Chapter 3

Operator Theory and Semigroups

In this chapter we introduce some basic tools from operator and semigroup theory. The class of sectorial operators is studied in detail, its functional calculus is introduced, leading to analytic semigroups and complex powers. The classes $\mathcal{BIP}(X)$ and $\mathcal{H}^\infty(X)$ are defined and elementary properties are shown. Via trace theory for abstract Cauchy problems the connections to real interpolation are derived, and the relation of complex interpolation to powers of operators is shown. The chapter concludes with a first study of maximal L_p-regularity.

3.1 Sectorial Operators

The concept of sectorial operators introduced in Definition 3.1.1 below is basic in this book. Most closed linear operators appearing in applications have this property, at least after translation and rotation. We will meet many examples of such operators in later sections.

1.1 Sectorial Operators
We begin with the definition of sectorial operators.

Definition 3.1.1. *Let X be a complex Banach space, and A a closed linear operator in X. A is called* **sectorial** *if the following two conditions are satisfied.*

(S1) $\overline{\mathsf{D}(A)} = X,\ \overline{\mathsf{R}(A)} = X,\ (-\infty, 0) \subset \rho(A)$;

(S2) $|t(t+A)^{-1}| \leq M$ *for all* $t > 0$, *and some* $M < \infty$.

The class of sectorial operators in X will be denoted by $\mathcal{S}(X)$. If $(-\infty, 0) \subset \rho(A)$ and only **(S2)** *holds then A is said to be* **pseudo-sectorial**. *The class of pseudo-sectorial operators will be denoted by $\mathcal{PS}(X)$.*

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Suppose that A is a linear operator in X which is pseudo-sectorial. Then the operator family $\{A(t+A)^{-1}\}_{t>0} \in \mathcal{B}(X)$ is uniformly bounded as well. For $x \in \mathsf{D}(A)$ we have

$$t(t+A)^{-1}x - x = -A(t+A)^{-1}x = -(t+A)^{-1}Ax \to_{t\to\infty} 0,$$

hence $\lim_{t\to\infty} t(t+A)^{-1}x = x$ for all $x \in \overline{\mathsf{D}(A)}$, by **(S2)**. In particular, if $\mathsf{D}(A)$ is dense in X then

$$\lim_{t\to\infty} t(t+A)^{-1}x = x \quad \text{for all } x \in X.$$

Similarly, for $y = Ax \in \mathsf{R}(A)$ we have

$$A(t+A)^{-1}Ax - Ax = -t(t+A)^{-1}Ax = -tA(t+A)^{-1}x \to_{t\to 0} 0,$$

hence $\lim_{t\to 0} A(t+A)^{-1}y = y$ for all $y \in \overline{\mathsf{R}(A)}$, employing once more **(S2)**. In particular, if $\mathsf{R}(A)$ is dense in X then

$$\lim_{t\to 0} A(t+A)^{-1}x = x \quad \text{for all } x \in X.$$

On the other hand, if $x \in \mathsf{N}(A)$ then $A(t+A)^{-1}x = 0$, and this shows that we always have $\mathsf{N}(A) \cap \overline{\mathsf{R}(A)} = \{0\}$.

If $\mathsf{D}(A)$ is dense in X, then its dual A^* is well-defined. The relation

$$\mathsf{N}(A^*) = \mathsf{R}(A)^{\perp}$$

then shows that $A \in \mathcal{S}(X)$ iff $A \in \mathcal{PS}(X)$ and $\mathsf{N}(A^*) = 0$.

Next, let X be reflexive and A be pseudo-sectorial. Then any sequence $(\lambda_n) \subset \rho(A)$, $\lambda_n \to \infty$ contains a subsequence, which may depend on x, such that $\lambda_n(\lambda_n + A)^{-1}x \rightharpoonup y \in X$. This implies $\lambda_n(\lambda_n + A)^{-1}(\lambda + A)^{-1}x \rightharpoonup (\lambda + A)^{-1}y$, for any $\lambda > 0$. But by means of the resolvent equation

$$\lambda_n(\lambda_n + A)^{-1}(\lambda + A)^{-1}x = \frac{\lambda_n}{\lambda_n - \lambda}[(\lambda + A)^{-1} - (\lambda_n + A)^{-1}]x \rightharpoonup (\lambda + A)^{-1}x,$$

hence $(\lambda + A)^{-1}x = (\lambda + A)^{-1}y$, by uniqueness of weak limits. This implies $x = y$, hence $\lambda(\lambda + A)^{-1}x \rightharpoonup x$ as $\lambda \to \infty$. As a consequence of this we see that $\mathsf{D}(A)$ is weakly dense in X, hence also strongly dense, and then by what has been proved before $\lambda(\lambda + A)^{-1}x \to x$ as $\lambda \to \infty$, for each $x \in X$.

At $\lambda = 0$ we proceed similarly. Fix $x \in X$ and choose a sequence $(\lambda_n) \subset \rho(A)$, $\lambda_n \to 0$ such that $A(\lambda_n + A)^{-1}x \rightharpoonup y \in X$. Then $\lambda A(\lambda_n + A)^{-1}(\lambda + A)^{-1}x \rightharpoonup \lambda(\lambda + A)^{-1}y \in X$, hence the resolvent equation yields

$$y - \lambda(\lambda + A)^{-1}y = A(\lambda + A)^{-1}x = x - \lambda(\lambda + A)^{-1}x,$$

for any $\lambda > 0$. This identity shows $x - y \in \mathsf{N}(A)$, in particular $A(\lambda + A)^{-1}x = A(\lambda + A)^{-1}y$, hence $A(\lambda_n + A)^{-1}y \rightharpoonup y$ as well. Writing

$$x = (x - y) + A(\lambda_n + A)^{-1}x + \lambda_n(\lambda_n + A)^{-1}y$$

and observing $\lambda_n(\lambda_n + A)^{-1}y \rightharpoonup 0$ the latter implies that $\mathsf{N}(A) + \mathsf{R}(A)$ is weakly dense in X, hence also strongly dense. But from what we already proved above this implies $A(\lambda + A)^{-1}x \to Px \in X$ as $\lambda \to 0$, for each $x \in X$. Here $P \in \mathcal{B}(X)$, by the Banach-Steinhaus theorem, and $\mathsf{R}(P) \subset \overline{\mathsf{R}(A)}$, as well as $\mathsf{R}(I - P) \subset \mathsf{N}(A)$. Finally, $A(\lambda + A)^{-1}x = A(\lambda + A)^{-1}Px$ for all $x \in X$ implies $P^2 = P$, i.e., P is the projection onto $\overline{\mathsf{R}(A)}$ along $\mathsf{N}(A)$. We have proved in particular the direct sum decomposition $X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$. Thus in a reflexive space, $\mathsf{R}(A)$ is dense in X if and only if $\mathsf{N}(A) = \{0\}$.

Let us summarize what we have shown above in

Theorem 3.1.2. *Let X be a Banach space and A a pseudo-sectorial operator in X. Then*

(i) $\mathsf{N}(A) \cap \overline{\mathsf{R}(A)} = \{0\}$, *and*

$$\begin{aligned} \lim_{t\to\infty} t(t + A)^{-1}x = x \quad & \textit{for each } x \in \overline{\mathsf{D}(A)}, \\ \lim_{t\to 0+} A(t + A)^{-1}x = x \quad & \textit{for each } x \in \overline{\mathsf{R}(A)}. \end{aligned} \tag{3.1}$$

(ii) *If $\mathsf{D}(A)$ is dense in X, then $A \in \mathcal{S}(X)$ if and only if $\mathsf{N}(A^*) = 0$.*

(iii) *If X is reflexive then $\lim_{t\to\infty} t(t + A)^{-1}x = x$ and $\lim_{t\to 0+} A(t + A)^{-1}x = Px$ for each $x \in X$, where P is the projection onto $\overline{\mathsf{R}(\mathsf{A})}$ along $\mathsf{N}(\mathsf{A})$, and $X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$. Thus, if X is reflexive then any pseudo-sectorial operator A with $\mathsf{N}(A) = \{0\}$ is sectorial.*

(iv) *If X is a general Banach space and A is sectorial, then $\mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ is dense in X, for each $k \in \mathbb{N}$.*

Concerning the last assertion of Theorem 3.1.2, note that $(1 + n^{-1}A)^{-k}A^k(n^{-1} + A)^{-k}$ converges strongly to I as $n \to \infty$ and has range in $\mathsf{D}(A^k) \cap \mathsf{R}(A^k)$.

Let $\Sigma_\theta \subset \mathbb{C}$ denote the open sector with vertex 0, opening angle 2θ, which is symmetric w.r.t. the positive half-axis $\mathbb{R}_+$, i.e.,

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}.$$

If $A \in \mathcal{P}S(X)$ then $\rho(-A) \supset \Sigma_\theta$, for some $\theta > 0$, and

$$\sup\{|\lambda(\lambda + A)^{-1}| : \ |\arg \lambda| < \theta\} < \infty.$$

In fact, with $(d/dt)^n(t + A)^{-1} = (-1)^n n!(t + A)^{-(n+1)}$, for $t > 0$ the Taylor expansion

$$(\lambda + A)^{-1} = \sum_{n=0}^{\infty} (-1)^n(\lambda - t)^n(t + A)^{-(n+1)}$$

and (**S2**) yield the estimate

$$|(\lambda+A)^{-1}| \le \sum_{n=0}^{\infty} |\lambda-t|^n |(t+A)^{-(n+1)}| \le (M/t)\sum_{n=0}^{\infty}(M|\lambda-t|/t)^n.$$

This bound is finite provided $|\lambda/t-1|<1/M$, which by minimization over $t>0$ yields $|\sin\phi|<1/M$, where $\lambda = re^{i\phi}$.

Therefore it makes sense to define the *spectral angle* ϕ_A of $A\in\mathcal{PS}(X)$ by

$$\phi_A = \inf\{\phi : \rho(-A)\supset \Sigma_{\pi-\phi},\ \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}|<\infty\}. \tag{3.2}$$

Evidently, we have $\phi_A\in[0,\pi)$ and

$$\phi_A \ge \sup\{|\arg\lambda| : \lambda\in\sigma(A)\}. \tag{3.3}$$

If $A\in\mathcal{PS}(X)$ is bounded and $0\in\rho(A)$ then there is equality in (3.3). In fact, by holomorphy of $(\lambda-A)^{-1}$ on $\rho(A)$, $\lambda(\lambda-A)^{-1}$ is bounded in $\mathcal{B}(X)$ on each compact subset of $\rho(A)$, and for all $|\lambda|>|A|$ we have

$$|\lambda(\lambda-A)^{-1}| \le \frac{|\lambda|}{|\lambda|-|A|},$$

which is uniformly bounded, say for $|\lambda|\ge 2|A|$. But this implies uniform boundedness of $\lambda(\lambda+A)^{-1}$ on each sector $\Sigma_{\pi-\phi}$ with $\phi>\sup\{|\arg(\lambda)| : \lambda\in\sigma(A)\}$.

For $\phi\in(\phi_A,\pi)$ we frequently employ the notations

$$M_{\pi-\phi}(A) = \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}|, \quad C_{\pi-\phi}(A) = \sup_{\lambda\in\Sigma_{\pi-\phi}} |A(\lambda+A)^{-1}|. \tag{3.4}$$

It is not difficult to see that $C_{\pi-\phi}(A)\ge 1$ as well as $M_{\pi-\phi}(A)\ge 1$, for all $\phi\in(\phi_A,\pi]$. Observe the limiting case $\phi=\pi$:

$$M_0(A) = \sup_{r>0}|r(r+A)^{-1}|, \quad C_0(A) = \sup_{r>0}|A(r+A)^{-1}|. \tag{3.5}$$

1.2 Permanence Properties

The class of sectorial operators has a number of nice permanence properties which are summarized in the following

Proposition 3.1.3. *Let X be a complex Banach space. The class $\mathcal{S}(X)$ of sectorial operators has the following permanence properties.*

(i) *$A\in\mathcal{S}(X)$ iff $\mathsf{N}(A)=\{0\}$ and $A^{-1}\in\mathcal{S}(X)$; then $\phi_{A^{-1}}=\phi_A$;*

(ii) *$A\in\mathcal{S}(X)$ implies $rA\in\mathcal{S}(X)$ and $\phi_{rA}=\phi_A$ for all $r>0$;*

(iii) *$A\in\mathcal{S}(X)$ implies $e^{\pm i\psi}A\in\mathcal{S}(X)$ for all $\psi\in[0,\pi-\phi_A)$, and $\phi_{e^{\pm i\psi}A}=\phi_A+\psi$;*

(iv) *$A\in\mathcal{S}(X)$ implies $(\mu+A)\in\mathcal{S}(X)$ for all $\mu\in\Sigma_{\pi-\phi_A}$, and $\phi_{\mu+A}\le\max\{\phi_A,|\arg\mu|\}$;*

(v) *if* $\mathsf{D}(A)$ *is dense in* X *and* $\mathsf{D}(A^*)$ *dense in* X^*, *then* $A \in \mathcal{S}(X)$ *iff* $A^* \in \mathcal{S}(X^*)$, *and* $\phi_A = \phi_{A^*}$*;*

(vi) *if* Y *denotes another Banach space and* $T \in \mathcal{B}(X,Y)$ *is bijective, then* $A \in \mathcal{S}(X)$ *iff* $A_1 = TAT^{-1} \in \mathcal{S}(Y)$, *and* $\phi_A = \phi_{A_1}$.

Proof. Assertion (i) follows from the identity

$$\lambda(\lambda + A^{-1})^{-1} = \lambda A(1 + \lambda A)^{-1} = A(\lambda^{-1} + A)^{-1}.$$

Similarly, (ii) is a consequence of

$$\lambda(\lambda + rA)^{-1} = (\lambda/r)((\lambda/r) + A)^{-1}, \quad r > 0,$$

and (iii) follows from $|(\lambda+e^{i\phi}A)^{-1}| = |(\lambda e^{-i\phi}+A)^{-1}|$. If $\mu \in \Sigma_{\pi-\phi_A}$, $|\arg(\mu)| = \psi$, and $\lambda \in \Sigma_{\pi-\phi}$, then for $(\pi - \phi) + \psi < \pi$ we have

$$|\arg(\lambda + \mu)| \le \max\{|\arg(\lambda)|, |\arg(\mu)|\},$$

as well as

$$|\lambda + \mu| \ge c(|\lambda| + |\mu|), \quad \text{where } c = \cos((\pi - \phi + \psi)/2).$$

Therefore, $\phi > \max\{\phi_A, \psi\}$ implies

$$|(\lambda + \mu + A)^{-1}| \le \frac{M_{\pi-\phi}(A)}{|\lambda + \mu|} \le \frac{M_{\pi-\phi}(A)}{c(|\lambda| + |\mu|)}, \quad \text{for all } \lambda \in \Sigma_{\pi-\phi},$$

and this yields (iv). To prove (v) it is enough to observe that an operator $T \in \mathcal{B}(X)$ is invertible if and only if $T^* \in \mathcal{B}(X^*)$ is invertible, and $|T| = |T^*|$. Finally, to prove (vi) we verify that the relation

$$(\lambda + A_1)^{-1} = T(\lambda + A)^{-1}T^{-1}$$

is satisfied. □

Next we introduce approximations of a sectorial operator which are again sectorial, but in addition bounded and invertible. This will be achieved as follows. For a given pseudo-sectorial operator A and $\varepsilon > 0$ set

$$A_\varepsilon = (\varepsilon + A)(1 + \varepsilon A)^{-1}. \tag{3.6}$$

Then A_ε is bounded, invertible with inverse

$$A_\varepsilon^{-1} = (1 + \varepsilon A)(\varepsilon + A)^{-1} = ((1/\varepsilon) + A)(1 + (1/\varepsilon)A)^{-1} = A_{1/\varepsilon},$$

and, more generally,

$$\begin{aligned}(t + A_\varepsilon)^{-1} &= (t + (\varepsilon + A)(1 + \varepsilon A)^{-1})^{-1}\\ &= (1 + \varepsilon A)(t + \varepsilon + (1 + \varepsilon t)A)^{-1}\\ &= \frac{1}{1 + \varepsilon t}(1 + \varepsilon A)(\frac{t + \varepsilon}{1 + \varepsilon t} + A)^{-1}, \quad t, \varepsilon > 0.\end{aligned}$$

This implies $\rho(A_\varepsilon) \supset (-\infty, 0]$, and as $\varepsilon \to 0$, $(t+A_\varepsilon)^{-1} \to (t+A)^{-1}$ in $\mathcal{B}(X)$ for each $t > 0$, $A_\varepsilon x \to Ax$ for each $x \in \mathsf{D}(A)$, $A_\varepsilon^{-1}x \to A^{-1}x$ for each $x \in \mathsf{R}(A)$. Since

$$|t(t+A_\varepsilon)^{-1}| \le \frac{tM_0(A)}{t+\varepsilon} + \frac{\varepsilon t C_0(A)}{1+\varepsilon t} \le M_0(A) + C_0(A), \quad t, \varepsilon > 0,$$

we have $A_\varepsilon \in \mathcal{S}(X)$ for each $\varepsilon > 0$, and there is a constant M for **(S2)** which is independent of ε. Replacing $t > 0$ by $\lambda \in \Sigma_{\pi-\phi}$ and observing that the functions $\varphi_\varepsilon(\lambda) = (\varepsilon+\lambda)/(1+\varepsilon\lambda)$ are leaving all sectors Σ_ϕ invariant, we obtain the following result.

Proposition 3.1.4. *Suppose $A \in \mathcal{PS}(X)$, and let A_ε be defined according to* (3.6)*. Then A_ε is bounded, sectorial, and invertible, for each $\varepsilon > 0$. The spectral angle of A_ε satisfies $\phi_{A_\varepsilon} \le \phi_A$, and the bounds $C_{\pi-\phi}(A_\varepsilon)$ and $M_{\pi-\phi}(A_\varepsilon)$ are uniformly bounded w.r.t. $\varepsilon > 0$, for each fixed $\phi > \phi_A$. Moreover,*

$$\lim_{\varepsilon\to 0} (\lambda + A_\varepsilon)^{-1} = (\lambda + A)^{-1} \quad \textit{in } \mathcal{B}(X) \textit{ for each } \lambda \in \Sigma_{\pi-\phi_A}, \tag{3.7}$$

and in case A is sectorial,

$$\lim_{\varepsilon\to 0} A_\varepsilon x = Ax \quad \textit{for each } x \in \mathsf{D}(A), \tag{3.8}$$
$$\lim_{\varepsilon\to 0} A_\varepsilon^{-1} x = A^{-1}x \quad \textit{for each} x \in \mathsf{R}(A).$$

In later sections we shall frequently make use of the approximations A_ε.

1.3 Perturbation Theory

In this section we consider the behaviour of the class $\mathcal{S}(X)$ w.r.t. perturbations. For this purpose, suppose $A \in \mathcal{S}(X)$, and let B be a closed linear operator in X which is subordinate to A in the sense that $\mathsf{D}(A) \subset \mathsf{D}(B)$ and

$$|Bx| \le b|Ax|, \quad \text{for all } x \in \mathsf{D}(A), \tag{3.9}$$

with some constant $b \ge 0$. If $b < 1$ then $A+B$ defined by

$$(A+B)x = Ax + Bx, \quad x \in \mathsf{D}(A+B) = \mathsf{D}(A), \tag{3.10}$$

is also closed, densely defined, and $\mathsf{N}(A+B) = \{0\}$. In fact, if $(A+B)x = 0$ then $|Ax| = |Bx| \le b|Ax|$, hence $Ax = 0$, which by injectivity of A in turn implies $x = 0$. The operator $K := BA^{-1}$ with domain $\mathsf{D}(K) = \mathsf{R}(A)$ is densely defined and bounded by $b < 1$, hence by density of $\mathsf{R}(A)$ in X admits a unique bounded extension to all of X which we again denote by K. Then $A+B$ can be factored as $A+B = (I+K)A$, and $I+K$ is invertible, by $b < 1$. Therefore, if $x^* \perp \mathsf{R}(A+B)$ then $(I+K^*)x^* \perp \mathsf{R}(A)$, hence $(I+K^*)x^* = 0$ by density of $\mathsf{R}(A)$ in X, and then $x^* = 0$, by invertibility of $I+K^*$. This shows that $\mathsf{R}(A+B)$ is also dense in X. Moreover, for $r > 0$ we have

$$r + A + B = (1 + B(r+A)^{-1})(r+A),$$

hence $r+A+B$ is invertible, provided $|B(r+A)^{-1}|<1$, and then

$$(r+A+B)^{-1}=(r+A)^{-1}(1+B(r+A)^{-1})^{-1}. \tag{3.11}$$

This implies that $A+B$ is also sectorial, whenever $bC_0(A)<1$, where $C_0(A)$ is defined by (3.5), and then

$$|r(r+A+B)^{-1}|\leq \frac{M_0(A)}{1-bC_0(A)}, \quad \text{for all } r>0, \tag{3.12}$$

with $M_0(A)$ also given by (3.5). Replacing $r>0$ by $\lambda\in\Sigma_{\pi-\phi}$ in the above argument we also obtain an estimate for the spectral angle of $A+B$, namely

$$\phi_{A+B}\leq \inf\{\phi>\phi_A:\ bC_{\pi-\phi}(A)<1\}. \tag{3.13}$$

Thus the class of operators B satisfying (3.9) with $bC_0(A)<1$ forms an admissible class of perturbations for $A\in\mathcal{S}(X)$.

Theorem 3.1.5. *Suppose $A\in\mathcal{S}(X)$, B linear with $\mathsf{D}(A)\subset\mathsf{D}(B)$ such that* (3.9) *holds, and let $A+B$ be defined by* (3.10).

Then $bC_0(A)<1$ implies $A+B\in\mathcal{S}(X)$, and the spectral angle ϕ_{A+B} of $A+B$ satisfies (3.13).

Let us next consider perturbations B which instead of (3.9) are subject to

$$|Bx|\leq b|Ax|+a|x|, \quad \text{for all } x\in\mathsf{D}(A), \tag{3.14}$$

where $a,b\geq 0$. Then even for small b one cannot expect that $A\in\mathcal{S}(X)$ implies $A+B\in\mathcal{S}(X)$, in general. For example $Bx=-ax$ satisfies (3.14) with $b=0$, but $A+B\notin\mathcal{S}(X)$ unless $\sigma(A)\cap[0,a)=\emptyset$. However, $\mathcal{S}(X)$ is invariant w.r.t. right shifts, and therefore it is reasonable to ask whether $\mu+A+B$ is sectorial, for some $\mu\geq 0$. Now (3.14) implies

$$\begin{aligned}|B(\mu+A)^{-1}| &\leq a|(\mu+A)^{-1}|+b|A(\mu+A)^{-1}|\\ &\leq \frac{aM_0(A)}{\mu}+bC_0(A),\end{aligned} \tag{3.15}$$

hence $\mu+A+B$ is invertible provided $aM_0(A)/\mu+bC_0(A)<1$, i.e., if $bC_0(A)<1$ and $\mu>\mu_0:=aM_0(A)/(1-bC_0(A))$, and then

$$|(\mu+A+B)^{-1}|\leq \frac{M_0(A)}{1-bC_0(A)}\frac{1}{\mu-\mu_0}, \quad \text{for all } \mu>\mu_0. \tag{3.16}$$

This shows that $\mu+A+B\in\mathcal{S}(X)$ if $bC_0(A)<1$ and $\mu\geq\mu_0$.

Similarly, applying Theorem 3.1.5 to the pair $(\mu+A,B)$ instead of (A,B) we obtain the following result.

Corollary 3.1.6. *Suppose $A \in \mathcal{PS}(X)$, B linear with $\mathsf{D}(A) \subset \mathsf{D}(B)$ such that* (3.14) *holds, and let $A + B$ be defined by* (3.10).

Then there are numbers $b_0 > 0$ and $\mu_0 \geq 0$ such that $\mu + A + B \in \mathcal{S}(X)$, whenever $b < b_0$ and $\mu \geq \mu_0$.

It should be mentioned that the condition of *lower order type*

$$|Bx| \leq a|x| + b|A^\alpha x|, \quad \text{for all } x \in \mathsf{D}(A), \tag{3.17}$$

where $a, b \geq 0$ and $\alpha \in [0, 1)$, implies (3.14) via the moment inequality, see (3.55),

$$|A^\alpha x| \leq k|Ax|^\alpha |x|^{1-\alpha}, \quad x \in \mathsf{D}(A), \tag{3.18}$$

for any $b > 0$. For the definition of A^α as well as for (3.18) we refer to the next subsections. In fact, (3.17) and (3.18) yield

$$|Bx| \leq a|x| + b|A^\alpha x| \leq a|x| + bk|Ax|^\alpha |x|^{1-\alpha},$$

hence by means of Young's inequality

$$|Bx| \leq (a + bk(1-\alpha)\varepsilon^{-\alpha/(1-\alpha)})|x| + \alpha bk\varepsilon|Ax|, \quad x \in \mathsf{D}(A).$$

Since ε can be chosen arbitrarily small, Corollary 3.1.6 applies in particular to perturbations satisfying (3.17) without restrictions on a and b, provided $\alpha \in [0, 1)$.

Next we consider A-compact perturbations, i.e., operators B in X such that $B : X_A \to X$ is compact. For such perturbations we have

Lemma 3.1.7. *Let $A \in \mathcal{PS}(X)$, B a linear operator in X such that $B : X_A \to X$ is compact. Furthermore, assume either of the following two conditions*

(i) *B is closable in X,*

(ii) *X is reflexive.*

Then for each $b > 0$ there is $a > 0$ such that (3.14) *is valid.*

Proof. We may assume that A is invertible; replace A by $A+1$ otherwise. Suppose the assertion does not hold. Then there is a constant $b_0 > 0$ and a sequence $(x_n) \subset \mathsf{D}(A)$ with $|Ax_n| = 1$ such that

$$|Bx_n| \geq b_0|Ax_n| + n|x_n| = b_0 + n|x_n|, \quad n \in \mathbb{N}.$$

As B is A-compact, there is a convergent subsequence $Bx_{n_k} \to y$ in X, hence $x_{n_k} \to 0$ in X, and $|y| \geq b_0 > 0$.

If (i) holds, then $y = 0$ as B is closable in X, which yields a contradiction to $y \neq 0$.

If (ii) holds, then there is a weakly-convergent subsequence Ax_{n_k}, its limit is 0 as $x_{n_k} \to 0$ in X. Therefore (x_{n_k}) converges to 0 weakly in X_A, hence $Bx_{n_k} \to y = 0$ strongly in X by compactness, and so we again obtain a contradiction to $y \neq 0$. □

As another consequence of Theorem 3.1.5, let us consider multiplicative perturbations. So let $A \in \mathcal{S}(X)$ and suppose $C \in \mathcal{B}(X)$; then the operator CA with domain $\mathsf{D}(CA) = \mathsf{D}(A)$ is well- and densely defined, and it is closed if in addition C is invertible. Moreover, the latter property of C shows also that $\mathsf{R}(CA)$ is dense in X. It is more difficult to obtain $\rho(CA) \supset (-\infty, 0)$ and (**S2**) for CA. A very simple case arises if we require C to be such that $|C - I| < 1/C_0(A)$. In fact, then we may write $CA = A + (C - I)A$, and $B = (C - I)A$ is subject to the assumption of Theorem 3.1.5. Note that this condition on C necessarily implies that C is bounded but also invertible since $C_0(A) \geq 1$. Observing that $\mathcal{S}(X)$ is invariant under dilations, as a second corollary to Theorem 3.1.5 we obtain

Corollary 3.1.8. *Suppose $A \in \mathcal{S}(X)$ and that $C \in \mathcal{B}(X)$ satisfies the condition*

$$|C - r_C| < r_C/C_0(A), \quad \textit{for some } r_C > 0. \tag{3.19}$$

Then CA and AC with natural domains $\mathsf{D}(CA) = \mathsf{D}(A)$ and $\mathsf{D}(AC) = C^{-1}\mathsf{D}(A)$ belong to $\mathcal{S}(X)$.

The assertion for AC follows by the similarity transform $AC = C^{-1}(CA)C$ of CA.

1.4 The Dunford Functional Calculus

In this subsection we want to develop the functional calculus for pseudo-sectorial operators. For this purpose we first introduce the following function algebras. Let $\phi \in (0, \pi]$ and define the algebra of holomorphic functions on Σ_ϕ

$$H(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ is holomorphic}\}, \tag{3.20}$$

and

$$H^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} : \ f \text{ is holomorphic and bounded}\}. \tag{3.21}$$

$H^\infty(\Sigma_\phi)$ with norm

$$|f|_{H^\infty(\Sigma_\phi)} = \sup\{|f(\lambda)| : \ |\arg\lambda| < \phi\} \tag{3.22}$$

is a Banach algebra. First we assume $B \in \mathcal{S}(X)$ to be bounded and invertible, and fix $\phi > \phi_B$. Then the well-known Dunford calculus for bounded linear operators applies. In fact, in this situation the spectrum $\sigma(B)$ is a compact subset of Σ_ϕ, hence choosing a simple closed path Γ_B in Σ_ϕ surrounding $\sigma(B)$ counterclockwise we define

$$f(B) = \frac{1}{2\pi i}\int_{\Gamma_B} f(\lambda)(\lambda - B)^{-1}\, d\lambda, \quad f \in H(\Sigma_\phi). \tag{3.23}$$

Since Γ_B is compact there are no convergence problems with the integral in this formula, and it defines an algebra homomorphism from $H(\Sigma_\phi)$ to $\mathcal{B}(X)$.

(3.23) can be used as a starting point to define the functional calculus for arbitrary pseudo-sectorial operators A in X. To achieve this, a main idea is to take $B = A_\varepsilon$, the approximations of A introduced in (3.6), and to pass to the limit

$\varepsilon \to 0+$. But then we first have to make the integration path Γ_B independent of B. This can be done in several ways at the expense that we have to restrict the function algebra $H(\Sigma_\phi)$.

(i) A natural approach is to deform the integration path Γ_B into Γ defined by $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$, where $\phi_A < \psi < \phi$. We will do this in two steps. First we deform Γ_B into the path $\Gamma_{r,R}$ defined by

$$\Gamma_{r,R} = e^{-i\psi}[r,R] \cup Re^{i[-\psi,\psi]} \cup e^{i\psi}[R,r] \cup re^{i[\psi,-\psi]}. \tag{3.24}$$

Here the numbers $0 < r < R$ should be chosen such that $R > |B|$ and $r < |B^{-1}|^{-1}$. By means of Cauchy's theorem we then obtain

$$f(B) = \frac{1}{2\pi i}\int_{\Gamma_{r,R}} f(\lambda)(\lambda - B)^{-1}\,d\lambda, \quad f \in H(\Sigma_\phi), \tag{3.25}$$

since $\Gamma_{r,R}$ is also a simple compact Lipschitz curve surrounding $\sigma(B)$ counterclockwise. But we still have the dependence of the integration path in (3.25) on the norms of B and B^{-1}.

Next we let $r \to 0+$ and $R \to \infty$. This cannot be done for arbitrary $f \in H(\Sigma_\phi)$, but by means of Lebesgue's convergence theorem it works for the subspace $H_0(\Sigma_\phi)$ defined according to

$$H_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} H_{\alpha,\beta}(\Sigma_\phi), \quad \text{where} \tag{3.26}$$

$$H_{\alpha,\beta}(\Sigma_\phi) = \{f \in H(\Sigma_\phi) :\ |f|^\phi_{\alpha,\beta} < \infty\}, \quad \text{and} \tag{3.27}$$

$$|f|^\phi_{\alpha,\beta} = \sup_{|\lambda|\le 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)|. \tag{3.28}$$

With $\Gamma = (\infty,0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ this yields (3.23) with Γ_B replaced by the contour Γ which is independent of r, R.

Now let $A \in \mathcal{PS}(X)$ be arbitrary. Employing the approximations A_ε introduced before, setting $B = A_\varepsilon$ and using Proposition 3.1.4, we may pass to the limit $\varepsilon \to 0+$, to obtain the following result.

Proposition 3.1.9. *Let $A \in \mathcal{PS}(X)$, fix any $\phi \in (\phi_A, \pi]$, and let $H_0(\Sigma_\phi)$ be defined as above. Then, with $\Gamma = (\infty,0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$, the Dunford integral*

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - A)^{-1}\,d\lambda, \quad f \in H_0(\Sigma_\phi), \tag{3.29}$$

defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : H_0(\Sigma_\phi) \to \mathcal{B}(X)$ which is a bounded algebra homomorphism. Moreover, we have

$$\lim_{\varepsilon\to 0+} f(A_\varepsilon) = f(A) \quad \text{in } \mathcal{B}(X), \tag{3.30}$$

and $\{f(A_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{B}(X)$ is uniformly bounded, for each $f \in H_0(\Sigma_\phi)$.

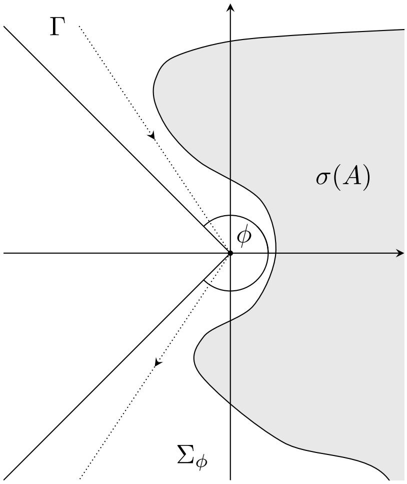

Figure 3.1: Integration path for the Dunford integral.

Observe that boundedness of Φ_A is understood in the sense of inductive limits. This means that we have estimates of the form

$$|f(A)| \leq C|f|^{\phi}_{\alpha,\beta}, \quad \text{for } f \in H_{\alpha,\beta}(\Sigma_\phi),$$

where C depends only on A, ϕ, α, and β. This follows directly from (3.29). In virtue of Proposition 3.1.4, a similar estimate holds also for A_ε, uniformly in $\varepsilon > 0$.

Remark 3.1.10. Consider the map $\varphi(\lambda) = 1/\lambda$ which maps Σ_ϕ onto itself. Then we have the identity

$$(f \circ \varphi)(A) = f(A^{-1}), \quad \text{for each } f \in H_0(\Sigma_\phi), \tag{3.31}$$

in case $\mathsf{N}(A) = 0$. In fact, the change of variable $\lambda \mapsto 1/\lambda$ yields

$$\begin{aligned}(f \circ \varphi)(A) &= \frac{1}{2\pi i}\int_\Gamma f(1/\lambda)(\lambda - A)^{-1}\, d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma f(\lambda)(1/\lambda - A)^{-1}\, d\lambda/\lambda^2 \\ &= -\frac{1}{2\pi i}\int_\Gamma f(\lambda)(A^{-1} - \lambda)^{-1}A^{-1}\, d\lambda/\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)[-1/\lambda + (\lambda - A^{-1})^{-1}]\, d\lambda = f(A^{-1}),\end{aligned}$$

where the last equality follows from Cauchy's theorem.

There is a simple but useful extensions of the Dunford calculus in Proposition 3.1.9. Namely, in case $f \in H(\Sigma_\phi)$ is holomorphic in a neighbourhood of zero and such that $\lambda^\alpha f(\lambda) \in H^\infty(\Sigma_\phi)$ for some $\alpha > 0$, then f belongs to $H_0(\Sigma_\phi)$ if and only if $f(0) = 0$. But in case $f(0) \neq 0$ we may write $f(\lambda) = f_0(\lambda) + f(0)/(1+\lambda)$, where $f_0 \in H_0^\infty(\Sigma_\phi)$, hence the definition $f(A) := f_0(A) + f(0)(1+A)^{-1}$ is reasonable. We want to derive a different representation formula for $f(A)$ in such situations. For this purpose we modify the integration path Γ_B in the representation (3.23) of $f(B)$ into

$$\Gamma_\delta = (\infty, \delta]e^{i\psi} \cup \delta e^{i[\psi, 2\pi-\psi]} \cup [\delta, \infty)e^{-i\psi},$$

and employing Cauchy's theorem we obtain

$$\begin{aligned} f(B) &= \frac{1}{2\pi i}\int_{\Gamma_\delta} f(\lambda)(\lambda - B)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_\delta} f_0(\lambda)(\lambda - B)^{-1}\, d\lambda + \frac{1}{2\pi i}\int_{\Gamma_\delta} f(0)(1+\lambda)^{-1}(\lambda - B)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma} f_0(\lambda)(\lambda - B)^{-1}\, d\lambda + f(0)(1+B)^{-1} \\ &= f_0(B) + f(0)(1+B)^{-1}. \end{aligned}$$

Setting again $B = A_\varepsilon$ and passing to the limit $\varepsilon \to 0+$, we get

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\delta} f(\lambda)(\lambda - A)^{-1}\, d\lambda, \tag{3.32}$$

where δ is small enough but arbitrary otherwise. Define the corresponding space by

$$H_a(\Sigma_\phi) = \{f \in \bigcup_{\beta<0} H_{0,\beta}(\Sigma_\phi) : f \text{ is holomorphic in a neighbourhood of } 0\}.$$

Then we have the following result.

Corollary 3.1.11. *Let $A \in \mathcal{PS}(X)$ with spectral angle ϕ_A, fix any $\phi > \phi_A$, and let $H_a(\Sigma_\phi)$ be defined as above.*

Then the Dunford map $\Phi : H_a(\Sigma_\phi) \to \mathcal{B}(X)$ defined via $\Phi(f) = f(A)$, where $f(A)$ is given by the Dunford integral (3.32)*, is well-defined and an algebra homomorphism. It coincides with the Dunford map of Proposition* 3.1.9*, and we have the relation*

$$f(A) = f_0(A) + f(0)(1+A)^{-1},$$

where $f_0(\lambda) = f(\lambda) - f(0)/(1+\lambda)$ belongs to $H_0(\Sigma_\phi)$. In particular, for the functions $g_\mu(\lambda) = 1/(\lambda - \mu)$ with $\mu \notin \overline{\Sigma_\phi}$ we have $g_\mu(A) = (A-\mu)^{-1}$. The convergence assertion (3.30) *of Proposition* 3.1.9 *is also valid for $H_a(\Sigma_\phi)$.*

Remark 3.1.12. (a) A similar result can be obtained for functions $f \in H(\Sigma_\phi)$ which are holomorphic at infinity and decay polynomially at zero. With $f_\infty(\lambda) = f(\lambda) - f(\infty)\lambda/(1+\lambda)$ we then have the relation

$$f(A) = f_\infty(A) + f(\infty)A(I+A)^{-1},$$

and there is an integral representation corresponding to (3.32) which we do not explicitly state here.

(b) If $f \in H(\Sigma_\phi)$ is holomorphic at infinity and at zero we have correspondingly

$$f(A) = f_{0,\infty}(A) + f(0)(I+A)^{-1} + f(\infty)A(I+A)^{-1}.$$

With $\delta > 0$ small and $\rho > \delta$ large one obtains alternatively

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\delta^\rho} f(\lambda)(\lambda - A)^{-1}\,d\lambda,$$

where

$$\Gamma_\delta^\rho = [\rho,\delta]e^{i\psi} \cup \delta e^{i[\psi,2\pi-\psi]} \cup [\delta,\rho]e^{-i\psi} \cup \rho e^{i[2\pi-\psi,\psi]}.$$

The proof of these facts is left to the reader.

(c) The functions $\varphi_\varepsilon(\lambda) = (\varepsilon+\lambda)/(1+\varepsilon\lambda)$ map Σ_ϕ into itself, and $\varphi(0) = \varepsilon$, $\varphi(\infty) = 1/\varepsilon$. This means that $f_\varepsilon = f \circ \varphi_\varepsilon$ belongs to $H(\Sigma_\phi)$ and is holomorphic at infinity and at zero, for any $f \in H(\Sigma_\phi)$. Therefore, (b) of this Remark applies and we obtain

$$(f \circ \varphi_\varepsilon)(A) = f(A_\varepsilon).$$

In fact, the identity

$$\begin{aligned}
(\lambda - A_\varepsilon)^{-1} &= (1+\varepsilon A)(\lambda - \varepsilon - (1-\lambda\varepsilon)A)^{-1}\\
&= (1+\varepsilon A)(1-\varepsilon\lambda)^{-1}(\frac{\lambda-\varepsilon}{1-\varepsilon\lambda} - A)^{-1}\\
&= \frac{1-\varepsilon^2}{(1-\varepsilon\lambda)^2}(\frac{\lambda-\varepsilon}{1-\varepsilon\lambda} - A)^{-1} - \frac{\varepsilon}{1-\varepsilon\lambda}
\end{aligned}$$

and the variable transformation $z = (\lambda-\varepsilon)/(1-\varepsilon\lambda)$, i.e., $\lambda = \varphi_\varepsilon(z)$ yield

$$\begin{aligned}
f(A_\varepsilon) &= \frac{1}{2\pi i}\int_{\Gamma_r^R} f(\lambda)(\lambda - A_\varepsilon)^{-1}\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_r^R} f(\lambda)(\frac{\lambda-\varepsilon}{1-\varepsilon\lambda} - A)^{-1}\frac{1-\varepsilon^2}{(1-\varepsilon\lambda)^2}\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\varphi_\varepsilon(\Gamma_r^R)} f(\varphi_\varepsilon(z))(z-A)^{-1}\,dz = (f\circ\varphi_\varepsilon)(A),
\end{aligned}$$

employing once more Cauchy's theorem.

3.2 The Derivation Operator

This section is devoted to the most elementary operator in analysis, the derivation operator d/dt. We will consider this operator on intervals $J = \mathbb{R}$, $J = \mathbb{R}_+$, and on $J = (0,a)$, in various spaces.

2.1. The Whole Line Case

Let $J = \mathbb{R}$. In the sequel we will use the notation $Y_p(\mathbb{R}) = L_p(\mathbb{R}; Y)$, where Y denotes a Banach space and $p \in [1, \infty]$, $Y_b(\mathbb{R}) = C_b(\mathbb{R}; Y)$, $Y_{ub}(\mathbb{R}) = C_{ub}(\mathbb{R}; Y)$, and $Y_0(\mathbb{R}) = C_0(\mathbb{R}; Y)$. Define B_p in $Y_p(\mathbb{R})$ by means of

$$(B_p u)(t) = \dot{u}(t), \quad t \in \mathbb{R}, \quad u \in \mathsf{D}(B_p) = H^1_p(\mathbb{R}; Y), \tag{3.33}$$

for $p \in [1, \infty]$ and $\mathsf{D}(B_p) = C^1_p(\mathbb{R}; Y)$ for $p \in \{0, b, ub\}$. It is easy to see that B_p is closed, and B_p is densely defined except for $p \in \{\infty, b\}$. Since $\dot{u}(t) = 0$ for all $t \in \mathbb{R}$ implies that u is constant, we have $\mathsf{N}(B_p) = \{0\}$ for all $p \in [1, \infty) \cup \{0\}$, while $\mathsf{N}(B_b) = \mathsf{N}(B_{ub}) = \mathsf{N}(B_\infty) \equiv Y$.

Next consider the range of B_p for $p \in (1, \infty) \cup \{0\}$. If $f \in C(\mathbb{R}; Y)$ has compact support and mean value $Mf = \int_{-\infty}^{\infty} f(s)\, ds = 0$, then the solution u of $\dot{u} = f$ on $\mathbb{R}$ belongs to $C^1(\mathbb{R}; Y)$ and has compact support as well. Since the set of such functions f is dense in $Y_p(\mathbb{R})$ for $1 < p < \infty$ and for $p = 0$, by the following lemma, we see that $\mathsf{R}(B_p)$ is dense in $Y_p(\mathbb{R})$, $1 < p < \infty$ and $p = 0$.

Lemma 3.2.1. *Let Y be a Banach space, $\varphi \in L_1(\mathbb{R}) \cap C_0(\mathbb{R})$ such that $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(t)\, dt = 1$, and define $\varphi_\varepsilon(t) = \varepsilon\varphi(\varepsilon t)$, $t \in \mathbb{R}$, $\varepsilon > 0$.*

Then for $f \in Y_1(\mathbb{R}) + Y_\infty(\mathbb{R})$ the approximations f_ε of f defined by $f_\varepsilon = \varphi_\varepsilon * f$ have the following properties.

(i) *$f_\varepsilon \to_{\varepsilon\to\infty} f$ in $Y_p(\mathbb{R})$, for each $f \in Y_p(\mathbb{R})$, $p \in [1, \infty) \cup \{0, ub\}$;*

(ii) *$f_\varepsilon \to_{\varepsilon\to 0+} 0$ in $Y_p(\mathbb{R})$, for each $f \in Y_p(\mathbb{R})$, $p \in (1, \infty) \cup \{0\}$.*

Proof. (i) Let $T(t)$ denote the translation group defined by

$$[T(t)f](s) = f(t + s), \quad t, s \in \mathbb{R}.$$

Then for $p \in [1, \infty) \cup \{0, ub\}$ we have $T(t)f \to f$ in $Y_p(\mathbb{R})$ as $t \to 0$, for each $f \in Y_p(\mathbb{R})$. Therefore with $\int_{\mathbb{R}} \varphi(t)\, dt = 1$ we obtain

$$\begin{aligned}
|f_\varepsilon - f|_p &= |\int_{\mathbb{R}} \varphi_\varepsilon(s)([T(-s)f] - f)\, ds|_p \\
&\leq \int_{|s|\leq R} \varphi_\varepsilon(s)|T(-s)f - f|_p\, ds + \int_{|s|\geq R} \varphi_\varepsilon(s)(|T(-s)f|_p + |f|_p)\, ds \\
&\leq \sup_{|s|\leq R} |T(-s)f - f|_p + 2|f|_p \int_{|s|\geq R\varepsilon} |\varphi(s)|\, ds.
\end{aligned}$$

Now, given an arbitrary number $\eta > 0$, choose first $R > 0$ such that $|T(s)f - f|_p \leq \eta/2$ for all $|s| \leq R$, and then for this fixed R a number $\varepsilon_\eta > 0$ such that $2|f|_p \int_{|s|\geq R\varepsilon_\eta} |\varphi(s)|\, ds < \eta/2|f|_p$. Then $|f_\varepsilon - f|_p \leq \eta$ for all $\varepsilon \geq \varepsilon_\eta$, which implies assertion (i).

(ii) To prove the second assertion, note that by Young's inequality $|f_\varepsilon|_p \leq |f|_p$, for each $f \in Y_p(\mathbb{R})$. On the other hand, $|f_\varepsilon|_\infty \leq \varepsilon|\varphi|_\infty|f|_1$. This implies $|f_\varepsilon|_\infty \to 0$

as $\varepsilon \to 0+$, for each $f \in Y_1(\mathbb{R})$, hence also

$$|f_\varepsilon|_p \le |f_\varepsilon|_\infty^{1-1/p} |f_\varepsilon|_1^{1/p} \le [|\varphi|_\infty \varepsilon]^{1-1/p} |f|_1 \to 0+$$

as $\varepsilon \to 0+$, for each $f \in Y_1(\mathbb{R}) \cap Y_0(\mathbb{R})$. By (i) and a cut off procedure such functions are dense in $Y_p(\mathbb{R})$, $p \in (1,\infty) \cup \{0\}$, and so assertion (ii) follows. □

For $p = 1$, $Mf = 0$ is a necessary condition for $f \in \mathsf{R}(B_1)$, hence $\mathsf{R}(B_1) \subset \mathsf{N}(M)$ and because M is bounded, $\mathsf{N}(M) \neq Y_1(\mathbb{R})$ is closed and so $\mathsf{R}(B_1)$ is not dense in $Y_1(\mathbb{R})$.

The kernel $\mathsf{N}(B_p)$ consists of the constant functions for $p \in \{b, ub, \infty\}$, hence $\dim \mathsf{N}(B_p) = 1$, and B_p is pseudo-sectorial as we shall see below, so $\mathsf{R}(B_p)$ cannot be dense for these p, by Theorem 3.1.2.

To compute the spectrum of B_p, we consider the equation

$$\lambda u(t) + \dot{u}(t) = f(t), \quad t \in \mathbb{R}. \tag{3.34}$$

For $\operatorname{Re}\lambda > 0$ a solution is given by

$$u_\lambda(t) = \int_0^\infty e^{-\lambda s} f(t-s)\, ds = \int_{-\infty}^t e^{-\lambda(t-s)} f(s)\, ds, \quad t \in \mathbb{R},$$

and we have the estimate

$$|u_\lambda|_p \le |f|_p / \operatorname{Re}\lambda, \quad \operatorname{Re}\lambda > 0.$$

On the other hand, for $\operatorname{Re}\lambda < 0$ a solution is

$$u_\lambda(t) = -\int_{-\infty}^0 e^{-\lambda s} f(t-s)\, ds = -\int_t^\infty e^{-\lambda(t-s)} f(s)\, ds, \quad t \in \mathbb{R},$$

and

$$|u_\lambda|_p \le |f|_p / |\operatorname{Re}\lambda|, \quad \operatorname{Re}\lambda < 0.$$

Since the general solution of (3.34) is given by $u(t) = u_\lambda(t) + ce^{-\lambda t}$, and for $\operatorname{Re}\lambda \neq 0$ the function $e^{-\lambda t}$ is not in $Y_p(\mathbb{R})$, we have $\mathsf{N}(\lambda + B_p) = 0$ for all $\operatorname{Re}\lambda \neq 0$. Summarizing we have

Proposition 3.2.2. *Let $J = \mathbb{R}$. Then the operators B_p and $-B_p$ defined above are pseudo-sectorial in $Y_p(\mathbb{R})$ with spectral angles $\phi_{B_p} = \phi_{-B_p} = \pi/2$, for all $p \in [1,\infty] \cup \{0, b, ub\}$. The domains of B_p are dense for all $p \in [1,\infty) \cup \{0, ub\}$, their kernels are trivial for all $p \in [1,\infty) \cup \{0\}$, and $\mathsf{R}(B_p)$ is dense for all $p \in (1,\infty) \cup \{0\}$. Consequently, B_p and $-B_p$ are sectorial iff $p \in (1,\infty) \cup \{0\}$.*

2.2 The Half-Line Case

Next we consider the operator B_p on $J = \mathbb{R}_+$. This time we let $Y_p(\mathbb{R}_+) = L_p(\mathbb{R}_+; Y)$ for $p \in [1,\infty]$, $Y_p(\mathbb{R}_+) = {}_0C_p(\bar{\mathbb{R}}_+; Y)$ for $p \in \{0, b, ub\}$, where the subscript 0 indicates zero trace at $t=0$. Define

$$(B_p u)(t) = \dot{u}(t),\ t \in J,\ u \in \mathsf{D}(B_p) = {}_0H^1_p(\mathbb{R}_+;Y), \tag{3.35}$$

for $p \in [1,\infty]$ and $\mathsf{D}(B_p) = {}_0C_p(\mathbb{R}_+;Y) \cap C^1_p(\mathbb{R}_+;Y)$ for $p \in \{0,b,ub\}$. As in the case of $J=\mathbb{R}$, it is easy to see that B_p is closed, and that B_p is densely defined except for $p \in \{\infty, b\}$. Since $\dot{u}(t)=0$ for all $t \in \mathbb{R}_+$ implies that u is constant hence $u(t) \equiv u(0) = 0$, we have $\mathsf{N}(B_p) = 0$ for all $p \in [1,\infty] \cup \{0,b,ub\}$.

To compute the spectrum of B_p for $J = \mathbb{R}_+$, consider the problem

$$\lambda u(t) + \dot{u}(t) = f(t),\ t > 0, \quad u(0) = 0.$$

For all $\lambda \in \mathbb{C}$ its solution is given by

$$u_\lambda(t) = \int_0^t e^{-\lambda s} f(t-s)\, ds, \quad t \in \mathbb{R}_+,$$

and we have the estimate

$$|u_\lambda|_p \le |f|_p / \mathrm{Re}\,\lambda, \quad \mathrm{Re}\,\lambda > 0.$$

Concerning the range of B_p, note that necessarily $(B_p^{-1}f)(t) = \int_0^t f(s)\,ds$ whenever $f \in \mathsf{R}(B_p)$. Since the set of continuous functions f with compact support in $(0,\infty)$ and mean value $Mf = \int_0^\infty f(s)\,ds = 0$ is dense in $Y_p(\mathbb{R}_+)$ for each $p \in (1,\infty) \cup \{0\}$, we see that the range of B_p for such p is dense. On the other hand, as in the case of $J = \mathbb{R}$ we see that $\mathsf{R}(B_1)$ is not dense, and this is also the case for $p \in \{\infty, b\}$. In fact, consider a Hahn-Banach extension of the limit functional $\langle l|f\rangle := \lim_{t\to\infty} f(t)$ from the closed subspace $C_l(\bar{\mathbb{R}}_+;Y)$ of $Y_{ub}(\mathbb{R}_+)$ to $Y_b(\mathbb{R}_+)$. Then for $f \in \mathsf{R}(B_p)$, $p \in \{b, ub\}$, $f \in C_l(\bar{\mathbb{R}}_+;Y)$ we must necessarily have $\langle l|f\rangle = 0$, which means $\mathsf{R}(B_p) \subset \mathsf{N}(l)$. From these considerations we obtain

Proposition 3.2.3. *Let $J = \mathbb{R}_+$. Then the operator B_p defined by* (3.35) *is injective and pseudo-sectorial in $Y_p(\mathbb{R}_+)$ with spectral angle $\phi_{B_p} = \pi/2$, for all $p \in [1,\infty] \cup \{0,b,ub\}$. The domain of B_p is dense for all $p \in [1,\infty) \cup \{0,ub\}$, and $\mathsf{R}(B_p)$ is dense for all $p \in (1,\infty) \cup \{0\}$. Consequently, B_p is sectorial iff $p \in (1,\infty) \cup \{0\}$.*

2.3 Finite Interval

Here we consider the operator B_p on the finite interval $J = (0,a)$. This time we let $Y_p(J) = L_p(J;Y)$ for $p \in [1,\infty]$, $Y_p(J) = {}_0C_p(\bar{J};Y)$ for $p \in \{0,b,ub\}$, where as before the subscript 0 indicates trace zero at $t=0$. Define

$$(B_p u)(t) = \dot{u}(t),\ t \in J,\ u \in \mathsf{D}(B_p) = {}_0H^1_p(J;Y), \tag{3.36}$$

for $p \in [1,\infty]$ and $\mathsf{D}(B_p) = {}_0C_p(J;Y) \cap C^1_p(J;Y)$ for $p \in \{0,b,ub\}$. As in the case of $J = \mathbb{R}_+$, it is easy to see that B_p is closed, injective, and that B_p is densely defined except for $p = \infty$.

This time the spectrum of B_p is empty for each p, in fact we have the relation

$$(\lambda + B_p)^{-1} f(t) = u_\lambda(t) = \int_0^t e^{-\lambda s} f(t-s)\, ds, \quad t \in J,\ \lambda \in \mathbb{C},$$

$$|u_\lambda|_p \le |f|_p (1 - e^{-\mathrm{Re}\,\lambda a})/\mathrm{Re}\,\lambda, \quad \mathrm{Re}\,\lambda \neq 0,$$

and

$$|u_\lambda|_p \le |f|_p a, \quad \mathrm{Re}\,\lambda = 0.$$

Therefore, although $\sigma(B_p) = \emptyset$, B_p still has spectral angle $\pi/2$. More precisely we have

Proposition 3.2.4. *Let $J = (0, a)$. Then the operator B_p defined by (3.36) is invertible and pseudo-sectorial in $Y_p(J)$ with spectral angle $\phi_{B_p} = \pi/2$, for all $p \in [1, \infty] \cup \{0, b, ub\}$. The domain of B_p is dense for all $p \in [1, \infty) \cup \{0, b, ub\}$, hence, B_p is sectorial iff $p \neq \infty$.*

It is instructive to have a look at the functional calculus for B_p. Since the resolvent of B_p admits the kernel representation

$$(\lambda - B_p)^{-1} w(t) = -\int_J e_\lambda(t-s) w(s)\, ds, \quad t \in J,$$

where $e_\lambda(t) = e^{\lambda t}$ for $t > 0$, $e_\lambda(t) = 0$ for $t \le 0$, for a function $f \in H_0(\Sigma_\phi)$, $\phi > \pi/2$, the operators $f(B_p)$ admit a kernel representation as well, namely

$$[f(B_p)w](t) = \int_J k_f(t-s) w(s)\, ds, \quad t \in J.$$

The kernel $k_f(t)$ is obtained as the contour integral

$$k_f(t) = -\frac{1}{2\pi i} \int_\Gamma f(\lambda) e_\lambda(t)\, d\lambda,$$

in particular $k_f(t) = 0$ for $t \le 0$. The contour Γ is chosen as in Section 3.1.4. This is precisely the inversion formula for the Laplace transform, i.e., f and k_f are related by $\widehat{k_f}(\lambda) = f(\lambda)$, for $\lambda > 0$, say.

The approximations $(B_p)_\varepsilon$ of B_p introduced in Section 3.1.2 also admit a kernel representation. In fact, the functions $f_\varepsilon(\lambda) = (\varepsilon + \lambda)/(1 + \varepsilon\lambda)$ are the Laplace transforms of $k_\varepsilon(t) = \delta_0(t)/\varepsilon + (1 - 1/\varepsilon^2) e^{-t/\varepsilon} \eta_0(t)$, where η_0 denotes the Heaviside function, and δ_0 its derivative, the Dirac measure. This implies

$$[(B_p)_\varepsilon w](t) = \varepsilon^{-1} w(t) + (1 - \varepsilon^{-2}) \int_0^t w(t-s) e^{-s/\varepsilon}\, ds, \quad t \in J,\ \varepsilon > 0,$$

the kernel representation of $(B_p)_\varepsilon$.

2.4 Weighted L_p-Spaces

Let Y be a Banach space and assume that $p \in (1,\infty)$ and $1/p < \mu \le 1$. We set

$$L_{p,\mu}(\mathbb{R}_+;Y) := \{f : \mathbb{R}_+ \to Y : \ t^{1-\mu} f \in L_p(\mathbb{R}_+;Y)\}$$

and equip it with the norm $|f|_{L_{p,\mu}(\mathbb{R}_+;Y)} := (\int_0^\infty |t^{1-\mu} f(t)|^p \, dt)^{1/p}$. We also define

$$H^1_{p,\mu}(\mathbb{R}_+;Y) := \{u \in L_{p,\mu}(\mathbb{R}_+;Y) \cap H^1_{1,\mathrm{loc}}(\mathbb{R}_+;Y) : \ \dot{u} \in L_{p,\mu}(\mathbb{R}_+;Y)\}.$$

$H^1_{p,\mu}(\mathbb{R}_+;Y)$ will always be given the norm

$$|u|_{H^1_{p,\mu}} = |u|^p_{L_{p,\mu}(\mathbb{R}_+;Y)} + |\dot{u}|^p_{L_{p,\mu}(\mathbb{R}_+;Y)})^{1/p},$$

which turns it into a Banach space.

Lemma 3.2.5. *Suppose $p \in (1,\infty)$ and $1/p < \mu \le 1$. Then*

(a) $L_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow L_{1,\mathrm{loc}}(\bar{\mathbb{R}}_+;Y)$;

(b) $H^1_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow W^1_{1,\mathrm{loc}}(\bar{\mathbb{R}}_+;Y)$;

(c) *Every function $u \in H^1_{p,\mu}(\mathbb{R}_+;Y)$ has a well-defined trace, that is, $u(0)$ is well-defined in Y.*

Proof. (a) The first assertion follows from

$$\int_0^T |f(t)| \, dt \le (\int_0^T t^{-p'(1-\mu)} \, dt)^{1/p'} (\int_0^T |t^{1-\mu} f(t)|^p \, dt)^{1/p} \le c|f|_{L_{p,\mu}(\mathbb{R}_+;Y)}$$

which is valid provided that $\mu > 1/p$.

(b) This follows from the definition of $H^1_{p,\mu}(\mathbb{R}_+;Y)$ and from (a).

(c) We conclude from (b) that every function $u \in H^1_{p,\mu}(\mathbb{R}_+;Y)$ is locally absolutely continuous, and this yields the assertion in (c). □

In the following we set

$${}_0H^1_{p,\mu}(\mathbb{R}_+;Y) := \{u \in H^1_{p,\mu}(\mathbb{R}_+;Y) : u(0) = 0\}.$$

Then the derivation operator

$$B_{p,\mu} u(t) := \dot{u}(t) := \frac{d}{dt} u(t), \quad t > 0, \quad \mathsf{D}(B_{p,\mu}) := {}_0H^1_{p,\mu}(\mathbb{R}_+;Y) \tag{3.37}$$

is well-defined on $L_{p,\mu}(\mathbb{R}_+;Y)$. It is natural to introduce the mapping

$$\Phi_\mu : L_{p,\mu}(\mathbb{R}_+;Y) \to L_p(\mathbb{R}_+;Y), \quad (\Phi_\mu u)(t) := t^{1-\mu} u(t), \quad t > 0.$$

Next we show that the operator Φ_μ also maps ${}_0H^1_{p,\mu}(\mathbb{R}_+;Y)$ into ${}_0H^1_p(\mathbb{R}_+;Y)$, provided $\mu > 1/p$.

Proposition 3.2.6. *Let $p \in (1, \infty)$ and let $1/p < \mu \leq 1$. Then*

(a) $\Phi_\mu : L_{p,\mu}(\mathbb{R}_+; Y) \to L_p(\mathbb{R}_+; Y)$ *is an isometric isomorphism.*

(b) $\Phi_\mu : {}_0H^1_{p,\mu}(\mathbb{R}_+; Y) \to {}_0H^1_p(\mathbb{R}_+; Y)$ *is a (topological) isomorphism.*

Proof. (a) The assertion in (a) is clear.

(b) (i) We will first show that Φ_μ^{-1} maps ${}_0H^1_p(\mathbb{R}_+; Y)$ into ${}_0H^1_{p,\mu}(\mathbb{R}_+; Y)$. In order to see this, let $v \in {}_0H^1_p(\mathbb{R}_+; Y)$ be given. An easy computation shows that the function $t^{\mu-1}v$ is in $H^1_{p,\mathrm{loc}}(\mathbb{R}_+; Y)$ and that

$$t^{1-\mu}\frac{d}{dt}[t^{\mu-1}v](t) = \dot{v}(t) - (1-\mu)\frac{v(t)}{t}, \quad t > 0. \tag{3.38}$$

By means of Hardy's inequality (see Proposition 3.4.5 below) we can verify that the function v/t belongs to $L_p(\mathbb{R}_+; Y)$. Indeed, we infer from $v(t) = \int_0^t \dot{v}(s)\, ds$ that

$$\Big(\int_0^\infty |t^{-1}v(t)|^p\, dt\Big)^{1/p} = \Big(\int_0^\infty |t^{-1}\int_0^t \dot{v}(s)ds|^p\, dt\Big)^{1/p} \leq p'\Big(\int_0^\infty |\dot{v}(s)|^p ds\Big)^{1/p}. \tag{3.39}$$

We conclude from (3.38)–(3.39) that $\Phi_\mu^{-1}v$ belongs to $H^1_{p,\mu}(\mathbb{R}_+; Y)$, and also that the mapping Φ_μ^{-1} is linear and bounded between the indicated spaces.

(ii) Next we show that $u = \Phi_\mu^{-1}v$ has trace zero. Observing that

$$u(t) = t^{\mu-1}v(t) = t^{\mu-1}\int_0^t \dot{v}(s)\, ds$$

we obtain by Hölder's inequality that $|u(t)| \leq t^{\mu-1/p}(\int_0^t |\dot{v}(s)|^p\, ds)^{1/p}$. This shows that $u(t) \to 0$ as $t \to 0+$.

(iii) Similar arguments show that Φ_μ maps ${}_0H^1_{p,\mu}(\mathbb{R}_+; Y)$ into ${}_0H^1_p(\mathbb{R}_+; Y)$, and that the mapping is bounded and linear. $\square$

We will now consider the derivation operator $B_{p,\mu}$ defined in (3.37). Thanks to Proposition 3.2.6 the operator

$$\bar{B}_{p,\mu} := \Phi_\mu B_{p,\mu}\Phi_\mu^{-1}, \quad \mathsf{D}(\bar{B}_{p,\mu}) := {}_0H^1_p(\mathbb{R}_+; Y), \tag{3.40}$$

which acts on the function space $L_p(\mathbb{R}_+; Y)$, is well-defined. It follows from (3.38) that

$$\bar{B}_{p,\mu} = B_{p,1} + B_0, \quad \text{where} \quad (B_0v)(t) := -(1-\mu)v(t)/t. \tag{3.41}$$

Observe that $\bar{B}_{p,\mu}$ and $B_{p,\mu}$ coincide if $\mu = 1$. Moreover, note that $B_{\mu,p}$ in $L_{p,\mu}(\mathbb{R}_+; Y)$ is similar to $B_{p,1} + B_0$ in $L_p(\mathbb{R}_+; Y)$. It follows from equation (3.39) that B_0 is relatively bounded with respect to $B_{p,1}$, with bound smaller than 1, provided $(1-\mu)p' < 1$, i.e., for $1 \geq \mu > 1/p$. It is now easy to see that the operators $B_{p,\mu}$ and $\bar{B}_{p,\mu}$ share the following properties.

Proposition 3.2.7. *Suppose* $1 < p < \infty$ *and* $1/p < \mu \le 1$. *Then*

(i) $\bar{B}_{p,\mu}$ *is closed and densely defined in* $L_p(\mathbb{R}_+; Y)$. *Moreover,* $\mathsf{N}(\bar{B}_{p,\mu}) = 0$, *and* $\mathsf{R}(\bar{B}_{p,\mu})$ *is dense in* $L_p(\mathbb{R}_+; Y)$.

(ii) $B_{p,\mu}$ *is closed and densely defined in* $L_{p,\mu}(\mathbb{R}_+; Y)$. *Moreover,* $\mathsf{N}(B_{p,\mu}) = 0$, *and* $\mathsf{R}(B_{p,\mu})$ *is dense in* $L_{p,\mu}(\mathbb{R}_+; Y)$.

Proof. (i) It has been proved above that $B_{p,1}$ has all the properties listed in the proposition. Since B_0 is relatively bounded with respect to $B_{p,1}$ with relative bound strictly smaller than 1, we obtain from (3.41) that $\bar{B}_{p,\mu}$ enjoys the same properties, see Section 3.1.3.

(ii) The assertions in (ii) follow from (i) by employing the isomorphism Φ_μ. □

In the sequel we take the liberty to work with $B_{p,\mu}$ and $\bar{B}_{p,\mu}$ interchangeably, that is, we will use the representation that is the most convenient one.

Lemma 3.2.8. *Let* $1/p < \mu \le 1$ *and suppose that* $k \in L_1(\mathbb{R}_+; \mathcal{B}(X, Y))$ *satisfies* $|k(t)| \le \kappa(t)$, *where* $\kappa \in L_1(\mathbb{R}_+)$ *is nonnegative and nonincreasing, and where* X, Y *are Banach spaces. Then we have*

(i) $\Big|\int_0^t k(t-s)(t/s)^{1-\mu} v(s)\, ds\Big|_p \le c_{p,\mu}|\kappa|_1 |v|_p$ *for* $v \in L_p(\mathbb{R}_+; X)$, *where* $c_{p,\mu} = 2^{1-\mu}[1 + (1 - p'(1-\mu))^{-p/p'}]^{1/p}$.

(ii) *The convolution operator* $K := k*$ *belongs to* $\mathcal{B}(L_{p,\mu}(\mathbb{R}_+; X), L_{p,\mu}(\mathbb{R}_+; Y))$ *and* $|K| \le c_{p,\mu}|\kappa|_1$.

Proof. (i) Let $v \in L_p(\mathbb{R}_+; X)$ be given. Then Hölder's inequality implies

$$\begin{aligned}
&\Big|\int_0^t k(t-s)(t/s)^{1-\mu} v(s)\, ds\Big|_p^p \le \int_0^\infty \Big[\int_0^t \kappa(t-s)(t/s)^{1-\mu}|v(s)|ds\Big]^p dt \\
&\le \int_0^\infty \Big[\int_0^t \kappa(t-r) r^{-p'(1-\mu)} dr\Big]^{p/p'} t^{p(1-\mu)} \int_0^t \kappa(t-s)|v(s)|^p\, dsdt \\
&= \int_0^\infty |v(s)|^p \Big\{\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\Big[\int_0^t \kappa(t-r) r^{-p'(1-\mu)} dr\Big]^{p/p'} dt\Big\}\, ds \\
&\le c_{p,\mu}^p |\kappa|_1^p |v|_p^p,
\end{aligned}$$

as the following estimates show. On the one hand, we have

$$\begin{aligned}
&\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\Big[\int_{t/2}^t \kappa(t-r) r^{-p'(1-\mu)} dr\Big]^{p/p'} dt \\
&\le 2^{p(1-\mu)} \int_s^\infty \kappa(t-s)\Big[\int_{t/2}^t \kappa(t-r) dr\Big]^{p/p'} dt \\
&\le 2^{p(1-\mu)} |\kappa|_1^{1+p/p'} = 2^{p(1-\mu)}|\kappa|_1^p.
\end{aligned}$$

Since $\kappa(t)$ is nonincreasing and $(1-\mu)p' < 1$ we have, on the other hand,

$$\begin{aligned}
&\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\Big[\int_0^{t/2}\kappa(t-r)r^{-p'(1-\mu)}dr\Big]^{p/p'}dt \\
&\le \int_s^\infty t^{p(1-\mu)}\kappa(t-s)\Big[\kappa(t/2)\int_0^{t/2} r^{-p'(1-\mu)}dr\Big]^{p/p'}dt \\
&= (1-p'(1-\mu))^{-p/p'}2^{p(1-\mu)}\int_s^\infty \kappa(t-s)[\kappa(t/2)(t/2)]^{p/p'}dt \\
&\le (1-p'(1-\mu))^{-p/p'}2^{p(1-\mu)}|\kappa|_1^p.
\end{aligned}$$

Note that the last inequality follows from

$$\kappa(t/2)(t/2) = \int_0^{t/2}\kappa(t/2)\,d\tau \le \int_0^{t/2}\kappa(\tau)\,d\tau \le |\kappa|_1,$$

where we have once more used that κ is nonincreasing.
(ii) We conclude from (i) that

$$\begin{aligned}
|Kv|_{L_{p,\mu}} &= \Big(\int_0^\infty t^{(1-\mu)p}|Kv(t)|^p\,dt\Big)^{1/p} \\
&= \Big(\int_0^\infty\Big|\int_0^t k(t-s)(t/s)^{1-\mu}s^{1-\mu}v(s)\,ds\Big|^p\,dt\Big)^{1/p} \\
&\le c_{p,\mu}|\kappa|_1|s^{1-\mu}v|_p = c_{p,\mu}|\kappa|_1|v|_{L_{p,\mu}}\,,
\end{aligned}$$

and the proof of Lemma 3.2.8 is complete. □

We already know that the operator $-B_{p,1}$ generates a positive C_0-semigroup $\{T(t) : t \in \mathbb{R}_+\}$ of contractions on $L_p(\mathbb{R}_+;Y)$ which is given by

$$[T(t)u](s) := \begin{cases} u(s-t) & \text{if } s > t, \\ 0 & \text{if } s < t. \end{cases} \tag{3.42}$$

This implies the resolvent estimate

$$|(\lambda + B_{p,1})^{-1}|_{\mathcal{B}(L_p(\mathbb{R}_+;Y))} \le \frac{1}{\operatorname{Re}\lambda}, \quad \operatorname{Re}\lambda > 0.$$

However, note that this semigroup is not of class C_0 in $L_{p,\mu}(\mathbb{R}_+;Y)$ for $\mu < 1$, as $T(t)$ does not map $L_{p,\mu}(\mathbb{R}_+;Y)$ into $L_{p,\mu}(\mathbb{R}_+;Y)$ for $t > 0$. Nevertheless, we now prove a resolvent estimate for $B_{p,\mu}$, which is best possible.

Proposition 3.2.9. *Let $1/p < \mu \le 1$. Then the resolvent set $\rho(B_{p,\mu})$ contains the open negative half-plane $\mathbb{C}_- = -\Sigma_{\pi/2}$, and there is a constant $c_{p,\mu} > 1$ such that*

$$|(\lambda + B_{p,\mu})^{-1}|_{\mathcal{B}(L_{p,\mu}(\mathbb{R}_+;Y))} \le \frac{c_{p,\mu}}{\operatorname{Re}\lambda}, \quad \operatorname{Re}\lambda > 0, \tag{3.43}$$

holds. In particular, $B_{p,\mu}$ is sectorial in $L_{p,\mu}(\mathbb{R}_+;Y)$ with $\phi_{B_{p,\mu}} = \pi/2$.

Proof. (i) Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ be fixed and set

$$(K_\lambda f)(t) := \int_0^t e^{-\lambda(t-s)} f(s)\, ds, \quad f \in L_{p,\mu}(\mathbb{R}_+; Y).$$

Moreover, let $\kappa(t) := e^{-t\operatorname{Re}\lambda}$. Then K_λ satisfies the assertions of Lemma 3.2.8, with $|\kappa|_1 = 1/\operatorname{Re}\lambda$. Consequently, Lemma 3.2.8 shows that K_λ is a bounded linear operator in $L_{p,\mu}(\mathbb{R}_+; Y)$, and that

$$|K_\lambda|_{\mathcal{B}(L_{p,\mu}(\mathbb{R}_+;Y))} \le \frac{c_{p,\mu}}{\operatorname{Re}\lambda}. \tag{3.44}$$

(ii) We verify that $(\lambda + B_{p,\mu}) : \mathsf{D}(B_{p,\mu}) \to L_{p,\mu}(\mathbb{R}_+; Y)$ is invertible for $\operatorname{Re}\lambda > 0$, with

$$[(\lambda + B_{p,\mu})^{-1} f](t) = \int_0^t e^{-\lambda(t-s)} f(s)\, ds, \quad f \in L_{p,\mu}(\mathbb{R}_+; Y). \tag{3.45}$$

Indeed, let $f \in L_{p,\mu}(\mathbb{R}_+; Y)$ be given and recall that $L_{p,\mu}(\mathbb{R}_+; Y)$ is embedded into $L_{1,\mathrm{loc}}(\mathbb{R}_+; Y)$. It is then not difficult to see that the differential equation

$$(\lambda + \frac{d}{dt})u = f, \quad u(0) = 0,$$

has a unique solution $u = u_\lambda$ in $H^1_{1,\mathrm{loc}}(\bar{\mathbb{R}}_+; Y)$. It is given by the right-hand side of equation (3.45). It remains to show $u_\lambda \in \mathsf{D}(B_{p,\mu})$. For this we note that $u_\lambda = K_\lambda f$ and $\dot u_\lambda = f - \lambda K_\lambda u_\lambda$. Hence we obtain from (i) that u_λ as well as $\dot u_\lambda$ belong to the space $L_{p,\mu}(\mathbb{R}_+; Y)$. Since $u_\lambda(0) = 0$ we conclude $u_\lambda \in \mathsf{D}(B_{p,\mu})$, and this establishes equation (3.45). We have shown that $\rho(B_{p,\mu})$ contains $\mathbb{C}_-$, and the resolvent estimate (3.43) is now a direct consequence of (3.44)–(3.45).

(iii) It follows from (3.43) that $\phi_{B_{p,\mu}} \le \pi/2$. On the other hand, $\phi_{B_{p,\mu}}$ cannot be strictly smaller than $\pi/2$, as this would imply that $B_{p,\mu}$ generates a (strongly continuous analytic) semigroup on $L_{p,\mu}(\mathbb{R}_+; Y)$, which is not possible. The assertion follows now from Proposition 3.2.7. □

3.3 Analytic Semigroups and Fractional Powers

3.1 Holomorphic Semigroups

Typical examples of functions in $H_a(\Sigma_\phi)$ with $\phi < \pi/2$ are the functions $e_t(z) = e^{-zt}$ for each $t > 0$. Provided $\phi_A < \pi/2$, the Dunford calculus from Section 3.1.4 gives rise to the family of operators $e_t(A) =: e^{-tA}$, $t > 0$, which because of the multiplicativity of the the calculus yields the semigroup property

$$e^{-A(t+s)} = e^{-At} e^{-As}, \quad t, s > 0.$$

Therefore it is called a *semigroup of operators.*

Definition 3.3.1. *A family of operators* $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ *in a Banach space* X *is called a* **semigroup**, *if*

$$T(t+s) = T(t)T(s), \quad t,s > 0, \quad T(0) = I,$$

is satisfied. The semigroup is called of **class** C_0, *if in addition*

$$\lim_{t\to 0+} T(t)x = x, \quad x \in X,$$

holds.

We prove the following result which is basic in semigroup theory and for parabolic partial differential equations.

Theorem 3.3.2. *Let* A *be a closed densely defined operator in a Banach space* X. *Then the following assertions are equivalent.*

(a) A *is pseudo-sectorial with spectral angle less than* $\pi/2$;

(b) $-A$ *generates a* C_0*-semigroup* $T(t)$ *which admits a bounded and holomorphic extension to a sector* Σ_ψ;

(c) $-A$ *generates a* C_0*-semigroup* $T(t)$ *such that* $\mathsf{R}(T(t)) \subset \mathsf{D}(A)$, *and there is a constant* $M_0 > 0$ *such that* $|T(t)| + |tAT(t)| \leq M_0$, *for each* $t > 0$.

Proof. (c) $\Rightarrow$ (b). Suppose $-A$ generates a C_0-semigroup such that the conditions of (c) are satisfied. Define $T(z)$ by means of the power series

$$T(t+z) = \sum_{n=0}^\infty \frac{z^n}{n!} T^{(n)}(t).$$

Because of $T^{(n)}(t) = A^n T(t) = [AT(t/n)]^n$ we obtain $|T^{(n)}(t)| \leq [M_0 n/t]^n$, for all $t > 0$ and $n \in \mathbb{N}_0$. These estimates imply

$$|T(t+z)| \leq \sum_{n=0}^\infty \frac{[n|z|M_0]^n}{t^n n!} < \infty,$$

provided

$$\overline{\lim}_{n\to\infty} [(n|z|M_0)^n / t^n n!]^{1/n} = M_0|z|e/t < 1,$$

which means $|z| < t/M_0 e$ or $|\arg z| < \psi_T := \arcsin(1/M_0 e)$. On each smaller sector Σ_ψ, $\psi < \psi_T$, $T(z)$ is then holomorphic, bounded, and has the semigroup property $T(z_1)T(z_2) = T(z_1+z_2)$, and $|T(z)| \leq M_\psi$.

(b) $\Rightarrow$ (a). Now let $T(z)$ be holomorphic on Σ_{ψ_T} and bounded on each smaller sector Σ_ψ. Then for each $\lambda > 0$, Cauchy's theorem applied to the closed contour $\Gamma_R = [0,R] \cup Re^{i[0,\psi]} \cup e^{i\psi}[R,0]$ implies with $R \to \infty$

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt = \int_0^\infty e^{-\lambda t e^{i\psi}} T(te^{i\psi})\, dt, \tag{3.46}$$

for each $|\psi| < \psi_T$, by virtue of

$$\Big|\int_0^\psi T(Re^{i\varphi})e^{-\lambda Re^{i\varphi}} iRe^{i\varphi}\,d\varphi\Big| \le M_\psi R\int_0^\psi e^{-R\lambda\cos\varphi}\,d\varphi \to 0$$

as $R\to\infty$. Because of the estimate

$$\begin{aligned}\Big|\int_0^\infty e^{-\lambda te^{i\psi}}T(te^{i\psi})\,dt\Big| &\le M_\psi\int_0^\infty e^{-t\mathrm{Re}(\lambda e^{i\psi})}\,dt \\ &\le \frac{M_\psi}{|\lambda|\cos(\psi+\arg\lambda)},\end{aligned} \tag{3.47}$$

formula (3.46) allows for holomorphic extension of the resolvent of A to the sector $-\Sigma_{\pi/2+\psi_T}$, and implies $\sigma(A)\subset\overline{\Sigma}_{\pi/2-\psi_T}$, and (3.46) holds for all $\lambda\in\Sigma_{\pi/2+\psi_T}$. Moreover, estimate (3.47) yields $\sup_{\lambda\in\Sigma_{\pi-\phi}}|\lambda(\lambda+A)^{-1}|<\infty$ for all $\phi>\pi/2-\psi_T$, and therefore $A\in\mathcal{PS}(X)$ and $\phi_A\le\pi/2-\psi_T$.

(a) $\Rightarrow$ (c). Suppose $A\in\mathcal{PS}(X)$ satisfies $\phi_A<\frac{\pi}{2}$, and let $\phi_A<\phi<\frac{\pi}{2}$. Then for $z\in\Sigma_\psi$, the functions $e_z(\lambda)=e^{-z\lambda}$ are holomorphic in $\mathbb{C}$ and belong to $H_a(\Sigma_\phi)$, as long as $\psi<\pi/2-\phi$. Therefore, the functional calculus for pseudo-sectorial operators yields bounded linear operators $T(z)=e_z(A)=e^{-zA}$, which satisfy the semigroup property

$$T(z_1+z_2)=T(z_1)T(z_2),\quad z_1,z_2\in\Sigma_{\frac{\pi}{2}-\phi}.$$

Since the map $z\mapsto f_z$ is holomorphic on $\Sigma_{\frac{\pi}{2}-\phi}$ with derivative $\partial_z e_z(\lambda)=-\lambda e_z(\lambda)$ which even belongs to $H_0(\Sigma_\phi)$, we may conclude that the family $\{T(z)\}_{z\in\Sigma_{\frac{\pi}{2}-\phi}}\subset\mathcal{B}(X)$ is holomorphic and $\frac{d}{dz}T(z)=-AT(z)$. In particular, $-A$ is the generator of $T(z)$ and the operators $T(z)$ have ranges contained in $\mathsf{D}(A)$, for each $z\in\Sigma_{\frac{\pi}{2}-\phi}$. Let us next derive bounds for $|T(z)|$. For this purpose we take the representation of $e_z(A)$ from (3.32).

$$T(z)=\frac{1}{2\pi i}\int_{\Gamma_\delta}e^{-z\lambda}(\lambda-A)^{-1}\,d\lambda.$$

With $|\arg z|\le\psi<\pi/2-\phi$ a straightforward estimate yields

$$\begin{aligned}|T(z)| &\le \frac{M_{\pi-\phi}(A)}{2\pi}\int_{\Gamma_\delta}e^{-\mathrm{Re}(z\lambda)}\frac{|\,d\lambda|}{|\lambda|}\\ &\le \frac{M_{\pi-\phi}(A)}{\pi}\Big[\int_\delta^\infty e^{-|z|r\cos(\phi+\psi)}\frac{dr}{r}+\int_\psi^\pi e^{|z|\delta}\,d\varphi\Big]\le K^0_\psi(A),\end{aligned}$$

by the choice $\delta=1/|z|$. This shows that the semigroup $T(z)$ is uniformly bounded on Σ_ψ. Similarly, choosing $\delta=0$ we obtain

$$|AT(z)|\le M_{\pi-\phi}(A)\int_0^\infty e^{-|z|r\cos(\phi+\psi)}dr=\frac{K^1_\psi(A)}{|z|},\quad z\in\Sigma_{\frac{\pi}{2}-\phi}.$$

To see that $T(z) \to I$ strongly as $z \to 0$, let $x \in \mathsf{D}(A)$ and fix $\delta > 0$. Then the identity $(\lambda - A)^{-1}x = x/\lambda + (\lambda - A)^{-1}Ax/\lambda$ yields

$$T(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{-z\lambda}[x + (\lambda - A)^{-1}Ax]\frac{d\lambda}{\lambda}.$$

By means of residue calculus the first part of this integral can be evaluated to the result

$$T(z)x = x + \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{-z\lambda}(\lambda - A)^{-1}Ax\frac{d\lambda}{\lambda},$$

and passing to the limit $z \to 0$, contracting the contour in $-\Sigma_{\pi-\phi}$ we conclude

$$T(z)x \to x + \frac{1}{2\pi i}\int_{\Gamma_\delta} (\lambda - A)^{-1}Ax\frac{d\lambda}{\lambda} = x,$$

by Cauchy's theorem. Since $\mathsf{D}(A)$ is dense in X and $T(z)$ is uniformly bounded we obtain $T(z) \to I$ strongly as $z \to 0$. The theorem is proved. $\square$

3.2 Extended Functional Calculus

We consider now a method to define $f(A)$ for all $A \in \mathcal{PS}(X)$ and all functions $f \in H(\Sigma_\phi)$ which grow at most polynomially at infinity and zero. More precisely, suppose $f \in H_{\alpha,\alpha}(\Sigma_\phi)$ for some $\alpha \in \mathbb{R}_+$. Define $\psi(\lambda) = \lambda/(1+\lambda)^2$; this function is rational and belongs to $H_0(\Sigma_\phi)$. Contracting the contour Γ, by residue calculus we obtain $\psi(A) = A(I + A)^{-2}$. This operator is bounded and injective, its range equals $\mathsf{D}(A) \cap \mathsf{R}(A)$ and its inverse is given by $\psi(A)^{-1} = 2 + A + A^{-1}$. If $k \in \mathbb{N}$ is such that $k > \alpha$ then $\psi^k f \in H_0(\Sigma_\phi)$ and so the Dunford calculus of Proposition 3.1.9 applies and yields a bounded operator $(\psi^k f)(A)$. We then set

$$\begin{aligned} f(A) &= \psi(A)^{-k}(\psi^k f)(A), \quad \text{and} \\ \mathsf{D}(f(A)) &= \{x \in X : \ (\psi^k f)(A)x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)\}. \end{aligned} \tag{3.48}$$

This definition of $f(A)$ is independent of $k > \alpha$; in fact, if $l > k > \alpha$ then $\psi^l f = \psi^{l-k}\psi^k f$, hence $(\psi^l f)(A) = \psi^{l-k}(A)(\psi^k f)(A)$ since ψ^{l-k} and also $\psi^k f$ belong to $H_0(\Sigma_\phi)$. Therefore we may always choose $k = [\alpha]+1$, the smallest integer larger than α. $f(A)$ defined this way is closed and densely defined. Moreover, we have

Theorem 3.3.3. *Let X be a complex Banach space and $A \in \mathcal{PS}(X)$. Then the functional calculus Φ_A defined by $\Phi_A(f) = f(A)$ with $f(A)$ given by* (3.48) *is well-defined for all functions in $\bigcup_{\alpha \in \mathbb{R}} H_{\alpha,\alpha}(\Sigma_\phi)$. For $\alpha \geq 0$ and $f \in H_{\alpha,\alpha}(\Sigma_\phi)$, $f(A)$ is a closed linear operator in X with domain*

$$\mathsf{D}(f(A)) = \{x \in X : \ (f\psi^k)(A)x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)\},$$

where $k > \alpha$. The inclusion $\mathsf{D}(f(A)) \supset \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ is valid, and

$$f(A)x = (f\psi^k)(A)\psi^{-k}(A)x, \quad x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k).$$

In particular, $f(A)$ is densely defined if A is sectorial. Φ_A is an algebra homomorphism in the sense that

$$(af+bg)(A)x = af(A)x + bg(A)x, \quad \text{for all } f,g \in H_{\alpha,\alpha}(\Sigma_\phi),\ x \in \mathsf{D}(A^k)\cap \mathsf{R}(A^k),$$

and all $a, b \in \mathbb{C}$, with $k > \alpha$, and

$$(fg)(A)x = f(A)g(A)x, \quad f \in \mathcal{H}_{\alpha,\alpha}(\Sigma_\phi),\ g \in H_{\beta,\beta}(\Sigma_\phi),\ x \in \mathsf{D}(A^k)\cap \mathsf{R}(A^k),$$

for $k > \alpha+\beta$. The approximations A_ε of A satisfy

$$\lim_{\varepsilon\to 0+} f(A_\varepsilon)x = f(A)x, \quad \text{for all } f \in H_{\alpha,\alpha}(\Sigma_\phi),\ x \in \mathsf{D}(A^k)\cap \mathcal{R}(A^k),\ k > \alpha.$$

It is useful to have a representation of $f(A)x$ as a contour integral, for $f \in H_{\alpha,\beta}(\Sigma_\phi)$ and $x \in \mathsf{D}(A^k)\cap\mathsf{R}(A^l)$, with $k > \alpha$ and $l > \beta$. To this aim we use again (3.25) for a bounded and invertible $B \in \mathcal{S}(X)$. Split the contour as $\Gamma_{r,R} = \Gamma_1^R \cup \Gamma_2^r$, where

$$\Gamma_1^R = e^{-i\psi}[1,R] \cup Re^{i[-\psi,\psi]} \cup e^{i\psi}[R,1], \quad \Gamma_2^r = [1,r]e^{i\psi} \cup re^{i[\psi,-\psi]} \cup [r,1]e^{-i\psi}. \tag{3.49}$$

Fix any $l \in \mathbb{N}_0$. On Γ_1^R we write

$$(\lambda - B)^{-1} = \sum_{j=1}^{l} \lambda^{-j}B^{j-1} + \lambda^{-l}(\lambda-B)^{-1}B^l,$$

and then we have

$$\begin{aligned}\int_{\Gamma_1^R} f(\lambda)(\lambda-B)^{-1}\,d\lambda &= \int_{\Gamma_1^R} \lambda^{-l} f(\lambda)(\lambda-B)^{-1}B^l\,d\lambda \\ &\quad + \sum_{j=1}^{l}\int_{\Gamma_1^R} f(\lambda)\lambda^{-j}B^{j-1}\,d\lambda.\end{aligned}$$

Deforming the contour Γ_1^R into $\Gamma_0 = e^{i[-\psi,\psi]}$ in Σ_ϕ, we may employ Cauchy's theorem to see that the contributions from the terms $\lambda^{l-j}B^{j-1}$ are independent of R.

The integral over Γ_2^r can be treated similarly. On this path we replace the resolvent $(\lambda-B)^{-1}$ according to the identity

$$(\lambda - B)^{-1} = \lambda^k(\lambda-B)^{-1}B^{-k} - \sum_{j=1}^{k}\lambda^{j-1}B^{-j},$$

to the result

$$\begin{aligned}\int_{\Gamma_2^r} f(\lambda)(\lambda-B)^{-1}\,d\lambda &= \int_{\Gamma_2^r} \lambda^{k} f(\lambda)(\lambda-B)^{-1}B^{-k}\,d\lambda \\ &\quad - \sum_{j=1}^{k}\int_{\Gamma_2^r} f(\lambda)\lambda^{j-1}B^{-j}\,d\lambda.\end{aligned}$$

Again by Cauchy's theorem we may deform the contributions from the terms $\lambda^{j-1}B^{-j}$ into an integral over Γ_0 which is independent of $r > 0$.

This way, we obtain the following representation formula for $f(B)$.

$$\begin{aligned} f(B) &= \frac{1}{2\pi i}\int_{\Gamma_1^R} \lambda^{-l} f(\lambda)(\lambda - B)^{-1}B^l \, d\lambda \\ &+ \frac{1}{2\pi i}\int_{\Gamma_2^r} \lambda^{k} f(\lambda)(\lambda - B)^{-1}B^{-k} \, d\lambda \\ &+ \frac{1}{2\pi i}\int_{\Gamma_0} f(\lambda)[\sum_{j=1}^{k} \lambda^{j-1}B^{-j} + \sum_{j=1}^{l} \lambda^{-j}B^{j-1}] \, d\lambda, \end{aligned} \tag{3.50}$$

where the contours Γ_1^R, Γ_2^r are defined by (3.49), and $\Gamma_0 = e^{i[-\psi,\psi]}$. Observe that the last integral is of the form

$$\sum_{j=-k}^{l-1} c_j(f)B^j, \quad \text{with} \tag{3.51}$$

$$c_{-j}(f) = \frac{1}{2\pi i}\int_{\Gamma_0} \lambda^{-(j+1)} f(\lambda)\, d\lambda, \quad c_j(f) = \frac{1}{2\pi i}\int_{\Gamma_0} \lambda^{-(j+1)} f(\lambda)\, d\lambda.$$

This shows that the coefficients $c_j(f)$ depend on f linearly and boundedly, in fact we have

$$|c_j(f)| \le 2\phi \sup\{|f(e^{it})| : \, |t| \le \phi\}, \quad \text{for all } j \in \mathbb{Z}.$$

For functions $f \in H(\Sigma_\phi)$ which grow at most polynomially at infinity and at zero we may now pass to the limits $R \to \infty$ and $r \to 0+$.

$$\begin{aligned} f(B) &= \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^{-l} f(\lambda)(\lambda - B)^{-1}B^l \, d\lambda \\ &+ \frac{1}{2\pi i}\int_{\Gamma_2} \lambda^{k} f(\lambda)(\lambda - B)^{-1}B^{-k} \, d\lambda + \sum_{j=-k}^{l-1} c_j(f)B^j \end{aligned} \tag{3.52}$$

where $k, l \in \mathbb{N}_0$ denote any numbers such that $\alpha < k$ and $\beta < l$.

Now consider an arbitrary operator $A \in \mathcal{S}(X)$ such that $\phi > \phi_A$. Then for any $\varepsilon > 0$ we let A_ε denote the approximations of A introduced in Section 3.1.2, and we may set $B = A_\varepsilon$ in formula (3.52). With Proposition 3.1.4 we have $(\lambda - A_\varepsilon)^{-1} \to (\lambda - A)^{-1}$ as $\varepsilon \to 0+$ in $\mathcal{B}(X)$, as well as $A_\varepsilon^j x \to A^j x$ for all $x \in \mathsf{D}(A^l)$, $0 \le j \le l$, and $A_\varepsilon^{-j} x \to A^{-j} x$ for all $x \in \mathsf{R}(A^k)$, $0 \le j \le k$. Since the function $|\lambda^{-(l+1)} f(\lambda)|$ is integrable over Γ_1, $|\lambda^{k-1} f(\lambda)|$ has this property on Γ_2, we may pass to the limit $\varepsilon \to 0+$ to the result

$$\begin{aligned} f(A)x &= \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^{-l} f(\lambda)(\lambda - A)^{-1}A^l x \, d\lambda \\ &+ \frac{1}{2\pi i}\int_{\Gamma_2} \lambda^{k} f(\lambda)(\lambda - A)^{-1}A^{-k} x \, d\lambda + \sum_{j=-k}^{l-1} c_j(f)A^j x, \end{aligned} \tag{3.53}$$

for any $x \in \mathsf{D}(A^l) \cap \mathsf{R}(A^k)$. This is the representation formula of $f(A)x$ we have been looking for.

3.3 Complex Powers of Sectorial Operators

For $z \in \mathbb{C}$ the functions $h_z(\lambda) = \lambda^z$ are holomorphic on Σ_π, the sliced complex plane and the estimate

$$|h_z(\lambda)| = |e^{z \log \lambda}| = e^{\mathrm{Re}\, z \log |\lambda| - \mathrm{Im} z \arg \lambda} \leq |\lambda|^{\mathrm{Re}\, z} e^{\phi |\mathrm{Im} z|}, \quad \lambda \in \Sigma_\phi,$$

shows that h_z belongs to $H_{\alpha,\alpha}(\Sigma_\phi)$ for $\alpha = \mathrm{Re}\, z$. Therefore, we may apply the extended functional calculus for sectorial operators to obtain the following result.

Proposition 3.3.4. *Suppose $A \in \mathcal{S}(X)$, let A^z be defined by $A^z = h_z(A)$, and $|\mathrm{Re} z| < k$, $k \in \mathbb{N}$. Then*

(i) *$A^z x$ is holomorphic on the strip $|\mathrm{Re}\, z| < k$, for each $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$;*

(ii) *A^z is closed for each $z \in \mathbb{C}$;*

(iii) *$A^{z+w} x = A^z A^w x$ for all $z, w \in \mathbb{C}$, $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$, where $k > |\mathrm{Re}\, z|, |\mathrm{Re}\, w|, |\mathrm{Re}\,(z+w)|$;*

(iv) *$A^z x = \lim_{\varepsilon \to 0} A_\varepsilon^z x$, $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$, $|\mathrm{Re}\, z| < k$.*

Because of Proposition 3.3.4, the operators A^z are linear, closed, densely defined and, because of $A^z A^{-z} x = x = A^{-z} A^z x$ for x in a dense subset of X, have also dense ranges and trivial kernels. If $A \in \mathcal{S}(X)$ is invertible then $\{A^{-z}$, $\mathrm{Re}\, z > 0\}$ forms a bounded holomorphic C_0-semigroup on $\Sigma_{\pi/2}$. This can be seen from formula (3.53) with $l = 0$ and $k = 1$ which in this case makes sense for all $x \in X$.

It turns out that for real α with $|\alpha| < \pi/\phi_A$ the powers A^α are sectorial as well, and the power law $(A^\alpha)^z x = A^{\alpha z} x$ is valid.

Theorem 3.3.5. *Let $A \in \mathcal{S}(X)$ and $\alpha \in \mathbb{R}$ be such that $|\alpha| < \pi/\phi_A$. Then A^α is also sectorial and $\phi_{A^\alpha} \leq |\alpha| \phi_A$. If $z \in \mathbb{C}$ and $k > |\mathrm{Re}\, z||\alpha|$, then*

$$(A^\alpha)^z x = A^{\alpha z} x, \quad \text{for all } x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k). \tag{3.54}$$

For any real numbers $\alpha < \beta < \gamma$ with $\gamma - \alpha < \pi/\phi_A$, the **moment inequality**

$$|A^\beta x| \leq k |A^\alpha x|^{\frac{\gamma-\beta}{\gamma-\alpha}} |A^\gamma x|^{\frac{\beta-\alpha}{\gamma-\alpha}}, \quad x \in \mathsf{D}(A^\alpha) \cap \mathcal{D}(A^\gamma), \tag{3.55}$$

is valid, where k denotes a constant depending only on α, β, γ and A.

Proof. Since $A^{-\alpha} = (A^{-1})^\alpha$, it is enough to consider positive α. So let $\alpha \in (0, \pi/\phi_A)$ be fixed. We want to show that the operators $\mu + A^\alpha$ are invertible for $\mu \in \Sigma_{\pi - \alpha\phi_A}$, and that the resolvent estimate

$$\sup_{\mu \in \Sigma_{\phi_\alpha}} |\mu(\mu + A^\alpha)^{-1}| \leq M_{\phi_\alpha} < \infty$$

is valid for each $\phi_\alpha < \pi - \alpha\phi_A$. For this purpose we consider the functions $g_\mu(\lambda) = \mu/(\mu + \lambda^\alpha)$, which are holomorphic and bounded on Σ_ϕ, uniformly w.r.t. μ, as long as $\mu \in \Sigma_{\phi_\alpha}$, and $\phi_\alpha + \alpha\phi < \pi$. By means of the extended functional calculus we have $g_\mu(A) = \mu(\mu + A^\alpha)^{-1}$, the problem is to show that these operators are bounded with a bound which is uniform in $\mu \in \Sigma_{\phi_\alpha}$. Observe that although the functions $g_\mu(\lambda)$ are uniformly bounded, they are neither holomorphic at zero nor at infinity, due to the presence of the power λ^α.

As a starting point we use formula (3.29) for the approximations A_ε of A which are bounded and invertible. Contract the contour Γ by means of Cauchy's theorem and by residue calculus to the halfray $\Gamma_\alpha = [0, \infty)e^{i\theta}$, with $\pi \geq \theta \geq \phi > \phi_A$, where the branch cut of λ^α is put on this ray. This is possible if the function $\mu + \lambda^\alpha$ has no zeros on this ray, which means that with $\varphi = \arg\mu$ we have $\varphi - \alpha\theta \neq (2k+1)\pi$ and $\varphi + 2\alpha\pi - \alpha\theta \neq (2k+1)\pi$, for all $k \in \mathbb{Z}$. Let λ_j, $j = 1, \dots, n$ denote the zeros of $\mu + \lambda^\alpha$; note that there are only finitely many of them, and $n = 0$ means that there are none. n is bounded from above in terms of α and ϕ_A. Then we obtain

$$\begin{aligned} g_\mu(A_\varepsilon) &= \mu \frac{1}{2\pi i} \int_0^\infty \Big[\frac{e^{i(\theta - 2\pi)}}{\mu + r^\alpha e^{i\alpha(\theta - 2\pi)}} - \frac{e^{i\theta}}{\mu + r^\alpha e^{i\alpha\theta}} \Big] (re^{i\theta} - A_\varepsilon)^{-1}\, dr \\ &\quad + \mu \sum_{j=1}^n \lambda_j^{1-\alpha} (\lambda_j - A_\varepsilon)^{-1}/\alpha \\ &= \frac{\mu e^{i\theta}}{2\pi i} \int_0^\infty \Big[\frac{e^{i(\theta\alpha)} - e^{i(\theta - 2\pi)\alpha}}{(\mu + r^\alpha e^{i\alpha(\theta - 2\pi)})(\mu + r^\alpha e^{i\alpha\theta})} \Big] r^\alpha (re^{i\theta} - A_\varepsilon)^{-1}\, dr \\ &\quad + \mu \sum_{j=1}^n \lambda_j^{1-\alpha} (\lambda_j - A_\varepsilon)^{-1}/\alpha. \end{aligned}$$

Estimating this expression we get

$$\begin{aligned} |g_\mu(A_\varepsilon)| &\leq C|\mu| \int_0^\infty \frac{r^{\alpha-1}\, dr}{|\mu e^{-i\alpha\theta} + r^\alpha||\mu e^{i\alpha(2\pi - \theta)} + r^\alpha|} + C \\ &\leq C\Big\{ 1 + \int_0^\infty \frac{dr}{|e^{i(\varphi - \alpha\theta)} + r||e^{i(\varphi - \alpha\theta + 2\alpha\pi)} + r|} \Big\} \leq C. \end{aligned}$$

Therefore we have uniform bounds on $g_\mu(A_\varepsilon)$, hence with $\varepsilon \to 0+$ also on $g_\mu(A)$, in virtue of $g_\mu(A_\varepsilon)x \to g_\mu(A)x$ as $\varepsilon \to 0+$ on a dense subset of X, and of the Banach-Steinhaus theorem. This proves that A^α is sectorial and $\phi_{A^\alpha} \leq \alpha\phi_A$ if $\alpha < \pi/\phi_A$.

The identity $(A_\varepsilon^\alpha)^z = A_\varepsilon^{\alpha z}$ is obviously valid, hence passing to the limit we obtain (3.54).

To prove the moment inequality, let us observe that it is enough to consider the case $\alpha = 0$ and $\gamma = 1$; in fact, replace x by $A^\alpha x$, β by $(\beta - \alpha)/(\gamma - \alpha)$, A by $A^{\gamma - \alpha}$, to see this; observe that by the restriction $\gamma - \alpha < \pi/\phi_A$, the operator

$A^{\gamma-\alpha}$ is again sectorial, by the first part of this proof. Contracting the contour Γ in the representation of $A_\varepsilon^{\beta-1}$ to the negative half-axis we obtain

$$A_\varepsilon^{\beta-1} = \frac{\sin(\beta\pi)}{\pi}\int_0^\infty r^{\beta-1}(r+A_\varepsilon)^{-1}\,dr.$$

Application of this formula to Ax for $x \in \mathsf{D}(A)$ and passing to the limit $\varepsilon \to 0+$ leads to

$$A^\beta x = A^{\beta-1}Ax = \frac{\sin(\beta\pi)}{\pi}\int_0^\infty r^{\beta-1}(r+A)^{-1}Ax\,dr;$$

observe that this integral is absolutely convergent. We split the range of integration at $\delta > 0$ and estimate as follows.

$$\begin{aligned}|A^\beta x| &\le C\int_0^\delta r^{\beta-1}\,dr|x| + C\int_\delta^\infty r^{\beta-2}\,dr|Ax|\\ &= C|x|\delta^\beta/\beta + C|Ax|\delta^{\beta-1}/(1-\beta) = C|x|^{1-\beta}|Ax|^\beta,\end{aligned}$$

by the choice $\delta = |Ax|/|x|$. This completes the proof of Theorem 3.3.5. □

3.4 Operators with Bounded Imaginary Powers

Proposition 3.3.4 shows that the following definition makes sense.

Definition 3.3.6. *Suppose $A \in \mathcal{S}(X)$. Then A is said to admit* **bounded imaginary powers** *if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$, and there is a constant $C > 0$ such that $|A^{is}| \le C$ for $|s| \le 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$.*

Since by Proposition 3.3.4, A^{is} has the group property, it is clear that A admits bounded imaginary powers if and only if $\{A^{is} : s \in \mathbb{R}\}$ forms a strongly continuous group of bounded linear operators in X. The growth bound θ_A of this group, i.e.,

$$\theta_A = \overline{\lim}_{|s|\to\infty}\frac{1}{|s|}\log|A^{is}| \tag{3.56}$$

will be called the **power angle** of A. Then for each $\omega > \theta_A$ there is a constant $M \ge 1$ such that

$$|A^{it}|_{\mathcal{B}(X)} \le Me^{\omega|t|}, \quad t \in \mathbb{R}.$$

It is in general not easy to verify that a given $A \in \mathcal{S}(X)$ belongs to $\mathcal{BIP}(X)$, although quite a few classes of operators are known for which the answer is positive; cf. the next subsections.

For a first application of the class $\mathcal{BIP}(X)$, consider the fractional power spaces

$$X_\alpha = X_{A^\alpha} = (\mathsf{D}(A^\alpha), |\cdot|_\alpha), \quad |x|_\alpha = |x| + |A^\alpha x|, \quad 0 < \alpha < 1,$$

where $A \in \mathcal{S}(X)$; the embeddings

$$X_A \hookrightarrow X_\beta \hookrightarrow X_\alpha \hookrightarrow X, \quad 1 > \beta > \alpha > 0,$$

are well-known. If A belongs to $\mathcal{BIP}(X)$, a characterization of X_α in terms of *complex interpolation spaces* can be derived.

Theorem 3.3.7. *Suppose $A \in \mathcal{BIP}(X)$. Then*

$$X_\theta \cong (X, X_A)_\theta, \quad \theta \in (0,1), \tag{3.57}$$

where $(X, X_A)_\theta$ denotes the complex interpolation space between X and $X_A \hookrightarrow X$ of order θ.

We recall the definition of the complex interpolation space $(X, X_A)_\theta$, $\theta \in (0,1)$. Consider the strip $S \subset \mathbb{C}$ given by $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Then $x \in (X, X_A)_\theta$ iff there is an $f \in H^\infty(S; X) \cap C(\bar{S}; X)$ with $\sup_{t\in\mathbb{R}} |f(1+it)|_{X_A} < \infty$, such that $f(\theta) = x$. The norm in $(X, X_A)_\theta$ is defined in the canonical way. More precisely,

$$|x|_{(X,X_A)_\theta} := \inf\{|h(i\cdot)|_{L_\infty(\mathbb{R};X)} + |h(1+i\cdot)|_{L_\infty(\mathbb{R};X_A)} : \ h \in H^\infty(S;X), \ h(\theta) = x\}.$$

The spaces $(X, X_A)_\theta$ are well-known to be Banach spaces such that $X_A \hookrightarrow (X, X_A)_\theta \hookrightarrow X$, with both embeddings dense if $\mathsf{D}(A)$ is dense in X.

Proof. We may assume w.l.o.g. that $A \in \mathcal{BIP}(X)$ is invertible. In fact, the functions $h_1(z) = (1+z)^\alpha(1+z^\alpha)^{-1} - 1$ and $h_2(z) = (1+z^\alpha)/(1+z)^\alpha - 1$ both belong to $H_0(\Sigma_\phi)$, for any $\phi < \pi$. This implies that $(1+A)^\alpha(1+A^\alpha)^{-1}$ and $(1+A^\alpha)(1+A)^{-\alpha}$ are bounded, and so $\mathsf{D}(A^\alpha) = \mathsf{D}((A+1)^\alpha)$.

Let $x \in \mathsf{D}(A)$ and let

$$f(z) = e^{z^2-\theta^2} A^{-z+\theta} x, \quad z \in S.$$

Then f is continuous on $\bar{S}$, holomorphic in S and bounded in X, since

$$|f(\sigma+it)| \le M e^{1-\theta^2} e^{\omega|t|-t^2} |A^{-\sigma+\theta} x| \le C|Ax|,$$

with some constant $C > 0$, as by assumption $A \in \mathcal{BIP}(X)$ is invertible, and employing the moment inequality. Moreover, for $\sigma = 0, 1$ we have

$$|f(it)|_X \le C|A^\theta x|, \quad |Af(1+it)| \le C|A^\theta x|,$$

hence

$$|x|_{(X,X_A)_\theta} \le C|A^\theta x|,$$

by definition of the complex interpolation spaces. As $\mathsf{D}(A)$ is dense in $\mathsf{D}(A^\theta)$ as well as in $(X, X_A)_\theta$, this yields the embedding $\mathsf{D}(A^\theta) \hookrightarrow (X, X_A)_\theta$.

To obtain the converse inclusion, fix $x \in \mathsf{D}(A)$, and let $f : \bar{S} \to X$ be bounded, continuous, and holomorphic in S, $f(\theta) = x$, and such that

$$|f(i\cdot)|_\infty, |Af(1+i\cdot)|_\infty \le 2|x|_{(X,X_A)_\theta}.$$

Set $g_\varepsilon(z) = e^{z^2-\theta^2} A^z (1+\varepsilon A)^{-1} f(z)$, $z \in S$. Then

$$g_\varepsilon(\theta) = A^\theta(1+\varepsilon A)^{-1} f(\theta) = A^\theta(1+\varepsilon A)^{-1} x \to A^\theta x \text{ as } \varepsilon \to 0,$$

as A^θ is closed and commutes with the resolvent of A. Obviously, g_ε is continuous and bounded on $\bar{S}$, holomorphic in S and

$$|g_\varepsilon(it)| \le Me^{\omega|t|-t^2}|(1+\varepsilon A)^{-1}||f(it)| \le C|x|_{(X,X_A)_\theta},$$

as well as

$$|g_\varepsilon(1+it)| \le Me\,e^{\omega|t|-t^2}|(1+\varepsilon A)^{-1}||Af(1+it)| \le C|x|_{(X,X_A)_\theta}.$$

Hadamard's three lines theorem then implies

$$|A^\theta(1+\varepsilon A)^{-1}x| = |g_\varepsilon(\theta)| \le |g_\varepsilon(i\cdot)|_\infty^{1-\theta}|g_\varepsilon(1+i\cdot)|_\infty^\theta \le C|x|_{(X,X_A)_\theta}.$$

Passing to the limit $\varepsilon \to 0$, this yields the inclusion $(X, X_A)_\theta \hookrightarrow \mathsf{D}(A^\theta)$, using once more density of $\mathsf{D}(A)$ in $\mathsf{D}(A^\theta)$ and in $(X, X_A)_\theta$. $\square$

The importance of Theorem 3.3.7 is twofold. It shows on one hand that X_α is largely independent of A; for instance if $A, B \in \mathcal{BIP}(X)$ are such that $\mathsf{D}(A) = \mathsf{D}(B)$ then $\mathsf{D}(A^\alpha) = \mathsf{D}(B^\alpha)$ for all $\alpha \in (0,1)$. On the other hand, (3.57) makes the tools of complex interpolation theory available for fractional power spaces and it becomes possible to characterize X_α in many cases. For example, the reiteration theorem yields the relation

$$(X_\alpha, X_\beta)_\theta = X_{\alpha(1-\theta)+\theta\beta}, \qquad \text{for all } 0 \le \alpha < \beta \le 1,\ \theta \in (0,1),$$

for complex interpolation of fractional power spaces of operators $A \in \mathcal{BIP}(X)$.

Some permanence properties for the class $\mathcal{BIP}(X)$ are collected in the next proposition.

Proposition 3.3.8. *Let X be a complex Banach space. The class $\mathcal{BIP}(X)$ has the following permanence properties.*

(i) *$A \in \mathcal{BIP}(X)$ iff $A^{-1} \in \mathcal{BIP}(X)$; then $\theta_{A^{-1}} = \theta_A$;*

(ii) *$A \in \mathcal{BIP}(X)$ implies $rA \in \mathcal{BIP}(X)$ and $\theta_{rA} = \theta_A$ for all $r > 0$;*

(iii) *$A \in \mathcal{BIP}(X)$ implies $e^{\pm i\psi}A \in \mathcal{BIP}(X)$ for all $\psi \in [0, \pi - \theta_A)$, and $\theta_{e^{\pm i\psi}A} \le \theta_A + \psi$;*

(iv) *$A \in \mathcal{BIP}(X)$ implies $(\mu + A) \in \mathcal{BIP}(X)$ for all $\mu \in \Sigma_{\pi-\phi_A}$, and $\theta_{\mu+A} \le \max\{\theta_A, |\arg\mu|\}$;*

(v) *if $\mathsf{D}(A^*)$ is dense in X^*, then $A \in \mathcal{BIP}(X)$ iff $A^* \in \mathcal{BIP}(X^*)$, and $\theta_A = \theta_{A^*}$;*

(vi) *if Y denotes another Banach space and $T \in \mathcal{B}(X,Y)$ is bijective, then $A \in \mathcal{BIP}(X)$ iff $A_1 = TAT^{-1} \in \mathcal{BIP}(Y)$, and $\theta_A = \theta_{A_1}$.*

Proof. Using the extended functional calculus and suitable variable transformations these permanence properties are abtained as in the proof of Proposition 3.1.3, except for (iv) which is a little more tricky. In fact, (iv) is very much related to the perturbation theory for the class $\mathcal{BIP}(X)$, it follows from our next proposition with $B = \mu$ and $h(z) = z^{is}$. $\square$

Proposition 3.3.9. *Suppose $A \in \mathcal{S}(X)$, B is a linear operator in X with $\mathsf{D}(B) \supset \mathsf{D}(A^\alpha)$, and*

$$|Bx| \le a|x| + b|A^\alpha x|, \quad x \in \mathsf{D}(A^\alpha),$$

holds with constants $a, b > 0$ and $\alpha \in [0,1)$. Assume that $A+B$ is sectorial and invertible.

Then $h(A) \in \mathcal{B}(X)$ implies $h(A+B) \in \mathcal{B}(X)$, for any $h \in H^\infty(\Sigma_\phi)$, where $\phi > \phi_A, \phi_{A+B}$. In particular, if $A \in \mathcal{BIP}(X)$ then $A+B \in \mathcal{BIP}(X)$, and

$$\theta_{A+B} \le \max\{\theta_A, \phi_{A+B}\}.$$

Proof. Fix h according to the assumptions of this proposition and let $f = \psi h$ with ψ as in Section 3.2.2. Then

$$h(A+B) = \psi^{-1}(A+B)f(A+B) = (2 + (A+B)^{-1} + A + B)f(A+B),$$

and with $B = B(1+A)^{-1}(1+A)$ this gives

$$h(A+B) = (2 + (A+B)^{-1} + B(1+A)^{-1} + (1 + B(1+A)^{-1})A)f(A+B).$$

Now, $(A+B)^{-1}$ and $B(1+A)^{-1}$ are bounded by assumption and $f(A+B)$ is bounded since $f \in H_0(\Sigma_\phi)$, hence we only need to show that $Af(A+B)$ is bounded. Choosing a standard contour Γ, the resolvent equation implies

$$Af(A+B) = Af(A) + \frac{1}{2\pi i}\int_\Gamma f(\lambda)A(\lambda - A)^{-1}B(\lambda - (A+B))^{-1}\, d\lambda.$$

Since by assumption $h(A)$ is bounded, $Af(A) = A\psi(A)h(A)$ is bounded as well, and the integral is absolutely convergent since B is of lower order. $\square$

In connection with operators with bounded imaginary powers another functional calculus is very useful and will be crucial. For this purpose recall the *Mellin transform* defined by

$$F(z) = \int_0^\infty f(t)t^{z-1}\, dt.$$

Mellin's inversion formula reads

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)t^{-z}\, dz.$$

The inverse Mellin transform can be used to define a functional calculus for $A \in \mathcal{BIP}(X)$ as follows. Set

$$M_\theta(\mathbb{R}) = \{\mu \in M_0(\mathbb{R}) : \ |\mu|_\theta := \frac{1}{2\pi}\int_\mathbb{R} e^{\theta|s|}|\, d\mu(s)| < \infty\},$$

where $M_0(\mathbb{R})$ denotes the space of all finite complex Borel measures on $\mathbb{R}$. $M_\theta(\mathbb{R})$ becomes a Banach algebra with unit, the convolution of measures, scaled by the factor $1/2\pi$ as multiplication. Evidently the Dirac masses δ_s with unit mass in $s \in \mathbb{R}$ belong to $M_\theta(\mathbb{R})$, and $2\pi\delta_0$ is the unit. For measures $\mu \in M_\theta(\mathbb{R})$ we define

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} z^{-is}\, d\mu(s), \quad z \in \Sigma_\theta.$$

This yields an algebra homomorphism from $M_\theta(\mathbb{R})$ into the Banach algebra $H^\infty(\Sigma_\theta)$, and it gives rise to the algebra homomorphism from $M_\theta(\mathbb{R})$ to $\mathcal{B}(X)$ defined by the formula

$$f(A) = \frac{1}{2\pi} \int_{\mathbb{R}} A^{-is}\, d\mu(s),$$

for any operator $A \in \mathcal{BIP}(X)$ with $\theta_A < \theta$. In fact, this formula is precisely the Phillips calculus for the C_0-group A^{-is}. We summarize these observations as

Theorem 3.3.10. *Let $A \in \mathcal{BIP}(X)$ and $\theta > \theta_A$. Then the formula*

$$f(A) = \frac{1}{2\pi} \int_{\mathbb{R}} A^{-is}\, d\mu(s)$$

defines an algebra homomorphism from $M_\theta(\mathbb{R})$ to $\mathcal{B}(X)$, where f and μ are related by

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} z^{-is}\, d\mu(s).$$

In particular, $f(z) = z^{-is}$ is mapped to A^{-is}, for each $s \in \mathbb{R}$. Moreover, there is a constant $K > 0$ such that

$$|f(A)|_{\mathcal{B}(X)} \leq K|\mu|_\theta, \quad \text{for all } \mu \in \mathcal{M}_\theta(\mathbb{R}),$$

where $K = \sup_{s\in\mathbb{R}} e^{-\theta|s|}|A^{is}|_{\mathcal{B}(X)}$.

Proof. The only thing left to prove is the multiplication property. Here we need to recall the convolution theorem for the Mellin transform, i.e., if $f_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-is}\, d\mu_j(s)$, then

$$f_1(t)f_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(\mu_1 * \mu_2)(s), \quad t > 0.$$

This identity implies

$$\begin{aligned}
(f_1f_2)(A) &= \frac{1}{2\pi} \int_{\mathbb{R}} A^{-is}\, d(\mu_1 * \mu_2)(s) \\
&= \frac{1}{(2\pi)^2} \int_{R} A^{-is} \int_{\mathbb{R}} d\mu_1(s-\tau)\, d\mu_2(\tau) \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} A^{-is}\, d\mu_1(s) \int_{\mathbb{R}} A^{-i\tau}\, d\mu_2(\tau) \\
&= f_1(A)f_2(A).
\end{aligned}$$

$\square$

It is not obvious how to get the resolvent of an operator A from its imaginary powers. This is due to the fact that the Mellin transform of the function $1/(1+t)$ has poles at 0 and 1. However, since such representations are useful and in particular show that the functional calculus from Theorem 3.3.10 is consistent with the Dunford calculus, we comment on this.

For this purpose observe that

$$(1+t)^{-1} = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} t^{-z}\frac{dz}{\sin(\pi z)}, \quad t>0,$$

where $0<c<1$ is arbitrary. Therefore,

$$Tx = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} A^{-z}x\frac{dz}{\sin(\pi z)}$$

is well-defined since the integral is absolutely convergent for $x \in \mathsf{D}(A)\cap\mathsf{R}(A)$. By Cauchy's theorem, the integral is independent of c. Using again Cauchy's theorem, we obtain by an easy computation $T=(1+A)^{-1}$. In fact, apply $1+A$ to Tx to the result

$$(1+A)Tx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A^{-z}x\frac{\pi dz}{\sin(\pi z)} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A^{1-z}x\frac{\pi dz}{\sin(\pi z)}.$$

Deforming the contour in the first integral to

$$\Gamma_0 = (-i\infty,-i\varepsilon]\cup\varepsilon e^{i[-\pi/2,\pi/2]}\cup[i\varepsilon,i\infty)$$

and the second one to

$$\Gamma_1 = (1-i\infty,1-i\varepsilon]\cup(1-\varepsilon e^{i[-\pi/2,\pi/2]})\cup[1+i\varepsilon,1+i\infty),$$

observing that the contributions on the straight lines cancel, and passing to the limit $\varepsilon\to 0+$ there follows $(1+A)Tx = x$ for each $x\in\mathsf{D}(A)\cap\mathsf{R}(A)$. Since by assumption A is sectorial this implies $Tx=(1+A)^{-1}x$ for each $x\in\mathsf{D}(A)\cap\mathsf{R}(A)$.

Replacing A by sA, $s>0$, and shifting the contour to the imaginary axis we get the formula

$$(1+sA)^{-1}x = \frac{1}{2}x + \frac{1}{2i}\mathrm{PV}\int_{-\infty}^{\infty}(sA)^{-i\rho}\frac{d\rho}{\sinh(\pi\rho)}, \quad s>0, \tag{3.58}$$

where PV means the principal value.

To deduce the second formula, recall the identity

$$\frac{1}{1+\lambda t} = \frac{1}{1+rt} + \frac{1}{2i}\int_{-\infty}^{\infty}(rt)^{-i\rho}\frac{(e^{\phi\rho}-1)}{\sinh(\pi\rho)}\,d\rho,$$

where $\lambda = re^{i\phi}$, $|\phi| < \pi$. Since the measure with density $(e^{i\phi\rho} - 1)r^{-i\rho}/\sinh(\pi\rho)$ belongs to $M_\theta(\mathbb{R})$, provided $|\phi| < \pi - \theta$, we get by Theorem 3.3.10 the identity

$$(1 + \lambda A)^{-1} = (1 + |\lambda|A)^{-1} + \frac{1}{2i}\int_{-\infty}^{\infty} (|\lambda|A)^{-i\rho}\frac{(e^{\phi\rho} - 1)}{\sinh(\pi\rho)}\, d\rho, \tag{3.59}$$

whenever $\phi = \arg(\lambda) \in (-\pi + \theta, \pi - \theta)$. As a consequence we have

Corollary 3.3.11. *Suppose $A \in \mathcal{BIP}(X)$, $\theta_A < \pi$. Then $\phi_A \leq \theta_A$.*

3.5 Operators with Bounded $\mathcal{H}^\infty$-Calculus

There is another important concept related to the Dunford calculus for a sectorial operator.

Definition 3.3.12. *A sectorial operator A is said to admit a* **bounded $\mathcal{H}^\infty$-calculus** *if there are $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that*

$$|f(A)| \leq K_\phi |f|_{H^\infty(\Sigma_\phi)}, \quad \textit{for all } f \in H_0(\Sigma_\phi). \tag{3.60}$$

The class of sectorial operators A which admit an $\mathcal{H}^\infty$-calculus will be denoted by $\mathcal{H}^\infty(X)$. The **$\mathcal{H}^\infty$-angle** *of A is defined by*

$$\phi_A^\infty = \inf\{\phi > \phi_A : \ (3.60) \textit{ is valid}\}. \tag{3.61}$$

If this is the case, then the functional calculus for A on $H_0(\Sigma_\phi)$ extends uniquely to $H^\infty(\Sigma_\phi)$. This can be seen by formula (3.53) with $k = l = 1$, which is valid for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. If $f \in H^\infty(\Sigma_\phi)$ and $(f_n) \subset H_0(\Sigma_\phi)$ is uniformly bounded and converges to f, uniformly on compact subsets of Σ_ϕ, then (3.53) for f_n and Lebesgue's dominated convergence theorem show $f_n(A)x \to f(A)x$ as $n \to \infty$, for each $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. Since $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, (3.53) and the Banach-Steinhaus theorem then yield $f_n(A) \to f(A)$ in the strong operator topology. This is a special case of the so-called convergence lemma.

Lemma 3.3.13. *Let $A \in \mathcal{S}(X)$ and $\phi > \phi_A$. Suppose $(f_n)_{n\geq 0} \subset H^\infty(\Sigma_\phi)$ is such that $f_n \to f_0$ uniformly on compact subsets of Σ_ϕ.*

Then $\sup_{n\geq 1} |f_n(A)|_{\mathcal{B}(X)} < \infty$ implies $f_n(A) \to f_0(A)$ strongly. In particular, this assertion holds if $|f_n|_{H^\infty(\Sigma_\phi)} \leq M < \infty$ and A admits a bounded $\mathcal{H}^\infty$-calculus on Σ_ϕ.

Well-known examples for general classes of sectorial operators with bounded $\mathcal{H}^\infty$-calculus are

- **(a)** normal sectorial operators in Hilbert spaces;
- **(b)** m-accretive operators in Hilbert spaces;
- **(c)** generators of bounded C_0-groups on L_p-spaces;
- **(d)** negative generators of positive contraction semigroups in L_p-spaces.

Here (a) follows from the functional calculus for normal operators in Hilbert spaces, see e.g. Dunford-Schwartz [91], while by the Cayley transform, (b) is a consequence of the Foias-Nagy calculus for contractions in Hilbert spaces; see Foias-Nagy [273]. (c) and (d) and some vector-valued extensions are implied by the theory of Coifman and Weiss [69].

Since the functions $f_s(z) = z^{is}$ belong to $H^\infty(\Sigma_\phi)$, for any $s \in \mathbb{R}$ and $\phi \in (0,\pi)$, we obviously have the inclusions

$$\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X), \tag{3.62}$$

and the inequalities

$$\phi_A^\infty \geq \theta_A \geq \phi_A \geq \sup\{|\arg\lambda| : \ \lambda \in \sigma(A)\}. \tag{3.63}$$

The permanence properties of the class $\mathcal{H}^\infty(X)$ are like those for general sectorial operators.

Proposition 3.3.14. *Let X be a complex Banach space. The class $\mathcal{H}^\infty(X)$ has the following permanence properties.*

(i) *$A \in \mathcal{H}^\infty(X)$ iff $A^{-1} \in \mathcal{H}^\infty(X)$; then $\phi_{A^{-1}}^\infty = \phi_A^\infty$;*

(ii) *$A \in \mathcal{H}^\infty(X)$ implies $rA \in \mathcal{H}^\infty(X)$ and $\phi_{rA}^\infty = \phi_A^\infty$ for all $r > 0$;*

(iii) *$A \in \mathcal{H}^\infty(X)$ implies $e^{\pm i\psi}A \in \mathcal{H}^\infty(X)$ for all $\psi \in [0, \pi - \phi_A^\infty)$, and $\phi_{e^{\pm i\psi}A}^\infty = \phi_A^\infty + \psi$;*

(iv) *$A \in \mathcal{H}^\infty(X)$ implies $(\mu + A) \in \mathcal{H}^\infty(X)$ for all $\mu \in \Sigma_{\pi-\phi_A}$, and $\phi_{\mu+A}^\infty \leq \max\{\phi_A^\infty, |\arg\mu|\}$;*

(v) *if $\mathsf{D}(A^*)$ is dense in X^*, then $A \in \mathcal{H}^\infty(X)$ iff $A^* \in \mathcal{H}^\infty(X^*)$, and $\phi_A^\infty = \phi_{A^*}^\infty$;*

(vi) *if Y denotes another Banach space and $T \in \mathcal{B}(X,Y)$ is bijective, then $A \in \mathcal{H}^\infty(X)$ iff $A_1 = TAT^{-1} \in \mathcal{H}^\infty(Y)$, and $\phi_A^\infty = \phi_{A_1}^\infty$.*

Following the lines of the proof of Proposition 3.1.3, the proof of this result is evident. Concerning perturbations, we have the following result which is a direct consequence of Proposition 3.3.9.

Corollary 3.3.15. *Suppose $A \in \mathcal{H}^\infty(X)$, B is a linear operator in X with $\mathsf{D}(B) \supset \mathsf{D}(A^\alpha)$, and*

$$|Bx| \leq a|x| + b|A^\alpha x|, \quad x \in \mathsf{D}(A^\alpha),$$

holds with constants $a, b > 0$ and $\alpha \in [0,1)$. Assume that $A + B$ is sectorial and invertible.

Then $A + B \in \mathcal{H}^\infty(X)$, and $\phi_{A+B}^\infty \leq \max\{\phi_A^\infty, \phi_{A+B}\}$.

3.4 Trace Spaces: Real Interpolation

4.1 Trace Spaces of L_p-Type
Consider the homogeneous Cauchy problem

$$\dot{u} + Au = 0, \quad t > 0, \quad u(0) = x, \tag{3.64}$$

in a Banach space X, where A is a densely defined pseudo-sectorial operator with spectral angle $\phi_A < \pi/2$. Then $-A$ generates a bounded holomorphic C_0-semigroup in X and the solution $u(t)$ of (3.64) is given by $u(t) = T(t)x$, for all $t \geq 0$, where $T(t) = e^{-At}$ denotes the semigroup generated by $-A$. In this subsection, we study again regularity properties of $u(t)$. More specifically, we ask for which initial values x the solution $u(t)$ is such that $u(t) \in \mathsf{D}(A)$ for a.a. $t > 0$ and $Au \in L_{p,\mu}(\mathbb{R}_+; X)$, $\mu \in (1/p, 1]$. In virtue of (3.64) this is equivalent to $u \in W^1_{p,loc}(\mathbb{R}_+; X)$ and $\dot{u} \in L_{p,\mu}(\mathbb{R}_+; X)$.

Suppose that u has this property. Then the initial value $x \in X$ satisfies $\int_0^\infty |AT(t)x|^p t^{p(1-\mu)}\, dt < \infty$. Let us introduce the following trace spaces.

Definition 3.4.1. *Let A be a densely defined pseudo-sectorial operator in X with spectral angle $\phi_A < \pi/2$, let $\alpha \in (0,1)$ and $p \in [1,\infty)$. The spaces $D_A(\alpha,p)$ are defined by means of*

$$D_A(\alpha,p) = \Big\{x \in X : \; [x]_{\alpha,p} := \Big(\int_0^\infty |t^{1-\alpha} AT(t)x|^p \, dt/t \Big)^{1/p} < \infty \Big\}.$$

When equipped with the norm

$$|x|_{\alpha,p} := |x| + [x]_{\alpha,p}, \quad x \in D_A(\alpha,p),$$

$D_A(\alpha,p)$ becomes a Banach space. For $k \in \mathbb{N}$ the spaces $D_A(k+\alpha,p)$ are defined by

$$D_A(k+\alpha,p) := \{x \in \mathsf{D}(A^k) : \; A^k x \in D_A(\alpha,p)\}.$$

We can now give a complete answer to the question raised at the beginning of this subsection.

Proposition 3.4.2. *Suppose A is a densely defined invertible sectorial operator in X with spectral angle $\phi_A < \pi/2$, $p \in (1,\infty)$ and $\mu \in (1/p, 1]$.*

Then for the solution u of (3.64) the following assertions are equivalent.

- **(a)** *$u(t) \in \mathsf{D}(A)$ for a.a. $t > 0$, and $u \in L_{p,\mu}(\mathbb{R}_+; X_A)$;*
- **(b)** *$u \in H^1_{p,\mu}(\mathbb{R}_+; X)$;*
- **(c)** *$x \in D_A(\mu - 1/p, p)$.*

In this case there is a constant $C_{p,\mu} > 0$ depending only on A, p and μ, such that

$$|\dot{u}|_{L_{p,\mu}(\mathbb{R}_+;X)} + |Au|_{L_{p,\mu}(\mathbb{R}_+;X)} \leq C_{p,\mu} |x|_{\mu-1/p,p},$$

for all $x \in D_A(\mu - 1/p, p)$.

Proof. By assumption, $-A$ generates the holomorphic semigroup $T(t) = e^{-At}$ which is bounded on $\mathbb{R}_+$, satisfies $T(t)X \subset \mathsf{D}(A)$ and, with some $\omega > 0$,

$$|T(t)| + t|AT(t)| \leq Me^{-\omega t}, \quad t > 0.$$

Let $x \in X$ and $u(t) = T(t)x$. Then $u(t) \in \mathsf{D}(A)$ for $t > 0$. By definition, $x \in D_A(\mu - 1/p, p)$ implies $Au \in L_{p,\mu}(\mathbb{R}_+; X)$, hence (c) implies (a). Since $T(t)$ is holomorphic and $\dot{T}(t) = AT(t)$ for $t > 0$, (a) implies (b). On the other hand, (b) yields $Au = -\dot{u} \in L_{p,\mu}(\mathbb{R}_+; X)$, hence

$$[x]^p_{\mu-1/p,p} = |Au|^p_{L_{p,\mu}(\mathbb{R}_+;X)}$$

shows that (b) implies (c). □

We will also use frequently the following result which extends the previous proposition to fractional orders.

Proposition 3.4.3. *Suppose A is a densely defined invertible sectorial operator in X with spectral angle $\phi_A < \pi/2$, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, and $\alpha - 1 + \mu - 1/p > 0$.*

Then for the solution u of (3.64) *the following assertions are equivalent.*

(a) $u \in L_{p,\mu}(\mathbb{R}_+; D_A(\alpha, p))$;

(b) $x \in D_A(\alpha - 1 + \mu - 1/p, p)$.

In this case, we have in addition

(c) $u \in W^\alpha_{p,\mu}(\mathbb{R}_+; X) \cap H^\alpha_{p,\mu}(\mathbb{R}_+; X) \cap L_{p,\mu}(\mathbb{R}_+; \mathsf{D}(A^\alpha))$,
and there is a constant $C_{p,\mu} > 0$ depending only on A, p and μ, such that

$$\begin{aligned}|u|_{W^\alpha_{p,\mu}(\mathbb{R}_+;X)} + |u|_{H^\alpha_{p,\mu}(\mathbb{R}_+;X)} + |u|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} + |u|_{L_{p,\mu}(\mathbb{R}_+;\mathsf{D}(A^\alpha))}\\ \leq C_{p,\mu}|x|_{\alpha-1+\mu-1/p,p}, \quad \textit{for all } x \in D_A(\alpha - 1 + \mu - 1/p, p).\end{aligned}$$

Note that for $\alpha - 1 + \mu - 1/p < 0$ assertions (a) and (c) hold for all $x \in X$. The spaces W^α and H^α are defined via interpolation; see Section 3.4.5 below.

Proof. Observe that (a) holds if and only if $I := \int_0^\infty |u(t)|^p_{D_A(\alpha,p)} t^{p(1-\mu)}\, dt < \infty$. We have by Fubini's theorem

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty |\tau^{1-\alpha} Ae^{-A\tau} u(t)|^p \frac{d\tau}{\tau} t^{p(1-\mu)}\, dt \\ &= \int_0^\infty \int_0^\infty |Ae^{-A(\tau+t)}x|^p t^{p(1-\mu)}\, dt \tau^{p(1-\alpha)-1} d\tau \\ &= \int_0^\infty \int_\tau^\infty |Ae^{-As}x|^p (s-\tau)^{p(1-\mu)}\, ds \tau^{p(1-\alpha)-1} d\tau, \end{aligned}$$

therefore applying Fubini another time

$$\begin{aligned} I &= \int_0^\infty |Ae^{-As}x|^p \int_0^s (s-\tau)^{p(1-\mu)}\tau^{p(1-\alpha)-1} d\tau ds \\ &= C_0(\alpha,\mu,p)\int_0^\infty |Ae^{-As}x|^p s^{p(1-\alpha+1-\mu)}\, ds \\ &\le C_0(\alpha,\mu,p)|x|^p_{D_A(\alpha-1+\mu-1/p,p)}, \end{aligned}$$

with $C_0(\alpha,\mu,p) = \mathsf{B}(p(1-\alpha), p(1-\mu)+1)$, where B denotes the Beta function.

The assertions in **(c)** will be proved in Section 3.4.6. $\square$

4.2 Trace Spaces and Real Interpolation

We present now some other characterizations of the trace spaces $D_A(\alpha,p)$.

For this, we first recall the definition of the *real interpolation spaces* $(X,X_A)_{\alpha,p}$ of order $\alpha\in(0,1)$ and exponent $p\in[1,\infty)$. $x\in(X,X_A)_{\alpha,p}$ iff there exist a function $w\in C([0,1];X)\cap C((0,1];X_A)\cap C^1((0,1];X)$ with $w(0)=x$, such that

$$[[w]]_{\alpha,p} := \Big[\int_0^1 |t^{1-\alpha}\dot w(t)|^p\, dt/t\Big]^{1/p} + \Big[\int_0^1 |t^{1-\alpha}Aw(t)|^p\, dt/t\Big]^{1/p} < \infty. \quad (3.65)$$

The norm in $(X,X_A)_{\alpha,p}$ is then defined as $|x|_{(X,X_A)_{\alpha,p}} := |x| + \inf[[w]]_{\alpha,p}$, where the infimum is taken over all functions w with the described properties.

Proposition 3.4.4. *Let A be a densely defined pseudo-sectorial operator in a Banach space X with spectral angle $\phi_A<\pi/2$, let $\alpha\in(0,1)$, and $p\in[1,\infty)$. Then for $x\in X$ the following assertions are equivalent.*

(a) $x\in D_A(\alpha,p)$;

(b) $[x]'_{\alpha,p} := [\int_0^\infty |t^{-\alpha}(T(t)x-x)|^p\, dt/t]^{1/p} < \infty$;

(c) $[x]''_{\alpha,p} := [\int_0^\infty |\lambda^\alpha A(\lambda+A)^{-1}x|^p\, d\lambda/\lambda]^{1/p} < \infty$;

(d) $x\in(X,X_A)_{\alpha,p}$.

The norms

$$|\cdot|_{\alpha,p},\ |\cdot|'_{\alpha,p} = |\cdot| + [\cdot]'_{\alpha,p},\ |\cdot|''_{\alpha,p} = |\cdot| + [\cdot]''_{\alpha,p},\ |\cdot|_{(X,X_A)_{\alpha,p}}$$

are equivalent.

To prove this result we need some preparation. Firstly, Note that (d) in the proposition makes sense for all closed linear operators in X, while (c) is well-defined if A is pseudo-sectorial, in contrast to (a) which requires $\phi_A<\pi/2$, and (b) where $-A$ must be the generator of a bounded C_0-semigroup.

Secondly, recall Jensen's inequality

$$\phi\Big(\int_\Omega g(\omega)\, d\mu(\omega)\Big) \le \int_\Omega \phi(g(\omega))\, d\mu(\omega), \quad (3.66)$$

which is valid for each probability measure μ on Ω, for each integrable function g on Ω, and $\phi : \mathbb{R} \to \mathbb{R}$ convex.

Thirdly, we shall need *Hardy's inequality.*

Lemma 3.4.5 (Hardy's inequality)**.** *Let* $p \in [1, \infty)$, $0 < T \le \infty$, *and* $f : \mathbb{R}_+ \to X$ *be measurable and such that* $\int_0^T |t^\beta f(t)|^p \, dt < \infty$, *for some* $\beta < 1/p' = 1 - 1/p$. *Then*

$$\int_0^T \Big| t^{\beta-1} \int_0^t f(s) ds \Big|^p \, dt \le c(\beta, p)^p \int_0^T |t^\beta f(t)|^p \, dt < \infty,$$

where $c(\beta, p) = (1/p' - \beta)^{-1}$.

Proof. The change of variables $t = e^\tau$, $s = e^\sigma$ yields

$$\begin{aligned}\int_0^T \Big| t^{\beta-1} \int_0^t f(s) ds \Big|^p \, dt &= \int_{-\infty}^{\log(T)} \Big| e^{(\beta-1)\tau} \int_{-\infty}^{\tau} f(e^\sigma) e^\sigma d\sigma \Big|^p e^\tau d\tau \\ &\le \int_{-\infty}^{\log(T)} \Big[\int_{-\infty}^{\tau} |f(e^\sigma)| e^{(\beta+1/p)\sigma} \cdot e^{(\beta-1+1/p)(\tau-\sigma)} d\sigma \Big]^p d\tau,\end{aligned}$$

hence by Young's inequality for convolutions

$$\begin{aligned}\int_0^T \Big| t^{\beta-1} \int_0^t f(s) ds \Big|^p \, dt &\le \Big[\int_0^\infty e^{(\beta-1/p')\sigma} d\sigma \Big]^p \cdot \Big[\int_{-\infty}^{\log(T)} |f(e^\tau) e^{(\beta+1/p)\tau}|^p d\tau \Big] \\ &= (1/p' - \beta)^{-p} \Big[\int_0^T |t^\beta f(t)|^p \, dt \Big],\end{aligned}$$

which proves the lemma. □

Proof of Proposition 3.4.4.
(a) ⇒ (b). Let $x \in D_A(\alpha, p)$; then the identity

$$T(t)x - x = -\int_0^t AT(s)x \, ds$$

and Lemma 3.4.5 with $\beta = 1 - \alpha - 1/p$ yield

$$\begin{aligned}\int_0^\infty |t^{-\alpha}(T(t)x - x)|^p \, dt/t &= \int_0^\infty t^{(\beta-1)p} \Big| \int_0^t AT(s)x \, ds \Big|^p \, dt \\ &\le \alpha^{-p} \int_0^\infty s^{\beta p} |AT(s)x|^p \, ds \\ &= \alpha^{-p} \int_0^\infty |t^{1-\alpha} AT(t)x|^p \, dt/t \\ &= \alpha^{-p} [x]_{\alpha,p}^p.\end{aligned}$$

This implies $[x]'_{\alpha,p} \le \alpha^{-1} [x]_{\alpha,p}$.

(b) $\Rightarrow$ (c). To prove this implication we employ the identity

$$A(\lambda + A)^{-1}x = x - \lambda(\lambda + A)^{-1}x = \int_0^\infty \lambda e^{-\lambda t}[x - T(t)x]\,dt, \quad \lambda > 0,$$

which yields by Jensen's inequality (3.66) and Fubini's theorem

$$\begin{aligned}\int_0^\infty |\lambda^\alpha A(\lambda + A)^{-1}x|^p\,d\lambda/\lambda &= \int_0^\infty \lambda^{\alpha p}\Big|\int_0^\infty (T(t)x - x)\lambda e^{-\lambda t}\,dt\Big|^p\,d\lambda/\lambda \\ &\le \int_0^\infty \lambda^{\alpha p}\Big[\int_0^\infty |T(t)x - x|^p \lambda e^{-\lambda t}\,dt\Big]\,d\lambda/\lambda \\ &= \int_0^\infty |T(t)x - x|^p\Big[\int_0^\infty \lambda^{\alpha p}e^{-\lambda t}\,d\lambda\Big]\,dt \\ &= \int_0^\infty |T(t)x - x|^p\Gamma(\alpha p + 1)t^{-\alpha p - 1}\,dt\end{aligned}$$

where $\Gamma(z)$ denotes the Gamma function. This yields $[x]''_{\alpha,p} \le (\Gamma(\alpha p + 1))^p[x]'_{\alpha,p}$.

(c) $\Rightarrow$ (d). Suppose $[x]''_{\alpha,p} < \infty$. Define $u(t) = (1 + tA)^{-1}x$ for $t \in [0,1]$; then $u \in C([0,1];X) \cap C((0,1];X_A) \cap C^1((0,1];X)$, $u(0) = x$, and $\dot{u}(t) = -A(1+tA)^{-2}x$ for $t \in (0,1]$. The variable transformation $t = 1/\lambda$ gives

$$\begin{aligned}[[u]]_{\alpha,p} &= \Big[\int_0^1 |t^{1-\alpha}A(1 + tA)^{-2}x|^p\,dt/t\Big]^{1/p} + \Big[\int_0^1 |t^{1-\alpha}A(1 + tA)^{-1}x|^p\,dt/t\Big]^{1/p} \\ &\le C\Big[\int_0^1 |t^{1-\alpha}A(1 + tA)^{-1}x|^p\,dt/t\Big]^{1/p} \\ &= C\Big[\int_1^\infty |\lambda^\alpha A(\lambda + A)^{-1}x|^p\,d\lambda/\lambda\Big]^{1/p} \\ &\le C[x]''_{\alpha,p}.\end{aligned}$$

This proves $x \in (X, X_A)_{\alpha,p}$ and $|x|_{(X,X_A)_{\alpha,p}} \le C|x|''_{\alpha,p}$.

(d) $\Rightarrow$ (a). Let $x \in (X, X_A)_{\alpha,p}$ and $w \in C([0,1];X) \cap C((0,1];X_A) \cap C^1((0,1];X)$ with $w(0) = x$, be such that

$$[[w]]_{\alpha,p} = \Big[\int_0^1 |t^{1-\alpha}\dot{w}(t)|^p\,dt/t\Big]^{1/p} + \Big[\int_0^1 |t^{1-\alpha}Aw(t)|^p\,dt/t\Big]^{1/p} < \infty.$$

Then the identity

$$x = w(0) = w(t) - \int_0^t \dot{w}(s)\,ds$$

implies by Lemma 3.4.5 with $\beta = 1/p' - \alpha$

$$
\begin{aligned}
&\Big[\int_0^1 |t^{1-\alpha}AT(t)x|^p\,dt/t\Big]^{1/p}\\
&\le \Big[\int_0^1 |t^{1-\alpha}T(t)Aw(t)|^p\,dt/t\Big]^{1/p} + \Big[\int_0^1 \Big|t^{1-\alpha}AT(t)\int_0^t \dot w(s)ds\Big|^p\,dt/t\Big]^{1/p}\\
&\le C\Big[\int_0^1 |t^{1-\alpha}Aw(t)|^p\,dt/t\Big]^{1/p} + C\Big[\int_0^1 \Big|t^{-\alpha}\int_0^t \dot w(s)ds\Big|^p\,dt/t\Big]^{1/p}\\
&\le C\Big[\int_0^1 |t^{1-\alpha}Aw(t)|^p\,dt/t\Big]^{1/p} + C\alpha^{-p}\Big[\int_0^1 |t^{1-\alpha-1/p}\dot w(t)|^p\,dt\Big]^{1/p}\\
&\le C\Big[\int_0^1 |t^{1-\alpha}Aw(t)|^p\,dt/t\Big]^{1/p} + C\Big[\int_0^1 |t^{1-\alpha}\dot w(t)|^p\,dt/t\Big]^{1/p}.
\end{aligned}
$$

Because of boundedness of $tAT(t)$ on $\mathbb{R}_+$ we also have

$$\int_1^\infty |t^{1-\alpha}AT(t)x|^p\,dt/t \le C|x|^p\int_1^\infty t^{-\alpha p-1}\,dt = C|x|^p/\alpha p,$$

hence we obtain $[x]_{\alpha,p} \le C(|x| + [[w]]_{\alpha,p})$, and since w has been arbitrary it is also clear that $[x]_{\alpha,p} \le C|x|_{(X,X_A)_{\alpha,p}}$ holds, for some constant C independent of x. The proof is complete. □

4.3 Embeddings

We continue the study of the trace spaces $D_A(\alpha,p)$ with some essential embedding results. For this purpose we extend the definition of $D_A(\alpha,p)$ to the cases $p=\infty,0$.

$$D_A(\alpha,\infty) := \{x\in X:\ [x]_{D_A(\alpha,\infty)} := \sup_{\lambda>0}\lambda^\alpha|A(\lambda+A)^{-1}x| < \infty\},$$

and

$$D_A(\alpha,0) := \{x\in D_A(\alpha,\infty):\ \lim_{\lambda\to\infty}\lambda^\alpha A(\lambda+A)^{-1}x = 0\}.$$

These definitions make sense for any pseudo-sectorial operator A in X. The norm in these spaces are

$$|x|_{D_A(\alpha,\infty)} = |x| + [x]_{D_A(\alpha,\infty)}.$$

Obviously the *continuous interpolation space* $D_A(\alpha,0)$ is a closed subspace of $D_A(\alpha,\infty)$.

Proposition 3.4.6. *Let A be a pseudo-sectorial operator in X with dense domain. Then for all $0<\alpha<\beta<1$, $1\le p<q<\infty$, $r\in[1,\infty]\cup\{0\}$, we have*

(i) $\mathsf{D}(A)\hookrightarrow D_A(\beta,r)\hookrightarrow D_A(\alpha,r)\hookrightarrow X$;

(ii) $D_A(\beta,\infty)\hookrightarrow D_A(\alpha,1)$;

(iii) $D_A(\alpha,1)\hookrightarrow D_A(\alpha,p)\hookrightarrow D_A(\alpha,q)\hookrightarrow D_A(\alpha,0)\hookrightarrow D_A(\alpha,\infty)$;

(iv) $D_A(\alpha,1) \hookrightarrow \mathsf{D}(A^\alpha) \hookrightarrow D_A(\alpha,0)$*;*

(v) $\mathsf{D}(A) \subset D_A(\alpha,r)$ *is dense for each* $r \neq \infty$*;*

(vi) *if* $-A$ *generates a bounded* C_0*-semigroup in* X*, then its restriction to* $D_A(\alpha,r)$ *is also a bounded* C_0*-semigroup, for each* $r \neq \infty$*.*

Proof. (i) Since for $x \in \mathsf{D}(A)$, $t > 0$, we have

$$t^\alpha |A(t+A)^{-1}x| \leq Ct^{\alpha-1}|Ax|,$$

so the first inclusion is obvious. The second one follows from assertion (ii) and (iii), while the third one is trivial by definition of $D_A(\alpha,p)$.

(ii) Let $x \in D_A(\beta,\infty)$, $\beta > \alpha$; then

$$\int_1^\infty t^\alpha |A(t+A)^{-1}x| \frac{dt}{t} \leq |x|_{\beta,\infty} \int_1^\infty t^{\alpha-\beta-1}\, dt = \frac{|x|_{\beta,\infty}}{\beta-\alpha},$$

which implies assertion (ii).

(iii) Let $p \in [1,\infty)$, $x \in D_A(\alpha,p)$; then choosing a standard contour we obtain

$$t^\alpha A(t+A)^{-1}x = \frac{1}{2\pi i}\int_\Gamma \frac{t^\alpha \lambda^{1-\alpha}}{t+\lambda} \cdot \lambda^\alpha A(\lambda-A)^{-1}x \frac{d\lambda}{\lambda}.$$

For $p > 1$, by means of Hölder's inequality this gives

$$t^\alpha |A(t+A)^{-1}x| \leq \frac{1}{2\pi}\left[\int_\Gamma \left|\frac{t^\alpha\lambda^{1-\alpha}}{t+\lambda}\right|^{p'} \left|\frac{d\lambda}{\lambda}\right|\right]^{1/p'} \left[\int_\Gamma |\lambda^\alpha A(\lambda-A)^{-1}x|^p \left|\frac{d\lambda}{\lambda}\right|\right]^{1/p}.$$

Next observe that from the resolvent equation

$$(\lambda - A)^{-1} = (|\lambda| + A)^{-1}[-1 + (\lambda + |\lambda|)(\lambda - A)^{-1}]$$

we obtain

$$|A(\lambda-A)^{-1}x| \leq (1 + 2|\lambda(\lambda-A)^{-1}|)|A(|\lambda|+A)^{-1}x| \leq C|A(|\lambda|+A)^{-1}x|.$$

Since by the variable transformation $\lambda = tz$

$$\int_\Gamma \left|\frac{t^\alpha\lambda^{1-\alpha}}{t+\lambda}\right|^{p'} \left|\frac{d\lambda}{\lambda}\right| = \int_\Gamma \left|\frac{z^{1-\alpha}}{1+z}\right|^{p'} \left|\frac{dz}{z}\right| < \infty,$$

we conclude

$$|t^\alpha A(t+A)^{-1}x| \leq C|x|_{\alpha,p},$$

which yields the embedding $D_A(\alpha,p) \hookrightarrow D_A(\alpha,\infty)$ in case $p > 1$. For $p = 1$ we use boundedness of $t^\alpha|\lambda|^{1-\alpha}/|t+\lambda|$ instead.

For $q > p$ we have from this

$$
\begin{aligned}
([x]''_{\alpha,q})^q &= \int_0^\infty |t^\alpha A(t+A)^{-1}x|^q \frac{dt}{t} \\
&\le \sup_{t>0} |t^\alpha A(t+A)^{-1}x|^{q-p} \int_0^\infty |t^\alpha A(t+A)^{-1}x|^p \frac{dt}{t} \\
&\le [x]^{q-p}_{D_A(\alpha,\infty)} ([x]''_{\alpha,p})^p \le C[x]^q_{\alpha,p},
\end{aligned}
$$

which yields $D_A(\alpha,p) \hookrightarrow D_A(\alpha,q)$.

Finally, since $D_A(\alpha,0) \subset D_A(\alpha,\infty)$ is closed, the embedding $D_A(\alpha,p) \subset D_A(\alpha,0)$ follows from (v).

(iv) Let $x \in \mathsf{D}(A)$; then we know from Section 3.3.3

$$A^\alpha x = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty r^\alpha A(r+A)^{-1}x \frac{dr}{r}.$$

This easily implies the first inclusion in (iv), as $\mathsf{D}(A)$ is dense in $\mathsf{D}(A^\alpha)$.

On the other hand, for $x \in \mathsf{D}(A^\alpha)$ and $r > 0$ we have by the moment inequality

$$r^\alpha |A(r+A)^{-1}x| = r^\alpha |A^{1-\alpha}(r+A)^{-1}A^\alpha x| \le r^\alpha C r^{-\alpha} |A^\alpha x|.$$

This proves the second embedding in (iv), by density of $\mathsf{D}(A)$ in $D_A(\alpha,0)$.

(v) Since $\mathsf{D}(A) \subset X$ is dense by assumption, we have $x_\varepsilon := (1+\varepsilon A)^{-1}x \to x$ as $\varepsilon \to 0$, for each $x \in X$. Therefore $t^\alpha A(t+A)^{-1}(x - x_\varepsilon) \to 0$ for each $t > 0$. Since

$$|t^\alpha A(t+A)^{-1}(x - x_\varepsilon)| \le C|t^\alpha A(t+A)^{-1}x|,$$

for $x \in D_A(\alpha,p)$, Lebesgue's theorem implies $x_\varepsilon \to x$ also in $D_A(\alpha,p)$, i.e., $\mathsf{D}(A)$ is dense in $D_A(\alpha,p)$. To prove density of $\mathsf{D}(A)$ in $D_A(\alpha,0)$, observe that the set $\{t^\alpha A(t+A)^{-1}x :\ t > 0\}$ is relatively compact in X, in case $x \in D_A(\alpha,0)$. But this implies

$$t^\alpha A(t+A)^{-1}x_\varepsilon = (1+\varepsilon A)^{-1} t^\alpha A(t+A)^{-1}x \to t^\alpha A(t+A)^{-1}x$$

uniformly in $t > 0$, which shows $x_\varepsilon \to x$ also in $D_A(\alpha,0)$.

(vi) If $-A$ generates a bounded C_0-semigroup in X, it follows from the definition of the spaces $D_A(\alpha,r)$ that $T(t)$ is also bounded in $D_A(\alpha,p)$. Since $T(\cdot)x$ is continuous in $\mathsf{D}(A)$ for each $x \in \mathsf{D}(A)$, the density of the embedding $\mathsf{D}(A) \hookrightarrow D_A(\alpha,r)$ for $r \neq \infty$ implies that $T(t)$ is strongly continuous also in $D_A(\alpha,r)$, $r \neq \infty$. $\square$

4.4 Interpolation of Intersections
The following result on real interpolation of intersections is very useful.

Theorem 3.4.7. *Let $A, B \in \mathcal{PS}(X)$ be densely defined and resolvent-commuting, $\alpha \in (0,1)$, $1 \le p < \infty$.*

Then $(X, \mathsf{D}(A) \cap \mathsf{D}(B))_{\alpha,p} \cong (X, \mathsf{D}(A))_{\alpha,p} \cap (X, \mathsf{D}(B))_{\alpha,p}$.
In particular, if $A + B$ with natural domain $\mathsf{D}(A+B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ is pseudo-sectorial then

$$D_{A+B}(\alpha,p) \cong D_A(\alpha,p) \cap D_B(\alpha,p).$$

Proof. We may assume that A, B are sectorial and invertible. The inclusion " $\subset$ " is trivial. To prove the converse inclusion, let $x \in (X, \mathsf{D}(A))_{\alpha,p} \cap (X, \mathsf{D}(B))_{\alpha,p}$ be given. Define $u(t) = (I + tA)^{-1}(I + tB)^{-1}x$. As the resolvents of A and B commute, it is clear that $u \in C([0,1]; X) \cap C((0,1]; \mathsf{D}(A) \cap \mathsf{D}(B))$, and

$$|t^{1-\alpha-1/p} Au(t)|_p = |t^{1-\alpha-1/p}(I + tB)^{-1} A(I + tA)^{-1}x|_p \le M_B |x|_{D_A(\alpha,p)},$$

as well as

$$|t^{1-\alpha-1/p} Bu(t)|_p = |t^{1-\alpha-1/p}(I + tA)^{-1} B(I + tB)^{-1}x|_p \le M_A |x|_{D_B(\alpha,p)}.$$

Next we have $\dot{u}(t) = -(I+tB)^{-1}(I+tA)^{-1}(A(I+tA)^{-1}x + B(I+tB)^{-1}x)$, hence in the same way as above we obtain

$$|t^{1-\alpha-1/p}\dot{u}(t)|_p \le M_A M_B (|x|_{D_A(\alpha,p)} + |x|_{D_B(\alpha,p)}).$$

This shows the converse inclusion. □

4.5 Vector-Valued Fractional Sobolev, Besov and Bessel-Potential Spaces
(i) Let Y be a Banach space and $1 < p < \infty$, $\omega > 0$. Then B_p is sectorial in $X_0 := L_p(\mathbb{R}_+; Y)$ with domain $X_1 = {}_0H^1_p(\mathbb{R}_+; Y)$, and spectral angle $\pi/2$, according to Section 3.2.3. Then we define the *vector-valued Besov spaces* by

$${}_0B^\alpha_{pq}(\mathbb{R}_+; Y) := D_{B_p}(\alpha, q) = (X_0, X_1)_{\alpha,q}, \quad \alpha \in (0,1),\ q \in [1,\infty] \cup \{0\}, \tag{3.67}$$

and the *vector-valued fractional Sobolev spaces* by

$${}_0W^\alpha_p(\mathbb{R}_+; Y) := {}_0B^\alpha_{pp}(\mathbb{R}_+; Y) = D_{B_p}(\alpha, p) = (X_0, X_1)_{\alpha,p}, \quad \alpha \in (0,1). \tag{3.68}$$

(ii) This definition extends to the weighted spaces $X_{0,\mu} = L_{p,\mu}(\mathbb{R}_+; Y)$ for $1/p < \mu \le 1$, as $B_{p,\mu}$ is also sectorial in this space, with domain $X_{1,\mu} = {}_0H^1_{p,\mu}(\mathbb{R}_+; Y)$, by Proposition 3.2.9. So we set

$${}_0B^\alpha_{pq,\mu}(\mathbb{R}_+; Y) := D_{B_{p,\mu}}(\alpha, q) = (X_{0,\mu}, X_{1,\mu})_{\alpha,q}, \tag{3.69}$$

for $\alpha \in (0,1)$, $q \in [1,\infty] \cup \{0\}$, and

$${}_0W^\alpha_{p,\mu}(\mathbb{R}_+; Y) := {}_0B^\alpha_{pp,\mu}(\mathbb{R}_+; Y) = D_{B_{p,\mu}}(\alpha, p) = (X_{0,\mu}, X_{1,\mu})_{\alpha,p} \tag{3.70}$$

for $\alpha \in (0,1)$. We recall the isomorphism Φ_μ from Section 3.2.4 defined by $\Phi_\mu(u)(t) = t^{1-\mu}u(t)$ which maps $X_{j,\mu}$ onto X_j for $j = 0,1$, by Proposition 3.2.6. Interpolating these isomorphisms by the real method implies that

$$\Phi_\mu : {}_0B^\alpha_{pq,\mu}(\mathbb{R}_+;Y) \to {}_0B^\alpha_{pq}(\mathbb{R}_+;Y)$$

is an isomorphism as well, hence we have the characterizations

$$u \in {}_0B^\alpha_{pq,\mu}(\mathbb{R}_+;Y) \quad \Leftrightarrow \quad t^{1-\mu}u \in {}_0B^\alpha_{pq}(\mathbb{R}_+;Y),$$

and

$$u \in {}_0W^\alpha_{p,\mu}(\mathbb{R}_+;Y) \quad \Leftrightarrow \quad t^{1-\mu}u \in {}_0W^\alpha_{p}(\mathbb{R}_+;Y),$$

for all $\alpha \in (0,1)$, $q \in [1,\infty] \cup \{0\}$.

(iii) Similarly, as B_p is also sectorial in $L_p(\mathbb{R};Y)$, we define

$$B^\alpha_{pq}(\mathbb{R};Y) := (L_p(\mathbb{R};Y), H^1_p(\mathbb{R};Y))_{\alpha,q}, \quad W^\alpha_p(\mathbb{R};Y) := B^\alpha_{pp}(\mathbb{R};Y),$$

for $p \in (1,\infty)$, $\alpha \in (0,1)$, and $q \in [1,\infty] \cup \{0\}$. Next we let $B^\alpha_{pq,\mu}(\mathbb{R}_+;Y)$ be defined by

$$B^\alpha_{pq,\mu}(\mathbb{R}_+;Y) = (L_{p,\mu}(\mathbb{R}_+;Y), H^1_{p,\mu}(\mathbb{R}_+;Y))_{\alpha,q}.$$

(iv) The *vector-valued Bessel-potential spaces* $H^\alpha_p(\mathbb{R};Y)$, $H^\alpha_p(\mathbb{R}_+;Y)$, as well as ${}_0H^\alpha_p(\mathbb{R}_+;Y)$ and ${}_0H^\alpha_{p,\mu}(\mathbb{R}_+;Y)$ are defined in an analogous way, employing the complex interpolation method. From the isomorphism Φ_μ we deduce

$$u \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+;Y) \quad \Leftrightarrow \quad t^{1-\mu}u \in {}_0H^\alpha_{p}(\mathbb{R}_+;Y),$$

for all $p \in (1,\infty)$ and $\alpha \in (0,1)$.

(v) *Sobolev Embeddings.* Consider the operator $B = -d/dt$ in $X_0 = L_{p,\mu}(\mathbb{R}_+;Y)$ with maximal domain

$$X_1 = \mathsf{D}(B) = H^1_{p,\mu}(\mathbb{R}_+;Y).$$

Here we take $p \in (1,\infty)$, $\mu \in (1/p,1]$, $\alpha \in (0,1]$ and set $\beta := \alpha - 1 + \mu - 1/p$. Then for $\beta > 0$ the *Sobolev embedding* $\mathsf{D}(B^\alpha) \hookrightarrow C_0(\bar{\mathbb{R}}_+;Y)$ is valid. More precisely, there a is constant $C > 0$ such that

$$|u(t)|_Y \le C|u|_{\mathsf{D}(B^\alpha)}, \quad t \ge 0, \quad u \in \mathsf{D}(B^\alpha).$$

By Section 3.4.3 and general interpolation theory, this shows that $K^\alpha_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow C_0(\bar{\mathbb{R}}_+;Y)$ for $K \in \{W,H\}$, as long as $\beta > 0$.

In fact, it is easy to verify the identity

$$u(t) = \int_t^\infty e^{-(s-t)} \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} (B+1)^\alpha u(s)\, ds, \quad s > 0,$$

for, say, $u \in \mathsf{D}(B)$. Applying Hölder's inequality, this relation implies

$$|u(t)|_Y \leq \varphi_0(t)|(B+1)^\alpha u|_{X_0} \leq C\varphi_0(t)|u|_{\mathsf{D}(B^\alpha)},$$

where

$$\varphi_0(t) = [\Gamma(\alpha)^{-1}\int_t^\infty e^{-p'(s-t)}(s-t)^{p'(\alpha-1)}s^{p'(\mu-1)}\,ds]^{1/p'}.$$

In case $\beta > 0$, an easy estimate yields

$$\sup_{t\geq 0}(1+t)^{(1-\mu)}\varphi_0(t) < \infty,$$

which proves the assertion, by density of $\mathsf{D}(B)$ in $\mathsf{D}(B^\alpha)$, and the embedding $H^1_p(\mathbb{R}_+;Y) \hookrightarrow C_0(\bar{\mathbb{R}}_+;Y)$.

We note that in case $\mu < 1$, $u(t)$ has even uniform polynomial decay as $t \to \infty$.

(vi) *Hölder Embeddings.* For $\beta > 0$ the *Hölder embedding* $\mathsf{D}(B^\alpha) \hookrightarrow \dot{C}^\beta_b(\bar{\mathbb{R}}_+;Y)$ is valid. More precisely, there is a constant $C > 0$ such that

$$|u(t+h) - u(t)|_Y \leq Ch^\beta |B^\alpha u|_{X_0}, \quad t \geq 0, \quad u \in \mathsf{D}(B^\alpha).$$

By Section 3.4.3 and general interpolation theory, this shows $K^\alpha_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow C^{\beta-\varepsilon}_b(\bar{\mathbb{R}}_+;Y)$ for $K \in \{W, H\}$, as long as $\beta > \varepsilon > 0$. We observe that in case Y belongs to the class $\mathcal{H}T$, we may set $\varepsilon = 0$. In fact, in this case $\mathsf{D}(B^\alpha) = (X_0, X_1)_\alpha$ by Theorems 3.3.7 and by the analogue of Theorem 4.3.14 for B.

To prove the claim, as in **(v)** we use the identity

$$u(t) = \int_t^\infty \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} B^\alpha u(s)\,ds, \quad s > 0,$$

where $u \in \mathsf{D}(B)$. Then for $t, h \geq 0$,

$$\begin{aligned} u(t+h) - u(t) &= \Gamma(\alpha)^{-1}\int_{t+h}^\infty [(s-(t+h))^{\alpha-1} - (s-t)^{\alpha-1}]B^\alpha u(s)\,ds \\ &\quad - \Gamma(\alpha)^{-1}\int_t^{t+h}(s-t)^{\alpha-1}B^\alpha u(s)ds =: I_1 + I_2. \end{aligned}$$

We estimate separately by Hölder's inequality.

$$\begin{aligned} |I_1| &\leq [\Gamma(\alpha)^{-1}\int_{t+h}^\infty |(s-(t+h))^{\alpha-1} - (s-t)^{\alpha-1}|^{p'} s^{p'(\mu-1)}\,ds]^{1/p'}|B^\alpha u|_{X_0} \\ &=: \varphi_1(h)|B^\alpha u|_{X_0}, \end{aligned}$$

and

$$|I_2| \leq [\Gamma(\alpha)^{-1}\int_t^{t+h}(s-t)^{p'(\alpha-1)}s^{p'(\mu-1)}\,ds]^{1/p'}|B^\alpha u|_{X_0} =: \varphi_2(h)|B^\alpha u|_{X_0}.$$

Next, we have

$$\varphi_1(h) \le c\Big[\int_0^\infty (\tau^{\alpha-1} - (\tau+h)^{\alpha-1})^{p'} (\tau+h)^{p'(\mu-1)}\, d\tau\Big]^{1/p'}$$
$$= ch^\beta\Big[\int_0^\infty (r^{\alpha-1} - (r+1)^{\alpha-1})^{p'} (r+1)^{p'(\mu-1)}\, dr\Big]^{1/p'},$$

and

$$\varphi_2(h) \le \Big[\int_0^h \tau^{p'(\alpha+\mu-2)}\, d\tau\Big]^{1/p'} = ch^\beta.$$

Both integrals are absolutely convergent as $p'(\alpha+\mu-2) = p'(\beta - 1/p') > -1$, provided $\beta > 0$. This proves the assertion.

4.6 A General Trace Theorem
We consider functions in the class $K^\alpha_{p,\mu}(\mathbb{R}_+;Y) \cap L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))$, where $K \in \{W,H\}$, $1 \ge \mu > 1/p$, and $\alpha \in (0,1]$ (recall that $W^1_p = H^1_p$ for $p \in (1,\infty)$). For $\beta := \alpha - 1 + \mu - 1/p > 0$ we have $K^\alpha_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow C(\bar{\mathbb{R}}_+;Y)$, so the question is what regularity the initial value $u_0 := u(0)$ of the function u enjoys. We want to prove the following result, which is employed at many places in subsequent sections.

Theorem 3.4.8. *Suppose A is a densely defined invertible sectorial operator in Y with spectral angle $\phi_A < \pi/2$, $p \in (1,\infty)$, $\mu \in (1/p,1]$, and $\beta := \alpha-1+\mu-1/p > 0$. Let $K \in \{H,W\}$, and set $Y_\alpha = D_A(\alpha,p)$ or $Y_\alpha = \mathsf{D}(A^\alpha)$.*

Then the trace map

$$\mathsf{tr} : K^\alpha_{p,\mu}(\mathbb{R}_+;Y) \cap L_{p,\mu}(\mathbb{R}_+;Y_\alpha) \to D_A(\beta,p), \quad \mathsf{tr} : u \mapsto u(0),$$

is linear and bounded. In particular, if $u \in K^\alpha_{p,\mu}(\mathbb{R}_+;Y)$ then the function $v = u - e^{-At}u_0$ belongs to ${}_0K^\alpha_{p,\mu}(\mathbb{R}_+;Y)$, and the trace map tr is surjective.

Note that the second assertion follows from Proposition 3.4.3.

Proof. (i) Observe that Hardy's inequality implies

$${}_0H^1_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow L_{p,\mu+1}(\mathbb{R}_+;Y),$$

hence interpolating with the trivial embedding

$$L_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow L_{p,\mu}(\mathbb{R}_+;Y)$$

we obtain by the complex method

$${}_0H^\alpha_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow L_{p,\mu+\alpha}(\mathbb{R}_+;Y),$$

and by the real method

$${}_0W^\alpha_{p,\mu}(\mathbb{R}_+;Y) \hookrightarrow L_{p,\mu+\alpha}(\mathbb{R}_+;Y),$$

for all $\alpha \in (0,1)$ and $1 \geq \mu > 1/p$.

(ii) We can now prove assertion **(c)** of Proposition 3.4.3. For this purpose, let $x \in D_A(\alpha - 1 + \mu - 1/p, p)$; then $u(t) = e^{-At}x - e^{-t}x \in {}_0H^1_{p,\mu+\alpha-1}(\mathbb{R}_+; X)$. Step (i) implies $u \in L_{p,\mu+\alpha}(\mathbb{R}_+; X)$, hence by complex interpolation $u \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; X)$, hence $e^{-At}x \in H^\alpha_{p,\mu}(\mathbb{R}_+; X)$. On the other hand, using real interpolation of type (α, p) we obtain $u \in {}_0W^\alpha_{p,\mu}(\mathbb{R}_+; X)$, hence $e^{-At}x \in W^\alpha_{p,\mu}(\mathbb{R}_+; X)$. For the last assertion, observe that $v(t) = e^{-At}x - e^{-t}A^{-1}x$ as before belongs to $L_{p,\mu+\alpha}(\mathbb{R}_+; X)$, but it is also in $L_{p,\mu+\alpha-1}(\mathbb{R}_+; X_A)$ by Proposition 3.4.2. Hence complex interpolation yields $u \in L_{p,\mu}(\mathbb{R}_+; \mathsf{D}(A^\alpha))$, which proves the last statement in **(c)** of Proposition 3.4.3.

(iii) Let $u \in K^\alpha_{p,\mu}(\mathbb{R}_+; Y) \cap L_{p,\mu}(\mathbb{R}_+; D_A(\alpha, p))$ be given and set $u_0 := u(0)$. We decompose u_0 as

$$u_0 = \frac{1}{t}\int_0^t u(s)\,ds + \frac{1}{t}\int_0^t (u_0 - u(s))\,ds = u_1 + u_2.$$

This decomposition leads to

$$|u_0|_{D_A(\beta,p)} \leq |u_1|_{D_A(\beta,p)} + |u_2|_{D_A(\beta,p)} = I_1^{1/p} + I_2^{1/p}.$$

We first estimate I_1.

$$\begin{aligned}
I_1 &\leq \int_0^1 t^{-1-\beta p}\Big[\int_0^t |Ae^{-At}u(s)|ds\Big]^p\,dt \\
&\leq \int_0^1 t^{-1-\beta p}\Big[\int_0^t s^{p'(\mu-1)}\,ds\Big]^{p/p'}\int_0^t s^{p(1-\mu)}|Ae^{-At}u(s)|^p\,ds]dt \\
&= c_{p,\mu}\int_0^1 t^{-1-\beta p+p/p'+p\mu-p}\int_0^t s^{p(1-\mu)}|Ae^{-At}u(s)|^p\,ds]dt \\
&= c_{p,\mu}\int_0^1 s^{p(1-\mu)}\Big[\int_s^1 (t^{1-\alpha}|Ae^{-At}u(s)|)^p\,dt/t\Big]\,ds \leq c_{p,\mu}|u|^p_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))},
\end{aligned}$$

where $c_{p,\mu} = (1 + p'(\mu - 1))^{-p/p'}$.

In case $Y_\alpha = \mathsf{D}(A^\alpha)$, we use the moment inequality to obtain the estimate $|t^{1-\alpha}A^{1-\alpha}e^{-At}| \leq C$, and employ once more Hardy's inequality, to the result

$$\begin{aligned}
I_1 &\leq C\int_0^1 t^{-\mu p}\Big[\int_0^t |A^\alpha u(s)|\,ds\Big]^p\,dt \\
&\leq C\int_0^1 |A^\alpha u(s)|^p s^{p(1-\mu)}\,ds = C|u|_{L_{p,\mu}(\mathbb{R}_+;\mathsf{D}(A^\alpha))}.
\end{aligned}$$

Next we estimate I_2 by the bound C for tAe^{-At} and Hardy's inequality

$$I_2 = \int_0^1 t^{p(1-\beta)} \Big| Ae^{-At} t^{-1} \int_0^t (u(s)-u_0)\,ds\Big|^p \,dt/t$$
$$\le C\int_0^1 t^{-1-\beta p - p}\Big[\int_0^t |u(s)-u_0|\,ds\Big]^p dt \le C\int_0^1 |u(s)-u_0|^p \frac{ds}{s^{1+\beta p}}.$$

By the embeddings in part (i), the last term is bounded by $|u-u_0|^p_{K^\alpha_{p,\mu}((0,1);Y)}$. This completes the proof. □

Example 3.4.9. In this example Σ will always denote a compact sufficiently smooth hypersurface.

(i) Consider as a base space Y the space $Y = L_p(\Sigma)$. Let $A = 1-\Delta_\Sigma$, $\mu \in (1/p,1]$. Then for all $\alpha \in (0,1]$ we have

$$\mathsf{tr}[W^\alpha_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2\alpha}_p(\Sigma))] = W_p^{2\alpha-2+2\mu-2/p}(\Sigma).$$

This will later on be used for $\alpha = 1$, $\alpha = 1-1/2p$, and $\alpha = 1/2-1/2p$.

(ii) Consider as a base space Y again the space $Y = L_p(\Sigma)$. Let $A = (1-\Delta_\Sigma)^2$, $\mu \in (1/p,1]$. Then we have

$$\mathsf{tr}[W^{1/2-1/2p}_{p,\mu}(\mathbb{R}_+; L_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2-2/p}_p(\Sigma))] = W_p^{4\mu-2-6/p}(\Sigma).$$

This result will be used in Section 6.6.

(iii) Consider as a base space Y the space $Y = H^2_p(\Sigma)$. Let $A = 1-\Delta_\Sigma$, $\mu \in (1/p,1]$. Then we have

$$\mathsf{tr}[W^{1-1/2p}_{p,\mu}(\mathbb{R}_+; H^2_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{4-1/p}_p(\Sigma))] = W_p^{2+2\mu-3/p}(\Sigma).$$

This result will be also used in Section 6.6.

(iv) Consider as a base space Y the space $Y = W^{2-1/p}_p(\Sigma)$. Let $A = (1-\Delta_\Sigma)^{1/2}$, $\mu \in (1/p,1]$. Then we have

$$\mathsf{tr}[H^1_{p,\mu}(\mathbb{R}_+; W^{2-1/p}_p(\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; W^{3-1/p}_p(\Sigma))] = W_p^{2+\mu-2/p}(\Sigma).$$

This result will be used in Chapter 8.

3.5 Maximal L_p-Regularity

5.1 Maximal L_p-Regularity

Let $J = \mathbb{R}_+$ or $(0,a)$ for some $a > 0$ and let $f : J \to X$. We consider the inhomogeneous initial value problem

$$\dot u(t) + Au(t) = f(t), \quad t \in J, \quad u(0) = u_0, \tag{3.71}$$

in $L_p(J;X)$ for $p \in (1,\infty)$.

The definition of *maximal L_p-regularity* for (3.71) is as follows.

Definition 3.5.1. *Suppose $A : \mathsf{D}(A) \subset X \to X$ is closed and densely defined. Then A is said to belong to the class $\mathcal{MR}_p(J;X)$ – and we say that there is* **maximal L_p-regularity** *for (3.71) – if for each $f \in L_p(J;X)$ there exists a unique $u \in H^1_p(J;X) \cap L_p(J;X_A)$ satisfying (3.71) a.e. in J, with $u_0 = 0$.*

The closed graph theorem implies then that there exists a constant $C > 0$ such that

$$|u|_{L_p(J;X)} + |\dot{u}|_{L_p(J;X)} + |Au|_{L_p(J;X)} \le C|f|_{L_p(J;X)}. \tag{3.72}$$

Combining L_p-maximal regularity with Section 3.4.1 we then obtain for the solution of (3.71) the estimate

$$|u|_{L_p(J;X)} + |\dot{u}|_{L_p(J;X)} + |Au|_{L_p(J;X)} \le C\big(|u_0|_{D_A(1-1/p,p)} + |f|_{L_p(J;X)}\big). \tag{3.73}$$

We denote the solution operator $f \mapsto u$ by $\mathcal{R}$. It is well known that there is maximal L_p regularity for (3.71) only if $-A$ generates an analytic semigroup. If $J = \mathbb{R}_+$, then the semigroup is even of negative exponential type. We state this as

Proposition 3.5.2. *Let $A \in \mathcal{MR}_p(J;X)$ for some $p \in (1,\infty)$.*

Then the following assertions are valid.

(i) *If $J = (0,a)$ then there are constants $\omega \ge 0$ and $M \ge 1$ such that*

$$\{z \in \mathbb{C} : \mathrm{Re}\, z \le -\omega\} \subset \rho(A) \quad \textit{and} \quad |z(z+A)^{-1}|_{\mathcal{B}(X)} \le M, \quad \mathrm{Re}\, z \ge \omega,$$

is valid. In particular, $\omega + A$ is sectorial with spectral angle $< \pi/2$.

(ii) *If $J = \mathbb{R}_+$ then $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\} \subset \rho(A)$ and there is a constant $M \ge 1$ such that*

$$|(z+A)^{-1}|_{\mathcal{B}(X)} \le \frac{M}{1+|z|}, \quad \mathrm{Re}\, z > 0,$$

is valid. In particular, A is sectorial with spectral angle $< \pi/2$ and $0 \in \rho(A)$.

Proof. Consider first the case $J = (0,a)$. We show that there are constants $\omega_1 \ge 0$ and $M \ge 1$ such that

$$|\mu||x|_X + |x|_{X_A} \le M|(\mu + A)x|_X, \quad x \in \mathsf{D}(A), \ \mathrm{Re}\, \mu > \omega_1. \tag{3.74}$$

In particular, $\mu + A$ is injective for each $\mathrm{Re}\, \mu > \omega_1$. Indeed, choose $\mu \in \mathbb{C}_+$, $x \in \mathsf{D}(A)$, and and let $v_\mu(t) := e^{\mu t}x$. Then v_μ satisfies $\dot{v}_\mu + Av_\mu = g_\mu(t)$ and $v_\mu(0) = x$, where $g_\mu(t) = e^{\mu t}(\mu + A)x \in L_p(J;X)$. The maximal regularity estimate (3.73) implies

$$|e^{t\mathrm{Re}\,\mu}|_{L_p(J;X)}\big(\mu|x|_X + |x|_{X_A}\big) \le C\big(|e^{t\mathrm{Re}\,\mu}|_{L_p(J;X)}|(\mu + A)x|_X + |x|_{X_A}\big).$$

Choosing ω_1 large enough such that $2C \le |e^{t\mathrm{Re}\,\mu}|_{L_p(J;X)}$ yields (3.74).

In a next step, which is more involved, we show that there is a constant $\omega_2 \ge 0$ such that $\mu + A$ is surjective for $\mathrm{Re}\, \mu > \omega_2$. Choose $\mu \in \mathbb{C}_+$, $x \in X$, and define $f_\mu \in L_p(\mathbb{R}_+; X)$ by $f_\mu(t) = e^{-\mu t}x$. Let $u_\mu(t;x) = \mathcal{R}(f_\mu)(t)$, where $\mathcal{R}$ denotes the solution operator for (3.71) with $u_0 = 0$. Set

$$U_\mu x := 2\mathrm{Re}\,\mu \int_0^a e^{-\bar{\mu}t} u_\mu(t;x)\,dt = \frac{2\mathrm{Re}\,\mu}{\bar{\mu}}\Big[\int_0^a e^{-\bar{\mu}t}\dot{u}_\mu(t;x)\,dt - e^{-\bar{\mu}a}u_\mu(a;x)\Big].$$

The maximal regularity property for (3.71) implies that there exists a constant $C > 0$ such that

$$|U_\mu|_{\mathcal{B}(X)} \le C(1+|\mu|)^{-1}, \quad \mathrm{Re}\,\mu > 0,$$

where ω is sufficiently large. In fact, we have with Hölder's inequality and the maximal regularity estimate (3.72)

$$|U_\mu x| \le 2(p'\mathrm{Re}\,\mu)^{1-1/p'}|u_\mu|_{L_p(J;X)} \le C(\mathrm{Re}\,\mu)^{1/p}|f_\mu|_{L_p(J;X)} \le C|x|,$$

as well as

$$\begin{aligned}|U_\mu x| &\le 2\mathrm{Re}\,\mu|\mu|^{-1}\big[(p'\mathrm{Re}\,\mu)^{-1/p'} + e^{-a\mathrm{Re}\,\mu}a^{1/p'}\big]|\dot{u}_\mu|_{L_p(J;X)}\\ &\le C|\mu|^{-1}(\mathrm{Re}\,\mu)^{1/p}|f_\mu|_{L_p(J;X)} \le |\mu|^{-1}C|x|.\end{aligned}$$

Next we multiply (3.71) with $f = f_\mu$ by $e^{-\bar{\mu}t}$ and integrate over J. This yields by closedness of A and an integration by parts

$$\begin{aligned}(1-e^{-2a\mathrm{Re}\,\mu})x &= 2\mathrm{Re}\,\mu\int_0^a e^{-\bar{\mu}t}f_\mu(t)\,dt = 2\mathrm{Re}\,\mu\int_0^a e^{-\bar{\mu}t}[\dot{u}_\mu(t;x) + Au_\mu(t;x)]\,dt\\ &= (\bar{\mu}+A)U_\mu x + 2(\mathrm{Re}\,\mu)e^{-\bar{\mu}a}u_\mu(a;x),\end{aligned}$$

which after rearrangement becomes

$$(\bar{\mu}+A)U_\mu x = x - V_\mu x, \quad V_\mu x := e^{-2a\mathrm{Re}\,\mu}x + 2(\mathrm{Re}\,\mu)e^{-\bar{\mu}a}u_\mu(a;x).$$

Estimating as before we obtain

$$|V_\mu x| \le \big[e^{-2a\mathrm{Re}\,\mu} + Ce^{-a\mathrm{Re}\,\mu}(a\mathrm{Re}\,\mu)^{1/p'}\big]|x|,$$

from which we see that there is $\omega_2 > 0$ such that $|V_\mu|_{\mathcal{B}(X)} \le 1/2$, for each $\mathrm{Re}\,\mu \ge \omega_2$. This then shows that $\bar{\mu}+A$ is surjective for all such μ. Setting $\omega = \max\{\omega_1, \omega_2\}$ we conclude that $\mu + A : D(A) \to X$ is invertible, and

$$(\bar{\mu}+A)^{-1} = U_\mu(1-V_\mu)^{-1}, \quad \mathrm{Re}\,\mu > \omega.$$

The estimate on U_μ (or the a priori estimate in (3.74)) then shows that $\omega + A$ is sectorial with spectral angle $< \pi/2$.

For the case $J = \mathbb{R}_+$ the proof is simpler; one deduces in the same way the relation $(\bar{\mu}+A)^{-1} = U_\mu$ with $\omega = 0$. □

There is variant of maximal L_p-regularity if one requires for the solution of (3.71) only $u \in C(\bar{\mathbb{R}}_+; X)$ and $\dot{u}, Au \in L_p(\mathbb{R}_+; X)$. We call the class of operators with this weaker property ${}_0\mathcal{MR}_p(\mathbb{R}_+; X)$. The proof of Proposition 3.5.2 shows that then in (ii) the condition $0 \in \rho(A)$ is dropped. More precisely we have

Corollary 3.5.3. *Suppose $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X)$.*

Then A is pseudo-sectorial in X with spectral angle $< \pi/2$.
Moreover, $A \in \mathcal{MR}_p(\mathbb{R}_+; X)$ if and only if $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X)$ and $0 \in \rho(A)$.

Proposition 3.5.2 shows that for a finite interval $J = (0, a)$ its length $a > 0$ plays no role for maximal L_p-regularity, and up to a shift of A, without loss of generality, we may consider $J = \mathbb{R}_+$ and may assume that $-A$ is the generator of an analytic semigroup of negative exponential type. Therefore, in the sequel we mostly consider $J = \mathbb{R}_+$ and abbreviate $\mathcal{MR}_p(X) = \mathcal{MR}_p(\mathbb{R}_+; X)$ as well as ${}_0\mathcal{MR}_p(X) = {}_0\mathcal{MR}_p(\mathbb{R}_+; X)$.

Unfortunately, the converse of Proposition 3.5.2 is false. Actually, it is a formidable task to prove that a given operator A belongs to $\mathcal{MR}_p(X)$. We want to explain the difficulty in more detail. Obviously, the variation of parameters formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s)\, ds, \quad t \geq 0,$$

implies that there is maximal L_p-regularity for (3.71) if and only if the operator $\mathcal{R}$ defined by

$$\mathcal{R}f := A \int_0^t e^{-A(t-s)} f(s)\, ds$$

acts as a bounded operator on $L_p(\mathbb{R}_+; X)$. It is nontrivial to show this since the kernel of this convolution operator on the half-line is Ae^{-At} which has a non-integrable singularity near $t = 0$, behaving like $1/t$, as follows from the well-known, best possible estimate

$$|Ae^{-At}|_{\mathcal{B}(X)} \leq \frac{Me^{-\eta t}}{t}, \quad t > 0,$$

valid for exponentially stable analytic semigroups. Therefore, $\mathcal{R}$ is a *singular* integral operator on $L_p(\mathbb{R}_+; X)$ with operator-valued kernel. This calls for vector-valued harmonic analysis and we take up this topic in the next chapter.

5.2 Maximal Regularity in Weighted L_p-Spaces

We next study maximal regularity in spaces $L_{p,\mu}$. The main result of this section reads as follows.

Theorem 3.5.4. *Let X be a Banach space, $p \in (1, \infty)$, and $1/p < \mu \leq 1$. Then*

$$A \in \mathcal{MR}_p(X) \text{ if and only if } A \in \mathcal{MR}_{p,\mu}(X).$$

Proof. In the following we shall use the notation $X_0 := X$ and $X_1 := X_A$. It follows that X_1 is a Banach space which is densely embedded in X_0.

(i) Suppose that $A \in \mathcal{MR}_p(X)$. Then we know by Proposition (3.5.2) that $-A$ generates an exponentially stable analytic semigroup $\{e^{-tA} : t \geq 0\}$ on X_0. Let $f \in L_{p,\mu}(\mathbb{R}_+; X_0)$ be given. Let us consider the function u defined by the variation of constants formula

$$u(t) := \int_0^t e^{-(t-s)A} f(s)\, ds, \quad t > 0. \tag{3.75}$$

It follows from Lemma 3.2.5(a) that this integral exists in X_0. We will now rewrite equation (3.75) in the following way

$$\begin{aligned} u(t) &= t^{\mu-1} \int_0^t e^{-(t-s)A} s^{1-\mu} f(s)\, ds + t^{\mu-1} \int_0^t e^{-(t-s)A} [(t/s)^{1-\mu} - 1] s^{1-\mu} f(s)\, ds \\ &= \Phi_\mu^{-1}[(B_p + A)^{-1} \Phi_\mu f + T_A \Phi_\mu f] = \Phi_\mu^{-1}[v_1 + v_2]. \end{aligned}$$

Here we use the same notation for A and its canonical extension on $L_p(\mathbb{R}_+; X_0)$, given by $(Au)(t) := Au(t)$ for $t > 0$. By definition, T_A is the integral operator

$$(T_A g)(t) := \int_0^t e^{-(t-s)A} [(t/s)^{1-\mu} - 1] g(s)\, ds, \quad g \in L_p(\mathbb{R}_+; X_0).$$

Observe that the kernel $K_A(t) := Ae^{-tA}$ satisfies the assumptions of Proposition 4.3.13 below with $Y = X_1$. We conclude that

$$T_A \in \mathcal{B}(L_p(\mathbb{R}_+; X_0), L_p(\mathbb{R}_+; X_1)). \tag{3.76}$$

It is a consequence of (3.76) that v_2 has a derivative almost everywhere, given by

$$\dot{v}_2 = -AT_A \Phi_\mu f + (1-\mu) t^{-\mu} \int_0^t e^{-(t-s)A} f(s)\, ds.$$

It follows from Hardy's inequality, Lemma 3.4.5, that

$$\int_0^\infty \Big| t^{-\mu} \int_0^t e^{-(t-s)A} f(s)\, ds \Big|^p dt \leq M \int_0^\infty \Big(t^{-\mu} \int_0^t |f(s)|\, ds \Big)^p dt \leq cM |f|_{L_{p,\mu}}^p$$

and we infer that

$$v_2 \in {}_0H_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1). \tag{3.77}$$

It follows from our assumption that v_1 enjoys the same regularity properties as v_2 and consequently, v satisfies (3.77) as well. Proposition 3.2.6 then shows that

$$u \in {}_0H_{p,\mu}^1(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1). \tag{3.78}$$

It is now easy to verify that u is in fact a solution of the Cauchy problem (3.71) with initial value 0. We have thus shown that $A \in \mathcal{MR}_{p,\mu}(X)$.

(b) Suppose now that $A \in \mathcal{MR}_{p,\mu}(X_0)$. As in the case $\mu = 1$ one shows that A generates a bounded analytic C_0-semigroup $\{e^{-tA}; t \geq 0\}$ on X_0. Let $f \in L_p(\mathbb{R}_+; X_0)$ be given. Here we use the representation

$$\begin{aligned} u(t) &= t^{1-\mu} \int_0^t e^{-(t-s)A} s^{\mu-1} f(s)\, ds - \int_0^t e^{-(t-s)A} [(t/s)^{1-\mu} - 1] f(s)\, ds \\ &= \Phi_\mu (B_{p,\mu} + A)^{-1} \Phi_\mu^{-1} f - T_A f, \end{aligned}$$

with T_A as above. The assertion follows now by similar arguments as in (a). $\square$

We will now consider the Cauchy problem (3.71) in $L_{p,\mu}(\mathbb{R}_+; X)$. Define the function spaces

$$\begin{aligned} \mathbb{E}_{0,\mu} &:= \mathbb{E}_{0,\mu}(\mathbb{R}_+) : = L_{p,\mu}(\mathbb{R}_+; X_0), \\ \mathbb{E}_{1,\mu} &:= \mathbb{E}_{1,\mu}(\mathbb{R}_+) : = H^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1), \end{aligned}$$

where $X_0 := X$ and $X_1 := X_A$. It is not difficult to verify that the norm

$$|u|_{\mathbb{E}_{1,\mu}} := \left(|u|^p_{L_{p,\mu}(\mathbb{R}_+;X_1)} + |\dot{u}|^p_{L_{p,\mu}(\mathbb{R}_+;X_0)}\right)^{1/p} \tag{3.79}$$

turns $\mathbb{E}_{1,\mu}(\mathbb{R}_+)$ into a Banach space. The result reads as follows

Theorem 3.5.5. *Let $p \in (1, \infty)$ and $1/p < \mu \leq 1$. Suppose that $A \in \mathcal{MR}_p(X)$. Then*

$$\left(\frac{d}{dt} + A, \mathsf{tr}\right) \in \mathrm{Isom}(\mathbb{E}_{1,\mu}(\mathbb{R}_+), \mathbb{E}_{0,\mu}(\mathbb{R}_+) \times X_{\gamma,\mu}),$$

where $\mathsf{tr}(u) := u(0)$ denotes the trace operator, and $X_{\gamma,\mu} = D_A(\mu - 1/p, p)$.

Proof. We observe that $(\frac{d}{dt} + A) \in \mathcal{B}(\mathbb{E}_{1,\mu}, \mathbb{E}_{0,\mu})$ and $\mathsf{tr} \in \mathcal{B}(\mathbb{E}_{1,\mu}, X_{\gamma,\mu})$ yield boundedness of $(\frac{d}{dt} + A, \mathsf{tr})$. Theorem 3.5.4 shows that the operator $(B_{p,\mu} + A)$ with domain

$$\mathsf{D}(B_{p,\mu} + A) = \mathsf{D}(B_{p,\mu}) \cap \mathsf{D}(A) = \{u \in \mathbb{E}_{1,\mu}(\mathbb{R}_+) :\, u(0) = 0\}$$

is invertible. Let $(f, u_0) \in \mathbb{E}_{0,\mu} \times X_{\gamma,\mu}$ be given and let

$$u := (B_{p,\mu} + A)^{-1} f + e^{-tA} u_0. \tag{3.80}$$

Clearly, u solves the Cauchy problem (3.71). Therefore, $(\frac{d}{dt} + A, \mathsf{tr})$ is surjective. The assertion follows now from the open mapping theorem. $\square$

If $1 < p < \infty$ and $\mu = 1$ the semigroup of translations $T(\tau)u(t) = u(t + \tau)$ is strongly continuous in $\mathbb{E}_{1,1}$, which implies that the time-trace tr maps $\mathbb{E}_{1,1}$ into $C(\bar{\mathbb{R}}_+; X_{\gamma,1})$, with bound

$$\sup_{t \geq \tau} |u(t)|_{X_{\gamma,1}} \leq C |T(\tau)u|_{\mathbb{E}_{1,1}} \to 0 \quad \text{as } \tau \to \infty.$$

Therefore, we have the embedding

$$\mathbb{E}_{1,1}(\mathbb{R}_+) \hookrightarrow C_0(\bar{\mathbb{R}}_+; X_{\gamma,1}). \tag{3.81}$$

On the other hand, as the time weights $t^{1-\mu}$ act only near $t = 0$ we obtain

$$\mathbb{E}_{1,\mu}(\mathbb{R}_+) \hookrightarrow \mathbb{E}_{1,1}(\delta, \infty), \quad \text{for each } \delta > 0.$$

This implies

$$\mathbb{E}_{1,\mu}(\mathbb{R}_+) \hookrightarrow C(\bar{\mathbb{R}}_+; X_{\gamma,\mu}) \cap C_0(\mathbb{R}_+; X_{\gamma,1}), \tag{3.82}$$

which shows parabolic regularization. This will be very useful in later chapters.

It is sometimes important to also have solvability results for the non-autonomous problem

$$\dot{u} + A(t)u = f(t), \quad t > 0, \quad u(0) = u_0.$$

This is the content of the next proposition.

Proposition 3.5.6. *Suppose $A \in C(J, \mathcal{B}(X_1, X_0))$ and $A(t) \in \mathcal{M}_p(J, X_0)$ for each $t \in J = [0, a]$. Then*

$$\Big(\frac{d}{dt} + A(\cdot), \mathsf{tr}\Big) \in \mathrm{Isom}(\mathbb{E}_{1,\mu}(J), \mathbb{E}_{0,\mu}(J) \times X_{\gamma,\mu}).$$

In particular, the non-autonomous problem

$$\dot{u} + A(t)u = f(t), \quad t \in \dot{J}, \quad u(0) = u_0,$$

admits for each $(f, u_0) \in \mathbb{E}_{0,\mu}(J) \times X_{\gamma,\mu}$ a unique solution $u \in \mathbb{E}_{1,\mu}(J)$.

Proof. (ii) As $(\frac{d}{dt} + A(\cdot), \mathsf{tr}) \in \mathcal{B}(\mathbb{E}_{1,\mu}(J), \mathbb{E}_{0,\mu}(J) \times X_{\gamma,\mu})$ it suffices to show that $(\frac{d}{dt} + A(\cdot), \mathsf{tr})$ is bijective, thanks to the open mapping theorem. By a perturbation and compactness argument one shows that there is a constant M such that

$$\Big|\Big(\frac{d}{dt} + A(s), \mathsf{tr}\Big)^{-1}\Big|_{\mathcal{B}(\mathbb{E}_{1,\mu}(J), \mathbb{E}_{0,\mu}(J) \times X_{\gamma,\mu})} \leq M, \quad s \in J.$$

By compactness of J we can choose points $0 = s_0 < s_1 \cdots < s_{m+2} = a$ such that

$$\max_{s_j \leq t \leq s_{j+2}} |A(t) - A(s_j)|_{\mathcal{B}(X_1, X_0)} \leq 1/2M, \quad j = 0, \ldots, m.$$

A Neumann series argument then yields with $J_j = (s_j, s_{j+1})$

$$\Big(\frac{d}{dt} + A(\cdot), \mathsf{tr}\Big) \in \mathrm{Isom}(\mathbb{E}_{1,\mu}(J_j), \mathbb{E}_{0,\mu}(J_j) \times X_{\gamma,\mu}), \quad j = 0, \ldots, m. \tag{3.83}$$

Let $(f, x) \in \mathbb{E}_{0,\mu}(J) \times X_{\gamma,\mu}$ be given. Then we solve the problem with maximal $L_{p,\mu}$-regularity on the first interval J_0. The final value $u(s_1)$ then belongs to X_γ, hence we solve the problem on J_1 with this initial value and maximal L_p-regularity, and then by induction on all of the remaining intervals. $\square$

5.3 Maximal $L_{2,\mu}$-Regularity in Hilbert Spaces
Let X be a Hilbert space and let A be pseudo-sectorial with $\phi_A < \frac{\pi}{2}$. Then $-A$ is the generator of a bounded holomorphic C_0-semigroup, in particular the domain of A is also dense in X. In this subsection we want to consider the L_2-theory of the abstract Cauchy problem

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0, \tag{3.84}$$

where $f \in L_{2,\mu}(\mathbb{R}_+; X)$. It is the purpose of this subsection to give a simple proof of maximal-L_2-regularity in this case.

Theorem 3.5.7. *Let X be a Hilbert space and $A \in \mathcal{PS}(X)$ and such that $\phi_A < \frac{\pi}{2}$. Then $A \in {}_0\mathcal{MR}_2(X)$.*

Proof. The proof of the result follows by the vector-valued Paley-Wiener theorem on the halfline which is valid in a Hilbert space setting. This result states that in case X is a Hilbert space, the Laplace transform is an isometric isomorphism from $L_2(\mathbb{R}_+; X)$ onto the vector-valued Hardy space $H_2(\mathbb{C}_+; X)$ equipped with the norm

$$|u|^2_{H_2(\mathbb{C}_+;X)} = \frac{1}{2\pi} \int_{\mathbb{R}} |u(i\rho)|^2 d\rho.$$

Let $f \in \mathcal{D}(\mathbb{R}_+; X)$ first. Then (3.84) admits a unique strong solution u. Laplace transform yields

$$\widehat{u}(\lambda) = (\lambda + A)^{-1} \widehat{f}(\lambda), \quad \operatorname{Re} \lambda > 0.$$

Uniform boundedness of $\lambda(\lambda + A)^{-1}$ on $\mathbb{C}_+$ then implies

$$|\lambda \widehat{u}(\lambda)| + |A\widehat{u}(\lambda)| \le C|\widehat{f}(\lambda)|, \quad \operatorname{Re} \lambda > 0,$$

with a constant $C > 0$ depending only on A, hence by the Paley-Wiener theorem

$$|\dot{u}|_{L_2(\mathbb{R}_+;X)} + |Au|_{L_2(\mathbb{R}_+;X)} \le C|f|_{L_2(\mathbb{R}_+;X)}. \tag{3.85}$$

Now $\mathcal{D}(\mathbb{R}_+; X)$ is dense in $L_2(\mathbb{R}_+; X)$, hence a standard approximation argument applies to obtain this estimate also for arbitrary $f \in L_2(\mathbb{R}_+; X)$. □

5.4 Maximal L_p-Regularity in Real Interpolation Spaces
It is a remarkable fact that maximal L_p-regularity holds in the real interpolation spaces $D_A(\alpha, p)$ if $-A$ generates an analytic C_0-semigroup in X. This is the content of the following result.

Theorem 3.5.8. *Let X be a Banach space, $A \in \mathcal{S}(X)$ invertible with $\phi_A < \pi/2$, let $\alpha \in (0, 1)$, and $p \in [1, \infty)$.*
Then $A \in \mathcal{MR}_p(D_A(\alpha, p))$.

Proof. Let $f \in L_p(\mathbb{R}_+; D_A(\alpha, p))$ be given and set $u = e^{-At} * f$; we have to prove

$$|Au|_{L_p(\mathbb{R}_+;D_A(\alpha,p))} \le C|f|_{L_p(\mathbb{R}_+;D_A(\alpha,p))},$$

for some constant $C > 0$ independent of f. For this purpose, note that

$$|Ae^{-A\tau}Au(t)| \leq \int_0^t |A^2 e^{-A(\tau+s)} f(t-s)|\, ds \leq M \int_0^t |Ae^{-A(\tau+s)} f(t-s)|(\tau+s)^{-1}\, ds,$$

hence by Hölder's inequality

$$|Ae^{-A\tau}Au(t)|^p \leq M\Big[\int_0^t (\tau+s)^{-ap'}\, ds\Big]^{p/p'} \int_0^t |Ae^{-A(\tau+s)} f(t-s)|^p (\tau+s)^{-bp}\, ds,$$

where $a + b = 1$ and $a > 1/p'$ to ensure

$$\Big[\int_0^t (\tau+s)^{-ap'} ds\Big]^{p/p'} \leq \Big[\int_0^\infty (\tau+s)^{-ap'} ds\Big]^{p/p'} = c_1 \tau^{p(1/p'-a)} < \infty.$$

Integrating over $t > 0$ and using Fubini's theorem, this yields

$$\begin{aligned}
|Ae^{-\tau A}Au|^p_{L_p(\mathbb{R}_+;X)} &\leq c_1 M \tau^{p(1/p'-a)} \int_0^\infty \int_s^\infty |Ae^{-A(\tau+s)} f(t-s)|^p (\tau+s)^{-bp}\, dtds \\
&= c_1 M \tau^{p(1/p'-a)} \int_0^\infty \int_0^\infty |Ae^{-A(\tau+s)} f(t)|^p (\tau+s)^{-bp}\, dtds.
\end{aligned}$$

From this estimate we obtain integrating over $\tau > 0$ with weight $\tau^{p(1-\alpha)-1}$, using again Fubini's theorem

$$\begin{aligned}
|Au|^p_{L_p(\mathbb{R}_+;D_A(\alpha,p))} &\leq c_1 M \int_0^\infty \int_0^\infty \int_0^\infty \tau^{\beta-1} |Ae^{-A(\tau+s)} f(t)|^p (\tau+s)^{-bp} ds d\tau dt \\
&= c_1 M \int_0^\infty \int_0^\infty \int_\tau^\infty \tau^{\beta-1} |Ae^{-As} f(t)|^p s^{-bp} ds d\tau dt \\
&= c_1 M \int_0^\infty \int_0^\infty |Ae^{-As} f(t)|^p \int_0^s \tau^{\beta-1} d\tau s^{-bp} ds dt \\
&= c_1 M \beta^{-1} \int_0^\infty \int_0^\infty |Ae^{-As} f(t)|^p s^{\beta-bp} ds dt \\
&= c_1 M \beta^{-1} |f|_{L_p(\mathbb{R}_+;D_A(\alpha,p))},
\end{aligned}$$

with $\beta = (1-\alpha)p + p/p' - ap > 0$ provided $a < 1 - \alpha + 1/p'$, and then $\beta - bp = (1-\alpha)p - 1$. The argument for $p = 1$ is similar and even simpler. □