lambda\}. \tag{6.19}$$

The commutators $[\mathcal{A}(x, D), \psi_j]$ are differential operators of order $m-1$, hence the second term on the right-hand side of (6.19) will be small for large $|\lambda|$ which as above shows that (6.19) gives rise to a right inverse of $\lambda + A$; in particular $\lambda + A$ is surjective for large $|\lambda|$.

Next, with

$$R_0(\lambda) = \sum_{j=0}^{N} \psi_j (\lambda + A_j)^{-1} \varphi_j, \quad R_1(\lambda) = \sum_{j=0}^{N} \psi_j (\lambda + A_j)^{-1} \mathcal{C}_j(x, D),$$

the resolvent of A may be written as the Neumann series

$$(\lambda + A)^{-1} = \sum_{k=0}^{\infty} R_1(\lambda)^k R_0(\lambda), \quad \lambda \in \Sigma_{\pi-\phi},\ |\lambda| \geq \lambda_0.$$

For $j, k = 0, \ldots, N$ we obtain by the contraction principle

$$\begin{aligned} &\mathcal{R}\{\mathcal{C}_j(x, D)(\lambda + A_k)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ |\lambda| \geq \lambda_0\} \\ &\quad \leq \sum_{|\beta|<m} |c^j_\beta|_{L_\infty(\mathbb{R}^n;E)} \mathcal{R}\{D^\beta (\lambda + A_k)^{-1}\} \\ &\quad \leq \sum_{|\beta|<m} |c^j_\beta|_{L_\infty(\mathbb{R}^n;E)} \lambda_0^{-1+|\beta|/m} \mathcal{R}\{\lambda^{1-|\beta|/m} D^\beta (\lambda + A_k)^{-1}\} \leq C\varepsilon, \end{aligned} \tag{6.20}$$

provided λ_0 is sufficiently large. This then implies

$$\begin{aligned} &\mathcal{R}\{\lambda^{1-|\alpha|/m} D^\alpha (\lambda + A)^{-1} :\ \lambda \in \Sigma_{\pi-\phi},\ |\lambda| \geq \lambda_0,\ |\alpha| \leq m\} \\ &\quad \leq (N+1)C \sum_{k=0}^{\infty} ((N+1)C\varepsilon)^k = (N+1)C/(1 - (N+1)C\varepsilon) < \infty, \end{aligned} \tag{6.21}$$

in particular, $\mu + A$ is $\mathcal{R}$-sectorial for all $\mu \geq \lambda_0$.

(c) Let us consider now the case where $a_\beta \in L_{r_k}(\mathbb{R}^n; \mathcal{B}(E))$, with $|\beta| = k < m$ and $r_k \geq q$, $m - k > n/r_k$. Then we estimate the terms $a_\beta(x) D^\beta (\lambda + A_l)^{-1}$ as follows. With $qr = r_k$, $1/r + 1/r' = 1$, the *Gagliardo-Nirenberg inequality* yields

$$\begin{aligned} &\Big| \sum_j \varepsilon_j a_\beta D^\beta (\lambda_j + A_l)^{-1} f_j \Big|_{L_q(\mathbb{R}^n;E)} \\ &\quad \leq |a_\beta|_{L_{qr}(\mathbb{R}^n;\mathcal{B}(E))} \Big| \sum_j \varepsilon_j D^\beta (\lambda_j + A_l)^{-1} f_j \Big|_{L_{qr'}(\mathbb{R}^n;E))} \\ &\quad \leq C |a_\beta|_{L_{qr}(\mathbb{R}^n;\mathcal{B}(E))} \Big[\sum_{|\alpha|=m} \Big| \sum_j \varepsilon_j D^\alpha (\lambda_j + A_l)^{-1} f_j \Big|_{L_q(\mathbb{R}^n;E))} \Big]^a \cdot \\ &\qquad \cdot \Big| \sum_j \varepsilon_j (\lambda_j + A_l)^{-1} f_j \Big|_{L_q(\mathbb{R}^n;E))} \Big]^{1-a} \end{aligned}$$

$$\leq C|a_\beta|_{L_{qr}(\mathbb{R}^n;\mathcal{B}(E))}\Big[\sum_{|\alpha|=m}\Big|\sum_j \varepsilon_j D^\alpha(\lambda_j+A_l)^{-1}f_j\Big|_{L_q(\mathbb{R}^n;E))}\Big]^a \cdot$$
$$\cdot\,\lambda_0^{-(1-|\beta|/m)(1-a)}\Big[\Big|\sum_j \varepsilon_j\lambda_j^{1-|\beta|/m}(\lambda_j+A_l)^{-1}f_j\Big|_{L_q(\mathbb{R}^n;E))}\Big]^{1-a},$$

where $am-k = n/qr = n/r_k$, in particular $a<1$ by assumption **(ra 3)**. Integrating over Ω this yields

$$\Big|\sum_j \varepsilon_j a_\beta D^\beta(\lambda_j+A_l)^{-1}f_j\Big|_{L_q(\Omega\times\mathbb{R}^n;E)}$$
$$\leq C\lambda_0^{-(1-|\beta|/m)(1-a)}|a_\beta|_{L_{qr}(\mathbb{R}^n;\mathcal{B}(E))}\Big|\sum_j\varepsilon_j f_j\Big|_{L_q(\mathbb{R}^n;E)}$$
$$\leq C\varepsilon\Big|\sum_j\varepsilon_j f_j\Big|_{L_q(\mathbb{R}^n;E)},$$

whenever λ_0 is sufficiently large, and consequently we have

$$\mathcal{R}\{a_\beta(x)D^\beta(\lambda+A_k)^{-1}:\ \lambda\in\Sigma_{\pi-\phi},\ |\lambda|\geq\lambda_0\}\leq C\varepsilon.$$

We now may proceed as in Step (b) to obtain the result in the general case. □

As a consequence of the results on maximal regularity from Section 4.5 we obtain for the time-dependent parabolic equation

$$\partial_t u+\omega u+Au=f,\quad t>0,\quad u(0)=u_0, \tag{6.22}$$

the following result.

Theorem 6.1.11. *Let Condition* **(ra)** *hold,* $1<p,q<\infty$, $\mu\in(1/p,1]$, *let* $\mathcal{A}(x,D)$ *be uniformly normally elliptic and,* $\omega\geq\omega_0>\mathsf{s}(-A)=\sup\operatorname{Re}\sigma(-A)$, *the spectral bound of* $-A$.

Then (6.22) *has maximal regularity of type* $L_{p,\mu}-L_q$ *on* $\mathbb{R}_+$. *More precisely,* (6.22) *admits a solution* u *in the class*

$$u\in H^1_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n;E))\cap L_{p,\mu}(\mathbb{R}_+;H^2_q(\mathbb{R}^n;E))=:\mathbb{E}_{1\mu}$$

if and only if

$$f\in L_{p,\mu}(\mathbb{R}_+;L_q(\mathbb{R}^n;E))=:\mathbb{E}_{0\mu}\quad\textit{and}\quad u_0\in B^{m(\mu-1/p)}_{qp}(\mathbb{R}^n;E)=:X_{\gamma,\mu}.$$

Moreover, there is a constant $C>0$ *such that*

$$|u|_{\mathbb{E}_{1\mu}}+\omega|u|_{\mathbb{E}_{0\mu}}\leq C(|u_0|_{X_{\gamma,\mu}}+|f|_{\mathbb{E}_{0\mu}}),$$

for all $(f,u_0)\in\mathbb{E}_{0\mu}\times X_{\gamma,\mu}$, *and all* $\omega\geq\omega_0$.

We observe that via the exponential shifts $u_\omega = e^{\omega t}u$ and $f_\omega = e^{\omega t}f$, u is a solution of (6.22) if and only if u_ω solves

$$\partial_t u_\omega + Au_\omega = f_\omega, \quad t > 0, \quad u_\omega(0) = u_0. \tag{6.23}$$

This way the following result is obtained.

Corollary 6.1.12. *Let Condition* **(ra)** *hold,* $1 < p, q < \infty$, $\mu \in (1/p, 1]$, *and let* $\mathcal{A}(x, D)$ *be uniformly normally elliptic and,* $\omega > \mathsf{s}(-A)$.

Then (6.23) *admits a unique solution* u *in the class*

$$e^{-\omega t}u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\mathbb{R}^n; E))$$

if and only if

$$e^{-\omega t}f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; E)) \quad \textit{and} \quad u_0 \in B^{m(\mu-1/p)}_{qp}(\mathbb{R}^n; E).$$

Consequently, on finite intervals (6.22) *has maximal* $L_{p,\mu} - L_q$*-regularity, for each* $\omega \in \mathbb{R}$.

1.5 Different Spatial Orders

Many times one is in need of maximal regularity results with different spatial regularity. In this subsection we briefly discuss this topic. We assume below that $\mathcal{A}(x, D)$ satisfies properties **(ra1), (ra2), (ra3)**.

(i) Higher Order Regularity

Here we want to replace the base space $L_q(\mathbb{R}^n; E)$ by $K^s_q(\mathbb{R}^n; E)$ for $s > 0$ and $K \in \{H, W\}$, where $s \notin \mathbb{N}$ in case $K = W$. For this purpose we fix any $k \in \mathbb{N}$ and consider the operator $\mathcal{A}(x, D)$ in $H^k_q(\mathbb{R}^n; E)$. Differentiating the equations

$$(\lambda + \omega + \mathcal{A}(x, D))u = f \ \text{ in } \ \mathbb{R}^n,$$

or

$$(\partial_t + \omega + \mathcal{A}(x; D))u = f, \ \ t > 0, \ \ u(0) = 0, \ \text{ in } \ \mathbb{R}^n$$

k times in space leads to the problems

$$(\lambda + \omega + \mathcal{A}(x, D))D^\beta u - [\mathcal{A}(x, D), D^\beta]u = D^\beta f \ \text{ in } \ \mathbb{R}^n,$$

or

$$(\partial_t + \omega + \mathcal{A}(x; D))D^\beta u - [\mathcal{A}(x, D), D^\beta]u = D^\beta f, \ \ t > 0, \ \ D^\beta u(0) = 0, \ \text{ in } \ \mathbb{R}^n.$$

As the commutator $[\mathcal{A}(x, D), D^\beta]$ is of lower order, this yields with Proposition 4.4.3 the analogues of Theorems 6.1.10 and 6.1.11 with base space $L_q(\mathbb{R}^n; E)$ replaced by $H^k_q(\mathbb{R}^n; E)$, provided the coefficients of $\mathcal{A}(x, D)$ have enough regularity. Computing the relevant commutator shows that Condition **(ra3)** must be replaced by

$(\mathbf{ra3_k})$ $a_\alpha \in H^k_{r_l}(\mathbb{R}^n; \mathcal{B}(E)) + W^k_\infty(\mathbb{R}^n; \mathcal{B}(E))$, $|\alpha| = l \le m$, $r_l \ge q$, $m+k-l > n/r_l$.

Then employing real or complex interpolation, we see that Theorems 6.1.10 and 6.1.11 are also valid for the base spaces $K^s_q(\mathbb{R}^n;E)$, for all $0\le s\le k$, $s\notin\mathbb{N}_0$ in case $K=W$. Note that for the parabolic problem we first choose $p=q$, $\mu=1$ to obtain $\mathcal{R}$-sectoriality, and then use Theorems 4.4.4 and 3.5.4 for the general case.

(ii) Lower Order Regularity
Here we want to replace the base space $L_q(\mathbb{R}^n;E)$ by $K^{-s}_q(\mathbb{R}^n;E)$ where $s>0$ and $K\in\{H,W\}$, $s\notin\mathbb{N}$ in case $K=W$. Consider first the space $H^{-2}_q(\mathbb{R}^n;E)$. As $I-\Delta: L_q(\mathbb{R}^n;E)\to H^{-2}_q(\mathbb{R}^n;E)$ is an isomorphism, it is reasonable to apply $(I-\Delta)^{-1}$ to the equations under consideration to obtain problems in L_q. This yields equations for $v=(I-\Delta)^{-1}u$ in $L_q(\mathbb{R}^n;E)$,

$$(\lambda+\omega+\mathcal{A}(x,D))v-[\mathcal{A}(x,D),(I-\Delta)^{-1}]u=(I-\Delta)^{-1}f \ \text{ in } \mathbb{R}^n,$$

or

$$(\partial_t+\omega+\mathcal{A}(x;D))v-[\mathcal{A}(x,D),(I-\Delta)^{-1}]u=(I-\Delta)^{-1}f,\ t>0,\ u(0)=0,\ \text{ in } \mathbb{R}^n.$$

Looking at the commutator we find

$$[\mathcal{A}(x,D),(I-\Delta)^{-1}]u=(I-\Delta)^{-1}[\Delta,\mathcal{A}(x,D)](I-\Delta)^{-1}u=(I-\Delta)^{-1}[\Delta,\mathcal{A}(x,D)]v.$$

Now we have

$$[\Delta,a_\alpha D^\alpha]=\sum_{j=1}^n(\partial_j^2 a_\alpha)D^\alpha+2(\partial_j a_\alpha)\partial_j D^\alpha,$$

which implies that the commutator is of lower order in $L_q(\mathbb{R}^n;E)$, provided the coefficients a_α are subject to $(\mathbf{ra3_2})$. Therefore, in this case Theorems 6.1.10 and 6.1.11 are also valid for the base space $H^{-2}_q(\mathbb{R}^n;E)$. Induction yields the same result for $H^{-2k}_q(\mathbb{R}^n;E)$ provided the coefficients satisfy $(\mathbf{ra3_{2k}})$, for all $k\in\mathbb{N}$. Interpolation finally shows that Theorems 6.1.10 and 6.1.11 hold for the base space $K^{-s}_q(\mathbb{R}^n;E)$, for all $s\in[0,2k]$, provided $(\mathbf{ra3_{2k}})$ holds; here $s\in\mathbb{N}_0$ is excluded in case $K=W$.

Remark 6.1.13. A more refined analysis shows that Theorems 6.1.10 and 6.1.11 are valid in $K^{\pm s}_q(\mathbb{R}^n;E)$, $s>0$, if the coefficients merely satisfy

$(\mathbf{a3_s})$ $a_\alpha\in H^s_{r_l}(\mathbb{R}^n;\mathcal{B}(E))+W^s_\infty(\mathbb{R}^n;\mathcal{B}(E))$, $|\alpha|=l\le m$, $r_l\ge q$, $m+s-l>n/r_l$.

However, this assertion is more elaborate, and so we refrain here from a proof.

6.2 Elliptic and Parabolic Systems on $\mathbb{R}^n_+$

Let E be a Banach space of class $\mathcal{HT}$, and consider the parabolic problem

$$\begin{aligned}\partial_t u+\omega u+\mathcal{A}(x,D)u&=f &&\text{in } \mathbb{R}^n_+,\\ \mathcal{B}_j(x,D)u&=g_j &&\text{on } \partial\mathbb{R}^n_+,\ j=1,\dots,m,\\ u(0)&=u_0 &&\text{in } \mathbb{R}^n_+.\end{aligned}\tag{6.24}$$

Here $\mathcal{A}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha D^\alpha$ is a differential operator of degree $2m$, $\mathcal{B}_j(x,D) = \sum_{|\beta|\le m_j} b_{j\beta}D^\beta$ are differential operators of degree $m_j < 2m$, and the data (f, g_j) and u_0 are given. This problem may be reduced to a homogeneous problem with inhomogeneous boundary conditions as follows. Extend the function $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$ trivially to a function $\bar{f} \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; E))$, the coefficients of $\mathcal{A}(x,D)$ by symmetry to all of $\mathbb{R}^n$, and extend the initial value $u_0 \in B^{2m(\mu-1/p)}_{qp}(\mathbb{R}^n_+; E)$ to some $\bar{u}_0 \in B^{2m(\mu-1/p)}_{qp}(\mathbb{R}^n; E)$. Then we may apply the results from the previous section, in particular Theorem 6.1.11, to obtain the solution

$$\bar{u} \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^n; E)))$$

of the full space problem

$$\begin{aligned} \partial_t \bar{u} + \omega \bar{u} + \mathcal{A}(x,D)\bar{u} &= \bar{f} \quad \text{in } \mathbb{R}^n, \\ \bar{u}(0) &= \bar{u}_0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{6.25}$$

Then the function $\tilde{u} = u - \bar{u}$ satisfies (6.24) with $(f, u_0) = 0$ and g_j replaced by $\tilde{g}_j = g_j - \mathcal{B}_j(x,D)\bar{u}$. This way we have reduced the problem to a homogeneous parabolic equation with trivial initial data, but inhomogeneous boundary data. Note that the natural compatibility conditions

$$\mathcal{B}_j(x,D)u_0 = g_j(0), \quad j = 1, \ldots m,$$

become $\tilde{g}_j(0) = 0$. Below we will therefore always consider the case $(f, u_0) = 0$.

Similarly for the elliptic problem

$$\begin{aligned} \lambda u + \omega u + \mathcal{A}(x,D)u &= f \quad \text{in } \mathbb{R}^n_+, \\ \mathcal{B}_j(x,D)u &= g_j \quad \text{on } \partial\mathbb{R}^n_+,\ j = 1, \ldots, m. \end{aligned} \tag{6.26}$$

We may assume $f = 0$, by Theorem 6.1.10 of the previous section.

2.1 The Boundary Symbol

We begin with the constant coefficient case, i.e., we consider

$$\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha, \quad \mathcal{B}_j(D) = \sum_{|\beta|=m_j} b_{j\beta} D^\beta$$

with coefficients $a_\alpha, b_{j\beta} \in \mathcal{B}(E)$. It is convenient to replace x by (x, y), where $x \in \mathbb{R}^{n-1}$ are tangential variables and $y > 0$ is the normal variable. Taking the Laplace transform in time with covariable λ and Fourier transform in the tangential direction with covariable $\xi \in \mathbb{R}^{n-1}$, with $\nu = e_n$ we obtain the transformed problem

$$\begin{aligned} (\lambda + \omega)v_1(y) + \mathcal{A}(\xi + \nu D_y)v_1(y) &= 0, \quad y > 0, \\ \mathcal{B}_j(\xi + \nu D_y)v_1(0) &= h_j, \quad j = 1, \ldots, m. \end{aligned} \tag{6.27}$$

This is a boundary value problem for an ordinary differential equation on $\mathbb{R}_+$, where the covariables λ and ξ are parameters. We may rewrite the differential operators in the following form.

$$\mathcal{A}(\xi + \nu D_y) = \sum_{k=0}^{2m} a_k(\xi) D_y^{2m-k}, \quad \mathcal{B}_j(\xi + \nu D_y) = \sum_{k=0}^{m_j} b_{jk}(\xi) D_y^{m_j-k}.$$

Observe that $a_k(\xi)$ as well as $b_{jk}(\xi)$ are homogeneous polynomials of degree k.

We shall assume from now on that $\mathcal{A}(D)$ is parameter-elliptic with angle $\phi_{\mathcal{A}}$. Then $a_0 = \mathcal{A}(0,\dots,0,1)$ is invertible. For $\lambda \in \Sigma_{\pi-\phi}$, $\phi > \phi_{\mathcal{A}}$, we introduce the new variables $v = [v_j]$, and the scaling parameter $\rho = (w + \lambda + |\xi|^{2m})^{1/2m}$

$$v_j(y) = \rho^{-j+1} D_y^{j-1} v_1(y), \quad j = 1, \dots, 2m,$$

we may rewrite the differential equation in (6.27) as

$$\partial_y v(y) = i\rho A_0(b,\sigma) v(y), \quad y > 0,$$

with $\sigma = (\omega + \lambda)/\rho^{2m}$, $b = \xi/\rho$ and

$$A_0(b,\sigma) = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I \\ c_{2m}(b,\sigma) & c_{2m-1}(b) & \dots & c_2(b) & c_1(b) \end{pmatrix},$$

where $c_j(b) = -a_0^{-1} a_j(b)$, $j = 1, \dots, 2m-1$ and $c_{2m}(b,\sigma) = -a_0^{-1}(\sigma + a_{2m}(b))$. Similarly, for homogeneity reasons the boundary conditions become

$$B_j^0(b) v(0) = \rho^{-m_j} h_j =: \tilde{h}_j, \quad j = 1, \dots, m,$$

with $B_j^0(b) : E^{2m} \to E$ defined by

$$B_j^0(b) = (b_{jm_j}(b), \dots, b_{j0}(b), 0, \dots, 0), \quad j = 1, \dots, m.$$

This way the boundary value problem (6.27) is transformed to the first-order system

$$\begin{aligned} \partial_y v(y) &= i\rho A_0(b,\sigma) v(y), \quad && y > 0, \\ B_j^0(b) v(0) &= \tilde{h}_j, && j = 1, \dots, m. \end{aligned} \tag{6.28}$$

To solve this boundary value problem we need some preparation.

Lemma 6.2.1. *Let b, σ and $A_0(b,\sigma)$ be defined as above. Then*

$$\sigma(A_0(b,\sigma)) \cap \mathbb{R} = \emptyset.$$

Proof. We first prove that $\sigma_p(A_0(b,\sigma))\cap\mathbb{R}$ is empty, where $\sigma_p(A_0(b,\sigma))$ denotes the point spectrum of $A_0(b,\sigma)$. To this end, suppose that $\eta\in\mathbb{R}$ is an eigenvalue of $A_0(b,\sigma)$ with eigenvector $x=[x_0,\ldots,x_{2m-1}]^{\mathsf{T}}\neq 0$. Then

$$\begin{aligned} &\eta x_0 = x_1, \quad \ldots \quad \eta x_{2m-2} = x_{2m-1}, \\ &\eta x_{2m-1} = -a_0^{-1}((\sigma + a_{2m}(b))x_0 + a_{2m-1}(b)x_1 + \ldots + a_1(b)x_{2m-1}). \end{aligned} \tag{6.29}$$

This implies $(\sigma+\sum_{k=0}^{2m} a_k(b)\eta^{2m-k})x_0=0$. It follows from the first line of (6.29) that $x_0\neq 0$. Therefore, $-\sigma$ is an eigenvalue for $\mathcal{A}(b,\eta)$ with eigenvector x_0. But as $\mathcal{A}$ is parameter-elliptic this implies $-\sigma\in\Sigma_\phi$, which contradicts the assumption $\lambda\in\Sigma_{\pi-\phi}$.

Next, assume that $\eta\in\mathbb{R}$ belongs to the residual spectrum $\sigma_r(A_0(b,\sigma))$. Then $\eta\in\sigma_p(A_0^*(b,\sigma))$, hence there is $x^*=(x_0^*,\ldots,x_{2m-1}^*)^{\mathsf{T}}\neq 0$ such that $A_0^*(b,\sigma)x^*=\eta x^*$. This implies as before $x_{2m-1}^*\neq 0$ and

$$(\sigma + \sum_{k=0}^{2m} a_k(b)^*\eta^{2m-k})x_{2m-1}^* = 0.$$

This shows that $-\sigma$ is an eigenvalue of $\mathcal{A}^*(b,\eta)$, hence belongs to $\sigma_r(\mathcal{A}(b,\eta))$, which is not possible.

Finally, assume that $\eta\in\mathbb{R}$ is in the continuous spectrum $\sigma_c(A_0(b,\sigma))$. Then we find $x_n=(x_{n,0},\ldots,x_{n,2m-1})^{\mathsf{T}}$ with $|x_n|_{E^{2m}}=1$ such that $A_0(b,\sigma)x_n=\eta x_n+y_n$, with $y_n\to 0$ as $n\to\infty$. As above this yields

$$(\sigma + \sum_{k=0}^{2m} a_k(b)\eta^{2m-k})x_{n,0} \to 0,$$

hence $-\sigma$ belongs to $\sigma_c(\mathcal{A}(b,\eta))$ which yields a contradiction as before. □

This lemma shows that the spectrum of $iA_0(b,\sigma)\in\mathcal{B}(E^{2m})$ splits into two parts, $s_-(b,\sigma)$ contained in the open left half-plane, and $s_+(b,\sigma)$ contained in the open right half-plane. By compactness, there are constants $c_\pm>0$ such that

$$\sup \operatorname{Re} s_-(b,\sigma) \le -c_- < 0 < c_+ \le \inf \operatorname{Re} s_+(b,\sigma),$$

for all relevant b,σ. Let $P_\pm(b,\sigma)\in\mathcal{B}(E^{2m})$ denote the associated spectral projections of $iA_0(b,\sigma)$; these are holomorphic and bounded, uniformly in (b,σ). The boundary value problem (6.28) admits precisely one solution $v\in C_0(\mathbb{R}_+;E^{2m})$ if and only if the system

$$\begin{aligned} B_j^0(b)w &= \tilde{h}_j, \quad j=1,\ldots,m, \\ P_+(b,\sigma)w &= 0 \end{aligned} \tag{6.30}$$

admits a unique solution $w\in E^{2m}$. The solution v of (6.28) is then given by

$$v(y) = e^{iy\rho A_0(b,\sigma)}w, \quad y \ge 0.$$

To ensure this solvability property we assume the equivalent

Lopatinskii-Shapiro Condition (LS)
For each $\xi, \nu \in \mathbb{R}^n$, $\lambda \in \Sigma_{\pi-\phi}$ for some $\phi > \phi_{\mathcal{A}}$, where $(\lambda, \xi) \neq (0,0)$, $|\nu| = 1$, $(\xi|\nu) = 0$, the problem

$$\begin{aligned} \lambda u(y) + \mathcal{A}(\xi + \nu D_y)u(y) &= 0, \quad y > 0, \\ \mathcal{B}_j(\xi + \nu D_y)u(0) &= g_j, \quad j = 1, \dots, m, \end{aligned}$$

has exactly one solution $u \in C_0(\mathbb{R}_+; E)$, for any given vectors $g_j \in E$, $j = 1, \dots, m$.

Remark 6.2.2. (i) It is obvious that also the Lopatinskii-Shapiro condition is invariant under orthogonal transformations. But even more, it is invariant w.r.t. general coordinate transformations as well. In fact, under the coordinate transformation $Tu(x) = u(Qx)$ with invertible $Q \in \mathbb{R}^{n\times n}$, the normal ν transforms to $\nu_Q = Q^{-\mathsf{T}}\nu$. Therefore,

$$\mathcal{A}_Q(\xi' + \nu_Q D_y) = \mathcal{A}(Q^{\mathsf{T}}\xi' + \nu D_y) = \mathcal{A}(\xi + \alpha\nu + \nu D_y),$$

where $(\xi|\nu) = 0$ and $\alpha = (\xi'|Q\nu)$. The same applies to the boundary operators $\mathcal{B}_j$. The exponential shift $v(y) = e^{i\alpha y}w(y)$ then shows that we may assume $\alpha = 0$. This reduces **(LS)** for the transformed problem to **(LS)** for the original one.

(ii) The shift argument also shows that the condition $(\xi|\nu) = 0$ in **(LS)** is redundant, only $|\nu| = 1$ is essential.

(iii) There are versions of the Lopatinskii-Shapiro condition for more refined boundary value problems which also appear in applications. Each of the m boundary operators may be split into finitely many ones of different order. More precisely, for fixed $j \in \{1, \dots, m\}$, we let $E = \oplus_{k=0}^{n_j} E_{jk}$, and replace the condition $\mathcal{B}_j(D)u = g_j$ by

$$\mathcal{B}_{jk}(D)u = g_{jk}, \quad k = 0, \dots, n_j,$$

where the coefficients of $\mathcal{B}_{jk}(D)$ satisfy $b_{jk\beta} \in \mathcal{B}(E, E_{jk})$, and their orders are $m_{jk} \in \{0, \dots, 2m-1\}$. Condition **(LS)** extends literally to such cases, and the analysis presented here carries over.

(iv) If $E \simeq \mathbb{C}^N$ is finite-dimensional, then the kernel of P_+ has dimension mN, hence if we prescribe mN scalar boundary conditions, it is enough to have uniqueness in **(LS)**, by a dimensional argument.

The Lopatinskii-Shapiro condition implies the following result.

Proposition 6.2.3. *Suppose that $\mathcal{A}(D)$ is parameter-elliptic with angle $\phi_{\mathcal{A}}$, and assume the Lopatinskii-Shapiro Condition for some $\phi > \phi_{\mathcal{A}}$. Then for each $\tilde{h} = [\tilde{h}_j] \in E^m$, $j = 1, \dots, m$, problem* (6.30) *admits a unique solution $w \in E^{2m}$. This solution is represented as $w = M_0(b, \sigma)\tilde{h}$, where the map $M_0 : U \to \mathcal{B}(E^m, E^{2m})$ is holomorphic on a neighbourhood $U \subset \mathbb{C}^{n+1}$ of $\{(b, \sigma) : (\lambda, \xi) \in \Sigma_{\pi-\phi} \times \mathbb{R}^{n-1}\}$.*

Proof. Existence, uniqueness and linearity are clear, so we need to show holomorphy of M_0. For this purpose set $z = (b, \sigma) \in U$ and $B(z) = (B_1^0(z), \dots, B_m^0(z))$. Then $u(z) = M_0(z)g$ defines the unique solution of the system

$$P_+(z)u = 0, \quad B(z)u = g.$$

Let D denote a compact subset of U. By means of the closed graph theorem, we obtain uniform boundedness of the maps $M_0(z) \in \mathcal{B}(E^m, E^{2m})$. In fact, the map $g \mapsto u(z)$ is a closed linear map from E^m into $B(D; E^{2m})$, the space of bounded functions from D to E^{2m}, hence bounded, i.e., $\sup_{z\in D} |M_0(z)| =: C_D < \infty$. By compactness and continuity this also holds on an open neighbourhood – which we again call U – of D.

Next we use the fact that $P_+(z)$ as well as $B(z)$ are holomorphic on U. Fix any $z \in U$, $h \in \mathbb{C}^n$ and let $0 \neq t \in \mathbb{C}$ be small. Then for fixed $g \in E^m$ we have

$$P_+(z + th)w(z + th) = 0 = P_+(z)w(z),$$

and

$$B(z + th)w(z + th) = g = B(z)w(z),$$

hence

$$\begin{aligned} P_+(z + th)[w(z + th) - w(z)] &= -[P_+(z + th) - P_+(z)]w(z) \\ B(z + th)[w(z + th) - w(z)] &= -[B(z + th) - B(z)]w(z). \end{aligned}$$

Now, $P_+(z)^2 = P_+(z)$ implies

$$\begin{aligned} &P_+(z + th) - P_+(z) = P_+(z + th)^2 - P_+(z)^2 \\ &= P_+(z + th)[P_+(z + th) - P_+(z)] + [P_+(z + th) - P_+(z)]P_+(z), \end{aligned}$$

which by $P_+(z)w(z) = 0$ yields

$$[P_+(z + th) - P_+(z)]w(z) = P_+(z + th)[P_+(z + th) - P_+(z)]w(z).$$

From this identity we obtain

$$P_+(z + th)[w(z + th) - w(z) + (P_+(z + th) - P_+(z))w(z) = 0,$$

and

$$\begin{aligned} &B(z + th)[w(z + th) - w(z) + (P_+(z + th) - P_+(z))w(z)] \\ &= B(z + th)[P_+(z + th) - P_+(z)]w(z) - [B(z + th) - B(z)]w(z), \end{aligned}$$

which implies

$$\begin{aligned} &w(z + th) - w(z) + [P_+(z + th) - P(z)]w(z) \\ &= M_0(z + th)[B(z + th)(P_+(z + th) - P_+(z))w(z) - (B(z + th) - B(z))w(z)]. \end{aligned} \tag{6.31}$$

By continuity of P_+ and B as well as boundedness of M_0, this shows continuity of w on complex lines. Thus $M_0(z)$ has this property as well. Dividing (6.31) by t we get

$$\begin{aligned}\frac{w(z+th)-w(z)}{t} = &-\frac{P_+(z+th)-P(z)}{t}w(z)\\ &+M_0(z+th)B(z+th)\frac{P_+(z+th)-P_+(z)}{t}w(z)\\ &-M_0(z+th)\frac{B(z+th)-B(z)}{t}w(z),\end{aligned}$$

which shows that $w(z)$ is complex differentiable on U, thanks to holomorphy of P_+ and B. Therefore, M_0 is also holomorphic on U. □

2.2 Harmonic Analysis

The last subsection shows that the unique solution v of (6.28) is given by

$$v = e^{iy\rho A_0(b,\sigma)}M_0(b,\sigma)\tilde{h}.$$

To invert the Laplace and Fourier transforms in the right regularity class, we rewrite this equation as

$$\rho^{2m}v = M(y,\rho,b,\sigma)\rho e^{-\eta y\rho}\rho^{2m-1}\tilde{h} = M(y,\rho,b,\sigma)\tilde{g}, \tag{6.32}$$

where $\eta > 0$ is small,

$$M(y,\rho,\sigma,b) = e^{iy\rho A_0(b,\sigma)+\eta y\rho}M_0(b,\sigma)$$

and

$$\tilde{g} = \rho e^{-\eta y\rho}\rho^{2m-1}\tilde{h}.$$

Here we need a result on analytic C_0-semigroups and the vector-valued Triebel-Lizorkin spaces $F^{\alpha}_{pq,\mu}$, which we state now. Define $L_0 = (\omega + \partial_t + (-\Delta_x)^m)$ in the space $X_0 = L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))$ with domain

$$\mathsf{D}(L_0) = {}_0H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^{n-1}; E)).$$

This operator, by the Dore-Venni theorem, belongs to the class $\mathcal{S}(X_0)$ with angle $\pi/2$. Therefore, its root $L_0^{1/2m}$ is also in this class, with angle $\pi/4m < \pi/2$. This implies that $L_0^{1/2m}$ is the negative generator of an analytic C_0-semigroup $e^{-yL_0^{1/2m}}$. In the sequel, we denote by L the canonical extension of L_0 to the space $\mathbb{E}_{0\mu} = L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$. We are here interested in the question for which boundary values $g \in X_0$ the extension $u(y) = e^{-yL_0^{1/2m}}g$ satisfies $L^{1/2m}u \in \mathbb{E}_{0\mu}$. The result is surprising; it is the content of the following proposition.

Proposition 6.2.4. *Let $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and E be a Banach space with property $\mathcal{HT}(\alpha)$. Moreover, let L_0 and L be defined as above, and let $u(y) = e^{-yL_0^{1/2m}} g$, $g \in X_0$, $y > 0$.*

Then the following assertions are equivalent.

(i) $u \in {}_0H^{1/2m}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E))$;

(ii) $L^{1/2m} u \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)) = \mathbb{E}_{0\mu}$;

(iii) $g \in {}_0F^{1/2m-1/2mq}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)) =: {}_0\mathbb{F}_{0\mu}$.

Similar statements are valid on $\mathbb{R}$; replace the symbols $L_{p,\mu}(\mathbb{R}_+; \cdot)$ and ${}_0K_{p,\mu}(\mathbb{R}_+; \cdot)$ by $L_p(\mathbb{R}; \cdot)$ and $K_p(\mathbb{R}; \cdot)$, respectively.

Proof. (i)⇒(iii). As the trace operator $(\mathsf{tr}\, u)(t, x) := u(t, 0, x)$ maps the space $H^1_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$ boundedly into $B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)$ we see that $g \in L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E))$. To obtain the time regularity of g we may concentrate on the variables (t, y), and hide x in $\tilde{E} = L_q(\mathbb{R}^{n-1}; E)$ which belongs to the class $\mathcal{HT}$ as $E \in \mathcal{HT}$. Then with $\alpha = 1/2m$, we have

$$u \in \mathbb{E}_{\alpha\mu} := {}_0H^{\alpha}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E})) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}_+; \tilde{E})).$$

Define an operator A in $\mathbb{E}_{0,\mu} = L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E}))$ by means of $Au = \partial_y u$ with domain $\mathsf{D}(A) = L_{p,\mu}(\mathbb{R}_+; {}_0H^1_q(\mathbb{R}_+; \tilde{E}))$ and B by means of $Bu = (\omega + \partial_t)^{\alpha} u$ with domain $\mathsf{D}(B) = {}_0H^{\alpha}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E}))$. Both operators are in $\mathcal{H}^\infty$ with $\mathcal{H}^\infty$-angles $\pi/2$, $\alpha\pi/2$, respectively, and B is invertible. They commute in the resolvent sense and $\phi^\infty_A + \phi^\infty_B = (1+\alpha)\pi/2 < \pi$. Therefore, by Corollary 4.5.11, $A+B$ with domain $\mathsf{D}(A+B) = \mathsf{D}(A) \cap \mathsf{D}(B) = \mathbb{E}_{\alpha\mu}$ belongs to the class $\mathcal{H}^\infty$, as well. Next we solve the problem $Av + Bv = \partial_y u + Bu \in \mathbb{E}_{0\mu}$ with maximal regularity to obtain a unique solution $v \in \mathsf{D}(A+B) = \mathbb{E}_{\alpha\mu}$. Then $w = u - v$ satisfies $\partial_y w = -Bw$ hence $w = e^{-By} g \in \mathbb{E}_{\alpha\mu} \subset \mathsf{D}(B)$. Therefore, Lemma 6.7.5 in the Appendix to this section yields $g \in {}_0F^{\alpha}_{pq,\mu}(\mathbb{R}_+; \tilde{E})$, which proves (iii).

(i)⇔(ii). We know that $L = \omega + \partial_t + (-\Delta_x)^m$ belongs to $\mathcal{H}^\infty$ with $\mathcal{H}^\infty$-angle $\pi/2$. Its domain is given by

$$\begin{aligned} \mathsf{D}(L) &= {}_0H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))) \cap L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^{n-1}; E))) \\ &= \mathsf{D}(B^{2m}) \cap \mathsf{D}((-\Delta_x)^m). \end{aligned}$$

Then by complex interpolation we have

$$\begin{aligned} \mathsf{D}(L^{1/2m}) &= \mathsf{D}(B) \cap \mathsf{D}((-\Delta_x)^{1/2}) \\ &= {}_0H^{\alpha}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))) \cap L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}_+; H^1_q(\mathbb{R}^{n-1}; E))), \end{aligned}$$

hence $L^{1/2m} u \in \mathbb{E}_{0\mu}$ if and only if $u \in \mathsf{D}(L^{1/2m})$. Furthermore, the representation $u = e^{-L_0^{1/2m} y} g$ implies also $\partial_y u \in \mathbb{E}_{0\mu}$. This proves the equivalence in question.

(iii)⇒(ii). Suppose

$$g \in {}_0F^{1/2m-1/2mq}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)) =: {}_0\mathbb{F}_{0\mu}.$$

Set $A_0 = (-\Delta_x)^{1/2}$ with $\mathsf{D}(A_0) = L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^{n-1}; E))$ and $B_0 = (\omega + \partial_t)^\alpha$ with domain $\mathsf{D}(B_0) = {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))$. These operators are of class $\mathcal{H}^\infty$ in the base space $X_0 = L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))$, with $\mathcal{H}^\infty$ angles 0 and $\alpha\pi/2$, respectively, and they commute in the resolvent sense. Then by Lemma 6.7.5 we see that $e^{-B_0 y} g \in \mathsf{D}(B) = {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$. On the other hand, $e^{-A_0 y} g \in L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+; E))$. Define $v = e^{-\eta(A_0+B_0)y} g$; then $(A_0 + B_0)v \in \mathbb{E}_{0,\mu}$, as $e^{-A_0 y}$ and $e^{-B_0 y}$ act boundedly in $\mathbb{E}_{0,\mu}$.
$A_0 + B_0$ is equivalent to $L_0^{1/2m}$ as $\mathsf{D}(L_0^{1/2m}) = \mathsf{D}(A_0) \cap \mathsf{D}(B_0)$. Moreover, by perturbation, $L_0^{1/2m} - \eta(A_0 + B_0)$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\alpha\pi/2$, provided $\eta > 0$ is sufficiently small. By means of Fourier multipliers it is not difficult to see that $e^{-(L_0^{1/2m} - \eta(A_0+B_0))y}$ acts boundedly on $\mathbb{E}_{0\mu}$.

In fact, we show that the symbol

$$m(\lambda, \xi, y) = e^{-y(\lambda+\omega+|\xi|^{2m})^{1/2m} - \eta((\omega+\lambda)^{1/2m}+|\xi|)}$$

is a Fourier multiplier for $\mathbb{E}_{0\mu}$. To prove this, we first observe that m is uniformly bounded and holomorphic in $(\lambda, \xi) \in \Sigma_{\pi/2+\varepsilon} \times (\Sigma_\varepsilon \cup -\Sigma_\varepsilon)^n$, provided $\eta, \varepsilon > 0$ are small. This implies the Mikhlin-condition w.r.t. ξ, uniformly in (λ, y), hence we first invert the Fourier transform, to obtain an $\mathcal{R}$-bounded family of operators $\mathcal{T}(\lambda, y)$ on $L_q(\mathbb{R}^{n-1}; E)$, provided E is of class $\mathcal{HT}$ and has property (α). Uniformity then shows that the family $T_m(\lambda) = \mathcal{T}_m(\lambda, \cdot)$ is also $\mathcal{R}$-bounded in $L_q(\mathbb{R}^n_+; E)$ and then trivially also in $\mathbb{E}_{0\mu}$. Finally, by the Kalton-Weis theorem, $T(\partial_t + \omega)$ is bounded in $\mathbb{E}_{0\mu}$.

Therefore

$$L^{1/2m} e^{-L_0^{1/2m} y} g = L^{1/2m}(A + B)^{-1} e^{-(L_0^{1/2m} - \eta(A_0+B_0))y} \cdot (A_0 + B_0)v \in \mathbb{E}_{0,\mu},$$

which proves the implication (iii) ⇒ (ii). □

Now we may continue the argumentation preceding Proposition 6.2.4. As h_j is the transform of a function in

$${}_0\mathbb{F}_{j\mu} = {}_0F^{1-m_j/2m-1/2mq}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m-m_j-1/q}_{qq}(\mathbb{R}^{n-1}; E))$$

we see that $\rho^{2m-1}\tilde{h}_j = \rho^{2m-m_j-1} h_j$ is the transform of a function in ${}_0\mathbb{F}_{0\mu}$, for each $j = 1, \ldots, m$. Proposition 6.2.4 then implies that $\rho e^{-\eta y \rho} \rho^{2m-1} \tilde{h}_j$ is the transform a function $g_j \in \mathbb{E}_{0\mu} := L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$.

Therefore, we need to know that $M(y, \rho, b, \sigma)$ is a Fourier multiplier for X_0. To prove this, we first observe that M is uniformly bounded and holomorphic in (λ, ξ). This implies the Mikhlin-condition w.r.t. ξ, hence we first invert the Fourier transform, to obtain an $\mathcal{R}$-bounded family of operators $\mathcal{T}(\lambda, y)$ on $L_q(\mathbb{R}^{n-1}; E)$, provided E is of class $\mathcal{HT}$ and has property (α). Uniformity then shows that the family $T(\lambda) = \mathcal{T}(\lambda, \cdot)$ is also $\mathcal{R}$-bounded in $L_q(\mathbb{R}^n_+; E)$ and then trivially also in X_0. Finally, by the Kalton-Weis theorem, $T(\partial_t + \omega)$ is bounded in X_0.

Summarizing we have proved the sufficiency part of the following result for the original parabolic half-space problem (6.24).

Theorem 6.2.5. *Let $1 < p, q < \infty$, $\omega > 0$, $\mu \in (1/p, 1]$, and E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $\mathcal{A}(D)$ is a normally elliptic differential operator of order $2m$, let $\mathcal{B}_j(D)$, $j = 1, \dots, m$, denote differential operators of order $m_j < 2m$, and suppose the Lopatinskii-Shapiro condition* **(LS)** *is satisfied, for some angle $\phi < \pi/2$.*

Then (6.24) admits a unique solution u in the class

$$u \in \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^n_+; E)),$$

if and only if the data are subject to the following conditions.

(a) $f \in \mathbb{E}_{0\mu} = L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$, $u_0 \in X_{\gamma,\mu} = B^{2m(\mu-1/p)}_{qp}(\mathbb{R}^n_+; E)$;

(b) $g_j \in \mathbb{F}_{j\mu} = F^{\kappa_j}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j}_{qq}(\mathbb{R}^{n-1}; E))$;

(c) $\mathcal{B}_j(D)u_0 = g_j(0)$ *if* $\kappa_j > 1/p + 1 - \mu$, $j = 1, \dots, m$.

Here $\kappa_j = 1 - m_j/2m - 1/2mq$. The solution depends continuously on the data in the corresponding spaces.

Remark 6.2.6. **(i)** Note that $\kappa_j > 1/p + 1 - \mu$ if and only $m_j < 2m(\mu - 1/p) - 1/q$.
(ii) In the case $p = q$ we have $F^{\kappa_j}_{pp,\mu} = B^{\kappa_j}_{pp,\mu} = W^{\kappa_j}_{p,\mu}$ as well as $B^{2m\kappa_j}_{pp} = W^{2m\kappa_j}_p$.

Proof. Necessity. We still need to prove the necessity part of Theorem 6.2.5. Suppose $u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^n_+; E))$ is a solution of (6.24). Then inserting u into (6.24) we clearly have $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$. To obtain the regularity of the time trace u_0 of u at time $t = 0$, we extend u in space by means of a usual extension operator to obtain a function $\bar{u} \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^n; E))$. Applying the trace theorem for the semigroup $e^{-(-\Delta)^m t}$ with base space $L_q(\mathbb{R}^n; E)$ this yields

$$\bar{u}_{|t=0} \in (L_q(\mathbb{R}^n; E), H^{2m}_q(\mathbb{R}^n; E))_{\mu-1/p,p} = B^{2m(\mu-1/p)}_{qp}(\mathbb{R}^n; E),$$

which implies by restriction $u_0 \in B^{2m(\mu-1/p)}_{qp}(\mathbb{R}^n_+; E)$. Next we consider the lateral traces at $y = 0$. For this purpose we first replace u by $v = t^{1-\mu}u$ and extend v in time by symmetry to $\mathbb{R}$. Then $v \in H^1_p(\mathbb{R}; L_q(\mathbb{R}^n_+; E)) \cap L_p(\mathbb{R}; H^{2m}_q(\mathbb{R}^n_+; E))$, hence $w = (\omega + \partial_t)^\alpha \partial^k_y D^\beta_x u$ belongs to $H^{1/2m}_p(\mathbb{R}; L_q(\mathbb{R}^n_+; E)) \cap L_p(\mathbb{R}; H^1_q(\mathbb{R}^n_+; E))$ if $2m\alpha + k + |\beta| = 2m - 1$. Next we solve the problem

$$\partial_y \bar{w} + L_0^{1/2m} \bar{w} = \partial_y w + L_0^{1/2m} w, \ y > 0, \ \bar{w}(0) = 0,$$

with maximal regularity, which shows that $\bar{w}$ has the same regularity as w, hence $w - \bar{w} = e^{-yL_0^{1/2m}} w|_{y=0}$ has as well. Then Proposition 6.2.4 implies that the trace of w at $y = 0$ belongs to $F_{pq}^{1/2m-1/2mq}(\mathbb{R}; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}; B_{qq}^{1-1/q}(\mathbb{R}^{n-1}; E))$. By the definition of w and proper choices of β and k, this yields

$$t^{1-\mu} g_j = \mathcal{B}_j(D) t^{1-\mu} v \in {}_0F_{pq}^{\kappa_j}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}_+; B_{qq}^{2m\kappa_j}(\mathbb{R}^{n-1}; E)),$$

by restriction to $t > 0$; therefore we finally obtain $g_j \in F_{pq,\mu}^{\kappa_j}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B_{qq}^{2m\kappa_j}(\mathbb{R}^{n-1}; E))$. This proves the necessity of the conditions in Theorem 6.2.5. □

It is of importance to have estimates on the solution which are also uniform in ω. This is the content of

Corollary 6.2.7. *Let the assumptions of Theorem* 6.2.5 *be satisfied, and fix any* $\omega_0 > 0$. *Then there is a constant* $C > 0$ *such that the solution of* (6.24) *satisfies the estimate*

$$|u|_{\mathbb{E}_{1\mu}} + \omega |u|_{\mathbb{E}_{0\mu}} \le C\Big(|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + \sum_{j=1}^{m} (|g_j|_{\mathbb{F}_{j\mu}} + \omega^{1-m_j/2m} |e^{-yL_\omega} g_j|_{\mathbb{E}_{0\mu}}) \Big), \tag{6.33}$$

for all $\omega \ge \omega_0$, $(f, g_j, u_0) \in \mathbb{E}_{0\mu} \times \mathbb{F}_{j\mu} \times X_{\gamma,\mu}$, $j = 1, \ldots, m$. *Here* L_ω *is defined by* $L_\omega = (\partial_t + \omega + (-\Delta)^m)^{1/2m}$.

Proof. To derive the inequality (6.33) we proceed in a similar way as in the proof of Theorem 6.2.5. We again work in frequency domain. Recall that the symbol of L_ω is $\rho = (\lambda + \omega + |\xi|^{2m})^{1/2m}$, and set $\rho_0 = (\lambda + \omega_0 + |\xi|^{2m})^{1/2m}$. Here we decompose as

$$\rho^{2m} v = M_1 \cdot M_2 \cdot \rho_0 e^{-\eta y \rho_0} \rho_0^{2m-m_j-1} h_j + M \cdot M_2 \omega^{1-m_j/2m} e^{-\eta y \rho} h_j,$$

with

$$M_1 = e^{i\rho A_0(b,\sigma) + \eta y \rho_0} M_0(b, \sigma), \quad M_2 = \frac{\rho^{2m-m_j}}{\rho_0^{2m-m_j} + \omega^{1-m_j/2m}}.$$

By the arguments at the end of the proof of Theorem 6.2.5, M as well as M_1 and M_2 are bounded Fourier multipliers for $\mathbb{E}_{0\mu}$, uniformly for $\omega \ge \omega_0 > 0$, hence the result follows by the same arguments. □

Estimate (6.33) is sharp for the half-space case. However, the last term involves a norm which is specific for a half-space. Observing that with some $\delta > 0$,

$$|e^{-yL_\omega} g_j|_{\mathbb{E}_{0\mu}} \le C |e^{-\delta \omega^{1/2m} y} g_j|_{\mathbb{E}_{0\mu}} \le C \omega^{-1/2mq} |g_j|_{L_{p,\mu}(L_q)},$$

we obtain the slightly weaker estimate

$$|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}} \le C\big(|u_0|_{X_{\gamma,\mu}} + |f|_{\mathbb{E}_{0\mu}} + \sum_{j=1}^{m}(|g_j|_{\mathbb{F}_{j\mu}} + \omega^{\kappa_j}|g_j|_{L_{p,\mu}(L_q)})\big). \quad (6.34)$$

The advantage of (6.34) lies in the fact that it only involves the norms of the boundary data. It is not good enough to cover boundary perturbations of highest order, but it is well suited to handle such of lower order, and is in particular useful for the localization process in domains.

2.3 Perturbed Coefficients

To consider the case of variable coefficients, on the boundary we have to work in Besov spaces. Here a result on pointwise multipliers is essential. Therefore we begin with this topic.

Lemma 6.2.8. *Let $1 \le p, q \le \infty$, $s > 0$, E a Banach space, and assume*

$$a \in B^s_{rq}(\mathbb{R}^n; \mathcal{B}(E)) + B^s_{\infty q}(\mathbb{R}^n; \mathcal{B}(E)), \quad (6.35)$$

with $r \ge p$ and $s > n/r$.

Then the multiplication operator $v \mapsto av$ is bounded in $B^s_{pq}(\mathbb{R}^n; E)$. Moreover, there are constants $\alpha \in [0, 1)$ and $C > 0$ such that

$$|av|_{B^s_{pq}} \le |a|_{L_\infty}|v|_{B^s_{pq}} + C|v|^{\alpha}_{B^s_{pq}}|v|^{1-\alpha}_{L_p}, \quad (6.36)$$

for all $v \in B^s_{pq}(\mathbb{R}^n; E)$. The constant C depends linearly on the norm of the space of multipliers defined by (6.35)*.*

Proof. We concentrate on the case $s \in (0, 1]$, as the general case can be reduced to this one by differentiation.

We will use the following norm on $B^s_{pq}(\mathbb{R}^n; E)$:

$$|v|_{B^s_{pq}} = |v|_{L_p} + [v]_{s,p,q},$$

where

$$[v]_{s,p,q} = \Big(\int_{|h|\le 1}(|h|^{-s}|\tau_h v - v|_{L_p})^q\, dh/|h|^n\Big)^{1/q}, \quad 1 \le q < \infty,$$

and

$$[v]_{s,p,\infty} = \sup_{|h|\le 1}|h|^{-s}|\tau_h v - v|_{L_p}.$$

Here $\{\tau_h\}_{h\in\mathbb{R}^n}$ denotes the group of translations defined by

$$(\tau_h v)(x) = v(x+h), \quad x, h \in \mathbb{R}^n.$$

Obviously we have $|av|_{L_p} \le |a|_{L_\infty}|v|_{L_p}$, so we concentrate on the estimation of $[av]_{s,p,q}$. The identity

$$\tau_h(av) - av = \tau_h a(\tau_h v - v) + (\tau_h a - a)v$$

yields with Hölder's inequality and Remark 6.2.9(ii)

$$[av]_{s,p,q} \le |a|_{L_\infty}[v]_{s,p,q} + [a]_{s,r,q}|v|_{L_{p\rho'}},$$

where $r = p\rho$, $1/\rho + 1/\rho' = 1$, and $s - n/p > -n/p\rho'$. The Gagliardo-Nirenberg inequality implies

$$|v|_{L_{p\rho'}} \le C|v|^{\alpha}_{B^s_{pq}}|v|^{1-\alpha}_{L_p},$$

with some constants $C > 0$ and $\alpha \in [0, 1)$. Alternatively, we may estimate like

$$[av]_{s,p,q} \le |a|_{L_\infty}[v]_{s,p,q} + [a]_{s,\infty,q}|v|_{L_p}.$$

In both cases (6.36) follows. □

Remark 6.2.9. (i) This lemma shows that $B^s_{pq}(\mathbb{R}^n)$ is a Banach algebra w.r.t. pointwise multiplication, provided $s > n/p$, i.e., provided it embeds into L_∞.

(ii) Observe that the multiplier space defined in (6.35) embeds into the uniform Hölder spaces $C_b^{s-n/r}(\mathbb{R}^n; \mathcal{B}(E))$.

We now consider problem (6.24) with variable coefficients, applying perturbation arguments. Thus we look at the case

$$\mathcal{A}(x, D) = \mathcal{A}^0(D) + \mathcal{A}^1(x, D), \quad \mathcal{B}_j(x, D) = \mathcal{B}^0_j(D) + \mathcal{B}^1_j(x, D),$$

where the system $(\mathcal{A}^0(D), \mathcal{B}^0_1(D), \ldots, \mathcal{B}^0_m(D))$ is normally elliptic and subject to the Lopatinskii-Shapiro condition.

For perturbations of $\mathcal{A}^0(D)$ the arguments of Section 6.1.4 apply again, so we require

$$a^1_\alpha \in L_{r_k}(\mathbb{R}^n_+; \mathcal{B}(E)) + L_\infty(\mathbb{R}^n_+; \mathcal{B}(E)), \quad |\alpha| = k < 2m,\ r_k \ge q,\ 2m - k > n/r_k,$$

and in addition the smallness condition

$$|a^1_\alpha|_{L_\infty} \le \eta, \quad |\alpha| = 2m.$$

The essential perturbations to be considered here are the boundary perturbations. In the sequel we assume

$$b^1_{j\beta} \in B^{2m\kappa_j}_{r_{jk}q}(\mathbb{R}^{n-1}; \mathcal{B}(E)) + B^{2m\kappa_j}_{\infty q}(\mathbb{R}^{n-1}; \mathcal{B}(E)),$$
$$|\beta| = k \le m_j,\ r_{jk} \ge q,\ 2m\kappa_j > (n-1)/r_{jk},$$

and the smallness condition

$$|b^1_{j\beta}|_{L_\infty} \le \eta, \quad |\beta| = m_j, \quad j = 1, \ldots, m.$$

Recall the definition $\kappa_j = 1 - m_j/2m - 1/2mq$, and observe that

$$b^1_{j\beta} \in C_b^{2m\kappa_j-(n-1)/r_{jm_j}}(\mathbb{R}^{n-1}; \mathcal{B}(E)), \quad |\beta| = m_j.$$

We estimate the boundary perturbations as follows, employing Lemma 6.2.8. For the highest order terms we get

$$\begin{aligned}|b^1_{j\beta}D^\beta u|_{B^{2m\kappa_j}_{qq}} &\leq |b^1_{j\beta}|_{L_\infty}|D^\beta u|_{B^{2m\kappa_j}_{qq}} + C|D^\beta u|^\alpha_{B^{2m\kappa_j}_{qq}}|D^\beta u|^{1-\alpha}_{L_p}\\ &\leq 2\eta|u|_{H^{2m}_q} + C_\eta|u|_{L_q}.\end{aligned}$$

This implies

$$|\mathcal{B}^1_{j\#}(x,D)u|_{L_{p,\mu}(\mathbb{R}_+;B^{2m\kappa_j}_{qq})} \leq 2\eta|u|_{\mathbb{E}_{1\mu}} + C_\eta|u|_{\mathbb{E}_{0\mu}}.$$

In a similar way we can dominate the lower order terms, without any smallness condition.

Next we need to estimate the terms $|e^{-L_\omega y}b^1_{j\beta}D^\beta v|_{L_q(\mathbb{R}^{n-1})}$, where $v = u_{|y=0}$ denotes the trace of u on the boundary. For this purpose we write

$$\begin{aligned}e^{-\omega^{1/2m}y}b^1_{j\beta}D^\beta v &= -\int_0^\infty \partial_s(e^{-\omega^{1/2m}(y+s)}b^1_{j\beta}D^\beta u(s))\,ds\\ &= \int_0^\infty \omega^{1/2m}e^{-\omega^{1/2m}(y+s)}b^1_{j\beta}D^\beta u(s)\,ds\\ &\quad - \omega^{-1/2m}\int_0^\infty \omega^{1/2m}e^{-\omega^{1/2m}(y+s)}b^1_{j\beta}\partial_s D^\beta u(s)\,ds.\end{aligned}$$

This implies

$$|e^{-\omega^{1/2m}y}b^1_{j\beta}D^\beta v|_{L_q} \leq C|b^1_{j\beta}|_{L_\infty}\int_0^\infty (|D^\beta u(s)|_{L_q} + \omega^{-1/2m}|\partial_s D^\beta u(s)|_{L_q})\frac{ds}{y+s},$$

and as the scalar Hilbert transform is bounded in $L_q(\mathbb{R}_+)$,

$$|e^{-\omega^{1/2m}y}b^1_{j\beta}D^\beta v|_{L_q(\mathbb{R}^n_+)} \leq C|b^1_{j\beta}|_{L_\infty}(|D^\beta u|_{L_q(\mathbb{R}^n_+)} + \omega^{-1/2m}|\partial_s D^\beta u|_{L_q(\mathbb{R}^n_+)}),$$

which yields by the Gagliardo-Nirenberg inequality

$$\omega^{1-m_j/2m}|e^{-\omega^{1/2m}y}b^1_{j\beta}D^\beta u|_{L_q(\mathbb{R}^n_+)} \leq C|b^1_{j\beta}|_{L_\infty}\sum_{i=1,2}|u|^{\gamma_i}_{H^{2m}_q(\mathbb{R}^n_+)}(\omega|u|_{L_q(\mathbb{R}^n_+)})^{1-\gamma_i},$$

with some constants $C > 0$ and $\gamma_i \in [0,1]$. As the coefficients $b^1_{j\beta}$ do not depend on time, this estimate implies

$$\omega^{1-m_j/2m}|e^{-\omega^{1/2m}y}\mathcal{B}^1_{j\#}v|_{\mathbb{E}_{0\mu}} \leq C\eta[|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}].$$

Finally, as $e^{-(L_\omega - \delta\omega^{1/2m})y}$ is bounded in $\mathbb{E}_{0\mu}$, for some $\delta > 0$, this implies

$$\omega^{1-m_j/2m}|e^{-L_\omega y}\mathcal{B}^1_{j\#}v|_{\mathbb{E}_{0\mu}} \leq C\eta[|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}].$$

We now turn to the perturbed initial-boundary value problem. Without loss of generality, we may assume $u_0 = 0$, solving a whole-space problem. We write the half-space problem in abstract form as

$$L_0 u + L_1 u = F,$$

where

$$L_0 u = (\partial_t u + \omega u + \mathcal{A}^0(D)u, \mathcal{B}^0_1(D)u, \ldots, \mathcal{B}^1_m(D)u)$$

defines an isomorphism between the spaces ${}_0\mathbb{E}_{1,\mu}$ and $\mathbb{E}_{0,\mu} \times \Pi^m_{j=1} {}_0\mathbb{F}_{j\mu}$,

$$L_1 u = (\mathcal{A}^1(x,D)u, \mathcal{B}^1(x,D)u, \ldots, \mathcal{B}^1_m(x,D)u),$$

and $F = (f, g_1, \ldots, g_m) \in \mathbb{E}_{0,\mu} \times \Pi^m_{j=1} {}_0\mathbb{F}_{j\mu}$. If $\eta > 0$ is small enough, choosing $\omega > 0$ large enough, we see by the above estimates that $L_0 + L_1$ is also an isomorphism. This way we obtain the following result on (6.24).

Theorem 6.2.10. *Let E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $\mathcal{A}^0(D)$ is a normally elliptic differential operator of order $2m$, let $\mathcal{B}^0_j(D)$, $j = 1, \ldots, m$, denote differential operators of order $m_j < 2m$, and suppose the Lopatinskii-Shapiro condition for $(\mathcal{A}^0(D), \mathcal{B}^0_j(D))$ is satisfied, with some angle $\phi < \pi/2$. Let*

$$\mathcal{A}(x,D) = \mathcal{A}^0(D) + \mathcal{A}^1(x,D), \quad \mathcal{B}_j(x,D) = \mathcal{B}^0_j + \mathcal{B}^1_j(x,D),$$

where the coefficients $a^1_\alpha(x)$, $b^1_{j\beta}(x)$ satisfy the following conditions.

$$\begin{aligned}
&|a^1_\alpha|_{L_\infty}, |b^1_{j\beta}|_{L_\infty} \le \eta, \quad |\alpha| = 2m,\ |\beta| = m_j,\ j = 1, \ldots, m;\\
&a^1_\alpha \in L_{r_k}(\mathbb{R}^n_+; \mathcal{B}(E)) + L_\infty(\mathbb{R}^n_+; \mathcal{B}(E)), \quad |\alpha| = k < 2m,\ r_k \ge q,\ 2m - k > n/r_k;\\
&b^1_{j\beta} \in B^{2m\kappa_j}_{r_{jk}q}(\mathbb{R}^{n-1}; \mathcal{B}(E)) + B^{2m\kappa_j}_{\infty q}(\mathbb{R}^{n-1}; \mathcal{B}(E)),\\
&|\beta| = k,\ r_{jk} \ge q,\ 2m\kappa_j > (n-1)/r_{jk}.
\end{aligned}$$

Then there is $\eta_0 > 0$ such that the assertions of Theorem 6.2.5 *and estimate* (6.33) *remain valid for the perturbed problem, provided $\eta \le \eta_0$.*

2.4 Localization

Here we assume that the top order coefficients a_α with $|\alpha| = 2m$, and $b_{j\beta}$ with $|\beta| = m_j$ are continuous, with limits at infinity. This replaces the smallness condition of the previous subsection. Choose a large ball $B(0,R) \subset \mathbb{R}^n$ such that

$$\begin{aligned}
|a_\alpha(x) - a_\alpha(\infty)| &\le \eta, \quad x \in \bar{\mathbb{R}}^n_+ \setminus B(0,R),\ |\alpha| = 2m,\\
|b_{j\beta}(x) - b_{j\beta}(\infty)| &\le \eta, \quad x \in \mathbb{R}^{n-1},\ |x| \ge R,\ |\beta| = m_j,\ j = 1, \ldots, m.
\end{aligned}$$

Observe that $R > 0$ exists, as the top order coefficients are continuous and have limits at infinity. Next we cover the boundary $\bar{B}(0,R) \cap \mathbb{R}^{n-1}$ by N_1 balls $B(x_k, r/2) \subset \mathbb{R}^n$ such that

$$\begin{aligned}
|a_\alpha(x) - a_\alpha(x_k)| &\le \eta, \quad x \in B(x_k, 2r),\ |\alpha| = 2m,\\
|b_{j\beta}(x) - b_{j\beta}(x_k)| &\le \eta, \quad x \in B(x_k, 2r) \cap \mathbb{R}^{n-1},\ |\beta| = m_j,\ j = 1, \ldots, m.
\end{aligned}$$

Finally, we cover the compact set $\bar{B}(0,R)\setminus\big(\cup_{k=1}^{N_1}B(x_k,r/2)\big)$ by balls $B(x_k,r/2)$, $k=N_1+1,\ldots,N_2$. We then set $U_0=\mathbb{R}^n\setminus\bar{B}(0,R)$, and $U_k=B(x_k,r)$, for $k=1,\ldots,N_2$. Then $\{U_k\}_{k=0}^{N_2}$, forms an open covering of $\bar{\mathbb{R}}^n_+$. Fix a partition of unity $\{\varphi_k\}_{k=0}^{N}$ of class C^∞ subordinate to this open covering, and let $\psi_k\in C^\infty(\mathbb{R}^n)$ be such that $\psi_k=1$ on $\operatorname{supp}\varphi_k$ and $\operatorname{supp}\psi_k\subset U_k$.

We assume in the sequel that the operator $\mathcal{A}_\#(x_0,D)$ is normally elliptic, for each $x_0\in\bar{\mathbb{R}}^n_+\cup\{\infty\}$, and that the system $(\mathcal{A}_\#(x_0,D),\mathcal{B}_{j\#}(x_0,D))$ satisfies the Lopatinskii-Shapiro Condition **(LS)**, for each $x_0\in\mathbb{R}^{n-1}\cup\{\infty\}$, with angle $\phi(x_0)<\pi/2$. Then the maximal regularity constants for the problems with frozen coefficients will be uniform in $x_0\in\bar{\mathbb{R}}^n_+\cup\{\infty\}$, by continuity and compactness, hence η_0 in Theorem 6.2.10 will be uniform in x_0 as well. Now we fix any $\eta\in(0,\eta_0]$.

Next we define for each k local operators $\mathcal{A}_k(x,D)$ on the half-space $\mathbb{R}^n_+$ and $\mathcal{B}_{jk}(x,D)$ on the boundary $\mathbb{R}^{n-1}$ in the following way. Choose a function $\chi\in\mathcal{D}(\mathbb{R})$ such that $\chi(s)=1$ for all $|s|\le 1$, $0\le\chi(s)\le 1$ and $\chi(s)=0$ for $|s|\ge 2$. Then we set

$$\begin{aligned}a^k_\alpha(x)&=a_\alpha(x_k+\chi(|x-x_k|^2/r^2)(x-x_k)),\quad x\in\bar{\mathbb{R}}^n_+,\ |\alpha|=2m,\ k=1,\ldots,N_2,\\ b^k_{j\beta}(x)&=b_{j\beta}(x_k+\chi(|x-x_k|^2/r^2)(x-x_k)),\quad x\in\mathbb{R}^{n-1},\ |\beta|=m_j,\end{aligned}$$

$j=1,\ldots,m$, and

$$\begin{aligned}a^0_\alpha(x)&=a_\alpha(\infty)+\chi(R^2/|x|^2)(a_\alpha(x)-a_\alpha(\infty)),\quad x\in\bar{\mathbb{R}}^n_+,\ |\alpha|=2m,\\ b^0_{j\beta}(x)&=b_{j\beta}(\infty)+\chi(R^2/|x|^2(b_{j\beta}(x)-b_{j\beta}(\infty)),\quad x\in\mathbb{R}^{n-1},\ |\beta|=m_j.\end{aligned}$$

Here we set $a^0_\alpha(0)=a_\alpha(\infty)$ and $b^0_{j\beta}(0)=b^0_{j\beta}(\infty)$. Then we define the local operators by means of

$$\mathcal{A}^k(x,D)=\sum_{|\alpha|=2m}a^k_\alpha(x)D^\alpha,\quad \mathcal{B}^k_j(x,D)=\sum_{|\beta|=m_j}b^k_{j\beta}(x)D^\beta.$$

By solving a full space problem, by Theorem 6.1.11, extending all coefficients of $\mathcal{A}(x,D)$ by symmetry to all of $\mathbb{R}^n$, we may assume $u_0=0$. Now let the data g_j be given and let $u\in{}_0\mathbb{E}_{1\mu}$ be a solution of (6.24) in $\mathbb{R}^n_+$. We set $u^k=\varphi_k u$, $f^k=\varphi_k f$, and $g^k_j=\varphi_k g_j$. Then we obtain the following localized problems. For the interior charts $k=N_1+1,\ldots,N_2$, the functions u^k satisfy

$$\begin{aligned}\partial_t u^k+\omega u^k+\mathcal{A}^k(x,D)u^k&=f^k+[\mathcal{A}_\#(x,D),\varphi_k]u-\varphi_k\mathcal{A}_1(x,D)u\ \text{ in }\ \mathbb{R}^n,\\ u^k(0)&=0,\end{aligned}$$

where $\mathcal{A}_1(x,D)=\mathcal{A}(x,D)-\mathcal{A}_\#(x,D)$ denotes the lower order part of $\mathcal{A}(x,D)$. Note that $\mathcal{A}^k(x,D)\varphi_k=\mathcal{A}_\#(x,D)\varphi_k$ by construction, and observe that the commutators $[\mathcal{A}_\#(x,D),\varphi_k]$ are of lower order as well. The boundary charts $k=0,\ldots,N_1$ lead to the following half-space problems.

$$\begin{aligned}\partial_t u^k+\omega u^k+\mathcal{A}^k(x,D)u^k&=f^k+[\mathcal{A}_\#(x,D),\varphi_k]u-\varphi_k\mathcal{A}_1(x,D)u\quad\text{in}\quad\mathbb{R}^n_+,\\ \mathcal{B}^k_j(x,D)u^k&=g^k_j+[\mathcal{B}_{j\#}(x,D),\varphi_k]u-\varphi_k\mathcal{B}_{j1}(x,D)u\quad\text{on}\quad\mathbb{R}^{n-1},\\ u^k(0)&=0,\end{aligned}$$

where $\mathcal{B}_{j1}(x,D) = \mathcal{B}_j(x,D) - \mathcal{B}_{j\#}(x,D)$ as well as the commutator $[\mathcal{B}_{j\#}(x,D), \varphi_k]$ are of order $m_j - 1$, these are trivial in case $m_j = 0$. We write these problems abstractly as

$$L_k u^k = G_k u + F_k, \quad k = 0, \dots, N_2,$$

where the operators L_k are defined by the left-hand sides of the localized equations, $G_k u$ are the lower order perturbations on the right-hand side, and F_k collects the data coming from the inhomogeneities (f, g_j). More precisely,

$$G_k u = ([\mathcal{A}_\#(x,D), \varphi_k]u - \varphi_k \mathcal{A}_1(x,D)u, [\mathcal{B}_{j\#}(x,D), \varphi_k]u - \varphi_k \mathcal{B}_{j1}(x,D)u)$$

and $F_k = \varphi_k F = \varphi_k(f, g_j)$. By Theorem 6.2.10, the operators L_k are invertible for ω large, hence we obtain

$$u^k = L_k^{-1} F_k + L_k^{-1} G_k u, \quad k = 0, \dots, N_2, \tag{6.37}$$

and so the following representation of the solution u. We first write

$$u = \sum_{k=0}^{N_2} \varphi_k u = \sum_{k=0}^{N_2} \psi_k \varphi_k u = \sum_{k=0}^{N_2} \psi_k u^k,$$

and then

$$u = \sum_{k=0}^{N_2} \psi_k L_k^{-1} F_k + \Big(\sum_{k=0}^{N_2} \psi_k L_k^{-1} G_k \Big) u.$$

We estimate in the following way, employing Theorem 6.1.11 for the interior charts and (6.34) for the boundary charts.

$$\begin{aligned} &|\psi_k L_k^{-1} G_k u|_{\mathbb{E}_{1\mu}} + \omega |\psi_k L_k^{-1} G_k u|_{\mathbb{E}_{0\mu}} \\ &\le C\Big(|G_k^i u|_{\mathbb{E}_{0\mu}} + \sum_{j=1}^m (|G_{kj}^b u|_{\mathbb{F}_{j\mu}} + \omega^{\kappa_j} |G_{kj}^b u|_{L_{p,\mu}(L_q)} \Big). \end{aligned}$$

Here the boundary terms are absent for the interior charts $k = N_1 + 1, \dots, N_2$. For the interior operators G_k^i defined by

$$G_k^i u = [\mathcal{A}_\#(x,D), \varphi_k]u - \varphi_k \mathcal{A}_1(x,D)u,$$

we obtain by the Gagliardo-Nirenberg inequality

$$|G_k^i u|_{\mathbb{E}_{0\mu}} \le C |u|_{\mathbb{E}_{1\mu}}^{\gamma} |u|_{\mathbb{E}_{0,\mu}}^{1-\gamma},$$

with some constants $C > 0$ and $\gamma \in (0,1)$, hence

$$|G_k^i u|_{\mathbb{E}_{0\mu}} \le \frac{C}{\omega^{1-\gamma}} \big(|u|_{\mathbb{E}_{1\mu}} + \omega |u|_{\mathbb{E}_{0\mu}} \big).$$

The boundary terms are of the form

$$G^b_{kj}u = [\mathcal{B}_{j\#}(x,D),\varphi_k]u - \varphi_k\mathcal{B}_{j1}(x,D)u.$$

Therefore, as in the previous subsection

$$|G^b_{kj}u|_{\mathbb{F}_{j\mu}} \le C_j|u|^{\gamma_j}_{\mathbb{E}_{1\mu}}|u|^{1-\gamma_j}_{\mathbb{E}_{0\mu}} \le \frac{C_j}{\omega^{1-\gamma_j}}\big(|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}\big),$$

with constants $C_j > 0$ and $\gamma_j \in (0,1)$. Finally, applying once more arguments of the previous subsection, we also obtain

$$\omega^{\kappa_j}|G^b_{kj}u|_{L_{p,\mu}(L_q)} \le \frac{C_j}{\omega^{1-\gamma_j}}\big(|u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}\big),$$

with possibly different constants $C_j > 0$ and $\gamma_j \in [0,1)$.

Summarizing, we see that for ω sufficiently large, the operator $G^L := \sum_{k=0}^{N_2}\psi_k L_k^{-1}G_k$ on ${}_0\mathbb{E}_{1\mu}$ satisfies the estimate

$$|G^L u|_{\mathbb{E}_{1\mu}} + \omega|G^L u|_{\mathbb{E}_{0\mu}} \le \frac{C}{\omega^{1-\gamma}}\big(|u|_{\mathbb{E}_{1\mu}} + \omega|u_{\mathbb{E}_{0\mu}}\big)$$

with appropriate constants C and γ that do not depend on ω. Equipping ${}_0\mathbb{E}_{1\mu}$ with the parameter-dependent norm $|u|^\omega_{\mathbb{E}_{1\mu}} := |u|_{\mathbb{E}_{1\mu}} + \omega|u|_{\mathbb{E}_{0\mu}}$ we conclude that the operator $I - G^L$ is invertible in $({}_0\mathbb{E}_{1\mu}, |\cdot|^\omega_{\mathbb{E}_{1\mu}})$, provided ω is sufficiently large. This yields a left inverse S of (6.24), which is given by

$$S(f,g_j) = (I - G^L)^{-1}\sum_{k=0}^{N_2}\psi_k L_k^{-1}\varphi_k(f,g_j).$$

In particular, the operator L defined by the left-hand side of (6.24) is injective and has closed range. So it remains to prove that L is also surjective. To show this we construct a right inverse which then by algebra equals its left inverse.

For this purpose we apply $L_\# := (\partial_t + \omega + \mathcal{A}_\#(x,D), \mathcal{B}_{j\#}(x,D))$ to $u = SF$, observing $L_\# = L_k$ in U_k. This yields with (6.37)

$$L_\# u = L_\#\sum_{k=0}^{N_2}\psi_k u^k = \sum_{k=0}^{N_2}[L_\#,\psi_k]L_k^{-1}(F_k + G_k u) + \sum_{k=0}^{N_2}\psi_k(F_k + G_k u).$$

Next, as $\psi_k = 1$ on the support of φ_k, we may drop ψ_k in the second term, which implies in the interior

$$\sum_{k=0}^{N_2}\psi_k(F_k + G_k u)^i = \sum_k(f_k + [\mathcal{A}_\#(x,D),\varphi_k]u - \varphi_k\mathcal{A}_1(x,D)u) = f - \mathcal{A}_1(x,D)u,$$

and on the boundary

$$\sum_{k=0}^{N_2} \psi_k (F_k + G_k u)^b = \sum_k g_{jk} + [\mathcal{B}_{j\#}(x,D), \varphi_k]u - \varphi_k \mathcal{B}_{j1}(x,D)u = g_j - \mathcal{B}_{j1}(x,D)u.$$

Replacing $u = SF$, this yields

$$LS = I + \Big(\sum_{k=0}^{N_2} [L_\#, \psi_k] L_k^{-1} \varphi_k\Big) + \Big(\sum_{k=0}^{N_2} [L_\#, \psi_k] L_k^{-1} G_k\Big) S =: I + G^R.$$

As the commutator $[L_\#, \psi_k] = ([\mathcal{A}_\#(x,D), \psi_k], [\mathcal{B}_{j\#}(x,D), \psi_k])$ is lower order, we see as above that the norm of G^R in $\mathbb{E}_{0\mu}$ is smaller than 1, provided ω is chosen large. Therefore $I + G^R$ is invertible, and so $R := S(I + G^R)^{-1}$ is a right inverse of L. This implies the following result for the half-space.

Theorem 6.2.11. *Let $1 < p, q < \infty$, $\mu \in (1/p, 1]$ and E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $\mathcal{A}(x,D)$ is a differential operator of order $2m$, let $\mathcal{B}_j^0(D)$, $j = 1, \ldots, m$, denote differential operators of order $m_j < 2m$. Suppose that the coefficients $a_\alpha(x)$, $b_{j\beta}(x)$ satisfy the following conditions.*

$$\begin{aligned}
&a_\alpha \in C_l(\bar{\mathbb{R}}^n_+; \mathcal{B}(E)), \quad b_{j\beta} \in C_l(\mathbb{R}^{n-1}; \mathcal{B}(E)) \quad |\alpha| = 2m,\ |\beta| = m_j,\ j = 1, \ldots, m;\\
&a_\alpha \in L_{r_k}(\mathbb{R}^n_+; \mathcal{B}(E)) + L_\infty(\mathbb{R}^n_+; \mathcal{B}(E)), \quad |\alpha| = k < 2m,\ r_k \geq q,\ 2m - k > n/r_k;\\
&b_{j\beta} \in B^{2m\kappa_j}_{r_{jk}q}(\mathbb{R}^{n-1}; \mathcal{B}(E)) + B^{2m\kappa_j}_{\infty q}(\mathbb{R}^{n-1}; \mathcal{B}(E)),\\
&|\beta| = k \leq m_j,\ r_{jk} \geq q,\ 2m\kappa_j > (n-1)/r_{jk}.
\end{aligned}$$

Assume that $\mathcal{A}_\#(x,D)$ is normally elliptic for each $x \in \bar{\mathbb{R}}^n_+ \cup \{\infty\}$, and that $(\mathcal{A}_\#(x,D), \mathcal{B}_{j\#}(x,D))$ satisfies the Lopatinskii-Shapiro Condition **(LS)** *with some angle $\phi(x) < \pi/2$, for each $x \in \mathbb{R}^{n-1} \cup \{\infty\}$.*

Then the assertions of Theorem 6.2.5 and Corollary 6.2.7 remain valid for the half-space problem with variable coefficients.

2.5 Normal Strong Ellipticity

We now consider the special case of strongly elliptic second-order operators in a Hilbert space E with so-called *natural boundary conditions*. This means, we consider $\mathcal{A}(D) = a^{ij} D_i D_j$, where $a^{ij} = a^{ji}$, with boundary operator either of Dirichlet type, i.e., $\mathcal{B}(D) = I$, or of co-normal (Neumann) type $\mathcal{B}(D) = \nu_i a^{ij} D_j$; here we employ the Einstein summation convention. Assuming that $\mathcal{A}(D)$ is strongly elliptic, what more conditions are needed for the Lopatinskii-Shapiro condition to be valid for these natural boundary operators?

To answer this question, let $u \in L_2(\mathbb{R}_+; E)$ be a solution of the ODE-boundary value problem

$$\begin{aligned}
\lambda u(y) + \mathcal{A}(\xi + \nu D_y) u(y) &= 0, \quad y > 0,\\
\mathcal{B}(\xi + \nu D_y) u(0) &= 0.
\end{aligned} \tag{6.38}$$

Here $\operatorname{Re}\lambda \geq 0$, $\xi, \nu \in \mathbb{R}^n$ are fixed, with $(\lambda, \xi) \neq (0,0)$, $|\nu| = 1$, $(\xi|\nu) = 0$. Take the inner product with u in E, integrate over $\mathbb{R}_+$, and take real parts. By means of the natural boundary conditions this yields the identity

$$\operatorname{Re}\lambda |u|_2^2 + \int_0^\infty \operatorname{Re}(a^{ij}(\xi_j + \nu_j D_y)u|(\xi_i + \nu_i D_y)u)\,dy = 0. \tag{6.39}$$

To be able to conclude from this identity that $u = 0$, the following condition is natural.

Definition 6.2.12. *A differential operator $\mathcal{A}(D) = a^{ij} D_i D_j$, with $a^{ij} = a^{ji} \in \mathcal{B}(E)$, is called* **normally strongly elliptic***, if its is strongly elliptic and there is a constant $c > 0$ such that*

$$\operatorname{Re}(a^{ij}(\xi_j u + \nu_j v)|\xi_i u + \nu_i v) \geq c|\operatorname{Im}(u|v)|, \quad u, v \in E,$$

for all $\xi, \nu \in \mathbb{R}^n$, $|\xi| = |\nu| = 1$, $(\xi|\nu) = 0$.

From this condition we may then conclude $\operatorname{Im}(u(y)|D_y u(y)) = 0$ for all $y > 0$, which implies

$$\frac{d}{dy}|u(y)|^2 = 2\operatorname{Re}(u(y)|\partial_y u(y)) = 2\operatorname{Im}(u(y)|D_y u(y)) = 0,$$

hence $|u|$ is constant on $\mathbb{R}_+$, and so must be 0 as $u \in L_2(\mathbb{R}_+; E)$.

In case E is finite-dimensional, we are finished, as by strong ellipticity the dimension of the space of solutions of the homogeneous differential equation (6.38) has dimension $\dim E$. The map $T : u \mapsto \mathcal{B}(\xi + \nu D_y)u(0)$ is injective, hence also surjective, and so the Lopatinskii-Shapiro condition holds. If E is infinite-dimensional we have to work a little harder to obtain this result.

For this purpose observe first that the operator T defined above is injective, but also has dense range, as with $\mathcal{A}(D)$ also $\mathcal{A}^*(D)$ is normally strongly elliptic. Therefore we need to show that the range of T is closed. So let $u \in L_2(\mathbb{R}_+; E)$ be a solution of the ODE-problem

$$\begin{aligned} \lambda u(y) + \mathcal{A}(\xi + \nu D_y)u(y) &= 0, \quad y > 0, \\ \mathcal{B}(\xi + \nu D_y)u(0) &= g \in E. \end{aligned} \tag{6.40}$$

(i) We first consider the Neumann case. Multiplying the equation for u in (6.40) with $u(y)$, integrating over $\mathbb{R}_+$ and integrating by parts, we get by normal strong ellipticity

$$c|u_0|^2 \leq c\int_0^\infty |\partial_y |u(y)|^2|\,dy \leq 2|g||u_0|,$$

where $u_0 = u(0)$. This implies $|u_0| \leq C|g|$. Hence we may restrict our attention to the Dirichlet case, and the goal is to prove that there is a constant $C > 0$ such that $|u|_2 \leq C|u_0|$, for each L_2-solution u of the homogeneous problem

$$\lambda u(y) + \mathcal{A}(\xi + \nu D_y)u(y) = 0, \quad y > 0.$$

(ii) We begin estimating the L_2-norm of $u'(y) := \partial_y u(y)$ as follows, employing an integration by parts.

$$|u'|_2^2 = -(u_1|u_0) - (u|u'')_2 \leq |u_1||u_0| + |u|_2|u''|_2$$
$$\leq |u_1||u_0| + C|u|_2(|u|_2 + |u'|_2).$$

Here $u_1 = u'(0)$ and we used the equation for u, as well as the fact that the operator $a^{ij}\nu_i\nu_j$ is invertible in E, by strong ellipticity. This implies by Young's inequality

$$|u'|_2^2 \leq 2|u_1||u_0| + C_1|u|_2^2. \tag{6.41}$$

(iii) Next we write

$$|u_1|^2 = -2Re\int_0^\infty (u''(y)|u'(y))\,dy,$$

to obtain

$$|u_1|^2 \leq 2|u'|_2|u''|_2 \leq C|u'|_2(|u|_2 + |u'|_2),$$

hence by Young's inequality and (6.41)

$$|u_1|^2 \leq C_2(|u|_2^2 + |u_0|^2). \tag{6.42}$$

(iv) Now we employ once more normal strong ellipticity, to obtain as in (i) the estimate

$$|u(y)|^2 \leq C|u_0|(|u_0| + |u_1|) \leq (C_\varepsilon|u_0| + \varepsilon|u_1|)^2, \tag{6.43}$$

again using Young's inequality.

The final estimate comes from strong ellipticity. Taking the Laplace transform of $\lambda u(y) + \mathcal{A}(\xi + \nu D_y)u(y) = 0$ w.r.t. the variable y we obtain

$$\mathcal{L}u(z) = -(\lambda + \mathcal{A}(\xi - iz\nu))^{-1}[(a^{kl}\nu_k\nu_l(zu_0 + u_1) + 2ia^{kl}\xi_k\nu_l u_0].$$

As $u \in L_2(\mathbb{R}_+; E)$, by strong ellipticity, the function $\mathcal{L}u(z)$ has only singularities in a compact subset of the negative half-plane, which only depends on (λ, ξ, ν). So choosing a contour Γ_- surrounding these singularities and lying entirely in the left half-plane, we obtain the representation

$$u(y) = \frac{1}{2\pi i}\int_{\Gamma_-} e^{zy}\mathcal{L}u(z)\,dz, \quad y > 0.$$

This implies

$$e^{\omega y}|u(y)| \leq C_3(|u_0| + |u_1|), \tag{6.44}$$

with some fixed constants $\omega > 0$ and $C_3 > 0$ independent of u. Interpolating (6.43) and (6.44) and integrating over $y > 0$, this implies

$$|u|_2^2 \leq \frac{C_3}{\omega}(|u_0| + |u_1|)(C_\varepsilon|u_0| + \varepsilon|u_1|)$$
$$\leq \frac{C_3}{\omega}(C'_\varepsilon|u_0|^2 + 2\varepsilon|u_1|^2),$$

applying once more Young's inequality. Finally, choosing $\varepsilon > 0$ small enough, combining the last estimate with (6.42) yields $|u|_2^2 \le C|u_0|^2$, which is what we wanted to prove.

(v) Finally we consider *mixed boundary conditions* which are also important in applications. For this purpose let $P \in \mathcal{B}(E)$ be an orthogonal projection, i.e., $P = P^* = P^2$, and consider the boundary conditions

$$Pu(0) = g_0, \quad (I - P)\mathcal{B}(D)u(0) = g_1.$$

Then the energy argument yields an estimate of the form

$$c|u_0|^2 \le C|g_0|(|u_0| + |u_1|) + |g_1||u_0|,$$

which implies

$$|u_0|^2 \le C(|g_0|^2 + |g_1|^2) + \varepsilon|u_1|^2,$$

and so by (6.42)

$$|u_0|^2 \le C(|g_0|^2 + |g_1|^2) + \varepsilon|u|_2^2,$$

and finally

$$|u|_2^2 \le C(|g_0|^2 + |g_1|^2).$$

This shows that also the case of mixed boundary conditions is covered.

We summarize the result obtained above.

Proposition 6.2.13. *Let E be a Hilbert space and suppose that $\mathcal{A}(D)$ is a second-order, normally strongly elliptic differential operator in E.*

Then the Lopatinskii-Shapiro condition is satisfied for the natural boundary conditions, i.e., for Dirichlet, Neumann, or mixed conditions.

The following proposition deals with a very special case which, however, is frequently met in applications.

Proposition 6.2.14. *Let $a^{ij} = \alpha^{ij}b$, where the matrix $[\alpha^{ij}]$ is real, symmetric, and positive definite, and $b \in \mathcal{B}(E)$ is strongly accretive in the Hilbert space E, i.e.,*

$$\mathrm{Re}(bu|u) \ge c|u|^2, \quad u \in E,$$

for some positive constant $c > 0$.

Then $\mathcal{A}(D)$ is normally strongly elliptic.

We leave the proof of this proposition to the interested reader, as it only involves the Cauchy-Schwarz inequality.

Remark. (i) For $E = \mathbb{C}^n$ there is another stronger concept of ellipticity. We say that $a \in \mathcal{B}(E)^{n\times n}$ satisfies the *strong Legendre condition*, if there is a constant $C > 0$ such that

$$\mathrm{Re}\, a_{kl}^{ij} d_j^l \bar{d}_i^k \ge C|d|_2^2, \quad \text{for all } d \in \mathcal{B}(\mathbb{C}^n).$$

This condition means that a is strongly accretive on $\mathcal{B}(\mathbb{C}^n)$.

Obviously, the strong Legendre condition implies normal strong ellipticity, as for $d = \xi \otimes u + \nu \otimes v$ with $\xi \cdot \nu = 0$ we have

$$|d|_2^2 = |\xi|^2|u|^2 + |\nu|^2|v|^2 \geq 2|\xi||\nu||(u|v)|.$$

(ii) For many applications, however, the strong Legendre condition is too strong. This comes from the fact that the tensor a usually has symmetries like

$$a_{kl}^{ij} = a_{ij}^{kl} = a_{kj}^{il} = a_{il}^{kj}.$$

These symmetries are called *hyperelastic* and mean that a only acts on the symmetric part of a matrix and yields again a symmetric matrix. This is quite common in elasticity theory and also in compressible fluids, as there a represents *stress-strain* relations like $S = aD$, where D means the symmetric part of a deformation gradient, or of a velocity gradient. Then the stress S will also be symmetric. In this case the operator a maps the space of symmetric matrices $\mathrm{Sym}(\mathbb{C}^n)$ into itself. For this situation, the appropriate condition – which we call the *Legendre condition* – reads

$$\mathrm{Re}\, a_{kl}^{ij} e_j^l \bar{e}_i^k \geq C|e|_2^2, \quad \text{for all } e \in \mathrm{Sym}(\mathbb{C}^n).$$

This means that a is strongly accretive on $\mathrm{Sym}(\mathbb{C}^n)$, and it will be even selfadjoint in case $a_{kl}^{ij} = \bar{a}_{lk}^{ji}$.

Obviously, the Legendre condition implies strong ellipticity, but also normal strong ellipticity. In fact, for $d = \xi \otimes u + \nu \otimes v$ and $e = (d + d^{\mathsf{T}})/2$ we have with $|\xi| = |\nu| = 1$, $\xi \cdot \nu = 0$, and

$$u = (u|\xi)\xi + (u|\nu)\nu + u_\perp, \quad v = (v|\xi)\xi + (v|\nu)\nu + v_\perp, \quad u_\perp, v_\perp \perp \xi, \nu,$$

the identity

$$|e|_2^2 = \frac{1}{2}\{|u_\perp|^2 + |v_\perp|^2 + 2|(u|\xi)|^2 + 2|(v|\nu)|^2 + |(u|\nu) + (v|\xi)|^2\}.$$

This shows $e = 0$ if and only if $u_\perp = v_\perp = 0$, $(u|\xi) = (v|\nu) = 0$, $(u|\nu) = -(v|\xi)$, which implies $u = (u|\nu)\nu$, $v = (v|\xi)\xi$, in particular $(u|v) = 0$. In other words, if $|\xi| = |\nu| = 1$, $\xi \cdot \nu = 0$, and $\mathrm{Im}(u|v) \neq 0$, then $e \neq 0$. Therefore, the Legendre condition implies normal strong ellipticity.

(iii) In summary, we have the following implications for a second-order differential operator $\mathcal{A}(D) = a^{ij}D_iD_j$, with $a^{ij} = a^{ji} \in \mathcal{B}(\mathbb{C}^n)$:

$$\begin{aligned} &\mathcal{A}(D) \text{ satisfies the strong Legendre condition} \\ &\quad \Rightarrow \mathcal{A}(D) \text{ satisfies the Legendre condition} \\ &\quad \Rightarrow \mathcal{A}(D) \text{ is normally strongly elliptic} \\ &\quad \Rightarrow \mathcal{A}(D) \text{ is strongly elliptic} \\ &\quad \Rightarrow \mathcal{A}(D) \text{ is normally elliptic.} \end{aligned}$$

(iv) As an example we consider the well-known *Lamé operator* L, which is defined by

$$\begin{aligned}\mathsf{L}u :=& - \operatorname{div}[\mu_s(\nabla u + \nabla u^{\mathsf{T}}) + \mu_b(\operatorname{div} u)I] \\ =& - \mu_s \Delta u - (\mu_s + \mu_b)\nabla \operatorname{div} u,\end{aligned}$$

which yields

$$[\mathsf{L}u]_k = -a^{ij}_{kl}\partial_i\partial_j u_l, \quad \text{with } a^{ij}_{kl} = \mu_s(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + \mu_b\delta_{ik}\delta_{jl}.$$

The tensor a is easily checked to be hyperelastic and selfadjoint, and the Legendre condition is equivalent to

$$\mu_s > 0, \quad 2\mu_s + n\mu_b > 0.$$

On the other hand, a is strongly elliptic if and only if

$$\mu_s > 0, \quad 2\mu_s + \mu_b > 0,$$

and a is normally strongly elliptic if and only if

$$\mu_s > 0, \quad \mu_s + \mu_b > 0.$$

This can be shown by elementary linear algebra.

6.3 General Domains

Let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary $\partial\Omega$ of class C^{2m}. So Ω may be an interior or an exterior domain. In this section we consider the following general parabolic initial-boundary problem which is completely inhomogeneous. Let E be a Banach space of class $\mathcal{HT}$, and consider the parabolic problem

$$\begin{aligned}\partial_t u + \omega u + \mathcal{A}(x,D)u &= f && \text{in } \Omega,\\ \mathcal{B}_j(x,D)u &= g_j && \text{on } \partial\Omega,\ j = 1,\dots,m,\\ u(0) &= u_0 && \text{in } \Omega.\end{aligned} \tag{6.45}$$

Here $\mathcal{A}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$ is a differential operator of order $2m$, $\mathcal{B}_j(x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(x)D^\beta$ are differential operators of order $m_j < 2m$, $\omega \in \mathbb{R}$, and the data (f, g_j, u_0) are given. We are interested in maximal $L_{p,\mu}$–L_q-regularity of (6.45).

3.1 The Main Result

We formulate the assumptions of the main theorem in the following way. The most essential is the ellipticity assumption.

Definition 6.3.1. *We call the system* $(\mathcal{A}(x,D),\mathcal{B}_1(x,D),\dots,\mathcal{B}_m(x,D))$ **uniformly normally elliptic** *if*

(i) $\mathcal{A}(x,D)$ *is normally elliptic, for each* $x\in\bar\Omega\cup\{\infty\}$;

(ii) *The Lopatinskii-Shapiro condition* **(LS)** *holds, for each* $x\in\partial\Omega$.

This assumption is crucial, and even necessary, for the main result stated below; see the Bibliographical Comments.

Next we state the regularity assumptions on the coefficients.

Condition (rA)

(rA1) $a_\alpha\in C_l(\bar\Omega;\mathcal{B}(E))$ for each $|\alpha|=2m$;

(rA2) $a_\alpha\in L_{r_k}(\Omega;\mathcal{B}(E))+L_\infty(\Omega;\mathcal{B}(E))$ for each $|\alpha|=k<2m$, with $r_k\geq q$ and $2m-k>n/r_k$.

For the regularity of the coefficients on the boundary we recall the definition $\kappa_j=1-m_j/2m-1/2mq$.

Condition (rB)

(rB) $b_{j\beta}\in B^{2m\kappa_j}_{r_{jk}q}(\partial\Omega;\mathcal{B}(E))$ for each $|\beta|=k\leq m_j$, with $r_{jk}\geq q$, and $2m\kappa_j>(n-1)/r_{jk}$.

With these assumptions we can state the main theorem of this section.

Theorem 6.3.2. *Let* $\Omega\subset\mathbb{R}^n$ *be open with compact boundary* $\partial\Omega$ *of class* C^{2m}, $1<p,q<\infty$, $\mu\in(1/p,1]$, *and let* E *be a Banach space of class* $\mathcal{HT}(\alpha)$. *Assume that* $(\mathcal{A}(x,D),\mathcal{B}_1(x,D),\dots\mathcal{B}_m(x,D))$ *is uniformly normally elliptic, and satisfies the regularity conditions* **(rA)** *and* **(rB)**. *Let* $\kappa_j\neq 1/p+1-\mu$ *for all* j.

Then there is $\omega_0\in\mathbb{R}$ *such that for each* $\omega>\omega_0$, *equation* (6.45) *admits a unique solution* u *in the class*

$$u\in\mathbb{E}_{1\mu}:=H^1_{p,\mu}(\mathbb{R}_+;L_q(\Omega;E))\cap L_{p,\mu}(\mathbb{R}_+;H^{2m}_q(\Omega;E)),$$

if and only if the data are subject to the following conditions.

(a) $f\in\mathbb{E}_{0\mu}=L_{p,\mu}(\mathbb{R}_+;L_q(\Omega;E))$, $u_0\in X_{\gamma,\mu}=B^{2m(\mu-1/p)}_{qp}(\Omega;E)$;

(b) $g_j\in\mathbb{F}_{j\mu}=F^{\kappa_j}_{pq,\mu}(\mathbb{R}_+;L_q(\partial\Omega;E))\cap L_{p,\mu}(\mathbb{R}_+;B^{2m\kappa_j}_{qq}(\partial\Omega;E)),\quad j=1,\dots,m.$

(c) $\mathcal{B}_j(D)u_0=g_j(0)$ *if* $\kappa_j>1/p+1-\mu$, $j=1,\dots,m$.

The solution depends continuously on the data in the corresponding spaces.

The proof of this result is given in the next subsections.

3.2 Coordinate Transformations

(a) Let $\Phi\in C^{2m}_b(\mathbb{R}^n;\mathbb{R}^n)$ be such that

$$c\leq|\det\partial\Phi(x)|\leq c^{-1},\quad x\in\mathbb{R}^n,$$

for some constant $c > 0$, and $\partial\Phi(x) \to I$ as $|x| \to \infty$. Define the coordinate transform T by means of

$$(Tv)(x) = v(\Phi(x)), \quad x \in \mathbb{R}^n.$$

Then $T : H_p^k(\mathbb{R}^n; E) \to H_p^k(\mathbb{R}^n; E)$ is an isomorphism for each $0 \le k \le 2m$. For the derivative $D = (D_1, \cdots, D_n)$ we obtain the transformation law

$$DTv(x) = \partial\Phi^{\mathsf{T}}(x)(Dv)(\Phi(x)),$$

hence the differential operator $\mathcal{A}(x, D)$ tranforms to $\mathcal{A}^\Phi(y, D)$, given by

$$\mathcal{A}^\Phi(y, D) = T^{-1}\mathcal{A}(x, D)T = \sum_{|\alpha|\le 2m} a_\alpha^\Phi(y)D = \sum_{|\alpha|\le 2m} a_\alpha(\Phi^{-1}(y))(\partial\Phi^{\mathsf{T}}(\Phi^{-1}(y))D)^\alpha.$$

Therefore, the coefficients a_α^Φ enjoy the same regularity conditions as a_α, and the principal symbol of $\mathcal{A}^\Phi$ is given by

$$\mathcal{A}_\#^\Phi(y, \xi) = \mathcal{A}_\#(\Phi^{-1}(y), \partial\Phi^{\mathsf{T}}(\Phi^{-1}(y))\xi), \quad y, \xi \in \mathbb{R}^n.$$

This shows that parameter-ellipticity of $\mathcal{A}^\Phi$ is equivalent to that of $\mathcal{A}$, with the same angle of ellipticity.

(b) We consider now the situation of a *bent half-space*. Replacing the variable $x \in \mathbb{R}_+^n$ by $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$, a bent half-space is defined by a coordinate transformation of the form $\Phi(x, y) = (x, y + h(x))$, with

$$h \in C_b^{2m}(\mathbb{R}^{n-1}; \mathbb{R}), \qquad \lim_{|x|\to\infty} \partial h(x) = 0. \tag{6.46}$$

Note that the boundary of the transformed domain is the graph $(x, h(x))$. Clearly, $\Phi \in C_b^{2m}(\mathbb{R}^n; \mathbb{R}^n)$, and with

$$\partial\Phi(x, y) = \begin{bmatrix} I & 0 \\ \partial h(x) & 1 \end{bmatrix}, \quad \partial\Phi(x, y)^{-1} = \begin{bmatrix} I & 0 \\ -\partial h(x) & 1 \end{bmatrix}$$

satisfies $\lim_{|x|+|y|\to\infty} \partial\Phi(x, y) = I$. Moreover, $\det \partial\Phi(x, y) = 1$. Hence we see that (a) applies. In a similar way, the boundary operators $\mathcal{B}_j(x, D)$ are transformed to $\mathcal{B}^\Phi(\cdot, D) = T^{-1}\mathcal{B}_j(\cdot, D)T$, hence their principal parts become

$$\mathcal{B}_{j\#}^\Phi(y, \xi) = \mathcal{B}_{j\#}(\Phi^{-1}(y), \partial\Phi^{\mathsf{T}}(\Phi^{-1}(y))\xi), \quad y, \xi \in \mathbb{R}^n.$$

Note that the normal of $\mathbb{R}_+^n$ at (x, y) transforms to

$$\nu = \frac{-\partial\Phi^{-\mathsf{T}} e_n}{|\partial\Phi^{-\mathsf{T}} e_n|} = \frac{1}{\sqrt{1 + |\nabla_x h|^2}} [\nabla_x h(x), -1]^{\mathsf{T}}.$$

This shows, by the remarks following the definition of the Lopatinskii-Shapiro Condition **(LS)**, that **(LS)** holds for the transformed problem $(\mathcal{A}^\Phi(x,D),\mathcal{B}_1^\Phi(x,D),\ldots,\mathcal{B}_m^\Phi(x,D))$ if and only it holds for the original problem.

(c) As the boundary spaces for the half-space are transformed to the corresponding boundary spaces on the bent half-space, these considerations show that the main result for the half-space, Theorem 6.2.11 as well as the estimate (6.34) remain valid for bent half-spaces.

3.3 Localization
If $\Omega \subset \mathbb{R}^n$ is unbounded, i.e., an exterior domain, we choose a large ball $B(0,R) \supset \Omega^c$ and define $U_0 = \mathbb{R}^n \setminus \bar{B}(0,R)$. If Ω is bounded then $U_0 = \emptyset$. We cover the compact set $\partial\Omega \subset \mathbb{R}^n$ by balls $B(x_k,r/2)$ with $x_k \in \partial\Omega$, $k=1,\ldots,N_1$, such that each part $\partial\Omega \cap B(x_k,2r)$ of the boundary $\partial\Omega$ can be parameterized by a function $h_k \in C^{2m}$ as a C^{2m}-graph over the tangent space $T_{x_k}\partial\Omega$. We extend this function h_k to a global function on $T_{x_k}\partial\Omega$ by a cut-off procedure, and denote the resulting bent half-space by H_k. This is possible by the regularity assumption $\partial\Omega \in C^{2m}$ as well as by compactness of $\partial\Omega$. We set $U_k = B(x_k,r)\cap\Omega$, $k=1,\ldots,N_1$. We cover the compact set $\bar{\Omega}\setminus\cup_{k=0}^{N_1}U_k$ by finitely many balls $B(x_k,r/2)$, $k=N_1+1,\ldots,N_2$, and set $U_k = B(x_k,r)$. Then $\{U_k\}_{k=0}^{N_2}$ is a finite open covering of $\bar{\Omega}$. Fix a C^∞-partition of unity $\{\varphi_k\}_{k=1}^{N_2}$ subordinate to the open covering $\{U_k\}_{k=0}^{N_2}$ of $\bar{\Omega}$, and let ψ_k denote C^∞-functions with $\psi_k = 1$ on $\operatorname{supp}\varphi_k$, $\operatorname{supp}\psi_k \subset U_k$.

To define local operators $\mathcal{A}^k(x,D)$ and $\mathcal{B}_j^k(x,D)$ we proceed as follows. For the interior charts $k=0$, $k=N_1+1,\ldots,N_2$, we define the coefficients of $\mathcal{A}^k(x,D)$ by reflection of the top order coefficients at the boundary of U_k. This is the same trick as in Section 6.1.4. For the boundary charts $k=1,\ldots,N_1$ we first transform the top order coefficients of $\mathcal{A}(x,D)$ and $\mathcal{B}_j(x,D)$ in U_k to a half-space, extend them as in the Section 6.2.4, and then transform them back to the bent half space H_k.

Having defined the local differential operators, we may proceed as in Section 6.2.4, introducing local problems for the functions $u^k = \varphi_k u$, which for the interior charts $k=0$, and $k=N_1+1,\ldots,N_2$ are problems on $\mathbb{R}^n$, and for the boundary charts $k=1,\ldots,N_1$ are problems on the bent half-spaces H_k. For the latter, instead of using Theorem 6.2.10 we employ the extension of Theorem 6.2.11 to bent half-spaces. This completes the proof of Theorem 6.3.2.

3.4 The Semigroup
To define the semigroup associated with (6.45), we introduce the base space $X_0 := L_q(\Omega;E)$, as well as the operator A by means of

$$(Au)(x) := \mathcal{A}(x,D)u(x), \quad x\in\Omega,$$
$$u \in \mathsf{D}(A) := \{u \in H_q^{2m}(\Omega;E);\ \mathcal{B}_j(x,D)u = 0 \text{ on } \partial\Omega,\ j=1,\ldots,m\},$$

and we set $X_1 = \mathsf{D}(A)$ equipped with the graph norm. Then the problem

$$\dot{u} + Au = f, \quad t>0, \quad u(0)=u_0,$$

has maximal L_p-regularity, by Theorem 6.3.2, hence $\omega_0 + A \in \mathcal{MR}_p(X_0)$, for some $\omega_0 > 0$, and so $-A$ generates an analytic C_0-semigroup in X_0, by Proposition 3.5.2. This implies that $\omega + A$ is $\mathcal{R}$-sectorial for all $\omega > \mathsf{s}(-A)$, the spectral bound of $-A$. We note that the time-trace space $X_{\gamma,\mu}$ is given by

$$X_{\gamma,\mu} = \{u \in B^{2m(\mu-1/p)}_{qp}(\Omega;E);\ \mathcal{B}_j(x,D)u = 0,\ \text{if}\ \kappa_j > 1/p+1-\mu,\ j=1,\ldots,m\},$$

where we exclude the degenerate cases $\kappa_j = 1/p + 1 - \mu$.

To determine the smallest value ω_0 in Theorem 6.3.2, we fix some large number ω_1 and solve (6.45) with ω replaced by ω_1 which results in some function $\bar{u} \in \mathbb{E}_{1\mu}$. Setting $\tilde{u} = u - \bar{u}$, the new function $\tilde{u}$ must solve the problem

$$\begin{aligned} \partial_t\tilde{u} + \omega\tilde{u} + \mathcal{A}(x,D)\tilde{u} &= (\omega_1 - \omega)\bar{u} && \text{in } \Omega,\\ \mathcal{B}_j(x,D)\tilde{u} &= 0 && \text{on } \partial\Omega,\ j=1,\ldots,m,\\ \tilde{u}(0) &= 0 && \text{in } \Omega, \end{aligned}$$

for $t > 0$. But this means

$$\dot{\tilde{u}} + \omega\tilde{u} + A\tilde{u} = (\omega_1 - \omega)\bar{u}, \quad t > 0, \quad \tilde{u}(0) = 0,$$

and so we see that $\omega > \mathsf{s}(-A)$ is sufficient, i.e., $\omega_0 = \mathsf{s}(-A)$.

3.5 Higher Order Space Regularity

In many problems maximal L_p-regularity in $H^s_q(\Omega;E)$ is required, where $s > 0$. In this subsection we consider the case $s = 1$, and comment later on other values of s. By localization, coordinate transformation and perturbation, it is again enough to restrict to the half-space case with constant coefficients. We have to distinguish two cases, namely (i) $m_j \geq 1$ for all j, and (ii) $m_j = 0$ for at least one j. We begin with the first case.

(i) $m_j \geq 1$ for all $j = 1,\ldots,m$.

This case is the easy one. So suppose that we have a solution of (6.24) in the class

$$u \in H^1_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+;E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m+1}_q(\mathbb{R}^n_+;E)). \tag{6.47}$$

Then necessarily

$$f \in L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+;E)), \quad u_0 \in B^{2m(\mu-1/p)+1}_{qp}(\mathbb{R}^n_+;E),$$

and

$$D^\beta u \in H^{1-k/2m+1/2m}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+;E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m+1-k}_q(\mathbb{R}^n_+;E)),$$

for $|\beta| = k$; hence

$$g_j \in F^{\kappa_j+1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1};E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j+1}_{qq}(\mathbb{R}^{n-1};E)),$$

and the compatibility conditions

$$\mathcal{B}_j(D)u_0 = g_j(0), \quad \kappa_j > 1/p + 1 - \mu - 1/2m,\ j = 1, \ldots, m,$$

are satisfied.

Conversely, let data (f, g_j, u_0) with these properties be given, and let $\mathcal{A}(D)$ be normally elliptic and assume that $(\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}(D))$ satisfies the Lopatinskii-Shapiro condition. Then we can show that (6.24) admits a unique solution in the class (6.47). In fact, extending f and u_0 to all of $\mathbb{R}^n$, we obtain a solution of the full-space problem in the right class. Thus we may restrict attention to the case $(f, u_0) = 0$. Looking at the crucial equation for the half-space (6.32), we see that the solution in this case has regularity (6.47), as we may multiply $\tilde{g}$ in (6.32) by ρ.

Obviously, for variable coefficients and general domains with compact boundary we need to require additional smoothness of the coefficients and Ω. These turn out to be

(rA1+) $a_\alpha \in C_l(\bar{\Omega}; \mathcal{B}(E))$ for each $|\alpha| = 2m$;

(rA2+) $a_\alpha \in H^1_{r_k}(\Omega; \mathcal{B}(E)) + W^1_\infty(\Omega; \mathcal{B}(E))$ for each $|\alpha| = k \le 2m$, with $r_k \ge q$ and $2m + 1 - k > n/r_k$;

(rB+) $b_{j\beta} \in B^{2m\kappa_j+1}_{r_{jk}q}(\partial\Omega; \mathcal{B}(E))$ for each $|\beta| = k \le m_j$, with $r_{jk} \ge q$, and $2m\kappa_j + 1 > (n-1)/r_{jk}$.

With these assumptions, we have the following result which parallels Theorem 6.3.2.

Theorem 6.3.3. *Let $\Omega \subset \mathbb{R}^n$ be open with compact boundary $\partial\Omega$ of class C^{2m+1}, $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and let E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), \ldots \mathcal{B}_m(x, D))$ is uniformly normally elliptic, and satisfies* **(rA1+)**, **(rA2+)** **(rB+)**. *Let $\kappa_j \ne 1/p + 1 - \mu - 1/2m$ for all j, and $m_j \ge 1$.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, equation (6.45) *admits a unique solution u in the class*

$$u \in \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m+1}_q(\Omega; E)),$$

if and only if the data are subject to the following conditions.

(a) *$f \in L_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega; E))$, $u_0 \in B^{2m(\mu-1/p)+1}_{qp}(\Omega; E)$;*

(b) *$g_j \in F^{\kappa_j+1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\partial\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j+1}_{qq}(\partial\Omega; E))$;*

(c) *$\mathcal{B}_j(D)u_0 = g_j(0)$ if $\kappa_j > 1/p + 1 - \mu - 1/2m$, $j = 1, \ldots, m$.*

The solution depends continuously on the data in the corresponding spaces.

(ii) $m_j = 0$, for some j.
So let for simplicity $\mathcal{B}_1(D) = I$, a Dirichlet condition, and $m_j \ge 1$ for $j = 2, \ldots, m$. This case is more involved than **(i)**, as an additional compatibility condition shows

up. In fact, we have $\kappa_1 + 1/2m = 1 + (1 - 1/q)/2m > 1$, hence $\partial_t u$ has a time trace on the boundary, which by taking the time derivative of the first boundary condition yields

$$\partial_t g_1 = \partial_t u = f_{|\partial\Omega} - [\mathcal{A}(D)u]_{|\partial\Omega}.$$

This suggests

$$g_1 \in H^1_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m+1-1/q}_{qq}(\mathbb{R}^{n-1}; E)).$$

On the other hand, we have

$$\mathcal{A}(D)u \in H^{1/2m}_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^n_+; E)),$$

which yields for its trace on $\partial\Omega$

$$[\mathcal{A}(D)u]_{|\partial\Omega} \in F^{(1-1/q)/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)).$$

This implies the additional regularity

$$\partial_t g_1 - f_{|\partial\Omega} \in F^{(1-1/q)/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m+1-1/q}_{qq}(\mathbb{R}^{n-1}; E)),$$

and the additional compatibility condition

$$\partial_t g_1(0) + [\mathcal{A}(D)u_0]_{|\partial\Omega} = f(0)_{|\partial\Omega}, \quad \text{if } (1-1/q)/2m > 1/p + 1 - \mu.$$

The regularity and compatibility of g_j for $j \geq 2$ is the same as in **(i)**, and $g_1(0) = u_0$ on $\partial\Omega$ must be satisfied, as well.

Having worked out these *higher order compatibilities*, we now may proceed as in **(i)** to see that these conditions yield also sufficiency for solutions of (6.24) in the class (6.47).

Theorem 6.3.4. *Let $\Omega \subset \mathbb{R}^n$ be open with compact boundary $\partial\Omega$ of class C^{2m+1}, $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and let E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), \dots \mathcal{B}_m(x, D))$ is uniformly normally elliptic, and satisfies* **(rA1+)**, **(rA2+)**, **(rB1+)**, *for $j = 2, \dots, m$. Let $\kappa_j \neq 1/p + 1 - \mu - 1/2m$ for all $j \geq 1$. Further assume that $\mathcal{B}_1(x, D)u = u$, i.e., $\mathcal{B}_1$ is a Dirichlet boundary condition.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, equation (6.45) *admits a unique solution u in the class*

$$u \in \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m+1}_q(\Omega; E)),$$

if and only if the data are subject to the following conditions.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; H^1_q(\Omega; E))$, $u_0 \in B^{2m(\mu-1/p)+1}_{qp}(\Omega; E)$;

(b) $g_j \in F^{\kappa_j+1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\partial\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j+1}_{qq}(\partial\Omega; E))$;

(c) $\mathcal{B}_j(D)u_0 = g_j(0)$ *if* $\kappa_j > 1/p + 1 - \mu - 1/2m$, $j = 1, \dots, m$;

(d) $\partial_t g_1 - f_{|\partial\Omega} \in F^{(1-1/q)/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E))$;

(f) $\partial_t g_1(0) + [\mathcal{A}(D)u_0]_{|\partial\Omega} = f(0)_{|\partial\Omega}$, *if* $(1-1/q)/2m > 1/p + 1 - \mu$.

The solution depends continuously on the data in the corresponding spaces.

(iii) General $s > 0$.
Extending the observations in **(i)** and **(ii)**, we are able to study solutions in the class

$$u \in H^1_{p,\mu}(\mathbb{R}_+; H^s_q(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m+s}_q(\mathbb{R}^n_+; E)), \tag{6.48}$$

for any $s > 0$ excluding the special values $s_i = m_i + 1/q$, and imposing the natural additional regularities $\partial\Omega \in C^{2m+s}$, as well as

$$a_\alpha \in H^s_{r_k}(\Omega) + H^s_\infty(\Omega), \quad r_k \geq q,\ 2m + s - k > n/r_k,\ 0 \leq |\alpha| = k \leq 2m,$$

and

$$b_{j\beta} \in B^{2m\kappa_j+s}_{r_{jk}q}(\partial\Omega), r_{jk} \geq q,\ 2m\kappa_j + s > (n-1)/r_{jk},\ 0 \leq |\beta| = k \leq m_j,$$

and imposing the higher order compatibilities as explained above. More precisely, let $m_1^0 < m_2^0 < \ldots < m^0_{i_{max}}$ be defined by the different orders m_j. Then for $0 \leq s < m_1^0 + 1/q$ we have no higher order compatibilities, for $m_1^0 + 1/q < s < m_2^0 + 1/q$ we have first (time-) order compatibilities, and with increasing s the number and the order of these higher compatibilities increases, whenever s crosses one the exceptional numbers s_i. So if s is large, this leads to a very complicated set of higher order compatibilities, which one clearly would like to avoid.

As a summary, in parabolic problems, such higher order compatibilities do not occur if $s < \min\{m_j\} + 1/q$, i.e., if the time derivatives of the boundary conditions do not have a space trace. For second-order problems this means in the Dirichlet case if $s < 1/q$, and in the Neumann case if $s < 1 + 1/q$.

(iv) The elliptic case.
Finally, we note that for *elliptic problems* this phenomenon does not occur. If $f \in H^s_q(\Omega)$ and $g_j \in B^{2m\kappa_j+s}_{qq}(\partial\Omega)$, then the solution of the elliptic problem

$$(\omega + \mathcal{A}(x, D)u = f \text{ in } \Omega, \quad \mathcal{B}_j(x, D)u = g_j \text{ on } \partial\Omega,\ j = 1, \ldots, m$$

has a unique solution in $H^{s+2m}_q(\Omega)$, provided $\mathcal{A}(x, D)$ is normally elliptic, the Lopatinskii-Shapiro condition holds, $\omega > \mathsf{s}(-A)$, $\partial\Omega \in C^{2m+s}$, and the coefficients satisfy the regularity conditions in **(iii)**.

3.6 Lower Order Space Regularity
In many problems, maximal L_p-regularity in $H^s_p(\Omega; E)$ is required, where $s < 0$. In this subsection we consider the case $s = -1$, i.e., we want to consider *weak solutions*. By localization, coordinate transformation and perturbation, it is again

enough to prove the results for the half-space case with constant coefficients. For all of this, we make the structural assumption

$$\mathcal{A}(x,D) = -i\sum_{\ell=1}^{n} \partial_\ell \mathcal{A}_\ell(x,D) = -i \operatorname{div} \mathsf{A}(x,D),$$

where $\mathcal{A}_\ell(x,D) = \sum_{|\alpha|\le 2m-1} a_{\ell\alpha}(x)D^\alpha$ are differential operators of order $2m-1$. We have to distinguish two cases:

(i) $m_j \le 2m-2$ for all $j = 1,\dots,m$.

(ii) $m_j \le 2m-2$ for all $j = 1,\dots,m-1$, but $m_m = 2m-1$; in this case we require

$$\mathcal{B}_m(x,D) = i\nu \cdot \mathsf{A}(x,D).$$

We begin with the first case.

(i) $m_j \le 2m-2$ for all $j = 1,\dots,m$.
We assume that $\mathcal{A}$ is normally elliptic, and that the system $(\mathcal{A},\mathcal{B}_1,\dots,\mathcal{B}_m)$ satisfies the Lopatinskii-Shapiro condition. The operator

$$\mathrm{Grad}_0 : {}_0H^1_{q'}(\Omega) \to L_{q'}(\Omega;\mathbb{C}^n), \quad \mathrm{Grad}_0\,\phi := \nabla\phi,$$

is well-defined, linear, bounded, and injective. Therefore, its dual

$$\mathrm{Div}_0 = -\mathrm{Grad}_0^* : L_q(\Omega;\mathbb{C}^n) \to {}_0H^1_{q'}(\Omega)^* =: H^{-1}_q(\Omega)$$

is also well-defined, bounded and has dense range. Note that in case Ω is bounded, by the Poincaré inequality $\mathsf{R}(\mathrm{Grad}_0)$ is closed, and hence Div_0 is surjective. Problem (6.45) can now be rewritten as

$$\begin{aligned} \partial_t(u|\phi)_\Omega + \omega(u|\phi)_\Omega + i(\mathsf{A}(x,D)u|\nabla\phi)_\Omega &= (f|\phi)_\Omega, \ \ \phi \in {}_0H^1_{q'}(\Omega)\\ \mathcal{B}_j(x,D)u = g_j &\qquad \text{on } \partial\Omega,\ j=1,\dots,m,\\ u(0) = u_0 &\qquad \text{in } \Omega. \end{aligned} \tag{6.49}$$

Abstractly, the first equation in (6.49) can be written as

$$\partial_t u + \omega u - i\,\mathsf{Div}_0(\mathsf{A}(x,D)u) = f \quad \text{in} \ \ H^{-1}_q(\Omega;E).$$

So we are looking for solutions in the class

$$u \in H^1_{p,\mu}(\mathbb{R}_+; H^{-1}_q(\Omega;E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m-1}_q(\Omega;E)). \tag{6.50}$$

This implies the following necessary regularity conditions for the data.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; H^{-1}_q(\Omega;E))$, $\quad u_0 \in B^{2m(\mu-1/p)-1}_{qp}(\Omega;E)$;

(b) $g_j \in F^{\kappa_j-1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\partial\Omega;E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j-1}_{qq}(\Omega;E))$, for all $j = 1, \ldots, m$. Here we require $1 \geq \mu > 1/p + 1/2m$. The compatibility conditions now read

$$\mathcal{B}_j(x, D)u_0 = g_j(0), \quad \kappa_j > 1/p + 1 - \mu + 1/2m, \ j = 1, \ldots, m.$$

The assumptions on the coefficients are changed slightly, they read

(rA1-) $a_{\ell\alpha} \in C_l(\bar{\Omega}; \mathcal{B}(E))$, $\ell = 1, \ldots, n$, $|\alpha| = 2m - 1$;

(rA2-) $a_{\ell\alpha} \in [L_{r_k} + L_\infty](\Omega; \mathcal{B}(E))$, $\ell = 1, \ldots, n$, $k = |\alpha| < 2m - 1$, with $r_k \geq q$, $2m - k > n/r_k$;

(rB-) $b_{j\beta} \in B^{2m\kappa_j - 1}_{r_{jk}q}(\partial\Omega; \mathcal{B}(E))$, $|\beta| = k \leq m_j$, with $r_{jk} \geq q$, and $2m\kappa_j - 1 > (n-1)/r_{jk}$.

Finally, in this situation we only need to require $\partial\Omega \in C^{2m-1}$ (in case $m > 1$ it is even enough to require $\partial\Omega \in C^{(2m-1)-}$).

Theorem 6.3.5. *Let $\Omega \subset \mathbb{R}^n$ be open with compact boundary $\partial\Omega$ of class C^{2m-1}, $1 < p, q < \infty$, $\mu \in (1/p, 1]$, and let E be a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), \ldots \mathcal{B}_m(x, D))$, with $\mathcal{A}(x, D) = -i\sum_{\ell=1}^n \partial_\ell \mathcal{A}_\ell(x, D)$, is uniformly normally elliptic, and* **(rA1-)**, **(rA2-)** *and* **(rB-)**. *Let $m_j \leq 2m-2$ and $\kappa_j \neq 1/p + 1 - \mu + 1/2m$ for all j.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, equation (6.45) *admits a unique solution u in the class*

$$u \in \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+; H^{-1}_q(\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m-1}_q(\Omega; E)),$$

if and only if the data are subject to the following conditions.

(a) $f \in \mathbb{E}_{0\mu} = L_{p,\mu}(\mathbb{R}_+; H^{-1}_q(\Omega; E))$, $u_0 \in B^{2m(\mu-1/p)-1}_{qp}(\Omega; E)$;

(b) $g_j \in \mathbb{F}_{j\mu} = F^{\kappa_j - 1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\partial\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j-1}_{qq}(\partial\Omega; E))$;

(c) $\mathcal{B}_j(D)u_0 = g_j(0)$ *if* $\kappa_j > 1/p + 1 - \mu - 1/2m$, $j = 1, \ldots, m$.

The solution depends continuously on the data in the corresponding spaces.

(ii) $m_j \leq 2m - 2$ for all $j = 1, \ldots, m-1$, $m_m = 2m - 1$.
In this case, as has been said before, we only consider $\mathcal{B}_m = i\nu \cdot \mathsf{A}$. Here we set

$$\mathsf{Grad} : \tilde{H}^1_{q'}(\Omega) \to L_{q'}(\Omega; \mathbb{C}^n), \quad \mathsf{Grad}\,\phi := \nabla\phi,$$

where $\tilde{H}$ means factorization over the constants, and we define

$$-\mathsf{Div} := \mathsf{Grad}^* : L_q(\Omega; \mathbb{C}^n) \to {}_0H^{-1}_q(\Omega) := \tilde{H}^1_{q'}(\Omega)^*.$$

As Grad is bounded, linear, injective, its dual Div is bounded, linear, and has dense range. Note that in case Ω is bounded, by the Poincaré-Wirtinger inequality

$\mathsf{R}(\mathsf{Grad})$ is closed, and hence Div is surjective. Problem (6.45) with f replaced by f_0 can now be rewritten as

$$\begin{aligned} \partial_t(u|\phi)_\Omega + \omega(u|\phi)_\Omega + i(\mathsf{A}(x,D)u|\nabla\phi)_\Omega = \langle f|\phi\rangle,\ \ \phi \in \tilde{H}^1_{q'}(\Omega), \\ \mathcal{B}_j(x,D)u = g_j \ \text{ on } \partial\Omega,\ j=1,\ldots,m-1, \\ u(0) = u_0 \ \text{ in } \Omega, \end{aligned} \tag{6.51}$$

with the function $f \in L_{p,\mu}(\mathbb{R}_+; {}_0H^{-1}_q(\Omega;E))$ defined by

$$\langle f|\phi\rangle := (f_0|\phi)_\Omega + (g_m|\phi)_{\partial\Omega}.$$

Abstractly, the first equation in (6.49) can be written as

$$\partial_t u + \omega u - i\,\mathsf{Div}(\mathsf{A}(x,D)u) = f \quad \text{in } {}_0H^{-1}_q(\Omega).$$

So we are looking for solutions in the class

$$u \in H^1_{p,\mu}(\mathbb{R}_+; {}_0H^{-1}_q(\Omega;E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m-1}_q(\Omega;E)). \tag{6.52}$$

The necessary regularity conditions on the data (g_j, u_0) as well as the compatibility and regularity conditions on the coefficients are the same as in **(i)**, where here $j = 1,\ldots,m-1$. The condition for f changes in an obvious way.

Theorem 6.3.6. *Let $\Omega \subset \mathbb{R}^n$ be open with compact boundary $\partial\Omega$ of class C^{2m-1}, $1 < p,q < \infty$, $\mu \in (1/p, 1]$, and let E a Banach space of class $\mathcal{HT}(\alpha)$. Assume that $(\mathcal{A}(x,D), \mathcal{B}_1(x,D), \ldots \mathcal{B}_m(x,D))$, with $\mathcal{A}(x,D) = -i\sum_{\ell=1}^n \partial_\ell \mathcal{A}_\ell(x,D)$ and $\mathcal{B}_m(x,D) = i\nu \cdot \mathsf{A}(x,D)$, is uniformly normally elliptic, and* **(rA1-)**, **(rA2-)**, **(rB-)**, *$m_j \le 2m-2$, $\kappa_j \ne 1/p + 1 - \mu + 1/2m$ for $j = 1,\ldots,m-1$.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, equation (6.45) *admits a unique solution u in the class*

$$u \in \mathbb{E}_{1\mu} := H^1_{p,\mu}(\mathbb{R}_+; {}_0H^{-1}_q(\Omega;E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m-1}_q(\Omega;E)),$$

if and only if the data are subject to the following conditions.

(a) $f \in L_{p,\mu}(\mathbb{R}_+; {}_0H^{-1}_q(\Omega;E))$, $u_0 \in B^{2m(\mu-1/p)-1}_{qp}(\Omega;E)$;

(b) $g_j \in F^{\kappa_j - 1/2m}_{pq,\mu}(\mathbb{R}_+; L_q(\partial\Omega;E)) \cap L_{p,\mu}(\mathbb{R}_+; B^{2m\kappa_j - 1}_{qq}(\partial\Omega;E))$, $j = 1,\ldots,m-1$;

(c) $\mathcal{B}_j(D)u_0 = g_j(0)$ *if* $\kappa_j > 1/p - 1 - \mu + 1/2m$, $j = 1,\ldots,m-1$.

The solution depends continuously on the data in the corresponding spaces.

(iii) Sufficiency of the conditions in Theorems 6.3.5 and 6.3.6 for the half-space case with constant coefficients.
We first reduce to the case $(f, u_0) = 0$ in the usual way: extend $u_0 \in B^{2m(\mu-1/p)-1}_{qp}(\Omega)$ to all of $\mathbb{R}^n$ and f trivially by zero in case **(i)** and symmetrically in case **(ii)**. Solve the resulting problem in $\mathbb{R}^n$ in the proper class, and

subtract this function from u. Then we consider the central identity (6.32) in the form

$$\rho^{2m-1}v = M(y, \rho, b, \sigma)\tilde{g}/\rho,$$

to see that the solution has regularity (6.50) in case **(i)** and (6.52) for **(ii)**. As a result, $\mathsf{A}(D)u \in L_{p,\mu}(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E^n))$, hence by construction

$$\partial_t u = i\,\mathrm{div}\,\mathsf{A}(D)u = i\,\mathsf{Div}_0\mathsf{A}(D)u \in L_{p,\mu}(\mathbb{R}_+; H^{-1}_q(\mathbb{R}^n_+; E)),$$

in case **(i)**, and similarly in case **(ii)** we have

$$\partial_t u = i\,\mathrm{div}\,\mathsf{A}(D)u = i\,\mathsf{Div}\mathsf{A}(D)u \in L_{p,\mu}(\mathbb{R}_+; {}_0H^{-1}_q(\mathbb{R}^n_+; E)).$$

(iv) The corresponding analytic C_0-semigroups.
Having maximal L_p-regularity of the problems (6.49) and (6.51) at our disposal, we may now argue as in Section 6.3.4 to derive the corresponding analytic C_0-semigroups in $H^{-1}_q(\Omega; E)$ resp. in ${}_0H^{-1}_q(\Omega; E)$. We omit the details here, however, note that these semigroups yield also corresponding semigroups in $L_q(\Omega; E)$, defining A_0 as the part of A in $L_q(\Omega; E)$. Note that $\mathsf{D}(A) \subset H^{2m-1}_q(\Omega; E)$, but $\mathsf{D}(A_0)$ is not explicitly known. Therefore it is an interesting question how the spectra of these extensions change, in particular the spectral bound. Then as $L_q(\Omega; E) \subset H^{-1}_q(\Omega; E)$, it is easy to see that $\rho(A) \subset \rho(A_0)$. But the converse is also true. In fact, suppose $f \in H^{-1}_q(\Omega; E)$ is given and $\lambda \in \rho(A_0)$. Set $J_\varepsilon = (I + \varepsilon A)^{-1}$; then $f_\varepsilon = J_\varepsilon f \in H^{2m-1}_q(\Omega; E)$ and $f_\varepsilon \to f$ in $H^{-1}_q(\Omega; E)$ as $\varepsilon \to 0$. Let $u_\varepsilon = (\lambda - A_0)^{-1} f_\varepsilon$, and choose ω large. Then we have

$$\begin{aligned} u_\varepsilon &= (\omega + A_0)^{-1}[-f_\varepsilon + (\omega + \lambda)u_\varepsilon] \\ &= -(\omega + A_0)^{-1} f_\varepsilon + (\omega + \lambda)(\lambda - A_0)^{-1}(\omega + A_0)^{-1} f_\varepsilon \\ &= -(\omega + A)^{-1} f_\varepsilon + (\omega + \lambda)(\lambda - A_0)^{-1}(\omega + A)^{-1} f_\varepsilon, \end{aligned}$$

as $(\omega + A)^{-1} f_\varepsilon = (\omega + A_0)^{-1} f_\varepsilon$. But this implies

$$u_\varepsilon \to u := (-I + (\omega + \lambda)(\lambda - A_0)^{-1})(\omega + A)^{-1} f.$$

Since $\mathsf{D}(A_0) \subset \mathsf{D}(A)$, we obtain $u \in \mathsf{D}(A)$ and then $(\lambda - A)u = f$. Hence $\lambda \in \rho(A)$. Therefore $\rho(A) = \rho(A_0)$ in case **(i)**, and by the same argument also in case **(ii)**.

6.4 Elliptic and Parabolic Problems on Hypersurfaces

Suppose that Σ is a compact hypersurface without boundary in $\mathbb{R}^n$ of class C^l. It is the purpose of this section to derive solvability results for elliptic and parabolic problems on Σ.

Let $\mathcal{A} : C^m(\Sigma; E) \to C(\Sigma; E)$ be a linear operator, where E denotes a Banach space of class $\mathcal{HT}$. Then $\mathcal{A}$ is a differential operator of order m on Σ if all representations of $\mathcal{A}$ in local coordinates (U, φ) are given by

$$\varphi_* \mathcal{A} u = \mathcal{A}_{(U,\varphi)}(x, D)\varphi_* u := \sum_{|\alpha| \le m} a^\alpha_{(U,\varphi)}(x) D^\alpha \varphi_* u, \tag{6.53}$$

where the coefficients $a^\alpha_{(U,\varphi)}$ are defined on the open set $\varphi(U)$ in $\mathbb{R}^{n-1}$, and $\varphi_* v = v \circ \varphi^{-1}$. $\mathcal{A}$ is said to be of class C^k if all coefficients are in this class. We may assume that the charts are normalized in such a way that $\varphi(U) = B_{\mathbb{R}^{n-1}}(0, 1)$.

The typical examples we have in mind, and which are used below, are the negative Laplace-Beltrami operator $-\Delta_\Sigma$ and Δ^2_Σ; see Section 2.1. A more involved operator is

$$\mathcal{A} = -\mathrm{div}_\Sigma(a(x)\nabla_\Sigma), \quad a \in C^1(\Sigma; \mathcal{B}(T\Sigma \otimes E)).$$

By using the language of covariant derivatives one can show that a differential operator defined on Σ is completely determined by the local representations (6.53).

Definition 6.4.1. *A differential operator $\mathcal{A}$ of order m on Σ is called* parameter-elliptic *if all local representations $\mathcal{A}_{(U,\varphi)}$ have this property. This means that for any local representation $\mathcal{A}_{(U,\varphi)}$ there is $\phi < \pi$ such that*

$$\sigma(\mathcal{A}^\#_{(U,\varphi)}(x, \xi)) \subset \Sigma_\phi, \quad (x, \xi) \in B_{\mathbb{R}^{n-1}}(0, 1) \times \mathbb{S}^{n-1}, \tag{6.54}$$

where

$$\mathcal{A}^\#_{(U,\varphi)}(x, \xi) := \sum_{|\alpha| = m} a^\alpha_{(U,\varphi)}(x)\xi^\alpha, \quad (x, \xi) \in B_{\mathbb{R}^{n-1}}(0, 1) \times \mathbb{S}^{n-1}.$$

By compactness, we then obtain

$$\phi_\mathcal{A} = \sup_{(U,\varphi)} \inf\{\phi \in (0, \pi) : (6.54) \textit{ holds}\} < \pi.$$

$\phi_\mathcal{A}$ is called the angle of ellipticity *of $\mathcal{A}$. Finally, $\mathcal{A}$ is called normally elliptic if it is parameter-elliptic with angle $\phi_\mathcal{A} < \pi/2$.*

It is not difficult to show that the definition of the angle of ellipticity $\phi_\mathcal{A}$ is independent of the local representations. Moreover, $\mathcal{A}_{(U,\varphi)}(x, \xi)$ is continuous and invertible, hence by compactness of Σ, $\mathcal{A}_{(U,\varphi)}(x, \xi)$ as well as $\mathcal{A}_{(U,\varphi)}(x, \xi)^{-1}$ are uniformly bounded on $B_{\mathbb{R}^{n-1}}(0, 1) \times \mathbb{S}^{n-1}$.

By compactness of Σ we find a family of charts $\{(U_j, \varphi_j) : 1 \le j \le N\}$ such that $\{U_j\}_{j=1}^N$ covers Σ. Let $\{\pi_j : 1 \le j \le N\} \subset C^l(\Sigma)$ be a family of functions on Σ such that $\{(U_j, \pi_j^2) : 1 \le j \le N\}$ is a partition of unity subordinate to the open cover $\{U_j : 1 \le j \le N\}$, i.e.,

$$\mathrm{supp}(\pi_j) \subset U_j, \quad \sum_{j=1}^N \pi_j^2 = 1 \quad \text{on } \Sigma. \tag{6.55}$$

Then we call $\{(U_j, \varphi_j, \pi_j) : 1 \le j \le N\}$ a *localization system* for Σ.

Definition 6.4.2. *Given a localization system $\{(U_j, \varphi_j, \pi_j) : 1 \le j \le N\}$ for Σ, let*

$$\begin{aligned} &R^c : L_1(\Sigma; E) \to L_1(\mathbb{R}^{n-1}; E)^N, \quad R^c u := (\psi_j^*(\pi_j u)), \\ &R : L_1(\mathbb{R}^{n-1}; E)^N \to L_1(\Sigma; E), \quad R((u_j)) := \sum_{j=1}^{N} \pi_j \varphi_j^* u_j, \end{aligned} \tag{6.56}$$

where $\varphi_j^* v := v \circ \varphi$ and $\psi_j := \varphi_j^{-1}$. Moreover, we set

$$\mathcal{A}_j := \mathcal{A}_{(U_j, \varphi_j)}(x, D), \quad 1 \le j \le N. \tag{6.57}$$

We extend the coefficients in the usual way (e.g. as in Section 6.2) to obtain an extension of $\mathcal{A}_j$ to all of $\mathbb{R}^{n-1}$ with coefficients which have a limit at infinity, so that we may apply the results of Section 6.1.

It follows that R is a *retraction* with R^c a *co-retraction*, i.e., we have

$$RR^c u = u, \quad u \in L_1(\Sigma; E). \tag{6.58}$$

In the sequel, we set $\mathsf{u} = R^c u$, so $R\mathsf{u} = u$. Moreover,

$$\psi_j^* \mathcal{A} u = \mathcal{A}_j \psi_j^* u, \quad 1 \le j \le N,$$

and

$$\psi_j^* \pi_j \mathcal{A} u = \mathcal{A}_j \psi_j^* \pi_j u + \psi_j^* [\pi_j, \mathcal{A}] u =: \mathcal{A}_j \psi_j^* \pi_j u + B_j u.$$

Set $\mathsf{A} = \mathrm{diag}[\mathcal{A}_j]$ and $\mathsf{B} = [B_j R]$; then we obtain with (6.58)

$$R^c(\lambda + \omega + \mathcal{A}) u = (\lambda + \omega + \mathsf{A} + \mathsf{B}) \mathsf{u}. \tag{6.59}$$

By Theorem 6.1.10, $\omega + \mathcal{A}_j$ is $\mathcal{R}$-sectorial in $L_q(\mathbb{R}^{n-1}; E)$ for ω sufficiently large, $j = 1, \ldots, N$, and $\omega + \mathsf{A}$ is $\mathcal{R}$-sectorial for such ω as well. As B_j are of lower order, it follows by perturbation arguments (choosing ω even larger) that

$$\lambda + \omega + \mathsf{A} + \mathsf{B} : H_q^m(\mathbb{R}^{n-1}; E)^N \to L_q(\mathbb{R}^{n-1}; E)^N, \quad \lambda \in \Sigma_\phi,$$

is invertible, and $\lambda(\lambda + \omega + \mathsf{A} + \mathsf{B})^{-1}$ is $\mathcal{R}$-bounded in $L_q(\mathbb{R}^{n-1}; E)^N$, where $\phi > \phi_{\mathcal{A}}$ is fixed. Therefore, the operators

$$L_{\lambda,\omega} := R(\lambda + \omega + \mathsf{A} + \mathsf{B})^{-1} R^c : L_q(\Sigma; E) \to H_q^m(\Sigma; E), \quad \lambda \in \Sigma_\phi, \tag{6.60}$$

are well-defined, and with (6.58) and (6.59) we obtain

$$L_{\lambda,\omega}(\lambda + \omega + \mathcal{A}) u = RR^c u = u, \quad u \in H_q^m(\Sigma; E),$$

i.e., $L_{\omega,\lambda}$ is a left-inverse for $(\lambda + \omega + \mathcal{A})$ and in addition, the family $\{L_{\lambda,\omega}\}_{\lambda \in \Sigma_\phi}$ is $\mathcal{R}$-bounded in $L_q(\Sigma)$.

On the other hand, we also have

$$\mathcal{A}(\pi_j\varphi_j^* u_j) = \pi_j\varphi_j^*\mathcal{A}_j u_j + \varphi_j^*[\mathcal{A}_j, \psi_j^*\pi_j]u_j =: \pi_j\varphi_j^*\mathcal{A}_j u_j + C_j u_j$$

and this yields

$$(\lambda+\omega+\mathcal{A})R\mathsf{u} = R(\lambda+\omega+\mathsf{A}+\mathsf{C})\mathsf{u}, \quad \mathsf{C}\mathsf{u} := R^c\sum_{j=1}^N C_j u_j. \tag{6.61}$$

For ω sufficiently large, we can again conclude that

$$\lambda+\omega+\mathsf{A}+\mathsf{C} : H_q^m(\mathbb{R}^{n-1};E))^N \to L_q(\mathbb{R}^{n-1};E)^N, \quad \lambda\in\Sigma_\phi,$$

is invertible, and hence

$$R_{\lambda,\omega} := R(\lambda+\omega+\mathsf{A}+\mathsf{C})^{-1}R^c$$

is well-defined. It follows from (6.58) and (6.61) that

$$(\lambda+\omega+\mathcal{A})R_{\lambda,\omega}u = RR^c u = u, \quad u\in H_q^m(\Sigma;E),$$

and this shows that $R_{\lambda,\omega}$ is a right-inverse for $\lambda+\omega+\mathcal{A}$. This implies

$$R_{\lambda,\omega} = L_{\lambda,\omega} = (\lambda+\omega+\mathcal{A})^{-1},$$

and $\{\lambda(\lambda+\omega+\mathcal{A})^{-1} : \lambda\in\Sigma_\phi\} \subset \mathcal{B}(L_q(\Sigma))$ is $\mathcal{R}$-bounded. Therefore $\omega+\mathcal{A}$ is $\mathcal{R}$-sectorial, which in case $\phi_{\mathcal{A}} < \pi/2$ implies, by Theorems 4.4.4 and 3.5.4, $\mathcal{A}\in\mathcal{MR}_{p,\mu}(L_q(\Sigma))$ for all $p,q\in(1,\infty)$, $1/p<\mu\le 1$.

Replacing in the above arguments the base space $L_q(\Sigma;E)$ by $K_q^s(\Sigma;E)$ and the regularity space $H_q^m(\Sigma;E)$ by $K_q^{s+m}(\Sigma;E)$, where $K=H$ or $K=W$, we obtain the same result, provided we have the corresponding result in $\mathbb{R}^{n-1}$. Employing Section 6.1.5, this yields the following maximal regularity result.

Theorem 6.4.3. *Let Σ be a compact hypersurface of class C^l without boundary in $\mathbb{R}^n$, $3\le l\le\infty$, $E\in\mathcal{HT}$, and let $p,q\in(1,\infty)$, $\mu\in(1/p,1]$. Suppose that $\mathcal{A}$ is a differential operator on Σ of order $m\in\mathbb{N}$ with coefficients in C^{2k}, where $k\in\mathbb{N}$, $2k+m\le l$. Define the realization A of $\mathcal{A}$ in $K_q^s(\Sigma;E)$ by means of*

$$Au := \mathcal{A}u \;\text{ on }\; \Sigma, \quad \mathrm{u}\in\mathsf{D}(\mathrm{A}) := \mathrm{K_q^{s+m}(\Sigma;E)},$$

where $K\in\{H,W\}$, $|s|\le 2k$, $s\notin\mathbb{N}_0$ for $K=W$. Then we have

(i) *Suppose that $\mathcal{A}$ is parameter-elliptic. Then there is $\omega_0\ge 0$ such that the equation*

$$(\lambda+\omega+A)u = f \quad \text{in } K_q^s(\Sigma;E)$$

admits a unique solution $u\in K_q^{s+m}(\Sigma;E)$ for each $\omega\ge\omega_0$ and each $f\in K_q^s(\Sigma;E)$. For any $\phi>\phi_{\mathcal{A}}$ there is a constant M_ϕ such that the resolvent estimate

$$|\lambda(\lambda+\omega+A)^{-1}|_{\mathcal{B}(K_q^s(\Sigma;E))} \le M_\phi, \quad \lambda\in\Sigma_\phi,\ \omega\ge\omega_0,\ |s|\le 2k,$$

is valid. In addition, we have $\omega_0 + A \in \mathcal{RS}(K^s_q(\Sigma;E))$ *with* $\phi^R_A \leq \phi_{\mathcal{A}}$.

(ii) *Suppose that* $\mathcal{A}$ *is normally elliptic. Then there is* $\omega_0 \geq 0$ *such that the equation*

$$(\partial_t + \omega + A)u = f, \quad t > 0, \quad u(0) = 0,$$

admits a unique solution $u \in H^1_{p,\mu}(\mathbb{R}_+ K^s_q(\Sigma;E)) \cap L_{p,\mu}(\mathbb{R}_+; K^{s+m}_q(\Sigma;E))$ *for each* $\omega \geq \omega_0$ *and each* $f \in L_{p,\mu}(\mathbb{R}_+; K^s_q(\Sigma;E))$. *Moreover, there is a constant* $C > 0$ *independent of* ω *and* s *such that*

$$\omega|u|_{L_{p,\mu}(K^s_q)} + |\partial_t u|_{L_{p,\mu}(K^s_q)} + |u|_{L_{p,\mu}(K^{s+m}_q)} \leq C|u|_{L_{p,\mu}(K^s_q)},$$

for all $f \in L_{p,\mu}(K^s_q(\Sigma;E))$. *In particular,* $\omega_0 + A \in \mathcal{MR}_p(K^s_q(\Sigma;E))$.

This result will be used frequently below, to understand moving boundaries analytically via the Hanzawa transform, and to handle dynamics on moving interfaces.

6.5 Transmission Problems

Elliptic and parabolic transmission conditions are present everywhere in mathematical physics, but one hardly finds citable references on this topic in the literature. For this reason, and also since we need results on transmission problems below, we consider such problems here, restricting to the second-order but vector-valued case.

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2-boundary, consisting of two parts Ω_1 and Ω_2 which are also open and such that that Ω_1 is separated from the boundary of Ω. Then we call Ω_2 the *continuous phase* and Ω_1 the *disperse phase*. Let $\Sigma = \partial\Omega_1$ be the *interface* separating Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Sigma \cup \Omega_2$. This is the typical two-phase situation. We consider in this section the following *transmission problem*.

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(x,\nabla_x))u &= f && \text{in } \Omega\setminus\Sigma,\\ \mathcal{B}(x,\nabla_x)u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] = g_\Sigma, \quad [\![\mathcal{B}(x,\nabla_x)u]\!] &= g && \text{on } \Sigma,\\ u(0) &= u_0 && \text{on } \Omega \end{aligned} \tag{6.62}$$

for $t > 0$. Here u lives in a finite-dimensional Hilbert space E and

$$\mathcal{A}(x,\nabla_x) = -\mathrm{div}(a(x)\nabla_x), \quad \mathcal{B}(x,\nabla_x) = -(\nu(x)|a(x)\nabla_x),$$

where $\nu(x)$ denotes the outer unit normal at $x \in \Sigma$ directed into the interior of Ω_2 (resp. the outer unit normal of Ω at $x \in \partial\Omega$) and $a \in C^1_{ub}(\Omega\setminus\Sigma;\mathcal{B}(E))^{n\times n}$. The data (f, g_Σ, g, u_0) are given.

The purpose of this section is to prove the following result.

Theorem 6.5.1. *Let $1 < p, q < \infty$ and $1 \geq \mu > 1/p$, let E be a finite-dimensional Hilbert space, and assume that $a \in C^1_{ub}(\Omega \setminus \Sigma; \mathcal{B}(E))^{n\times n}$ is uniformly normally strongly elliptic.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, problem (6.62) admits exactly one solution u in the class

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Omega \setminus \Sigma; E)),$$

if and only if

(a) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\Omega; E))$;

(b) $g_\Sigma \in F^{1-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma; E)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_q(\Sigma; E))$;

(c) $g \in F^{1/2-1/2q}_{pq,\mu}(\mathbb{R}_+; L_q(\Sigma; E)) \cap L_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_q(\Sigma; E))$;

(d) $u_0 \in B^{2\mu-2/p}_{qp}(\Omega \setminus \Sigma; E)$;

(e) $[\![u_0]\!] = g_\Sigma(0)$ *for* $\mu > 3/2p$, *and* $[\![\mathcal{B}(x, \nabla)u_0]\!] = g(0)$ *for* $\mu > 1/2 + 3/2p$.

The solution map is continuous between the corresponding spaces.

The next subsections deal with the proof of this result.

5.1 The Model Problem

We consider the constant coefficient case with flat interface $\Sigma = \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$, and $\Omega = \mathbb{R}^n \setminus \Sigma$. As before, it is convenient to replace the variable $x \in \mathbb{R}^n$ by $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then the problem reads

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(\nabla_x + \nu\partial_y))u = f, &\quad y \neq 0,\\ [\![u]\!] = g_\Sigma, \quad [\![\mathcal{B}(\nabla_x + \nu\partial_y)u]\!] = g, &\quad y = 0,\\ u(0) = u_0, &\quad y \neq 0, \end{aligned} \tag{6.63}$$

for $t > 0$, with $\nu = e_n$ the outer unit normal of $\Omega_1 = \mathbb{R}^n_-$. We first verify the Lopatinskii-Shapiro condition for this case. For this purpose let $u \in L_2(\mathbb{R}; E)$ be a solution of the ode-problem

$$\lambda u(y) + \mathcal{A}(i\xi + \nu\partial_y)u(y) = 0, \quad y \neq 0,$$

such that

$$[\![u]\!] = 0, \quad [\![\mathcal{B}(i\xi + \nu\partial_y)u]\!] = 0 \quad \text{for } y = 0.$$

Here $\operatorname{Re}\lambda \geq 0$, $\xi \in \mathbb{R}^n$ and $(\xi|\nu) = 0$. Taking the inner product with $u(y)$, integrating over $\mathbb{R}$, and employing an integration by parts we obtain

$$0 = \lambda|u|_2^2 + \int_{\mathbb{R}} \sum_{k,l=1}^n (a^{kl}(\xi_l u(y) - i\nu_l\partial_y u(y)|(\xi_k u(y) - i\nu_k\partial_y u(y))_E\, dy,$$

as the boundary terms disappear by the jump conditions. Taking real parts, by normal strong ellipticity this yields

$$\operatorname{Re}(a^{kl}(\xi_l u(y) - i\nu_l\partial_y u(y)|(\xi_k u(y) - i\nu_k\partial_y u(y))_E = 0, \quad y \neq 0.$$

Using normal strong ellipticity once more we obtain

$$\partial_y |u(y)|_E^2 = 2\mathrm{Re}(\partial_y u(y)|u(y))_E = 0, \quad y \neq 0,$$

hence u is constant on $(0,\infty)$ and also on $(-\infty,0)$ which implies $u=0$ as $u \in L_2(\mathbb{R};E)$ by assumption. Thus the Lopatinskii-Shapiro condition for the two-phase problem is valid.

To obtain solvability of the problem in the right regularity class, perform a transformation to the half-space case as follows. Set

$$\begin{aligned} \tilde{u}(t,x,y) &= [u(t,x,y), u(t,x,-y)]^{\mathsf{T}}, & \tilde{u}_0(x,y) &= [u_0(x,y), u_0(x,-y)]^{\mathsf{T}}, \\ \tilde{f}(t,x,y) &= [f(t,x,y), f(t,x,-y)]^{\mathsf{T}}, & &\text{for } t \in (0,\infty),\ x \in \mathbb{R}^{n-1},\ y \in (0,\infty), \end{aligned}$$

and consider the problem

$$\begin{aligned} (\partial_t + \omega + \tilde{\mathcal{A}}(\nabla_x + \nu\partial_y))\tilde{u} &= \tilde{f} \quad \text{in } \mathbb{R}^n_+, \\ \tilde{u}(0) &= \tilde{u}_0 \quad \text{on } \mathbb{R}^{n-1}, \end{aligned} \tag{6.64}$$

with $t>0$, where $\tilde{\mathcal{A}}(\nabla_x + \nu\partial_y) = \mathrm{diag}[\mathcal{A}_2(\nabla_x+\nu\partial_y), \mathcal{A}_1(\nabla_x - \nu\partial_y)]$, with subscripts $2,1$ referring to the coefficients in the upper resp. lower half-plane. The boundary conditions now become

$$\begin{aligned} \tilde{u}_2(t,x,0) - \tilde{u}_1(t,x,0) &= g_\Sigma(t,x), \\ \mathcal{B}_2(\nabla_x+\nu\partial_y)\tilde{u}_2(t,x,0) + \mathcal{B}_1(\nabla_x+\nu\partial_y)\tilde{u}_1(t,x,0) &= g(t,x). \end{aligned}$$

Then with these boundary conditions, (6.64) is normally strongly elliptic and satisfies the Lopatinskii-Shapiro condition for the half-space. By the results of the previous section this problem is uniquely solvable in the right class, hence the transmission problem (6.63) has this property as well. This proves Theorem 6.5.1 for the constant coefficient case with flat interface.

5.2 Proof of Theorem 6.5.1

To complete the proof of Theorem 6.5.1, we may now proceed as in the one-phase case.

1. By perturbation, the result for the flat interface with constant coefficients remains valid for variable coefficients with small deviation from constant ones.

2. By another perturbation argument, a proper coordinate transformation transfers the result to the case of a bent interface.

3. The localization technique finally yields the result for the case of general domains and general coefficients.

One may then employ perturbation arguments another time to include lower order terms, at the expense of possibly enlarging ω_0.

5.3 The Steady Case

A result like Theorem 6.5.1 also holds for the steady case, i.e., for elliptic transmission problems. We consider here the corresponding result for the problem

$$\begin{aligned} (\omega + \mathcal{A}(x,\nabla_x))u &= f \quad \text{in } \Omega\setminus\Sigma,\\ \mathcal{B}(x,\nabla)u &= 0 \quad \text{on } \partial\Omega,\\ [\![u]\!] = g_\Sigma, \quad [\![\mathcal{B}(x,\nabla)u]\!] &= g \quad \text{on } \Sigma. \end{aligned} \tag{6.65}$$

Here the data (f, g_Σ, g) are given. For this problem we have

Theorem 6.5.2. *Let $1 < p < \infty$, let E be a finite-dimensional Hilbert space, and assume that $a \in C^1_{ub}(\Omega\setminus\Sigma;\mathcal{B}(E))^{n\times n}$ is uniformly normally strongly elliptic.*

Then there is $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, problem (6.65) admits exactly one solution u in the class

$$u \in H^2_p(\Omega\setminus\Sigma;E),$$

if and only if $(f, g_\Sigma, g) \in L_p(\Omega;E)\times W^{2-1/p}_p(\Sigma;E)\times W^{1-1/p}_p(\Sigma;E)$. The solution map is continuous between the corresponding spaces.

Remark 6.5.3. Higher regularity can be obtained for transmission problems in the same way as in Section 6.3.5 for the one-phase case, whereas lower regularity is obtained in the same way as in Section 6.3.6.

A natural question which arises is to determine the minimal value of ω_0. For this purpose, we first solve (6.65) for a large value $\omega = \bar\omega$, to obtain a function $\bar u$. Then we set $\tilde u = u - \bar u$; $\tilde u$ then must satisfy the problem

$$\begin{aligned} (\omega + \mathcal{A}(x,\nabla_x))\tilde u &= (\bar\omega - \omega)\bar u \quad \text{in } \Omega\setminus\Sigma,\\ \mathcal{B}(x,\nabla)\tilde u &= 0 \quad \text{on } \partial\Omega\setminus\Sigma,\\ [\![\tilde u]\!] = 0, \quad [\![\mathcal{B}(x,\nabla)\tilde u]\!] &= 0 \quad \text{on } \Sigma. \end{aligned} \tag{6.66}$$

This means that $-\omega$ should belong to the resolvent set of the operator A in $L_p(\Omega;E)$ defined by

$$Au(x) = \mathcal{A}(x,\nabla_x)u(x), \quad x \in \Omega\setminus\Sigma, \tag{6.67}$$
$$\mathsf{D}(A) = \{u \in H^2_p(\Omega\setminus\Sigma;E) : [\![u]\!] = [\![\mathcal{B}(x,\nabla_x)u]\!] = 0 \text{ on } \Sigma,\ \mathcal{B}(x,\nabla_x)u = 0 \text{ on } \partial\Omega\}.$$

In virtue of Theorem 6.5.1, this operator has maximal L_p-regularity, hence $-A$ generates an analytic C_0-semigroup. Therefore, ω_0 is the spectral bound $\mathsf{s}(-A)$ of $-A$. By a similar argument, the same is valid for the number ω_0 in Theorem 6.5.1.

5.4 Dirichlet-to-Neumann Operators

Dirichlet-to-Neumann operators appear frequently in mathematical physics and also at several places in this book. Such operators map Dirichlet boundary data to Neumann boundary data in several possible ways, and the goal is to obtain properties of such maps. In this subsection we assume throughout that $\mathcal{A}(x, \nabla_x)$ is uniformly normally strongly elliptic and that $\mathcal{B}$ is the corresponding co-normal derivative, as in the previous subsections.

(i) We begin with the elliptic case. Here there are two types of Dirichlet-to-Neumann operators, namely one- and two-phase operators. In the following, we always consider the elliptic problem

$$\begin{aligned} (\omega + \mathcal{A}(x, \nabla_x))u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\ \mathcal{B}(x, \nabla_x)u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.68}$$

where at first $\omega \geq 0$ is sufficiently large. We may now assign Dirichlet data on the interface.

$$[\![u]\!] = 0, \quad u = g \quad \text{on } \Sigma, \tag{6.69}$$

to obtain a unique solution $u \in H^2_p(\Omega \setminus \Sigma; E)$ provided $g \in W^{2-1/p}_p(\Sigma; E)$. These are actually two one-phase problems, one in Ω_1 and one in Ω_2. We then may compute the Neumann-boundary values $\mathcal{B}(x, \nabla_x)u$ on either side of Σ. We set $u_k = u|_{\Omega_k}$ for $k = 1, 2$ in the following definition.

Definition 6.5.4. *We call the maps $S_k : W^{2-1/p}_p(\Sigma; E) \to W^{1-1/p}_p(\Sigma; E)$ defined by the one-sided traces of the conormal derivative at Σ*

$$S_1 g := -\mathcal{B}(x, \nabla_x)u_1|_\Sigma, \quad S_2 g := \mathcal{B}(x, \nabla_x)u_2|_\Sigma,$$

the **one-phase Dirichlet-to-Neumann operators** *of* (6.68)–(6.69).

The operators S_k for $k = 1, 2$ are well-defined whenever the corresponding boundary value problem (6.68) with Dirichlet condition on Σ is well-posed. Clearly, S_k only depends on Ω_k, so that these operators are really one-phase. Considering (6.68) in Ω_k with Neumann condition $\mathcal{B}(x, \nabla_x)u = h$ on Σ, it becomes apparent that each S_k, $k = 1, 2$, is invertible if the corresponding boundary value problem with Neumann condition on Σ is well-posed. So in this situation S_1 and S_2 are isomorphisms.

On the other hand, there are two typical two-phase Dirichlet-to-Neumann operators for (6.68). The first one, called S_{d}, is obtained by solving the transmission problem

$$\begin{aligned} (\omega + \mathcal{A}(x, \nabla_x))u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\ \mathcal{B}(x, \nabla_x)u &= 0 \quad \text{on } \partial\Omega, \\ [\![u]\!] = 0, \quad u &= g \quad \text{on } \Sigma, \end{aligned} \tag{6.70}$$

and setting $S_{\mathsf{d}} g := [\![\mathcal{B}(x, \nabla_x)u]\!]$. Actually we have $S_{\mathsf{d}} = S_1 + S_2$, as the normals of Ω_k on Σ have opposite directions. To obtain the inverse of S_{d}, one has to solve problem (6.68) with transmission conditions

$$[\![u]\!] = 0, \quad [\![\mathcal{B}(x, \nabla_x)u]\!] = h \quad \text{on } \Sigma,$$

yielding $g = u_{|\Sigma} = S_{\mathsf{d}}^{-1} h$. Hence S_{d} is an isomorphism as well.

To define the second two-phase Dirichlet-to-Neumann operator S_{n} we solve the transmission problem

$$\begin{aligned} (\omega + \mathcal{A}(x,\nabla_x))u = 0 \quad & \text{in } \Omega\setminus\Sigma, \\ \mathcal{B}(x,\nabla_x)u = 0 \quad & \text{on } \partial\Omega, \\ [\![u]\!] = g, \quad [\![\mathcal{B}(x,\nabla_x)u]\!] = 0 \quad & \text{on } \Sigma, \end{aligned} \tag{6.71}$$

and set $S_{\mathsf{n}} g := \mathcal{B}(x,\nabla_x)u$. To obtain the inverse of S_{n} we have to solve (6.68) with boundary condition

$$[\![\mathcal{B}(x,\nabla_x)u]\!] = 0, \quad \mathcal{B}(x,\nabla_x)u = h \quad \text{on } \Sigma,$$

yielding $g = [\![u]\!] = S_{\mathsf{n}}^{-1} h$. An easy computation shows the relation

$$S_{\mathsf{n}} = S_1 S_{\mathsf{d}}^{-1} S_2 = S_2 S_{\mathsf{d}}^{-1} S_1.$$

The two-phase Dirichlet-to-Neumann operators

$$S_{\mathsf{d}}, S_{\mathsf{n}} : W_p^{2-1/p}(\Sigma; E) \to W_p^{1-1/p}(\Sigma; E)$$

are well-defined and at the same time isomorphisms if ω is large enough. Observe that S_k, $k \in \{1, 2, \mathsf{d}, \mathsf{n}\}$, are pseudo-differential operators of order 1, while S_k^{-1} typically are integral operators on Σ with weakly singular kernels.

(ii) In the parabolic case one proceeds similarly. We begin with the problem in the bulk

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(x,\nabla_x))u = 0 \quad & \text{in } \Omega\setminus\Sigma, \\ \mathcal{B}(x,\nabla_x)u = 0 \quad & \text{on } \partial\Omega, \\ u(0) = 0 \quad & \text{in } \Omega, \end{aligned} \tag{6.72}$$

with $t > 0$. Here we have to distinguish the case of a finite interval $J = [0, a]$, from that of the half-line $J = \mathbb{R}_+$. We concentrate on the case of the half-line and assume $\omega \geq 0$ to be sufficiently large. For a finite interval $J = [0, a]$, no restrictions on $\omega \in \mathbb{R}$ are necessary. To avoid compatibility conditions here, we assume initial value $u(0) = 0$.

Imposing conditions on Σ as for the elliptic case in (i), we obtain the corresponding parabolic Dirichlet-to-Neumann operators, which we call again S_k, for $k \in \{1, 2, \mathsf{d}, \mathsf{n}\}$. The same assertions as in (i) are valid, but now the spaces are of course also time-dependent. We have isomorphisms

$$\begin{aligned} S_k : {}_0W_{p,\mu}^{1-1/2p}(\mathbb{R}_+; & L_p(\Sigma; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2-1/p}(\Sigma; E)) \\ & \to {}_0W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+; L_p(\Sigma; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{1-1/p}(\Sigma; E)) \end{aligned}$$

for $k \in \{1, 2, \mathsf{d}, \mathsf{n}\}$, provided ω is sufficiently large. Note that in this case S_k are pseudo-differential operators jointly in time and space, of order 1/2 in time and order 1 in space. These assertions remain valid if $\mathcal{A}$ and $\mathcal{B}$ are perturbed by lower order operators, at the expense that one possibly has to enlarge ω.

(iii) We now look closer at the possible values of ω. If $\mathcal{A}(x,\nabla_x) = -\partial_i a^{ij}(x)\partial_j$ and $\mathcal{B}(x,\nabla_x) = -\nu_i(x)a^{ij}(x)\partial_j$ such that $\mathcal{A}(x,\nabla_x)$ is normally strongly elliptic, uniformly in $x \in \Omega$ and $a^{ij} \in C^1_{ub}(\Omega \setminus \Sigma; \mathcal{B}(E))$, then $\omega > 0$ is sufficient. This follows from the fact that, as E is finite-dimensional, $\mathcal{A}(x,\nabla_x)$ with Neumann condition on $\partial\Omega$ and with each of the interface conditions (6.69), (6.70), (6.71) has compact resolvent, hence its spectrum consists only of discrete eigenvalues of finite multiplicity, and is independent of $p \in (1,\infty)$. By the standard energy argument it follows that the corresponding spectral bounds are in each case 0. The case $\omega = 0$ is more involved, as 0 is an eigenvalue. We postpone this case to Chapter 10, where $\omega = 0$ is essential.

6.6 Linearized Stefan Problems

The following linear problem is essential for the understanding of Problems **(P1)**, **(P3)**, **(P5)** and many other problems with moving interface. For its formulation, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^2. As before, we assume that Ω consists of two parts, Ω_1 and Ω_2 such that $\Sigma = \partial\Omega_1$ does not touch $\partial\Omega$. We assume that the hypersurface Σ is a C^3-manifold in $\mathbb{R}^n$. Note that in this section $E = \mathbb{C}$. Consider

$$\begin{aligned}
(\partial_t + \omega + \mathcal{A}(x,\nabla_x))u &= f_u && \text{in } \Omega \setminus \Sigma,\\
\mathcal{B}(x,\nabla_x)u &= 0 && \text{on } \partial\Omega,\\
[\![u]\!] = 0, \quad u - \mathcal{C}(x,\nabla_\Sigma)h &= g && \text{on } \Sigma,\\
(\partial_t + \omega)h + [\![\mathcal{B}(x,\nabla_x)u]\!] &= f_h && \text{on } \Sigma,\\
u(0) = u_0 \ \text{ in } \Omega, \quad h(0) &= h_0 && \text{on } \Sigma.
\end{aligned} \tag{6.73}$$

for $t > 0$. Here $\omega \geq 0$,

$$\mathcal{A}(x,\nabla_x) = -\mathrm{div}(a(x)\nabla),\ \mathcal{B}(x,\nabla_x) = -\nu(x)\cdot a(x)\nabla_x,\ \mathcal{C}(x,\nabla_\Sigma) = -\mathrm{div}_\Sigma(c(x)\nabla_\Sigma).$$

We assume that the coefficients $a \in C^1_{ub}(\Omega \setminus \Sigma; \mathcal{B}(\mathbb{R}^n))$ and $c \in C^3(\Sigma; \mathcal{B}(T\Sigma))$ are symmetric and uniformly positive definite. Note that the coefficients of $\mathcal{A}$ are allowed to jump across the interface Σ. The unit normal $\nu(x)$ at $x \in \Sigma$ is pointing from Ω_1 into Ω_2.

For Problems **(P1)**, **(P3)**, and **(P5)**, the prototype operators will be $\mathcal{A} = -\Delta$, $\mathcal{B} = -\partial_\nu$ and $\mathcal{C} = -\Delta_\Sigma$. The main result for this problem in the L_p-setting, $3 < p < \infty$, is the following.

Theorem 6.6.1. *Let $p > 3$ and $1 \geq \mu > 1/2 + 3/2p$. There exists $\omega_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, Problem (6.73) admits exactly one solution (u,h) in the class*

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_p(\Omega \setminus \Sigma)) =: \mathbb{E}_u,$$

$$h \in W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{4-1/p}_p(\Sigma)) =: \mathbb{E}_h,$$

if and only if the data (f_u, g, f_h, u_0, h_0) *are subject to the following conditions:*

(a) $f_u \in L_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) =: \mathbb{F}_u$;

(b) $g \in W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_p(\Sigma)) =: \mathbb{F}$;

(c) $f_h \in W^{1/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_p(\Sigma)) =: \mathbb{F}_h$;

(d) $u_0 \in W^{2\mu-2/p}_p(\Omega \setminus \Sigma)$, $h_0 \in W^{2+2\mu-3/p}_p(\Sigma)$;

(e) $u_0 - \mathcal{C}(x, \nabla_\Sigma)h_0 = g(0)$, $[\![\mathcal{B}(x, \nabla_x)u_0]\!] - f_h(0) \in W^{4\mu-2-6/p}_p(\Sigma)$, $\mathcal{B}(x, \nabla_x)u_0 = 0$ *on* $\partial\Omega$.

The solution map is continuous between the corresponding spaces.

6.1 Solution Spaces

To show necessity of the conditions in Theorem 6.6.1 and to explain the choice of the space for h which is illustrated in Figure 6.1, we begin with the regularity of u, which is the desired regularity in the bulk phases $\Omega \setminus \Sigma$. So let $(u, h) \in \mathbb{E}_u \times \mathbb{E}_h$ be a solution of (6.73). Then $f_u \in \mathbb{F}_u$ and the trace theory for second-order parabolic problems yields $u_0 \in W^{2\mu-2/p}_p(\Omega \setminus \Sigma)$, and

$$\begin{aligned} u_{|\Sigma} &\in W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_p(\Sigma)) = \mathbb{F}, \\ \nabla u_{|\Sigma} &\in W^{1/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma))^n \ \cap L_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_p(\Sigma))^n = \mathbb{F}^n_h. \end{aligned}$$

This implies (a), and it is natural to assume $\mathcal{C}(\nabla_\Sigma)h \in \mathbb{F}$ as well, which then implies (b) and suggests

$$h \in W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{4-1/p}_p(\Sigma)).$$

Example 3.4.9(iii) then yields $h_0 \in W^{2+2\mu-3/p}_p(\Sigma)$. Looking at the equation for h this implies $f_h \in \mathbb{F}_h$, hence (c), and suggests

$$h \in W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap H^1_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_p(\Sigma)).$$

By Example 4.5.16(ii) we have

$$W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \hookrightarrow H^1_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)), \quad (6.74)$$

and we arrive at the natural space $\mathbb{E}_h$ for h.

The first compatibility condition in (e) is obviously necessary if the corresponding traces exist, i.e., if $2\mu > 3/p$. The second compatibility condition is somewhat hidden, coming from the trace of $\partial_t h$. In fact we have by (6.74) and Example 3.4.9(ii)

$$W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \hookrightarrow C^1_{ub}(\mathbb{R}_+; W^{4\mu-2-6/p}_p(\Sigma)),$$

hence the trace of $\partial_t h$ at $t = 0$ exists if $\mu > 1/2 + 3/2p$. This yields the second compatibility condition in (e). Note that the time trace of the class $\mathbb{F}_h$ merely

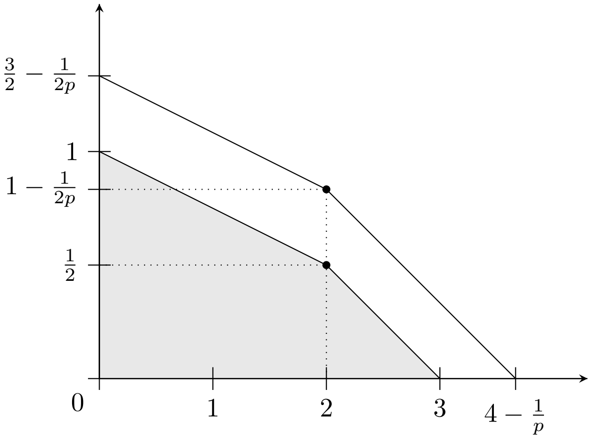

Figure 6.1: Regularity diagram for the Stefan problem.

belongs to $W_p^{2\mu-1-3/p}(\Sigma)$, as follows from Example 3.4.9(i). We remark that later on for the nonlinear problems we even have to require $\mu > 1/2+(n+2)/2p$, hence we cannot avoid this compatibility condition. The next subsections deal with the proof of sufficiency in Theorem 6.6.1.

6.2 Reductions

It is convenient to reduce problem (6.73) to the homogeneous conditions $(u_0, h_0, f_u, g) = 0$ and $f_h \in {}_0\mathbb{F}_h$, to simplify the problem and in particular to trivialize the compatibility conditions. For this purpose we define the operators $A = 1+\omega-\Delta_\Sigma$ and $B = 1+\omega+\Delta_\Sigma^2$; these are negative generators of exponentially stable analytic C_0-semigroups with maximal L_p-regularity on $L_p(\Sigma)$, hence also on $H_p^s(\Sigma)$ and on $W_p^s(\Sigma)$. We then define

$$\bar{h}(t) = (2e^{-At} - e^{-2At})h_0 + (e^{-Bt} - e^{-2Bt})B^{-1}h_1,$$

where $h_0 \in W_p^{2+2\mu-3/p}(\Sigma)$ and $h_1 = f_h(0) - [\![\mathcal{B}(x, \nabla_x)u_0]\!] - \omega h_0 \in W_p^{4\mu-2-6/p}(\Sigma)$. Obviously we have

$$\bar{h}(0) = h_0, \quad (\partial_t + \omega)\bar{h}(0) = h_1 + \omega h_0,$$

hence $\tilde{h} = h - \bar{h}$ has vanishing traces at $t = 0$.

We have to show that $\bar{h}$ belongs to $\mathbb{E}_h$. For this purpose we only need to consider the functions $e^{-At}h_0$ and $e^{-Bt}h_1$.

(i) Choosing as a base space $X_0 = H_p^2(\Sigma)$, Proposition 3.4.3 yields

$$e^{-At}h_0 \in W_{p,\mu}^{1-1/2p}(\mathbb{R}_+; H_p^2(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{4-1/p}(\Sigma)) \;\Leftrightarrow\; h_0 \in W_p^{2+2\mu-3/p}(\Sigma).$$

This then implies

$$\partial_t e^{-At}h_0 = -Ae^{-At}h_0 \in W_{p,\mu}^{1-1/2p}(\mathbb{R}_+; L_p(\Sigma)),$$

which yields $e^{-At}h_0 \in \mathbb{E}_h$.
(ii) Next we look at $e^{-Bt}B^{-1}h_1$ in the base space $X_0 = L_p(\Sigma)$. Proposition 3.4.3 yields

$$e^{-Bt}h_1 \in W^{1/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)) \;\Leftrightarrow\; h_1 \in W^{4\mu-2-6/p}_p(\Sigma).$$

This implies

$$e^{-Bt}B^{-1}h_1 \in W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap H^1_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{6-2/p}_p(\Sigma)),$$

which is easily seen to embed into $\mathbb{E}_h$.

Having the function $\bar h$ at our disposal, we solve the problem

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(x,\nabla_x))\bar u &= f_u && \text{in } \Omega\setminus\Sigma,\\ \mathcal{B}(x,\nabla_x)\bar u &= 0 && \text{on } \partial\Omega,\\ [\![\bar u]\!] = 0, \quad \bar u - \mathcal{C}(x,\nabla_\Sigma)\bar h &= g && \text{on } \Sigma,\\ \bar u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

in the class $\mathbb{E}_u$. Then the pair $(\tilde u, \tilde h) = (u - \bar u, h - \bar h)$ must satisfy (6.73) with data $(f_u, g, u_0, h_0) = 0$ and f_h replaced by $\tilde f_h$, defined by

$$\tilde f_h = f_h - [\![\mathcal{B}(x,\nabla_x))\bar u]\!] - (\partial_t + \omega)\bar h \in {}_0\mathbb{F}_h.$$

6.3 The Boundary Symbol

In this subsection we consider the constant coefficient case in $\Omega = \mathbb{R}^n$ with flat interface $\Sigma = \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$. This means that we consider the problem

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(\nabla_x))u &= f_u && \text{in } \dot{\mathbb{R}}^n,\\ [\![u]\!] = 0, \quad u - \mathcal{C}(\nabla_\Sigma)h &= g && \text{on } \mathbb{R}^{n-1},\\ (\partial_t + \omega)h + [\![\mathcal{B}(\nabla_x)u]\!] &= f_h && \text{on } \mathbb{R}^{n-1},\\ u(0) = u_0 \;\text{ in } \dot{\mathbb{R}}^n,\; h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{6.75}$$

Here once more we use the notation $\dot{\mathbb{R}}^n = \mathbb{R}^{n-1} \times \dot{\mathbb{R}}$. As explained in the previous subsection, we may assume $(f_u, g, u_0, h_0) = 0$. We want to show that this problem admits a unique solution $h \in \mathbb{E}_h$ once we have $f_h \in {}_0\mathbb{F}_h$; then u is determined by its boundary value $u_\Sigma = \mathcal{C}(\nabla_x)h$ as explained in the previous subsection. It is also convenient to replace the variable $x \in \mathbb{R}^n$ by $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, which means that we split into the tangential variable x and the normal variable y.

Taking Laplace transforms in time and Fourier transforms in the tangential variables we obtain the problem

$$\begin{aligned} (\lambda + a(\xi,\xi))\tilde u - 2ia(\xi,\nu)\partial_y \tilde u - a(\nu,\nu)\partial_y^2 \tilde u &= 0, && y > 0,\\ [\![\tilde u]\!] = 0, \quad \tilde u - \mathcal{C}(\xi)\tilde h &= 0, && y = 0,\\ \lambda \tilde h - [\![a(\nu,\nu)\partial_y \tilde u + ia(\xi,\nu)\tilde u]\!] &= \tilde f_h, && y = 0, \end{aligned} \tag{6.76}$$

where the tilde indicates Laplace transform in t with λ the co-variable of $\partial_t+\omega$ and Fourier transform in the tangential variable x with co-variable ξ. Here we employed the notation $\nu = e_n$ for the normal at the interface; observe that $\xi \perp \nu$. Note that the coefficients of $\mathcal{A}(\nabla_x)$ may jump across the interface. As the forms a_k, $k = 1, 2$ defining $\mathcal{A}(\nabla_x)$ are real symmetric and positive-definite, given $u_\Sigma = \mathcal{C}(\xi)\tilde{h}$, we may solve the equations in the region $y \neq 0$ to the result

$$\tilde{u}(y) = e^{-yr_2(\lambda,\xi)}u_\Sigma, \quad y > 0,$$

and

$$\tilde{u}(y) = e^{yr_1(\lambda,\xi)}u_\Sigma, \quad y < 0.$$

The symbols $r_k$are defined by $r_k(\lambda,\xi) = a_k(\nu|\nu)^{-1}[n_k(\lambda,\xi)+(-1)^k ia_k(\xi,\nu)]$, with

$$n_k(\lambda,\xi) = \sqrt{(\lambda + a_k(\xi,\xi))a_k(\nu,\nu) - a_k(\xi,\nu)^2}, \quad k = 1, 2.$$

This implies

$$-[\![a(\nu,\nu)\partial_y\tilde{u} + ia(\xi,\nu)\tilde{u}]\!] = \big(n_1(\lambda,\xi) + n_2(\lambda,\xi)\big)u_\Sigma.$$

For the equation on the boundary this yields

$$s(\lambda,\xi)\tilde{h} = \tilde{f}_h, \quad \text{with} \quad s(\lambda,\xi) = \lambda + \mathcal{C}(\xi)\big(n_1(\lambda,\xi) + n_2(\lambda,\xi)\big). \tag{6.77}$$

So the main task is to show that this boundary symbol is invertible, and to obtain lower bounds of the form

$$|s(\lambda,\xi)| \geq c(|\lambda| + |\xi|^2\sqrt{\lambda + |\xi|^2}|), \quad \lambda \in \Sigma_{\pi/2}, \quad \xi \in \mathbb{R}^{n-1}.$$

Observe that a multiple of the lower bound in the line above yields trivially also an upper bound for $s(\lambda,\xi)$. Actually, as $|a_k(\xi,\nu)|^2 \leq a_k(\xi,\xi)a_k(\nu,\nu)$, with equality only if ξ and ν are linearly dependent - which is not possible as $\xi \perp \nu$ - this is very easy since the second and third terms in the definition of $s(\lambda,\xi)$ lie in the sector $\Sigma_{\pi/4}$ if $\lambda \in \Sigma_{\pi/2}$, and $\mathcal{C}(\xi)$ is positive and scales like $|\xi|^2$. As a consequence, the symbol

$$m(\lambda,\xi) := \frac{\lambda + |\xi|^2\sqrt{\lambda + |\xi|^2}}{s(\lambda,\xi)}$$

is bounded from above and below even on a larger set

$$\lambda \in \Sigma_{\pi/2+\varepsilon}, \quad \xi \in \Sigma_\varepsilon^{n-1} \cup -\Sigma_\varepsilon^{n-1},$$

and it is a holomorphic function in λ and ξ. Therefore, m satisfies the scalar Mikhlin-condition w.r.t. ξ, uniformly w.r.t. $\lambda \in \Sigma_{\pi/2+\varepsilon}$. Inverting the Fourier transform, we obtain a holomorphic family of operators $M(\lambda)$ on $L_p(\mathbb{R}^{n-1})$, hence also on $W_p^s(\mathbb{R}^{n-1})$ for any real number s. The Kalton-Weis Theorem implies that $M(\partial_t + \omega)$ is bounded in each space ${}_0H^m_{p,\mu}(\mathbb{R}_+; W_p^s(\mathbb{R}^{n-1}))$, $m \geq 0$, hence by real interpolation also on ${}_0W^r_{p,\mu}(\mathbb{R}_+; W_p^s(\mathbb{R}^{n-1}))$, $r > 0$, and so Theorem 6.6.1 is valid for this model problem.

Remark 6.6.2. The argument given above shows that the boundary symbol $s(\lambda, \xi)$ is equivalent to the *essential symbol* of the problem which is given by

$$s_{ess}(\lambda, \xi) = \lambda + |\xi|^2\sqrt{\lambda + |\xi|^2}, \quad \operatorname{Re}\lambda > 0,\ \xi \in \mathbb{R}^{n-1}.$$

The essential symbol is responsible for the 'strange' solution space of h. The symbol does not come from an evolution equation, but from an evolutionary integral equation. In fact, $s_{ess}(\lambda, \xi)$ is the symbol of the pseudo-differential operator

$$L_{ess} = \partial_t + (-\Delta_x)\sqrt{\partial_t - \Delta_x},$$

which in different form may be written as

$$L_{ess} = \partial_t + (-\Delta_x)(\partial_t - \Delta_x)k_t \star,$$

where k_t denotes the heat kernel and $\star$ convolution in space and time.

6.4 General Coefficients and Domains

To complete the proof of Theorem 6.6.1, we may now proceed as before.

1. By perturbation, the result for the flat interface with constant coefficients remains valid for variable coefficients with the required regularity and small deviation from constant ones.
2. By another perturbation argument, the usual coordinate transformation transfers the result to the case of a bent interface.
3. The localization technique yields the case of general domains and general coefficients.
4. Employing perturbation arguments another time, we may include lower order terms, at the expense of possibly enlarging ω_0.

We refrain here from working out details, this is left to the interested reader.

6.5 The Stefan Semigroup

As problem (6.73) is a linear well-posed system of differential equations, there should be an underlying semigroup. However, it is not straightforward to formulate this, and to show that its negative generator has maximal regularity. To extract the semigroup, we indeed need another type of maximal regularity. For this purpose observe that by (6.74)

$$W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \hookrightarrow H^1_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)).$$

Therefore it makes sense to consider as the base space

$$(u, h) \in X_0 := L_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) \times L_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)),$$

and to ask for solutions

$$(u,h)\in \mathbb{E}_u\times \mathbb{E}_h^{sg},\quad \text{with } \mathbb{E}_h^{sg}=H^1_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma))\cap L_{p,\mu}(\mathbb{R}_+;W_p^{4-1/p}(\Sigma)).$$

This means that, given $(f_u,g,u_0,h_0)=0$, but now with $f_h\in L_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma))$ instead of $f_h\in\mathbb{F}_h$, we want to find a unique solution $(u,h)\in\mathbb{E}_u\times\mathbb{E}_h^{sg}$ satisfying (6.73). Clearly, if such a solution exists then the extra condition

$$[\![\mathcal{B}(x,\nabla_x)u]\!]\in L_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma)) \tag{6.78}$$

must be satisfied. As we also have $[\![\mathcal{B}(x,\nabla_x)u]\!]\in W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+;L_p(\Sigma))$, by Example 3.4.9(ii) we obtain the compatibility condition $[\![\mathcal{B}(x,\nabla_x)u_0]\!]\in W_p^{4\mu-2-6/p}(\Sigma)$. This property allows again reduction to the case $(f_u,g,u_0,h_0)=0$, by first solving (6.73) by means of Theorem 6.6.1 with $f_h=0$ and (f_u,g,u_0,h_0) satisfying the assumptions of the theorem, to obtain functions $(\bar u,\bar h)\in\mathbb{E}_u\times\mathbb{E}_h$. The residual functions $(\tilde u,\tilde h)=(u-\bar u,h-\bar h)$ must then satisfy (6.73) with $(f_u,g,u_0,h_0)=0$, as contemplated. Note that $\bar u$ has the property (6.78), hence $\tilde u$ will also have this property if $\tilde h\in\mathbb{E}_h^{sg}$ and $f_h\in L_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma))$. Thus we need to show that for such f_h, problem (6.73) admits a unique solution in $\mathbb{E}_u\times\mathbb{E}_h^{sg}$. Actually, this follows immediately from the mapping properties of the symbol $s(\lambda,\xi)$ for the constant coefficient case with flat interface, and by perturbation and localization in general, as in the previous subsections. As a result we obtain

Theorem 6.6.3. *Let $p>3$ and $1\ge\mu>1/2+3/2p$. There exists $\omega_0\in\mathbb{R}$ such that for each $\omega>\omega_0$, Problem (6.73) admits exactly one solution (u,h) in the class*

$$u\in H^1_{p,\mu}(\mathbb{R}_+;L_p(\Omega))\cap L_{p,\mu}(\mathbb{R}_+;H^2_p(\Omega\setminus\Sigma)) =: \mathbb{E}_u,$$
$$[\![\mathcal{B}(x,\nabla_x)u]\!]\in L_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma)),$$
$$h\in H^1_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma))\cap W_{p,\mu}^{1-1/2p}(\mathbb{R}_+;H^2_p(\Sigma))\cap L_{p,\mu}(\mathbb{R}_+;W_p^{4-1/p}(\Sigma)),$$

if and only if the data (f_u,g,f_h,u_0,h_0) are subject to the following conditions:

(a) $f_u\in L_{p,\mu}(\mathbb{R}_+;L_p(\Omega))=:\mathbb{F}_u$*;*

(b) $g\in W_{p,\mu}^{1-1/2p}(\mathbb{R}_+;L_p(\Sigma))\cap L_{p,\mu}(\mathbb{R}_+;W_p^{2-1/p}(\Sigma))=:\mathbb{F}$*;*

(c) $f_h\in L_{p,\mu}(\mathbb{R}_+;W_p^{2-2/p}(\Sigma))=:\mathbb{F}_h^{sg}$*;*

(d) $u_0\in W_p^{2\mu-2/p}(\Omega\setminus\Sigma)$, $h_0\in W_p^{2+2\mu-3/p}(\Sigma)$*;*

(e) $u_0-\mathcal{C}(x,\nabla_\Sigma)h_0=g(0)$, $[\![\mathcal{B}(x,\nabla_x)u_0]\!]\in W_p^{4\mu-2-6/p}(\Sigma)$,
$\mathcal{B}(x,\nabla_x)u_0=0$ *on* $\partial\Omega$.

The solution map is continuous between the corresponding spaces.

By means of Theorem 6.6.3, we may define the *Stefan semigroup* in X_0 in the following way. We set $z=[u,h]^{\mathsf{T}}$, $X_1=H^2_p(\Omega\setminus\Sigma)\times W_p^{4-1/p}(\Sigma)$, and define an operator A in $X_0 = L_p(\Omega) \times W_p^{2-2/p}(\Sigma)$ by means of

$$\begin{aligned} A &= \begin{bmatrix} \mathcal{A}(x,\nabla_x) & 0 \\ [\![\mathcal{B}(x,\nabla_x)]\!] & 0 \end{bmatrix}, \\ \mathsf{D}(A) &= \{z \in X_1 : \mathcal{B}(x,\nabla_x)u = 0 \text{ on } \partial\Omega,\ u - \mathcal{C}(x,\nabla_\Sigma)h = 0 \text{ on } \Sigma, \\ &\qquad [\![\mathcal{B}(x,\nabla_x)u]\!] \in W_p^{2-2/p}(\Sigma)\}. \end{aligned} \tag{6.79}$$

Problem (6.73) for $g = 0$ is equivalent to the abstract evolution equation

$$\dot{z} + Az = f, \quad t > 0, \quad z(0) = z_0, \tag{6.80}$$

where we employed the abbreviations $z_0 = [u_0, h_0]^\mathsf{T}$ and $f = [f_u, f_h]^\mathsf{T}$. Then maximal L_p-regularity of (6.80) is equivalent to maximal L_p-regularity of (6.73) for $g = 0$ in the modified setting. Theorem 6.6.3 and Proposition 3.5.2 imply that $-A$ is the generator of an analytic C_0-semigroup with maximal L_p-regularity. This completes the construction of the semigroup.

Again we are interested in the smallest possible value of ω in Theorem 6.6.3. For this purpose we first solve the problem for a large value of ω, say $\bar{\omega}$, to obtain a solution $(\bar{u}, \bar{h}) \in \mathbb{E}_u \times \mathbb{E}_h^{sg}$, and we set $\tilde{u} = u - \bar{u}$, $\tilde{h} = h - \bar{h}$. Then we obtain the reduced system for these new functions

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(x,\nabla_x))\tilde{u} &= (\bar{\omega} - \omega)\bar{u} && \text{in } \Omega \setminus \Sigma, \\ \mathcal{B}(x,\nabla_x)\tilde{u} &= 0 && \text{on } \partial\Omega, \\ [\![\tilde{u}]\!] = 0, \quad u - \mathcal{C}(x,\nabla_\Sigma)\tilde{h} &= 0 && \text{on } \Sigma, \\ (\partial_t + \omega)\tilde{h} + [\![\mathcal{B}(x,\nabla_x)\tilde{u}]\!] &= (\bar{\omega} - \omega)\bar{h} && \text{on } \Sigma, \\ \tilde{u}(0) = 0 \text{ in } \Omega, \quad \tilde{h}(0) &= 0 && \text{on } \Sigma. \end{aligned} \tag{6.81}$$

Employing the semigroup this yields

$$\dot{\tilde{z}} + \omega\tilde{z} + A\tilde{z} = \tilde{f},\ t > 0, \quad \tilde{z}(0) = 0,$$

with $\tilde{z} = [\tilde{u}, \tilde{h}]^\mathsf{T}$ and $\tilde{f} = (\bar{\omega} - \omega)[\bar{u}, \bar{h}]^\mathsf{T}$. Therefore, the lower bound of ω is the spectral bound $\omega_0 = \mathsf{s}(-A)$. We are going to discuss this number in more detail in Chapter 10.

6.6 The Linearized Mullins-Sekerka Problem

In this subsection we consider the quasi-steady problem

$$\begin{aligned} (\eta + \mathcal{A}(x,\nabla_x))u &= f_u && \text{in } \Omega \setminus \Sigma, \\ \mathcal{B}(x,\nabla_x)u &= 0 && \text{on } \partial\Omega, \\ [\![u]\!] = 0, \quad u - \mathcal{C}(x,\nabla_\Sigma)h &= g && \text{on } \Sigma, \\ (\partial_t + \omega)h + [\![\mathcal{B}(x,\nabla_x)u]\!] &= f_h && \text{on } \Sigma, \\ h(0) &= h_0 && \text{on } \Sigma. \end{aligned} \tag{6.82}$$

Here $\omega, \eta \geq 0$, $\mathcal{A}(x, \nabla_x) = -\mathrm{div}(a(x)\nabla_x)$, $\mathcal{B}(x, \nabla_x) = -(\nu(x)|a(x)\nabla_x)$ and $\mathcal{C}(x, \nabla_\Sigma) = -\mathrm{div}_\Sigma(c(x)\nabla_\Sigma)$ are differential operators with $a \in C^1_{ub}(\Omega \setminus \Sigma; \mathcal{B}(\mathbb{R}^n))$, $c \in C^3(\Sigma; \mathcal{B}(T\Sigma))$, with both a and c symmetric and uniformly positive definite. Note that the coefficients of $\mathcal{A}$ are allowed to jump across the interface Σ. Here the unit normal $\nu(x)$ at $x \in \Sigma$ is pointing from Ω_1 into Ω_2.

The main result for this problem in the L_p-setting, $1 < p < \infty$, is the following.

Theorem 6.6.4. *Let $p \in (1, \infty)$ and $1 \geq \mu > 1/p$. There exists $\omega_0, \eta_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, $\eta > \eta_0$, problem* (6.82) *admits exactly one solution (u, h) in the class*

$$u \in L_{p,\mu}(\mathbb{R}_+; H^2_p(\Omega \setminus \Sigma)) =: \mathbb{E}_u,$$
$$h \in H^1_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{4-1/p}_p(\Sigma)) =: \mathbb{E}_h,$$

if and only if the data (f_u, g, f_h, h_0) are subject to the following conditions:

(a) $f_u \in L_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) =: \mathbb{F}_u$;

(b) $g \in L_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_p(\Sigma)) =: \mathbb{F}$;

(c) $f_h \in L_{p,\mu}(\mathbb{R}_+; W^{1-1/p}_p(\Sigma)) =: \mathbb{F}_h$;

(d) $h_0 \in W^{1+3\mu-4/p}_p(\Sigma)$.

The solution map is continuous between the corresponding spaces.

This result is proved in the same way as Theorem 6.6.1. As the bulk problem is stationary, the proof is even simpler, so we skip the details here.

We are interested in the parameters η and ω. For this purpose we define an operator A in $X = L_p(\Omega)$ by means of

$$Au(x) = \mathcal{A}(x, \nabla_x)u(x), \quad x \in \Omega \setminus \Sigma, \tag{6.83}$$
$$\mathsf{D}(A) = \{u \in H^2_p(\Omega \setminus \Sigma) : \ u = 0 \text{ on } \Sigma, \ \mathcal{B}(x, \nabla_x)u = 0 \text{ on } \partial\Omega\}.$$

As $\mathcal{A}$ is uniformly strongly elliptic by assumption, Theorem 6.5.1 shows that $-A$ is the generator of an analytic C_0-semigroup with maximal L_p-regularity. Moreover, as Ω is bounded and Σ and $\partial\Omega$ are of class C^2 and do not intersect, the semigroup as well as the resolvent of A are compact. Therefore, the spectrum of A consist only of eigenvalues of finite algebraic multiplicity, and is independent of p. So we only need to consider $p = 2$. If z is an eigenvalue of A with eigenfunction $u \neq 0$, the usual energy argument yields

$$z|u|^2_{L_2} = \int_\Omega a^{ij}\partial_j u\, \overline{\partial_i u}\, dx,$$

we see that z must be real, and employing uniform strong ellipticity,

$$z|u|^2_{L_2} \geq c|\nabla u|^2_{L_2},$$

hence $z \geq 0$. If $z = 0$ then $\nabla u = 0$ in Ω hence u is constant, as Ω is connected, and u has no jump across Σ, and so $u = 0$. This shows that $0 \in \rho(A)$.

We now may proceed as follows. Solve the problem

$$\begin{aligned} (\eta + \mathcal{A}(x, \nabla_x))u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\ \mathcal{B}(x, \nabla_x)u &= 0 \quad \text{on } \partial\Omega, \\ [\![u]\!] = 0,\ u &= g \quad \text{on } \Sigma, \end{aligned}$$

and denote the solution by $u_\eta = T_\eta g$. The Dirichlet-to-Neumann operator for this problem is given by $S_{\mathsf{d},\eta} g = [\![\mathcal{B}(x, \nabla_x) T_\eta g]\!]$. Then we define A_η in $X_0 := W_p^{1-1/p}(\Sigma)$ by means of

$$A_\eta h = S_{\mathsf{d},\eta} \mathcal{C}(x, \nabla_\Sigma) h, \quad X_1 := \mathsf{D}(A_\eta) = W_p^{4-1/p}(\Sigma). \tag{6.84}$$

It is clear that (6.82) with $\eta = 0$, and $(f_u, g) = 0$ is equivalent to the evolution equation

$$\partial_t h + \omega h + A_0 h = f_h, \quad t > 0, \quad h(0) = h_0.$$

We can easily show that $-A_0$ generates an analytic C_0-semigroup with maximal L_p-regularity, the *Mullins-Sekerka semigroup*. In fact, for this purpose note that by Theorem 6.6.4, A_η has maximal L_p-regularity for η large. Now we have the identity

$$T_0 g = T_\eta g + \eta(\eta + A)^{-1} T_0 g,$$

which follows from

$$\begin{aligned} \eta(\eta + A)^{-1} T_0 g &= (\eta + A)(\eta + A)^{-1} T_0 g - A(\eta + A)^{-1} T_0 g \\ &= T_0 g - (\eta + A)^{-1} A (T_0 g - T_\eta g) - A(\eta + A)^{-1} T_\eta g \\ &= T_0 g + (\eta + A)^{-1} \mathcal{A}(x, \nabla_x) T_\eta g - A(\eta + A)^{-1} T_\eta g \\ &= T_0 g - (\eta + A)(\eta + A)^{-1} T_\eta g. \end{aligned}$$

Hence,

$$A_0 = S_{\mathsf{d},0} \mathcal{C}(x, \nabla_\Sigma) = A_\eta + \eta [\![\mathcal{B}(x, \nabla_x)]\!] (\eta + A)^{-1} T_0 \mathcal{C}(x, \nabla_x).$$

As the second term is a compact perturbation of the first one, the claim follows. We summarize these considerations.

Corollary 6.6.5. *The* Mullins-Sekerka operator A_0 *defined above is the negative generator of an analytic C_0-semigroup $e^{-A_0 t}$, the* Mullins-Sekerka semigroup*, with maximal L_p-regularity in the base space $X_0 = W_p^{1-1/p}(\Sigma)$ and domain $X_1 = \mathsf{D}(A_0) = W_p^{4-1/p}(\Sigma)$.*

We note that A_0 is a pseudo-differential operator of order three. The spectrum of this operator will be considered in Chapter 12.

6.7 The Linearized Verigin Problem

The following linear problem arises as the linearization of the *Verigin problem.* It can be treated analytically in the same way as the linearized Stefan problem with surface tension. Therefore we will keep this section quite short. For the formulation, as in the previous section, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^2. Ω consists of two parts, Ω_1 and Ω_2 such that $\Sigma = \partial\Omega_1$ does not touch $\partial\Omega$. We assume that the hypersurface Σ is a C^3-manifold in $\mathbb{R}^n$. Consider

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(x,\nabla_x))u &= f_u && \text{in } \Omega\setminus\Sigma,\\ \mathcal{B}(x,\nabla_x)u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] + \mathcal{C}(x,\nabla_\Sigma)h &= g && \text{on } \Sigma,\\ [\![\mathcal{B}(x,\nabla_x)u]\!] &= 0 && \text{on } \Sigma,\\ (\partial_t+\omega)h - \mathcal{B}(x,\nabla_x)u &= f_h && \text{on } \Sigma,\\ u(0) = u_0 \ \text{ in } \Omega, \quad h(0) &= h_0 && \text{on } \Sigma. \end{aligned} \tag{6.85}$$

Here $\omega \geq 0$, $\mathcal{A}(x,\nabla_x) = -\mathrm{div}(a(x)\nabla)$, $\mathcal{B}(x,\nabla_x) = -(\nu(x)|a(x)\nabla_x)$ and $\mathcal{C}(x,\nabla_\Sigma) = -\mathrm{div}_\Sigma(c(x)\nabla_\Sigma)$ are differential operators with $a \in C^1_{ub}(\Omega\setminus\Sigma;\mathcal{B}(\mathbb{R}^n))$, $c \in C^3(\Sigma;\mathcal{B}(T\Sigma))$, where a and c are both symmetric and uniformly positive definite. The coefficients of $\mathcal{A}$ are allowed to jump across the interface Σ. The unit normal $\nu(x)$ at $x \in \Sigma$ is pointing from Ω_1 into Ω_2.

The main result for this problem in the L_p-setting, $3 < p < \infty$, is the following.

Theorem 6.7.1. *Let $p > 3$ and $1 \geq \mu > 1/2 + 3/2p$. There exists $\omega_0 \in \mathbb{R}$ such that for each $\omega \geq \omega_0$, problem* (6.85) *admits exactly one solution (u,h) in the class*

$$u \in H^1_{p,\mu}(\mathbb{R}_+;L_p(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+;H^2_p(\Omega\setminus\Sigma)) =: \mathbb{E}_u,$$
$$h \in W^{3/2-1/2p}_{p,\mu}(\mathbb{R}_+;L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+;H^2_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+;W^{4-1/p}_p(\Sigma)) =: \mathbb{E}_h,$$

if and only if the data (f_u, g, f_h, u_0, h_0) are subject to the following conditions:

(a) $f_u \in L_{p,\mu}(\mathbb{R}_+;L_p(\Omega)) =: \mathbb{F}_u$;
(b) $g \in W^{1-1/2p}_{p,\mu}(\mathbb{R}_+;L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+;W^{2-1/p}_p(\Sigma)) =: \mathbb{F}$;
(c) $f_h \in W^{1/2-1/2p}_{p,\mu}(\mathbb{R}_+;L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+;W^{1-1/p}_p(\Sigma)) =: \mathbb{F}_h$;
(d) $u_0 \in W^{2\mu-2/p}_p(\Omega\setminus\Sigma)$, $h_0 \in W^{2+2\mu-3/p}_p(\Sigma)$;
(e) $[\![u_0]\!] + \mathcal{C}(x,\nabla_\Sigma)h_0 = g(0)$, $\mathcal{B}(x,\nabla_x)u_0 + f_h(0) \in W^{4\mu-2-6/p}_p(\Sigma)$,
$[\![\mathcal{B}(x,\nabla_x)u]\!] = 0$, $\mathcal{B}(x,\nabla_x)u_0 = 0$ *on* $\partial\Omega$.

The solution map is continuous between the corresponding spaces.

There is no need to discuss the solution spaces, as they are the same as in the previous section, similar reductions are available, and the process of localization will also be the same. Therefore we will concentrate on the model problem.

7.1 The Boundary Symbol

In this subsection we consider the constant coefficient case in $\Omega = \mathbb{R}^n$ with flat interface $\Sigma = \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$, for short. This means that we consider the problem which is already in reduced form

$$\begin{aligned} (\partial_t + \omega + \mathcal{A}(\nabla_x))u &= 0 &&\text{in } \hat{\mathbb{R}}^n, \\ [\![u]\!] + \mathcal{C}(\nabla_\Sigma)h &= 0 &&\text{on } \mathbb{R}^{n-1}, \\ [\![\mathcal{B}(\nabla_x)u]\!] &= 0 &&\text{on } \mathbb{R}^{n-1}, \\ (\partial_t + \omega)h - \mathcal{B}(\nabla_x)u &= f_h &&\text{on } \mathbb{R}^{n-1}, \\ u(0) = 0 \ \text{ in } \hat{\mathbb{R}}^n, \quad h(0) &= 0 &&\text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{6.86}$$

As in the previous section, it is convenient to replace the variable $x \in \mathbb{R}^n$ by $(x, y) \in \hat{\mathbb{R}} := \mathbb{R}^{n-1} \times \dot{\mathbb{R}}$, which means that we split into the tangential variables x and the normal variable y.

Taking Laplace transform in time and Fourier transform in the tangential variables we obtain the problem

$$\begin{aligned} (\lambda + a(\xi, \xi))\tilde{u} - 2ia(\xi, \nu)\partial_y \tilde{u} - a(\nu, \nu)\partial_y^2 \tilde{u} &= 0, && y > 0, \\ [\![\tilde{u}]\!] + \mathcal{C}(\xi)\tilde{h} &= 0, && y = 0, \\ [\![a(\nu, \nu)\partial_y \tilde{u} + ia(\xi, \nu)\tilde{u}]\!] &= 0, && y = 0, \\ \lambda\tilde{h} + (a(\nu, \nu)\partial_y \tilde{u} + ia(\xi, \nu)\tilde{u}) &= \tilde{f}_h, && y = 0, \end{aligned} \tag{6.87}$$

where, as before, the tilde indicates Laplace transform in t with co-variable τ, $\lambda = \tau + \omega$, and Fourier transform in the tangential variable x with co-variable ξ, and $\nu = e_n$ is the normal at the interface. Note that the coefficients of $\mathcal{A}(\nabla_x)$ may jump across the interface. As the forms a_k, $k = 1, 2$, defining $\mathcal{A}(\nabla_x)$ are real symmetric and positive definite, and given $u_\Sigma = \mathcal{C}(\xi)\tilde{h}$, we may solve the equations in the region $y \neq 0$ to the result

$$\tilde{u}(y) = e^{-yr_2(\lambda,\xi)} u_\Sigma^2, \ \ y > 0, \quad \text{and} \quad \tilde{u}(y) = e^{yr_1(\lambda,\xi)} u_\Sigma^1, \ \ y < 0,$$

where u_Σ^k denote the unknown boundary values of u in Ω_k. The symbols r_k, $k = 1, 2$, are defined as in Section 6.6.3. The interface conditions imply

$$u_\Sigma^2 - u_\Sigma^1 = -\mathcal{C}(\xi)\tilde{h},$$

and with the notation

$$n_k(\lambda, \xi) = \sqrt{(\lambda + a_k(\xi, \xi))a_k(\nu, \nu) - a_k(\xi, \nu)^2},$$

the second interface condition reads

$$n_1(\lambda, \xi)u_\Sigma^1 + n_2(\lambda, \xi)u_\Sigma^2 = 0.$$

For the equation on the boundary this yields

$$s(\lambda,\xi)\tilde{h} = \tilde{f}_h, \quad \text{with} \quad s(\lambda,\xi) = \lambda + \mathcal{C}(\xi)\frac{n_1(\lambda,\xi)n_2(\lambda,\xi)}{n_1(\lambda,\xi)+n_2(\lambda,\xi)}. \tag{6.88}$$

As the harmonic mean $n_1n_2/(n_1+n_2) = 1/(1/n_1+1/n_2)$ is leaving each sector Σ_θ, $\theta \le \pi/2$, invariant we may conclude as in Section 6.6.3 that the symbol

$$m(\lambda,\xi) := \frac{\lambda + |\xi|^2\sqrt{\lambda+|\xi|^2}}{s(\lambda,\xi)}$$

is bounded from above and below even on a larger set $\lambda \in \Sigma_{\pi/2+\varepsilon}$, $\xi \in \Sigma_\varepsilon^{n-1} \cup -\Sigma_\varepsilon^{n-1}$, and as in Section 6.6.3 this proves the assertion for the case of constant coefficients and flat interface. Note that the essential symbol of the Verigin problem is the same as that for the Stefan problem considered in the previous section.

7.2 The Verigin Semigroup
As problem (6.85) is a linear well-posed system of differential equations there should be an underlying semigroup. This semigroup can be constructed in a similar way as the Stefan semigroup in the previous section.

Theorem 6.7.2. *Let $p > 3$ and $1 \ge \mu > 1/2 + 3/2p$. There exists $\omega_0 \in \mathbb{R}$ such that for each $\omega \ge \omega_0$, Problem (6.85) admits exactly one solution (u,h) in the class*

$$\begin{aligned}
&u \in H^1_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_p(\Omega\setminus\Sigma)) =: \mathbb{E}_u,\\
&\mathcal{B}(x,\nabla_x)u \in L_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)),\\
&h \in H^1_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{4-1/p}_p(\Sigma)),
\end{aligned}$$

if and only if the data (f_u, g, f_h, u_0, h_0) are subject to the following conditions:

(a) $f_u \in L_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) =: \mathbb{F}_u$;

(b) $g \in W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_p(\Sigma)) =: \mathbb{F}$;

(c) $f_h \in L_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma)) =: \mathbb{F}_h$;

(d) $u_0 \in W^{2\mu-2/p}_p(\Omega\setminus\Sigma)$, $h_0 \in W^{2+2\mu-3/p}_p(\Sigma)$;

(e) $[\![u_0]\!] + \mathcal{C}(x,\nabla_\Sigma)h_0 = g(0)$.

The solution map is continuous between the corresponding spaces.

Proof. The proof of this result involves similar ideas as the proof of Theorem 6.6.1 and we will hence skip the details. □

By means of Theorem 6.7.2, we may define the *Verigin semigroup* in X_0 in the following way. We set $z = [u,h]^\mathsf{T}$, $X_1 = H^2_p(\Omega\setminus\Sigma) \times W^{4-1/p}_p(\Sigma)$, and define an operator A in $X_0 = L_p(\Omega) \times W_p^{2-2/p}(\Sigma)$ by means of

$$A = \begin{bmatrix} \mathcal{A}(x, \nabla_x) & 0 \\ \mathcal{B}(x, \nabla_x) & 0 \end{bmatrix}, \tag{6.89}$$

$$\begin{aligned}\mathsf{D}(A) = \{z \in X_1 : &\mathcal{B}(x, \nabla_x)u = 0 \text{ on } \partial\Omega,\ [\![u]\!] + \mathcal{C}(x, \nabla_\Sigma)h = 0 \text{ on } \Sigma,\\ &\mathcal{B}(x, \nabla_x)u \in W_p^{2-2/p}(\Sigma)\}.\end{aligned}$$

Then (6.85) for $g = 0$ is equivalent to the abstract evolution equation

$$\dot{z} + Az = f, \quad t > 0, \quad z(0) = z_0, \tag{6.90}$$

where we employed the abbreviations $z_0 = [u_0, h_0]^\mathsf{T}$ and $f = [f_u, f_h]^\mathsf{T}$. Maximal L_p-regularity of (6.90) is equivalent to maximal L_p-regularity of (6.85) for $g = 0$ in the modified setting. Theorem 6.7.2 and Proposition 3.5.2 then imply that $-A$ is the generator of an analytic C_0-semigroup with maximal L_p-regularity. This completes the construction of the Verigin semigroup.

In the same way as in the previous section, employing the semigroup this yields that the lower bound of ω is the spectral bound $\omega_0 = \mathsf{s}(-A)$.

7.3 The Linearized Muskat Problem

In this subsection we consider the quasi-steady problem

$$\begin{aligned}
(\eta + \mathcal{A}(x, \nabla_x))u &= f_u && \text{in } \Omega \setminus \Sigma,\\
\mathcal{B}(x, \nabla_x)u &= 0 && \text{on } \partial\Omega,\\
[\![u]\!] + \mathcal{C}(x, \nabla_\Sigma)h &= g && \text{on } \Sigma,\\
[\![\mathcal{B}(x, \nabla_x)u]\!] &= 0 && \text{on } \Sigma,\\
(\partial_t + \omega)h - \mathcal{B}(x, \nabla_x)u &= f_h && \text{on } \Sigma,\\
h(0) &= h_0 && \text{on } \Sigma.
\end{aligned} \tag{6.91}$$

The main result for this problem in the L_p-setting, $3 < p < \infty$, is the following.

Theorem 6.7.3. *Let $p \in (1, \infty)$ and $1 \geq \mu > 1/p$. There exists $\omega_0, \eta_0 \in \mathbb{R}$ such that for each $\omega > \omega_0$, $\eta > \eta_0$, Problem* (6.91) *admits exactly one solution (u, h) in the class*

$$\begin{aligned}
u &\in L_{p,\mu}(\mathbb{R}_+; H_p^2(\Omega \setminus \Sigma)) =: \mathbb{E}_u,\\
h &\in H^1_{p,\mu}(\mathbb{R}_+; W_p^{1-1/p}(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{4-1/p}(\Sigma)) =: \mathbb{E}_h,
\end{aligned}$$

if and only if the data (f_u, g, f_h, h_0) are subject to the following conditions:

(a) $f_u \in L_{p,\mu}(\mathbb{R}_+; L_p(\Omega)) =: \mathbb{F}_u$*;*

(b) $g \in L_{p,\mu}(\mathbb{R}_+; W_p^{2-1/p}(\Sigma)) =: \mathbb{F}$*;*

(c) $f_h \in L_{p,\mu}(\mathbb{R}_+; W_p^{1-1/p}(\Sigma)) =: \mathbb{F}_h$.

(d) $h_0 \in W_p^{1+3\mu-4/p}(\Sigma)$*;*

The solution map is continuous between the corresponding spaces.

Proof. This result is proved in the same way as Theorem 6.7.1. □

We are interested in the parameters η and ω. For this purpose we define the operator A in $X = L_p(\Omega)$ by means of

$$Au(x) = \mathcal{A}(x, \nabla_x)u(x), \quad x \in \Omega \setminus \Sigma, \tag{6.92}$$
$$\mathsf{D}(A) = \{u \in H_p^2(\Omega \setminus \Sigma) : \mathcal{B}(x, \nabla_x)u = 0 \text{ on } \partial\Omega,\ [\![\mathcal{B}(x, \nabla_x)u]\!] = [\![u]\!] = 0 \text{ on } \Sigma\}.$$

As $\mathcal{A}$ is uniformly strongly elliptic by assumption, Theorem 6.5.1 shows that $-A$ is the generator of an analytic C_0-semigroup e^{-At} with maximal L_p-regularity. The semigroup as well as the resolvent of A are compact. Therefore the spectrum of A consists only of eigenvalues of finite algebraic multiplicity, which do not depend on p. By the energy argument, we obtain $\sigma(A) \subset \mathbb{R}_+$. However, in contrast to the case of the linearized Mullins-Sekerka problem, here 0 is an eigenvalue of A, it is algebraically simple and spanned by the function e which is constant 1, $\mathsf{e} \perp \mathsf{R}(A)$ as the divergence theorem shows. To circumvent this difficulty in the construction of the Muskat semigroup, we observe that in Theorem 6.7.3 the solution u has mean value 0 if f_u has this property. So instead of $X = L_p(\Omega)$ we employ

$$X = L_{p,0}(\Omega) = \{u \in L_p(\Omega) : (u|\mathsf{e})_\Omega = 0\}.$$

This removes 0 from the spectrum of A. Then we proceed as in Section 6.6.6 to construct the *Muskat operator* as follows.

Define the *Muskat operator* A_0 in $X_0 := W_p^{1-1/p}(\Sigma)$ with help of the Dirichlet-to-Neumann operator S_n by means of

$$A_0 h = S_\mathsf{n}\mathcal{C}(x, \nabla_\Sigma)h, \quad X_1 := \mathsf{D}(A_0) = W_p^{4-1/p}(\Sigma). \tag{6.93}$$

Then it is obvious that (6.91) with $\eta = 0$, and $(f_u, g) = 0$ is equivalent to the evolution equation

$$\partial_t h + \omega h + A_0 h = f_h, \quad t > 0, \quad h(0) = h_0.$$

As for the Mullins-Sekerka case, we can show that $-A_0$ generates an analytic C_0-semigroup with maximal L_p-regularity.

Corollary 6.7.4. *The* Muskat operator A_0 *defined above is the negative generator of an analytic C_0-semigroup $e^{-A_0 t}$, the* Muskat semigroup, *with maximal L_p-regularity in $X_0 = W_p^{1-1/p}(\Sigma)$ and domain $X_1 = \mathsf{D}(A_0) = W_p^{4-1/p}(\Sigma)$.*

The spectrum of this operator will be considered in Chapter 12.

Appendix

The Triebel-Lizorkin spaces $F^\alpha_{pq}(\mathbb{R}; E)$ and ${}_0F^\alpha_{pq,\mu}(\mathbb{R}_+; E)$ for $\alpha \in (0,1)$, $p, q \in (1, \infty)$, and $1/p < \mu \leq 1$ can be characterized as follows.

Lemma 6.7.5. *Let $1 < p,q < \infty$, $1/p < \mu \le 1$, $\alpha \in (0,1)$, and suppose E is a Banach space of class $\mathcal{HT}$. Define $B = (\partial_t)^\alpha$ in $L_{p,\mu}(\mathbb{R}_+; L_q((0,1);E))$ with domain $\mathsf{D}(B) = {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q((0,1);E))$.*

Then, for any $g \in L_{p,\mu}(\mathbb{R}_+;E)$,

$$w := e^{-By} g \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q((0,1);E))$$

if and only if $g \in {}_0F^{\alpha(1-1/q)}_{pq,\mu}(\mathbb{R}_+;E)$.

The same result is valid for the whole line case, i.e.,

$$w \in H^\alpha_p(\mathbb{R}; L_q((0,1);E)) \quad \Leftrightarrow \quad g \in F^{\alpha(1-1/q)}_{pq}(\mathbb{R};E).$$

These results hold for $\mathbb{R}_+$ instead of $(0,1)$ if we replace ∂_t^α by $(\omega+\partial_t)^\alpha$, for some $\omega > 0$.

Actually, we might have taken the assertion of this lemma for the whole line case as a definition for the vector-valued spaces $F^\alpha_{pq}(\mathbb{R};E)$. However, to draw the connection with the definition of F^α_{pq} given in Triebel [284], we add a proof. Observe that

$$u \in {}_0F^\alpha_{pq,\mu}(\mathbb{R}_+;E) \quad \Leftrightarrow \quad t_+^{1-\mu} u \in F^\alpha_{pq}(\mathbb{R};E),$$

where $t_+^{1-\mu} = \max\{t^{1-\mu}, 0\}$. Therefore we may concentrate on the whole line case, and we restrict to the case $\omega = 0$.

Proof. For $E = \mathbb{C}$, Theorem 2.4.1 of [284] proves Lemma 6.7.5 with the choices $\phi(x) = (ix)^\alpha e^{-(ix)^\alpha}$ and $\phi_0(x) = 1$, $s_0 = 0$, $s_1 = \alpha$. The proof given there carries over to the vector-valued case since E is assumed to be of class $\mathcal{HT}$, provided $\alpha > a > 1/\min\{p,q\}$. For general $p,q \in (1,\infty)$ Theorem 2.4.1 of [284] does not apply since the moment condition (8) in that reference does not hold.

To see sufficiency of the condition in the general case, assume that $w_0 := Be^{-By} g \in L_p(\mathbb{R}; L_q((0,1);E))$. Using maximal regularity we solve successively the problems

$$\partial_y w_k + Bw_k = Bw_{k-1}, \quad w_k|_{y=0} = 0,$$

to obtain

$$Bw_k = y^k B^{k+1} e^{-yB} g \in L_p(\mathbb{R}; L_q((0,1);E)), \quad k \in \mathbb{N}_0.$$

Now we have with the variable transformation $y = \tau^\alpha$

$$\begin{aligned}\int_0^1 |y^k B^{k+1} e^{-yB} g|_E^q \, dy &= \alpha \int_0^1 \tau^{-q\alpha(1-1/q)} |(\tau^\alpha B)^{k+1} e^{-(\tau^\alpha B)} g|_E^q \, \frac{d\tau}{\tau} \\ &= \alpha \int_0^1 \tau^{-q\alpha(1-1/q)} |\phi(\tau D) g|_E^q \, \frac{d\tau}{\tau},\end{aligned}$$

where we used the notation in [284], Section 2.4.1, with $\phi(\xi) = (i\xi)^{\alpha(k+1)} e^{-(i\xi)^\alpha}$. It is not difficult to check that the relevant conditions (7) and (9) are valid for all $k \in \mathbb{N}_0$ with $s_0 = 0$. On the other hand, (8) holds in case $\alpha k \ge 1$. In fact, the inverse Fourier transform $p^{k+1}(t)$ of $\phi(i\xi)$, with contour $\Gamma = e^{-i\theta}(\infty, 0] \cup e^\theta[0,\infty)$, $\theta \in (\pi/2, \pi)$, $\alpha\theta < \pi/2$, becomes

$$p^{k+1}(t) = \frac{1}{2\pi i} \int_\Gamma z^{\alpha k+1} e^{-z^\alpha} e^{zt} \, dz, \quad t \ge 0.$$

Note that the support of p^{k+1} is contained in $\mathbb{R}_+$, thanks to holomorphy. This formula is valid for all $\alpha(k+1) > -1$, and it implies that $p^{k+1}(t)$ is bounded and behaves asymptotically like $t^{-(1+\alpha(k+1))}$ as $t \to \infty$. Therefore $(1+t^a)p^{k+1} \in L_1(\mathbb{R}_+)$ if and only if $a < \alpha(k+1)$. Choosing $s_1 = \alpha$ and $1/\min\{p,q\} < a < 1$, and $k \geq 1/\alpha$, the vector-valued version of Theorem 2.4.1 of [284] implies $g \in F_{pq}^{\alpha(1-1/q)}(\mathbb{R}; E)$.

For the converse statement we need to choose $k = 0$. Since the critical condition (8) does not hold, we have to modify Steps 1 and 4 of the proof of Theorem 2.4.1 of [284], the only places where (8) is used. We concentrate on the modification of Step 1, and employ the notation used there. Let $s = \alpha(1-1/q)$ and fix a resolution of unity $\{\rho_j\}_{j\in\mathbb{N}_0}$ in the sense of [284] Section 2.3.1. Then by definition, $g \in {}_0F_{pq}^s(\mathbb{R}; E)$ if and only if

$$(2^{sj}\rho_j(D)g)_{j\in\mathbb{N}_0} \in L_p(\mathbb{R}; l_q(\mathbb{N}_0; E)).$$

Now we have as in [284], proof of Theorem 2.4.1, Step 1

$$2^{js}\mathcal{F}^{-1}\mathcal{L}p^1(2^{-j}i\xi)\mathcal{F} = \sum_{l=-\infty}^{\infty} 2^{js}\mathcal{F}^{-1}\mathcal{L}p^1(2^{-j}i\xi)\rho_{l+j}(\xi)\mathcal{F}g.$$

Here $\mathcal{L}$ denotes the Laplace transform. Splitting the sum into two parts, we have to estimate in Step 1 the part running from $l = -\infty$ to $l = k$. We write

$$\begin{aligned} &2^{js}\mathcal{F}^{-1}\mathcal{L}p^1(2^{-j}i\xi)\rho_{l+j}(\xi)\mathcal{F}g \\ &\quad = 2^{\alpha l/q}\mathcal{F}^{-1}\mathcal{L}p^0(2^{-j}i\xi)\cdot(2^{-(j+l)}i\xi)^\alpha\chi(2^{-(j+l)}\xi)\cdot 2^{s(j+l)}\rho_{j+l}\mathcal{F}g, \end{aligned}$$

where $\chi(r)$ denotes a cut off function which is 1 on $|r| \leq 2$. Since $\sum_{l=-\infty}^k 2^{\alpha l/q} < \infty$, it suffices to estimate

$$\mathcal{F}^{-1}\mathcal{L}p^0(2^{-j}i\xi)\cdot(2^{-(j+l)}i\xi)^\alpha\chi(2^{-(j+l)}\xi)\cdot 2^{s(j+l)}\rho_{j+l}\mathcal{F}g$$

in $L_p(\mathbb{R}; l_q(\mathbb{N}_0; E))$, uniformly w.r.t. l. By assumption we have

$$|(2^{s(j+l)}\mathcal{F}^{-1}\rho_{j+l}\mathcal{F}g)_{j\geq 0}|_{L_p(\mathbb{R};l_q(\mathbb{N}_0;E))} \leq |g|_{F_{pq}^s(\mathbb{R}_+;E)},$$

hence its is enough to show that the sequences $(\mathcal{L}p^0(2^{-j}i\xi))_{j\in\mathbb{N}_0}$ and $((2^{-(j+l)}i\xi)^\alpha\chi(2^{-(j+l)}\xi))_{j\in\mathbb{N}_0}$ define Fourier multipliers for $L_p(\mathbb{R}; l_q(\mathbb{N}_0; E))$ with bounds independent of l.

For the first sequence, observe that $\mathcal{L}p^0(\lambda) = e^{-\lambda^\alpha}$ is completely monotonic, hence $p^0(t)$ is nonnegative and integrable with integral equal to 1, i.e., p^0 is a probability density. Therefore, the operator defined by the first sequence is given by

$$(T_1f)_j(t) = 2^jp^0(2^j\cdot) * f_j(t), \quad t>0,\ j\in\mathbb{N}_0.$$

Thus we obtain

$$|(T_1f)_j(t)|_E \leq M|f_j|_E(t), \quad t>0,\ j\in\mathbb{N}_0,$$

where M denotes the usual maximal operator. Since M is bounded in $L_p(\mathbb{R}; l_q(\mathbb{N}_0))$, the assertion follows for the first sequence, i.e., T_1 is bounded in $L_p(\mathbb{R}; l_q(\mathbb{N}_0; E))$.

The second sequence is treated in a similar way. We write

$$(i\xi)^\alpha\chi(\xi) = \frac{(i\xi)^\alpha}{(1+i\xi)^2} + \frac{(i\xi)^\alpha}{(1+i\xi)^2}(\chi(\xi)-1) + \frac{(i\xi)^{1+\alpha}(2+i\xi)}{(1+i\xi)^2}\chi(\xi).$$

The first term belongs to the Hardy space $\mathcal{H}^\infty(\mathbb{C}_+)$ and its derivative belongs to $\mathcal{H}^1(\mathbb{C}_+)$, therefore by Hardy's inequality it is the Laplace transform of a function $k_1 \in L_1(\mathbb{R}_+)$. The second and the third terms belong to $L_2(\mathbb{R})$ as well as their derivatives, hence by means of Bernstein's theorem they are Fourier transforms of functions $k_j \in L_1(\mathbb{R})$, $j = 2, 3$. This shows that $(i\xi)^\alpha \chi(\xi) = \mathcal{F}k(\xi)$, for some $k \in L_1(\mathbb{R})$. Now we may argue as before to see that also the second sequence defines a bounded operator T_2 in $L_p(\mathbb{R}; l_q(\mathbb{N}_0; E))$, with bound independent of l. This completes the proof of Lemma 6.7.5. □