The Urysohn Extension Theorem for Bishop Spaces

Iosif Petrakis(✉)

University of Munich, Theresienstr. 39, 80333 Munich, Germany
petrakis@math.lmu.de

Abstract. Bishop's notion of function space, here called Bishop space, is a function-theoretic analogue to the classical set-theoretic notion of topological space. Bishop introduced this concept in 1967, without exploring it, and Bridges revived the subject in 2012. The theory of Bishop spaces can be seen as a constructive version of the theory of the ring of continuous functions. In this paper we define various notions of embeddings of one Bishop space to another and develop their basic theory in parallel to the classical theory of embeddings of rings of continuous functions. Our main result is the translation within the theory of Bishop spaces of the Urysohn extension theorem, which we show that it is constructively provable. We work within Bishop's informal system of constructive mathematics BISH, inductive definitions with countably many premises included.

Keywords: Constructive topology · Bishop spaces · Embeddings · Urysohn extension theorem

1 Introduction

The theory of Bishop spaces (TBS) is a constructive approach to general topology based on the notion of function space, here called Bishop space, that it was introduced by Bishop in [1], p. 71, but it was not really studied until Bridges's paper [7], that was followed by Ishihara'a paper [16], and our development of TBS in [22–24]. The main characteristics of TBS are the following:

1. Points are accepted from the beginning, hence it is not a point-free approach to topology.
2. Most of its notions are function-theoretic. Set-theoretic notions are avoided or play a secondary role to its development.
3. It is constructive. We work within Bishop's informal system of constructive mathematics BISH (see [4,5]), inductive definitions with rules of countably many premises included, a system connected to Martin-Löf's constructivism [17] and type theory [18]. The underlying logic of BISH is intuitionistic, while Myhill's system CST* of constructive Set Theory with inductive definitions, or Martin-Löf's extensional type theory, can be considered as formalizations of its underlying set theory.

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4. It has simple foundation and it follows the style of standard mathematics.

In other words, TBS is an approach to constructive point-function topology. The main motivation behind the introduction of Bishop spaces is that function-based concepts suit better to constructive study rather than set-based ones. Instead of having space-structures on a set X and $\mathbb{R}$, that determine a posteriori which functions of type $X \to \mathbb{R}$ are continuous with respect to them, we start from a given class of "continuous" functions of type $X \to \mathbb{R}$ that determines a posteriori a topological space-structure on X. "Continuity" in TBS is a primitive notion, a starting point similar to Spanier's theory of quasi-topological spaces in [27], or to the theory of limit spaces of Fréchet in [13].

TBS permits a "communication" with the classical theory of the rings of continuous functions, since many concepts, questions and results from the classical theory of $C(X)$, where X is a topological space, can be translated into TBS. Although this communication does not imply a direct translation from the theory of $C(X)$ to TBS, since the logic of TBS is intuitionistic, it is one of the features of TBS which makes it, in our view, so special as an approach to constructive topology. One could see TBS as an abstract, constructive version of the classical theory of $C(X)$, which we hope to be of interest to a classical mathematician too.

In this paper we develop the constructive basic theory of embeddings of Bishop spaces in parallel to the classical basic theory of embeddings of rings of continuous functions which is found in the book [11] of Gillman and Jerison. Our main result is the incorporation of the fundamental Urysohn extension theorem within the theory of embeddings of Bishop spaces.

2 Basic Definitions and Facts

In order to be self-contained we include in this section some basic definitions and facts necessary to the rest of the paper, that are partly found in [23]. For all proofs not included in this paper we refer to [24].

If X, Y are sets and $\mathbb{R}$ is the set of the constructive reals, we denote by $\mathbb{F}(X, Y)$ the functions of type $X \to Y$, by $\mathbb{F}(X)$ the functions of type $X \to \mathbb{R}$, by $\mathbb{F}_b(X)$ the bounded elements of $\mathbb{F}(X)$, and by $\mathrm{Const}(X)$ the subset of $\mathbb{F}(X)$ of all constant functions $\overline{a}$, where $a \in \mathbb{R}$. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called *Bishop-continuous*, if ϕ is uniformly continuous on every bounded subset of $\mathbb{R}$, and we denote their set by $\mathrm{Bic}(\mathbb{R})$. If $f, g \in \mathbb{F}(X)$, $\epsilon > 0$, and $\Phi \subseteq \mathbb{F}(X)$, we define $U(g, f, \epsilon)$ and $U(\Phi, f)$ by

$$U(g, f, \epsilon) := \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) := \forall_{\epsilon > 0} \exists_{g \in \Phi}(U(g, f, \epsilon)).$$

Definition 1. *A Bishop space is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$, a Bishop topology on X, or simply a topology on X, satisfies the following conditions:*

(BS$_1$) $a \in \mathbb{R} \rightarrow \overline{a} \in F$.
(BS$_2$) $f \in F \rightarrow g \in F \rightarrow f + g \in F$.
(BS$_3$) $f \in F \rightarrow \phi \in \mathrm{Bic}(\mathbb{R}) \rightarrow \phi \circ f \in F$,

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & \mathbb{R} \\ & \searrow^{F \ni \phi \circ f} & \Big\downarrow {\scriptstyle \phi \in \mathrm{Bic}(\mathbb{R})} \\ & & \mathbb{R}. \end{array}$$

(BS$_4$) $f \in \mathbb{F}(X) \rightarrow U(F, f) \rightarrow f \in F$.

Bishop used the term *function space* for $\mathcal{F}$ and *topology* for F. Since the former is used in many different contexts, we prefer the term Bishop space for $\mathcal{F}$, while we use the latter, as the *topology of functions* F on X corresponds nicely to the standard *topology of opens* $\mathcal{T}$ on X. Using BS$_2$ and BS$_3$ we get that if F is a topology on X, then fg, λf, $-f$, $\max\{f, g\} = f \vee g$, $\min\{f, g\} = f \wedge g$ and $|f| \in F$, for every $f, g \in F$ and $\lambda \in \mathbb{R}$. By BS$_4$ F is closed under uniform limits, where $f_n \xrightarrow{u} f$ denotes that f is the uniform limit of $(f_n)_{n\in\mathbb{N}}$. Moreover, $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$, where $\mathrm{Const}(X)$ is the *trivial* topology on X and $\mathbb{F}(X)$ is the *discrete* topology on X. If F is a topology on X, the set F_b of all bounded elements of F is also a topology on X that corresponds to the ring $C^*(X)$ of the bounded elements of $C(X)$, for some topological space X. It is easy to see that $\mathrm{Bic}(\mathbb{R})$ is a topology on $\mathbb{R}$, and the structure $\mathcal{R} = (\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$ is the *Bishop space of reals.*

The importance of the notion of a Bishop topology lies on Bishop's inductive concept of the least topology including a given subbase F_0, found in [1], p. 72, and in [4], p. 78, where the definitional clauses of a Bishop topology are turned into inductive rules.

Definition 2. *The least topology $\mathcal{F}(F_0)$ generated by a set $F_0 \subseteq \mathbb{F}(X)$, called a subbase of $\mathcal{F}(F_0)$, is defined by the following inductive rules:*

$$\frac{f_0 \in F_0}{f_0 \in \mathcal{F}(F_0)}, \quad \frac{a \in \mathbb{R}}{\overline{a} \in \mathcal{F}(F_0)}, \quad \frac{f, g \in \mathcal{F}(F_0)}{f + g \in \mathcal{F}(F_0)},$$

$$\frac{f \in \mathcal{F}(F_0),\ \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \mathcal{F}(F_0)}, \quad \frac{(g \in \mathcal{F}(F_0),\ U(g, f, \epsilon))_{\epsilon > 0}}{f \in \mathcal{F}(F_0)}.$$

If F_0 is inhabited, then the rule of the inclusion of the constant functions is redundant to the rule of closure under composition with $\mathrm{Bic}(\mathbb{R})$. The most complex inductive rule above can be replaced by the rule

$$\frac{g_1 \in \mathcal{F}(F_0) \ \wedge \ U(g_1, f, \frac{1}{2}), \ g_2 \in \mathcal{F}(F_0) \ \wedge \ U(g_2, f, \frac{1}{2^2}), \ldots}{f \in \mathcal{F}(F_0)},$$

which has the "structure" of Brouwer's F-inference with countably many conditions in its premiss (see e.g., [19]). The above rules induce the following induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$:

$$\begin{aligned}
&\forall_{f_0 \in F_0}(P(f_0)) \rightarrow \\
&\forall_{a \in \mathbb{R}}(P(\overline{a})) \rightarrow \\
&\forall_{f,g \in \mathcal{F}(F_0)}(P(f) \rightarrow P(g) \rightarrow P(f+g)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)}\forall_{\phi \in \mathrm{Bic}(\mathbb{R})}(P(f) \rightarrow P(\phi \circ f)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)}(\forall_{\epsilon > 0}\exists_{g \in \mathcal{F}(F_0)}(P(g) \ \wedge \ U(g, f, \epsilon)) \rightarrow P(f)) \rightarrow \\
&\forall_{f \in \mathcal{F}(F_0)}(P(f)),
\end{aligned}$$

where P is any property on $\mathbb{F}(X)$. Hence, starting with a constructively acceptable subbase F_0 the generated least topology $\mathcal{F}(F_0)$ is a constructively graspable set of functions exactly because of the corresponding principle $\mathrm{Ind}_{\mathcal{F}}$. Despite the seemingly set-theoretic character of the notion of a Bishop space the core of TBS is the study of the inductively generated Bishop spaces. For example, since $\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, where $\mathrm{id}_{\mathbb{R}}$ is the identity on $\mathbb{R}$, we get by the closure of $\mathcal{F}(\mathrm{id}_{\mathbb{R}})$ under BS_3 that $\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}})$. Moreover, most of the new Bishop spaces generated from old ones are defined through the concept of the least topology. A property P on $\mathbb{F}(X)$ is *lifted* from a subbase F_0 to the generated topology $\mathcal{F}(F_0)$, if

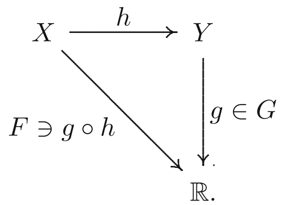
$$\forall_{f_0 \in F_0}(P(f_0)) \rightarrow \forall_{f \in \mathcal{F}(F_0)}(P(f)).$$

It is easy to see inductively that boundedness is a lifted property. If (X, d) is a metric space and the elements of F_0 are bounded and uniformly continuous functions, then uniform continuity is also a lifted property.

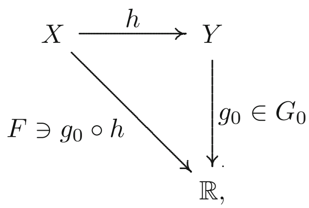
Since Bishop did not pursue a constructive reconstruction of topology in [1], he didn't mention $\mathrm{Ind}_{\mathcal{F}}$, or some related lifted property. Apart from the notion of a Bishop space, Bishop introduced in [1], p. 68, the inductive notion of the least algebra $\mathcal{B}(B_{0,F})$ of Borel sets generated by a given set $B_{0,F}$ of F-complemented subsets, where F is an arbitrary subset of $\mathbb{F}(X)$. Since this notion was central to the development of constructive measure theory in [1], Bishop explicitly mentioned there the corresponding induction principle $\mathrm{Ind}_{\mathcal{B}}$ on $\mathcal{B}(B_{0,F})$ and studied specific lifted properties in that setting. Brouwer's inductive definition of the countable ordinals in [8] and Bishop's inductive notion of a Borel set were the main non-elementary inductively defined classes of mathematical objects used in constructive mathematics and motivated the formal study of inductive definitions in the 60 s and the 70 s (see [9]). Since then the use of inductive definitions in constructive mathematics and theoretical computer science became a common practice. In [3] Bishop and Cheng developed though, a reconstruction of constructive measure theory independently from the inductive definition of Borel sets, that replaced the old theory in [4]. In [2] Bishop, influenced by

Gödel's Dialectica interpretation, discussed a formal system Σ that would "efficiently express" his informal system of constructive mathematics. Since the new measure theory was already conceived and the theory of Bishop spaces was not elaborated at all, Bishop found no reason to extend Σ to subsume inductive definitions. In [20] Myhill proposed instead the formal theory CST of sets and functions to codify [1]. He also took Bishop's inductive definitions at face value and showed that the existence and disjunction properties of CST persist in the extended with inductive definitions system CST*.

Definition 3. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a Bishop morphism, or simply a morphism, from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \to Y$ such that $\forall_{g \in G}(g \circ h \in F)$*



We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of morphisms from $\mathcal{F}$ to $\mathcal{G}$, which are the arrows in the category of Bishop spaces **Bis**. It is easy to see that if $\mathcal{F} = (X, F)$ is a Bishop space, then $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$. If $\mathcal{F} = (X, F)$ and $\mathcal{G}_0 = (Y, \mathcal{F}(G_0))$ are Bishop spaces, a function $h : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_0)$ if and only if $\forall_{g_0 \in G_0}(g_0 \circ h \in F)$



a very useful property that it is proved inductively and we call the *lifting of morphisms*. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is onto Y, then h is called a *set-epimorphism*, and we denote their set by $\mathrm{setEpi}(\mathcal{F}, \mathcal{G})$. We call some $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ *open*, if $\forall_{f \in F}\exists_{g \in G}(f = g \circ h)$. Clearly, if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that h is 1-1 and onto Y, then $h^{-1} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ if and only if h is open. In this case h is called an *isomorphism* between $\mathcal{F}$ and $\mathcal{G}$. In [23] we showed that in the case of a set-epimorphism h, openness of h is also a lifted property.

Definition 4. *If $\mathcal{F} = (X, F)$ is a Bishop space and $A \subseteq X$ is inhabited, the relative Bishop space of $\mathcal{F}$ on A is the structure $\mathcal{F}_{|A} = (A, F_{|A})$, where $F_{|A} := \mathcal{F}(\{f_{|A} \mid f \in F\})$. We also call $\mathcal{F}_{|A}$ a subspace of $\mathcal{F}$. If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are given Bishop spaces, their product is the structure $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where $F \times G := \mathcal{F}(\{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\})$, and π_1, π_2 are the projections of $X \times Y$ to X and Y, respectively.*

If F_0 is a subbase of F, we get inductively that $F_{|A} = \mathcal{F}(\{f_{0|A} \mid f_0 \in F_0\})$. It is straightforward to see that $\mathcal{F} \times \mathcal{G}$ satisfies the universal property for products and that $F \times G$ is the least topology which turns the projections π_1, π_2 into morphisms. If F_0 is a subbase of F and G_0 is a subbase of G, then we get inductively that $\mathcal{F}(F_0) \times \mathcal{F}(G_0) = \mathcal{F}(\{f_0 \circ \pi_1 \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\})$. Consequently, $\mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\{\mathrm{id}_{\mathbb{R}} \circ \pi_1\} \cup \{\mathrm{id}_{\mathbb{R}} \circ \pi_2\}) = \mathcal{F}(\pi_1, \pi_2)$. The arbitrary product $\prod_{i \in I} \mathcal{F}_i$ of a family $(\mathcal{F}_i)_{i \in I}$ of Bishop spaces indexed by some I is defined similarly. Using the lifting of morphisms it is easy to show the following proposition.

Proposition 1. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ are Bishop spaces and $A \subseteq X$, $B \subseteq Y$.*

(i) $j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$ *if and only if* $\pi_1 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ *and* $\pi_2 \circ j \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$.
(ii) *If* $e : X \to B$, *then* $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \leftrightarrow e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_{|B})$.
(iii) $(F \times G)_{|A \times B} = F_{|A} \times G_{|B}$.

Note that Proposition 1(i) and (iii) hold for arbitrary products too. If $\mathcal{F}_i = (X_i, F_i)$ is a family of Bishop spaces indexed by some inhabited set I and $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$, then the *slice* $S(x; j)$ through x parallel to x_j, where $j \in I$, is the set $S(x; j) := X_j \times \prod_{i \neq j} \{x_i\} \subseteq \prod_{i \in I} X_i$ of all I-tuples where all components other the j-component are the ones of x, while the j-component ranges over X_j. The next fact is used in the proof of the Proposition 11 and it is a direct consequence of the Proposition 1.

Proposition 2. *If $\mathcal{F}_i = (X_i, F_i)$ is a family of Bishop spaces indexed by some inhabited set I and $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$, then the function $s_j : X_j \to S(x; j)$, defined by $x_j \mapsto x_j \times \prod_{i \neq j} \{x_i\}$, where $S(x; j)$ is the slice through x parallel to x_j, is an isomorphism between $\mathcal{F}_j$ and $\mathcal{S}(x; j) = (S(x; j), F(x; j))$, where $F(x; j) = (\prod_{i \in I} F_i)_{|S(x;j)}$.*

Definition 5. *If $\mathcal{G} = (Y, G)$ is a Bishop space, X is an inhabited set and $\theta : X \to Y$, the weak topology $F(\theta)$ on X induced by θ is defined as $F(\theta) := \mathcal{F}(\{g \circ \theta \mid g \in G\})$. The space $\mathcal{F}(\theta) = (X, F(\theta))$ is called the weak Bishop space on X induced by θ. If $\mathcal{F} = (X, F)$ is a Bishop space, Y is an inhabited set and $e : X \to Y$ is onto Y, the set of functions $G_e := \{g \in \mathbb{F}(Y) \mid g \circ e \in F\}$ is a topology on Y. We call $\mathcal{G}_e = (Y, G_e)$ the quotient Bishop space, and G_e the quotient topology on Y, with respect to e.*

The weak topology $F(\theta)$ is the least topology on X which makes θ a morphism. If θ is onto Y, then $\theta \in \mathrm{setEpi}(\mathcal{F}(\theta), \mathcal{G})$, and by the lifting of openness we get that $F(\theta) = \{g \circ \theta \mid g \in G\}$, a fact that we use in the proof of the Proposition 6. In analogy to classical topology, the quotient topology G_e is the largest topology on Y which makes e a morphism.

In [4], pp. 91–92, it is shown[1] that if $D \subseteq X$ is a dense subset of the metric space X, Y is a complete metric space, and $f : D \to Y$ is uniformly continuous

[1] The uniqueness property is included, for example, in [21], p. 238.

with modulus of continuity ω, then there exists a unique uniform continuous extension $g : X \to Y$ of f with modulus of continuity $\frac{1}{2}\omega$. The next lemma is a useful generalization of it[2] that we proved in [24] and we use it here in the proof of the Proposition 3(vi).

Lemma 1. *Suppose that X is an inhabited metric space, $D \subseteq X$ is dense in X and Y is a complete metric space. If $f : D \to Y$ is uniformly continuous on every bounded subset of D, then there exists a unique extension $g : X \to Y$ of f which is uniformly continuous on every bounded subset of X with modulus of continuity $\omega_{g,B}(\epsilon) = \frac{1}{2}\omega_{f,B\cap D}(\epsilon)$, for every inhabited, bounded and metric-open subset B of X. Moreover, if f is bounded by some $M > 0$, then g is also bounded by M.*

Within BISH a *compact* metric space is defined as a complete and totally bounded space. A *locally compact* metric space X is a space in which every bounded subset of X is included in a compact one. If X is locally compact, the set $\mathrm{Bic}(X)$, defined like $\mathrm{Bic}(\mathbb{R})$, is a topology on X. Using the definition of a continuous function on a locally compact metric space, given in [4], p. 110, Bishop's formulation of the Tietze theorem for metric spaces becomes as follows.

Theorem 1. *Let Y be a locally compact subset of a metric space X and $I \subset \mathbb{R}$ an inhabited compact interval. Let $f : Y \to I$ be uniformly continuous on the bounded subsets of Y. Then there exists a function $g : X \to I$ which is uniformly continuous on the bounded subsets of X, and which satisfies $g(y) = f(y)$, for every $y \in Y$.*

Corollary 1. *If Y is a locally compact subset of $\mathbb{R}$ and $g : Y \to I \in \mathrm{Bic}(Y)$, where $I \subset \mathbb{R}$ is an inhabited compact interval, then there exists a function $\phi : \mathbb{R} \to I \in \mathrm{Bic}(\mathbb{R})$ which satisfies $\phi(y) = g(y)$, for every $y \in Y$.*

We use the Corollary 1 in the proof of the Propositions 3(v) and 8, while in [24] we used it to show the following fundamental fact, which is used here in the proof of the Proposition 9.

Theorem 2. *Suppose that (X, F) is a Bishop space and $f \in F$ such that $f \geq \overline{c}$, for some $c > 0$. Then, $\frac{1}{f} \in F$.*

[2] According to Bishop and Bridges [4], p. 85, if $B \subseteq X$, where (X, d) is an inhabited metric space, B is a *bounded subset* of X, if there is some $x_0 \in X$ such that $B \cup \{x_0\}$ with the induced metric is a bounded metric space. If we suppose that the inclusion map of a subset is the identity (see [4], p. 68), the induced metric on $B \cup \{x_0\}$ is reduced to the relative metric on $B \cup \{x_0\}$. We may also denote a bounded subset B of an inhabited metric space X by (B, x_0, M), where $M > 0$ is a bound for $B \cup \{x_0\}$. If (B, x_0, M) is a bounded subset of X then $B \subseteq \mathcal{B}(x_0, M)$, and $(\mathcal{B}(x_0, M), x_0, 2M)$ is also a bounded subset of X. I.e., a bounded subset of X is included in an inhabited bounded subset of X which is also metric-open i.e., it includes an open ball of every element of it.

Definition 6. *If (X, F) is a Bishop space, the relations defined by*

$$x_1 \bowtie_F x_2 :\leftrightarrow \exists_{f \in F}(f(x_1) \bowtie_{\mathbb{R}} f(x_2)),$$

$$A \bowtie_F B :\leftrightarrow \exists_{f \in F} \forall_{a \in A} \forall_{b \in B}(f(a) = 0 \wedge f(b) = 1)$$

where $x_1, x_2 \in X$, $a \bowtie_{\mathbb{R}} b :\leftrightarrow a > b \vee a < b \leftrightarrow |a - b| > 0$, for every $a, b \in \mathbb{R}$, and $A, B \subseteq X$, are the canonical point-point and set-set apartness relations on X. If $\bowtie$ is a point-point apartness relation on X[3]*, F is called $\bowtie$-Hausdorff, if $\bowtie \subseteq \bowtie_F$. The F-zero sets $Z(F)$ of (X, F) are the subsets of X of the form $\zeta(f) = \{x \in X \mid f(x) = 0\}$, where $f \in F$.*

In [24] we showed within BISH that $Z(F)$ is closed under countably infinite intersections, and the sets $[f \leq \overline{a}] = \{x \in X \mid f(x) \leq a\}, [f \geq \overline{a}] = \{x \in X \mid f(x) \geq a\}$, where $a \in \mathbb{R}$, are in $Z(F)$. We also used the Theorem 2 to show the Urysohn lemma for the zero sets of a Bishop space. According to the classical Urysohn lemma for $C(X)$-zero sets, the disjoint zero sets of any topological space X are separated by some $f \in C(X)$ (see [11], p. 17). Constructively, we need to replace the negative notion of disjointness of two zero sets by a positive notion.

Theorem 3 (Urysohn Lemma for F-Zero Sets). *If (X, F) is a Bishop space and $A, B \subseteq X$, then $A \bowtie_F B \leftrightarrow \exists_{f,g \in F} \exists_{c>0}(A \subseteq \zeta(f) \;\wedge\; B \subseteq \zeta(g) \;\wedge\; |f| + |g| \geq \overline{c})$.*

3 Embeddings of Bishop Spaces

If $\mathcal{G}, \mathcal{F}$ are Bishop spaces, the notions "$\mathcal{G}$ is embedded in $\mathcal{F}$" and "$\mathcal{G}$ is bounded-embedded in $\mathcal{F}$" translate into TBS the notions "Y is C-embedded in X" and "Y is C^*-embedded in X", for some $Y \subseteq X$ and a given topology of opens $\mathcal{T}$ on X (see [11], p. 17). If F is a topology on X, $f \in F$ and $a, b \in \mathbb{R}$ such that $a \leq b$, we say that a, b *bound* f, if $\forall_{x \in X}(a \prec f(x) \prec b)$, where $\prec \in \{<, \leq\}$.

Definition 7. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$, then*

(i) $\mathcal{G}$ is embedded in $\mathcal{F}$, if $\forall_{g \in G} \exists_{f \in F}(f_{|Y} = g)$.
(ii) $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, if $\mathcal{G}_b$ is embedded in $\mathcal{F}_b$.
(iii) $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, if $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, and for every $g \in G_b$, if a, b bound g, then a, b bound some extension f of g in F_b.
(iv) $\mathcal{G}$ is dense-embedded in $\mathcal{F}$, if $\forall_{g \in G} \exists !_{f \in F}(f_{|Y} = g)$.
(v) $\mathcal{G}$ is dense-bounded-embedded in $\mathcal{F}$ and $\mathcal{G}$ is dense-full bounded-embedded in $\mathcal{F}$ are defined similarly to (iv).
(vi) $\mathcal{F}$ extends $\mathcal{G}$, if $\forall_{f \in F}(f_{|Y} \in G)$.

[3] See definition 2.1 in [4], p. 72. It is also easy to see that $a \bowtie_{\mathbb{R}} b \leftrightarrow a \bowtie_{\mathrm{Bic}(\mathbb{R})} b$, for every $a, b \in \mathbb{R}$.

Clearly, (X, G) is embedded in (X, F) if and only if $G \subseteq F$. The Definition 7(vi) is necessary, since a topology F on some X does not necessarily behave like $C(X)$, where every $f \in C(X)$ restricted to Y belongs to $C(Y)$. By the definition of the relative Bishop space we get immediately that $\mathcal{F}$ extends $\mathcal{F}_{|Y}$. If $\mathcal{G}$ is embedded in $\mathcal{F}$, then $\mathcal{G}'$ is embedded in $\mathcal{F}$, where $\mathcal{G}' = (Y, G')$ and $G' \subseteq G$. If (X, F) is a Bishop space and $Y \subseteq X$, a *retraction* of X onto Y is a function $r : X \to Y$ such that $r(y) = y$, for every $y \in Y$, and $r \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}_{|Y})$. In this case Y is called a *retract* of X. For example, the Cantor space with the product topology on $(2, \mathbb{F}(2))$ is a retract of the Baire space with the product topology on $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$.

Proposition 3. *Suppose that $Y \subseteq X$ and $\bowtie$ is a point-point apartness relation on X.*

(i) $(Y, \mathrm{Const}(Y))$ *is embedded in every Bishop space* (X, F)*.*
(ii) If $\forall_{x \in X}(x \in Y \vee x \notin Y)$*, then* $(Y, \mathbb{F}(Y))$ *is embedded in* $(X, \mathbb{F}(X))$*.*
(iii) If $Y = \{x_1, \ldots, x_n\}$*, where* $x_i \bowtie x_j$*, for every* $i \neq j \in \{1, \ldots, n\}$*, and* F *is a topology on* X *which is* $\bowtie$*-Hausdorff, then* $(Y, \mathbb{F}(Y))$ *is full bounded-embedded in* (X, F)*.*
(iv) $(\mathbb{N}, \mathbb{F}(\mathbb{N}))$ *is full bounded-embedded in* $(\mathbb{Q}, \mathrm{Bic}(\mathbb{Q}))$*.*
(v) If $X = \mathbb{R}$ *and* Y *is locally compact, then* $(Y, \mathrm{Bic}(Y))$ *is bounded-embedded in* $\mathcal{R}$*.*
(vi) If X *is a locally compact metric space and* Y *is dense in* X*, then* $(Y, \mathrm{Bic}(Y))$ *is dense-embedded and dense-bounded-embedded in* $(X, \mathrm{Bic}(X))$*.*
(vii) If F *is a topology on* X *and* Y *is a retract of* X*, then* $\mathcal{F}_{|Y}$ *is embedded in* $\mathcal{F}$*.*

Proof. (i) and (ii) are trivial. To show (iii) we fix some $g \in \mathbb{F}(Y)$ and let $g(x_i) = a_i$, for every i. If we consider the $(n-1) + (n-2) + \ldots + 1$ functions $f_{ij} \in F$ such that $f_{ij}(x_i) \bowtie_{\mathbb{R}} f_{ij}(x_j)$, for every $i < j$, then the function f on X, defined by $f(x) := \sum_{i=1}^{n} a_i A_i(x)$, where

$$A_i(x) := \prod_{k=i+1}^{n} \frac{f_{ik}(x) - f_{ik}(x_k)}{f_{ik}(x_i) - f_{ik}(x_k)} \prod_{k=1}^{i-1} \frac{f_{ki}(x_k) - f_{ki}(x)}{f_{ki}(x_k) - f_{ki}(x_i)},$$

is in F and $A_i(x_j) = 1$, if $j = i$, $A_i(x_j) = 0$, if $j \neq i$. Hence, f extends g, and clearly $(Y, \mathbb{F}(Y))$ is full-bounded embedded in (X, F). We need the $\bowtie$-Hausdorff condition on F so that $(f_{ij}(x_i) - f_{ij}(x_j)) \bowtie_{\mathbb{R}} 0$ and then $(f_{ij}(x_i) - f_{ij}(x_j)^{-1}$ is well-defined, for every $i < j$.

(iv) If q is a rational such that $q \geq 0$, there is a unique $n \in \mathbb{N}$ such that $q \in [n, n+1)$. If $g : \mathbb{N} \to \mathbb{R}$, we define $\phi^*(q) = \gamma_n(q)$, where $\gamma_n : \mathbb{Q} \cap [n, n+1) \to \mathbb{R}$ is defined by $\gamma_n(q) = (g(n+1) - g(n))q + (n+1)g(n) - g(n+1)n$ i.e., $\gamma_n(\mathbb{Q} \cap [n, n+1))$ is the set of the rational values in the linear segment between $g(n)$ and $g(n+1)$. Of course, $\phi^*(n) = g(n)$. Next we define $\phi^*(q) = g(0)$, for every $q < 0$. To show that $\phi^* \in \mathrm{Bic}(\mathbb{Q})$, and since ϕ^* is constant on $\mathbb{Q}_-$, it suffices to show that $\phi^* \in \mathrm{Bic}(\mathbb{Q}_+)$. For that we fix a bounded subset (B, q_0, M) of $\mathbb{Q}_+$, where without loss of generality $M \in \mathbb{N}$. Since $B \subseteq \mathcal{B}(q_0, M)$, we have that $B \subseteq [n, N]$, where $n, N \in \mathbb{N}$, $n < N$, $q_0 - M \in [n, n+1)$ and $q_0 + M \in [N, N+1)$. Each γ_i is uniformly continuous on $[i, i+1) \cap \mathbb{Q}$ with modulus of continuity $\omega_i(\epsilon) = \frac{\epsilon}{|g(i+1)-g(i)|+1}$, for every $\epsilon > 0$. Hence, ϕ^* is uniformly continuous on B with modulus of continuity $\omega_{\phi^*,B}(\epsilon) = \min\{\omega_i(\epsilon) \mid n \leq i \leq N\}$, for every $\epsilon > 0$. If g is bounded, then by its definition ϕ^* is also bounded and if a, b bound g, then a, b bound ϕ^*.

(v) If $M > 0$ such that $f(Y) \subseteq [-M, M]$, then we use the Corollary 1.

(vi) Since $\mathbb{R}$ is a complete metric space, we use the Lemma 1.

(vii) We show first that r is a quotient map i.e., $F_{|Y} = G_r = \{g : Y \to \mathbb{R} \mid g \circ r \in F\}$. By the definition of $r \in \text{Mor}(\mathcal{F}, \mathcal{F}_{|Y})$, we have that $\forall_{g \in F_{|Y}} (g \circ r \in F)$ i.e., $F_{|Y} \subseteq G_r$. For that we can also use our remark in Sect. 2 that the quotient topology G_r is the largest topology such that r is a morphism. If $g \in G_r$, then $(g \circ r)_{|Y} = g \in F_{|Y}$ i.e., $F_{|Y} \supseteq G_r$. Hence, if $g \in F_{|Y} = G_r$, the function $g \circ r \in F$ extends g.

Proposition 4. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$. If $\mathcal{G}$ is embedded in $\mathcal{F}$, then $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$.*

Proof. We show that if $g \in G_b$ and $\exists_{f \in F}(f_{|Y} = g)$, then $\exists_{f \in F_b}(f_{|Y} = g)$; if f extends g and $|g| \leq M$, then $h = (-\overline{M} \vee f) \wedge \overline{M} \in F_b$ and $h_{|Y} = g$. I.e., $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, if $\forall_{g \in G_b} \exists_{f \in F}(f_{|Y} = g)$. Since $G_b \subseteq G$ and G is embedded in $\mathcal{F}$, $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$.

There are trivial counterexamples to the converse of the previous proposition; if Y is an unbounded locally compact subset of $\mathbb{R}$, then by the Proposition 3(v) $(Y, \text{Bic}(Y))$ is full bounded-embedded in $\mathcal{R}_b$, while $(Y, \text{Bic}(Y))$ is not embedded in $\mathcal{R}_b$, since $\text{id}_Y \in \text{Bic}(Y)$ and any extension of id_Y is an unbounded function.

Proposition 5. *If $Z \subseteq Y \subseteq X$, $\mathcal{H} = (Z, H)$, $\mathcal{G} = (Y, G)$, $\mathcal{F} = (X, F)$ are Bishop spaces, $\mathcal{F}$ extends $\mathcal{G}$ and $\mathcal{G}$ is embedded in $\mathcal{F}$, then $\mathcal{H}$ is embedded in $\mathcal{F}$ if and only if $\mathcal{H}$ is embedded in $\mathcal{G}$.*

Proof. If $\forall_{h \in H} \exists_{f \in F}(f_{|Z} = h)$, we show that $\forall_{h \in H} \exists_{g \in G}(g_{|Z} = h)$. If $h \in H$ and we restrict some $f \in F$ which extends h to Y, we get an extension of h in G. For the converse if $h \in H$, we extend it to some $g \in G$, and g is extended to some $f \in F$, since $\mathcal{G}$ is embedded in $\mathcal{F}$.

The next three propositions show how the embedding of $\mathcal{G}$ in $\mathcal{F}$ generates new embeddings under the presence of certain morphisms.

Proposition 6. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ and $\mathcal{H} = (B, H)$ are Bishop spaces, where $B \subseteq Y$. If $\mathcal{H}$ is embedded in $\mathcal{G}$ and $e \in \text{setEpi}(\mathcal{F}, \mathcal{G})$, then the weak Bishop space $\mathcal{F}(e_{|A})$ on $A = e^{-1}(B)$ induced by $e_{|A}$ is embedded in $\mathcal{F}$.*

Proof. Since $e : X \to Y$ is onto Y, we have that $e_{|A} : A \to B$ is onto B and $e_{|A} \in \text{setEpi}(\mathcal{F}(e_{|A}), \mathcal{H})$, where by a remark following the definition of weak topology in Sect. 2 we have that $F(e_{|A}) = \{h \circ e_{|A} \mid h \in H\}$. If we fix some $h \circ e_{|A} \in F(e_{|A})$, where $h \in H$, then, since $\mathcal{H}$ is embedded in $\mathcal{G}$, there is some $g \in G$ such that $g_{|B} = h$. Since $e \in \text{setEpi}(\mathcal{F}, \mathcal{G}) \subseteq \text{Mor}(\mathcal{F}, \mathcal{G})$, we get that $g \circ e \in F$. If $a \in A$, then $(g \circ e)(a) = g(b) = h(b)$, where $b = e(a)$. Since $(h \circ e_{|A})(a) = h(e(a)) = h(b)$, we get that $(g \circ e)_{|A} = h \circ e_{|A}$ i.e., $\mathcal{F}(e_{|A})$ is embedded in $\mathcal{F}$.

Proposition 7. *If $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ $\mathcal{H} = (Z, H)$ are Bishop spaces, $Y \subseteq X$, $\mathcal{G}$ is embedded in $\mathcal{F}$ and $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$ is open, then the quotient Bishop space $\mathcal{G}_{e_{|Y}} = (e(Y), G_{e_{|Y}})$ is embedded in $\mathcal{H}$.*

Proof. Let $g' : e(Y) \to \mathbb{R} \in G_{e_{|Y}}$ i.e., $g' \circ e_{|Y} \in G$. Since $\mathcal{G}$ is embedded in $\mathcal{F}$, there exists some $f \in F$ such that $f_{|Y} = g' \circ e_{|Y}$. Since e is open, there exists some $h \in H$ such that $f = h \circ e$. We show that $h_{|e(Y)} = g'$; if $b = e(y) \in e(Y)$, for some $y \in Y$, then $h(b) = h(e(y)) = f(y) = (g' \circ e_{|Y})(y) = g'(e(y)) = g'(b)$.

Next we translate to TBS the classical fact that if an element of $C(X)$ carries a subset of X homeomorphically onto a closed set S in $\mathbb{R}$, then S is C-embedded in X (see [11], p. 20).

Proposition 8. *Suppose that A is a locally compact subset of $\mathbb{R}$, $\mathcal{F} = (X, F)$ is a Bishop space, $Y \subseteq X$ and $f \in F$ such that $f_{|Y} : Y \to A$ is an isomorphism between $\mathcal{F}_{|Y}$ and $(A, \mathrm{Bic}(A)_b)$. Then $\mathcal{F}_{|Y}$ is embedded in $\mathcal{F}$.*

Proof. Since $f_{|Y}$ is an isomorphism between $\mathcal{F}_{|Y}$ and $(A, \mathrm{Bic}(A)_b)$, its inverse θ is an isomorphism between $(A, \mathrm{Bic}(A)_b)$ and $\mathcal{F}_{|Y}$. We fix some $g \in F_{|Y}$. Since $\theta \in \mathrm{Mor}((A, \mathrm{Bic}(A)_b), \mathcal{F}_{|Y})$, we have that $g \circ \theta \in \mathrm{Bic}(A)_b$. By the Corollary 1 there exists some $\phi \in \mathrm{Bic}(\mathbb{R})$ which extends $g \circ \theta$. By BS_3 we have that $\phi \circ f \in F$ and for every $y \in Y$ we have that $(\phi \circ f)(y) = ((g \circ \theta) \circ f)(y) = (g \circ (\theta \circ f))(y) = (g \circ (\theta \circ f_{|Y}))(y) = (g \circ \mathrm{id}_{|Y})(y) = g(y)$.

If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ is C^*-embedded in X, then if Y is also C-embedded in X, it is (completely) separated in $C(X)$ from every $C(X)$-zero set disjoint from it (see [11], pp. 19–20). If we add within TBS a positive notion of disjointness between Y and $\zeta(f)$ though, we avoid the corresponding hypothesis of $\mathcal{G}$ being embedded in $\mathcal{F}$.

Definition 8. *If F is a topology on X, $f \in F$ and $Y \subseteq X$, we say that Y and $\zeta(f)$ are separated,* $\mathrm{Sep}(Y, \zeta(f))$, *if $\forall_{y \in Y}(|f(y)| > 0)$, and Y and $\zeta(f)$ are uniformly separated,* $\mathrm{Usep}(Y, \zeta(f))$, *if there is some $c > 0$ such that $\forall_{y \in Y}(|f(y)| \geq c)$.*

Of course, $\mathrm{Usep}(Y, \zeta(f)) \to \mathrm{Sep}(Y, \zeta(f))$. If $f, g \in F$ such that $|f| + |g| \geq \overline{c}$ (see the formulation of the Theorem 3), then we get $\mathrm{Usep}(\zeta(g), \zeta(f))$ and $\mathrm{Usep}(\zeta(f), \zeta(g))$. Since the sets $U(f) = \{x \in X \mid f(x) > 0\}$, where $f \in F$, are basic open sets in the induced neighborhood structure on X by F (see [4], p. 77), we call Y a *uniform G_δ-set*, if there exists a sequence $(f_n)_n$ in F such that $Y = \bigcap_{n \in \mathbb{N}} U(f(n))$ and $\mathrm{Usep}(Y, \zeta(f_n))$, for every $n \in \mathbb{N}$.

Proposition 9. *If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, $\mathcal{F}$ extends $\mathcal{G}$, $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, and $f \in F$, then* $\mathrm{Usep}(Y, \zeta(f)) \to Y \bowtie_F \zeta(f)$.

Proof. Since $|f| \in F$ and $\mathcal{F}$ extends $\mathcal{G}$, we have that $|f|_{|Y} \in G$, and $|f|_{|Y} \geq \overline{c}$. By Theorem 2 we get that $\frac{1}{|f|_{|Y}} \in G$. Since $\overline{0} < \frac{1}{|f|_{|Y}} \leq \overline{\frac{1}{c}}$, we actually have that $\frac{1}{|f|_{|Y}} \in G_b$. Since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there exists $h \in F$ such that

$h_{|Y} = \frac{1}{|f|_{|Y}}$. Since $|h| \in F$ satisfies $|h|_{|Y} = \frac{1}{|f|_{|Y}}$ too, we suppose without loss of generality that $h \geq \overline{0}$. If we define $g := h|f|$, then $g \in F$, $g(y) = h(y)|f(y)| = \frac{1}{|f(y)|}|f(y)| = 1$, for every $y \in Y$, and $g(x) = h(x)|f(x)| = h(x)0 = 0$, for every $x \in \zeta(f)$.

Corollary 2. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, and $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$. If Y is a uniform G_δ-set, then Y is an F-zero set.*

Proof. Suppose that $Y = \bigcap_{n\in\mathbb{N}} U(f_n)$ and $\forall_{y\in Y}(|f_n(y)| \geq c_n)$, where $c_n > 0$, for every $n \in \mathbb{N}$. Since $U(f) = U(f \vee \overline{0})$ and $\mathrm{Usep}(Y, \zeta(f)) \to \mathrm{Usep}(Y, \zeta(f \vee \overline{0}))$, we assume without loss of generality that $f_n \geq \overline{0}$, for every $n \in \mathbb{N}$. By the proof the Proposition 9 we have that there is a function $h_n \in F$ such that $h_n \geq \overline{0}$, $(h_n f_n)(Y) = 1$ and $(h_n f_n)(\zeta(f_n)) = 0$, for every $n \in \mathbb{N}$. Therefore, $Y \subseteq \zeta(g_n)$, where $g_n = (h_n f_n - \frac{2}{3}) \wedge \overline{0}$, for every $n \in \mathbb{N}$. Next we show that $\zeta(g_n) \subseteq U(f_n)$, for every $n \in \mathbb{N}$. Since $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$ and according to the proof the Proposition 9, $\overline{0} < \frac{1}{f_{n|Y}} \leq \overline{\frac{1}{c_n}}$, we get that $\overline{0} < h_n \leq \overline{\frac{1}{c_n}}$. If $z \in X$ such that $g_n(z) = 0$, then $h_n(z) f_n(z) \geq \frac{2}{3}$, and since $h_n(z) > 0$, we conclude that $f_n(z) \geq \frac{2}{3h_n(z)} > 0$. Thus, $Y \subseteq \bigcap_{n\in\mathbb{N}} \zeta(g_n) \subseteq \bigcap_{n\in\mathbb{N}} U(f_n) = Y$, which implies that $Y = \bigcap_{n\in\mathbb{N}} \zeta(g_n) = \zeta(g)$, for some $g \in F$, since $Z(F)$ is closed under countably infinite intersections.

Without the condition of $\mathcal{G}$ being full bounded-embedded in $\mathcal{F}$ in the previous proposition we can show only that $\neg(f_n(z) = 0)$. Although $f_n(z) \geq 0$, we cannot infer within BISH that $f_n(z) > 0$; the property of the reals $\forall_{x,y\in\mathbb{R}}(\neg(x \geq y) \to x < y)$ is equivalent to Markov's principle (MP) (see [5], p. 14), and it is easy to see that this property is equivalent to $\forall_{x\in\mathbb{R}}(x \geq 0 \to \neg(x = 0) \to x > 0)$. Next we translate to TBS the classical result that if Y is C^*-embedded in X such that Y is (completely) separated from every $C(X)$-zero set disjoint from it, then Y is C-embedded in X. Constructively it is not clear, as it is in the classical case, how to show that the expected positive formulation of the previous condition provides an inverse to Proposition 4. The reason is that if (X, F) is an arbitrary Bishop space, it is not certain that $\tan \circ f \in F$, for some $f : X \to (-\frac{\pi}{2}, \frac{\pi}{2}) \in F$ (note that $\tan^{-1} = \arctan \in \mathrm{Bic}(\mathbb{R})$). If $\Phi_1, \Phi_2 \subseteq \mathbb{F}(X)$, we denote by $\Phi_1 \vee \Phi_2$ the least topology including them. The proof of the interesting case of the next theorem is in BISH + MP.

Theorem 4. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$, $a > 0$, $e : (-a, a) \to \mathbb{R}$ such that $e^{-1} : \mathbb{R} \to (-a, a) \in \mathrm{Bic}(\mathbb{R})$, and $\mathcal{F}(a) = (X, F(a))$, where*

$$F(a) = F \vee \{e \circ f \mid f \in F \text{ and } f(X) \subseteq (-a, a)\}.$$

(i) If $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, then $\mathcal{G}$ is embedded in $\mathcal{F}(a)$.
(ii) (MP) *If $\forall_{f\in F}(\mathrm{Sep}(Y, \zeta(f)) \to Y \bowtie_F \zeta(f))$ and $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, then $\mathcal{G}$ is embedded in $\mathcal{F}(a)$.*

Proof. We fix some $g \in G$. Since $e^{-1} \in \mathrm{Bic}(\mathbb{R})$, by the condition BS_3 we have that $e^{-1} \circ g : Y \to (-a, a) \in G_b$. Since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there is some $f \in F_b$ such that $f_{|Y} = e^{-1} \circ g$.

(i) If $\mathcal{G}$ is full bounded-embedded in $\mathcal{F}$, then we have that $f : X \to (-a, a)$. Hence, $e \circ f \in F(a)$, and $(e \circ f)_{|Y} = e \circ f_{|Y} = e \circ (e^{-1} \circ g) = g$.

(ii) In [24] we showed within BISH that $[|f| \geq \overline{a}] = \{x \in X \mid |f|(x) \geq a\} = \zeta(f^*)$, where $f^* = (|f| - \overline{a}) \wedge \overline{0} \in F$. If $y \in Y$, then $|f^*(y)| = |(|f(y)| - a) \wedge 0| = |(|(e^{-1} \circ g)(y)| - a) \wedge 0| = ||(e^{-1} \circ g)(y)| - a| = a - |(e^{-1} \circ g)(y)| > 0$, since $|(e^{-1} \circ g)(y)| \in [0, a)$ (if $-a < x < a$, then $|x| < a$). Since $\mathrm{Sep}(Y, \zeta(f^*))$, by our hypothesis there exists some $h \in F$ such that $0 \leq h \leq 1$, $h(Y) = 1$ and $h(\zeta(f^*)) = 0$. There is no loss of generality if we assume that $0 \leq h \leq 1$, since if $h \in F$ separates Y and $\zeta(f^*)$, then $|h| \wedge \overline{1} \in F$ separates them too. We define $J := f \cdot h \in F$. If $y \in Y$, we have that $J(y) = f(y)h(y) = f(y)$. Next we show that $\forall_{x \in X}(\neg(|J(x)| \geq a))$. If $x \in X$ such that $|J(x)| \geq a$, then $|f(x)| \geq |f(x)||h(x)| = |j(x)| \geq a$, therefore $x \in \zeta(f^*)$. Consequently, $h(x) = 0$, and $0 = |J(x)| \geq a > 0$, which leads to a contradiction. Because of MP we get that $\forall_{x \in X}(|J(x)| < a))$, in other words, $J : X \to (-a, a)$. Hence $e \circ J \in F(a)$, and $(e \circ J)_{|Y} = e \circ J_Y = e \circ f = e \circ (e^{-1} \circ g) = g$.

4 The Urysohn Extension Theorem

In this section we show the Urysohn extension theorem within TBS, an adaptation of Urysohn's theorem that any closed set in a normal topological space is C^*-embedded (see [11], p. 266). As Gillman and Jerison note in [11], p. 18, it is "the basic result about C^*-embedding". According to it, a subspace Y of a topological space X is C^*-embedded in X if and only if any two (completely) separated sets in Y are (completely) separated in X. Here we call Urysohn extension theorem the appropriate translation to TBS of the non-trivial sufficient condition. Next follows the translation to TBS of the trivial necessity condition. The hypothesis "$\mathcal{F}$ extends $\mathcal{G}$" of the Theorem 5 is not necessary to its proof.

Proposition 10. *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $Y \subseteq X$. If $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, then $\forall_{A, B \subseteq Y}(A \bowtie_{G_b} B \to A \bowtie_{F_b} B)$.*

Proof. If $A, B \subseteq Y$ such that A, B are separated by some $g \in G_b$, then, since $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$, there is some $f \in F_b$ which extends g, hence f separates A and B.

Next we show that the proof of the classical Urysohn extension theorem can be carried out within BISH. Recall that if $x \in \mathbb{R}$, then $x = (x_n)_{n \in \mathbb{N}}$, where $x_n \in \mathbb{Q}$, for every $n \in \mathbb{N}$, such that $\forall_{n, m \in \mathbb{N}^+}(|x_m - x_n| \leq m^{-1} + n^{-1})$. Moreover, $x > 0 :\leftrightarrow \exists_{n \in \mathbb{N}}(x_n > \frac{1}{n})$, and $x \geq 0 :\leftrightarrow \forall_{n \in \mathbb{N}}(x_n \geq -\frac{1}{n})$ (see [4], pp. 18–22). If $q \in \mathbb{Q}$, then $q = (q_n)_{n \in \mathbb{N}} \in \mathbb{R}$, where $q_n = q$, for every $n \in \mathbb{N}$. Using MP one shows immediately that $\neg(x \leq -q) \to \neg(x \geq q) \to |x| < q$, where $x \in \mathbb{R}$ and $q \in \mathbb{Q}$. Without MP and completely within BISH, we show that under the same hypotheses one gets that $|x| \leq q$, which is what we need in order to get a constructive proof of the Urysohn extension theorem.

Lemma 2. $\forall_{q\in\mathbb{Q}}\forall_{x\in\mathbb{R}}(\neg(x \leq -q) \rightarrow \neg(x \geq q) \rightarrow |x| \leq q)$.

Proof. We fix some $q \in \mathbb{Q}$, $x = (x_n)_n \in \mathbb{R}$ and we suppose that $\neg(x \leq -q)$ and $\neg(x \geq q)$. Since $|x| = (\max\{x_n, -x_n\})_{n\in\mathbb{N}}$, we show that $q \geq |x| \leftrightarrow q - |x| \geq 0 \leftrightarrow \forall_n(q - \max\{x_n, -x_n\} \geq -\frac{1}{n})$. If we fix some $n \in \mathbb{N}$, and since $x_n \in \mathbb{Q}$, we consider the following case distinction.

(i) $x_n \geq 0$: Then $q - \max\{x_n, -x_n\} = q - x_n$ and we get that $q - x_n < -\frac{1}{n} \rightarrow x_n - q > \frac{1}{n} \rightarrow x > q \rightarrow x \geq q \rightarrow \bot$, by our second hypothesis. Hence, $q - x_n \geq -\frac{1}{n}$.
(ii) $x_n \leq 0$: Then $q - \max\{x_n, -x_n\} = q + x_n$ and we get that $q + x_n < -\frac{1}{n} \rightarrow -q - x_n > \frac{1}{n} \rightarrow -q > x \rightarrow -q \geq x \rightarrow \bot$, by our first hypothesis. Hence, $q + x_n \geq -\frac{1}{n}$.

Theorem 5 (Urysohn Extension Theorem for Bishop Spaces). *Suppose that $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces, $Y \subseteq X$ and $\mathcal{F}$ extends $\mathcal{G}$. If $\forall_{A,B\subseteq Y}(A \bowtie_{G_b} B \rightarrow A \bowtie_{F_b} B)$, then $\mathcal{G}$ is bounded-embedded in $\mathcal{F}$.*

Proof. We fix some $g \in G_b$, and let $|g| \leq \overline{M}$, for some natural $M > 0$. In order to find an extension of g in F_b we define a sequence $(g_n)_{n\in\mathbb{N}^+}$, such that $g_n \in G_b$ and

$$|g_n| \leq \overline{3r_n}, \qquad r_n := \frac{M}{2}(\frac{2}{3})^n,$$

for every $n \in \mathbb{N}^+$. For $n = 1$ we define $g_1 = g$, and we have that $|g_1| \leq \overline{M} = \overline{3r_1}$. Suppose next that we have defined some $g_n \in G_b$ such that $|g_n| \leq \overline{3r_n}$. We consider the sets

$$A_n = [g_n \leq \overline{-r_n}] = \{y \in Y \mid g_n(y) \leq -r_n\},$$

$$B_n = [g_n \geq \overline{r_n}] = \{y \in Y \mid g_n(y) \geq r_n\}.$$

Clearly, $g_n^*(A_n) = -r_n$ and $g_n^*(B_n) = r_n$, where $g_n^* = (\overline{-r_n} \vee g_n) \wedge \overline{r_n} \in G_b$. Since $g_n^*(A_n) \bowtie_{\mathbb{R}} g_n^*(B_n)$, we get that $A_n \bowtie_{G_b} B_n$, therefore there exists some $f \in F_b$ such that $A_n \bowtie_f B_n$. Without loss of generality we assume that $f_n(A_n) = -r_n$, $f_n(B_n) = r_n$ and $|f_n| \leq \overline{r_n}$. Next we define

$$g_{n+1} := g_n - f_{n|Y} \in G_b,$$

since $\mathcal{F}$ extends $\mathcal{G}$. If $y \in A_n$ we have that

$$|g_{n+1}(y)| = |(g_n - f_{n|Y})(y)| = |g_n(y) - (-r_n)| = |g_n(y) + r_n| \leq 2r_n,$$

since $-3r_n \leq g_n(y) \leq -r_n \rightarrow -2r_n \leq g_n(y) + r_n \leq 0$. If $y \in B_n$ we have that

$$|g_{n+1}(y)| = |(g_n - f_{n|Y})(y)| = |g_n(y) - r_n| = g_n(y) - r_n \leq 2r_n,$$

since $r_n \leq g_n(y) \leq 3r_n \rightarrow 0 \leq g_n(y) - r_n \leq 2r_n$. Next we show that

$$\forall_{y\in Y}(|g_{n+1}(y)| \leq 2r_n).$$

We fix some $y \in Y$ and we suppose that $|g_{n+1}(y)| > 2r_n$. This implies that $y \notin A_n \cup B_n$, since if $y \in A_n \cup B_n$, then by the previous calculations we get that $|g_{n+1}(y)| \leq 2r_n$, which contradicts our hypothesis. Hence we have that $\neg(g_n(y) \leq -r_n)$ and $\neg(g_n(y) \geq r_n)$. By the Lemma 2 we get that $|g_n(y)| \leq r_n$, therefore $|g_{n+1}(y)| \leq |g_n(y)| + |f_n(y)| \leq r_n + r_n = 2r_n$, which contradicts our assumption $|g_{n+1}(y)| > 2r_n$. Thus we get that $|g_{n+1}(y)| \leq 2r_n$, and since y is arbitrary we get

$$|g_{n+1}| \leq \overline{2r_n} = \overline{3r_{n+1}}.$$

By the condition BS_4 the function $f := \sum_{n=1}^{\infty} f_n$ belongs to F, since the partial sums converge uniformly to f. Note that the infinite sum is well-defined by the Weierstrass comparison test (see [4], p. 32). Note also that

$$(f_1 + \ldots + f_n)_{|Y} = (g_1 - g_2) + (g_2 - g_3) + \ldots + (g_n - g_{n+1}) = g_1 - g_{n+1}.$$

Since $r_n \xrightarrow{n} 0$, we get $g_{n+1} \xrightarrow{n} 0$, hence $f_{|Y} = g_1 = g$. Note that f is also bounded by M:

$$|f| = |\sum_{n=1}^{\infty} f_n| \leq \sum_{n=1}^{\infty} |f_n| \leq \sum_{n=1}^{\infty} \frac{M}{2}(\frac{2}{3})^n = \frac{M}{2} \sum_{n=1}^{\infty} (\frac{2}{3})^n = \frac{M}{2} 2 = M.$$

The main hypothesis of the Urysohn extension theorem

$$\forall_{A,B \subseteq Y}(A \bowtie_{G_b} B \rightarrow A \bowtie_{F_b} B)$$

requires quantification over the power set of Y, therefore it is against the practice of predicative constructive mathematics. It is clear though by the above proof that we do not need to quantify over all the subsets of Y, but only over the ones which have the form of A_n and B_n. If we replace the initial main hypothesis by the following

$$\forall_{g,g' \in G_b} \forall_{a,b \in \mathbb{R}}([g \leq \overline{a}] \bowtie_{G_b} [g' \geq \overline{b}] \rightarrow [g \leq \overline{a}] \bowtie_{F_b} [g' \geq \overline{b}]),$$

we get a stronger form of the Urysohn extension theorem, since this is the least condition in order the above proof to work. Actually, this stronger formulation of the Urysohn extension theorem applies to the classical setting too. A slight variation of the previous new main hypothesis, which is probably better to use, is

$$\forall_{g,g' \in G_b}(\zeta(g) \bowtie_{G_b} \zeta(g') \rightarrow \zeta(g) \bowtie_{F_b} \zeta(g')),$$

since the sets of the form A_n and B_n are G_b-zero sets.

Definition 9. *If (X, F) is a Bishop space and $Y \subseteq X$ is inhabited, we say that Y is a Urysohn subset of X, if $\forall_{g,g' \in (F_{|Y})_b}(\zeta(g) \bowtie_{(F_{|Y})_b} \zeta(g') \rightarrow \zeta(g) \bowtie_{F_b} \zeta(g'))$.*

Next follows a direct corollary of the Theorem 5 and the previous remark.

Corollary 3. *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space, $Y \subseteq X$ is a Urysohn subset of X and $g : Y \to \mathbb{R}$ is in $(F_{|Y})_b$. Then there exists $f : X \to \mathbb{R}$ in F_b which extends g.*

An absolute retract for normal topological spaces is a space that can be substituted for $\mathbb{R}$ in the formulation of the Tietze theorem, according to which a continuous real-valued function on a closed subset of a normal topological space has a continuous extension (see [10], p. 151).

Definition 10. *If Q is a property on sets, a Bishop space $\mathcal{H} = (Z, H)$ is called an absolute retract with respect to Q, or $\mathcal{H}$ is $\mathrm{AR}(Q)$, if for every Bishop space $\mathcal{F} = (X, F)$ and $Y \subseteq X$ we have that*

$$Q(Y) \to \forall_{e \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H})} \exists_{e^* \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})} (e^*_{|Y} = e).$$

Clearly, the Corollary 3 says that $\mathcal{R}$ is AR(Urysohn). The next proposition shows that there exist many absolute retracts. In particular, the products $\mathcal{R}^n, \mathcal{R}^\infty$ are AR(Urysohn).

Proposition 11. *Suppose that $\mathcal{H}_i = (Z_i, H_i)$ is a Bishop space, for every $i \in I$. Then $\prod_{i \in I} \mathcal{H}_i$ is $\mathrm{AR}(Q)$ if and only if $\mathcal{H}_i$ is $\mathrm{AR}(Q)$, for every $i \in I$.*

Proof. ($\leftarrow$) If $Y \subseteq X$ such that $Q(Y)$ and if $\mathcal{H}_i$ is $\mathrm{AR}(Q)$, for every $i \in I$, then by the Proposition 1(i) we have that

$$e : Y \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I}) \leftrightarrow \forall_{i \in I} (\pi_i \circ e \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H}_i))$$

$$\to \forall_{i \in I} (\exists_{e_i^* \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)} (e^*_{i\,|Y} = \pi_i \circ e)).$$

We define $e^* : X \to \prod_{i \in I} Z_i$ by $x \mapsto (e_i^*(x))_{i \in I}$. Clearly, $e^*(y) = e_i^*(y))_{i \in I} = ((\pi_i \circ e)(y))_{i \in I} = e(y)$ and $e^* \in \mathrm{Mor}(\mathcal{F}, \prod_{i \in I} \mathcal{H}_{i \in I})$, by the Proposition 1(i) and the fact that $e_i^* = \pi_i \circ e^* \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)$, for every $i \in I$.

($\to$) Suppose that $\prod_{i \in I} \mathcal{H}_i$ is $\mathrm{AR}(Q)$ and $e_i : Y \to Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \mathcal{H}_i)$. If we fix $z = (z_i)_{i \in I} \in \prod_{i \in I} Z_i$, then by the Proposition 2 the function

$$s_i : Z_i \to S(z; i) = Z_i \times \prod_{j \neq i} \{z_j\} \subseteq \prod_{i \in I} Z_i$$

$$z_i \mapsto z_i \times \prod_{j \neq i} \{z_j\}$$

is an isomorphism between $\mathcal{H}_i$ and the slice space $\mathcal{S}(z; i) = (S(z; i), H(z; i))$, where $H(z; i) = (\prod_{i \in I} H_i)_{|S(z;i)}$. Hence, the mapping $s_i \circ e_i : Y \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I})$. By our hypothesis there exists some $e^* : X \to \prod_{i \in I} Z_i \in \mathrm{Mor}(\mathcal{F}_{|Y}, \prod_{i \in I} \mathcal{H}_{i \in I})$ which extends $s_i \circ e_i$. Thus, $\pi_i \circ e^* : X \to Z_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{H}_i)$, for every $i \in I$. But $\pi_i \circ e^* = e_i$, since for every $y \in Y$ we have that $(\pi_i \circ e^*)(y) = \pi(e^*(y)) = \pi_i((s_i \circ e_i)(y)) = \pi_i(e_i(y) \times \prod_{j \neq i} \{z_j\}) = e_i(y)$.

5 Concluding Comments

In this paper we presented the basic theory of embeddings of Bishop spaces and we showed that the classical proof of the Urysohn extension theorem for topological spaces generates a constructive proof of the Urysohn extension theorem for Bishop spaces. Our results form only the very beginning of a theory of embeddings of Bishop spaces. If we look at the classical theory of embeddings of rings of continuous functions, we will see too many topics that at first sight it seems difficult, to say the least, to develop constructively. The Stone-Čech compactification and Hewitt's realcompactification depend on the existence of non-trivial ultrafilters, while many facts in the characterizations of the maximal ideals of $C(X)$ or $C^*(X)$ depend on non-constructive formulations of compactness.

Nevertheless, we find encouraging that quite "soon" one can start developing a theory of embeddings within TBS, and also rewarding that non-trivial theorems, like the Urysohn extension theorem, belong to it. Behind these partial "successes" lies, in our view, the function-theoretic character of TBS which offers the direct "communication" between TBS and the theory of $C(X)$ that we mentioned in the Introduction. Maybe, this is the main advantage of TBS with respect to other approaches to constructive topology.

The apartness relations mentioned already here show the connection of TBS with the theory of apartness spaces of Bridges and Vîţă in [6]. Both these theories start from a notion of space that differs from a topological space treated intuitionistically, as in [28] or [12], or from a constructive variation of the notion of a base of a topological space, the starting point of the point-free formal topology of Martin-Löf and Sambin (see [25,26]) and Bishop's theory of neighborhood spaces, as it is developed mainly by Ishihara in [14,15]. In our opinion, if the notion of space in constructive topology "mimics" that of topological space, then it is more difficult to constructivise topology than starting from a notion of space which by its definition is more suitable to constructive study. The function-theoretic character of the notion of Bishop space and of Bishop morphism, in contrast to the set-theoretic character of an apartness space and of a strongly continuous function, seems to facilitate a constructive reconstruction of topology and a possible future translation of TBS to type theory.

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