Chapter 2

Tools from Differential Geometry

In this chapter we introduce the necessary background in differential geometry of closed compact hypersurfaces in $\mathbb{R}^n$. We investigate the differential geometric properties of embedded hypersurfaces in n-dimensional Euclidean space, introducing the notions of Weingarten tensor, principal curvatures, mean curvature, tubular neighbourhood, surface gradient, surface divergence, and Laplace-Beltrami operator. The main emphasis lies in deriving representations of these quantities for hypersurfaces $\Gamma = \Gamma_\rho$ that are given as parameterized surfaces in normal direction of a fixed reference surface Σ by means of a height function ρ. We derive all of the aforementioned geometric quantities for Γ_ρ in terms of ρ and Σ. It is also important to study the mapping properties of these quantities in dependence of ρ, and to derive expressions for their variations. For instance, we show that

$$\kappa'(0) = \operatorname{tr} L_\Sigma^2 + \Delta_\Sigma,$$

where $\kappa = \kappa(\rho)$ denotes the mean curvature of Γ_ρ, L_Σ the Weingarten tensor of Σ, and Δ_Σ the Laplace-Beltrami operator on Σ. This is done in Section 2. We also study the first and second variations of the area and volume functional, respectively. In Section 3 we show, among other things, that C^2-hypersurfaces can be approximated in a suitable topology by smooth (i.e., analytic) hypersurfaces. This leads, in particular, to the existence of parameterizations. In Section 4 we show that the class of compact embedded hypersurfaces in $\mathbb{R}^n$ gives rise to a new manifold (whose points are the compact embedded hypersufaces). We also show that the class $\mathcal{M}^2(\Omega, r)$ of all compact embedded hypersurfaces contained in a bounded domain $\Omega \subset \mathbb{R}^n$, and satisfying a uniform ball condition with radius $r > 0$, can be identified with a subspace of $C^2(\bar{\Omega})$. This is important, as it allows us to derive compactness and embedding properties for $\mathcal{M}^2(\Omega, r)$. Finally, in Section 5 we consider moving hypersurfaces and prove various transport theorems.

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J. Prüss and G. Simonett, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, Monographs in Mathematics 105,
DOI 10.1007/978-3-319-27698-4_2

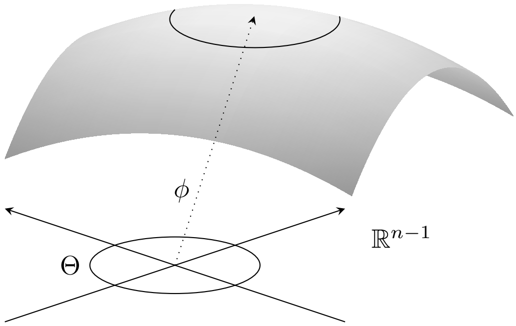

Figure 2.1: A typical chart for Σ.

2.1 Differential Geometry of Hypersurfaces

We consider a closed embedded hypersurface Σ of class C^k, $k \geq 3$, enclosing a bounded domain Ω in $\mathbb{R}^n$. Thus for each point $p \in \Sigma$ there is a ball $B(p,r) \subset \mathbb{R}^n$ and a diffemorphism $\Phi : B(p,r) \to U \subset \mathbb{R}^n$ such that $\Phi(p) = 0 \in U$ and

$$\Phi^{-1}(U \cap (\mathbb{R}^{n-1} \times \{0\})) = B(p,r) \cap \Sigma.$$

We may assume that Σ is connected; otherwise we would concentrate on one of its components. The points of Σ are denoted by p, and $\nu_\Sigma = \nu_\Sigma(p)$ means the outer unit normal of Σ at p. Locally at $p \in \Sigma$ we have the parameterization

$$p = \phi(\theta) := \Phi^{-1}(\theta, 0),$$

where θ runs through an open parameter set $\Theta \subset \mathbb{R}^{n-1}$. We denote the tangent vectors generated by this parameterization by

$$\tau_i = \tau_i(p) = \frac{\partial}{\partial \theta_i} \phi(\theta) = \partial_i \phi, \quad i = 1, \ldots, n-1. \tag{2.1}$$

These vectors τ_i form a basis of the *tangent space* $T_p\Sigma$ of Σ at p. Note that $(\tau_i|\nu_\Sigma) = 0$ for all i, where $(\cdot|\cdot) := (\cdot|\cdot)_{\mathbb{R}^n}$ denotes the Euclidean inner product in $\mathbb{R}^n$. Similarly, we set $\tau_{ij} = \partial_i\partial_j\phi$, $\tau_{ijk} = \partial_i\partial_j\partial_k\phi$, and so on. In the sequel we employ Einstein's summation convention, which means that equal lower and upper indices are to be summed, and δ^i_j are the entries of the unit matrix I. For two vectors $a, b \in \mathbb{R}^n$ the tensor product $a \otimes b \in \mathcal{B}(\mathbb{R}^n)$ is defined by $[a \otimes b](x) = (b|x)a$ for $x \in \mathbb{R}^n$. If a belongs to the tangent space $T_p\Sigma$, we may represent a as a linear combination of the basis vectors of $T_p\Sigma$, i.e., $a = a^i\tau_i$. The coefficients a^i are called the *contravariant components* of a. On the other hand, this vector a is also uniquely characterized by its *covariant components*, a_i defined by $a_i = (a|\tau_i)$, which means that the covariant components are the coefficients of the representation of a in the basis $\{\tau^i\}$ dual to the basis $\{\tau_j\}$, defined by the relations $(\tau^i|\tau_j) = \delta^i_j$. Similarly, if $K \in \mathcal{B}(T_p\Sigma)$ is a tensor we have the representations

$$K = k^{ij}\tau_i \otimes \tau_j = k_{ij}\tau^i \otimes \tau^j = k^i_j \tau_i \otimes \tau^j = k^j_i \tau^i \otimes \tau_j,$$

with e.g. $k_{ij} = (\tau_i|K\tau_j)$ and $k^i_j = (\tau^i|K\tau_j)$. Moreover, $\operatorname{tr} K$, the *trace* of K, is given by

$$\operatorname{tr} K = (K\tau_i|\tau^i) = (K\tau^i|\tau_i). \tag{2.2}$$

In particular, $\operatorname{tr}[a \otimes b] = (a|b) = a_i b^i = a^i b_i$.

1.1 The First Fundamental Form

Define

$$g_{ij} = g_{ij}(p) = (\tau_i(p)|\tau_j(p)) = (\tau_i|\tau_j), \quad i, j = 1, \ldots, n-1. \tag{2.3}$$

The matrix $G = [g_{ij}]$ is called *the first fundamental form* of Σ. Note that G is symmetric and also positive definite, since

$$(G\xi|\xi) = g_{ij}\xi^i\xi^j = (\xi^i\tau_i|\xi^j\tau_j) = |\xi^i\tau_i|^2 > 0, \quad \text{for all } \xi \in \mathbb{R}^{n-1},\ \xi \neq 0.$$

We let $G^{-1} = [g^{ij}]$, hence $g_{ik}g^{kj} = \delta^j_i$, and $g^{il}g_{lj} = \delta^i_j$. The determinant $g := \det G$ is positive. Let a be a tangent vector. Then $a = a^i\tau_i$ implies

$$a_k = (a|\tau_k) = a^i(\tau_i|\tau_k) = a^i g_{ik} \quad \text{and} \quad a^i = g^{ik}a_k.$$

Thus the fundamental form G allows for the passage from contra- to covariant components of a tangent vector and vice versa. If a, b are two tangent vectors, then

$$(a|b) = a^i b^j(\tau_i|\tau_j) = g_{ij}a^i b^j = a_j b^j = a^i b_i = g^{ij}a_i b_j =: (a|b)_\Sigma$$

defines an inner product on $T_p\Sigma$ in the canonical way, the *Riemannian metric*. By means of the identity

$$(g^{ik}\tau_k|\tau_j) = g^{ik}g_{kj} = \delta^i_j$$

we further see that

$$\tau^i = g^{ij}\tau_j \quad \text{and} \quad \tau_j = g_{ij}\tau^i.$$

This implies the relations

$$k^i_j = g^{ir}k_{rj} = g_{jr}k^{ri}, \quad k^{ij} = g^{ir}k^j_r, \quad k_{ij} = g_{ir}k^r_j,$$

for any tensor $K \in \mathcal{B}(T_p\Sigma)$. We set for the moment $\mathcal{G} = g^{ij}\tau_i \otimes \tau_j$ and have equivalently

$$\mathcal{G} = g^{ij}\tau_i \otimes \tau_j = g_{ij}\tau^i \otimes \tau^j = \tau_i \otimes \tau^i = \tau^j \otimes \tau_j.$$

Let $u = u^k\tau_k + (u|\nu_\Sigma)\nu_\Sigma$ be an arbitrary vector in $\mathbb{R}^n$. Then

$$\mathcal{G}u = g^{ij}\tau_i(\tau_j|u) = g^{ij}\tau_i u^k g_{jk} = u^k\tau_k,$$

i.e., $\mathcal{G}$ equals the orthogonal projection $\mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma$ of $\mathbb{R}^n$ onto the tangent space $T_p\Sigma$ at $p \in \Sigma$. Therefore, we have the relation

$$\mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma = \tau_i \otimes \tau^i = \tau^i \otimes \tau_i,$$

where I denotes the identity map on $\mathbb{R}^n$. These properties explain the meaning of the first fundamental form $[g_{ij}]$.

1.2 The Second Fundamental Form
Define

$$l_{ij} = l_{ij}(p) = (\tau_{ij}|\nu_\Sigma), \quad L = [l_{ij}]. \tag{2.4}$$

L is called *the second fundamental form* of Σ. Note that L is symmetric, and differentiating the relations $(\tau_i|\nu_\Sigma) = 0$ we derive

$$l_{ij} = (\tau_{ij}|\nu_\Sigma) = -(\tau_i|\partial_j\nu_\Sigma) = -(\tau_j|\partial_i\nu_\Sigma). \tag{2.5}$$

The matrix K with entries l^i_j, defined by

$$l^i_j = g^{ir}l_{rj}, \quad K = G^{-1}L,$$

is called the *shape matrix* of Σ. The eigenvalues κ_i of K are called the *principal curvatures* of Σ at p, and the corresponding eigenvectors η_i determine the *principal curvature directions*. Observe that $K\eta_i = \kappa_i\eta_i$ is equivalent to $L\eta_i = \kappa_i G\eta_i$, hence the relation

$$(L\eta_i|\eta_i) = \kappa_i(G\eta_i|\eta_i)$$

and symmetry of L and G show that the principal curvatures κ_i are real. Moreover,

$$\kappa_i(G\eta_i|\eta_j) = (L\eta_i|\eta_j) = (\eta_i|L\eta_j) = \kappa_j(\eta_i|G\eta_j) = \kappa_j(G\eta_i|\eta_j)$$

implies that principal directions corresponding to different principal curvatures are orthogonal with respect to the inner product $(G\cdot|\cdot)_{\mathbb{R}^{n-1}}$. We can always assume that eigenvectors associated to an eigenvalue κ_i are orthogonal w.r.t. $(G\cdot|\cdot)_{\mathbb{R}^{n-1}}$ in case κ_i has geometric multiplicity greater than one. The eigenvalues κ_i are semi-simple, i.e., $\mathsf{N}((\kappa_i - K)^2) = \mathsf{N}(\kappa_i - K)$. In fact, suppose $x \in \mathsf{N}((\kappa_i - K)^2)$. Then

$$(\kappa_i - K)x = \sum_{r=1}^{m_i} t_r\eta_{i,r},$$

with $t_r \in \mathbb{R}$, where $\{\eta_{i,r} : 1 \le r \le m_i\}$ is an (orthogonal) basis of $\mathsf{N}(\kappa_i - K)$. Therefore,

$$t_k(G\eta_{i,k}|\eta_{i,k}) = \Big(\sum_{r=1}^{m_i} t_rG\eta_{i,r}|\eta_{i,k}\Big) = (G(\kappa_i - K)x|\eta_{i,k}) = (x|(\kappa_iG - L)\eta_{i,k}) = 0$$

for $1 \le k \le m_i$. Since G is positive definite, $t_k = 0$, and hence $x \in \mathsf{N}(\kappa_i - K)$. This shows that K is diagonalizable.

The trace of K, i.e., the first invariant of K, is called the $(n-1)$-fold *mean curvature* κ of Σ at p, i.e., we have

$$\kappa_\Sigma := \operatorname{tr} K = l_i^i = g^{ij} l_{ij} = \sum_{i=1}^{n-1} \kappa_i. \tag{2.6}$$

The *Gaussian curvature* $\mathcal{K}_\Sigma$ is defined as the last invariant of K, i.e.,

$$\mathcal{K}_\Sigma = \det K = g^{-1} \det L = \Pi_{i=1}^{n-1} \kappa_i.$$

We define the *Weingarten tensor* L_Σ by means of

$$L_\Sigma = L_\Sigma(p) = l^{ij} \tau_i \otimes \tau_j = l_j^i \tau_i \otimes \tau^j = l_i^j \tau^i \otimes \tau_j = l_{ij} \tau^i \otimes \tau^j. \tag{2.7}$$

L_Σ is symmetric with respect to the inner product $(\cdot|\cdot)$ in $\mathbb{R}^n$. We note that $L_\Sigma \in \mathcal{B}(\mathbb{R}^n)$ leaves the tangent space $T_p\Sigma$ invariant and, moreover, $L_\Sigma \nu_\Sigma = 0$. This shows that L_Σ enjoys the decomposition

$$L_\Sigma = \begin{bmatrix} L_\Sigma|_{T_p\Sigma} & 0 \\ 0 & 0 \end{bmatrix} : \; T_p\Sigma \oplus \mathbb{R}\nu_\Sigma \to T_p\Sigma \oplus \mathbb{R}\nu_\Sigma. \tag{2.8}$$

In particular, we note

$$\operatorname{tr} L_\Sigma(p) = \operatorname{tr}[L_\Sigma|_{T_p\Sigma}], \quad \det[I + rL_\Sigma(p)] = \det[(I + rL_\Sigma(p))|_{T_p\Sigma}] \tag{2.9}$$

for $r \in \mathbb{R}$. We will in the following not distinguish between L_Σ and its restriction to $T_p\Sigma$. Observe that

$$\operatorname{tr} L_\Sigma = l_i^i = g^{ij} l_{ij} = \kappa_\Sigma, \tag{2.10}$$

and the eigenvalues of L_Σ in $T_p\Sigma$ are the principal curvatures, since

$$L_\Sigma \eta_k = l_j^i (\tau^j|\eta_k) \tau_i = l_j^i \eta_k^j \tau_i = \kappa_k \eta_k^i \tau_i = \kappa_k \eta_k.$$

The remaining eigenvalue of L_Σ in $\mathbb{R}^n$ is 0 with eigenvector ν_Σ.

1.3 The Third Fundamental Form

To obtain another property of the shape matrix K we differentiate the identity $|\nu_\Sigma|^2 = 1$ to the result $(\partial_i \nu_\Sigma|\nu_\Sigma) = 0$. This means that $\partial_i \nu_\Sigma$ belongs to the tangent space, hence $\partial_i \nu_\Sigma = \gamma_i^k \tau_k$ for some numbers γ_i^k. Taking the inner product with τ_j we get

$$\gamma_i^k g_{kj} = \gamma_i^k (\tau_k|\tau_j) = (\partial_i \nu_\Sigma|\tau_j) = -(\tau_{ij}|\nu_\Sigma) = -l_{ij},$$

hence

$$\gamma_i^r = \gamma_i^k g_{kj} g^{jr} = -l_{ij} g^{jr} = -g^{rj} l_{ji} = -l_i^r,$$

where we used symmetry of L and G. Therefore we have

$$\partial_i \nu_\Sigma = -l_i^r \tau_r = -L_\Sigma \tau_i, \quad i = 1, \ldots, n-1, \tag{2.11}$$

the *Weingarten relations*. Furthermore,

$$0 = \partial_i(\nu_\Sigma|\partial_j\nu_\Sigma) = (\partial_i\nu_\Sigma|\partial_j\nu_\Sigma) + (\nu_\Sigma|\partial_i\partial_j\nu_\Sigma)$$

implies

$$-(\partial_i\partial_j\nu_\Sigma|\nu_\Sigma) = (\partial_i\nu_\Sigma|\partial_j\nu_\Sigma) = l_i^r l_j^s(\tau_r|\tau_s) = l_i^r g_{rs} l_j^s = l_{is} g^{sr} l_{rj} = l_i^r l_{rj}, \quad (2.12)$$

which are the entries of the matrix $LG^{-1}L$, i.e., the covariant components of L_Σ^2. This is the so-called *third fundamental form* of Σ. In particular this implies the relation

$$\operatorname{tr} L_\Sigma^2 = (L_\Sigma\tau^i|L_\Sigma\tau_i) = -g^{ij}(\partial_i\partial_j\nu_\Sigma|\nu_\Sigma), \quad (2.13)$$

which will be useful later on. Moreover, we deduce from (2.12)

$$\operatorname{tr} L_\Sigma^2 = (L_\Sigma\tau^i|L_\Sigma\tau_i) = g^{ij} l_i^r l_{rj} = l_i^r l_r^i = \sum_{i=1}^{n-1} \kappa_i^2. \quad (2.14)$$

1.4 The Christoffel Symbols

The *Christoffel symbols* are defined according to

$$\Lambda_{ij|k} = (\tau_{ij}|\tau_k), \quad \Lambda_{ij}^k = g^{kr}\Lambda_{ij|r}. \quad (2.15)$$

Their importance stems from the representation of τ_{ij} in the basis $\{\tau_k, \nu_\Sigma\}$ of $\mathbb{R}^n$ via

$$\tau_{ij} = \Lambda_{ij}^k\tau_k + l_{ij}\nu_\Sigma. \quad (2.16)$$

Indeed, suppose $\tau_{ij} = a_{ij}^k\tau_k + b_{ij}\nu_\Sigma$. Then $l_{ij} = (\tau_{ij}|\nu_\Sigma) = b_{ij}$ and

$$\Lambda_{ij|k} = (\tau_{ij}|\tau_k) = (a_{ij}^r\tau_r|\tau_k) = g_{kr}a_{ij}^r.$$

Therefore, $a_{ij}^s = g^{sk}g_{kr}a_{ij}^r = g^{sk}\Lambda_{ij|k} = \Lambda_{ij}^s$. To express the Christoffel symbols in terms of the fundamental form G we use the identities

$$\begin{aligned}
\partial_k g_{ij} &= \partial_k(\tau_i|\tau_j) = (\tau_{ik}|\tau_j) + (\tau_i|\tau_{jk}),\\
\partial_i g_{kj} &= \partial_i(\tau_k|\tau_j) = (\tau_{ik}|\tau_j) + (\tau_k|\tau_{ij}),\\
\partial_j g_{ik} &= \partial_j(\tau_i|\tau_k) = (\tau_{ij}|\tau_k) + (\tau_i|\tau_{jk}),
\end{aligned}$$

which yield

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 2(\tau_{ij}|\tau_k),$$

i.e.,

$$\Lambda_{ij|k} = \frac{1}{2}[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}]. \quad (2.17)$$

Another important identity follows by differentiation of the relations $(\tau^j|\tau_k) = \delta_k^j$ and $(\tau^j|\nu_\Sigma) = 0$. We have

$$(\partial_i\tau^j|\tau_k) = -(\tau^j|\tau_{ik}) = -\Lambda_{ik}^r(\tau^j|\tau_r) = -\Lambda_{ik}^j,$$

and

$$(\partial_i \tau^j | \nu_\Sigma) = -(\tau^j | \partial_i \nu_\Sigma) = (\tau^j | L_\Sigma \tau_i) = l_i^j,$$

hence

$$\partial_i \tau^j = -\Lambda_{ik}^j \tau^k + l_i^j \nu_\Sigma. \tag{2.18}$$

This gives another interpretation of the Christoffel symbols and of the second fundamental form.

1.5 The Surface Gradient

Let ρ be a scalar field on Σ. The *surface gradient* $\nabla_\Sigma \rho$ at p is a vector which belongs to the tangent space of Σ at p. Thus it can be characterized by its

- *covariant components* a_i, i.e., $\nabla_\Sigma \rho = a_i \tau^i$, or by its
- *contravariant components* a^i, i.e., $\nabla_\Sigma \rho = a^i \tau_i$.

The chain rule

$$\partial_i (\rho \circ \phi) = (\nabla_\Sigma \rho | \tau_i)$$

yields $a_i = \partial_i (\rho \circ \phi) = \partial_i \rho$. This implies

$$a_i = (\nabla_\Sigma \rho | \tau_i) = a^k (\tau_k | \tau_i) = a^k g_{ki},$$

hence

$$\nabla_\Sigma \rho = \tau^i \partial_i \rho = (g^{ij} \partial_j \rho) \tau_i. \tag{2.19}$$

Suppose $\tilde{\rho}$ is a C^1-extension of ρ in a neighbourhood of Σ. We then have

$$\nabla \tilde{\rho} = (\nabla \tilde{\rho} | \nu_\Sigma) \nu_\Sigma + (\nabla \tilde{\rho} | \tau_i) \tau^i = (\nabla \tilde{\rho} | \nu_\Sigma) \nu_\Sigma + (\nabla_\Sigma \rho | \tau_i) \tau^i,$$

and hence, the surface gradient $\nabla_\Sigma \rho$ is the projection of $\nabla \tilde{\rho}$ onto $T_p \Sigma$, that is,

$$\nabla_\Sigma \rho = \mathcal{P}_\Sigma \nabla \tilde{\rho}. \tag{2.20}$$

For a vector field $f : \Sigma \to \mathbb{R}^m$ of class C^1 we define similarly

$$\nabla_\Sigma f := g^{ij} \tau_i \otimes \partial_j f = \tau^j \otimes \partial_j f. \tag{2.21}$$

In particular, this yields for the identity map id_Σ on Σ

$$\nabla_\Sigma \, \mathrm{id}_\Sigma = g^{ij} \tau_i \otimes \partial_j \phi = g^{ij} \tau_i \otimes \tau_j = \mathcal{P}_\Sigma,$$

and by the Weingarten relations

$$\nabla_\Sigma \nu_\Sigma = g^{ij} \tau_i \otimes \partial_j \nu_\Sigma = -g^{ij} l_j^r \tau_i \otimes \tau_r = -l^{ij} \tau_i \otimes \tau_j = -L_\Sigma.$$

For the surface gradient of tangent vectors we have

$$\begin{aligned}\nabla_\Sigma \tau_k &= g^{ij} \tau_i \otimes \partial_j \tau_k = g^{ij} \tau_i \otimes \tau_{jk} = g^{ij} \tau_i \otimes (\Lambda_{jk}^r \tau_r + l_{jk} \nu_\Sigma) \\ &= g^{ij} \Lambda_{jk}^r \tau_i \otimes \tau_r + l_k^i \tau_i \otimes \nu_\Sigma = \Lambda_{kj}^r \tau^j \otimes \tau_r + (L_\Sigma \tau_k) \otimes \nu_\Sigma.\end{aligned}$$

Finally we note that the *surface gradient for tensors* is defined according to

$$\nabla_\Sigma K = \tau^j \otimes \partial_j K. \tag{2.22}$$

1.6 The Surface Divergence

Let f be a tangential vector field on Σ. As before, $f^i = (f|\tau^i)$ denote the contravariant components of f, and $f_i = (f|\tau_i)$ the covariant components, respectively. The *surface divergence* of f is defined by

$$\mathrm{div}_\Sigma\, f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}f^i) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}f_j). \tag{2.23}$$

As before, $g := \det G$ denotes the determinant of $G = [g_{ij}]$. This definition ensures that partial integration can be carried out as usual, i.e., that the *surface divergence theorem* holds for tangential C^1-vector fields f:

$$\int_\Sigma (\nabla_\Sigma \rho|f)_\Sigma\, d\Sigma = -\int_\Sigma \rho\, \mathrm{div}_\Sigma f\, d\Sigma. \tag{2.24}$$

In fact, if e.g. ρ has support in a chart $\phi(\Theta)$ at p, then

$$\begin{aligned}\int_\Sigma (\nabla_\Sigma \rho|f)_\Sigma\, d\Sigma &= \int_\Theta \partial_i(\rho\circ\phi)[(f^i\circ\phi)\sqrt{g})]\, d\theta\\ &= -\int_\Theta (\rho\circ\phi)\frac{1}{\sqrt{g}}\partial_i[\sqrt{g}(f^i\circ\phi)]\sqrt{g}\, d\theta = -\int_\Sigma \rho\, \mathrm{div}_\Sigma f\, d\Sigma.\end{aligned}$$

Here we used that the surface measure in local coordinates is given by $d\Sigma = \sqrt{g}d\theta$. The general case follows from this argument by using a partition of unity. There is another useful representation of surface divergence, given by

$$\mathrm{div}_\Sigma f = g^{ij}(\tau_j|\partial_i f) = (\tau^i|\partial_i f). \tag{2.25}$$

It comes from

$$\mathrm{div}_\Sigma f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}f_j) = \frac{1}{\sqrt{g}}\partial_i[\sqrt{g}g^{ij}(\tau_j|f)],$$

since

$$(\partial_i(\sqrt{g}g^{ij}\tau_j)|\tau_k) = 0, \quad k = 1,\ldots,n-1. \tag{2.26}$$

Here (2.26) follows from

$$\begin{aligned}(\partial_i(\sqrt{g}g^{ij}\tau_j)|\tau_k) &= \partial_i(\sqrt{g}g^{ij}(\tau_j|\tau_k)) - \sqrt{g}g^{ij}(\tau_j|\tau_{ki}) = \partial_k\sqrt{g} - \sqrt{g}g^{ij}(\tau_j|\tau_{ki})\\ &= \partial_k\sqrt{g} - \frac{1}{2}\sqrt{g}g^{ij}\partial_k(\tau_j|\tau_i) = \frac{1}{2\sqrt{g}}\left(\partial_k g - gg^{ij}\partial_k g_{ij}\right)\end{aligned}$$

and the relation

$$\partial_k g = g\,\mathrm{tr}\,[G^{-1}\partial_k G] = g g^{ij}\partial_k g_{ij}. \tag{2.27}$$

The last assertion can be verified as follows:

$$\begin{aligned}\partial_k g = \partial_k \det G &= \sum_{j=1}^{n-1} \det\,[G_1,\cdots,\partial_k G_j,\cdots G_{n-1}] \\ &= (\det G)\sum_{j=1}^{n-1}\det\big(G^{-1}[G_1,\cdots,\partial_k G_j,\cdots,G_{n-1}]\big) = g\,\mathrm{tr}\,[G^{-1}\partial_k G],\end{aligned}$$

where $G = [g_{ij}] = [G_1,\cdots,G_{n-1}]$, with G_j the j-th column of G. From (2.25) follows

$$\mathrm{div}_\Sigma \tau_k = g^{ij}(\tau_j|\tau_{ki}) = g^{ij}\Lambda_{ki|j} = \Lambda^i_{ik}.$$

Equation (2.25) can be used as a definition of surface divergence for general, not necessarily tangential vector fields f, i.e., we have

$$\mathrm{div}_\Sigma f := g^{ij}(\tau_j|\partial_i f) = (\tau^i|\partial_i f), \quad f\in C^1(\Sigma,\mathbb{R}^n). \tag{2.28}$$

For example, consider $f=\nu_\Sigma$. Then $\partial_i\nu_\Sigma = -l_i^k\tau_k$ by the Weingarten relations and we obtain

$$\mathrm{div}_\Sigma\nu_\Sigma = g^{ij}(\tau_j|\partial_i\nu_\Sigma) = -g^{ij}l_{ij} = -\kappa_\Sigma.$$

This way we have derived the important relation

$$\kappa_\Sigma = -\mathrm{div}_\Sigma\nu_\Sigma. \tag{2.29}$$

With this in hand, we can now deduce the relation

$$\mathrm{div}_\Sigma f = \mathrm{div}_\Sigma\mathcal{P}_\Sigma f - (f|\nu_\Sigma)\kappa_\Sigma. \tag{2.30}$$

We remind that the surface divergence theorem (2.24) only holds for tangential vector fields. The *surface divergence theorem* for general vector fields reads as

$$\int_\Sigma(\nabla_\Sigma\rho|f)\,d\Sigma = -\int_\Sigma\rho(\,\mathrm{div}_\Sigma f + (f|\nu_\Sigma)\kappa_\Sigma)\,d\Sigma, \quad f\in C^1(\Sigma,\mathbb{R}^n). \tag{2.31}$$

This follows from (2.24) and (2.30) by means of

$$\int_\Sigma(\nabla_\Sigma\rho|f)_\Sigma\,d\Sigma = \int_\Sigma(\nabla_\Sigma\rho|\mathcal{P}_\Sigma f)_\Sigma\,d\Sigma = -\int_\Sigma\rho\,\mathrm{div}_\Sigma\mathcal{P}_\Sigma f\,d\Sigma.$$

Another representation of the surface divergence of a general vector field f is given by

$$\mathrm{div}_\Sigma f = (\tau^i|\partial_i f) = \mathrm{tr}[\tau^i\otimes\partial_i f] = \mathrm{tr}\,\nabla_\Sigma f. \tag{2.32}$$

Suppose that $f \in C^1(\Sigma, \mathbb{R}^n)$ admits a C^1-extension $\tilde{f}$ in a neighbourhood of Σ. Then

$$\mathrm{div}_\Sigma f = \mathrm{div}_x \tilde{f} - (\nu_\Sigma | [\nabla_x \tilde{f}]^\mathsf{T} \nu_\Sigma) = \mathrm{div}_x \tilde{f} - \Big(\nu_\Sigma \Big| \frac{\partial \tilde{f}}{\partial \nu_\Sigma}\Big),$$

as can be deduced from

$$\begin{aligned}\mathrm{div}_\Sigma f &= (\tau^i | \partial_i f) = (\tau^i | [\nabla_x \tilde{f}]^\mathsf{T} \tau_i) \\ &= (\tau^i | [\nabla_x \tilde{f}]^\mathsf{T} \tau_i) + (\nu_\Sigma | [\nabla_x \tilde{f}]^\mathsf{T} \nu_\Sigma) - (\nu_\Sigma | [\nabla_x \tilde{f}]^\mathsf{T} \nu_\Sigma).\end{aligned}$$

Suppose now that ν_Σ admits a C^1-extension $\tilde{\nu}_\Sigma$ in a neighbourhood of Σ such that $|\tilde{\nu}_\Sigma| = 1$ is this neighbourhood. Then we have

$$2(\nu_\Sigma(p) | [\nabla_x \tilde{\nu}_\Sigma(p)]^\mathsf{T} \nu_\Sigma(p)) = \frac{d}{dt} (\tilde{\nu}_\Sigma(p + t\nu_\Sigma(p) | \tilde{\nu}(p + t\nu_\Sigma(p)) \Big|_{t=0} = 0,$$

and we obtain

$$\mathrm{div}_x \tilde{\nu}_\Sigma = \mathrm{div}_\Sigma \nu_\Sigma = -\kappa_\Sigma. \tag{2.33}$$

Consequently, if Σ is given as the zero set of a C^2-level function φ with $\nabla_x \varphi \neq 0$, with $\nabla_x \varphi$ pointing in the direction of ν_Σ, we have the well-known formula

$$\kappa_\Sigma = -\mathrm{div}_x \left(\frac{\nabla_x \varphi}{|\nabla_x \varphi|} \right).$$

Finally, the *surface divergence for tensors* is given by

$$\mathrm{div}_\Sigma K = (\tau^j | \partial_j K) := (\partial_j K)^\mathsf{T} \tau^j. \tag{2.34}$$

This immediately yields the important relation

$$\mathrm{div}_\Sigma \mathcal{P}_\Sigma = \kappa_\Sigma \nu_\Sigma. \tag{2.35}$$

1.7 The Laplace-Beltrami Operator

The *Laplace-Beltrami* operator on Σ is defined for scalar fields by means of

$$\Delta_\Sigma \rho = \mathrm{div}_\Sigma \nabla_\Sigma \rho,$$

which in local coordinates reads

$$\Delta_\Sigma \rho = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g} g^{ij} \partial_j \rho].$$

Another representation of Δ_Σ is given by

$$\Delta_\Sigma \rho = g^{ij} \partial_i \partial_j \rho - g^{ij} \Lambda_{ij}^k \partial_k \rho. \tag{2.36}$$

This follows from (2.19), (2.25) and (2.18). Since at each point $p \in \Sigma$ we may choose a chart such that $g_{ij} = \delta_{ij}$ and $\Lambda_{ij}^k = 0$ at p, we see from this representation that the Laplace-Beltrami operator is equivalent to the Laplacian at the point p; see also Section 2.1.8 below.

To obtain another representation of Δ_Σ, for a scalar C^2-function we compute

$$\nabla_\Sigma^2 \rho = \nabla_\Sigma(\tau^j \partial_j \rho) = \tau^i \otimes \partial_i(\tau^j \partial_j \rho).$$

This yields with (2.18)

$$\begin{aligned}\nabla_\Sigma^2 \rho =& (\partial_i \partial_j \rho)\tau^i \otimes \tau^j + (\partial_j \rho)\tau^i \otimes \partial_i \tau^j \\ =& (\partial_i \partial_k \rho - \Lambda_{ik}^j \partial_j \rho)\tau^i \otimes \tau^k + (L_\Sigma \nabla_\Sigma \rho) \otimes \nu_\Sigma.\end{aligned}$$

Taking traces gives

$$\Delta_\Sigma \rho = \operatorname{tr} \nabla_\Sigma^2 \rho.$$

Similarly, the Laplace-Beltrami operator applies to general vector fields f according to

$$\Delta_\Sigma f = g^{ij}(\partial_i \partial_j f - \Lambda_{ij}^r \partial_r f).$$

For example, this yields, for the identity map id_Σ on Σ,

$$\Delta_\Sigma \,\mathrm{id}_\Sigma = g^{ij}(\partial_i \partial_j \phi - \Lambda_{ij}^r \partial_r \phi) = g^{ij}(\tau_{ij} - \Lambda_{ij}^r \tau_r),$$

and hence by (2.16)

$$\Delta_\Sigma \,\mathrm{id}_\Sigma = g^{ij} l_{ij} \nu_\Sigma = \kappa_\Sigma \nu_\Sigma.$$

Finally, we prove the important formula

$$\Delta_\Sigma \nu_\Sigma = -\nabla_\Sigma \kappa_\Sigma - [\operatorname{tr} L_\Sigma^2] \nu_\Sigma. \tag{2.37}$$

In fact, we have from (2.12)

$$(\Delta_\Sigma \nu_\Sigma | \nu_\Sigma) = g^{ij}(\partial_{ij} \nu_\Sigma - \Lambda_{ij}^r \partial_r \nu_\Sigma | \nu_\Sigma) = g^{ij}(\partial_{ij} \nu_\Sigma | \nu_\Sigma) = -\operatorname{tr} L_\Sigma^2.$$

Next observe that

$$\begin{aligned}&(\partial_k \partial_j \nu_\Sigma | \tau_i) - (\partial_i \partial_j \nu_\Sigma | \tau_k) = \partial_k(\partial_j \nu_\Sigma | \tau_i) - \partial_i(\partial_j \nu_\Sigma | \tau_k) \\ &= -\partial_k(\nu_\Sigma | \tau_{ij}) + \partial_i(\nu_\Sigma | \tau_{kj}) = \partial_k(\partial_i \nu_\Sigma | \tau_j) - \partial_i(\partial_k \nu_\Sigma | \tau_j) \\ &= (\partial_i \nu_\Sigma | \tau_{kj}) - (\partial_k \nu_\Sigma | \tau_{ij}) = \Lambda_{kj}^r(\partial_i \nu_\Sigma | \tau_r) - \Lambda_{ij}^r(\partial_k \nu_\Sigma | \tau_r) \\ &= \Lambda_{kj}^r(\partial_r \nu_\Sigma | \tau_i) - \Lambda_{ij}^r(\partial_r \nu_\Sigma | \tau_k),\end{aligned}$$

hence

$$(\partial_k \partial_j \nu_\Sigma - \Lambda_{kj}^r \partial_r \nu_\Sigma | \tau_i) = (\partial_i \partial_j \nu_\Sigma - \Lambda_{ij}^r \partial_r \nu_\Sigma | \tau_k).$$

This implies

$$(\Delta_\Sigma \nu_\Sigma | \tau_i) = g^{jk}(\partial_k \partial_j \nu_\Sigma - \Lambda_{kj}^r \partial_r \nu_\Sigma | \tau_i) = (\partial_i \partial_j \nu_\Sigma - \Lambda_{ij}^r \partial_r \nu_\Sigma | \tau^j).$$

On the other hand,

$$\begin{aligned}-(\nabla_\Sigma\kappa_\Sigma|\tau_i) &= -\partial_i\kappa_\Sigma = \partial_i(\partial_j\nu_\Sigma|\tau^j)\\ &= (\partial_i\partial_j\nu_\Sigma|\tau^j) + (\partial_r\nu_\Sigma|\partial_i\tau^r)\\ &= (\partial_i\partial_j\nu_\Sigma - \Lambda^r_{ij}\partial_r\nu_\Sigma|\tau^j).\end{aligned}$$

This proves formula (2.37).

1.8 The Case of a Graph over $\mathbb{R}^{n-1}$

Suppose that Σ is a graph over $\mathbb{R}^{n-1}$, i.e., there is a function $h \in C^2(\mathbb{R}^{n-1})$ such that the hypersurface Σ is given by the chart $\phi(x) = [x^\mathsf{T}, h(x)]^\mathsf{T}$, $x \in \mathbb{R}^{n-1}$. Then the tangent vectors are given by $\tau_i = [e_i^\mathsf{T}, \partial_i h]^\mathsf{T}$, where $\{e_i\}$ denotes the standard basis in $\mathbb{R}^{n-1}$. The (upward pointing) normal ν_Σ is given by

$$\nu_\Sigma(x) = \beta(x)[-\nabla_x h(x)^\mathsf{T}, 1]^\mathsf{T}, \quad \beta(x) = 1/\sqrt{1+|\nabla_x h(x)|^2}.$$

The first fundamental form becomes $g_{ij} = \delta_{ij} + \partial_i h \partial_j h$, hence

$$g^{ij} = \delta^{ij} - \beta^2\partial_i h\partial_j h.$$

This yields

$$\tau^i = [[e_i - \beta^2\partial_i h\nabla_x h]^\mathsf{T}, \beta^2\partial_i h]^\mathsf{T},$$

and with $\tau_{ij} = [0, \partial_i\partial_j h]^\mathsf{T}$,

$$l_{ij} = (\tau_{ij}|\nu_\Sigma) = \beta\partial_i\partial_j h,$$

and therefore

$$\kappa_\Sigma = g^{ij}l_{ij} = \beta[\Delta_x h - \beta^2(\nabla_x^2 h\nabla_x h|\nabla_x h)] = \mathrm{div}_x\left(\frac{\nabla_x h}{\sqrt{1+|\nabla_x h|^2}}\right).$$

The Christoffel symbols in this case are given by

$$\Lambda_{ij|k} = \partial_i\partial_j h\partial_k h, \quad \Lambda^k_{ij} = \beta^2\partial_i\partial_j h\partial_k h.$$

Suppose that $\mathbb{R}^{n-1}\times\{0\}$ is the tangent plane at $\phi(0) = 0 \in \Sigma$. Then $h(0) = 0$ and $\nabla_x h(0) = 0$, hence at this point we have $g_{ij} = \delta_{ij}$, $\tau_i = [e_i^\mathsf{T}, 0]^\mathsf{T}$, $\nu_\Sigma = [0,1]^\mathsf{T}$, $\beta = 1$, and $l_{ij} = \partial_i\partial_j h$. Thus the principal curvatures $\kappa_i(0)$ are the eigenvalues of $\nabla_x^2 h(0)$, the mean curvature is $\kappa_\Sigma(0) = \Delta_x h(0)$, and $\Lambda^k_{ij}(0) = 0$.

To obtain a representation of the surface gradient, let $\rho : \Sigma \to \mathbb{R}$. Then

$$\nabla_\Sigma\rho = \tau^j\partial_j\rho = [[\nabla_x\rho - \beta^2(\nabla_x\rho|\nabla_x h)\nabla_x h]^\mathsf{T}, \beta^2(\nabla_x\rho|\nabla_x h)]^\mathsf{T}.$$

Similarly, for $f = (\bar f, f^n) : \Sigma \to \mathbb{R}^{n-1}\times\mathbb{R}$ we obtain

$$\mathrm{div}_\Sigma f = (\tau^i|\partial_i f) = \mathrm{div}_x\bar f + \beta^2(\nabla_x h|\nabla_x f^n - (\nabla_x h\cdot\nabla_x)\bar f),$$

and for the Laplace-Beltrami

$$\Delta_\Sigma\rho = \Delta_x\rho - \beta^2(\nabla_x^2\rho\nabla_x h|\nabla_x h) - \beta^2[\Delta_x h - \beta^2(\nabla_x^2 h\nabla_x h|\nabla_x h)](\nabla_x h|\nabla_x\rho).$$

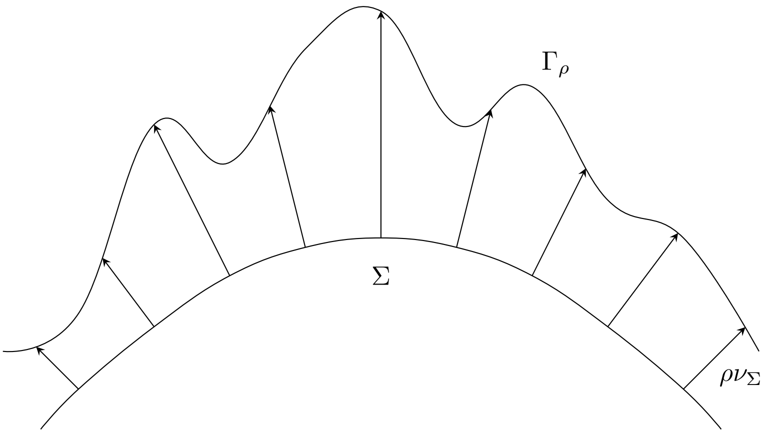

Figure 2.2: Parameterization of Γ over Σ.

2.2 Parameterized Hypersurfaces

We consider now a hypersurface $\Gamma = \Gamma_\rho$ which is parameterized over a fixed hypersurface Σ according to

$$q = \psi_\rho(p) = p + \rho(p)\nu_\Sigma(p), \quad p \in \Sigma, \tag{2.38}$$

where as before $\nu_\Sigma = \nu_\Sigma(p)$ denotes the outer unit normal of Σ at $p \in \Sigma$.

We want to derive the basic geometric quantities of Γ in terms of ρ and those of Σ. In the sequel we assume that ρ is of class C^1 and small enough. A precise bound on ρ will be given below.

2.1 The Fundamental Form
Differentiating (2.38) we obtain with the Weingarten relations (2.11)

$$\tau_i^\Gamma = \partial_i\psi_\rho = \tau_i + \rho\partial_i\nu_\Sigma + (\partial_i\rho)\nu_\Sigma = (I - \rho L_\Sigma)\tau_i + (\partial_i\rho)\nu_\Sigma. \tag{2.39}$$

We may then compute the fundamental form $G^\Gamma = [g_{ij}^\Gamma]$ of Γ to the result

$$\begin{aligned} g_{ij}^\Gamma = (\tau_i^\Gamma|\tau_j^\Gamma) &= ((I - \rho L_\Sigma)\tau_i + \partial_i\rho\nu_\Sigma|(I - \rho L_\Sigma)\tau_j + \partial_j\rho\nu_\Sigma) \\ &= ((I - \rho L_\Sigma)\tau_i|(I - \rho L_\Sigma)\tau_j) + \partial_i\rho\partial_j\rho, \\ &= (\tau_i|(I - \rho L_\Sigma)^2\tau_j) + (\tau_i|[\nabla_\Sigma\rho \otimes \nabla_\Sigma\rho]\tau_j) \end{aligned}$$

where we used that $((I - \rho L_\Sigma)\tau_k|\nu_\Sigma) = 0$. Hence

$$[g_{ij}^\Gamma] = [(I - \rho L_\Sigma)^2 + \nabla_\Sigma\rho \otimes \nabla_\Sigma\rho]_{ij}.$$

This yields the representation

$$\begin{aligned}[g_{ij}^{\Gamma}] &= [(I-\rho L_\Sigma)^2[I+(I-\rho L_\Sigma)^{-2}\nabla_\Sigma\rho\otimes\nabla_\Sigma\rho]]_{ij} \\ &= [g_{ik}][(I-\rho L_\Sigma)^2[I+(I-\rho L_\Sigma)^{-2}\nabla_\Sigma\rho\otimes\nabla_\Sigma\rho]]_j^k.\end{aligned} \tag{2.40}$$

We then have

$$g^\Gamma := \det G^\Gamma := \det[g_{ij}^\Gamma] = g\det[[(I-\rho L_\Sigma)^2[I+(I-\rho L_\Sigma)^{-2}\nabla_\Sigma\rho\otimes\nabla_\Sigma\rho]].$$

Since for any two vectors $a, b\in\mathbb{R}^n$,

$$\det(I+a\otimes b) = 1+(a|b),$$

we obtain

$$g^\Gamma = g\alpha^2(\rho)\mu^2(\rho), \tag{2.41}$$

where

$$\alpha(\rho) = \det(I-\rho L_\Sigma) = \det(I-\rho K) = \Pi_{i=1}^{n-1}(1-\rho\kappa_i),$$

and

$$\begin{aligned}\mu(\rho) &= (1+((I-\rho L_\Sigma)^{-2}\nabla_\Sigma\rho|\nabla_\Sigma\rho))^{1/2} \\ &= (1+((I-\rho L_\Sigma)^{-1}\nabla_\Sigma\rho|(I-\rho L_\Sigma)^{-1}\nabla_\Sigma\rho))^{1/2}.\end{aligned}$$

This yields for the surface measure $d\Gamma$ on Γ_ρ,

$$d\Gamma = \sqrt{g^\Gamma}d\theta = \alpha(\rho)\mu(\rho)\sqrt{g}\,d\theta = \alpha(\rho)\mu(\rho)\,d\Sigma, \tag{2.42}$$

hence

$$|\Gamma_\rho| = \int_{\Gamma_\rho} d\Gamma = \int_\Sigma \alpha(\rho)\mu(\rho)\,d\Sigma,$$

where $|\Gamma_\rho|$ denotes the surface area of Γ_ρ. Since

$$(I+a\otimes b)^{-1} = I - \frac{a\otimes b}{1+(a|b)},$$

we obtain for $[G^\Gamma]^{-1}$ the identity

$$[G^\Gamma]^{-1} = [g^{ij}_\Gamma] = [[I-\mu^{-2}(\rho)(I-\rho L_\Sigma)^{-2}\nabla_\Sigma\rho\otimes\nabla_\Sigma\rho](I-\rho L_\Sigma)^{-2}]_k^i[g_\Sigma^{kj}].$$

All of this makes sense only for functions ρ such that $I-\rho K$ is invertible, i.e., $\alpha(\rho)$ should not vanish. Thus the precise bound for ρ is determined by the principle curvatures of Σ, and we assume here and in the sequel that

$$|\rho|_\infty < \frac{1}{\max\{|\kappa_i(p)|:\, i=1,\dots,n-1,\, p\in\Sigma\}} =: \rho_0. \tag{2.43}$$

2.2 The Normal at Γ
We next compute the outer unit normal at Γ. For this purpose we set

$$\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho)),$$

where $\beta(\rho)$ is a scalar and $a(\rho) \in T_p\Sigma$. Then $\beta(\rho) = (1 + |a(\rho)|^2)^{-1/2}$ and

$$0 = (\nu_\Gamma|\tau_i^\Gamma) = \beta(\rho)(\nu_\Sigma - a(\rho)|(I - \rho L_\Sigma)\tau_i + \nu_\Sigma \partial_i \rho),$$

which yields

$$0 = \partial_i \rho - (a(\rho)|(I - \rho L_\Sigma)\tau_i) = \partial_i \rho - ((I - \rho L_\Sigma)a(\rho)|\tau_i),$$

by symmetry of L_Σ. But this implies $(I - \rho L_\Sigma)a(\rho) = \nabla_\Sigma \rho$, i.e., we have

$$\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho)) \tag{2.44}$$

with

$$a(\rho) = M_0(\rho)\nabla_\Sigma \rho, \quad M_0(\rho) = (I - \rho L_\Sigma)^{-1}, \quad \beta(\rho) = (1 + |a(\rho)|^2)^{-1/2}. \tag{2.45}$$

Note that $\mu(\rho) = \beta^{-1}(\rho)$, where $\mu(\rho)$ was introduced in the last section. By means of $a(\rho)$, $\beta(\rho)$ and $M_0(\rho)$ this leads to another representation of G^Γ and G_Γ^{-1}, namely

$$[g_{ij}^\Gamma] = [(I - \rho L_\Sigma)[I + a(\rho) \otimes a(\rho)](I - \rho L_\Sigma)]_{ij},$$

and

$$[g_\Gamma^{ij}] = [M_0(\rho)[I - \beta^2(\rho)a(\rho) \otimes a(\rho)]M_0(\rho)]^{ij}.$$

2.3 The Surface Gradient and the Surface Divergence on Γ
It is of importance to have a representation for the surface gradient on Γ in terms of Σ. For this purpose recall that

$$\mathcal{P}_\Gamma = I - \nu_\Gamma \otimes \nu_\Gamma = g_\Gamma^{ij} \tau_i^\Gamma \otimes \tau_j^\Gamma,$$

where $\nu_\Gamma = \beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho)$, and

$$\tau_i^\Gamma = (I - \rho L_\Sigma)\tau_i^\Sigma + \partial_i \rho \nu_\Sigma.$$

By virtue of $L_\Sigma \nu_\Sigma = 0$, the latter implies

$$\tau_i^\Gamma = (I - \rho L_\Sigma)(\tau_i^\Sigma + \partial_i \rho \nu_\Sigma).$$

As remarked before we do not distinguish between $L_\Sigma \in \mathcal{B}(\mathbb{R}^n)$ and its restriction to $T_p\Sigma$. With this identification, and by the fact that $(I - \rho L_\Sigma) = I$ on $\mathbb{R}\nu_\Sigma$, we have

$$(I - \rho L_\Sigma)(p) \in \text{Isom}\,(\mathbb{R}^n, \mathbb{R}^n) \cap \text{Isom}\,(T_p\Sigma, T_p\Sigma),$$

provided ρ satisfies (2.43). As before, ρL_Σ is the short form for $\rho(p)L_\Sigma(p)$. Hence,

$$M_0(\rho)(p) \in \mathrm{Isom}(\mathbb{R}^n, \mathbb{R}^n) \cap \mathrm{Isom}(T_p\Sigma, T_p\Sigma).$$

We conclude that

$$M_0(\rho)\tau_i^\Gamma = \tau_i^\Sigma + (\partial_i\rho)\nu_\Sigma,$$

and therefore

$$\mathcal{P}_\Sigma M_0(\rho)\tau_i^\Gamma = \tau_i^\Sigma. \tag{2.46}$$

On the other hand, we have

$$\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^r = g_\Gamma^{ij}\tau_i^\Gamma \otimes \tau_j^\Gamma M_0(\rho)\tau_\Sigma^r = \tau_\Gamma^j(\tau_j^\Gamma | M_0(\rho)\tau_\Sigma^r),$$

hence

$$\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^r = \tau_\Gamma^r. \tag{2.47}$$

(2.46) and (2.47) allow for an easy change between the bases of $T_p\Sigma$ and $T_q\Gamma$, where $q = \psi_\rho(p) = p + \rho(p)\nu_\Sigma(p)$. (2.47) implies for a scalar function φ on Γ,

$$\nabla_\Gamma\varphi = \tau_\Gamma^r\partial_r\varphi = \mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^r\partial_r\varphi_* = \mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma\varphi_*, \quad \varphi_* = \varphi\circ\psi_\rho$$

which leads to the identity

$$\nabla_\Gamma\varphi = \mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma\varphi_*.$$

Similarly, if f denotes a vector field on Γ, then

$$\nabla_\Gamma f = \mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma f_*,$$

and so

$$\mathrm{div}_\Gamma f = (\tau_\Gamma^r|\partial_r f) = (\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^r|\partial_r f) = \mathrm{tr}\,[\mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma f_*].$$

As a consequence, we obtain for the Laplace-Beltrami operator on Γ,

$$\Delta_\Gamma\varphi = \mathrm{tr}\,[\mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma(\mathcal{P}_\Gamma M_0(\rho)\nabla_\Sigma\varphi_*)],$$

which can be written as

$$\Delta_\Gamma\varphi = M_0(\rho)\mathcal{P}_\Gamma M_0(\rho) : \nabla_\Sigma^2\varphi_* + (b(\rho, \nabla_\Sigma\rho, \nabla_\Sigma^2\rho)|\nabla_\Sigma\varphi_*),$$

with $b = \partial_i(M_0(\rho)\mathcal{P}_\Gamma)\mathcal{P}_\Gamma M(\rho)\tau_\Sigma^i$. One should note that the structure of the Laplace-Beltrami operator on Γ in local coordinates is

$$\Delta_\Gamma\varphi = a^{ij}(\rho, \partial\rho)\partial_i\partial_j\varphi_* + b^k(\rho, \partial\rho, \partial^2\rho)\partial_k\varphi_*$$

with

$$a^{ij}(\rho, \partial\rho) = (\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^i|\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^j) = (\tau_\Gamma^i|\tau_\Gamma^j) = g_\Gamma^{ij}$$

and

$$b^k(\rho,\partial\rho,\partial^2\rho) = (\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^i|\partial_i(M_0(\rho)\mathcal{P}_\Gamma\tau^k)) = (\tau_\Gamma^i|\partial_i\tau_\Gamma^k) = -g_\Gamma^{ij}\Lambda_{\Gamma ij}^k.$$

This shows that $-\Delta_\Gamma$ is strongly elliptic on the reference manifold Σ as long as $|\rho|_\infty < \rho_0$.

2.4 Normal Variations
For $\rho, h \in C(\Sigma)$ sufficiently smooth and $F(\rho) : \Sigma \to \mathbb{R}^k$ we define

$$F'(\rho)h := \frac{d}{d\varepsilon}F(\rho+\varepsilon h)\Big|_{\varepsilon=0}.$$

First we have

$$M_0'(\rho) = M_0(\rho)L_\Sigma M_0(\rho), \quad M_0'(0) = L_\Sigma,$$

as $M_0(0) = I$. Next

$$\beta'(\rho)h = -\beta(\rho)^3\big(M_0(\rho)\nabla_\Sigma\rho\big|M_0'(\rho)h\nabla_\Sigma\rho + M_0(\rho)\nabla_\Sigma h\big),$$

which yields $\beta'(0) = 0$, as $\beta(0) = 1$. From this we get for the normal

$$\nu(\rho) = \nu_\Gamma = \beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma\rho)$$

the relation

$$\nu'(\rho)h = \beta'(\rho)h(\nu_\Sigma - M_0(\rho)\nabla_\Sigma\rho) - \beta(\rho)(M_0'(\rho)h\nabla_\Sigma\rho + M_0(\rho)\nabla_\Sigma h),$$

which yields

$$\nu'(0)h = -\nabla_\Sigma h.$$

This in turn implies for the projection $P(\rho) := \mathcal{P}_\Gamma$

$$P'(\rho)h = -\nu'(\rho)h\otimes\nu(\rho) - \nu(\rho)\otimes\nu'(\rho)h,$$

hence

$$P'(0)h = \nabla_\Sigma h\otimes\nu_\Sigma + \nu_\Sigma\otimes\nabla_\Sigma h =: [\nabla_\Sigma\otimes\nu_\Sigma + \nu_\Sigma\otimes\nabla_\Sigma]h.$$

Applying these relations to $\nabla(\rho) := \nabla_\Gamma = P(\rho)M_0(\rho)\nabla_\Sigma$ yields

$$\begin{aligned}(\nabla'(0)h)\varphi &= [P'(0)h + P(0)M'(0)h]\nabla_\Sigma\varphi \\ &= [\nabla_\Sigma h\otimes\nu_\Sigma + \nu_\Sigma\otimes\nabla_\Sigma h + hL_\Sigma]\nabla_\Sigma\varphi = [\nu_\Sigma\otimes\nabla_\Sigma h + hL_\Sigma]\nabla_\Sigma\varphi,\end{aligned}$$

and for a not necessarily tangent vector field f

$$(\nabla'(0)h)f = \nu_\Sigma\otimes(\nabla_\Sigma h|\nabla_\Sigma)f + hL_\Sigma\nabla_\Sigma f$$

where $(\nabla_\Sigma h|\nabla_\Sigma)f := (\nabla_\Sigma h|\tau^j)\partial_j f$. For the divergence of the vector field f this implies

$$[\mathrm{div}'(0)h]f = (\nu_\Sigma|(\nabla_\Sigma h|\nabla_\Sigma)f) + h\,\mathrm{tr}[L_\Sigma\nabla_\Sigma f].$$

Finally, the variation of the Laplace-Beltrami operator $\Delta(\rho) := \Delta_\Gamma$ becomes

$$(\Delta'(0)h)\varphi = h\,\mathrm{tr}[L_\Sigma \nabla_\Sigma^2 \varphi + \nabla_\Sigma(L_\Sigma \nabla_\Sigma \varphi)] + 2(L_\Sigma \nabla_\Sigma h|\nabla_\Sigma \varphi) - \kappa(\nabla_\Sigma h|\nabla_\Sigma \varphi).$$

Note that in local coordinates we have

$$\mathrm{tr}[L_\Sigma \nabla_\Sigma^2 \varphi] = l_\Sigma^{ij}(\partial_i \partial_j \varphi - \Lambda_{ij}^k \partial_k \varphi),$$

hence with

$$\mathrm{tr}[\nabla_\Sigma(L_\Sigma \nabla_\Sigma \varphi)] = \mathrm{tr}[L_\Sigma \nabla_\Sigma^2 \varphi] + (\mathrm{div}_\Sigma L_\Sigma|\nabla_\Sigma \varphi),$$

we may write alternatively

$$(\Delta'(0)h)\varphi = 2h\,\mathrm{tr}[L_\Sigma \nabla_\Sigma^2 \varphi] + (h\,\mathrm{div}_\Sigma L_\Sigma + [2L_\Sigma - \kappa_\Sigma]\nabla_\Sigma h|\nabla_\Sigma \varphi).$$

2.5 The Weingarten Tensor and the Mean Curvature of Γ

In invariant formulation we have

$$L(\rho) := L_\Gamma = -\nabla_\Gamma \nu_\Gamma = -P(\rho)M_0(\rho)\nabla_\Sigma\{\beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho)\}.$$

Thus for the variation of L_Γ at $\rho = 0$ we obtain with $P(0) = \mathcal{P}_\Sigma$, $\beta(0) = 1$, $M_0(0) = I$, and $P'(0) = \nabla_\Sigma \otimes \nu_\Sigma + \nu_\Sigma \otimes \nabla_\Sigma$, $\beta'(0) = 0$, $M_0'(0) = L_\Sigma$,

$$L'(0) = \nu_\Sigma \otimes L_\Sigma \nabla_\Sigma + L_\Sigma^2 + \nabla_\Sigma^2.$$

In particular, for $\kappa(\rho) := \kappa_\Gamma$ we have

$$\kappa(\rho) = -\mathrm{tr}[\nabla_\Gamma \nu_\Gamma] = \mathrm{tr}\,L(\rho),$$

hence

$$\kappa'(0) = \mathrm{tr}\,L_\Sigma^2 + \Delta_\Sigma. \tag{2.48}$$

Let us take another look at the mean curvature $\kappa(\rho) := \kappa_\Gamma$. By the relations $\tau_\Gamma^r = \mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^r$ and $\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho))$ we obtain

$$\begin{aligned}\kappa(\rho) &= -(\tau_\Gamma^j|\partial_j \nu_\Gamma) = -(\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^j|(\partial_j \beta(\rho)/\beta(\rho))\nu_\Gamma + \beta(\rho)(\partial_j \nu_\Sigma - \partial_j a(\rho)))\\ &= \beta(\rho)(\mathcal{P}_\Gamma M_0(\rho)\tau_\Sigma^j|L_\Sigma \tau_j^\Sigma + \partial_j a(\rho))\\ &= \beta(\rho)(M_0(\rho)\tau_\Sigma^j|L_\Sigma \tau_j^\Sigma + \partial_j a(\rho)) - \beta(\rho)(\nu_\Gamma|M_0(\rho)\tau_\Sigma^j)(\nu_\Gamma|L_\Sigma \tau_j^\Sigma + \partial_j a(\rho)).\end{aligned}$$

Since $(M_0(\rho)\tau_\Sigma^j|L_\Sigma \tau_j^\Sigma) = \mathrm{tr}[M_0(\rho)L_\Sigma]$ as well as

$$(M_0(\rho)\tau_\Sigma^j|\partial_j a(\rho)) = \mathrm{tr}[M_0(\rho)\nabla_\Sigma a(\rho)],$$

and $(\nu_\Gamma|M_0(\rho)\tau_\Sigma^j) = -\beta(\rho)[M_0(\rho)a(\rho)]^j$, we obtain

$$\begin{aligned}\kappa(\rho) = \beta(\rho)\big\{&\mathrm{tr}\big[M_0(\rho)(L_\Sigma + \nabla_\Sigma a(\rho))\big]\\ &+ \beta^2(\rho)\big[M_0(\rho)a(\rho)\big]^j\big[(\nu_\Sigma|\partial_j a(\rho)) - (a(\rho)|\partial_j a(\rho)) - (a(\rho)|L_\Sigma \tau_j^\Sigma)\big]\big\}\\ = \beta(\rho)\big\{&\mathrm{tr}\big[M_0(\rho)(L_\Sigma + \nabla_\Sigma a(\rho))\big] - \beta^2(\rho)(M_0(\rho)a(\rho)|\nabla_\Sigma a(\rho)a(\rho))\big\},\end{aligned}$$

as $(\nu_\Sigma|a(\rho)) = 0$ implies

$$(\nu_\Sigma|\partial_j a(\rho)) = -(\partial_j \nu_\Sigma|a(\rho)) = (L_\Sigma \tau_j^\Sigma|a(\rho)).$$

This yields the final form for the mean curvature of Γ.

$$\kappa(\rho) = \beta(\rho)\big\{\mathrm{tr}\big[M_0(\rho)(L_\Sigma + \nabla_\Sigma a(\rho))\big] - \beta^2(\rho)(M_0(\rho)a(\rho)|[\nabla_\Sigma a(\rho)]a(\rho))\big\}. \quad (2.49)$$

Recall that $a(\rho) = M_0(\rho)\nabla_\Sigma \rho$.

We can write the curvature of Γ in local coordinates in the following form.

$$\kappa(\rho) = c^{ij}(\rho, \partial\rho)\partial_i\partial_j\rho + g(\rho, \partial\rho),$$

with

$$c^{ij}(\rho, \partial\rho) = \beta(\rho)[M_0^2(\rho)]^{ij} - \beta^3(\rho)[M_0^2(\rho)\nabla_\Sigma\rho]^i[M_0^2(\rho)\nabla_\Sigma\rho]^j.$$

A simple computation yields for the symbol $c(\rho, \xi) = c^{ij}(\rho, \partial\rho)\xi_i\xi_j$ of the principal part of $-\kappa(\rho)$

$$c(\rho, \xi) = \beta(\rho)\{|M_0(\rho)\xi|^2 - \beta^2(\rho)(a(\rho)|M_0(\rho)\xi)^2\} \geq \beta^3(\rho)|M_0(\rho)\xi|^2 \geq \eta|\xi|^2,$$

for $\xi = \xi_k\tau_\Sigma^k \in T_p\Sigma$, as long as $|\rho|_\infty < \rho_0$. Therefore, $-\kappa(\rho)$ is a quasilinear strongly elliptic differential operator on Σ, acting on the parameterization ρ of Γ over Σ.

2.6 The Area Functional

As shown before, the area functional for the surface $\Gamma_\rho = \{p + \rho(p)\nu_\Sigma(p) : p \in \Sigma\}$ is given by

$$\Phi(\rho) = \int_{\Gamma_\rho} d\Gamma = \int_\Sigma \alpha(\rho)\mu(\rho)\, d\Sigma.$$

Here we use the notation

$$\alpha(\rho) = \det(I - \rho K) = \Pi_{i=1}^{n-1}(1 - \rho\kappa_i), \quad \mu(\rho) = (1 + |a(\rho)|^2)^{1/2},$$

with $a(\rho)$ defined in (2.45).

We compute its first variation to the result

$$\langle\Phi'(\rho)|h\rangle = \int_\Sigma [(\mu(\rho)\alpha'(\rho) + \alpha(\rho)\mu'(\rho)]h\, d\Sigma.$$

For the derivatives of α and μ we get

$$\alpha'(\rho) = \alpha(\rho)\sum_{i=1}^{n-1}\frac{-\kappa_i}{1 - \rho\kappa_i}, \quad \mu'(\rho)h = \mu(\rho)^{-1}(a(\rho)|a'(\rho)h).$$

In particular, at $\rho = 0$ we get with $\alpha(0) = \mu(0) = 1$ and $a(0) = 0$

$$\alpha'(0) = -\kappa_\Sigma, \quad \mu'(0) = 0.$$

This implies for the *first variation* of Φ at $\rho = 0$,

$$\langle \Phi'(0)|h\rangle = -\int_\Sigma \kappa_\Sigma h \, d\Sigma. \tag{2.50}$$

This shows, in particular, that the critical points of the area functional Φ are hypersurfaces with mean curvature $\kappa_\Sigma = 0$. Such surfaces are called *minimal surfaces.*

Similarly, the second variation becomes

$$\begin{aligned}\langle \Phi''(\rho)h|k\rangle = &\int_\Sigma [\mu(\rho)\alpha''(\rho) + \alpha(\rho)\mu''(\rho)]hk \, d\Sigma \\ &+ \int_\Sigma [\alpha'(\rho)h\mu'(\rho)k + \alpha'(\rho)k\mu'(\rho)h] \, d\Sigma.\end{aligned}$$

Since $\alpha(0) = \mu(0) = 1$ and $\mu'(0) = 0$ we get

$$\langle \Phi''(0)h|k\rangle = \int_\Sigma [\alpha''(0) + \mu''(0)]hk \, d\Sigma.$$

We have

$$\alpha''(\rho) = \alpha(\rho)\Big[\Big(\sum_{i=1}^{n-1} \frac{-\kappa_i}{1-\rho\kappa_i}\Big)^2 - \sum_{i=1}^{n-1} \frac{\kappa_i^2}{(1-\rho\kappa_i)^2}\Big],$$

hence

$$\alpha''(0) = \Big(\sum_{i=1}^{n-1} \kappa_i\Big)^2 - \sum_{i=1}^{n-1} \kappa_i^2 = (\operatorname{tr} K)^2 - \operatorname{tr} K^2,$$

which is the second invariant of the shape operator K.

In particular, in case Σ is a sphere of radius R we have $\kappa_i = -1/R$, hence $\alpha''(0) = (n-1)(n-2)/R^2$.

For the second derivative of μ at $\rho = 0$ we obtain

$$\mu''(0)hk = (a'(0)h|a'(0)k) = (\nabla_\Sigma h|\nabla_\Sigma k).$$

This yields the following representation for the *second variation* of Φ at $\rho = 0$,

$$\langle \Phi''(0)h|k\rangle = \int_\Sigma \{[(\operatorname{tr} K)^2 - \operatorname{tr} K^2]hk + (\nabla_\Sigma h|\nabla_\Sigma k)\} d\Sigma. \tag{2.51}$$

By means of the surface divergence theorem (2.24), this representation can be rewritten as

$$\langle \Phi''(0)h|k\rangle = \int_\Sigma \{[(\operatorname{tr} K)^2 - \operatorname{tr} K^2]h - \Delta_\Sigma h\}k \, d\Sigma,$$

and therefore

$$\Phi''(0)h = [(\operatorname{tr} K)^2 - \operatorname{tr} K^2]h - \Delta_\Sigma h, \tag{2.52}$$

i.e., $\Phi''(0)$ is the *Jacobi operator*, (sometimes also called the stability operator). Thus we see that

$$\Phi''(0) = -\kappa'(0) + \kappa_\Sigma^2.$$

In the next section we will come back to this relation.

2.7 The Volume Functional

Let Ω_ρ denote the domain bounded by the surface $\Gamma_\rho = \{p + \rho(p)\nu_\Sigma(p) : p \in \Sigma\}$. We define the volume functional Ψ by means of

$$\Psi(\rho) := |\Omega_\rho|. \tag{2.53}$$

In order to obtain the variation of $\Psi(\rho)$ we rewrite the volume functional by means of the divergence theorem as

$$n\Psi(\rho) = \int_{\Omega_\rho} \operatorname{div} x \, dx = \int_{\Gamma_\rho} (x|\nu_\Gamma) \, d\Gamma = \int_\Sigma (\mathrm{id}_\Sigma + \rho\nu_\Sigma|\nu_\Gamma)\alpha(\rho)\mu(\rho) \, d\Sigma,$$

which yields, with $\nu_\Gamma = \beta(\rho)(\nu_\Sigma - a(\rho))$,

$$n\Psi(\rho) = \int_\Sigma [\rho + (\mathrm{id}_\Sigma|\nu_\Sigma - a(\rho))]\alpha(\rho) \, d\Sigma,$$

where as before $\alpha(\rho) = \det(I - \rho K) = \Pi_{i=1}^{n-1}(1 - \rho\kappa_i)$. The first variation of Ψ then is

$$n\langle\Psi'(\rho)|h\rangle = \int_\Sigma \{[\rho + (\mathrm{id}_\Sigma|\nu_\Sigma - a(\rho))]\alpha'(\rho)h + [h - (\mathrm{id}_\Sigma|a'(\rho)h)]\alpha(\rho)\} \, d\Sigma.$$

From $\alpha(0) = 1$, $\alpha'(0) = -\kappa_\Sigma$ and $a'(0)h = \nabla_\Sigma h$ follows

$$\begin{aligned} n\langle\Psi'(0)|h\rangle &= \int_\Sigma [1 - (\mathrm{id}_\Sigma|\nu_\Sigma)\kappa_\Sigma]h \, d\Sigma - \int_\Sigma (\mathrm{id}_\Sigma|\nabla_\Sigma h) \, d\Sigma \\ &= \int_\Sigma (1 + \operatorname{div}_\Sigma \mathrm{id}_\Sigma)h \, d\Sigma, \end{aligned}$$

where we used the surface divergence theorem (2.31) in the last step. From

$$\operatorname{div}_\Sigma \mathrm{id}_\Sigma = (\tau^i|\partial_i \,\mathrm{id}_\Sigma) = (\tau^i|\tau_i) = (n-1)$$

follows the well-known formula for the *first variation* of the volume functional

$$\langle\Psi'(0)|h\rangle = \int_\Sigma h \, d\Sigma. \tag{2.54}$$

Now we reconsider the area functional Φ. We want to minimize surface area of Σ under the constraint that the volume of the domain bounded by Σ is a given constant Ψ_0. The method of Lagrange multipliers yields a number $\lambda \in \mathbb{R}$ such that $\Phi' - \lambda\Psi' = 0$. According to (2.50) and (2.54), this means

$$0 = \langle \Phi' - \lambda\Psi' | h\rangle = -\int_\Sigma (\kappa_\Sigma + \lambda) h \, d\Sigma = 0,$$

for all functions h. This implies $\kappa_\Sigma \equiv -\lambda$, i.e., Σ must be a sphere since Σ is an embedded closed and compact hypersurface. But then the value Φ is given by the constraint, i.e.,

$$\Phi(S_R(x_0)) = \omega_n R^{n-1}, \quad \kappa_\Sigma = -(n-1)/R, \quad \lambda = (n-1)/R, \quad (\omega_n/n)R^n = \Psi_0.$$

The *second variation* of Ψ can be computed as follows.

$$\begin{aligned} n\langle \Psi''(0)h|k\rangle = &\int_\Sigma (\mathrm{id}_\Sigma|\nu_\Sigma)\alpha''(0) hk \, d\Sigma \\ &+ \int_\Sigma \{[k - (\mathrm{id}_\Sigma|\nabla_\Sigma k)]h + [(h - (\mathrm{id}_\Sigma|\nabla_\Sigma h)]k\}\alpha'(0)\, d\Sigma \\ &- \int_\Sigma (\mathrm{id}_\Sigma|a''(0)hk)\alpha(0)\, d\Sigma. \end{aligned}$$

We observe that

$$(\mathrm{id}_\Sigma|\nabla_\Sigma k)h + (\mathrm{id}_\Sigma|\nabla_\Sigma h)]k = (\mathrm{id}_\Sigma|\nabla_\Sigma (hk))$$

and

$$a''(0)hk = M_0'(0)k\nabla_\Sigma h + M_0'(0)h\nabla_\Sigma k = L_\Sigma[k\nabla_\Sigma h + h\nabla_\Sigma k] = L_\Sigma \nabla_\Sigma(hk).$$

Collecting terms this yields

$$\begin{aligned} \langle \Psi''(0)h|k\rangle = &\frac{1}{n}\int_\Sigma [(\mathrm{id}_\Sigma|\nu_\Sigma)\alpha''(0) + 2\alpha'(0)]hk \, d\Sigma \\ &- \frac{1}{n}\int_\Sigma (\mathrm{id}_\Sigma|[\alpha'(0)I + L_\Sigma]\nabla_\Sigma(hk))\, d\Sigma. \end{aligned} \tag{2.55}$$

Here we recall that $\alpha'(0) = -\kappa_\Sigma$ and $\alpha''(0) = (\mathrm{tr}\, L_\Sigma)^2 - \mathrm{tr}\, L_\Sigma^2$.

In particular, for a sphere of radius R centered at the origin we get $\mathrm{id}_\Sigma = R\nu_\Sigma$, and hence

$$\langle \Psi''(0)h|k\rangle = \frac{1}{n}\int_\Sigma \left[\frac{R(n-1)(n-2)}{R^2} + \frac{2(n-1)}{R}\right] hk\, d\Sigma = \frac{n-1}{R}\int_\Sigma hk\, d\Sigma.$$

This implies at a stationary point of the surface functional Φ with constraint $\Psi(\rho) = c$ with $\Phi' + \lambda\Psi' = 0$ and $\lambda = \kappa_\Sigma$,

$$\Phi'' + \lambda\Psi'' = -\Delta_\Sigma - (n-1)/R^2 = -\kappa'(0).$$

2.3 Approximation of Hypersurfaces

3.1 The Tubular Neighbourhood of a Hypersurface
Let Σ be a compact connected C^2-hypersurface bounding a domain $\Omega \subset \mathbb{R}^n$, and let ν_Σ be the outer unit normal field on Σ with respect to Ω. Then Σ satisfies the *uniform interior and exterior ball condition* , i.e., there is a number $a > 0$ such that for each point $p \in \Sigma$ there are balls $B(x_1, a) \subset \Omega$ and $B(x_2, a) \subset \bar{\Omega}^c$, such that $\Sigma \cap \bar{B}(x_i, a) = \{p\}$. Choosing the radius a_0 maximal, we set $a = a_0/2$ in the sequel. Consider the mapping

$$\Lambda : \Sigma \times (-a, a) \to \mathbb{R}^n, \quad \Lambda(p, r) := p + r\nu_\Sigma(p). \tag{2.56}$$

We claim that Λ is a C^1-diffeomorphism onto its image

$$U_a := \mathrm{im}(\Lambda) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Sigma) < a\}.$$

Note that the centers of the balls $B(x_i, a)$ necessarily are equal to $x_1 = p - a\nu_\Sigma(p)$ and $x_2 = p + a\nu_\Sigma(p)$. To prove injectivity of Λ, suppose

$$p_1 + r_1\nu_\Sigma(p_1) = p_2 + r_2\nu_\Sigma(p_2),$$

where we may assume w.l.o.g. that $r_2 \le r_1 < a$. But then

$$p_2 - (p_1 + r_1\nu_\Sigma(p_1)) = -r_2\nu_\Sigma(p_2),$$

hence $p_2 \in \bar{B}(p_1 + r_1\nu_\Sigma(p_1), r_1) \cap \Sigma = \{p_1\}$, which shows $p_1 = p_2$ and then also $r_1 = r_2$. The set U_a will be called the *tubular neighbourhood* of Σ of of width a. To prove that Λ is a diffeomorphism, fix a point $(p_0, r_0) \in \Sigma \times (-a, a)$ and a chart ϕ for p_0. Then the function $f(\theta, r) = \Lambda(\phi(\theta), r)$ has derivative

$$Df(0, r_0) = [[I - r_0 L_\Sigma(p_0)]\phi'(0), \nu_\Sigma(p_0)].$$

It follows from (2.58) that $[I - r_0 L_\Sigma(p_0)] \in \mathcal{B}(T_{p_0}\Sigma)$ is invertible, and consequently, $Df(0, r_0) \in \mathcal{B}(\mathbb{R}^n)$ is invertible as well. The inverse function theorem implies that Λ is locally invertible with inverse of class C^1.

It will be convenient to decompose the inverse of Λ into $\Lambda^{-1} = (\Pi_\Sigma, d_\Sigma)$ such that

$$\Pi_\Sigma \in C^1(U_a, \Sigma), \quad d_\Sigma \in C^1(U_a, (-a, a)). \tag{2.57}$$

$\Pi_\Sigma(x)$ is the nearest point on Σ to x, $d_\Sigma(x)$ is the signed distance from x to Σ.

From the uniform interior and exterior ball condition follows that the number $1/a_0$ bounds the principal curvatures of Σ, i.e.,

$$\max\{\kappa_i(p) : p \in \Sigma,\ i = 1, \cdots, n-1\} \le 1/a_0. \tag{2.58}$$

A remarkable fact is that the signed distance d_Σ is even of class C^2. To see this, we use the identities

$$x - \Pi_\Sigma(x) = d_\Sigma(x)\nu_\Sigma(\Pi_\Sigma(x)), \quad d_\Sigma(x) = (x - \Pi_\Sigma(x)|\nu_\Sigma(\Pi_\Sigma(x)).$$

Differentiating w.r.t. x_k this yields

$$\begin{aligned}\partial_{x_k} d_\Sigma(x) &= (e_k - \partial_{x_k}\Pi_\Sigma(x)|\nu_\Sigma(\Pi_\Sigma(x))) + (x - \Pi_\Sigma(x)|\partial_{x_k}(\nu_\Sigma \circ \Pi_\Sigma)(x)) \\ &= \nu_\Sigma^k(\Pi_\Sigma(x)) + d_\Sigma(x)(\nu_\Sigma(\Pi_\Sigma(x))|\partial_{x_k}(\nu_\Sigma \circ \Pi_\Sigma(x))) \\ &= \nu_\Sigma^k(\Pi_\Sigma(x)),\end{aligned}$$

since $\partial_{x_k}\Pi_\Sigma(x)$ belongs to the tangent space $T_{\Pi_\Sigma(x)}\Sigma$, as does $\partial_{x_k}(\nu_\Sigma \circ \Pi_\Sigma(x))$, since $|\nu_\Sigma \circ \Pi_\Sigma(x)| \equiv 1$. Thus we have the formula

$$\nabla_x d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in U_a. \tag{2.59}$$

This shows, in particular, that d_Σ is of class C^2.

It is useful to also have a representation for the derivatve $\partial\Pi_\Sigma(x)$ of $\Pi_\Sigma(x)$. With

$$I - \partial\Pi_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)) \otimes \nabla_x d_\Sigma(x) + d_\Sigma(x)\partial\nu_\Sigma(\Pi_\Sigma(x))\partial\Pi_\Sigma(x),$$

and (2.59), we obtain

$$\partial\Pi_\Sigma(x) = M_0(d_\Sigma(x))(\Pi_\Sigma(x))\mathcal{P}_\Sigma(\Pi_\Sigma(x)), \tag{2.60}$$

where $M_0(r)(p) := (I - rL_\Sigma(p))^{-1}$. This shows that $\partial\Pi_\Sigma(p) = \nabla_x\Pi_\Sigma(p) = \mathcal{P}_\Sigma(p)$, the orthogonal projection onto the tangent space $T_p\Sigma$.

3.2 The Level Function

Let Σ be a compact connected hypersurface of class C^2 bounding the domain Ω in $\mathbb{R}^n$. According to the previous section, Σ admits a tubular neighbourhood U_a of width $a > 0$. We may assume w.l.o.g. $a \le 1$. The signed distance function $d_\Sigma(x)$ in this tubular neighbourhood is of class C^2 as well, and since

$$\nabla_x d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad x \in U_a,$$

we can view $\nabla_x d_\Sigma(x)$ as a C^1-extension of the normal field $\nu_\Sigma(x)$ from Σ to the tubular neighbourhood U_a of Σ. Computing the second derivatives $\nabla_x^2 d_\Sigma$ we obtain

$$\begin{aligned}\nabla_x^2 d_\Sigma(x) = \nabla_x\nu_\Sigma(\Pi_\Sigma(x)) &= -L_\Sigma(\Pi_\Sigma(x))\mathcal{P}_\Sigma(\Pi_\Sigma(x))(I - d_\Sigma(x)L_\Sigma(\Pi_\Sigma(x)))^{-1} \\ &= -L_\Sigma(\Pi_\Sigma(x))(I - d_\Sigma(x)L_\Sigma(\Pi_\Sigma(x)))^{-1},\end{aligned}$$

for $x \in U_a$, as $L_\Sigma(p) = L_\Sigma(p)\mathcal{P}_\Sigma(p)$. Taking traces then yields

$$\Delta d_\Sigma(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(\Pi_\Sigma(x))}{1 - d_\Sigma(x)\kappa_i(\Pi_\Sigma(x))}, \quad x \in U_a. \tag{2.61}$$

In particular, this implies

$$\nabla_x^2 d_\Sigma(p) = -L_\Sigma(p), \quad \Delta_x d_\Sigma(p) = -\kappa_\Sigma(p), \quad p \in \Sigma. \tag{2.62}$$

Therefore the norm of $\nabla_x^2 d_\Sigma$ is equivalent to the maximum of the moduli of the curvatures of Σ at a fixed point. Hence we find a constant c, depending only on n, such that

$$c|\nabla_x^2 d_\Sigma|_\infty \leq \max\{|\kappa_i(p)| : i = 1, \ldots, n-1, \, p \in \Sigma\} \leq c^{-1}|\nabla_x^2 d_\Sigma|_\infty.$$

Next we extend d_Σ as a function φ to all of $\mathbb{R}^n$. For this purpose we choose a C^∞-function $\chi(s)$ such that $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$, $0 \leq \chi(s) \leq 1$. Then we set

$$\varphi(x) := \begin{cases} d_\Sigma(x)\chi(3d_\Sigma(x)/a) + \operatorname{sign}(d_\Sigma(x))(1 - \chi(3d_\Sigma(x)/a)), & x \in U_a, \\ \chi_{\Omega_{\rm ex}}(x) - \chi_{\Omega_{\rm in}}(x), & x \notin U_a, \end{cases} \tag{2.63}$$

where $\Omega_{\rm ex}$ and $\Omega_{\rm in}$ denote the exterior and interior component of $\mathbb{R}^n \setminus U_a$, respectively. This function φ is then of class C^2, $\varphi(x) = d_\Sigma(x)$ for $x \in U_{a/3}$, and

$$\varphi(x) = 0 \quad \Leftrightarrow \quad x \in \Sigma.$$

Thus Σ is given as zero-level set of φ, i.e., $\Sigma = \varphi^{-1}(0)$. φ is called a *canonical level function* for Σ. It is a special level function for Σ, as

$$\nabla_x \varphi(x) = \nu_\Sigma(\Pi_\Sigma(x)) \quad \text{for } x \in U_{a/3}.$$

3.3 Existence of Parameterizations

Recall the Hausdorff metric on the set $\mathcal{K}$ of compact subsets of $\mathbb{R}^n$ defined by

$$d_H(K_1, K_2) = \max\{\sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1)\}. \tag{2.64}$$

Suppose Σ is a compact connected closed hypersurface of class C^2 bounding a bounded domain in $\mathbb{R}^n$. As before, let U_a be its tubular neighbourhood, $\Pi_\Sigma : U_a \to \Sigma$ the projection and $d_\Sigma : U_a \to \mathbb{R}$ the signed distance. We want to parameterize hypersurfaces Γ which are close to Σ as

$$\Gamma = \{p + \rho(p)\nu_\Sigma(p) : p \in \Sigma\},$$

where $\rho : \Sigma \to \mathbb{R}$ is then called the *normal parameterization* of Γ over Σ. For this to make sense, Γ must belong to the tubular neighbourhood U_a of Σ. Therefore, a natural requirement would be $d_H(\Gamma, \Sigma) < a$. We then say that Γ and Σ are C^0-*close* (of order ε) if $d_H(\Gamma, \Sigma) < \varepsilon$.

However, this condition is not enough to allow for existence of a normal parameterization, since it is not clear that the map Π_Σ is injective on Γ: small Hausdorff distance does not prevent Γ from folding within the tubular neighbourhood. We need a stronger assumption to prevent this. If Γ is a hypersurface of class C^1 we may introduce the *normal bundle* $\mathcal{N}\Gamma$ defined by

$$\mathcal{N}\Gamma := \{(q, \nu_\Gamma(q)) : q \in \Gamma\} \subset \mathbb{R}^{2n}.$$

Suppose Γ is a compact, connected C^1-hypersurface in $\mathbb{R}^n$. We say that Γ and Σ are *C^1-close* (of order ε) if $d_H(\mathcal{N}\Gamma, \mathcal{N}\Sigma) < \varepsilon$. We are going to show that C^1-hypersurfaces Γ which are C^1-close to Σ can in fact be parametrized over Σ.

For this purpose observe that, in case Γ and Σ are C^1-close of order ε, whenever $q \in \Gamma$, there is $p \in \Sigma$ such that $|q - p| + |\nu_\Gamma(q) - \nu_\Sigma(p)| < \varepsilon$. Hence $|q - \Pi_\Sigma q| < \varepsilon$, with $\Pi_\Sigma q := \Pi_\Sigma(q)$, and

$$|\nu_\Gamma(q) - \nu_\Sigma(\Pi_\Sigma q)| \le |\nu_\Gamma(q) - \nu_\Sigma(p)| + |\nu_\Sigma(\Pi_\Sigma q) - \nu_\Sigma(p)| \le \varepsilon + L|\Pi_\Sigma q - p|,$$

which yields with $|\Pi_\Sigma q - p| \le |\Pi_\Sigma q - q| + |p - q| < 2\varepsilon$,

$$|q - \Pi_\Sigma q| + |\nu_\Gamma(q) - \nu_\Sigma(\Pi_\Sigma q)| \le 2(1 + L)\varepsilon,$$

where L denotes the Lipschitz constant of the normal of Σ. In particular, the tangent space $T_q\Gamma$ is transversal to $\nu_\Sigma(\Pi_\Sigma q)$, for each $q \in \Gamma$, that is,

$$(\nu_\Sigma(\Pi_\Sigma q) \,|\, \nu_\Gamma(q)) \ne 0, \quad q \in \Gamma.$$

Now fix a point $q_0 \in \Gamma$ and set $p_0 = \Pi_\Sigma q_0$. Since the tangent space $T_{q_0}\Gamma$ is transversal to $\nu_\Sigma(p_0)$, we infer that $\Pi'_\Sigma(q_0) : T_{q_0}\Gamma \to T_{p_0}\Sigma$ is invertible. The inverse function theorem yields an open neighbourhood $V(p_0) \subset \Sigma$ and a C^1-map $g : V(p_0) \to \Gamma$ such that $g(p_0) = q_0$, $g(V(p_0)) \subset \Gamma$, and $\Pi_\Sigma g(p) = p$ in $V(p_0)$. Therefore we obtain

$$q = g(p) = \Lambda \circ (\Pi_\Sigma, d_\Sigma)g(p) = \Pi_\Sigma g(p) + d_\Sigma(g(p))\nu_\Sigma(\Pi_\Sigma g(p)) = p + \rho(p)\nu_\Sigma(p),$$

with

$$\rho(p) := d_\Sigma(g(p)).$$

Thus we have a local normal parameterization of Γ over Σ. We may extend g to a maximal domain $V \subset \Sigma$, e.g. by means of Zorn's lemma. Clearly V is open in Σ and we claim that $V = \Sigma$. If not, then the boundary of V in Σ is nonempty and hence we find a sequence $p_n \in V$ such that $p_n \to p_\infty \in \partial V$. Since $\rho_n = \rho(p_n)$ is bounded, we may assume w.l.o.g. that $\rho_n \to \rho_\infty$. But then $q_\infty = p_\infty + \rho_\infty \nu_\Sigma(p_\infty)$ belongs to Γ as Γ is closed. Now we may apply the inverse function theorem again to see that V cannot be maximal. Since the map $\Phi(p) = p + \rho(p)\nu_\Sigma(p)$ is a local C^1-diffeomorphism, it is also open. Hence $\Phi(\Sigma) \subset \Gamma$ is open and compact, i.e., $\Phi(\Sigma) = \Gamma$ by connectedness of Γ. The map Φ is therefore a C^1-diffeomorphism from Σ to Γ. In case Σ is of class C^{k+1} and Γ is of class C^k for $k \ge 1$ the proof above immediately implies that $\Phi \in \mathrm{Diff}^k(\Sigma, \Gamma)$.

Observe that because of $x = \Pi_\Sigma x + d_\Sigma(x)\nu_\Sigma(\Pi_\Sigma x)$ in U_a we have $x \in \Gamma$ if and only if $d_\Sigma(x) = \rho(\Pi_\Sigma x)$. This property can be used to construct a C^1-function ψ on $\mathbb{R}^n$ such that $\Gamma = \psi^{-1}(0)$, i.e., a level function for Γ. For example we may take

$$\psi(x) = \varphi(x) - \rho(\Pi_\Sigma x)\chi(3d_\Sigma(x)/a), \quad x \in \mathbb{R}^n,$$

provided $\varepsilon < a/3$, where φ and χ are defined in (2.63).

3.4 Approximation of Hypersurfaces
Suppose as before that Σ is a compact connected hypersurface of class C^2 bounding a bounded domain Ω in $\mathbb{R}^n$. We may use the level function $\varphi : \mathbb{R}^n \to \mathbb{R}$ introduced in (2.63) to construct a real analytic hypersurface Σ_ε such that Σ appears as a C^2-graph over Σ_ε. In fact, we show that there is $\varepsilon_0 \in (0, a/3)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is an analytic manifold Σ_ε and a function $\rho_\varepsilon \in C^2(\Sigma_\varepsilon)$ with the property that

$$\Sigma = \{p + \rho_\varepsilon(p)\nu_{\Sigma_\varepsilon}(p) : p \in \Sigma_\varepsilon\}$$

and

$$|\rho_\varepsilon|_\infty + |\nabla_{\Sigma_\varepsilon}\rho_\varepsilon|_\infty + |\nabla^2_{\Sigma_\varepsilon}\rho_\varepsilon|_\infty \le \varepsilon.$$

For this purpose, choose $R > 0$ such that $\varphi(x) = 1$ for $|x| > R/2$. Then define

$$\psi_k(x) = c_k\Big(1 - \frac{|x|^2}{R^2}\Big)^k_+, \quad x \in \mathbb{R}^n,$$

where $c_k > 0$ is chosen such that $\int_{\mathbb{R}^n} \psi_k(x)dx = 1$. Then $c_k \sim \alpha k^{n/2}$ as $k \to \infty$, with some number $\alpha = \alpha(n, R)$. Indeed, we have

$$\int_{B(0,R)} \Big(1 - \frac{|x|^2}{R^2}\Big)^k dx = \omega_n R^n \int_0^1 (1 - r^2)^k r^{n-1}\, dr = \frac{\omega_n R^n}{2} \int_0^1 (1 - t)^k t^{n/2-1}\, dt,$$

where $\omega_n = |\partial B(0, 1)|$. Using the well-known relations

$$\int_0^1 (1 - t)^k t^{n/2-1}\, dt = \mathsf{B}\big(\frac{n}{2}, k + 1\big) = \frac{\Gamma(\frac{n}{2})\Gamma(k + 1)}{\Gamma(k + 1 + \frac{n}{2})} \sim \Gamma(n/2)k^{-n/2}$$

with B the Beta function and Γ the Gamma function, the claim follows, with $\alpha = ((\omega_n R^n/2)\Gamma(n/2))^{-1} = (\pi R^2)^{-n/2}$.

Then as $k \to \infty$, we have $\psi_k(x) \to 0$, uniformly for $|x| \ge \eta > 0$, since $k^{n/2}q^k \to 0$ for any fixed $q \in (0, 1)$. Consequently, $\psi_k * f \to f$ in $C^m_{ub}(\mathbb{R}^n)$, whenever $f \in C^m_{ub}(\mathbb{R}^n)$. Let $\varphi_k = 1 + \psi_k * (\varphi - 1)$. Then

$$\varphi_k \to \varphi \quad \text{in} \quad C^2_{ub}(\mathbb{R}^n). \tag{2.65}$$

Moreover,

$$\psi_k * (\varphi - 1)(x) = \int_{\mathbb{R}^n} (\varphi(y) - 1)\psi_k(x - y)dy = \int_{|y|\le R/2} (\varphi(y) - 1)\psi_k(x - y)dy.$$

For $|x|, |y| < R/2$ follows $|x-y| < R$, and hence $\psi_k(x-y) = c_k(1-|x-y|^2/R^2)^k$ is polynomial in x, y. But then $\varphi_k(x)$ is a polynomial for such values of x; in particular, φ_k is real analytic in U_a. Choosing k large enough, we have $|\varphi - \varphi_k|_{C^2_b(\mathbb{R}^n)} < \varepsilon$.

Now suppose $\varphi_k(x) = 0$. Then $|\varphi(x)| < \varepsilon$, hence $x \in U_a$ and then $|d_\Sigma(x)| < \varepsilon$. This shows that the set $\Sigma_k := \varphi_k^{-1}(0)$ is in the ε-tubular neighbourhood around

Σ. Moreover, $|\nabla\varphi_k - \nabla\varphi|_\infty < \varepsilon$ yields $\nabla\varphi_k(x) \neq 0$ in U_a, and therefore Σ_k is a manifold, which is real analytic.

Next we show that Σ and Σ_k are C^1-diffeomorphic. For this purpose, fix a point $q_0 \in \Sigma_k$. Then $q_0 = p_0 + r_0\nu_\Sigma(p_0)$, where $p_0 = \Pi_\Sigma q_0 \in \Sigma$ and $r_0 = d_\Sigma(q_0)$. Consider the equation $g_k(p,r) := \varphi_k(p + r\nu_\Sigma(p)) = 0$ near (p_0, r_0). Since

$$\partial_r g_k(p,r) = (\nabla_x\varphi_k(p + r\nu_\Sigma(p))|\nu_\Sigma(p))$$

we have

$$\begin{aligned}\partial_r g_k(p_0,r_0) &= (\nabla_x\varphi_k(q_0)|\nabla_x\varphi(p_0))\\ &\geq 1 - |\nabla_x\varphi_k(q_0) - \nabla_x\varphi(q_0)| - |\nabla_x\varphi(q_0) - \nabla_x\varphi(p_0)|\\ &\geq 1 - |\varphi_k - \varphi|_{C_b^1(\mathbb{R}^n)} - \varepsilon|\nabla_x^2\varphi|_{C_b(\mathbb{R}^n)} > 0.\end{aligned}$$

Therefore, we may apply the implicit function theorem to obtain an open neighbourhood $V(p_0) \subset \Sigma$ and a C^1-function $r_k : V(p_0) \to \mathbb{R}$ such that $r_k(p_0) = r_0$ and $p + r_k(p)\nu_\Sigma(p) \in \Sigma_k$ for all $p \in V(p_0)$. We can now proceed as in the previous subsection to extend $r_k(\cdot)$ to a maximal domain $V \subset \Sigma$, which coincides with Σ by compactness and connectedness of Σ.

Thus we have a well-defined C^1-map $f_k : \Sigma \to \Sigma_k$, $f_k(p) = p + r_k(p)\nu_\Sigma(p)$, which is injective and a diffeomorphism from Σ to its range. We claim that f_k is also surjective. If not, there is some point $q \in \Sigma_k$, $q \notin f_k(\Sigma)$. Set $p = \Pi_\Sigma q$. Then $q = p + d_\Sigma(p)\nu_\Sigma(p)$ with $d_\Sigma(p) \neq r_k(p)$. Thus, there are at least two numbers $\beta_1, \beta_2 \in (-a, a)$ with $p + \beta_i\nu_\Sigma(p) \in \Sigma_k$. This implies with $\nu_\Sigma = \nu_\Sigma(p)$

$$0 = \varphi_k(p+\beta_2\nu_\Sigma)-\varphi_k(p+\beta_1\nu_\Sigma) = (\beta_2-\beta_1)\int_0^1 (\nabla_x\varphi_k(p+(\beta_1+t(\beta_2-\beta_1))\nu_\Sigma)|\nu_\Sigma)\,dt,$$

which yields $\beta_2 - \beta_1 = 0$ since

$$\int_0^1 (\nabla_x\varphi_k(p + (\beta_1 + t(\beta_2 - \beta_1))\nu_\Sigma)|\nu_\Sigma)\,dt \geq 1 - \varepsilon - \varepsilon|\nabla_x^2\varphi|_{C_b(\mathbb{R}^n)} > 0,$$

as above. Therefore, the map f_k is also surjective, and hence $f_k \in \mathrm{Diff}^1(\Sigma, \Sigma_k)$. This implies, in particular, that $\Sigma_k = f_k(\Sigma)$ is connected. For later use we note that

$$|r_k|_\infty + |\nabla_\Sigma r_k|_\infty \to 0 \;\text{ as }\; k \to \infty,$$

as can be inferred from $\partial_i r_k(p) = (\tau_i^{\Sigma_k}(p+r_k(p)\nu_\Sigma(p))|\nu_\Sigma(p))$ for $p \in \Sigma$, see (2.39).

Next we show that the mapping

$$\Lambda_k : \Sigma_k \times (-a/2, a/2) \to U(\Sigma_k, a/2), \quad \Lambda_k(q,s) := q + s\nu_k(q)$$

is a C^1-diffeomorphism for $k \geq k_0$, with $k_0 \in \mathbb{N}$ sufficiently large. In order to see this, we use the diffeomorphism f_k constructed above to rewrite Λ_k as

$$\begin{aligned}\Lambda_k(q,s) &= \Lambda_k(f_k(p), s)\\ &= p + s\,\nu_\Sigma(p) + r_k(p)\nu_\Sigma(p) + s[\nu_k(p + r_k(p)\nu_\Sigma(p)) - \nu_\Sigma(p)]\\ &=: \Lambda(p,s) + G_k(p,s) =: H_k(p,s).\end{aligned}$$

Clearly $H_k \in C^1(\Sigma \times (-a/2, a/2), \mathbb{R}^n)$ and $\Lambda \in \mathrm{Diff}^1(\Sigma \times (-a, a), U(\Sigma, a))$. It is not difficult to see that

$$|G_k(p,s)| + |DG_k(p,s)| \to 0 \ \text{ as } \ k \to \infty, \ \text{ uniformly in } (p,s) \in \Sigma \times [-a/2, a/2].$$

Consequently, $DH_k(p,s) : T_p\Sigma \times (-a/2, a/2) \to \mathbb{R}^n$ is invertible for $k \geq k_0$, and by the inverse function theorem, H_k is a local C^1-diffeomorphism. We claim that H_k is injective for all k sufficiently large. For this purpose, note that due to compactness of $\Sigma \times [-a/2, a/2]$ and injectivity of Λ there exists a constant $c > 0$ such that

$$|\Lambda(p,s) - \Lambda(\bar{p}, \bar{s})| \geq c\big(|p - \bar{p}| + |s - \bar{s}|\big), \quad (p,s),\ (\bar{p}, \bar{s}) \in \Sigma \times [-a/2, a/2].$$

The properties of G_k and compactness of $\Sigma \times [-a/2, a/2]$ imply, in turn, that the estimate above remains true for Λ replaced by H_k, and c replaced by $c/2$, provided $k \geq k_0$ with k_0 sufficiently large. Hence H_k is a C^1-diffeomorphism onto its image for k sufficiently large, as claimed. This shows that Σ_k has a uniform tubular neighbourhood of width $a/2$ for any $k \geq k_0$, and it follows that $\Sigma \subset U(\Gamma_k, a/2)$. Σ and Σ_k are compact connected closed C^1 hypersurfaces, and we may now apply the results of the previous subsection, showing that Σ can be parameterized over Σ_k by means of

$$p \mapsto p + \rho_k(p)\nu_k(p) \text{ with } \rho_k \in C^2(\Sigma_k, \mathbb{R}),$$

with $\nu_k := \nu_{\Sigma_k}$.

Finally we show that $|\rho_k|_\infty + |\nabla_{\Sigma_k}\rho_k|_\infty + |\nabla^2_{\Sigma_k}\rho_k|_\infty \leq \varepsilon$ for k sufficiently large. We already know from the construction that $|\rho_k|_\infty \to 0$ as $k \to \infty$. However, we need the following estimate on the rate of convergence: there exists $k_0 \in \mathbb{N}$ and a constant $C = C(k_0)$ such that

$$|\rho_k|_\infty \leq Ck^{-1/2}, \quad k \geq k_0. \tag{2.66}$$

In order to see this, we first observe that, for $|x| \leq R/2$,

$$\begin{aligned} |\varphi(x) - \varphi_k(x)| &= \Big|\int_{\mathbb{R}^n} [\varphi(x) - \varphi(x-y)]\psi_k(y)\,dy\Big| \leq |\nabla\varphi|_\infty \int_{|y|\leq R} |y|\psi_k(y)dy \\ &= |\nabla\varphi|_\infty C(n,R)c_k \mathsf{B}\big(\frac{n+1}{2}, k+1\big). \end{aligned}$$

Using similar arguments as above for the asymptotics of c_k and $B((n+1)/2, k+1)$ this yields constants $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that $|\varphi(x) - \varphi_k(x)| \leq Ck^{-1/2}$, whenever $|x| \leq R/2$ and $k \geq k_0$. Let $p \in \Sigma_k$ be given, and let $q = p + \rho_k(p)\nu_k(p)$. Then $|\varphi_k(q)| = |\varphi_k(q) - \varphi(q)| \leq Ck^{-1/2}$ for $k \geq k_0$. On the other hand,

$$|\varphi_k(q)| = |\varphi_k(q) - \varphi_k(p)| = \rho_k(p)\Big|\int_0^1 (\nabla\varphi_k(p + t\rho_k(p)\nu_k(p)|\nu_k(p))dt\Big| \geq \frac{1}{2}\rho_k(p),$$

provided k is sufficiently large, and this implies (2.66).

Next we show that there exists $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that $|\partial^\alpha \varphi_k(x)| \le Ck^{1/2}$ whenever $k \ge k_0$, $|x| \le R/2$, and $|\alpha| = 3$. Indeed this follows from

$$\begin{aligned}|\partial_\ell \partial_i \partial_j \varphi_k(x)| &= (2/R^2)kc_k \Big| \int_{|y|\le R} \partial_i\partial_j\varphi_k(x-y)y_\ell(1-|y|^2/R^2)^{k-1}dy\Big| \\ &\le Ckc_k \mathsf{B}\Big(\frac{n+1}{2}, k\Big) \sim ck^{1/2},\end{aligned}$$

where c is an appropriate constant. Combining with (2.66) we have shown that there are constants $k_0 \in \mathbb{N}$ and $C = C(k_0)$ such that

$$\rho_k(p)|\partial^\alpha \varphi_k(x)| \le C, \tag{2.67}$$

for $k \ge k_0$, $|\alpha| = 3$, $p \in \Sigma_k$, and $|x| \le R/2$.

In order to show smallness of $|\nabla_{\Sigma_k}\rho_k|_\infty + |\nabla^2_{\Sigma_k}\rho_k|_\infty$, we consider the relation

$$\varphi\big(\psi_k(\theta) + (\rho_k\nu_k)(\psi_k(\theta)\big) = 0, \quad \theta \in \Theta_k, \tag{2.68}$$

where $\psi_k : \Theta_k \to \Sigma_k$ is a C^2-parameterization of Σ_k around a point $p_k = f_k(q)$ for some $q \in \Sigma$. Since $\Sigma_k = \varphi_k^{-1}(0)$ and $\varphi_k \to \varphi$ in $C^2_{ub}(\mathbb{R}^n)$ one shows that $|\partial_j\psi_k(0)|$ is uniformly bounded in k for k sufficiently large.

Let $\tilde\nu_k(x) := \nabla_x\varphi_k(x)/|\nabla_x\varphi_k(x)|$ for $x \in \mathbb{R}^n$. Clearly, $\nu_k(\psi_k(\theta)) = \tilde\nu_k(\psi_k(\theta))$. Taking partial derivatives in (2.68) and using the orthogonality relation $(\nabla_x\varphi_k(\psi_k(\theta))\,|\,\partial_j\psi_k(\theta)) = 0$ yields

$$\begin{aligned}\partial_j(\rho_k\circ\psi_k)(\theta)&\big(\nabla_x\varphi(q_k(\theta))\,|\,(\nu_k\circ\psi_k)(\theta)\big)\\ &= \big(\nabla_x\varphi(q_k(\theta)) - \nabla_x\varphi_k(\psi_k(\theta))\,|\,\partial_j\psi_k(\theta)\big)\\ &\quad - (\rho_k\circ\psi_k)(\theta)\big(\nabla_x\varphi(q_k(\theta))\,|\,\partial_j(\tilde\nu_k\circ\psi_k)(\theta)\big)\end{aligned} \tag{2.69}$$

where, for brevity, we set $q_k(\theta) = \psi_k(\theta) + (\rho_k\nu_k)(\psi_k(\theta))$. It follows from (2.65) and uniform continuity that

$$(\nabla_x\varphi(q_k(\theta))|(\nu_k\circ\psi_k)(\theta)) \ge 1/2, \tag{2.70}$$

provided $k \ge k_0$ with k_0 sufficiently large. The fact that $\partial_j\psi_k(\theta)$ is uniformly bounded for $k \ge k_0$ and (2.65) implies that the right-hand side in (2.69) converges to zero as $k \to \infty$. We have shown that $|\partial_j\rho_k(p_k)| \le \varepsilon$, provided that $k \ge k_0$ with k_0 sufficiently large.

Next, we take an additional derivative $\partial_i = \partial_{\theta_i}$ in (2.69). This will produce the terms

$$\begin{aligned}&\partial_i\partial_j(\rho_k\circ\psi_k)(\theta)\big(\nabla_x\varphi(q_k(\theta))\,|\,(\nu_k\circ\psi_k)(\theta)\big)\\ &+ \partial_j(\rho_k\circ\psi_k)(\theta)\partial_i\big(\nabla_x\varphi(q_k(\theta))\,|\,(\nu_k\circ\psi_k)(\theta)\big)\end{aligned}$$

on the left-hand side. From the previous step for $\partial_j(\rho_k\circ\psi_k)$ we conclude that the second term converges to 0 as $k \to \infty$. Thus it follows from (2.70) that

$\partial_i\partial_j(\rho_k \circ \psi_k)$ converges to 0 as $k \to \infty$, provided we can show that the derivatives of the right-hand side in (2.69) converge to zero as $k \to \infty$. A moment of reflection shows that this is indeed the case, with the possible exception of the term $\rho_k(\psi_k(\theta))\big(\nabla_x\varphi(q_k(\theta)) \,|\, \partial_i\partial_j(\tilde\nu_k \circ \psi_k)(\theta)\big)$ which is problematic as $\partial_i\partial_j(\tilde\nu_k \circ \psi_k)$ involves third-order derivatives of φ_k. Since $((\tilde\nu_k \circ \psi_k)(\theta) \,|\, (\tilde\nu_k \circ \psi_k)(\theta)) = 1$ we get $((\tilde\nu_k \circ \psi_k)(\theta) \,|\, \partial_j(\tilde\nu_k \circ \psi_k)(\theta)) = 0$, and hence

$$((\tilde\nu_k \circ \psi_k)(\theta) \,|\, \partial_i\partial_j(\tilde\nu_k \circ \psi_k)(\theta)) = -(\partial_i(\tilde\nu_k \circ \psi_k)(\theta) \,|\, \partial_j(\tilde\nu_k \circ \psi_k)(\theta)).$$

With $\nabla_x\varphi(q_k(\theta)) = \nu_\Sigma(q_k(\theta))$ this yields

$$\begin{aligned}&\rho_k(\psi_k(\theta))\big(\nabla_x\varphi(q_k(\theta)) \,|\, \partial_i\partial_j(\tilde\nu \circ \psi_k))(\theta))\big)\\ &= \big(\nu_\Sigma(q_k(\theta)) - \tilde\nu_k(\psi_k(\theta)) \,|\, \rho_k(\psi_k(\theta))\partial_i\partial_j(\tilde\nu_k \circ \psi_k)(\theta)\big)\\ &\quad + \rho_k(\psi_k(\theta))\big(\partial_i(\tilde\nu_k \circ \psi_k)(\theta) \,|\, \partial_j(\tilde\nu_k \circ \psi_k)(\theta)\big).\end{aligned}$$

Convergence to 0 of the first term on the right-hand side follows from (2.67) and (2.65), while the second term converges to 0 since ρ_k has this property.

Since $f_k : \Sigma \to \Sigma_k$ is a bijection, the assertion holds true for any point $p_k \in \Sigma_k$, $k \geq k_0$, and hence the claim follows.

2.4 The Manifold of Hypersurfaces in $\mathbb{R}^n$

4.1 Compact Connected Hypersurfaces of Class C^2

Consider the set $\mathcal{MH}^2$ of all compact connected C^2-hypersurfaces Σ in $\mathbb{R}^n$. Let $\mathcal{N}\Sigma$ denote their associated normal bundles. The second normal bundle of Σ is defined by

$$\mathcal{N}^2\Sigma = \{(p, \nu_\Sigma(p), \nabla_\Sigma\nu_\Sigma(p)) : p \in \Sigma\}.$$

We introduce a metric $d_{\mathcal{MH}^2}$ on $\mathcal{MH}^2$ by means of $d_{\mathcal{MH}^2}(\Sigma_1, \Sigma_2) = d_H(\mathcal{N}^2\Sigma_1, \mathcal{N}^2\Sigma_2)$. This way $\mathcal{MH}^2$ becomes a metric space. We want to show that $\mathcal{MH}^2$ is a Banach manifold.

Fix a hypersurface $\Sigma \in \mathcal{MH}^2$ of class C^3. Then we define a chart over the Banach space $X_\Sigma := C^2(\Sigma, \mathbb{R})$ as follows. Σ has a tubular neighbourhood U_a of width a. For a given function $\rho \in B_{X_\Sigma}(0, a/3)$ we obtain a hypersurface Γ_ρ^Σ by means of the map

$$\Phi_\Sigma(\rho)(p) := p + \rho(p)\nu_\Sigma(p), \quad p \in \Sigma.$$

According to Section 2.3, this yields a hypersurface Γ_ρ^Σ of class C^2, diffeomorphic to Σ. Moreover, with some constant C_a^Σ, we have

$$d_{\mathcal{MH}^2}(\Gamma_\rho^\Sigma, \Sigma) \leq C_a^\Sigma |\rho|_{C_b^2(\Sigma)},$$

which shows that the map $\Phi_\Sigma : B_{X_\Sigma}(0, a/3) \to \mathcal{MH}^2$ is continuous. Conversely, given $\Gamma \in \mathcal{MH}^2$ which is C^2-close to Σ, the results in Section 2.3.3 show that Γ can be parameterized by a function $\rho \in C^2(\Sigma, \mathbb{R})$, such that $|\rho|_{C_b^2(\Sigma)} < a/3$.

We now determine the tangent space $T_\Sigma\mathcal{MH}^2$ at some fixed C^3-hypersurface $\Sigma \in \mathcal{MH}^2$. For this purpose we take a differentiable curve $\Gamma : (-\delta_0, \delta_0) \to \mathcal{MH}^2$ with $\Gamma(0) = \Sigma$. According to Section 2.3.3, there is $\delta \in (0, \delta_0)$ such that for each $t \in (-\delta, \delta)$ we find a normal parameterization $\rho(t) \in C^2(\Sigma, \mathbb{R})$ of $\Gamma(t)$. Then in these coordinates we have

$$v := \frac{d}{dt}\Gamma(0) = \frac{d}{dt}\rho(0)\nu_\Sigma \in C^2(\Sigma, T_\Sigma^\perp\mathcal{MH}^2).$$

On the other hand, if $v = \rho\nu_\Sigma$ is a normal field on Σ with $\rho \in X_\Sigma$ we obtain a curve $\Gamma : (-\delta, \delta) \to \mathcal{MH}^2$ by means of $\Gamma(t)(p) = p + t\rho(p)\nu_\Sigma(p)$. Clearly, $\Gamma(0) = \Sigma$ and $\rho\nu_\Sigma = \frac{d}{dt}\Gamma(0) \in T_\Sigma\mathcal{MH}^2$. In other words, the tangent space $T_\Sigma\mathcal{MH}^2$ consists of all normal fields v on Σ which are of class C^2.

There is one shortcoming with this approach, namely the need to require that $\Sigma \in C^3$. This is due to the fact that we are losing one derivative when forming the normal ν_Σ. However, since we may approximate a given hypersurface of class C^2 by a real analytic one in the second normal bundle, this defect can be avoided by only parameterizing over real analytic hypersurfaces, which will be sufficient below.

4.2 Compact Hypersurfaces with Uniform Ball Condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider a closed compact connected C^2-hypersurface $\Gamma \subset \Omega$. This hypersurface separates Ω into two disjoint open connected sets Ω_1 and Ω_2, the interior and the exterior of Γ w.r.t. Ω. By means of the level function φ_Γ of Γ we have $\Omega_1 = \varphi_\Gamma^{-1}(-\infty, 0)$ and $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. Then $\partial\Omega_1 = \Gamma$ and $\partial\Omega_2 = \partial\Omega \cup \Gamma$.

The hypersurface Γ satisfies the *ball condition*, i.e., there is a radius $r > 0$ such that for each point $p \in \Gamma$ there are balls $B(x_i, r) \subset \Omega_i$ such that $\Gamma \cap \bar{B}(x_i, r) = \{p\}$. The set of hypersurfaces of class C^2 contained in Ω satisfying the ball condition with radius $r > 0$ will be denoted by $\mathcal{MH}^2(\Omega, r)$. Note that hypersurfaces in this class have uniformly bounded principal curvatures.

The elements of $\mathcal{MH}^2(\Omega, r)$ have a tubular neighbourhood of width a larger than r. Therefore the construction of the level function φ_Γ of Γ from Section 2.3.2 can be carried out with the same a and the same cut-off function χ for each $\Gamma \in \mathcal{MH}^2(\Omega, r)$. More precisely, we have

$$\varphi_\Gamma(x) = g(d_\Gamma(x)), \quad x \in \Omega,$$

with

$$g(s) = s\chi(3s/a) + \operatorname{sgn}(s)(1 - \chi(3s/a)), \quad s \in \mathbb{R};$$

note that g is strictly increasing and equals ± 1 for $\pm s > 2a/3$. This induces an injective map

$$\Phi : \mathcal{MH}^2(\Omega, r) \to C^2(\bar{\Omega}), \quad \Phi(\Gamma) := \varphi_\Gamma. \tag{2.71}$$

Φ is in fact a homeomorphism of $\mathcal{MH}^2(\Omega, r)$ onto $\Phi(\mathcal{MH}^2(\Omega, r)) \subset C^2(\bar{\Omega})$.

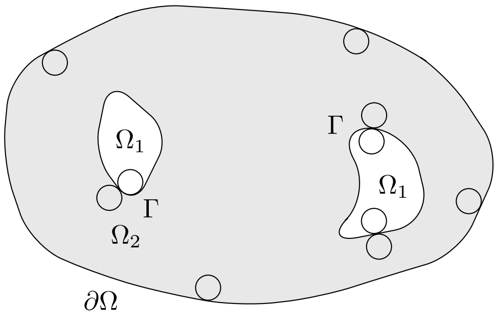

Figure 2.3: Illustration of the ball condition.

This can be seen as follows. Let $\varepsilon > 0$ be small enough. If $|\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{2,\infty} \le \varepsilon$, then $d_{\Gamma_1}(x) \le \varepsilon$ on Γ_2 and $d_{\Gamma_2}(x) \le \varepsilon$ on Γ_1, which implies $d_H(\Gamma_1, \Gamma_2) \le \varepsilon$. Moreover, we also have $|\nabla_x \varphi_{\Gamma_1}(x) - \nu_{\Gamma_2}(x)| \le \varepsilon$ on Γ_2 and $|\nabla_x \varphi_{\Gamma_2}(x) - \nu_{\Gamma_1}(x)| \le \varepsilon$ on Γ_1 which yields $d_H(\mathcal{N}\Gamma_1, \mathcal{N}\Gamma_2) \le C_0\varepsilon$. Then the hypersurfaces Γ_j can both be parameterized over a C^3-hypersurface Σ, and therefore $d_H(\mathcal{N}^2\Gamma_1, \mathcal{N}^2\Gamma_2) \le \varepsilon$ if and only if

$$|\rho_1 - \rho_2|_\infty + |\nabla_\Sigma(\rho_1 - \rho_2)|_\infty + |\nabla_\Sigma^2(\rho_1 - \rho_2)|_\infty \le C_1\varepsilon.$$

This in turn is equivalent to $|\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{2,\infty} \le C_2\varepsilon$.

Let $s - (n-1)/p > 2$. For $\Gamma \in \mathcal{MH}^2(\Omega, r)$ we then define

$$\Gamma \in W_p^s(\Omega, r) \quad \text{if} \quad \varphi_\Gamma \in W_p^s(\Omega), \tag{2.72}$$

and

$$\operatorname{dist}_{W_p^s(\Omega,r)}(\Gamma_1, \Gamma_2) := |\varphi_{\Gamma_1} - \varphi_{\Gamma_2}|_{W_p^s(\Omega)}. \tag{2.73}$$

In this case the local charts for Γ can be chosen of class W_p^s as well. A subset $A \subset W_p^s(\Omega, r)$ is said to be (relatively) compact, if $\Phi(A) \subset W_p^s(\Omega)$ is (relatively) compact. In particular, it follows from Rellich's theorem that $W_p^s(\Omega, r)$ is a compact subset of $W_q^\sigma(\Omega, r)$, whenever $s - n/p > \sigma - n/q$, and $s > \sigma$.

2.5 Moving Hypersurfaces and Domains

In this section we consider the situation of moving hypersurfaces, that is, hypersurfaces that are time dependent. We first introduce the notion of normal velocity, and we then prove a transport theorem for moving surfaces. A special case is the well-known formula for the change of surface area. In addition, we prove a transport theorem for moving domains, and derive the change of volume formula.

5.1 Moving Hypersurfaces

Let $\{\Gamma(t) : t \in I\}$ be a family of compact connected closed C^2-hypersurfaces in

$\mathbb{R}^n$ bounding domains $\Omega(t) \subset \mathbb{R}^n$, with $I \subset \mathbb{R}$ an open interval. In the following, we write $\nu_\Gamma(t,\cdot)$, $\kappa_\Gamma(t,\cdot)$, and $L_\Gamma(t,\cdot)$ for the unit normal, the mean curvature and the Weingarten tensor of $\Gamma(t)$, respectively. Let

$$\mathcal{M} = \bigcup_{t\in I} (\{t\} \times \Gamma(t)) . \tag{2.74}$$

By definition, $\mathcal{M}$ is of class $C^{1,2}$ if it is a C^1-hypersurface in $\mathbb{R}^{n+1}$ and, moreover, $\nu_\Gamma \in C^1(\mathcal{M}, \mathbb{R}^n)$.

We now show that for every $t_0 \in I$ there is a closed, compact, analytic hypersurface Σ, an interval $I_0 := (t_0 - \delta, t_0 + \delta) \subset I$ and a function $\rho : I_0 \times \Sigma \to \mathbb{R}$ with

$$\rho \in C^1(I_0 \times \Sigma), \quad \nabla_\Sigma \rho \in C^1(I_0 \times \Sigma, \mathbb{R}^n) \tag{2.75}$$

such that

$$\Gamma(t) = \{\xi + \rho(t,\xi)\nu_\Sigma(\xi) : t \in I_0,\ \xi \in \Sigma\}. \tag{2.76}$$

This is obtained as follows. Let $t_0 \in I$ be fixed. The assumption that $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{n+1}$ implies that for every $\varepsilon > 0$ there exits $\delta > 0$ such that $d_H(\Gamma(t_0), \Gamma(t)) \le \varepsilon$ whenever $|t - t_0| \le \delta$. In order to prove the assertion, it suffices to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathrm{dist}(p, \Gamma(t_0)) \le \varepsilon \quad \text{for all } p \in \Gamma(t) \text{ and all } |t - t_0| \le \delta.$$

Suppose the latter assertion is not true. Then there exists $a > 0$, a sequence $(p_n)_{n\in\mathbb{N}}$ in $\Gamma(t)$, and a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to t_0$ such that $\mathrm{dist}(p_n, \Gamma(t_n)) \ge 2a$ for all $n \in \mathbb{N}$. As $\Gamma(t_0)$ is compact, we find $p \in \Gamma(t_0)$ and a subsequence of $(p_n)_{n\in\mathbb{N}}$, again denoted by $(p_n)_{n\in\mathbb{N}}$, such that $p_n \to p$. Therefore, $\mathrm{dist}(p, \Gamma(t_n)) \ge a$ for $n \ge N$, with N sufficiently large. This shows that $\big(\{t_n\} \times \Gamma(t_n)\big) \cap \big(\mathbb{R} \times B_{\mathbb{R}^n}(p,a)\big) = \emptyset$ for $n \ge N$, contradicting the assumption that $\mathcal{M}$ is a manifold. As ν_Γ is continuous on $\mathcal{M}$ we conclude that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_H(\mathcal{N}\Gamma(t), \mathcal{N}\Gamma(t_0)) \le \varepsilon, \quad \text{whenever } |t - t_0| \le \delta.$$

According to the approximation result in Section 2.3.4 we can find an analytic hypersurface Σ which approximates $\Gamma(t_0)$. We can assume that $\Gamma(t) \subset U_{\varepsilon/2}(\Sigma)$ for $t \in I_0$, that is, $\Gamma(t)$ is contained in the tubular neighbourhood $U_{\varepsilon/2}(\Sigma)$ of Σ of width $\varepsilon/2$. By Section 2.3.3, for every $t \in I_0$ there exists a function $\rho(t,\cdot) \in C^2(\Sigma)$ such that (2.76) holds. It remains to show that ρ satisfies the regularity assumptions claimed in (2.75). In order to see this, let us consider the mapping

$$\hat{\Pi}_\Sigma : \mathcal{M}(I_0) \to I_0 \times \Sigma, \quad \hat{\Pi}_\Sigma(t,p) = (t, \Pi_\Sigma(p)), \text{ where } \mathcal{M}(I_0) := \bigcup_{t\in I_0} \big(\{t\} \times \Gamma(t)\big\}.$$

We note that $\hat{\Pi}_\Sigma$ is well-defined, as $\Gamma(t) \subset U_{\varepsilon/2}(\Sigma)$ for each $t \in I_0$. Moreover, we have

$$\hat{\Pi}_\Sigma \in C^1(\mathcal{M}(I_0), I_0 \times \Sigma), \quad \hat{\Pi}_\Sigma(t,\cdot) = \Pi_\Sigma|_{\Gamma(t)}.$$

An inspection of the proof in Section 2.3.3 shows that

$$\hat{\Pi}_\Sigma \in \mathrm{Diff}^1(\mathcal{M}(I_0), I_0 \times \Sigma), \quad (\hat{\Pi}_\Sigma)^{-1}(t,\xi) = \Phi(t,\xi) := (t, \xi + \rho(t,\xi)\nu_\Sigma(\xi)).$$

This yields, in particular, $\rho \in C^1(I_0 \times \Sigma)$ and it remains to show the additional regularity claimed in (2.75). We recall from (2.44) that

$$\nu_\Gamma(\Phi(t,\xi)) = \big(\beta(\rho)(\nu_\Sigma - M_0(\rho)\nabla_\Sigma \rho)\big)(t,\xi), \quad (t,\xi) \in I_0 \times \Sigma. \tag{2.77}$$

This representation, in conjuction with the regularity $\rho \in C^1(I_0 \times \Sigma)$ already established, implies that

$$\nu_\Gamma \in C^1(\mathcal{M}(I_0), \mathbb{R}^n) \iff \nabla_\Sigma \rho \in C^1(I_0 \times \Sigma, \mathbb{R}^n),$$

as we will see next. Clearly, $\nu_\Gamma \in C^1(\mathcal{M}(I_0), \mathbb{R}^n)$ iff $\nu_\Gamma \circ \Phi \in C^1(I_0 \times \Sigma, \mathbb{R}^n)$.

Suppose that $\nu_\Gamma \in C^1(\mathcal{M}(I_0), \mathbb{R}^n)$. Thanks to $\beta(\rho)(t,\xi) = (\nu_\Gamma(\Phi(t,\xi)) \mid \nu_\Sigma(\xi))$ we have $\beta(\rho) \in C^1(I_0 \times \Sigma)$ and this, in turn, implies

$$\nabla_\Sigma \rho = (I - \rho L_\Sigma)\big(\nu_\Sigma - (1/\beta(\rho))(\nu_\Gamma \circ \Phi)\big) \in C^1(I_0 \times \Sigma).$$

On the other hand, if ρ satisfies the regularity assumptions in (2.75) and the family $\{\Gamma(t) : t \in I_0\}$ is given by (2.76), then it is not difficult to verify that $\mathcal{M}(I_0)$ is a hypersurface of class $C^{1,2}$.

We now state a useful variant of (2.76). The result reads as follows: for every fixed $t \in I$ there exists a number $\delta > 0$ and a function $\rho \in C^1((-\delta,\delta) \times \Sigma)$, where $\Sigma = \Gamma(t)$, such that

$$\Gamma(t+s) = \{p + \rho(s,p)\nu_\Sigma(p) : s \in (-\delta,\delta),\ p \in \Sigma\}, \quad \Sigma := \Gamma(t). \tag{2.78}$$

This follows by an obvious modification of the arguments given above. In fact, the proof is less involved, as there is no need to generate a smooth approximation for $\Gamma(t)$.

5.2 The Normal Velocity

Let $\mathcal{M}$ be as above. Suppose I_0 is a subinterval of I and $\gamma : I_0 \to \mathbb{R}^n$ is a C^1-curve. Then γ is called a C^1*-curve on* $\mathcal{M}$ if $\gamma(t) \in \Gamma(t)$ for each $t \in I_0$. Hence, γ is a C^1-curve on $\mathcal{M}$ iff $(t, \gamma(t)) \in \mathcal{M}$ for $t \in I_0$. If γ is C^1-curve on $\mathcal{M}$, then

$$V_\Gamma(t,p) := (\gamma'(t) | \nu_\Gamma(t,p)), \quad p = \gamma(t), \tag{2.79}$$

is called the *normal velocity* of $\{\Gamma(t) : t \in I\}$ at the point (t,p). The normal velocity V_Γ is well-defined, that is, $V_\Gamma(t,p)$ does not depend on the choice of a C^1-curve on $\mathcal{M}$ through $p \in \Gamma(t)$. Indeed, let $\gamma : I_0 \to \mathbb{R}^n$ be an arbitrary C^1-curve on $\mathcal{M}$ and let $p = \gamma(t)$. We can assume, by possibly shrinking I_0, that the representation (2.76) holds. Therefore, the curve γ can be expressed by

$$\gamma(t) = \xi(t) + \rho(t, \xi(t))\nu_\Sigma(\xi(t)), \quad t \in I_0, \quad \xi(t) \in \Sigma,$$

and hence,

$$\gamma'(t) = (I - \rho(t,\xi(t))L_\Sigma(\xi(t)))\xi'(t) + (\partial_t\rho(t,\xi(t)) + (\nabla_\Sigma\rho(t,\xi(t))|\xi'(t))\nu_\Sigma(\xi(t)).$$

Using (2.77), and suppressing the variables, we obtain

$$V_\Gamma = (\gamma'|\nu_\Gamma) = \beta(\rho)\{\partial_t\rho + (\nabla_\Sigma\rho|\xi') - ((I - \rho L_\Sigma)\xi'|M_0(\rho)\nabla_\Sigma\rho)\} = \beta(\rho)\partial_t\rho,$$

or in more precise notation, $V_\Gamma(t,p) = (V_\Gamma \circ \Phi)(t,\xi) = \beta(\rho(t))(\xi)\partial_t\rho(t,\xi)$. This expression does not refer to the curve γ, and this shows that the definition (2.79) is independent of a particular curve. Moreover, this also shows that we can, alternatively, define the normal velocity by

$$V_\Gamma = \beta(\rho)\partial_t\rho, \tag{2.80}$$

provided $\{\Gamma(t) : t \in I_0\}$ is represented by (2.76), which can always be assumed.

For later use we note that

$$[1, V_\Gamma\nu_\Gamma]^\mathsf{T} \in T_{(t,p)}\mathcal{M}, \tag{2.81}$$

i.e., $[1,(V_\Gamma\nu_\Gamma)(t,p)]^\mathsf{T}$ is a tangent vector for $\mathcal{M}$ at the point (t,p). This can be seen as follows. Suppose $\gamma : I_0 \to \mathbb{R}^n$ is a C^1-curve on $\mathcal{M}$. Then $(t,\gamma(t)) \in \mathcal{M}$ for $t \in I_0$ and consequently, $[1,\gamma'(t)]^\mathsf{T} \in T_{(t,p)}\mathcal{M}$ with $p = \gamma(t)$. Hence, by (2.79),

$$\begin{aligned}[1,(V_\Gamma\nu_\Gamma)(t,p)]^\mathsf{T} &= [1,(\gamma'(t)|\nu_\Gamma(t,p))\nu_\Gamma(t,p)]^\mathsf{T}\\ &= [1,\gamma'(t)]^\mathsf{T} - [0,\mathcal{P}_{\Gamma(t)}(p)\gamma'(t)]^\mathsf{T} \in T_{(t,p)}\mathcal{M},\end{aligned}$$

as $[0,v]^\mathsf{T} \in T_{(t,p)}\mathcal{M}$ for any vector $v \in T_p\Gamma(t)$.

5.3 The Lagrange Derivative for Moving Surfaces

Suppose that $u_\Gamma(t,\cdot) := u_{\Gamma(t)}(\cdot) : \Gamma(t) \to \mathbb{R}^n$ is a vector field for each $t \in I$. Hence u_Γ is defined on $\mathcal{M}$ and we assume that $u_\Gamma \in C^1(\mathcal{M},\mathbb{R}^n)$. Then u_Γ is called a C^1-*velocity field* for the family $\{\Gamma(t) : t \in I\}$ if

$$V_\Gamma = (u_\Gamma|\nu_\Gamma), \tag{2.82}$$

or more precisely, if $V_\Gamma(t,p) = (u_\Gamma(t,p)|\nu_\Gamma(t,p))$ for $(t,p) \in \mathcal{M}$.

A velocity field u_Γ is called a *normal velocity field* for $\{\Gamma(t) : t \in I\}$ if $u_\Gamma(t,\cdot) \in T^\perp\Gamma(t)$, i.e., $u_\Gamma(t,\cdot)$ lies in the normal bundle of $\Gamma(t)$ for each $t \in I$. Hence,

$$u_\Gamma \text{ is a normal velocity field} \iff u_\Gamma = V_\Gamma\nu_\Gamma. \tag{2.83}$$

Although only normal velocity fields matter from a geometric point of view, we nevertheless need to consider general velocity fields in order to treat the motion of fluid particles in fluid flows subject to phase transitions.

We note that if u_Γ is a velocity field for $\{\Gamma(t) : t \in I\}$ then

$$[1, u_\Gamma]^\mathsf{T} \in T\mathcal{M}. \tag{2.84}$$

This can be deduced from (2.81), (2.82), and the decomposition

$$[1, u_\Gamma] = [1, (u_\Gamma|\nu_\Gamma)\nu_\Gamma] + [0, \mathcal{P}_\Gamma u_\Gamma] = [1, V_\Gamma \nu_\Gamma] + [0, \mathcal{P}_\Gamma u_\Gamma],$$

where, as before, we use the fact that $[0, v]^\mathsf{T} \in T\mathcal{M}$ for any vector $v \in T\Gamma(t)$.

Next we show that for every C^1-velocity field u_Γ and every $p \in \Gamma(t)$, with t fixed, there exists $\delta > 0$ and a unique C^1-curve $[s \mapsto x(t+s)] : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\begin{aligned} \frac{d}{ds} x(t+s) &= u_\Gamma(t+s, x(t+s)), \quad x(t+s) \in \Gamma(t+s), \;\; s \in (-\delta, \delta), \\ x(t) &= p. \end{aligned} \tag{2.85}$$

The solution to (2.85), in the sequel denoted by $x(t+s, t, p)$, is then called a *trajectory* or a *flow line* on $\mathcal{M}$ through $p \in \Gamma(t)$, generated by the velocity field u_Γ. The existence of such a trajectory can be seen by the following argument. Setting

$$z(s) := [t+s, x(t+s)]^\mathsf{T}$$

we see that $x(t+s) \in \Gamma(t+s)$ is equivalent to $z(s) \in \mathcal{M}$ for $s \in (-\delta, \delta)$. Therefore, (2.85) has a (unique) solution if and only if the differential equation

$$\dot{z}(s) = [1, u_\Gamma(z(s))]^\mathsf{T}, \quad s \in (-\delta, \delta), \quad z(0) = (t, p), \tag{2.86}$$

has a (unique) solution. Existence and uniqueness of a solution $z(s) = z(s, (t, p))$ to (2.86) follows from the fact that the vector field $[1, u_\Gamma]^\mathsf{T}$ is tangential to $\mathcal{M}$, see (2.84), and well-known results from the theory of differential equations. Moreover, we conclude that

$$[(s, (t, p)) \mapsto z(s, (t, p))] \in C^1((-\delta, \delta) \times \mathcal{M}, \mathcal{M}),$$

and this implies

$$[(s, p) \mapsto x(t+s, t, p)] \in C^1((-\delta, \delta) \times \Gamma(t), \Gamma(t)).$$

We note that

$$u_\Gamma \text{ is a } C^1\text{-velocity field} :\Longleftrightarrow V_\Gamma = (u_\Gamma|\nu_\Gamma) \Longleftrightarrow [1, u_\Gamma]^\mathsf{T} \in T\mathcal{M}. \tag{2.87}$$

The first equivalence follows by definition, while the second implication " $\Rightarrow$ " has been shown above. Suppose that $[1, u_\Gamma]^\mathsf{T} \in T\mathcal{M}$. Then (2.85) admits a C^1-solution $[s \mapsto x(t+s, t, p)]$, and the definition of V_Γ in (2.79) implies

$$V_\Gamma(t, p) = \Big(\frac{d}{ds} x(t+s, t, p)\big|_{s=0} \;\big|\; \nu_\Gamma(t, p) \Big) = (u_\Gamma(t, p)|\nu_\Gamma(t, p)).$$

It is illustrative to point out an alternative way to establish existence of solutions to (2.85). By (2.76) we can assume that $\{\Gamma(t+s) : s \in (-\delta,\delta)\}$ is given by

$$\Gamma(t+s) = \{\xi + \rho(s,\xi)\nu_\Sigma(\xi) : s \in (-\delta,\delta),\ \xi \in \Sigma\},$$

where Σ is a smooth hypersurface. Then the curve $x(s) = \xi(s)+\rho(s,\xi(s))\nu_\Sigma(\xi(s))$, with $\xi(s) \in \Sigma$, satisfies (2.76) if and only if

$$\begin{aligned} \xi'(s) &= (I-\rho L_\Sigma)^{-1}\mathcal{P}_\Sigma(\xi(s))u_\Gamma(s,\xi(s)+\rho(s,\xi(s))\nu_\Sigma(\xi(s))) \\ \xi(t) &= \xi_0, \end{aligned} \tag{2.88}$$

where $(I-\rho L_\Sigma)$ is the short form for $(I-\rho(s,\xi(s))L_\Sigma(\xi(s)))$. Indeed, applying the projection $\mathcal{P}_\Sigma$ to the equation

$$(I-\rho L_\Sigma)\xi'(s) + [\partial_s\rho(s,\xi(s)) + (\nabla_\Sigma\rho(s,\xi(s))|\xi'(s))]\nu_\Sigma(\xi(s)) = u_\Gamma(s,x(s))$$

yields (2.88), while the projection onto $T^\perp\Sigma$ trivializes, i.e., we automatically have

$$\partial_s\rho(s,\xi(s)) + (\nabla_\Sigma\rho(s,\xi(s))|\xi'(s)) = (u_\Gamma(s,x(s))|\nu_\Sigma(\xi(s))).$$

The last assertion follows from

$$\begin{aligned} &\beta(\rho(s))(u_\Gamma(s,x(s))|\nu_\Sigma(\xi(s))) \\ &\quad = (u_\Gamma(s,x(s))|\nu_\Gamma(s,x(s))) + \beta(\rho(s))(u_\Gamma(s,x(s))|M_0(\rho(s))\nabla_\Sigma\rho(s,\xi(s))) \\ &\quad = \beta(\rho(s))[\partial_s\rho(s,\xi(s)) + (M_0(\rho(s))\mathcal{P}_\Sigma(\xi(s))u_\Gamma(s,x(s))|\nabla_\Sigma\rho(s,\xi(s))) \\ &\quad = \beta(\rho(s))[\partial_s\rho(s,\xi(s)) + (\xi'(s)|\nabla_\Sigma\rho(s,\xi(s))), \end{aligned}$$

where we employed (2.77), (2.80) and (2.88). It remains to observe that the ordinary differential equation (2.88), defined on Σ, admits a unique solution as $(I-\rho L_\Sigma)^{-1}\mathcal{P}_\Sigma u_\Gamma \in T\Sigma$.

Suppose that u_Γ is a C^1-velocity field for $\{\Gamma(t) : t \in I\}$ and $f_\Gamma \in C^1(\mathcal{M},\mathbb{R})$. Then we define the *Lagrange derivative* of f_Γ (sometimes also called the *material derivative* of f_Γ) with respect to the velocity field u_Γ at the point $(t,p) \in \mathcal{M}$ by

$$\frac{D}{Dt}f_\Gamma(t,p) := \frac{D_{u_\Gamma}}{Dt}f_\Gamma(t,p) := \frac{d}{ds}\ f_\Gamma(t+s,x(t+s,t,p))\Big|_{s=0}$$

where $[s \mapsto x(s+t,t,p)]$ denotes the solution of (2.85). In case u_Γ is a normal C^1-velocity field, in which case $u_\Gamma = V_\Gamma\nu_\Gamma$, the Lagrange derivative is called the *normal derivative*, and we set

$$\frac{D_n}{Dt} := \frac{D_{V_\Gamma\nu_\Gamma}}{Dt}.$$

Then the following relation holds.

$$\frac{D_{u_\Gamma}}{Dt}f_\Gamma(t,p) = \frac{D_n}{Dt}f_\Gamma(t,p) + (u_\Gamma(t,p)|\nabla_{\Gamma(t)})f_\Gamma(t,p). \tag{2.89}$$

In order to see this, let us consider an extension $\tilde{f}_\Gamma$ of f_Γ in an open neighbourhood of $\mathcal{M}$ in $\mathbb{R}^{n+1}$. Such an extension can, for instance, be obtained on the neighbourhood

$$\mathcal{U}_a(\mathcal{M}) := \bigcup_{t\in I} \big(\{t\} \times U_a(\Gamma(t))\big),$$

where $U_a(\Gamma(t))$ is a tubular neighbourhood of $\Gamma(t)$ of with a, by setting

$$\tilde{f}_\Gamma(t,x) := f_\Gamma(t,p), \quad (t,x) \in \mathcal{U}_a(\mathcal{M}), \quad p = \Pi_{\Gamma(t)}(x).$$

Then one obtains

$$\frac{D_{u_\Gamma}}{Dt} f_\Gamma(t,p) = \partial_t \tilde{f}_\Gamma(t,p) + (u_\Gamma(t,p)|\nabla_x)\tilde{f}_\Gamma(t,p). \tag{2.90}$$

By the same argument one has

$$\begin{aligned}\frac{D_n}{Dt} f_\Gamma(t,p) &= \frac{d}{ds} f_\Gamma(t+s, y(t+s,t,p))\Big|_{s=0} = \frac{d}{ds}\tilde{f}_\Gamma(t+s, y(t+s,t,p))\Big|_{s=0} \\ &= \partial_t \tilde{f}_\Gamma(t,p) + V_\Gamma(t,p)(\nu_\Gamma(t,p)|\nabla_x)\tilde{f}_\Gamma(t,p),\end{aligned}$$

where $y(\cdot)$ is the solution of (2.85) with respect to the normal velocity field $V_\Gamma \nu_\Gamma$. Using the relation

$$\nabla_x \tilde{f}_\Gamma = (\nabla_x \tilde{f}_\Gamma|\nu_\Gamma)\nu_\Gamma + \mathcal{P}_\Gamma \nabla_x \tilde{f}_\Gamma = (\nabla_x \tilde{f}_\Gamma|\nu_\Gamma)\nu_\Gamma + \nabla_\Gamma f_\Gamma,$$

see (2.20), we conclude with (2.82)

$$\begin{aligned}\frac{D_{u_\Gamma}}{Dt} f_\Gamma(t,p) &= \partial_t \tilde{f}_\Gamma(t,p) + V_\Gamma(t,p)(\nu_\Gamma(t,p)|\nabla_x)\tilde{f}_\Gamma(t,p) + (u_\Gamma(t,p)|\nabla_{\Gamma(t)}) f_\Gamma(t,p) \\ &= \frac{D_n}{Dt} f_\Gamma(t,p) + (u_\Gamma(t,p)|\nabla_{\Gamma(t)}) f_\Gamma(t,p).\end{aligned}$$

5.4 The Transport Theorem for Moving Hypersurfaces
Suppose u_Γ is a C^1-velocity field for $\{\Gamma(t) : t \in I\}$ and $f_\Gamma \in C^1(\mathcal{M}, \mathbb{R}^n)$. The *transport theorem for moving surfaces* states that

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma(t)} f_\Gamma(t,x)\, d\Gamma &= \int_{\Gamma(t)} \Big[\frac{D_{u_\Gamma}}{Dt} f_\Gamma(t,x) + f_\Gamma(t,x)\,\mathrm{div}_\Gamma u_\Gamma(t,x)\Big]\, d\Gamma \\ &= \int_{\Gamma(t)} \Big[\frac{D_n}{Dt} f_\Gamma(t,x) - f_\Gamma(t,x)\kappa_\Gamma(t,x)V_\Gamma(t,x)\Big]\, d\Gamma.\end{aligned} \tag{2.91}$$

Proof. Let $(t,p) \in \mathcal{M}$ be fixed let $\phi(t,\cdot) : \Theta \subset \mathbb{R}^{n-1} \to \Gamma(t)$ be a sufficiently smooth parameterization of an open neighbourhood of p in $\Gamma(t)$. Then

$$\phi(t+s,\cdot) := x(t+s,t,\phi(t,\cdot)) : \Theta \to \Gamma(t+s), \quad s \in (-\delta,\delta),$$

defines a C^1-parameterization of a neighbourhood of $x(t+s,t,p)$ in $\Gamma(t+s)$. We first suppose that $\operatorname{supp} f_\Gamma \subset\subset U := \{\phi(t+s,\theta) : (s,\theta) \in (-\delta,\delta)\times\Theta\}$. Let

$$g_{ij}(t+s,\theta) := (\partial_i\phi(t+s,\theta) \mid \partial_j\phi(t+s,\theta)), \quad G(t+s,\theta) := [g_{ij}(t+s,\theta)].$$

Hence, $G(t+s,\theta)$ is the fundamental matrix of $\Gamma(t+s)$ with respect to the parameterization $\phi(t+s,\cdot)$. With $g(t+s,\cdot) := \det G(t+s,\cdot)$ we obtain

$$\int_{\Gamma(t+s)} f_\Gamma(t+s,y)\,d\Gamma = \int_\Theta f_\Gamma(t+s,\phi(t+s,\theta))\sqrt{g(t+s,\theta)}\,d\theta,$$

and hence

$$\begin{aligned}&\frac{d}{ds}\int_{\Gamma(t+s)} f_\Gamma(t+s,y)\,d\Gamma\,\Big|_{s=0}\\ &\quad= \int_\Theta \Big(\frac{D}{Dt}f_\Gamma(t,\phi(t,\theta))\Big)\sqrt{g(t,\theta)} + f_\Gamma(t,\phi(t,\theta))\frac{\partial}{\partial s}\sqrt{g(t+s,\theta)}\,\Big|_{s=0}\,d\theta.\end{aligned}$$

As in (2.27) we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\sqrt{g(t+s,\theta)} &= \frac{1}{2\sqrt{g(t+s,\theta)}}\,\frac{\partial}{\partial s}g(t+s,\theta)\\ &= \frac{1}{2}\sqrt{g(t+s,\theta)}g^{ij}(t+s,\theta)\frac{\partial}{\partial s}g_{ij}(t+s,\theta).\end{aligned}$$

From

$$\partial_s\partial_i x(t+s,t,\phi(t,\theta)) = \partial_i\partial_s x(t+s,t,\phi(t,\theta)) = \partial_i u_\Gamma(t+s,x(t+s,t,\phi(t,\theta)))$$

follows

$$\begin{aligned}&\frac{1}{2}g^{ij}(t+s,\theta)\frac{\partial}{\partial s}g_{ij}(t+s,\theta)\Big|_{s=0}\\ &= \frac{1}{2}g^{ij}(t,\theta)\,[(\partial_i u_\Gamma(t,\phi(t,\theta)) \mid \partial_j\phi(t,\theta)) + (\partial_i\phi(t,\theta) \mid \partial_j u_\Gamma(t,\phi(t,\theta)))]\\ &= \frac{1}{2}g^{ij}(t,\theta)\,\Big[\big(\partial_i u_\Gamma(t,\phi(t,\theta)) \mid \tau_j^{\Gamma(t)}(\phi(t,\theta))\big) + \big(\tau_i^{\Gamma(t)}(\phi(t,\theta)) \mid \partial_j u_\Gamma(t,\phi(t,\theta))\big)\Big]\\ &= \frac{1}{2}\Big[\big(\partial_i u_\Gamma(t,\phi(t,\theta)) \mid \tau^i_{\Gamma(t)}(\phi(t,\theta))\big) + \big(\tau^j_{\Gamma(t)}(\phi(t,\theta)) \mid \partial_j u_\Gamma(t,\phi(t,\theta))\big)\Big]\\ &= \operatorname{div}_{\Gamma(t)} u_\Gamma(t,\phi(t,\theta)).\end{aligned}$$

Combining all steps yields

$$\begin{aligned}&\frac{d}{ds}\int_{\Gamma(t+s)} f_\Gamma(t+s,y)\,d\Gamma\Big|_{s=0}\\ &\quad= \int_\Theta \Big[\frac{D}{Dt}f_\Gamma(t,\phi(t,\theta)) + f_\Gamma(t,\phi(t,\theta))\operatorname{div}_{\Gamma(t)}u_\Gamma(t,\phi(t,\theta))\Big]\sqrt{g(t,\theta)}\,d\theta\\ &\quad= \int_{\Gamma(t)} \Big[\frac{D}{Dt}f_\Gamma(t,y) + f_\Gamma(t,y)\operatorname{div}_{\Gamma(t)}u_\Gamma(t,y)\Big]\,d\Gamma.\end{aligned}$$

For a more general function f_Γ we can apply the result above in conjunction with a partition of unity for $\mathcal{M}$. Hence, we have shown that

$$\frac{d}{dt}\int_{\Gamma(t)} f_\Gamma(t,x)\,d\Gamma = \int_{\Gamma(t)} \Big[\frac{D_{u_\Gamma}}{Dt} f_\Gamma(t,x) + f_\Gamma(t,x)\,\mathrm{div}_\Gamma u_\Gamma(t,x)\Big]\,d\Gamma.$$

The second assertion in (2.91) follows from the surface divergence theorem (2.31) and (2.89). □

We note that (2.91) implies, in particular, the well-known *change of area formula*

$$\frac{d}{dt}|\Gamma(t)| = -\int_{\Gamma(t)} \kappa_\Gamma V_\Gamma\,d\Gamma. \tag{2.92}$$

It is worthwhile to point out that (2.92) can also be derived from (2.50). This can be obtained as follows. Using the representation (2.78) we have $|\Gamma(t+s)| = \Phi(\rho(s))$ for t fixed, and the change of area formula (2.50) in conjunction with the relation $\rho(0) = 0$ immediately yields

$$\frac{d}{ds}|\Gamma(t+s)|\Big|_{s=0} = \langle \Phi'(0), \partial_s\rho(0)\rangle = -\int_\Sigma \kappa_\Sigma \partial_s\rho(0)\,d\Sigma = -\int_{\Gamma(t)} \kappa_\Gamma V_\Gamma\,d\Gamma.$$

5.5 The Transport Theorem for Moving Domains

Suppose $\{\Gamma(t) : t\in I\}$ is a family of compact connected closed C^2-hypersurfaces in $\mathbb{R}^n$, bounding domains $\Omega(t)\subset\mathbb{R}^n$. We assume again that

$$\mathcal{M} = \bigcup_{t\in I}\big(\{t\}\times\Gamma(t)\big)$$

is a $C^{1,2}$-hypersurface in $\mathbb{R}^{n+1}$, and we set

$$\mathcal{Q} = \bigcup_{t\in I}\big(\{t\}\times\Omega(t)\big).$$

Let $f\in C^1(\overline{\mathcal{Q}})$. Then we have the *transport theorem for moving domains*:

$$\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega(t)} \partial_t f(t,x)\,dx + \int_{\Gamma(t)} f(t,x)V_\Gamma(t,x)\,d\Gamma. \tag{2.93}$$

Proof. We first show that for each fixed $t\in I$ there exists a family of mappings

$$\Phi(t+s,\cdot) : \overline{\Omega}(t)\to\overline{\Omega}(t+s), \quad s\in(-\delta,\delta),$$

such that

$$\Phi(t+s,\cdot)\in \mathrm{Diff}^1\big(\Omega(t),\Omega(t+s)\big)\cap\mathrm{Diff}^1\big(\Gamma(t),\Gamma(t+s)\big), \quad s\in(-\delta,\delta), \tag{2.94}$$

where $\mathrm{Diff}^1(U,V)$ denotes the set of all C^1-diffeomorphisms from U into V. The mappings $\Phi(t+s,\cdot)$ can, for instance, be constructed as follows. According to (2.78) we know that

$$\phi(t+s,p) := p + \rho(s,p)\nu_\Sigma(p), \quad p \in \Sigma := \Gamma(t), \;\; s \in (-\delta,\delta), \tag{2.95}$$

satisfies $\phi(t+s,\cdot) \in \mathrm{Diff}^1(\Gamma(t),\Gamma(t+s))$. By means of a Hanzawa transform we can extend $\phi(t+s,\cdot)$ to $\overline{\Omega}(t)$ such that (2.94) holds. In more detail, let

$$\Phi(t+s,x) = x + \chi(d_\Sigma(x)/a)\rho(s,\Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \overline{\Omega}(t).$$

Here d_Σ and Π_Σ have the same meaning as in (2.57), and χ is a suitable cut-off function, say $\chi \in \mathcal{D}(\mathbb{R})$, $0 \le \chi \le 1$, $\chi(r) = 1$ for $|r| < 1/3$, and $\chi(r) = 0$ for $|r| > 2/3$.

Clearly, $\Phi(t+s,p) = \phi(t+s,p)$ for $p \in \Gamma(t)$. Since $\Phi(t,\cdot) = \mathrm{id}_{\overline{\Omega}(t)}$ we can assume that $\det \partial_x\Phi(t+s,x) > 0$ for $(s,x) \in (-\delta,\delta) \times \Omega(t)$ by choosing δ small enough. Next we observe that by (2.27)

$$\frac{d}{ds}\det \partial_x\Phi(t+s,x) = \det \partial_x\Phi(t+s,x)\mathrm{tr}([\partial_y\Phi(t+s,x)]^{-1}[\partial_x\partial_s\Phi(t+s,x)]),$$

and therefore,

$$\frac{d}{ds}\det \partial_x\Phi(t+s,x)\Big|_{s=0} = \mathrm{tr}[\partial_x\partial_s\Phi(t,x)] = \mathrm{div}_x\,\partial_s\Phi(t,x).$$

Employing the transformation rule for integrals yields

$$\int_{\Omega(t+s)} f(t+s,y)\,dy = \int_{\Omega(t)} f(t+s,\Phi(t+s,x))\det \partial_x\Phi(t+s,x)\,dx,$$

and hence,

$$\begin{aligned}
&\frac{d}{ds}\int_{\Omega(t+s)} f(t+s,y)\,dy\,\Big|_{s=0}\\
&= \int_{\Omega(t)} \Big[\partial_t f(t,x) + (\nabla_x f(t,x)|\partial_s\Phi(t,x)) + f(t,x)\mathrm{div}_x\,\partial_s\Phi(t,x)\Big]\,dx\\
&= \int_{\Omega(t)} \Big[\partial_t f(t,x) + \mathrm{div}_x\,(f(t,x)\partial_s\Phi(t,x))\Big]\,dx\\
&= \int_{\Omega(t)} \partial_t f(t,x)\,dx + \int_{\Gamma(t)} f(t,x)(\partial_s\Phi(t,x)|\nu_\Gamma(t,x))d\Gamma\\
&= \int_{\Omega(t)} \partial_t f(t,x)\,dx + \int_{\Gamma(t)} f(t,x)V_\Gamma(t,x)d\Gamma,
\end{aligned} \tag{2.96}$$

where we used (2.79) in the last step. This completes the proof. □

The relation (2.93) immediately yields the well-known *change of volume formula*

$$\frac{d}{dt}|\Omega(t)| = \int_{\Gamma(t)} V_\Gamma \, d\Gamma. \tag{2.97}$$

We point out that (2.97) can also be derived from (2.54). Indeed, using once more the representation

$$\Gamma(t+s) = \{p + \rho(s,p)\nu_\Sigma(p) : s \in (-\delta,\delta),\ p \in \Sigma\}, \quad \Sigma := \Gamma(t),$$

we have $|\Omega(t+s)| = \Psi(\rho(s))$, with Ψ the volume functional introduced in Section 2.2.7. Then the first variation formula (2.54) and the relation $\rho(0) = 0$ imply

$$\frac{d}{ds}|\Omega(t+s)|\Big|_{s=0} = \langle \Psi'(0), \partial_s\rho(0)\rangle = \int_\Sigma \partial_s\rho(0)\, d\Sigma = \int_{\Gamma(t)} V_\Gamma \, d\Gamma.$$

We now consider the more special case where the moving domain $\Omega(t)$ is transported by a velocity field u. Suppose then that $J \subset \mathbb{R}$ is an open interval, $G \subset \mathbb{R}^n$ is an open set, and $u \in C^1(J \times \Omega, \mathbb{R}^n)$. We assume that solutions to the ordinary differential equation

$$y'(t) = u(t, y(t)), \quad y(\tau) = \xi,$$

exist on I for all $(\tau,\xi) \in J \times G$, and we denote the unique solution with initial value $y(\tau) = \xi$ by $y(t,\tau,\xi)$. Let $\Omega_0 \subset\subset G$ be a C^2-domain, $t_0 \in I$ a fixed number, and suppose that the family $\{\Omega(t) : t \in I\}$ of moving domains is given by

$$\Omega(t) = y(t,t_0,\cdot)|_{\Omega_0} = \{y(t,t_0,x_0) : x_0 \in \Omega_0\}, \quad t \in I.$$

Suppose that $f \in C^1(J \times G)$. Then the *Reynolds transport theorem* states that

$$\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx = \int_{\Omega(t)} \big[\partial_t f(t,x) + \operatorname{div}_x\left(f(t,x)u(t,x)\right)\big]\,dx. \tag{2.98}$$

Proof. Let $t \in I$ be fixed and let $\Phi(s,x) := y(s,t,x)$ for $(s,x) \in J \times G$. From the theory of ordinary differential equations follows that

$$\Phi(t+s,\cdot) \in \operatorname{Diff}^1\big(\Omega(t), \Omega(t+s)\big), \quad s \in (-\delta,\delta), \tag{2.99}$$

with $\phi^{-1}(t+s,\cdot) = y(t,t+s,\cdot)$. We can now follow the computations in (2.93) to the result

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx &= \int_{\Omega(t)} \Big[\partial_t f(t,x) + \operatorname{div}_x\left(f(t,x)\partial_s\Phi(t,x)\right)\Big]\,dx \\ &= \int_{\Omega(t)} \Big[\partial_t f(t,x) + \operatorname{div}_x\left(f(t,x)u(t,x)\right)\Big]\,dx\end{aligned}$$

and this completes the proof. □

5.6 The Transport Theorem for Two-Phase Moving Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain in $\mathbb{R}^n$ with C^2-boundary $\partial\Omega$. Suppose that $\{\Gamma(t) : t \in I\}$ is a family of closed compact C^2-hypersurfaces with $\Gamma(t) \subset \Omega$, such that $\Gamma(t)$ encloses a region $\Omega_1(t) \subset \Omega$, and such that $\partial\Omega_1(t) = \Gamma(t)$ for each $t \in I$. Let $\Omega_2(t) := \Omega \setminus \bar{\Omega}_1(t)$. Then

$$\overline{\Omega} = \overline{\Omega}_1(t) \cup \overline{\Omega}_2(t), \quad \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t) = \Gamma(t), \quad \partial\Omega_2(t) = \Gamma(t) \cup \partial\Omega, \quad t \in I.$$

Hence, $\Gamma(t)$ separates Ω into an 'inner' region $\Omega_1(t)$ and an 'outer' region $\Omega_2(t)$, with $\Omega_2(t)$ being in contact with the boundary $\partial\Omega$. Then $\nu_{\Gamma(t)}$ denotes the outward pointing unit normal field for $\Omega_1(t)$ on $\Gamma(t)$. Let

$$\mathcal{Q}_j = \bigcup_{t\in I} \big(\{t\} \times \Omega_j(t)\big), \quad j = 1, 2.$$

As above, we assume that $\mathcal{M}$ is a $C^{1,2}$-hypersurface. Let $f_j : \mathcal{Q}_j \to \mathbb{R}$ be given. Then we set

$$f(t,x) := \begin{cases} f_1(t,x), & x \in \Omega_1(t), \\ f_2(t,x), & x \in \Omega_2(t), \end{cases}$$

so that $f : \mathcal{Q}_1 \cup \mathcal{Q}_2 \to \mathbb{R}$. In case f_j admits a continuous extension $\bar{f}_j \in C(\overline{\mathcal{Q}}_j)$ we define the jump of f across $\Gamma(t)$ by means of

$$[\![f(t,p)]\!] := \bar{f}_2(t,p) - \bar{f}_1(t,p), \quad p \in \Gamma(t). \tag{2.100}$$

Suppose that the functions f_j admit extensions $\bar{f}_j \in C^1(\overline{\mathcal{Q}}_j)$, $j = 1, 2$. Then the *transport theorem for two-phase moving domains* states that

$$\frac{d}{dt}\int_{\Omega\setminus\Gamma(t)} f(t,x)\,dx = \int_{\Omega\setminus\Gamma(t)} \partial_t f(t,x)\,dx - \int_{\Gamma(t)} [\![f(t,x)]\!] V_\Gamma(t,x)\,d\Gamma. \tag{2.101}$$

Proof. Let $t \in I$ be fixed. As in the proof of (2.93) we extend the family of diffeormorphisms $\phi(t+s,\cdot) : \Gamma(t) \to \Gamma(t+s)$ given in (2.95) by means of

$$\Phi(t+s,x) = x + \chi(d_\Sigma(x)/a)\rho(s,\Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \overline{\Omega},$$

to a family of diffeomorphisms $\Phi(t+s,\cdot) : \Omega \to \Omega$ such that

$$\Phi_j(t+s,\cdot) := \Phi(t+s,\cdot)|_{\Omega_j(t+s)} \in \mathrm{Diff}^1\big(\Omega_j(t), \Omega_j(t+s)\big), \quad j = 1, 2, \;\; s \in (-\delta, \delta).$$

By choosing a small enough we can assume that a tubular neighbourhood of $\Gamma(t)$ of width a is contained in Ω, and hence that $\Phi(t+s,\cdot) = \mathrm{id}_{\mathbb{R}^n}$ in a neighbourhood of $\partial\Omega$. We can now proceed as in the proof of (2.93) to obtain

$$\frac{d}{dt}\int_{\Omega_j(t)} f(t,x)\,dx = \int_{\Omega_j(t)} \partial_t f(t,x)\,dx - \int_{\Gamma(t)} (-1)^j \bar{f}_j(t,x) V_\Gamma(t,x)\,d\Gamma,$$

and (2.101) then follows from (2.100). ∎