Chapter 1

Problems and Strategies

The purpose of this introductory chapter is to explain the problems to be considered in the main part of this book in some detail. We derive their physical origin from first principles, discuss some of the main structural properties of the models, and describe the strategies of our analytical approach. All the notions and properties relating to differential geometry of hypersurfaces will be introduced and explained in Chapter 2.

1.1 Modeling

Suppose a (fixed) container Ω – a bounded domain in $\mathbb{R}^n$ with smooth boundary – is filled with a material which is present in two phases that occupy the regions $\Omega_1(t)$ and $\Omega_2(t)$. The interface $\Gamma(t)$ separating these two phases will depend on time t, but should not be in contact with the outer boundary $\partial\Omega$ of the container in order to avoid the contact angle problem. Then the so-called *continuous phase* $\Omega_2(t)$ is in contact with the outer boundary, while the *diperse phase* $\Omega_1(t)$ is not, which means that $\partial\Omega_1(t) = \Gamma(t)$ and $\partial\Omega_2(t) = \partial\Omega \cup \Gamma(t)$. The outer unit normal of $\Gamma(t)$ w.r.t. $\Omega_1(t)$ will be denoted by ν_Γ, it depends on $p \in \Gamma(t)$ as well as on t; the outer unit normal of Ω is called ν, it only depends on $p \in \partial\Omega$. The *Weingarten tensor* L_Γ is defined by $L_\Gamma := -\nabla_\Gamma \nu_\Gamma$, where ∇_Γ means the surface gradient, and the $((n-1)$-fold) mean curvature H_Γ of Γ by

$$H_\Gamma = \operatorname{tr} L_\Gamma = -\operatorname{div}_\Gamma \nu_\Gamma,$$

where $\operatorname{div}_\Gamma$ means the surface divergence on Γ. In the sequel, the jump of a physical quantity ϕ across Γ will be denoted by

$$[\![\phi]\!](p) := \lim_{s \to 0+} [\phi(p + s\nu_\Gamma(p)) - \phi(p - s\nu_\Gamma(p))], \quad p \in \Gamma.$$

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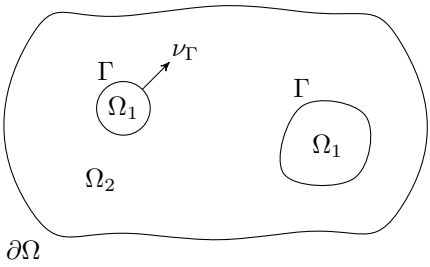

Figure 1.1: A two-phase domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$.

1.1 First Principles in the Bulk

We begin with the basic balance laws in the bulk.

Balance of Mass

Let $\varrho > 0$ denote the density and u the velocity in the bulk phases Ω_j, u_Γ the velocity and $V_\Gamma := u_\Gamma \cdot \nu_\Gamma$ the normal velocity of Γ, respectively. Note that ϱ and u may jump across the interface Γ and that u_Γ is in general not a tangent vector field to Γ. If there are no sources of mass in the bulk, then conservation of mass is given by the *continuity equation*

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0 \quad \text{in } \Omega \setminus \Gamma(t). \tag{1.1}$$

If there is no surface mass on Γ, we also have the jump condition

$$[\![\varrho(u - u_\Gamma) \cdot \nu_\Gamma]\!] = 0 \quad \text{on } \Gamma(t). \tag{1.2}$$

The interfacial mass flux j_Γ, *phase flux* for short, is defined by means of

$$j_\Gamma := \varrho(u - u_\Gamma) \cdot \nu_\Gamma, \quad \text{i.e.,} \quad [\![\frac{1}{\varrho}]\!] j_\Gamma = [\![u \cdot \nu_\Gamma]\!]. \tag{1.3}$$

Observe that j_Γ is well defined, as (1.2) shows. *Phase Transition* takes place if $j_\Gamma \not\equiv 0$. On the other hand, if $j_\Gamma \equiv 0$, then $u \cdot \nu_\Gamma = u_\Gamma \cdot \nu_\Gamma = V_\Gamma$, and in this case the interface is advected with the velocity field u.

Next we have by the transport theorem for moving domains

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_1(t)} \varrho \, dx &= \int_{\Gamma(t)} \varrho V_\Gamma \, d\Gamma + \int_{\Omega_1(t)} \partial_t \varrho \, dx \\
&= \int_{\Gamma(t)} \varrho V_\Gamma \, d\Gamma - \int_{\Omega_1(t)} \operatorname{div}(\varrho u) \, dx \\
&= \int_{\Gamma(t)} (\varrho u_\Gamma \cdot \nu_\Gamma - \varrho u \cdot \nu_\Gamma) \, d\Gamma = - \int_{\Gamma(t)} j_\Gamma \, d\Gamma,
\end{aligned}$$

and in case $u \cdot \nu = 0$ on $\partial\Omega$ in the same way

$$\frac{d}{dt}\int_{\Omega_2(t)} \varrho \, dx = \int_{\Gamma(t)} j_\Gamma \, d\Gamma,$$

proving conservation of total mass, i.e.,

$$\frac{d}{dt}\int_{\Omega} \varrho \, dx = 0. \tag{1.4}$$

In this book we mostly consider the *completely incompressible case* , i.e., we assume that the densities are constant in the phases Ω_j. Then conservation of mass reduces to

$$\operatorname{div} u = 0 \quad \text{in } \Omega \setminus \Gamma(t).$$

If only the latter property holds, we say that the material is *incompressible*. In case both phases are completely incompressible we have

$$\varrho_1|\Omega_1(t)| + \varrho_2|\Omega_2(t)| \equiv \varrho_1|\Omega_1(0)| + \varrho_2|\Omega_2(0)| =: c_0.$$

This implies

$$[\![\varrho]\!]|\Omega_1(t)| = \varrho_2|\Omega| - c_0,$$

hence $|\Omega_1(t)|$ is constant in the case of nonequal densities, i.e., the phase volumes are preserved. On the other hand, there is no preservation of phase volumes in general if one or both phases are compressible, or if the densities are constant and equal.

The Universal Balance Law Let ϕ be any (mass-specific) physical quantity, J its flux, and f its sources. Then the balance law for ϕ in the bulk reads

$$\partial_t(\varrho\phi) + \operatorname{div}(\varrho\phi u + J) = \varrho f \quad \text{in } \Omega \setminus \Gamma(t), \tag{1.5}$$

and if there is a source f_Γ for ϕ on the interface we have

$$[\![(\varrho\phi(u - u_\Gamma) + J) \cdot \nu_\Gamma]\!] = f_\Gamma \quad \text{on } \Gamma(t). \tag{1.6}$$

Employing balance of mass and the definition of the phase flux j_Γ this simplifies to

$$\begin{aligned} \varrho(\partial_t\phi + u \cdot \nabla\phi) + \operatorname{div} J &= \varrho f \quad \text{in } \Omega \setminus \Gamma(t), \\ [\![\phi]\!] j_\Gamma + [\![J \cdot \nu_\Gamma]\!] &= f_\Gamma \quad \text{on } \Gamma(t). \end{aligned} \tag{1.7}$$

By (2.101), the corresponding *universal transport theorem* becomes

$$\begin{aligned} \frac{d}{dt}\int_\Omega \varrho\phi \, dx &= \int_\Omega \partial_t(\varrho\phi)\, dx - \int_\Gamma [\![\varrho\phi]\!] V_\Gamma \, d\Gamma \\ &= \int_\Omega (\varrho f - \operatorname{div}(\varrho\phi u + J))\, dx - \int_\Gamma [\![\varrho\phi u_\Gamma \cdot \nu_\Gamma]\!] \, d\Gamma \\ &= \int_\Omega \varrho f \, dx + \int_\Gamma [\![(\varrho\phi(u - u_\Gamma) + J) \cdot \nu_\Gamma]\!] \, d\Gamma - \int_{\partial\Omega} (\varrho\phi u + J) \cdot \nu \, d(\partial\Omega) \\ &= \int_\Omega \varrho f \, dx + \int_\Gamma ([\![\phi]\!] j_\Gamma + [\![J \cdot \nu_\Gamma]\!])\, d\Gamma + \int_{\partial\Omega} g \, d(\partial\Omega), \end{aligned}$$

with $g = -(\varrho\phi u + J)\cdot\nu$ on $\partial\Omega$. Therefore, we obtain the conservation law

$$\frac{d}{dt}\int_\Omega \varrho\phi\,dx = \int_\Omega \varrho f\,dx + \int_\Gamma f_\Gamma\,d\Gamma + \int_{\partial\Omega} g\,d(\partial\Omega).$$

In particular, if $(f, f_\Gamma, g) = 0$, then the total amount of ϕ in Ω is conserved.

Balance of Momentum
Let π denote the pressure, T the (symmetric) stress tensor, and let f be a force field, say gravity. Then balance of momentum reads, employing (1.5) with $\phi = u$ and $J = -T$,

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u\otimes u) - \operatorname{div} T = \varrho f \quad \text{in } \Omega\setminus\Gamma(t).$$

Similarly, using (1.6) we get the following jump condition at the interface.

$$[\![(\varrho u\otimes(u-u_\Gamma) - T)\nu_\Gamma]\!] = \operatorname{div}_\Gamma T_\Gamma \quad \text{on } \Gamma(t).$$

Here T_Γ denotes the (symmetric) surface stress, a tensor field on Γ. Using balance of mass and the definition of the phase flux j_Γ we may rewrite these conservation laws as follows.

$$\begin{aligned} \varrho(\partial_t u + u\cdot\nabla u) - \operatorname{div} T = \varrho f \qquad & \text{in } \Omega\setminus\Gamma(t),\\ [\![u]\!]j_\Gamma - [\![T\nu_\Gamma]\!] = \operatorname{div}_\Gamma T_\Gamma \quad & \text{on } \Gamma(t). \end{aligned} \tag{1.8}$$

By the surface divergence theorem, total conservation of momentum reads as

$$\frac{d}{dt}\int_\Omega \varrho u\,dx = \int_\Omega \varrho f\,dx + \int_{\partial\Omega} g\,d(\partial\Omega),$$

with $g = -(\varrho u u\cdot\nu - T\nu)$ on $\partial\Omega$. Note that total momentum is in general not conserved as the boundary term g on $\partial\Omega$ need not be zero.

Balance of Energy
Let ϵ denote the (mass specific) internal energy density, $\theta > 0$ the absolute temperature, q the heat flux, and r an external (mass specific) heat source. Then with $\phi = |u|^2/2 + \epsilon$, $J = -Tu + q$ we obtain from the universal balance law (1.5) conservation of energy, which in the bulk reads

$$\partial_t\Big(\frac{\varrho}{2}|u|^2 + \varrho\epsilon\Big) + \operatorname{div}\{(\frac{\varrho}{2}|u|^2 + \varrho\epsilon)u\} - \operatorname{div}(Tu - q) = \varrho f\cdot u + \varrho r \quad \text{in } \Omega\setminus\Gamma(t).$$

On the interface we have, in accordance with (1.6),

$$\Big[\!\Big[\Big(\frac{\varrho}{2}|u|^2 + \varrho\epsilon\Big)(u-u_\Gamma) - Tu + q\Big]\!\Big]\cdot\nu_\Gamma = (\operatorname{div}_\Gamma T_\Gamma)\cdot u_\Gamma + r_\Gamma \quad \text{on } \Gamma(t),$$

where r_Γ denotes a heat source on Γ. Using (1.1), (1.8), and the definition of the phase flux j_Γ we may rewrite this conservation law as follows.

$$\begin{aligned} \varrho(\partial_t\epsilon + u\cdot\nabla\epsilon) + \operatorname{div} q - T:\nabla u = \varrho r \text{ in } \Omega\setminus\Gamma(t),\\ \Big([\![\epsilon]\!] + \Big[\!\Big[\frac{1}{2}|u-u_\Gamma|^2\Big]\!\Big]\Big)j_\Gamma - [\![T\nu_\Gamma\cdot(u-u_\Gamma)]\!] + [\![q\cdot\nu_\Gamma]\!] = r_\Gamma \text{ on } \Gamma(t). \end{aligned} \tag{1.9}$$

The total bulk energy is given by

$$\mathsf{E}(u,\epsilon,\Gamma) := \frac{1}{2}\int_{\Omega\setminus\Gamma} \varrho|u|^2\,dx + \int_{\Omega\setminus\Gamma} \varrho\epsilon\,dx.$$

For its time derivative we obtain from the universal balance law

$$\partial_t \mathsf{E} = \int_\Omega (\varrho f\cdot u + \varrho r)\,dx + \int_{\partial\Omega} g\,d(\partial\Omega) + \int_\Gamma \{\mathrm{div}_\Gamma T_\Gamma\cdot u_\Gamma + r_\Gamma\}\,d\Gamma,$$

where $g = -\big((\frac{\varrho}{2}|u|^2+\varrho\epsilon)u\cdot\nu - Tu\cdot\nu + q\cdot\nu\big)$ on $\partial\Omega$. In particular, if $(f,r)=0$ in Ω, $(u\cdot\nu, q\cdot\nu, T\nu\cdot u)=0$ on $\partial\Omega$ as well as $\mathrm{div}_\Gamma T_\Gamma\cdot u_\Gamma + r_\Gamma = 0$ on Γ, then

$$\frac{d}{dt}\mathsf{E}(u,\epsilon,\Gamma) = 0,$$

which means that the total bulk energy is preserved.

The Entropy
As is common in thermodynamics, we write

$$\epsilon(\varrho,\theta) = \psi(\varrho,\theta) + \theta\eta(\varrho,\theta), \quad \eta(\varrho,\theta) = -\partial_\theta\psi(\varrho,\theta), \tag{1.10}$$

where $\theta>0$ denotes the absolute temperature, and ψ the *Helmholtz free energy*. In this book it is considered given. η means the (mass specific) *entropy* density. Then the *Clausius–Duhem equation* holds in the bulk, which means

$$\begin{aligned}\partial_t(\varrho\eta) + \mathrm{div}(\varrho\eta u) + \mathrm{div}\,(q/\theta) = \frac{1}{\theta}S:\nabla u - \frac{1}{\theta^2}q\cdot\nabla\theta \\ + \frac{\varrho^2\partial_\varrho\psi - \pi}{\theta}\mathrm{div}\,u \quad \text{in } \Omega\setminus\Gamma(t),\end{aligned} \tag{1.11}$$

where $S := T+\pi$ denotes the viscous stress tensor. Therefore, entropy is non-decreasing locally in the bulk provided the right-hand side of (1.11) is nonnegative. This gives the well-known requirements

$$S:\nabla u \geq 0, \quad q\cdot\nabla\theta \leq 0, \tag{1.12}$$

and, since in general the last term will not have a sign, either $\mathrm{div}\,u \equiv 0$, which corresponds to the incompressible case, or

$$\pi = p(\varrho,\theta) := \varrho^2\partial_\varrho\psi(\varrho,\theta), \tag{1.13}$$

which is the famous *Maxwell relation* for compressible materials. Note that p should be an increasing function in both variables, ϱ and θ. Hence we require at least

$$\partial_\varrho\partial_\theta\psi \geq 0, \quad 2\partial_\varrho\psi + \varrho\partial_\varrho^2\psi \geq 0, \quad \varrho,\theta>0,$$

in the compressible case. The total bulk entropy is defined by

$$\mathsf{N}_b(\varrho,\theta,\Gamma)=\int_{\Omega\setminus\Gamma}\varrho\eta(\varrho,\theta)dx.$$

By the universal balance law we then obtain

$$\frac{d}{dt}\mathsf{N}_b(\varrho,\theta,\Gamma)=\int_{\Omega\setminus\Gamma}\Big\{\frac{1}{\theta}S:\nabla u-\frac{1}{\theta^2}q\cdot\nabla\theta\Big\}dx+\int_\Gamma\{[\![\eta]\!]j_\Gamma+[\![q/\theta]\!]\cdot\nu_\Gamma\}d\Gamma,$$

provided $u\cdot\nu=q\cdot\nu=0$ on $\partial\Omega$. In particular, there is no entropy production on the interface if

$$[\![\eta]\!]j_\Gamma+[\![q/\theta]\!]\cdot\nu_\Gamma=0\quad\text{on }\Gamma.$$

1.2 First Principles on the Interface

Throughout we assume that there is no surface mass, and therefore also no surface momentum on Γ. However, due to surface tension we have to take into account surface energy, and then also surface entropy. A basic principle of our approach is conservation of energy and entropy across the interface. We begin with

The Universal Balance Law on the Interface

Suppose ϕ is a scalar physical quantity which also lives on Γ with surface density ϕ_Γ and let J_Γ denote its flux. Thus, J_Γ is a tangent vector field to Γ. The basic balance law for ϕ_Γ reads

$$\frac{D}{Dt}\phi_\Gamma+\phi_\Gamma\mathrm{div}_\Gamma u_\Gamma+\mathrm{div}_\Gamma J_\Gamma=-f_\Gamma.\tag{1.14}$$

Here D/Dt means the Lagrangian derivative with respect to the vector field u_Γ which moves Γ, i.e.,

$$\frac{D}{Dt}\phi_\Gamma(t,\xi)=\frac{d}{ds}\phi_\Gamma(s+t,x(s+t,t,\xi))\Big|_{s=0},$$

with $x(s+t,t,\xi)$ the flow induced by the velocity field u_Γ, i.e.,

$$\frac{d}{ds}x(s+t,t,\xi)=u_\Gamma(s+t,x(s+t,t,\xi)),\quad x(t,t,\xi)=\xi,\ \xi\in\Gamma(t).$$

We emphasize again that the velocity field u_Γ is in general not tangent to Γ. The surface transport theorem then yields

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\phi_\Gamma\,d\Gamma&=\int_\Gamma\Big(\frac{D}{Dt}\phi_\Gamma+\phi_\Gamma\mathrm{div}_\Gamma u_\Gamma\Big)\,d\Gamma\\&=\int_\Gamma(-\mathrm{div}_\Gamma J_\Gamma-f_\Gamma)\,d\Gamma=-\int_\Gamma f_\Gamma\,d\Gamma,\end{aligned}$$

by the surface divergence theorem. Therefore, we obtain conservation of the total amount of ϕ in Ω, i.e., we have

$$\frac{d}{dt}\Big\{\int_\Omega \varrho\phi\,dx + \int_\Gamma \phi_\Gamma\,d\Gamma\Big\} = 0,$$

provided $(f, g) = 0$. Thus the balance law for ϕ on Γ reads

$$\frac{D}{Dt}\phi_\Gamma + \phi_\Gamma \mathrm{div}_\Gamma u_\Gamma + \mathrm{div}_\Gamma J_\Gamma = -([\![\phi]\!] j_\Gamma + [\![J]\!]\cdot\nu_\Gamma). \tag{1.15}$$

We apply this interface conservation law first to

Conservation of Energy on the Interface
Here we have $\phi_\Gamma = \epsilon_\Gamma$ and $J_\Gamma = -T_\Gamma u_\Gamma + q_\Gamma$. Then balance of surface energy reads

$$\frac{D}{Dt}\epsilon_\Gamma + \epsilon_\Gamma \mathrm{div}_\Gamma\, u_\Gamma + \mathrm{div}_\Gamma\,(q_\Gamma - T_\Gamma u_\Gamma) = -\{(\mathrm{div}_\Gamma\, T_\Gamma)\cdot u_\Gamma + r_\Gamma\}.$$

Hence

$$\frac{D}{Dt}\epsilon_\Gamma + \epsilon_\Gamma \mathrm{div}_\Gamma\, u_\Gamma + \mathrm{div}_\Gamma\, q_\Gamma = T_\Gamma : \nabla_\Gamma u_\Gamma - r_\Gamma.$$

By the conservation laws this implies conservation of total energy

$$\frac{d}{dt}\left\{\int_\Omega \varrho\left(\frac{|u|^2}{2} + \epsilon\right)dx + \int_\Gamma \epsilon_\Gamma\,d\Gamma\right\} = 0,$$

provided $(f, r) = 0$ in Ω, $u\cdot\nu = q\cdot\nu = 0$ and $T\nu\cdot u = 0$ on $\partial\Omega$.

Surface Entropy
As in the bulk we write

$$\epsilon_\Gamma(\theta_\Gamma) = \psi_\Gamma(\theta_\Gamma) + \theta_\Gamma\eta_\Gamma(\theta_\Gamma), \quad \eta_\Gamma(\theta_\Gamma) = -\psi_\Gamma'(\theta_\Gamma),$$

where we consider the free energy ψ_Γ as a given function of surface temperature θ_Γ. Similarly, we decompose

$$T_\Gamma = \sigma(\theta_\Gamma)\mathcal{P}_\Gamma + S_\Gamma,$$

where σ denotes the coefficient of surface tension, $\mathcal{P}_\Gamma = I - \nu_\Gamma\otimes\nu_\Gamma$ the orthogonal projection onto the tangent bundle of Γ, and S_Γ the interface viscous stress. Then surface force becomes

$$\mathrm{div}_\Gamma\, T_\Gamma = \sigma H_\Gamma\nu_\Gamma + \nabla_\Gamma\sigma + \mathrm{div}_\Gamma S_\Gamma.$$

The first term in this decomposition is surface tension which acts in a normal direction. The second is called the *Marangoni force* which acts tangentially, and the last one is the viscous surface force induced by surface viscosity.

The total surface entropy is given by

$$\mathsf{N}_\Gamma = \int_\Gamma \eta_\Gamma d\Gamma.$$

With the surface transport theorem (2.91) we get

$$\begin{aligned}\frac{d}{dt}\mathsf{N}_\Gamma &= \int_\Gamma \Big(\frac{D}{Dt}\eta_\Gamma + \eta_\Gamma \mathrm{div}_\Gamma u_\Gamma\Big)\, d\Gamma = \int_\Gamma \Big(\frac{D}{Dt}\epsilon_\Gamma + \theta_\Gamma\eta_\Gamma \mathrm{div}_\Gamma u_\Gamma\Big)/\theta_\Gamma\, d\Gamma \\ &= \int_\Gamma (-\mathrm{div}_\Gamma q_\Gamma - \psi_\Gamma \mathrm{div}_\Gamma u_\Gamma + T_\Gamma : \nabla_\Gamma u_\Gamma - r_\Gamma)/\theta_\Gamma\, d\Gamma \\ &= \int_\Gamma \big(S_\Gamma : \nabla_\Gamma u_\Gamma/\theta_\Gamma - q_\Gamma\cdot\nabla_\Gamma\theta_\Gamma/\theta_\Gamma^2 + (\sigma-\psi_\Gamma)\mathrm{div}_\Gamma u_\Gamma/\theta_\Gamma - r_\Gamma/\theta_\Gamma\big)\, d\Gamma.\end{aligned}$$

Now we argue as in the bulk case. To ensure entropy production on the interface we should have

$$S_\Gamma : \nabla_\Gamma u_\Gamma \geq 0, \quad q_\Gamma\cdot\nabla_\Gamma\theta_\Gamma \leq 0,$$

as well as

$$\psi_\Gamma = \sigma,$$

which is the analogue of the Maxwell relation on the interface. Thus in the situation considered here, the free energy on the interface is the coefficient of surface tension, which acts as a negative surface pressure.

For the total entropy in Ω we finally obtain

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega \varrho\eta\, dx + \int_\Gamma \eta_\Gamma\, d\Gamma\Big) &= \int_\Omega \Big\{\frac{1}{\theta}S : \nabla u - \frac{1}{\theta^2}q\cdot\nabla\theta\Big\}\, dx \\ &\quad + \int_\Gamma \Big\{\frac{1}{\theta_\Gamma}S_\Gamma : \nabla_\Gamma u_\Gamma - \frac{1}{\theta_\Gamma^2}q_\Gamma\cdot\nabla_\Gamma\theta_\Gamma\Big\}\, d\Gamma \qquad (1.16)\\ &\quad + \int_\Gamma \{[\![\eta]\!]j_\Gamma + [\![q/\theta]\!]\cdot\nu_\Gamma - r_\Gamma/\theta_\Gamma\}\, d\Gamma.\end{aligned}$$

Since the integrand in the last integral does not have a sign, we postulate that it vanishes. This means that the only sources for entropy is friction due to viscosity or heat conduction, also on the interface. In case $(S_\Gamma, q_\Gamma) = 0$ it means *conservation of entropy across the interface.*

This assumption implies by (1.9)

$$-\Big(\Big([\![\epsilon]\!] + \Big[\!\Big[\frac{1}{2}|u-u_\Gamma|^2\Big]\!\Big]\Big)j_\Gamma - [\![T\nu_\Gamma\cdot(u-u_\Gamma)]\!] + [\![q\cdot\nu_\Gamma]\!]\Big)/\theta_\Gamma + [\![\eta]\!]j_\Gamma + [\![q/\theta]\!]\cdot\nu_\Gamma = 0.$$

Assuming $[\![\theta]\!] = 0$, $\theta = \theta_\Gamma$ on Γ, the latter simplifies to

$$\Big([\![\psi]\!] + \Big[\!\Big[\frac{1}{2}|u-u_\Gamma|^2\Big]\!\Big]\Big)j_\Gamma - [\![T\nu_\Gamma\cdot(u-u_\Gamma)]\!] = 0. \qquad (1.17)$$

This is the *generalized Gibbs–Thomson relation*. Taking it for granted, balance of surface energy becomes

$$\frac{D}{Dt}\epsilon_\Gamma + \epsilon_\Gamma \mathrm{div}_\Gamma u_\Gamma + \mathrm{div}_\Gamma q_\Gamma = S_\Gamma : \nabla_\Gamma u_\Gamma + \sigma \mathrm{div}_\Gamma u_\Gamma - ([\![\theta\eta]\!] j_\Gamma + [\![q \cdot \nu_\Gamma]\!]). \quad (1.18)$$

On the interface the Clausius–Duhem equation reads

$$\begin{aligned}\frac{D}{Dt}\eta_\Gamma + \eta_\Gamma \mathrm{div}_\Gamma u_\Gamma + \mathrm{div}_\Gamma(q_\Gamma/\theta_\Gamma) = \frac{1}{\theta_\Gamma} S_\Gamma : \nabla_\Gamma u_\Gamma - \frac{1}{\theta_\Gamma^2} q_\Gamma \cdot \nabla_\Gamma \theta_\Gamma \\ - ([\![\eta]\!] j_\Gamma + [\![q/\theta]\!] \cdot \nu_\Gamma),\end{aligned} \quad (1.19)$$

showing that surface entropy production is nonnegative, locally on Γ.

Note that in case $(\eta_\Gamma, q_\Gamma, S_\Gamma) = 0$ on Γ this equation implies the famous *Stefan condition*

$$\theta[\![\eta]\!] j_\Gamma + [\![q \cdot \nu_\Gamma]\!] = 0.$$

$\eta_\Gamma \equiv 0$ means $\psi_\Gamma = \sigma \equiv$ *constant* and $\epsilon_\Gamma = \sigma$. In this case total surface energy becomes $\sigma|\Gamma|$, and surface energy balance is trivial.

1.3 Constitutive Laws

In the sequel we assume that there are no external sources for momentum and energy, i.e., $(f, r) = 0$.

Constitutive Laws on the Outer Boundary

$$q \cdot \nu = 0 \quad \text{and} \quad u = 0. \quad (1.20)$$

Actually, we could also consider a condition for u of Navier-type at the outer boundary, which means

$$u \cdot \nu = 0 \quad \text{and} \quad \mathcal{P}_{\partial\Omega} T\nu + ku = 0,$$

where $k \geq 0$, and $\mathcal{P}_{\partial\Omega}$ denotes the projection onto the tangent bundle of the hypersurface $\partial\Omega$. However, here we stay with the simplest case.

Constitutive Laws in the Phases

$$\begin{aligned}\epsilon(\varrho,\theta) &= \psi(\varrho,\theta) + \theta\eta(\varrho,\theta), \quad \eta(\varrho,\theta) = -\partial_\theta \psi(\varrho,\theta), \\ T &= 2\mu(\varrho,\theta)D + \lambda(\varrho,\theta)(\mathrm{div}\, u)I - \pi I, \quad D = \frac{1}{2}(\nabla u + (\nabla u)^{\mathsf{T}}), \\ q &= -d(\varrho,\theta)\nabla\theta.\end{aligned} \quad (1.21)$$

Here μ is called *shear viscosity*, λ *bulk viscosity*, and d is the coefficient of heat conduction or *heat conductivity*. μ, λ, d are functions depending on (ϱ, θ), and on the phase, and hence may jump across the interface $\Gamma(t)$. The second and the third equations are the classical laws of *Newton* and *Fourier*. To meet the requirements (1.12) we assume

$$\mu(\varrho,\theta), d(\varrho,\theta) > 0, \quad \lambda(\varrho,\theta) + 2\mu(\varrho,\theta)/n > 0, \quad \varrho, \theta > 0,$$

and in the compressible case also the Maxwell relation (1.13).

Constitutive Laws on the Interface

$$\begin{aligned}
\epsilon_\Gamma(\theta_\Gamma) &= \sigma(\theta_\Gamma) + \theta_\Gamma \eta_\Gamma(\theta_\Gamma), \quad \eta_\Gamma(\theta_\Gamma) = -\sigma'(\theta_\Gamma),\\
[\![\theta]\!] &= 0, \quad \theta_\Gamma = \theta,\\
\mathcal{P}_\Gamma[\![u]\!] &= 0, \quad \mathcal{P}_\Gamma(u - u_\Gamma) = 0,\\
T_\Gamma &= \sigma(\theta_\Gamma)\mathcal{P}_\Gamma + 2\mu_\Gamma(\theta_\Gamma) D_\Gamma + \lambda_\Gamma(\theta_\Gamma)(\mathrm{div}_\Gamma u_\Gamma)\mathcal{P}_\Gamma,\\
D_\Gamma &= \frac{1}{2}\mathcal{P}_\Gamma(\nabla_\Gamma u_\Gamma + [\nabla_\Gamma u_\Gamma]^{\mathsf{T}})\mathcal{P}_\Gamma, \quad q_\Gamma = -d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma,\\
0 &= \Big([\![\psi(\theta_\Gamma)]\!] + \Big[\!\Big[\frac{1}{2}|u - u_\Gamma|^2\Big]\!\Big]\Big) j_\Gamma - [\![T\nu_\Gamma(u - u_\Gamma)]\!].
\end{aligned} \tag{1.22}$$

The coefficient of surface tension σ and the surface viscosities $(\mu_\Gamma, \lambda_\Gamma)$ are functions of θ_Γ, which are subject to

$$\sigma, \mu_\Gamma > 0, \quad \lambda_\Gamma + \frac{2\mu_\Gamma}{n-1} > 0.$$

Recall the relation

$$V_\Gamma := u_\Gamma \cdot \nu_\Gamma = u \cdot \nu_\Gamma - \frac{1}{\varrho} j_\Gamma,$$

for the normal velocity of the interface. In case $[\![\varrho]\!] \neq 0$ this implies

$$[\![u]\!] = [\![1/\varrho]\!] j_\Gamma \nu_\Gamma, \quad j_\Gamma = [\![u \cdot \nu_\Gamma]\!]/[\![1/\varrho]\!], \quad V_\Gamma = [\![\varrho u \cdot \nu_\Gamma]\!]/[\![\varrho]\!], \tag{1.23}$$

and if $[\![\varrho]\!] = 0$ we have $[\![u]\!] = 0$. This shows a fundamental difference between theses cases: if the densities are not equal, then the phase flux enters directly the velocity jump on the interface, inducing what is called *Stefan current*. If the densities are equal, there is no Stefan current and the velocity field is continuous across the interface. On each side of the interface we have the identity

$$u = u_\Gamma + j_\Gamma \nu_\Gamma / \varrho,$$

which, in view of the definition of the phase flux j_Γ, is equivalent to the conditions

$$\mathcal{P}_\Gamma[\![u]\!] = 0, \quad \mathcal{P}_\Gamma(u - u_\Gamma) = 0, \quad [\![\varrho(u - u_\Gamma) \cdot \nu_\Gamma]\!] = 0.$$

Now we may rewrite

$$\begin{aligned}
\Big[\!\Big[\frac{1}{2}|u - u_\Gamma|^2\Big]\!\Big] &= \Big[\!\Big[\frac{1}{2\varrho^2}\Big]\!\Big] j_\Gamma^2,\\
-[\![T\nu_\Gamma \cdot (u - u_\Gamma)]\!] &= [\![-T\nu_\Gamma]\!] \cdot \mathcal{P}_\Gamma(u - u_\Gamma) + [\![-T\nu_\Gamma \cdot \nu_\Gamma/\varrho]\!] j_\Gamma\\
&= [\![-T\nu_\Gamma \cdot \nu_\Gamma/\varrho]\!] j_\Gamma,
\end{aligned}$$

hence the generalized Gibbs–Thomson relation becomes

$$\Big([\![\psi]\!] + \Big[\!\Big[\frac{1}{2\varrho^2}\Big]\!\Big] j_\Gamma^2 - [\![T\nu_\Gamma \cdot \nu_\Gamma/\varrho]\!]\Big) j_\Gamma = 0.$$

It holds trivially if $j_\Gamma \equiv 0$, i.e., if there is *no phase transition*, and otherwise we assume

$$[\![\psi]\!] + \left[\!\!\left[\frac{1}{2\varrho^2}\right]\!\!\right] j_\Gamma^2 - [\![T\nu_\Gamma \cdot \nu_\Gamma/\varrho]\!] = 0. \tag{1.24}$$

We define the heat capacity κ and the surface heat capacity κ_Γ as usual by

$$\kappa(\varrho,\theta) = \partial_\theta \epsilon(\varrho,\theta) = -\theta\partial_\theta^2\psi(\varrho,\theta), \quad \kappa_\Gamma(\theta_\Gamma) = \epsilon_\Gamma'(\theta_\Gamma) = -\theta_\Gamma\sigma''(\theta_\Gamma)$$

respectively. Moreover, we define the latent heat l and the surface latent heat l_Γ by

$$l(\varrho,\theta) = -[\![\theta\eta(\varrho,\theta)]\!] = [\![\theta\partial_\theta\psi(\varrho,\theta)]\!], \quad l_\Gamma(\theta_\Gamma) = -\theta_\Gamma\eta_\Gamma(\theta_\Gamma) = \theta_\Gamma\sigma'(\theta_\Gamma).$$

The conditions $\partial_\theta^2\psi \le 0$ as well as $\sigma'' \le 0$ will be needed for well-posedness.

Remark
(1.24) may be generalized to take into account *kinetic undercooling*. More precisely, we may replace (1.24) by the law

$$[\![\psi]\!] + \left[\!\!\left[\frac{1}{2\varrho^2}\right]\!\!\right] j_\Gamma^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] = -\gamma j_\Gamma + \mathrm{div}_\Gamma[\alpha\nabla_\Gamma(j_\Gamma/\theta_\Gamma)] + \theta_\Gamma \,\mathrm{div}_\Gamma[\beta\nabla_\Gamma j_\Gamma], \tag{1.25}$$

where $\alpha, \beta, \gamma \ge 0$ may depend on the surface temperature θ_Γ. In this case the entropy production on Γ is increased by

$$\int_\Gamma \{\gamma j_\Gamma^2/\theta_\Gamma + \alpha|\nabla_\Gamma(j_\Gamma/\theta_\Gamma)|_2^2 + \beta|\nabla_\Gamma j_\Gamma|^2\} d\Gamma,$$

and on the right-hand side of the surface energy balance the term

$$j_\Gamma(\gamma j_\Gamma - \mathrm{div}_\Gamma[\alpha\nabla_\Gamma(j_\Gamma/\theta_\Gamma)] - \theta_\Gamma \,\mathrm{div}_\Gamma[\beta\nabla_\Gamma j_\Gamma])$$

has to be added.

1.4 The Resulting Dynamic Problem
Summarizing we obtain the following initial-boundary value problem in the absence of external forces and heat sources.

$$\begin{aligned}
\partial_t\varrho + \mathrm{div}(\varrho u) &= 0 && \text{in } \Omega\setminus\Gamma(t),\\
\varrho(\partial_t u + u\cdot\nabla u) - \mathrm{div}\, S + \nabla\pi &= 0 && \text{in } \Omega\setminus\Gamma(t),\\
u &= 0 && \text{on } \partial\Omega,\\
[\![u]\!] &= [\![1/\varrho]\!] j_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\\
[\![1/\varrho]\!] j_\Gamma^2 \nu_\Gamma - [\![T\nu_\Gamma]\!] &= \mathrm{div}_\Gamma T_\Gamma && \text{on } \Gamma(t),\\
\varrho(0) = \varrho_0, \quad u(0) &= u_0 && \text{in } \Omega,
\end{aligned} \tag{1.26}$$

where $S = T + \pi$ and $S_\Gamma = \sigma(\theta_\Gamma)\mathcal{P}_\Gamma - T_\Gamma$ are defined above,

$$\begin{aligned} \varrho\kappa(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d\nabla\theta) &= S:\nabla u - \theta\partial_\theta p \operatorname{div} u && \text{in } \Omega\setminus\Gamma(t),\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ \theta &= \theta_\Gamma && \text{on } \Gamma(t),\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{1.27}$$

On the interface we have

$$\begin{aligned} &\kappa_\Gamma\frac{D}{Dt}\theta_\Gamma - \operatorname{div}_\Gamma(d_\Gamma\nabla_\Gamma\theta_\Gamma)\\ &\quad = S_\Gamma:\nabla_\Gamma u_\Gamma + \theta_\Gamma\sigma'(\theta_\Gamma)\operatorname{div}_\Gamma u_\Gamma - ([\![\theta\eta]\!]j_\Gamma + [\![q\cdot\nu_\Gamma]\!]) && \text{on } \Gamma(t),\\ &[\![\psi]\!] + [\![1/2\varrho^2]\!]j_\Gamma^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] = 0 && \text{on } \Gamma(t),\\ &V_\Gamma = u_\Gamma\cdot\nu_\Gamma = u\cdot\nu_\Gamma - j_\Gamma/\varrho && \text{on } \Gamma(t)\\ &\Gamma(0) = \Gamma_0. \end{aligned} \tag{1.28}$$

This system has to be supplemented with the constitutive laws for T and T_Γ from the previous subsection. Here the first system should be read as a problem for u and ϱ, resp. π, the second as one for θ, while the last set determines θ_Γ, the free boundary Γ, and the phase flux j_Γ. Note that in the absence of phase transitions, the Gibbs–Thomson relation has to be replaced by $j_\Gamma = 0$.

1.2 Entropy and Equilibria

2.1 The Entropy
We have seen above that the total entropy

$$\mathsf{N} := \int_\Omega \varrho\eta\,dx + \int_\Gamma \eta_\Gamma\,d\Gamma$$

satisfies

$$\begin{aligned} \frac{d}{dt}\Big(\int_\Omega \varrho\eta\,dx + \int_\Gamma \eta_\Gamma\,d\Gamma\Big) &= \int_\Omega \Big\{\frac{1}{\theta}S:\nabla u - \frac{1}{\theta^2}q\cdot\nabla\theta\Big\}\,dx\\ &\quad + \int_\Gamma \Big\{\frac{1}{\theta_\Gamma}S_\Gamma:\nabla_\Gamma u_\Gamma - \frac{1}{\theta_\Gamma^2}q_\Gamma\cdot\nabla_\Gamma\theta_\Gamma\Big\}\,d\Gamma. \end{aligned}$$

Hence the negative total entropy is a Lyapunov functional for the problem. We show now that it is even a strict one. To see this, assume that N is constant on some interval (t_1, t_2). Then $d\mathsf{N}/dt = 0$ in (t_1, t_2), hence $D = 0$ and $\nabla\theta = 0$ in $(t_1, t_2)\times\Omega$. Therefore, θ is constant, which implies $[\![d\partial_\nu\theta]\!] = 0$, and then from the interfacial boundary condition we obtain $j_\Gamma = 0$, provided $[\![\eta]\!] \neq 0$ on Γ; we assume this for the moment. This implies $[\![u]\!] = 0$, hence by Korn's inequality we have $\nabla u = 0$ and then $u = 0$ by the no-slip condition on $\partial\Omega$. Hence $V_\Gamma = 0$,

$u_\Gamma = 0$, and $(\partial_t\theta, \partial_t u, \partial_t\varrho, D\theta_\Gamma/Dt) = 0$, which means that we are at equilibrium. Further, $\nabla\pi = 0$, i.e., the pressure is constant in the components of the phases. If one or both phases are compressible, then assuming p_j to be strictly increasing in ϱ, we conclude that ϱ is constant in the components of $\Omega_j(t)$ as well. Actually ϱ is even constant in each phase. To see this, employing Maxwell's relation we rewrite the Gibbs–Thomson condition $[\![\psi]\!] + [\![\pi/\rho]\!] = 0$ as

$$\partial_\varrho(\varrho\psi_1(\varrho)) = \partial_\varrho(\varrho\psi_2(\varrho)).$$

Suppose ϱ_2 is known; then ϱ_1 is uniquely determined by ϱ_2 (and θ) since $\partial_\varrho(\varrho\psi_j(\varrho))$ is strictly increasing, for, by assumption, p_j has this property. Since θ is continuous across the interface, the last relation shows that π, and therefore ϱ, are constant in all of Ω_1, even if it is not connected. From this we finally deduce by the Young-Laplace law $[\![\pi]\!] = \sigma H_\Gamma$ that Ω_1 is a ball if it is connected, or otherwise a finite union of non-intersecting balls of equal radii, since Ω_1 is bounded by assumption.

If, by chance, $[\![\eta]\!] = 0$ on Γ, or only on part of it, we are not allowed to use Korn's inequality since u may have a jump across the interface. Nevertheless, $u = 0$ holds in this case as well, but the proof is a little more involved. For this we need

Lemma 1.2.1. *Suppose $u \in H_2^2(\Omega \setminus \Gamma)$ satisfies $u = 0$ on $\partial\Omega$ and $\mathcal{P}_\Gamma[\![u]\!] = 0$ on Γ. Then $D = 0$ implies $u = 0$ in Ω.*

Proof. Integrating by parts twice we obtain

$$\begin{aligned} 2|D|^2_{L_2(\Omega)} &= |\nabla u|^2_{L_2(\Omega)} + |\operatorname{div} u|^2_{L_2(\Omega)} + \int_\Gamma [\![u\cdot\nu_\Gamma \operatorname{div} u - \nu_\Gamma\cdot(u\cdot\nabla)u]\!]\, d\Gamma \\ &= |\nabla u|^2_{L_2(\Omega)} + |\operatorname{div} u|^2_{L_2(\Omega)} + \int_\Gamma 2[\![u\cdot\nu_\Gamma]\!]\operatorname{div}_\Gamma \mathcal{P}_\Gamma u - [\![(u\cdot\nu_\Gamma)^2]\!]H_\Gamma\, d\Gamma, \end{aligned}$$

since $u = 0$ on $\partial\Omega$ and $\mathcal{P}_\Gamma[\![u]\!] = 0$ on Γ. Here we employed the identities

$$\begin{aligned} \operatorname{div} u &= \operatorname{div}_\Gamma(\mathcal{P}_\Gamma u) - (u\cdot\nu_\Gamma)H_\Gamma + \nu_\Gamma\cdot\partial_\nu u, \\ \nu_\Gamma\cdot(u\cdot\nabla)u &= (\mathcal{P}_\Gamma u\cdot\nabla_\Gamma)u\cdot\nu_\Gamma + (u\cdot\nu_\Gamma)(\nu_\Gamma\cdot\partial_\nu u) + L_\Gamma\mathcal{P}_\Gamma u\cdot\mathcal{P}_\Gamma u \end{aligned}$$

on Γ as well as the surface divergence theorem. Now, if $D = 0$, then $\nu_\Gamma\cdot\partial_\nu u = 0$, and so the equation for the divergence of u on Γ yields

$$\operatorname{div}_\Gamma\mathcal{P}_\Gamma u = (u\cdot\nu_\Gamma)H_\Gamma,$$

hence $[\![(u\cdot\nu_\Gamma)^2]\!]H_\Gamma = 0$ which implies $\nabla u = 0$ in Ω. Therefore, u is constant in the phases, which yields $u = 0$ in Ω_2 by the no-slip condition on the outer boundary $\partial\Omega$. Further, $[\![u]\!] = \alpha\nu_\Gamma$ is constant on Γ which implies $\alpha = 0$, hence $[\![u]\!] = 0$ and so $u = 0$ in Ω_1, as well. □

Having shown that $u = 0$ we may proceed as before, provided $\varrho_1 \neq \varrho_2$. Actually, there is a problem if $[\![\varrho]\!] = [\![\eta]\!] = 0$; then we cannot conclude $j_\Gamma = 0$

which means that V_Γ may be nontrivial. We exclude this pathology in the sequel. It is absent anyway if kinetic undercooling is included.

If there is *no phase transition*, i.e., $j_\Gamma \equiv 0$, then $[\![u]\!] = 0$, and we obtain directly $u \equiv 0$ by Korn's inequality. In this case we conclude as above that the pressures are constant in the components of the phases, hence the densities are so as well, assuming as before that p_j is increasing. We further conclude from the interface stress condition that H_Γ is constant on each component of the interface, which implies that these components are spheres. But they may have differing sizes, as the Gibbs–Thomson relation is no longer available. If a phase transition is absent, constant temperature does no longer ensure that the spheres have equal size!

2.2 Equilibria as Critical Points of the Entropy

We want to determine the critical points of the total entropy N under the constraints of given total mass M_0 and given total energy E_0. With

$$\mathsf{M} = \int_\Omega \varrho \, dx, \quad \mathsf{E} = \int_\Omega \varrho(|u|^2/2 + \epsilon) \, dx + \int_\Gamma \epsilon_\Gamma \, d\Gamma,$$

the method of Lagrange multipliers then yields

$$\mathsf{N}' + \lambda \mathsf{M}' + \mu \mathsf{E}' = 0.$$

We compute the derivatives of the involved functionals, where $z = (\tau, v, \vartheta, \vartheta_\Gamma, h)$.

$$\begin{aligned}
\langle \mathsf{N}'|z\rangle &= \int_\Omega \{\partial_\varrho(\varrho\eta)\tau + \varrho\partial_\theta\eta\vartheta\} \, dx - \int_\Gamma \{[\![\varrho\eta]\!]h - \eta_\Gamma'\vartheta_\Gamma + \eta_\Gamma H_\Gamma h\} \, d\Gamma, \\
\langle \mathsf{M}'|z\rangle &= \int_\Omega \tau \, dx - \int_\Gamma [\![\varrho]\!]h \, d\Gamma, \\
\langle \mathsf{E}'|z\rangle &= \int_\Omega \{\varrho u \cdot v + \varrho\partial_\theta\epsilon\vartheta + (|u|^2/2 + \epsilon + \varrho\partial_\varrho\epsilon)\tau\} \, dx \\
&\quad - \int_\Gamma \{[\![\varrho|u|^2/2 + \varrho\epsilon]\!]h - \epsilon_\Gamma'\vartheta_\Gamma + \epsilon_\Gamma H_\Gamma h\} \, d\Gamma.
\end{aligned}$$

Varying first ϑ and ϑ_Γ this yields

$$\varrho\partial_\theta\eta + \mu\varrho\partial_\theta\epsilon = 0,$$

and

$$\eta_\Gamma' + \mu\epsilon_\Gamma' = 0,$$

hence $\partial_\theta\epsilon = \theta\partial_\theta\eta = \kappa > 0$ and $\epsilon_\Gamma' = \theta_\Gamma\eta_\Gamma' = \kappa_\Gamma > 0$ imply $\theta_\Gamma = \theta = -1/\mu > 0$ constant. Next we vary v to obtain $u = 0$ since $\mu \neq 0$. Variation of τ (when ϱ is not a priori constant) implies similarly

$$\eta + \varrho\partial_\varrho\eta + \lambda + \mu(\epsilon + \varrho\partial_\varrho\epsilon) = 0,$$

hence $\lambda = (\psi + \varrho\partial_\varrho\psi)/\theta$. As a consequence ϱ is constant, since

$$0 < \partial_\varrho p(\varrho,\theta)/\varrho = 2\partial_\varrho\psi(\varrho,\theta) + \varrho\partial_\varrho^2\psi(\varrho,\theta) = \partial_\varrho(\psi(\varrho,\theta) + \varrho\partial_\varrho\psi(\varrho,\theta))$$

in a phase where ϱ is not a priori constant. In particular, if both phases are compressible this yields $[\![\psi + p/\varrho]\!] = 0$, which is the generalized Gibbs–Thomson relation at equilibrium. Finally, we vary h to obtain

$$-[\![\varrho\eta]\!] - \eta_\Gamma H_\Gamma - \lambda[\![\varrho]\!] + ([\![\varrho\epsilon]\!] + \epsilon_\Gamma H_\Gamma)/\theta = 0,$$

which by the definition of ϵ and $\psi_\Gamma = \sigma$ yields

$$\sigma H_\Gamma + [\![\varrho\psi]\!] = \lambda\theta[\![\varrho]\!]$$

on the interface Γ. This implies that H_Γ is constant, hence Ω_1 consists of a finite number of balls with the same radius. If both phases are compressible we may further conclude $\sigma H_\Gamma = [\![p]\!]$, which is the normal stress condition on the interface.

In this derivation we assumed $\kappa_\Gamma > 0$. If instead $\kappa_\Gamma \equiv 0$, then $\eta_\Gamma' \equiv 0$ as well, hence we do not obtain information on θ_Γ. However, the remaining conclusions are valid as before. In this case $\sigma(\theta_\Gamma)$ is linear, and as there is no surface heat capacity it makes sense then to ignore surface diffusion as well.

In summary, we see that the critical points of the total entropy with the constraints of given mass and prescribed total energy are precisely the equilibria of the system.

2.3 Equilibria which are Maxima of Total Entropy

Suppose we have an equilibrium $e := (\varrho, u, \theta, \theta_\Gamma, \Gamma)$ where the total entropy has a local maximum, w.r.t. the constraints $\mathsf{M} = \mathsf{M}_0$ and $\mathsf{E} = \mathsf{E}_0$ constant. Then $\mathcal{D} := [\mathsf{N} + \lambda\mathsf{M} + \mu\mathsf{E}]''$ is negative semi-definite on the kernel of M' intersected with that of E', where (λ, μ) are the fixed Lagrange multipliers found in the previous subsection. The kernel of $\mathsf{M}'(e)$ is easily found to be characterized by the relation

$$\int_\Omega \tau\,dx = [\![\varrho]\!]\int_\Gamma h\,d\Gamma, \tag{1.29}$$

and that of $\mathsf{E}'(e)$ by

$$\int_\Omega \partial_\varrho(\varrho\eta)\tau\,dx + \int_\Omega(\varrho\kappa/\theta)\vartheta\,dx + (\kappa_\Gamma/\theta)\int_\Gamma \vartheta_\Gamma\,d\Gamma = ([\![\varrho\eta]\!] + \eta_\Gamma H_\Gamma)\int_\Gamma h\,d\Gamma. \tag{1.30}$$

On the other hand, a straightforward but somewhat lengthy calculation yields

$$\begin{aligned} -\theta\langle\mathcal{D}z|z\rangle = &\int_\Omega \varrho|v|^2\,dx + \int_\Omega \partial_\varrho^2(\varrho\psi)\tau^2\,dx + \int_\Omega(\varrho\kappa/\theta)\vartheta^2\,dx \\ &+ (\kappa_\Gamma/\theta)\int_\Gamma \vartheta_\Gamma^2\,d\Gamma - \sigma\int_\Gamma (H_\Gamma' h)h\,d\Gamma. \end{aligned} \tag{1.31}$$

As ϱ, κ, κ_Γ and

$$\partial_\varrho^2(\varrho\psi) = 2\partial_\varrho\psi + \varrho\partial_\varrho^2 = [\partial_\varrho p(\varrho)]/\varrho$$

are nonnegative, we see that the form $\langle \mathcal{D}z|z\rangle$ is negative semi-definite as soon as H'_Γ is negative semi-definite. We will see in the next chapter that

$$H'_\Gamma = (n-1)/R^2 + \Delta_\Gamma,$$

where Δ_Γ denotes the Laplace-Beltrami operator on Γ and R means the radius of the equilibrium spheres.

We want to derive necessary conditions for an equilibrium e to be a local maximum of entropy.

1. Suppose that Γ is not connected, i.e., Γ consists of a finite union of spheres Γ_k. Set $(\tau, v, \vartheta, \vartheta_\Gamma) = 0$, and let $h = h_k$ constant on Γ_k with $\sum_k h_k = 0$. Then the constraints (1.30) and (1.31) hold and

$$\langle \mathcal{D}z|z\rangle = \frac{\sigma(\theta)(n-1)}{\theta R^2}\sum_k \Gamma_k h_k^2 > 0,$$

hence $\mathcal{D}$ is not negative semi-definite in this case. Thus if e is an equilibrium with local maximal total entropy, then Γ must be connected, hence both phases are connected. This is related to the so-called *Ostwald ripening* effect.

2. Assume that Γ is connected and $\varrho_1 \neq \varrho_2$ are a priori constant. Then $\tau = 0$ and the first constraint (1.30) implies $\int_\Gamma h\, d\Gamma = 0$. As $H'_\Gamma(h)$ is negative semi-definite for functions with average zero, we see that in this case $\mathcal{D}$ is negative semi-definite.

3. Assume that Γ is connected and $\varrho_1 = \varrho_2 =: \varrho$ is constant. Then $\tau = 0$, but the first constraint gives no information. Setting $v = 0$, $\vartheta = \vartheta_\Gamma$ constant, as well as h constant, we see that $\mathcal{D}$ negative semi-definite on the kernel of $\mathsf{E}'(e)$ implies the condition

$$\frac{\sigma(\theta)(n-1)}{R^2} \leq \frac{l_0^2|\Gamma|}{\theta((\kappa|\varrho)_\Omega + \kappa_\Gamma|\Gamma|)}, \tag{1.32}$$

where $l_0 = l_0(\theta) = -\theta(\varrho[\![\eta]\!] + \eta_\Gamma H_\Gamma)$.

4. If e is an equilibrium which (locally) maximizes the total entropy, it is generically not isolated. If the sphere Γ does not touch the outer boundary, we may move it inside of Ω without changing the total entropy. This fact is reflected in $\mathcal{D}$ by choosing $\tau = \vartheta = \vartheta_\Gamma = 0$ and $h = Y_j$, the spherical harmonics for Γ, which satisfy $H'_\Gamma Y_j = 0$.

It is one of our purposes in this book to prove in the completely incompressible case that an equilibrium is stable if and only if the total entropy at this equilibrium is maximal. Thus in case $\varrho_1 \neq \varrho_2$ are a priori constant, an equilibrium is stable if and only if the interface is connected, and in case $\varrho_1 = \varrho_2$ if in addition the stability condition (1.32) is satisfied with strict inequality. (Here we exclude the limiting case where in (1.32) equality holds.)

2.4 The Manifold of Equilibria

As we have seen above, the equilibria of the system (1.26), (1.27), (1.28) are zero velocities, constant pressures in the phases, constant temperature, vanishing phase flux, and the dispersed phase Ω_1 consists of finitely many non-intersecting balls with the same radius if phase transition is present. We call an equilibrium *non-degenerate* if the balls do not touch the outer boundary $\partial\Omega$ and also do not touch each other. This set will be denoted by $\mathcal{E}$. We want to show that $\mathcal{E}$ is a manifold; it is not connected but has infinitely many finite dimensional components, the components are given by the number of spheres. The dimension of the component consisting of m spheres is $m(n+1)$, where n comes from the center and 1 from the radius of a particular sphere.

To show that $\mathcal{E}$ is a manifold, we just have to show how a neighbouring sphere is parameterized over a given one. In fact, let us assume that $\Sigma = S_R(0)$ is centered at the origin of $\mathbb{R}^n$. Suppose $\mathcal{S} \subset \Omega$ is a sphere that is sufficiently close to Σ. Denote by $(y_1, \ldots, y_n)$ the coordinates of its center and let y_0 be such that $R + y_0$ corresponds to its radius. Then the sphere $\mathcal{S}$ can be parameterized over Σ by the distance function

$$\delta(y) = \sum_{j=1}^{n} y_j Y_j - R + \sqrt{(\sum_{j=1}^{n} y_j Y_j)^2 + (R + y_0)^2 - \sum_{j=1}^{n} y_j^2},$$

where Y_j are the spherical harmonics of degree one. Obviously, this is a real analytic parametrization.

We summarize our considerations in

Theorem 1.2.2. **(a)** *The total mass* M *and the total energy* E *are preserved for smooth solutions.*

(b) *The negative total entropy* $-\mathsf{N}$ *is a strict Lyapunov functional except on the pathological points* (ϱ, θ) *constant,* $[\![\varrho]\!] = [\![\eta]\!] = 0$.

(c) *The critical points of the entropy functional for prescribed total mass and total energy are precisely the equilibria of the system.*

(d) *The* non-degenerate equilibria *are zero velocities, constant pressures in the components of the phases, and the interface is a union of non-intersecting spheres which do not touch the outer boundary* $\partial\Omega$. *If phase transition is present, then the spheres are of equal size.*

(e) *If the total entropy at an equilibrium is locally maximal, then the phases are connected and, in addition, in the case of equal constant densities the stability condition* (1.32) *holds.*

(f) *The set* $\mathcal{E}$ *of non-degenerate equilibria forms a real analytic manifold.*

This result shows that the models are thermodynamically consistent, hence are physically reasonable.

2.5 Equilibrium Temperatures
To determine θ, R and π at equilibrium, we have to solve the system

$$\begin{aligned} |\Omega_1|\varrho_1\epsilon_1 + |\Omega_2|\varrho_2\epsilon_2 + \epsilon_\Gamma|\Gamma| &= \mathsf{E}_0, \\ [\![\pi]\!] &= \sigma H_\Gamma, \\ [\![\psi]\!] &= -[\![\pi/\varrho]\!]. \end{aligned} \tag{1.33}$$

In addition, there is conservation of mass

$$\varrho_1|\Omega_1| + \varrho_2|\Omega_2| = c_0.$$

If the equilibrium densities are not equal, this equation can be employed to compute the radius of the balls, i.e., with $\omega_n = |\partial B(0,1)|$ we have

$$m(\omega_n/n)R^n = (\varrho_2|\Omega| - c_0)/[\![\varrho]\!]$$

in case there are m balls with common radius R. The energy equation then uniquely determines θ since ϵ_Γ is non-decreasing and $\epsilon_j(\theta)$ are strictly increasing. Finally, the last two conditions in (1.33) determine the pressures in the phases.

If there is no phase transition, then the dimension of the component $\mathcal{E}_m$ of $\mathcal{E}$, with $m \in \mathbb{N}$ the number of spheres, is $\dim \mathcal{E}_m = m(n+1)+1$. Here the variables are the centers of the balls, their radia, and the temperature. Prescribing total energy and individual volumes of the components of the dispersed phase reduces the dimension to mn.

On the other hand, if phase transition takes place and $\varrho_1 \neq \varrho_2$, then $\dim \mathcal{E}_m = mn + 2$. The variables are the centers of the balls, the common radius, and the temperature. If we prescribe phase volumes and total energy, then the radius of the balls and the temperature are fixed, resulting into $\dim \mathcal{E}_m = nm$.

But if the equilibrium densities are equal, $\varrho_1 = \varrho_2 =: \varrho$, then conservation of mass determines merely the value of the density ϱ, no information on the phase volumes at equilibrium is available. Hence only θ, R and the pressure jump

$$[\![\pi]\!] = [\![\pi]\!](\theta) = \sigma(\theta)H_\Gamma = -\frac{\sigma(\theta)(n-1)}{R(\theta)}$$

can be obtained from (1.33). This implies that the dimension of $\mathcal{E}_m$ is $mn+1$, and if we prescribe the total energy, then it will be nm.

In this case we get

$$R = R(\theta) = \frac{\sigma(\theta)(n-1)}{\varrho[\![\psi(\theta)]\!]}$$

for the radius $R > 0$, and system (1.33) reduces to a single equation for the temperature θ:

$$\mathsf{E}_e(\theta) := |\Omega|\varrho\epsilon_2(\theta) - m(\omega_n/n)R^n(\theta)\varrho[\![\epsilon(\theta)]\!] + \epsilon_\Gamma m\omega_n R^{n-1}(\theta) = \mathsf{E}_0.$$

We call the function $\mathsf{E}_e(\theta)$ the *equilibrium energy* function.

Note that only the temperature range $[\![\psi(\theta)]\!]/\sigma(\theta) > 0$ is relevant due to the requirement $R > 0$, and with

$$R_m^* = \sup\{R > 0 : \Omega \text{ contains } m \text{ disjoint balls of radius } R\}$$

we must also have $R < R_m^*$, i.e., with $\varphi(\theta) = \varrho[\![\psi(\theta)]\!]$

$$0 < \frac{\sigma(\theta)}{\varphi(\theta)} < \frac{R_m^*}{n-1}.$$

With $\epsilon(\theta) = \psi(\theta) - \theta\psi'(\theta)$ and $\epsilon_\Gamma = \sigma(\theta) - \theta\sigma'(\theta)$, after some calculations $\mathsf{E}_e(\theta)$ may be rewritten as

$$\begin{aligned}\mathsf{E}_e(\theta) &= |\Omega|\varrho\epsilon_2(\theta) + c_n\Big(\frac{\sigma(\theta)^n}{\varphi(\theta)^{n-1}} - \theta\frac{d}{d\theta}\frac{\sigma(\theta)^n}{\varphi(\theta)^{n-1}}\Big)\\ &= |\Omega|\varrho\epsilon_2(\theta) + c_n\Big(\frac{\sigma(\theta)^n}{\varphi(\theta)^{n-1}} + (n-1)\theta\frac{\sigma(\theta)^n\varphi'(\theta)}{\varphi(\theta)^n} - n\theta\frac{\sigma(\theta)^{n-1}\sigma'(\theta)}{\varphi(\theta)^{n-1}}\Big),\end{aligned}$$

where we have set $c_n = m\frac{\omega_n}{n}(n-1)^{n-1}$. Observe that the equilibrium energy function $\mathsf{E}_e(\theta)$ has the form

$$\mathsf{E}_e(\theta) = \Psi(\theta) - \theta\Psi'(\theta),$$

where $\Psi(\theta) = |\Omega|\varrho\psi_2(\theta) + c_n\sigma(\theta)^n/\varphi(\theta)^{n-1}$ plays the role of the equilibrium free energy. We have then

$$\mathsf{E}_e'(\theta) = -\theta\Psi''(\theta),$$

hence with

$$R'(\theta) = \frac{(n-1)\sigma'(\theta)}{\varphi(\theta)} - \frac{\sigma(\theta)(n-1)\varphi'(\theta)}{\varphi^2(\theta)},$$

after some more calculations

$$\mathsf{E}_e'(\theta) = (\kappa(\theta)|\varrho)_\Omega + \kappa_\Gamma(\theta)|\Gamma| - \frac{R(\theta)^2 l_0(\theta)^2|\Gamma|}{\theta\sigma(\theta)(n-1)},$$

with $l_0(\theta)$ defined in the previous subsection. Now recall the stability condition (1.32) to see that $\mathsf{E}_e'(\theta)$ is non-positive if and only if the stability condition holds. Thus, loosely speaking, total entropy is maximal at an equilibrium if and only if $\mathsf{E}_e'(\theta) \le 0$. We may write $\mathsf{E}_e'(\theta)$ yet in another form, namely

$$\mathsf{E}_e'(\theta) = (\kappa(\theta)|\varrho)_\Omega + \kappa_\Gamma(\theta)|\Gamma| - (n-1)|\Gamma|\sigma\theta(\frac{\varphi'(\theta))}{\varphi(\theta)} - \frac{\sigma'(\theta)}{\sigma(\theta)})^2.$$

In general it is not a simple task to analyze the equation for the temperature

$$\mathsf{E}_e(\theta) = \Psi(\theta) - \theta\Psi'(\theta) = \mathsf{E}_0,$$

unless more properties of the functions $\epsilon_j(\theta)$ and in particular of $\varphi(\theta)$ and $\sigma(\theta)$ are known. A natural assumption is that φ has exactly one positive zero $\theta_m > 0$, the so called *melting temperature*. Therefore we look at two examples.

Example 1. Suppose that ϵ_2 is increasing and convex, $\varrho = 1$, $\eta_\Gamma \equiv 0$, i.e., σ is constant, and that the heat capacities are identical, i.e., $[\![\kappa]\!] \equiv 0$. This implies

$$\theta\varphi''(\theta) = \theta[\![\psi''(\theta)]\!] = -[\![\kappa(\theta)]\!] \equiv 0,$$

which means that $\varphi(\theta) = \varphi_0 + \varphi_1\theta$ is linear. The melting temperature then is $0 < \theta_m = -\varphi_0/\varphi_1$, hence we have two cases.

Case 1. $\varphi_0 < 0$, $\varphi_1 > 0$. This means $l(\theta_m) > 0$.
Then the relevant temperature range is $\theta > \theta_m$ as φ is positive there. As $\theta \to \theta_m+$ we have $\varphi(\theta) \to 0$ hence $\mathsf{E}_e(\theta) \to \infty$, and also $\mathsf{E}_e(\theta) \to \infty$ for $\theta \to \infty$ as $\epsilon_2(\theta)$ is increasing and convex. Further, we have

$$\mathsf{E}_e'(\theta) = |\Omega|\epsilon_2'(\theta) - n(n-1)c_n\sigma(\theta)^n \frac{\varphi_1^2\theta}{(\varphi_0+\varphi_1\theta)^{n+1}},$$
$$\mathsf{E}_e''(\theta) = |\Omega|\epsilon_2''(\theta) + n(n-1)c_n\sigma(\theta)^n\varphi_1^2 \frac{-\varphi_0+n\varphi_1\theta}{(\varphi_0+\varphi_1\theta)^{n+2}} > 0,$$

which shows that $\mathsf{E}_e(\theta)$ is strictly convex for $\theta > \theta_m$. Thus $\mathsf{E}_e(\theta)$ has a unique minimum $\theta_0 > \theta_m$, $\mathsf{E}_e(\theta)$ is decreasing for $\theta_m < \theta < \theta_0$ and increasing for $\theta > \theta_0$. Thus there are precisely two equilibrium temperatures $\theta_*^+ \in (\theta_0, \infty)$ and $\theta_*^- \in (\theta_m, \theta_0)$ provided $\mathsf{E}_0 > \phi(\theta_0)$ and none if $\mathsf{E}_0 < \mathsf{E}(\theta_0)$. The smaller temperature leads to stable equilibria while the larger to unstable ones.

Case 2. $\varphi_0 > 0$, $\varphi_1 < 0$. This means $l(\theta_m) < 0$.
Then the relevant temperature range is $0 < \theta < \theta_m$ as h is positive there. As $\theta \to \theta_m-$ we have $\varphi(\theta) \to 0+$ hence $\mathsf{E}_e(\theta) \to -\infty$, and as $\theta \to 0+$ we have $\mathsf{E}_e(\theta) \to \mathsf{E}(0) = |\Omega|\epsilon_2(0) + c_n\sigma^n/\varphi_0^{n-1} > 0$, assuming that $\epsilon_2(0) = \lim_{s\to 0+}\epsilon_2(s)$ exists. Further, for θ close to 0 this implies $\mathsf{E}_e'(\theta) > 0$ and $\mathsf{E}_e'(\theta) \to -\infty$ as $\theta \to \theta_m-$. Therefore $\mathsf{E}_e'(\theta)$ admits at least one zero in $(0, \theta_m)$. But there may be more than one unless $\epsilon_2(\theta)$ is concave, so let us assume this. Let $\theta_0 \in (0, \theta_m)$ denote the absolute maximum of $\mathsf{E}_e(\theta)$ in $(0, \theta_m)$. Then there is exactly one equilibrium temperature $\theta_* \in (\theta_0, \theta_m)$ if $\mathsf{E}_0 < \mathsf{E}_e(0+)$ and it is stable; there are exactly two equilibria $\theta_*^- \in (0, \theta_0)$ and $\theta_*^+ \in (\theta_0, \theta_m)$ if $\mathsf{E}_e(0+) < \mathsf{E}_0 < \mathsf{E}_e(\theta_0)$, the first one is unstable the second stable. If $\mathsf{E}_0 > \mathsf{E}_e(\theta_0)$ there are no equilibria.

Note that in both cases these equilibrium temperatures give rise to equilibria only if the corresponding radius is smaller than R^*.

Example 2. Suppose $\eta_\Gamma \equiv 0$, $\varrho = 1$, and that the internal energies $\epsilon_j(\theta)$ are linear increasing, i.e.,

$$\epsilon_j(\theta) = a_j + \kappa_j\theta, \quad j = 1, 2,$$

where $\kappa_j > 0$, and now $[\![\kappa]\!] \neq 0$. The identity $\epsilon_j = \psi_j - \theta\psi_j'$ then leads to

$$\psi_j(\theta) = a_j + b_j\theta - \kappa_j\theta\log\theta, \quad j = 1, 2,$$

where b_j are arbitrary. This yields, with $\alpha = [\![a]\!]$, $\beta = [\![b]\!]$ and $\gamma = [\![\kappa]\!]$,

$$\varphi(\theta) = \alpha + \beta\theta - \gamma\theta\log\theta.$$

Scaling the temperature by $\theta = \theta_0\vartheta$ with $\beta - \gamma\log\theta_0 = 0$ and scaling φ we may assume $\beta = 0$ and $\gamma = \pm 1$. Then we have to investigate the equation $\mathsf{E}_e(\vartheta) = \mathsf{E}_1$, where

$$\mathsf{E}_e(\vartheta) = \delta\vartheta + \Big\{\frac{1}{\varphi^{n-1}(\vartheta)} + (n-1)\vartheta\frac{\varphi'(\vartheta)}{\varphi^n(\vartheta)}\Big\}, \quad \varphi(\vartheta) = \pm(\alpha + \vartheta\log\vartheta),$$

with $\delta > 0$ and $\alpha, \mathsf{E}_1 \in \mathbb{R}$. The requirement of existence of a melting temperature $\vartheta_m > 0$, i.e., a zero of $\varphi(\vartheta)$, leads to the restriction $\alpha \leq 1/e$. Also here we have to distinguish two cases, namely that of a plus-sign for φ where the relevant temperature range is $\vartheta > \vartheta_m$, and in case of a minus-sign it is $(0, \vartheta_m)$. Note that φ is convex in the first, and concave in the second case. In the case of $\varphi(\vartheta) = (\alpha + \vartheta\log\vartheta)$ we get

$$\begin{aligned}
\mathsf{E}_e'(\vartheta) &= \delta + (n-1)\Big\{\frac{\varphi(\vartheta) - n\vartheta\varphi'(\vartheta)^2}{\varphi^{n+1}(\vartheta)}\Big\},\\
\mathsf{E}_e''(\vartheta) &= n(n-1)\frac{\varphi'(\vartheta)}{\varphi^{n+2}(\vartheta)}\Big\{(n+1)\vartheta\varphi'(\vartheta)^2 - \varphi(\vartheta)(3 + \varphi'(\vartheta))\Big\}.
\end{aligned}$$

We have $\mathsf{E}_e(\vartheta) \to \infty$ for $\vartheta \to \infty$ and for $\vartheta \to \vartheta_m+$, hence $\mathsf{E}_e(\vartheta)$ has a global minimum θ_0 in (θ_m, ∞). Furthermore, $\mathsf{E}_e''(\vartheta) > 0$ in (θ_m, ∞), hence the minimum is unique and there are precisely two equilibrium temperatures $\vartheta_*^- \in (\vartheta_m, \vartheta_0)$ and $\vartheta_*^+ \in (\vartheta_0, \infty)$, provided $\mathsf{E}_1 > \mathsf{E}_e(\vartheta_0)$, the first one is stable, the second unstable.

To prove convexity of E_e we write

$$(n+1)\vartheta\varphi'(\vartheta)^2 - 3\varphi(\vartheta) - \varphi(\vartheta)\varphi'(\vartheta) = (n-1)\vartheta\varphi'(\vartheta)^2 + f(\vartheta),$$

where

$$f(\vartheta) = 2\vartheta\varphi'(\vartheta)^2 - \varphi(\vartheta)(3 + \varphi'(\vartheta)) = 2\vartheta(1 + \log\vartheta)^2 - (\alpha + \vartheta\log\vartheta)(4 + \log\vartheta).$$

We then have $f(\vartheta_m) = 2\vartheta_m(1 + \log\vartheta_m)^2 > 0$, and

$$f'(\theta) = (1 + \log\vartheta)^2 + 1 - \alpha/\vartheta > 1 - \alpha/\vartheta \geq 0,$$

for $\alpha \leq 1/e < \vartheta_m \leq \vartheta$.

Actually, the requirement that the melting temperature is unique, i.e., that φ has exactly one positive zero, implies $\alpha < 0$. Indeed, for $\alpha \in (0, 1/e)$ there is a

second zero $\vartheta_- > 0$ of φ, and φ is positive in $(0, \vartheta_-)$. Equilibrium temperatures in this range would not make sense physically.

Let us illustrate the sign in φ for the water-ice system, ignoring the density jump of water at freezing temperature. So suppose that Ω_2 consists of ice and Ω_1 of water. In this case we have $\kappa_1 > \kappa_2$, and hence $\gamma < 0$, which implies the plus-sign for φ. Here we obtain $\theta_*^{\pm} > \theta_m$, i.e., the ice is overheated. Equilibria only exist if ψ_0 is large enough, which means that there is enough energy in the system. If the energy in the system is very large, then the stable equilibrium temperature θ_*^- comes close to the melting temperature ϑ_m and then $R(\vartheta)$ will become large, eventually larger than R^*. This excludes equilibria in Ω, the physical interpretation being that everything will eventually melt.

On the other hand, if Ω_1 consists of ice and Ω_2 of water, we have the minus sign, which we want to consider next. Here we expect under-cooling of the water-phase, existence of equilibria only for low values of energy, and if the energy in the system is too small everything will freeze.

So assume that $\varphi(\vartheta) = -(\alpha + \vartheta \log \vartheta)$ and let $\alpha < 0$. Then the relevant temperature range is $(0, \vartheta_m)$. Here we have $\mathsf{E}_e(\vartheta) \to -\infty$ as $\vartheta \to \vartheta_m -$ and $\mathsf{E}_e(\vartheta) \to 1/|\alpha|^{n-1} > 0$. Moreover we have $\mathsf{E}_e'(0) = \delta + (n-1)/|\alpha|^n > 0$, and $\mathsf{E}_e'(\vartheta) \to -\infty$ for $\vartheta \to \vartheta_m -$. Therefore, $\mathsf{E}_e(\theta)$ has an absolute maximum in ϑ_0 in the interval $(0, \vartheta_m)$. If $\mathsf{E}_e(\vartheta)$ would be concave in $(0, \vartheta_m)$, then this maximum would be unique and there would be precisely two equilibrium temperatures $\vartheta_*^- \in (0, \vartheta_0)$ and $\vartheta_*^+ \in (\vartheta_0, \vartheta_m)$, provided $\mathsf{E}_1 \in (-\infty, \mathsf{E}_e(\vartheta_0))$, the first one unstable and the second stable. However, as we will see things are not as simple.

To investigate concavity of E_e in the interval $(0, \vartheta_m)$, we recompute the derivatives of E_e.

$$\mathsf{E}_e'(\vartheta) = \delta - (n-1)\Big\{\frac{1}{\varphi^n(\vartheta)} + n\frac{\vartheta\varphi'(\vartheta)^2}{\varphi^{n+1}(\vartheta)}\Big\},$$
$$\mathsf{E}_e''(\vartheta) = n(n-1)\frac{\varphi'(\vartheta)}{\varphi^{n+2}(\vartheta)}\Big\{(n+1)\vartheta\varphi'(\vartheta)^2 + \varphi(\vartheta)(3 - \varphi'(\vartheta))\Big\}.$$

Setting $\vartheta_+ = 1/e$, for $\vartheta \in (\vartheta_+, \vartheta_m)$ we have $\varphi(\vartheta) > 0$ and $\varphi'(\vartheta) < 0$, and hence $\mathsf{E}_e''(\vartheta) < 0$. On the other hand, for $\vartheta \in (0, \vartheta_+)$, both $\varphi(\vartheta)$ and $\varphi'(\vartheta)$ are positive. Then we rewrite

$$(n+1)\vartheta\varphi'(\vartheta)^2 + 3\varphi(\vartheta) - \varphi(\vartheta)\varphi'(\vartheta) = (n-1)\vartheta(1 + \log\vartheta)^2 + f(\vartheta),$$

where

$$\begin{aligned} f(\vartheta) &= 2\vartheta\varphi'(\vartheta)^2 + \varphi(\vartheta)(3 - \varphi'(\vartheta)) \\ &= 2\vartheta(1 + \log\vartheta)^2 - (\alpha + \vartheta\log\vartheta)(4 + \log\vartheta) \\ &= \vartheta(2 + \log^2(\theta)) - \alpha(4 + \log\vartheta), \\ f'(\vartheta) &= 2 + \log^2\vartheta + 2\log\vartheta - \alpha/\vartheta = (1 + \log\vartheta)^2 + 1 - \alpha/\vartheta \geq 0, \end{aligned}$$

provided $\alpha \le 0$. This shows that f is increasing, $f(\vartheta) \to -\infty$ as $\vartheta \to 0$, and $f(1/e^3) = 11/e^3 - \alpha > 0$. On the other hand, the function $\vartheta(1+\log\vartheta)^2$ is increasing in $(0, 1/e^3)$, hence $\psi''(\vartheta)$ has a unique zero $\vartheta_- \in (0, 1/e^3)$. Therefore, E_e is concave in $(0, \vartheta_-) \cup (\vartheta_+, \vartheta_m)$ and convex in $(\vartheta_-, \vartheta_+)$, and E_e' has a minimum at ϑ_- and a maximum at ϑ_+. Observe that $\mathsf{E}_e'(\vartheta) < \delta$, $\mathsf{E}_e'(\vartheta) \to -\infty$ for $\vartheta \to \vartheta_m-$ and $\mathsf{E}_e'(0+) = \delta - (n-1)/|\alpha|^n < \psi'(\vartheta_+)$. Therefore, E_0' may have no, one, two, or three zeros in $(0, \vartheta_m)$, depending on the value of $\delta > 0$. However, if $\delta > 0$ is large enough, then E_e' has only one zero ϑ_1 which lies in $(\vartheta_+, \vartheta_m)$. In this case E_e is increasing in $(0, \vartheta_1)$ and decreasing in $(\vartheta_1, \vartheta_m)$, hence for $\mathsf{E}_e \in (\psi(0), \psi(\vartheta_1))$ there are precisely two equilibrium temperatures, the smaller leads to unstable, the larger to a stable equilibrium. If $\mathsf{E}_1 < \mathsf{E}_e(0+)$ there is a unique equilibrium which is stable, and in case $\mathsf{E}_1 > \mathsf{E}_e(\vartheta_1)$ there is none. However, in general there may be up to four equilibrium temperatures.

1.3 Goals and Strategies

In this book we will consider only the *completely incompressible* case, i.e., the densities ϱ_1 and ϱ_2 are assumed to be constant. Throughout we neglect viscous surface stress, so we set $S_\Gamma \equiv 0$. Thus the only surface stress acting is the surface tension $T_\Gamma = \sigma \mathcal{P}_\Gamma$. We always assume the constitutive laws

$$T = S - \pi I, \quad S := 2\mu(\theta) D, \quad D = (\nabla u + [\nabla u]^{\mathsf{T}})/2.$$

In this book we want to consider the following main problems which are ordered by complexity. The main hypotheses for these problems are formulated as well. Throughout, Ω will be a bounded domain with boundary $\partial\Omega$ of class C^3.

3.1 The Main Models

Problem 1. The Stefan Problem with Surface Tension.
Here we assume $\varrho_1 = \varrho_2 =: \varrho > 0$, $\sigma > 0$, and $u \equiv 0$.
Then we have

$$V_\Gamma = -j_\Gamma/\varrho, \quad [\![-T\nu_\Gamma]\!] = \sigma H_\Gamma \nu_\Gamma,$$

hence the Gibbs–Thomson law becomes

$$[\![\psi(\theta)]\!] = \frac{1}{\varrho}[\![T\nu_\Gamma \cdot \nu_\Gamma]\!] = -\frac{\sigma}{\varrho} H_\Gamma,$$

and we have the Stefan law $-\varrho[\![\theta\eta(\theta)]\!]V_\Gamma - [\![d(\theta)\partial_\nu\theta]\!] = 0$ on Γ. Observing that at melting temperature θ_m there holds $[\![\psi(\theta_m)]\!] = 0$, by linearization of ψ one obtains with the relative temperature $\vartheta = (\theta - \theta_m)/\theta_m$

$$\vartheta = -\frac{\sigma}{l_m \varrho} H_\Gamma, \quad l_m = -\theta_m [\![\eta(\theta_m)]\!],$$

which is the standard constitutive relation for the classical Stefan problem with surface tension. Here l_m is the latent heat at melting temperature. Similarly, the

linearized Stefan law becomes $\varrho l_m V_\Gamma - [\![d(\theta)\partial_\nu \theta]\!] = 0$, which is the classical one. Note that these relations are only valid near melting temperature, and in particular exclude large curvatures of Γ. In this model, surface entropy is zero and balance of surface energy is trivial. The model equations read

$$\begin{aligned} \varrho\kappa(\theta)\partial_t\theta - \operatorname{div}(d(\theta)\nabla\theta) &= 0 \quad && \text{in } \Omega\setminus\Gamma(t),\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ [\![\theta]\!] = 0, \quad \varrho[\![\psi(\theta)]\!] + \sigma H_\Gamma &= 0 && \text{on } \Gamma(t),\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{1.34}$$

$$\begin{aligned} -\varrho[\![\theta\eta(\theta)]\!]V_\Gamma - [\![d(\theta)\partial_\nu\theta]\!] &= 0 \quad \text{on } \Gamma(t),\\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{1.35}$$

Concerning ψ and d we assume

$$(\mathbf{H1}) \quad \psi \in C^3(0,\infty),\ d \in C^2(0,\infty), \quad -\psi''(s), d(s) > 0 \ \text{ for all } s > 0.$$

Remark 1.3.1. If $\kappa \equiv 0$, i.e., if ψ is linear, we obtain the so-called quasi-stationary Stefan problem with surface tension, also called *Mullins–Sekerka problem* or *Mullins–Sekerka flow* in the literature. It has the same equilibria as in the case $\kappa \not\equiv 0$, but their stability properties are different.

Problem 2. The Two-Phase Navier–Stokes Problem with Surface Tension.
Here we assume $j_\Gamma \equiv 0$, $\sigma > 0$ constant.
This is the case without phase transitions. Then

$$[\![u]\!] = 0, \quad V_\Gamma = u\cdot\nu_\Gamma, \quad -[\![T\nu_\Gamma]\!] = \sigma H_\Gamma \nu_\Gamma,$$

which leads to the classical model for *incompressible two-phase flow without phase transitions.*

$$\begin{aligned} \varrho(\partial_t u + u\cdot\nabla u) - \operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ \operatorname{div} u &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] = 0, \quad -[\![S\nu_\Gamma]\!] + [\![\pi]\!]\nu_\Gamma &= \sigma H_\Gamma\nu_\Gamma && \text{on } \Gamma(t),\\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{1.36}$$

$$\begin{aligned} \varrho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) &= 2\mu(\theta)|D|_2^2 \quad && \text{in } \Omega\setminus\Gamma(t),\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ [\![\theta]\!] = 0, \quad [\![d(\theta)\partial_\nu\theta]\!] &= 0 && \text{on } \Gamma(t),\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{1.37}$$

$$V_\Gamma = u\cdot\nu_\Gamma \quad \text{on} \quad \Gamma(t), \quad \Gamma(0) = \Gamma_0. \tag{1.38}$$

Here we suppose

$$(\mathbf{H2}) \quad \psi \in C^3(0,\infty),\ d,\mu \in C^2(0,\infty), \quad -\psi''(s), d(s), \mu(s) > 0 \ \text{ for all } s > 0.$$

Remark 1.3.2. **(i)** If μ is constant, then the Navier-Stokes problem decouples from the heat problem. More generally, in the *isothermal case*, the temperature is assumed to be constant and the equation for the temperature, i.e., energy balance, is ignored. This means that the friction term $2\mu|D|_2^2$ is neglected. In this case the reduced energy E_0 defined by

$$\mathsf{E}_0(u,\Gamma) := \frac{1}{2}\int_{\Omega\setminus\Gamma} \varrho|u|^2\,dx + \sigma|\Gamma|$$

is a strict Lyapunov functional, as the identity

$$\frac{d}{dt}\mathsf{E}_0(u(t),\Gamma(t)) = -2\int_\Omega \mu|D|_2^2\,dx$$

and Korn's inequality show. Also in this case the equilibria are zero velocity and constant pressures in the components of the phases. The disperse phase Ω_1 is an at most countable union of disjoint balls, and the radia of the balls are related to the pressures according to the *Young-Laplace law*

$$[\![\pi]\!] = \sigma H_\Gamma = -\frac{\sigma(n-1)}{R}.$$

(ii) If θ is constant and ignoring inertia (i.e., the term $\varrho(\partial_t u + u\cdot\nabla u)$) we are left with a quasi-stationary problem, the *two-phase Stokes problem*, which generates the so-called *two-phase Stokes flow*. More precisely, this problem reads

$$\begin{aligned} -\operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ \operatorname{div} u &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] = 0,\quad -[\![S\nu_\Gamma]\!] + [\![\pi]\!]\nu_\Gamma &= \sigma H_\Gamma\nu_\Gamma && \text{on } \Gamma(t),\\ V_\Gamma &= u\cdot\nu_\Gamma && \text{on } \Gamma(t),\\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{1.39}$$

(iii) If $\sigma = 0$, then $u = 0$ is a solution of the Navier–Stokes problem. Then we end up with the standard transmission problem for the heat equation with fixed domain.

(iv) Modeling flows in porous media frequently relies on *Darcy's law*, which reads

$$u = -k\nabla\pi,$$

where $k = k(\pi) > 0$ may depend on π, and depends on the phases. The interface velocity then becomes

$$V_\Gamma = u\cdot\nu_\Gamma = -k(\pi)\partial_\nu\pi.$$

This is meaningful, provided

$$-[\![k(\pi)\partial_\nu\pi]\!] = [\![u\cdot\nu_\Gamma]\!] = 0.$$

Furthermore, the driving force for the evolution of the interface is surface tension, hence we require

$$[\![\pi]\!] = \sigma H_\Gamma,$$

where $\sigma > 0$ is constant. Finally, we have to take into account conservation of mass which results in the *porous medium equation*

$$\partial_t \varrho(\pi) - \operatorname{div}(\varrho(\pi)k(\pi)\nabla\pi) = 0.$$

Here $\varrho > 0$ is non-decreasing w.r.t. π, and depends on the phases. Summarizing we obtain the problem

$$\begin{aligned}
\varrho'(\pi)\partial_t\pi - \operatorname{div}(\varrho(\pi)k(\pi)\nabla\pi) &= 0 && \text{in } \Omega\setminus\Gamma(t),\\
\partial_\nu\pi &= 0 && \text{on } \partial\Omega,\\
[\![\pi]\!] &= \sigma H_\Gamma && \text{on } \Gamma(t),\\
[\![k(\pi)\partial_\nu\pi]\!] &= 0 && \text{on } \Gamma(t),\\
V_\Gamma + k(\pi)\partial_\nu\pi &= 0 && \text{on } \Gamma(t),\\
\Gamma(0) = \Gamma_0, \quad \pi(0) &= \pi_0.
\end{aligned} \tag{1.40}$$

This problem is called the *Verigin problem* in the literature, and its quasi-steady (i.e., incompressible) version, where ϱ is constant in the phases, is known as the *Muskat problem* or the *Muskat flow*, a geometric evolution equation.

(v) A variant of Darcy's law is *Forchheimer's law* which reads

$$g(|u|)u = -\nabla\pi,$$

where the function g is strictly positive and $s \mapsto sg(s)$ is strictly increasing. Solving this equation for u we obtain

$$u = -k(|\nabla\pi|^2)\nabla\pi,$$

where k is strictly positive and satisfies $k(t) + 2tk'(t) > 0$ on $\mathbb{R}_+$. These conditions ensure strong ellipticity of the operator $-\operatorname{div}(k(|\nabla\pi|^2)\nabla\pi)$.

Problem 3. Incompressible Two-Phase Fluid Flow with Phase Transition I.
Here we assume $\varrho_1 = \varrho_2 =: \varrho$, $\sigma > 0$ constant.
In this situation the Navier–Stokes problem is only weakly coupled to a Stefan problem. It can be treated by combining the methods developed for Problems 1 and 2. We call this case *temperature dominated*.

$$\begin{aligned}
\varrho(\partial_t u + u\cdot\nabla u) - \operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega\setminus\Gamma(t),\\
\operatorname{div} u &= 0 && \text{in } \Omega\setminus\Gamma(t),\\
u &= 0 && \text{on } \partial\Omega,\\
[\![u]\!] = 0, \quad -[\![S\nu_\Gamma]\!] + [\![\pi]\!]\nu_\Gamma &= \sigma H_\Gamma\nu_\Gamma && \text{on } \Gamma(t),\\
u(0) &= u_0 && \text{in } \Omega.
\end{aligned} \tag{1.41}$$

$$\begin{aligned} \varrho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) &= 2\mu(\theta)|D|_2^2 && \text{in } \Omega\setminus\Gamma(t),\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ [\![\theta]\!] = 0, \quad [\![\theta\eta(\theta)]\!]j_\Gamma - [\![d(\theta)\partial_\nu\theta]\!] &= 0 && \text{on } \Gamma(t),\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{1.42}$$

$$\begin{aligned} \varrho[\![\psi(\theta)]\!] + \sigma H_\Gamma &= 0 && \text{on } \Gamma(t),\\ V_\Gamma = u_\Gamma\cdot\nu_\Gamma = u\cdot\nu_\Gamma - j_\Gamma/\varrho &&& \text{on } \Gamma(t),\\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{1.43}$$

We set hypothesis (**H3**) := (**H2**). Recall that we can eliminate the phase flux j_Γ by

$$j_\Gamma = -[\![d(\theta)\partial_\nu\theta]\!]/l(\theta),$$

provided $l(\theta)\neq 0$. This will be one restriction for well-posedness of this model.

Remark 1.3.3. We will see that the Navier–Stokes problem is only weakly coupled to the Stefan problem with surface tension. Setting $u=0$ and ignoring the Navier–Stokes problem it reduces to Problem **(P1)**.

Problem 4. Incompressible Two-Phase Fluid Flow with Phase Transition II.
Here we assume $\varrho_1\neq\varrho_2$, $\sigma>0$ constant.
This case is more difficult than the previous one. Here the problem for θ is only weakly coupled with that for (u,π,h). We call this case *velocity dominated*.

$$\begin{aligned} \varrho(\partial_t u + u\cdot\nabla u) - \operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ \operatorname{div} u &= 0 && \text{in } \Omega\setminus\Gamma(t),\\ u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] &= [\![1/\varrho]\!]j_\Gamma\nu_\Gamma && \text{on } \Gamma(t),\\ [\![1/\varrho]\!]j_\Gamma^2\nu_\Gamma - [\![T\nu_\Gamma]\!] &= \sigma H_\Gamma\nu_\Gamma && \text{on } \Gamma(t),\\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{1.44}$$

$$\begin{aligned} \varrho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) &= 2\mu(\theta)|D|_2^2 && \text{in } \Omega\setminus\Gamma(t),\\ \partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\ [\![\theta]\!] &= 0 && \text{on } \Gamma(t),\\ [\![\theta\eta(\theta)]\!]j_\Gamma - [\![d(\theta)\partial_\nu\theta]\!] &= 0 && \text{on } \Gamma(t),\\ \theta(0) &= \theta_0 && \text{in } \Omega. \end{aligned} \tag{1.45}$$

$$\begin{aligned} [\![\psi(\theta)]\!] + [\![1/2\varrho^2]\!]j_\Gamma^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] &= 0 && \text{on } \Gamma(t),\\ V_\Gamma = u_\Gamma\cdot\nu_\Gamma = u\cdot\nu_\Gamma - j_\Gamma/\varrho &&& \text{on } \Gamma(t),\\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{1.46}$$

The main hypothesis here is (**H4**) := (**H2**). Here we can eliminate j_Γ as explained before by means of the identities

$$j_\Gamma = [\![u \cdot \nu_\Gamma]\!]/[\![1/\rho]\!], \quad V_\Gamma = [\![\rho u \cdot \nu_\Gamma]\!]/[\![\rho]\!].$$

Remark 1.3.4. (i) A variant of this problem concerns the situation where heat conduction is taken into account in both phases but only one phase is moving, the model for *melting and solidification*. This problem formally results by letting $\mu_1 \to \infty$. To obtain this model, for finite μ_1, let T_j denote the stress tensor in Ω_j. Set $u \equiv \pi \equiv 0$ in Ω_1, maintain the jump condition for u, drop the stress jump condition on the interface, but replace $T_1\nu_\Gamma \cdot \nu_\Gamma$ in the Gibbs–Thomson law from the normal stress jump, according to

$$T_1\nu_\Gamma \cdot \nu_\Gamma = T_2\nu_\Gamma \cdot \nu_\Gamma + \sigma H_\Gamma - [\![1/\varrho]\!] j_\Gamma^2$$

to the result

$$u_2 = [\![1/\varrho]\!] j_\Gamma \nu_\Gamma, \quad V_\Gamma = -j_\Gamma/\varrho_1,$$

and

$$[\![\psi(\theta)]\!] + (1/2)[\![1/\varrho]\!]^2 j_\Gamma^2 - [\![1/\varrho]\!] T_2\nu_\Gamma \cdot \nu_\Gamma + (\sigma/\varrho_1)H_\Gamma = 0.$$

These conditions on the interface do not contain the viscosity μ_1, hence we may formally pass to the limit $\mu_1 \to \infty$. Therefore, the resulting model reads

$$\begin{aligned}
\varrho(\partial_t u + u \cdot \nabla u) - \operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega_2(t),\\
\operatorname{div} u &= 0 && \text{in } \Omega_2(t),\\
u &= 0 && \text{on } \partial\Omega,\\
u &= [\![1/\varrho]\!] j_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\\
u(0) &= u_0 && \text{in } \Omega.
\end{aligned}$$

$$\begin{aligned}
\varrho\kappa(\theta)(\partial_t\theta + u \cdot \nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) &= 2\mu(\theta)|D|_2^2 && \text{in } \Omega \setminus \Gamma(t),\\
\partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\
[\![\theta]\!] &= 0 && \text{on } \Gamma(t),\\
[\![\theta\eta(\theta)]\!] j_\Gamma - [\![d(\theta)\partial_\nu\theta]\!] &= 0 && \text{on } \Gamma(t),\\
\theta(0) &= \theta_0 && \text{in } \Omega_2.
\end{aligned}$$

$$\begin{aligned}
[\![\psi(\theta)]\!] + (1/2)[\![1/\varrho]\!]^2 j_\Gamma^2 - [\![1/\varrho]\!] T_2\nu_\Gamma \cdot \nu_\Gamma + (\sigma/\varrho_1)H_\Gamma &= 0 && \text{on } \Gamma(t),\\
V_\Gamma &= -j_\Gamma/\varrho_1 && \text{on } \Gamma(t),\\
\Gamma(0) &= \Gamma_0.
\end{aligned}$$

This model also has conservation of total energy and production of total entropy is nonnegative, hence it is consistent with thermodynamics. Note, however, that momentum is not conserved across the interface, as at the outer boundary $\partial\Omega$. Furthermore, if the densities are equal, the viscosity is constant, and the initial

velocity is zero also in Ω_2, then $u \equiv 0$ and π is constant in Ω_2. In this situation the model reduces to Problem 1.

(ii) In the *isothermal case* the temperature θ is assumed to be constant and the heat problem is ignored. Then we obtain a model for isothermal two-phase flows with surface tension and phase transition, the latter is driven by pressure, only.

(iii) Again in the incompressible, isothermal case, ignoring inertia and j_Γ^2, we obtain the equations for the *Stokes flow with phase transition* which reads

$$\begin{aligned} -\operatorname{div} S + \nabla\pi &= 0 && \text{in } \Omega \setminus \Gamma(t),\\ \operatorname{div} u &= 0 && \text{in } \Omega \setminus \Gamma(t),\\ u &= 0 && \text{on } \partial\Omega,\\ [\![u]\!] &= [\![1/\varrho]\!] j_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\\ -[\![T\nu_\Gamma]\!] &= \sigma H_\Gamma \nu_\Gamma && \text{on } \Gamma(t),\\ -[\![T\nu_\Gamma \cdot \nu_\Gamma/\varrho]\!] &= c && \text{on } \Gamma(t),\\ V_\Gamma = u_\Gamma \cdot \nu_\Gamma &= u \cdot \nu_\Gamma - j_\Gamma/\varrho && \text{on } \Gamma(t),\\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{1.47}$$

Here $c = -[\![\psi]\!]$ is constant. The phase flux j_Γ can be eliminated from the normal component of the velocity jump, and so we have a transmission problem for the Stokes equation with $(n-1)$ jump conditions for the velocity and $(n+1)$ for the normal stresses. This leads to a geometric evolution equation where the interface is moved by surface tension as well as by stationary phase transitions due to the different densities.

(iv) Employing again Darcy's (or Forchheimer's) law $u = -k(\pi)\nabla\pi$, we obtain the *Verigin problem with phase transition*

$$\begin{aligned} \varrho'(\pi)\partial_t\pi - \operatorname{div}(\varrho(\pi)k(\pi)\nabla\pi) &= 0 && \text{in } \Omega \setminus \Gamma(t),\\ \partial_\nu \pi &= 0 && \text{on } \partial\Omega,\\ [\![\pi]\!] &= \sigma H_\Gamma && \text{on } \Gamma(t),\\ [\![\psi + \pi/\varrho]\!] &= 0 && \text{on } \Gamma(t),\\ [\![\varrho]\!] V_\Gamma + [\![\varrho(\pi)k(\pi)\partial_\nu\pi]\!] &= 0 && \text{on } \Gamma(t),\\ \Gamma(0) = \Gamma_0, \quad \pi(0) &= \pi_0. \end{aligned} \tag{1.48}$$

Note that here the pressure π is the independent variable, and Maxwell's law then reads $\psi'(\pi) = \pi\varrho'(\pi)/\varrho^2(\pi)$. Its quasi-steady version, where ϱ is constant in the phases, is the *Muskat flow with phase transition*, another geometric evolution equation.

Problem 5. Marangoni Forces I. Here we assume $\varrho_1 = \varrho_2 =: \varrho$, σ nonconstant. Experience shows that σ is strictly decreasing and positive at melting temperature

θ_m, and as σ is also concave, it has a unique zero $\theta_c > \theta_m$; we call θ_c the *critical temperature*. As beyond the critical temperature there is no phase separation anymore, we restrict to the temperature range $\theta \in (0, \theta_c)$.

Here the model equations read

$$\begin{aligned}
\varrho(\partial_t u + u \cdot \nabla u) - \operatorname{div} S + \nabla \pi &= 0 && \text{in } \Omega \setminus \Gamma(t),\\
\operatorname{div} u &= 0 && \text{in } \Omega \setminus \Gamma(t),\\
u &= 0 && \text{on } \partial\Omega,\\
[\![u]\!] = 0, \quad \mathcal{P}_\Gamma u_\Gamma &= \mathcal{P}_\Gamma u && \text{on } \Gamma(t),\\
-[\![T\nu_\Gamma]\!] &= \sigma(\theta_\Gamma) H_\Gamma \nu_\Gamma + \sigma'(\theta_\Gamma)\nabla_\Gamma \theta_\Gamma && \text{on } \Gamma(t),\\
u(0) &= u_0 && \text{in } \Omega.
\end{aligned} \tag{1.49}$$

$$\begin{aligned}
\varrho\kappa(\theta)(\partial_t \theta + u \cdot \nabla\theta) - \operatorname{div}(d(\theta)\nabla\theta) &= 2\mu(\theta)|D|_2^2 && \text{in } \Omega \setminus \Gamma(t),\\
\partial_\nu \theta &= 0 && \text{on } \partial\Omega,\\
\theta &= \theta_\Gamma && \text{on } \Gamma(t),\\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned} \tag{1.50}$$

$$\begin{aligned}
&\kappa_\Gamma(\theta_\Gamma)\frac{D}{Dt}\theta_\Gamma - \operatorname{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma)\\
&\qquad = \theta_\Gamma \sigma'(\theta_\Gamma)\operatorname{div}_\Gamma u_\Gamma - ([\![\theta\eta(\theta)]\!] j_\Gamma - [\![d(\theta)\partial_\nu\theta]\!]) && \text{on } \Gamma(t),\\
&\varrho[\![\psi(\theta)]\!] + \sigma(\theta) H_\Gamma = 0 && \text{on } \Gamma(t),\\
&V_\Gamma = u_\Gamma \cdot \nu_\Gamma = u \cdot \nu_\Gamma - j_\Gamma/\varrho && \text{on } \Gamma(t),\\
&\Gamma(0) = \Gamma_0.
\end{aligned} \tag{1.51}$$

We assume

(H5) $\psi, \sigma \in C^3(0,\theta_c)$, $d, d_\Gamma, \mu \in C^2(0,\theta_c)$,
$-\psi''(s), -\sigma''(s), -\sigma'(s), d(s), d_\Gamma(s), \mu(s) > 0$ for all $s \in (0,\theta_c)$.

In this problem, the Navier-Stokes problem is again only weakly coupled with a Stefan problem, modified by energy conservation on the interface. Note that

$$\operatorname{div}_\Gamma u_\Gamma = \operatorname{div}_\Gamma \mathcal{P}_\Gamma u - H_\Gamma V_\Gamma,$$

which eliminates u_Γ, but here it is not so easy to eliminate j_Γ, as for this problem it really is an implicit variable!

Remark 1.3.5. Setting $u = 0$ and ignoring the Navier-Stokes problem, the latter becomes the Stefan problem with surface tension and surface heat capacity, which reads

$$\begin{aligned}
\varrho\kappa(\theta)\partial_t\theta - \operatorname{div}(d(\theta)\nabla\theta) &= 0 && \text{in } \Omega \setminus \Gamma(t),\\
\partial_\nu\theta &= 0 && \text{on } \partial\Omega,\\
\theta &= \theta_\Gamma && \text{on } \Gamma(t),\\
\theta(0) &= \theta_0 && \text{in } \Omega.
\end{aligned} \tag{1.52}$$

$$\begin{aligned}
\kappa_\Gamma(\theta_\Gamma)\frac{D}{Dt}\theta_\Gamma - \mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma) & \\
= -\theta_\Gamma\sigma'(\theta_\Gamma)H_\Gamma V_\Gamma + \varrho[\![\theta\eta(\theta)]\!]V_\Gamma + [\![d(\theta)\partial_\nu\theta]\!] & \quad \text{on } \Gamma(t), \\
\varrho[\![\psi(\theta)]\!] + \sigma(\theta)H_\Gamma = 0 & \quad \text{on } \Gamma(t), \\
\Gamma(0) = \Gamma_0. &
\end{aligned} \tag{1.53}$$

This problem will be studied in Chapter 12.

Problem 6. Marangoni Forces II. Here we assume $\varrho_1 \neq \varrho_2$, σ nonconstant. This is the model of highest complexity considered in this book.

$$\begin{aligned}
\varrho(\partial_t u + u\cdot\nabla u) - \mathrm{div}\, S + \nabla\pi = 0 & \quad \text{in } \Omega\setminus\Gamma(t), \\
\mathrm{div}\, u = 0 & \quad \text{in } \Omega\setminus\Gamma(t), \\
u = 0 & \quad \text{on } \partial\Omega, \\
\mathcal{P}_\Gamma u_\Gamma = \mathcal{P}_\Gamma u, \quad [\![u]\!] = [\![1/\varrho]\!]j_\Gamma\nu_\Gamma & \quad \text{on } \Gamma(t), \\
[\![1/\varrho]\!]j_\Gamma^2\nu_\Gamma - [\![T\nu_\Gamma]\!] = \sigma(\theta_\Gamma)H_\Gamma + \sigma'(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma & \quad \text{on } \Gamma(t), \\
u(0) = u_0, & \quad \text{in } \Omega.
\end{aligned} \tag{1.54}$$

$$\begin{aligned}
\varrho\kappa(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \mathrm{div}(d(\theta)\nabla\theta) = 2\mu(\theta)|D|_2^2 & \quad \text{in } \Omega\setminus\Gamma(t), \\
\partial_\nu\theta = 0 & \quad \text{on } \partial\Omega, \\
\theta = \theta_\Gamma & \quad \text{on } \Gamma(t), \\
\theta(0) = \theta_0 & \quad \text{in } \Omega.
\end{aligned} \tag{1.55}$$

$$\begin{aligned}
\kappa_\Gamma(\theta_\Gamma)\frac{D}{Dt}\theta_\Gamma - \mathrm{div}_\Gamma(d_\Gamma(\theta_\Gamma)\nabla_\Gamma\theta_\Gamma) = & \\
= \theta_\Gamma\sigma'(\theta_\Gamma)\mathrm{div}_\Gamma u_\Gamma - ([\![\theta\eta(\theta)]\!]j_\Gamma - [\![d(\theta)\partial_\nu\theta]\!]) & \quad \text{on } \Gamma(t), \\
V_\Gamma = u_\Gamma\cdot\nu_\Gamma = u\cdot\nu_\Gamma - j_\Gamma/\varrho & \quad \text{on } \Gamma(t), \\
[\![\psi(\theta)]\!] + [\![1/2\varrho^2]\!]j_\Gamma^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] = 0 & \quad \text{on } \Gamma(t), \\
\Gamma(0) = \Gamma_0. &
\end{aligned} \tag{1.56}$$

The main assumption on the coefficients is (**H6**) := (**H5**). Here j_Γ can be eliminated as in Problem **(P4)**, and $\mathrm{div}_\Gamma u_\Gamma$ as in Problem **(P5)**.

3.2 Transformation to a Fixed Domain

A basic idea is to transform Problems **(P1)**–**(P6)** to a domain with a fixed interface Σ, where $\Gamma(t)$ is parameterized over Σ by means of a height function $h(t)$. For this we rely on the so-called *Hanzawa transform* which we will now explain.

(a) The Hanzawa Transform

We assume, as before, that $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^2, and that $\Gamma \subset \Omega$ is a hypersurface of class C^2, i.e., a C^2-manifold which

is the boundary of a bounded domain $\Omega_1 \subset \Omega$. As above, we set $\Omega_2 = \Omega \backslash \bar{\Omega}_1$. Note that Ω_2 typically is connected, while Ω_1 may be disconnected. In the later case, Ω_1 consists of finitely many components, since $\partial\Omega_1 = \Gamma \subset \Omega$ by assumption is a manifold, at least of class C^2. As will be shown in Section 2.4, the hypersurface Γ can be approximated by a real analytic hypersurafce Σ, in the sense that the Hausdorff distance of the second-order normal bundles is as small as we please. More precisely, given $\eta > 0$, there exists an analytic hypersurface Σ such that $d_H(\mathcal{N}^2\Sigma, \mathcal{N}^2\Gamma) \leq \eta$. If $\eta > 0$ is small enough, then Σ bounds a domain Ω_1^Σ with $\overline{\Omega_1^\Sigma} \subset \Omega$ and we set $\Omega_2^\Sigma = \Omega \setminus \overline{\Omega_1^\Sigma} \subset \Omega$.

In the sequel we will freely use results that are established in Chapter 2. In particular, it is shown in Section 2.3 that the C^2-hypersurface Σ admits a tubular neighbourhood, which means that there is $a_0 > 0$ such that the map

$$\begin{aligned} &\Lambda : \Sigma \times (-a_0, a_0) \to \mathbb{R}^n, \\ &\Lambda(p, r) := p + r\nu_\Sigma(p) \end{aligned}$$

is a diffeomorphism from $\Sigma \times (-a_0, a_0)$ onto $\mathrm{im}(\Lambda)$, the image of Λ. The inverse

$$\Lambda^{-1} : \mathrm{im}(\Lambda) \to \Sigma \times (-a_0, a_0)$$

of this map is conveniently decomposed as

$$\Lambda^{-1}(x) = (\Pi_\Sigma(x), d_\Sigma(x)), \quad x \in \mathrm{im}(\Lambda).$$

Here $\Pi_\Sigma(x)$ means the metric projection of x onto Σ and $d_\Sigma(x)$ the signed distance from x to Σ; so $|d_\Sigma(x)| = \mathrm{dist}(x, \Sigma)$ and $d_\Sigma(x) < 0$ if and only if $x \in \Omega_1^\Sigma$. In particular we have $\mathrm{im}(\Lambda) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Sigma) < a_0\}$. The maximal number a_0 is given by the radius $r_\Sigma > 0$, defined as the largest number r such that the exterior and interior ball conditions for Σ in Ω hold. In the following, we choose

$$a_0 = r_\Sigma/2 \quad \text{and} \quad a = a_0/3.$$

The derivatives of $\Pi_\Sigma(x)$ and $d_\Sigma(x)$ are given by

$$\nabla d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad \partial\Pi_\Sigma(x) = M_0(d_\Sigma(x))\mathcal{P}_\Sigma(\Pi_\Sigma(x)),$$

where, as before, $\mathcal{P}_\Sigma(p) = I - \nu_\Sigma(p) \otimes \nu_\Sigma(p)$ denotes the orthogonal projection onto the tangent space $T_p\Sigma$ of Σ at $p \in \Sigma$, and $M_0(r) = (I - rL_\Sigma)^{-1}$, with L_Σ the Weingarten tensor. Then

$$|M_0(r)| \leq 1/(1 - r|L_\Sigma|) \leq 3 \quad \text{for all } |r| \leq 2r_\Sigma/3.$$

If $\mathrm{dist}(\Gamma, \Sigma)$ is small enough, we may use the map Λ to parameterize the unknown free boundary $\Gamma(t)$ over Σ by means of a *height function* $h(t)$ via

$$\Gamma(t) = \{p + h(t, p)\nu_\Sigma(p) : p \in \Sigma\}, \quad t \geq 0,$$

for small $t \geq 0$, at least. Extend this diffeomorphism to all of $\bar{\Omega}$ by means of

$$\Xi_h(t,x) = x + \chi(d_\Sigma(x)/a)h(t,\Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)) =: x + \xi_h(t,x).$$

Here χ denotes a suitable cut-off function. More precisely, let $\chi \in \mathcal{D}(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(r) = 1$ for $|r| < 1$, and $\chi(r) = 0$ for $|r| > 2$. (We may choose χ in such a way that $1 < |\chi'|_\infty \leq 3$.) Note that $\Xi_h(t,x) = x$ for $|d_\Sigma(x)| > 2a$, and

$$\Pi_\Sigma(\Xi_h(t,x)) = \Pi_\Sigma(x), \quad |d_\Sigma(x)| < a,$$

as well as

$$d_\Sigma(\Xi_h(t,x)) = d_\Sigma(x) + \chi(d_\Sigma(x)/a)h(t,\Pi_\Sigma(x)), \quad |d_\Sigma(x)| < 2a.$$

This yields

$$\Xi_h^{-1}(t,x) = x - h(t,\Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)) \quad \text{for} \ \ |d_\Sigma(x)| < a,$$

in particular,

$$\Xi_h^{-1}(t,x) = x - h(t,x)\nu_\Sigma(x) \quad \text{for} \ \ x \in \Sigma.$$

Furthermore, we obtain

$$\partial\,\Xi_h = I + \partial\xi_h, \qquad (\partial\,\Xi_h)^{-1} = I - [I + \partial\xi_h]^{-1}\partial\xi_h =: I - M_1^{\mathsf{T}}(h),$$

where $\partial := \partial_x$ denotes the derivative with respect to $x \in \mathbb{R}^n$, and

$$\begin{aligned}&\partial\xi_h(t,x)\\ &\quad = \nu_\Sigma(\Pi_\Sigma(x)) \otimes M_0(d_\Sigma(x))\nabla_\Sigma h(t,\Pi_\Sigma(x)) - h(t,\Pi_\Sigma(x))M_0(d_\Sigma(x))L_\Sigma(\Pi_\Sigma(x))\end{aligned}$$

for $|d_\Sigma(x)| < a$, $\xi_h'(t,x) = 0$ for $|d_\Sigma(x)| > 2a$, and in general

$$\begin{aligned}\partial\xi_h(t,x) = \frac{1}{a}&\chi'(d_\Sigma(x)/a)h(t,\Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)) \otimes \nu_\Sigma(\Pi_\Sigma(x))\\ &+ \chi(d_\Sigma(x)/a)\nu_\Sigma(\Pi_\Sigma(x)) \otimes M_0(d_\Sigma(x))\nabla_\Sigma h(t,\Pi_\Sigma(x))\\ &- \chi(d_\Sigma(x)/a)h(t,\Pi_\Sigma(x))M_0(d_\Sigma(x))L_\Sigma(\Pi_\Sigma(x)).\end{aligned}$$

It is a matter of simple algebra to determine the inverse of $\partial\Xi_h$, to the result

$$\begin{aligned}&(\partial\Xi_h(t,x))^{-1}\\ &= I - \Big(\chi h L_\Sigma - \frac{\chi' h/a}{1+\chi' h/a}\nu_\Sigma \otimes \nu_\Sigma - \frac{\chi}{1+\chi' h/a}\nu_\Sigma \otimes \nabla_\Sigma h\Big)M_0(d_\Sigma + \chi h),\end{aligned}$$

where we dropped the obvious arguments. This implies

$$M_1(h) = \chi M_0(d_\Sigma + \chi h)\big(\frac{\nabla_\Sigma h \otimes \nu_\Sigma}{1+\chi' h/a} - hL_\Sigma\big) + \frac{\chi' h/a}{1+\chi' h/a}\nu_\Sigma \otimes \nu_\Sigma.$$

Note that $M_1(h)$ depends linearly on $\nabla_\Sigma h$. On the interface we then have

$$M_1(h) = M_0(h)\big(\nabla_\Sigma h \otimes \nu_\Sigma - hL_\Sigma\big).$$

In particular, $\partial\Xi_h$ is invertible, provided $M_0(d_\Sigma + \chi h) = (I - (d_\Sigma + \chi h)L_\Sigma)^{-1}$ exists, and $1 + \chi' h/a > 0$. This certainly holds if

$$|d_\Sigma + \chi h||L_\Sigma| \le 2/3 \quad \text{and} \quad |\chi'|_\infty |h|/a \le 1/2,$$

which leads to the restriction $|h|_\infty \le h_\infty := a/2|\chi'|_\infty$; note that $|\chi'|_\infty > 1$. Observe that at this place no restrictions on $\nabla_\Sigma h$ are required.

Next we have

$$\partial_t \Xi_h(t,x) = \chi(d_\Sigma(x)/a)\partial_t h(t, \Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \bar\Omega,$$

hence the relation $\Xi_h^{-1}(t, \Xi_h(t,x)) = x$ implies

$$\partial_t \Xi_h^{-1}(t, \Xi_h(t,x)) = -m_0(h)\partial_t h(t, \Pi_\Sigma(x))\nu_\Sigma(\Pi_\Sigma(x)), \quad x \in \bar\Omega,$$

where

$$m_0(h)(t,x) = \frac{\chi(d_\Sigma(x)/a)}{(1 + h(t, \Pi_\Sigma(x))\chi'(d_\Sigma(x)/a)/a}.$$

With the Weingarten tensor L_Σ and the surface gradient ∇_Σ we further have

$$\begin{aligned} \nu_\Gamma(h) &= \beta(h)(\nu_\Sigma - a(h)), & a(h) &= M_0(h)\nabla_\Sigma h, \\ M_0(h) &= (I - hL_\Sigma)^{-1}, & \beta(h) &= (1 + |a(h)|^2)^{-1/2}, \end{aligned}$$

and

$$V_\Gamma = \partial_t \Xi_h \cdot \nu_\Gamma = (\nu_\Sigma \cdot \nu_\Gamma)\partial_t h = \beta(h)\partial_t h.$$

It will be shown in Section 2.2 that the surface gradient of a function ϕ on Γ is given by

$$\nabla_\Gamma \phi = \mathcal{P}_\Gamma(h) M_0(h) \nabla_\Sigma \bar\phi =: \mathcal{G}_\Gamma(h)\bar\phi,$$

where $\bar\phi = \phi \circ \Xi_h$, the surface divergence of a vector field f on Γ becomes

$$\operatorname{div}_\Gamma f = \operatorname{tr}[\mathcal{P}_\Gamma(h) M_0(h) \nabla_\Sigma \bar f],$$

and the Laplace–Beltrami operator Δ_Γ reads

$$\Delta_\Gamma \varphi = \operatorname{tr}[\mathcal{P}_\Gamma(h) M_0(h) \nabla_\Sigma \mathcal{P}_\Gamma(h) M_0(h) \nabla_\Sigma \bar\varphi].$$

Finally, for the mean curvature $H_\Gamma(h)$ we have

$$H_\Gamma(h) = \beta(h)\{\operatorname{tr}[M_0(h)(L_\Sigma + \nabla_\Sigma a(h))] - \beta^2(h)(M_0(h)a(h)|[\nabla_\Sigma a(h)]a(h))\},$$

a differential expression involving second-order derivatives of h only linearly. We may write

$$H_\Gamma(h) = \mathcal{C}_0(h) : \nabla_\Sigma^2 h + \mathcal{C}_1(h),$$

where $\mathcal{C}_0(h)$ and $\mathcal{C}_1(h)$ depend on h and $\nabla_\Sigma h$, provided $|h| \le h_\infty$ holds. The linearization of $H_\Gamma(h)$ at $h = 0$ is given by

$$H_\Gamma'(0) = \operatorname{tr} L_\Sigma^2 + \Delta_\Sigma.$$

Here Δ_Σ denotes the Laplace–Beltrami operator on Σ.

(b) The Transformed Problem
Now we define the transformed quantities

$$\begin{aligned}
&\bar\varrho(t,x) = \varrho(t,\Xi_h(t,x)), && \bar u(t,x) = u(t,\Xi_h(t,x)) && \text{in } \Omega\backslash\Sigma,\\
&\bar\pi(t,x) = \pi(t,\Xi_h(t,x)), && \bar\theta(t,x) = \theta(t,\Xi_h(t,x)) && \text{in } \Omega\backslash\Sigma,\\
&\bar u_\Gamma(t,p) = u_\Gamma(t,\Xi_h(t,p)), && \bar j_\Gamma(t,p) = j_\Gamma(t,\Xi_h(t,p)) && \text{on } \Sigma,
\end{aligned} \tag{1.57}$$

the *pull back* of $(\varrho, u, \pi, \theta, u_\Gamma, j_\Gamma)$. This way we have transformed the time varying regions $\Omega \setminus \Gamma(t)$ to the fixed region $\Omega \setminus \Sigma$. This transforms the general problem (1.26), (1.27), (1.28) to the following problem for $(\bar\varrho, \bar u, \bar\pi, \bar\theta, \bar u_\Gamma, \bar j_\Gamma, h)$.

$$\begin{aligned}
\partial_t\bar\varrho + \mathcal{G}(h)\cdot\bar\varrho\bar u &= m_0(h)\partial_t h(\nu_\Sigma\cdot\nabla)\bar\varrho && \text{in } \Omega\setminus\Sigma,\\
\bar\varrho\partial_t\bar u - \mathcal{G}(h)\cdot\bar S + \mathcal{G}(h)\bar\pi &= \bar\varrho\mathcal{R}_u(\bar u,\bar\theta,h) && \text{in } \Omega\setminus\Sigma,\\
\bar u &= 0 && \text{on } \partial\Omega,\\
[\![1/\bar\varrho]\!]\bar j_\Gamma^2\nu_\Gamma(h) - [\![\bar S\nu_\Gamma(h)]\!] + [\![\bar\pi]\!]\nu_\Gamma(h) &= \mathcal{G}_\Gamma(h)\cdot(\sigma(\bar\theta_\Gamma)\mathcal{P}_\Gamma(h) + \bar S_\Gamma) && \text{on } \Sigma,\\
[\![\bar u]\!] - [\![1/\bar\varrho]\!]\bar j_\Gamma\nu_\Gamma(h) &= 0 && \text{on } \Sigma,\\
\bar\varrho(0) = \bar\varrho_0,\quad \bar u(0) &= \bar u_0,
\end{aligned} \tag{1.58}$$

where

$$\begin{aligned}
\bar S &= \mu(\bar\theta,\bar\varrho)(\mathcal{G}(h)\bar u + [\mathcal{G}(h)\bar u]^{\mathsf T}) + \lambda(\bar\theta,\bar\varrho)(\mathcal{G}(h)\cdot\bar u)I,\\
\bar S_\Gamma &= \mu_\Gamma(\theta_\Gamma)\mathcal{P}_\Gamma(h)(\mathcal{G}_\Gamma(h)\bar u_\Gamma + [\mathcal{G}_\Gamma(h)\bar u_\Gamma]^{\mathsf T})\mathcal{P}_\Gamma(h) + \lambda(\bar\theta_\Gamma)(\mathcal{G}_\Gamma(h)\cdot\bar u_\Gamma)\mathcal{P}_\Gamma(h),
\end{aligned}$$

$$\begin{aligned}
\bar\varrho\kappa(\bar\theta,\bar\varrho)\partial_t\bar\theta - \mathcal{G}(h)\cdot d(\bar\theta,\bar\varrho)\mathcal{G}(h)\bar\theta &= \bar\varrho\kappa(\bar\theta,\bar\varrho)\mathcal{R}_\theta(\bar u,\bar\theta,h) && \text{in } \Omega\backslash\Sigma,\\
\partial_\nu\bar\theta &= 0 && \text{on } \partial\Omega,\\
[\![\bar\theta]\!] = 0,\quad \bar\theta &= \bar\theta_\Gamma && \text{on } \Sigma,\\
\bar\theta(0) &= \bar\theta_0 && \text{in } \Omega,
\end{aligned} \tag{1.59}$$

$$\begin{aligned}
&\kappa_\Gamma(\bar\theta_\Gamma)\partial_t\bar\theta_\Gamma - (\mathcal{G}_\Gamma(h)|d_\Gamma(\bar\theta_\Gamma)\mathcal{G}_\Gamma(h)\bar\theta_\Gamma) - [\![\bar\theta\eta(\bar\theta,\bar\rho)]\!]\bar j_\Gamma\\
&+[\![d(\bar\theta,\bar\rho)\mathcal{G}(h)\bar\theta\cdot\nu_\Gamma(h)]\!] = \bar S_\Gamma : \mathcal{G}_\Gamma(h)\bar u_\Gamma + \sigma(\bar\theta)\mathcal{G}_\Gamma(h)\cdot\bar u_\Gamma + \mathcal{R}_\Gamma(\bar\theta_\Gamma,h) && \text{on } \Sigma\\
&[\![\psi(\bar\theta,\bar\rho)]\!] + [\![1/2\bar\rho^2]\!]\bar j_\Gamma^2 - [\![\bar S\nu_\Gamma\cdot\nu_\Gamma/\bar\rho]\!] + [\![\bar\pi/\bar\rho]\!]\nu_\Gamma(h) = 0 && \text{on } \Sigma,\\
&\beta(h)\partial_t h - (\bar u|\nu_\Gamma) + \bar j_\Gamma/\bar\rho = 0, && \text{on } \Sigma,\\
&\bar\theta_\Gamma(0) = \bar\theta_0,\quad h(0) = h_0.
\end{aligned} \tag{1.60}$$

Here $\mathcal{G}(h)$ and $\mathcal{G}_\Gamma(h)$ denote the transformed gradient resp. the transformed surface gradient. More precisely, we have the relations

$$[\nabla\pi]\circ\Xi_h = \mathcal{G}(h)\bar\pi = [(\partial\,\Xi_h^{-1})^{\mathsf{T}}\circ\Xi_h]\nabla\bar\pi = (I - M_1(h))\nabla\bar\pi$$

and

$$[\nabla\theta]\circ\Xi_h = (I - M_1(h))\nabla\bar\theta,$$

as well as

$$(\nabla\cdot u)\circ\Xi_h = (\mathcal{G}(h)|\bar u) = ((I - M_1(h))\nabla|\bar u).$$

Furthermore,

$$\frac{D}{Dt}\theta_\Gamma\circ\Xi_h = \partial_t\bar\theta_\Gamma + \bar u_\Gamma\cdot\nabla_\Sigma\bar\theta_\Gamma - \bar u_\Gamma\cdot M_1(h)\nabla_\Sigma\bar\theta_\Gamma,$$

and

$$[\partial_t u]\circ\Xi_h = \partial_t\bar u + \partial\bar u[(\partial_t\Xi_h^{-1})\circ\Xi_h] = \partial_t\bar u - m_0(h)\partial_t h(\nu_\Sigma\cdot\nabla)\bar u,$$

hence

$$\mathcal{R}_u(\bar u,\bar\theta,h) = -\bar u\cdot\mathcal{G}(h)\bar u + m_0(h)\partial_t h(\nu_\Sigma\cdot\nabla)\bar u.$$

Similarly we have

$$[\partial_t\theta]\circ\Xi_h = \partial_t\bar\theta - m_0(h)\partial_t h(\nu_\Sigma\cdot\nabla)\bar\theta,$$

and so

$$\mathcal{R}_\theta(\bar u,\bar\theta,h) = -\bar u\cdot\mathcal{G}(h)\bar\theta + m_0(h)\partial_t h(\nu_\Sigma\cdot\nabla)\bar\theta.$$

In the same way we get

$$R_\Gamma(\bar\theta_\Gamma,h) = -\bar u_\Gamma\cdot\nabla_\Sigma\bar\theta_\Gamma + \bar u_\Gamma\cdot M_1(h)\nabla_\Sigma\bar\theta_\Gamma + \bar\theta_\Gamma\sigma'(\bar\theta_\Gamma)\mathcal{G}_\Gamma(h)\cdot\bar u_\Gamma.$$

It is convenient to decompose the stress boundary condition into tangential and normal parts; here we set $S_\Gamma = 0$. For this purpose let $\mathcal{P}_\Sigma = I - \nu_\Sigma\otimes\nu_\Sigma$ denote the projection onto the tangent space of Σ. Multiplying the stress interface condition with ν_Σ/β we obtain

$$\begin{aligned}\llbracket 1/\bar\varrho\rrbracket\bar j_\Gamma^2 + \llbracket\bar\pi\rrbracket - \sigma H_\Gamma(h) = {}& (\llbracket\bar S\rrbracket(\nu_\Sigma - M_0(h)\nabla_\Sigma h)|\nu_\Sigma)\\ &+ \sigma'\beta(M_0\nabla_\Sigma h|M_0\nabla_\Sigma\bar\theta_\Gamma)\end{aligned}\tag{1.61}$$

for the normal part of the stress boundary condition. Substituting this expression into the stress interface condition and then applying the projection $\mathcal{P}_\Sigma$ yields, after some computation,

$$\begin{aligned}\mathcal{P}_\Sigma\llbracket\bar S\rrbracket(\nu_\Sigma - M_0(h)\nabla_\Sigma h) = {}& (\llbracket\bar S\rrbracket(\nu_\Sigma - M_0(h)\nabla_\Sigma h)|\nu_\Sigma)M_0(h)\nabla_\Sigma h\\ &+ (\sigma'/\beta)M_0(h)\nabla_\Sigma\bar\theta_\Gamma\end{aligned}\tag{1.62}$$

for the tangential part. Note that the latter neither contains the phase flux nor the pressure jump nor the curvature!

3.3 Goals and Strategies

The goal of this monograph is the exposition of a general theory for the models introduced above. We present in detail a rigorous analysis of these problems. It will become clear that the scope of our approach is much wider. It can be used for many other problems with moving interfaces, such as phase transitions driven by chemical potentials, two-phase flow problems with surface viscosities, multi-component two phase flows, as well as similar quasi-steady problems or purely geometric ones, to mention a few more applications. The essential restriction is that the problems in question ought to be of *parabolic nature*. In this book we will employ L_p-theory since it avoids higher order compatibility conditions. In addition, deep results of harmonic analysis are at our disposal. Nevertheless, one could also use other frameworks where maximal regularity is available, e.g. C^α-theory.

In particular, we address the following topics.

a) Local well-posedness and local semiflow;

b) Stability analysis of equilibria;

c) Long-time behaviour of solutions.

We now outline our approach, explaining the main ideas and tools to be employed.

a) Local-Well-posedness and Local Semiflow

To obtain local well-posedness we write the transformed problem in the form

$$\mathcal{L}z = (N(z), z_0).$$

Here $\mathcal{L}$ is the principal linear part of the problem in question, and N is the remaining nonlinear part which is small in the sense that N collects all lower order terms and contains only highest order terms which carry a factor $|\nabla_\Sigma h|$ which is small on small time intervals due to the choice of the Hanzawa transform. The variable z with initial value z_0 collects all essential variables of the problem under consideration.

The *first step* is to find function spaces $\mathbb{E}(J)$ and $\mathbb{F}(J)$, $J = (0, a)$ or $J = \mathbb{R}_+$, such that $\mathcal{L} : \mathbb{E}(J) \to \mathbb{F}(J) \times \mathbb{E}_\gamma$ is an isomorphism. Here $\mathbb{E}_\gamma$ denotes the time-trace space of $\mathbb{E}(J)$ which the initial value z_0 should belong to. This is the question of *maximal regularity*. These spaces differ from problem to problem and the question of maximal regularity has to be studied separately for each one. Here we will use the framework of L_p-spaces and rely on deep results from vector-valued harmonic analysis and operator theory which will be introduced and discussed in Chapter 4.

The *second step* then employs the contraction mapping principle to obtain local solutions, and the implicit function theorem to obtain smooth dependence of the solutions on the data. For this, estimates of the nonlinearity N are needed, eventually showing that $N : \mathbb{E}(J) \to \mathbb{F}(J)$ is continuously Fréchet-differentiable, at

least. This requires some smoothness of the coefficient functions in the constitutive laws. If these are, say, even real analytic then N will be so as well, and by a scaling argument and the implicit function theorem we will show that the solutions are real analytic jointly in time and space as well. In particular, the interface will become instantaneously real analytic, which shows the strong regularizing effect, characteristic for parabolic problems.

The *third step* consists in setting up the state manifold $\mathcal{SM}$ of the untransformed problem. It will be a truly nonlinear manifold which comes from the generic nonlinear structure, due to geometry and the involved nonlinear compatibility conditions of the problem. Charts for the state manifold are induced by the Hanzawa transform mentioned above. The local existence and regularity results for the transformed problem induce a local semiflow on the proper state manifold $\mathcal{SM}$ for the problem in question.

b) Stability Analysis of Equilibria

For the stability analysis of equilibria it is natural to employ again the Hanzawa transform, where the reference manifold Σ now is the equilibrium interface Γ_*, a union of finitely many disjoint spheres contained in Ω. As the linearized problem enjoys maximal L_p-regularity, an abstract result shows that the operator L associated with the fully linearized problem is the negative generator of a compact analytic C_0-semigroup. Therefore, the spectrum of L consists only of countably many isolated eigenvalues of finite algebraic multiplicity. Thus, it is natural to study these eigenvalues and to apply the *principle of linearized stability* for the nonlinear problem.

However, a major difficulty of this approach lies in the fact that the equilibria are not isolated in the state manifold, but form a finite-dimensional submanifold $\mathcal{E}$ of $\mathcal{SM}$. For the linearization of the transformed problem this implies that the kernel of L is nontrivial, i.e., the imaginary axis is not in the resolvent set of L, and so the standard principle of linearized stability is not applicable. Fortunately, 0 is the only eigenvalue of L on $i\mathbb{R}$ and it is nicely behaved: the kernel $\mathsf{N}(L)$ is isomorphic to the tangent space of $\mathcal{E}$ at this equilibrium, and 0 is semi-simple. This shows that 0 is *normally stable* if the remaining eigenvalues of L have positive real parts, and *normally hyperbolic* if some of them have negative real parts; these are only finitely many. Therefore, we can employ what we call the *generalized principle of linearized stability*, a method which is adapted to such a situation and has been worked out recently for quasilinear parabolic evolution equations by the authors. So our stability analysis of equilibria proceeds in two steps.

In the *first step* we analyze the eigenvalues of L and find conditions, if possible necessary and sufficient, which ensure that all eigenvalues of L except 0 have positive real parts; this is the *normally stable case*. In the *normally hyperbolic case* we determine the dimension of the unstable subspace of L. And of course, we have to show that 0 is semi-simple, to determine the kernel of L, and to prove that $\mathsf{N}(L)$ is isomorphic to the tangent space of $\mathcal{E}$.

In the *second step* we employ the generalized principle of linearized stability

to the nonlinear problem. This can be done simultaneously for all six problems in question, as the proof only uses the general structure of the problems under consideration. Here we employ once more the implicit function theorem.

c) Long-Time behaviour of Solutions
In general, solutions in $\mathcal{SM}$ will exist on a maximal time interval $[0, t_+(z_0))$ which typically will be finite, due to several obstructions, such as missing a priori bounds, loss of well-posedness, or topological changes in the moving interface. However, if a solution *does not develop singularities* in a sense to be specified, then we will prove that the solution exists globally, i.e., $t_+(z_0) = \infty$, and it converges in the topology of $\mathcal{SM}$ to an equilibrium. This essentially relies on a method using time weights to improve regularity and on compact Sobolev embeddings. Actually, we are able to *characterize* solutions which exist globally and converge as $t \to \infty$. This result is also proved simultaneously for all problems under consideration, as the proof only relies on general properties of semiflows, relative compactness of bounded orbits, the existence of a strict Lyapunov functional (the negative entropy), and the results on stability of equilibria.

On our way of presenting the tools which are needed to achieve these goals we will frequently discuss other problems to illustrate the main ideas. For example, the Laplacian, the Laplace–Beltrami operator, the heat operator, the Stokes operator, and several Dirichlet-to-Neumann operators will be studied in various frameworks. In Chapter 5 we develop an L_p-theory of abstract quasilinear parabolic evolution equations which serves as a guide for the more complex problems to be studied later on. In Chapter 12 we will present several applications of the main results of Chapter 5 to problems arising from generalized Newtonian flows, nematic liquid crystal flows, Maxwell-Stefan diffusion, and the Stefan problem with surface tension and surface heat capacity, as well as to geometric evolutions equations like the averaged mean curvature flow, the surface diffusion flow, the Mullins–Sekerka flow, the Muskat flow, the Stokes flow, and the Stokes flow with phase transition.