

Sequent Calculus for Intuitionistic Epistemic Logic IEL

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Abstract. The formal system of intuitionistic epistemic logic IEL was proposed by S. Artemov and T. Protopopescu. It provides the formal foundation for the study of knowledge from an intuitionistic point of view based on Brouwer-Heyting-Kolmogorov semantics of intuitionism. We construct a cut-free sequent calculus for IEL and establish that polynomial space is sufficient for the proof search in it. We prove that IEL is PSPACE-complete.

Keywords: Modal logic · Intuitionistic epistemic logic · Sequent calculus · Cut-elimination · PSPACE

1 Introduction

Modal logic IEL, the basic Intuitionistic Epistemic Logic, was proposed by S. Artemov and T. Protopopescu in [1]. It was defined by the following Hilbert system:

Axioms

1. Axioms of propositional intuitionistic logic,
2. $K(F \rightarrow G) \rightarrow (KF \rightarrow KG)$ (distribution),
3. $F \rightarrow KF$ (co-reflection),
4. $\neg K\perp$ (consistency).

Rule $F, F \rightarrow G \vdash G$ (Modus Ponens). Here knowledge modality K means verified truth, as suggested by T. Williamson in [2]. According to the Brouwer-Heyting-Kolmogorov semantics of intuitionistic logic, a proposition is true iff it is proved. The co-reflection principle states that any such proof can be verified.

The intuitionistic meaning of implication provides an effective proof checking procedure that produces a proof of KF given a proof of F . But the assumption that its output always contains a proof of F is too restrictive. The procedure may involve some trusted sources which do not necessarily produce explicit proofs of what they verify¹. So the backward implication which is the reflection principle

¹ For example, it can be some trusted database that stores true facts without proofs or some zero-knowledge proof.

$KF \rightarrow F$ used in the classical epistemic logic (see [3]) is wrong in the intuitionistic setting. In general, a proof of KF is less informative than a proof of F .

At the same time some instances of the reflection principle are true in IEL. In particular, it is the consistency principle which is equivalent to $K\perp \rightarrow \perp$. The proof of $K\perp$ contains the same information as the proof of \perp because there is no such proof at all. The more general example is the reflection principle for negative formulas: $K\neg F \rightarrow \neg F$. It is provable in IEL (see [1]).

In this paper we develop the proof theory for IEL. Our main contributions are the cut-free sequent formulation and the complexity bound for this logic. It is established that polynomial space is sufficient for the proof search, so IEL is PSPACE-complete.

Some other sequent cut-free formalization of intuitionistic reasoning about knowledge was proposed in [4].² It is based on a bimodal logic with reflexive knowledge modality \mathcal{K} that is different from the co-reflexive modality K used in IEL.

Our cut-elimination technique is syntactic (see [5]). We formulate a special cut-free sequent calculus $\text{IEL}_{\mathcal{G}}^-$ without structural rules (see Sect. 3) that is correct with respect to the natural translation into IEL. It has a specific K -introduction rule (KI_1) that also allows to contract a formula F in the presence of KF in antecedents. This choice makes it possible to prove the admissibility of the standard contraction rule as well as the admissibility of all natural IEL-correct modal rules (Sects. 4, 5). The admissibility of the cut-rule is proved by the usual induction on the cutrank (Sect. 6). As the result we obtain the equivalence between $\text{IEL}_{\mathcal{G}}^-$ and $\text{IEL}_{\mathcal{G}}^0$. (The latter is the straightforwardly formulated sequent counterpart for IEL with the cut-rule). Finally we formulate a light cut-free variant of $\text{IEL}_{\mathcal{G}}^-$ with the contraction rule and with modal rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow F}{\Gamma_1, K(\Gamma_2) \Rightarrow KF} (KI), \quad \frac{\Gamma \Rightarrow K\perp}{\Gamma \Rightarrow F} (U).$$

It is equivalent to $\text{IEL}_{\mathcal{G}}^-$.

The proof search for IEL can be reduced to the case of so-called minimal derivations (Sect. 7). We implement it as a game of polynomial complexity and use the characterization $\text{AP}=\text{PSPACE}$ (see [7]) to prove the upper complexity bound for IEL. The matching lower bound follows from the same bound for intuitionistic propositional logic [8].

2 Sequent Formulation of IEL

The definition of intuitionistic sequents is standard (see [5]). Formulas are build from propositional variables and \perp using \wedge , \vee , \rightarrow and K ; $\neg F$ means $F \rightarrow \perp$.

² Sequents in [4] are classical (multiconclusion) and contain special labels denoting worlds of a Kripke structure, so this formalization can be considered as a classical formulation of the theory of forcing relation in a Kripke structure that corresponds to the intuitionistic bimodal epistemic logic.

A sequent has the form $\Gamma \Rightarrow F$ where F is a formula and Γ is a multiset of formulas. $K(\Gamma)$ denotes KF_1, \dots, KF_n when $\Gamma = F_1, \dots, F_n$.

Let IEL_G^0 be the following calculus:

Axioms

$$\Gamma, A \Rightarrow A \quad (A \text{ is a variable}), \quad \Gamma, \perp \Rightarrow F, \quad \Gamma, K\perp \Rightarrow F.$$

Rules

$$\begin{array}{l} \frac{\Gamma, F, F \Rightarrow G}{\Gamma, F \Rightarrow G} \text{ (Contraction)} \\ \frac{\Gamma, F, G \Rightarrow H}{\Gamma, F \wedge G \Rightarrow H} (\wedge \Rightarrow) \\ \frac{\Gamma, F \Rightarrow H \quad \Gamma, G \Rightarrow H}{\Gamma, F \vee G \Rightarrow H} (\vee \Rightarrow) \\ \frac{\Gamma \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H} (\rightarrow \Rightarrow) \\ \frac{\Gamma \Rightarrow F}{K(\Gamma) \Rightarrow KF} (KI_0) \end{array} \quad \begin{array}{l} \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2, F \Rightarrow G}{\Gamma_1, \Gamma_2 \Rightarrow G} \text{ (Cut)} \\ \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G} (\Rightarrow \wedge) \\ \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2} (\Rightarrow \vee)_i \quad (i = 1, 2) \\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} (\Rightarrow \rightarrow) \\ \frac{\Gamma, F, KF \Rightarrow G}{\Gamma, F \Rightarrow G} (KC) \end{array}$$

Comment. IEL_G^0 is a straightforwardly formulated sequent counterpart of IEL. Axioms and rules without the modality correspond to the standard sequent formulation of the intuitionistic propositional logic (cf. system G2i from [5] with the cut-rule). The modal axiom corresponds to the consistency principle. Modal rules (KI_0) and (KC) reflect the distribution and co-reflection principles respectively. Instead of the K -contraction rule (KC) one can take the equivalent K -elimination rule:

$$\frac{\Gamma, KF \Rightarrow G}{\Gamma, F \Rightarrow G} (KE).$$

Theorem 1. $\text{IEL}_G^0 \vdash \Gamma \Rightarrow F$ iff $\text{IEL} \vdash \wedge \Gamma \rightarrow F$.

Proof. Straightforward induction on the derivations. □

Our goal is to eliminate the cut-rule. But the cut-elimination result for IEL_G^0 will not have the desirable consequences, namely, the subformula property and termination of the proof search procedure. Below we give a different formulation without these disadvantages.

3 Cut-Free Variant IEL_G^- with Rules (KI_1) and (U)

Axioms

$$\Gamma, A \Rightarrow A, \quad A \text{ is a variable or } \perp.$$

Rules

$$\begin{array}{c}
\frac{\Gamma, F, G \Rightarrow H}{\Gamma, F \wedge G \Rightarrow H} (\wedge \Rightarrow) \\
\frac{\Gamma, F \Rightarrow H \quad \Gamma, G \Rightarrow H}{\Gamma, F \vee G \Rightarrow H} (\vee \Rightarrow) \\
\frac{\Gamma, F \rightarrow G \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H} (\rightarrow \Rightarrow) \\
\frac{\Gamma, K(\Delta), \Delta \Rightarrow F}{\Gamma, K(\Delta) \Rightarrow KF} (KI_1)
\end{array}
\qquad
\begin{array}{c}
\frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G} (\Rightarrow \wedge) \\
\frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2} (\Rightarrow \vee)_i \quad (i = 1, 2) \\
\frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} (\Rightarrow \rightarrow) \\
\frac{\Gamma \Rightarrow K\perp}{\Gamma \Rightarrow F} (U)
\end{array}$$

In the rule (KI_1) we additionally require that Γ does not contain formulas of the form KG . (This requirement is unessential, see Corollary 3).

We define the main (occurrences of) formulas for axioms and for all inference rules except (KI_1) as usual — they are the displayed formulas in the conclusions (not members of Γ, H). For the rule (KI_1) all members of $K(\Delta)$ and the formula KF are main.

Comment. The propositional part of $\text{IEL}_{\mathcal{G}}^-$ is the same as in the system G3m from [5]. In the modal part we do not add (KE) or (KC) , but modify (KI_0) . In the presence of weakening (it is admissible, see Lemma 2) (KI_0) is derivable:

$$\frac{\frac{\Gamma \Rightarrow F}{K(\Gamma), \Gamma \Rightarrow F} (W)}{K(\Gamma) \Rightarrow KF} (KI_1) .$$

So one can derive all sequents of the forms $F \Rightarrow F$ for complex F and $F \Rightarrow KF$. It can be shown by induction on the complexity of the formula F . The latter also requires weakening in the case of $F = KG$:

$$\frac{\frac{F \Rightarrow F}{F \Rightarrow KF} (KI_1), F \neq KG,}{\frac{KG \Rightarrow KG}{KG, G \Rightarrow KG} (W)}{KG \Rightarrow KKG} (KI_1) .$$

Comment. (U) is necessary. There is no way to prove the sequent $K\perp \Rightarrow \perp$ in $\text{IEL}_{\mathcal{G}}^-$ without the rule (U) .

4 Structural Rules Are Admissible

We prove the depth-preserving admissibility of weakening and contraction. Our proof follows [5] except the case of the rule (KI_1) . The corresponding inductive step in the proof of Lemma 6 does not require the inversion of the rule. Instead of it, some kind of contraction is build-in in the rule itself³

We write $\vdash_n \Gamma \Rightarrow F$ for “ $\Gamma \Rightarrow F$ has a $\text{IEL}_{\mathcal{G}}^-$ -proof of depth at most n ”.

³ This method was introduced by Kleene in the construction of his G3 systems, see [6].

Lemma 2 (Weakening). *If $\vdash_n \Gamma \Rightarrow F$ then $\vdash_n \Gamma, G \Rightarrow F$.*

Proof. Induction on n , similar to the proof of Depth-preserving Weakening lemma from [5]. For example, consider the additional modal rule (KI_1) . Consider a derivation

$$\frac{\frac{\mathcal{D}}{\Gamma, K(\Delta), \Delta \Rightarrow F'}}{\Gamma, K(\Delta) \Rightarrow KF'} (KI_1)$$

with $\vdash_n \Gamma, K(\Delta), \Delta \Rightarrow F'$. If $G = KG'$ for some G' then, by the induction hypothesis, $\vdash_n \Gamma, K(\Delta), \Delta, KG', G' \Rightarrow F'$, so the rule (KI_1) can be applied to this premise and we have $\vdash_{n+1} \Gamma, K(\Delta), KG' \Rightarrow KF'$ by (KI_1) . In the remaining case the rule (KI_1) can be applied to $\vdash_n \Gamma, K(\Delta), \Delta, G \Rightarrow F'$, so we have $\vdash_{n+1} \Gamma, K(\Delta), G \Rightarrow KF'$ too. \square

Corollary 3. *The extended K-introduction rule*

$$\frac{\Gamma_1, K(\Delta), \Delta, \Gamma_2 \Rightarrow F}{\Gamma_1, K(\Delta, \Gamma_2) \Rightarrow KF} (KI_{ext})$$

is admissible in IEL_G^- and $\vdash_n \Gamma_1, K(\Delta), \Delta, \Gamma_2 \Rightarrow F$ implies $\vdash_{n+1} \Gamma_1, K(\Delta, \Gamma_2) \Rightarrow KF$.

Proof. Suppose $\vdash_n \Gamma_1, K(\Delta), \Delta, \Gamma_2 \Rightarrow F$ and $\Gamma_1 = \Gamma'_1, K(\Gamma''_1)$ where Γ'_1 does not contain formulas of the form KG . By Lemma 2,

$$\vdash_n \Gamma'_1, K(\Gamma''_1), K(\Delta), K(\Gamma_2), \Gamma''_1, \Delta, \Gamma_2 \Rightarrow F.$$

So,

$$\vdash_{n+1} \Gamma'_1, K(\Gamma''_1, \Delta, \Gamma_2) \Rightarrow KF$$

by (KI_1) . But $\Gamma'_1, K(\Gamma''_1, \Delta, \Gamma_2) = \Gamma_1, K(\Delta, \Gamma_2)$, so $\vdash_{n+1} \Gamma_1, K(\Delta, \Gamma_2) \Rightarrow KF$. \square

Corollary 4. *All axioms of IEL_G^0 are provable in IEL_G^- .*

Proof. It is sufficient to prove sequents $\Gamma, \perp \Rightarrow F$ and $\Gamma, K\perp \Rightarrow F$:

$$\frac{\frac{\Gamma, \perp \Rightarrow \perp}{\Gamma, \perp \Rightarrow K\perp} (KI_{ext})}{\Gamma, \perp \Rightarrow F} (U), \quad \frac{\frac{\Gamma, \perp \Rightarrow \perp}{\Gamma, K\perp \Rightarrow K\perp} (KI_{ext})}{\Gamma, K\perp \Rightarrow F} (U).$$

\square

Lemma 5 (Inversion lemma, cf. [5]). *Left rules are invertible in the following sense:*

- If $\vdash_n \Gamma, A \wedge B \Rightarrow C$ then $\vdash_n \Gamma, A, B \Rightarrow C$.*
- If $\vdash_n \Gamma, A_1 \vee A_2 \Rightarrow C$ then $\vdash_n \Gamma, A_i \Rightarrow C$, $i = 1, 2$.*
- If $\vdash_n \Gamma, A \rightarrow B \Rightarrow C$ then $\vdash_n \Gamma, B \Rightarrow C$.*

Proof. The proof is essentially the same as the proof of Inversion lemma from [5]. The additional cases of modal rules are straightforward. \square

Lemma 6 (Contraction). *If $\vdash_n \Gamma, F, F \Rightarrow G$ then $\vdash_n \Gamma, F \Rightarrow G$.*

Proof. Induction on n . Case $n = 1$. When the first sequent is an axiom, the second one is an axiom too.

Case $n + 1$. When the displayed two occurrences of F in $\Gamma, F, F \Rightarrow G$ are not main for the last rule of the derivation, apply the induction hypothesis to the premises of the rule and contract F there.

Suppose one of the occurrences is main. Only axioms may have atomic main formulas, so we treat atomic F as in case $n = 1$.

When F has one of the forms $A \wedge B$, $A \vee B$ or $A \rightarrow B$, we use the same proof as in [5]. It is based on the items of Inversion lemma formulated in Lemma 5.

Case $F = KA$ is new. The derivation of $\Gamma, F, F \Rightarrow G$ of depth $n + 1$ has the form

$$\frac{\mathcal{D}}{\frac{\Gamma', K(\Delta), \Delta \Rightarrow B}{\Gamma', K(\Delta) \Rightarrow KB} (KI_1)}$$

where $\Gamma, F, F = \Gamma', K(\Delta)$ and $G = KB$; the multiset Δ contains two copies of A . We have

$$\vdash_n \Gamma', K(\Delta), \Delta \Rightarrow B. \quad (1)$$

Let $(\)^-$ means to remove one copy of A from a multiset. We apply the induction hypothesis to (1) and obtain $\vdash_n \Gamma, K(\Delta^-), \Delta^- \Rightarrow B$. Then, by (KI_1) ,

$$\vdash_{n+1} \Gamma, K(\Delta^-) \Rightarrow KB.$$

But $\Gamma, F = \Gamma', K(\Delta^-)$, so $\vdash_{n+1} \Gamma, F \Rightarrow G$. \square

5 Admissible Modal Rules

We have already seen that (KI_0) is admissible in $\text{IEL}_{\mathcal{G}}^-$.

Lemma 7 (Depth-preserving K -elimination). *If $\vdash_n \Gamma, KF \Rightarrow G$ then $\vdash_n \Gamma, F \Rightarrow G$.*

Proof. Induction on n . Case $n = 1$. When the first sequent is an axiom, the second one is an axiom too.

Case $n + 1$. Consider a proof of depth $n + 1$ of a sequent $\Gamma, KF \Rightarrow G$. Let (R) be its last rule. When the displayed occurrence of KF is not main for (R) , apply the induction hypothesis to its premises and then apply (R) to reduced premises. It will give $\vdash_n \Gamma, F \Rightarrow G$.

Suppose the occurrence of KF is main. The derivation has the form

$$\frac{\vdash_n \Gamma', K(\Delta), KF, \Delta, F \Rightarrow G'}{\vdash_{n+1} \Gamma', K(\Delta, F) \Rightarrow KG'} (KI_1).$$

Apply the induction hypothesis to the premise and remove one copy of F . By Lemma 6, $\vdash_n \Gamma', K(\Delta), \Delta, F \Rightarrow G'$. Then apply an instance of (KI_{ext}) with $\Gamma_1 = \Gamma', F$ and empty Γ_2 . By Corollary 3, $\vdash_{n+1} \Gamma', K(\Delta), F \Rightarrow KG'$. \square

Corollary 8 (Depth-preserving K -contraction). *If $\vdash_n \Gamma, KF, F \Rightarrow G$ then $\vdash_n \Gamma, F \Rightarrow G$.*

Proof. Apply (KE) and contraction. Both rules are admissible and preserve the depth (Lemmas 7, 6). \square

6 Cut Is Admissible

Consider an IEL_G^- -derivation with additional cut-rule

$$\frac{\Gamma_1 \Rightarrow F \quad \Gamma_2, F \Rightarrow G}{\Gamma_1, \Gamma_2 \Rightarrow G} (Cut). \quad (2)$$

and some instance of (Cut) in it. The level of the cut is the sum of the depths of its premises. The rank of the cut is the length of F .

Lemma 9. *Suppose the premises of (Cut) are provable in IEL_G^- without (Cut) . Then the conclusion is also provable in IEL_G^- without (Cut) .*

Proof. We define the following well-ordering on pairs of natural numbers: $(k_1, l_1) > (k_2, l_2)$ iff $k_1 > k_2$ or $k_1 = k_2$ and $l_1 > l_2$ simultaneously. By induction on this order we prove that a single cut of rank k and level l can be eliminated.

As in [5], we consider three possibilities:

I. One of the premises is an axiom. In this case the cut-rule can be eliminated. If the left premise of (2) is an axiom,

$$\frac{\Gamma'_1, A \Rightarrow A \quad \Gamma_2, A \Rightarrow G}{\Gamma'_1, A, \Gamma_2 \Rightarrow G} (Cut),$$

then (Cut) is unnecessary. The conclusion can be derived from the right premise by weakening (Lemma 2).

Now suppose that the right premise is an axiom. If the cutformula F is not main for the axiom $\Gamma_2, F \Rightarrow G$ then the conclusion $\Gamma_1, \Gamma_2 \Rightarrow G$ is also an axiom, so (Cut) can be eliminated. If F is main for the right premise then $F = G = A$ where A is atomic, so (2) has the form

$$\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, A \Rightarrow A}{\Gamma_1, \Gamma_2 \Rightarrow A} (Cut).$$

The conclusion can be derived without (*Cut*) from the left premise by weakening (Lemma 2).

II. Both premises are not axioms and the cutformula is not main for the last rule in the derivation of at least one of the premises. In this case one can permute the cut upward and reduce the level of the cut. For example,

$$\frac{\Gamma_1 \Rightarrow F \quad \frac{\Phi, F, K(\Psi), \Psi \Rightarrow B}{\Phi, F, K(\Psi) \Rightarrow KB} (Cut)}{\Gamma_1, \Phi, K(\Psi) \Rightarrow KB} (Cut) \rightsquigarrow \frac{\Gamma_1 \Rightarrow F \quad \Phi, F, K(\Psi), \Psi \Rightarrow B}{\Gamma_1, \Phi, K(\Psi) \Rightarrow KB} (Cut) (KI_{ext}) .$$

The cutformula remains the same, so the cut rule can be eliminated by induction hypothesis (see [5]).

III. The cutformula F is main for the last rules in the derivations of both premises. In this case we reduce the rank of cut and apply the induction hypothesis.

Note that F is not atomic. (The atomic case is considered in I.) If the last rule in the derivation of the left premise is (U) then (*Cut*) can be eliminated:

$$\frac{\frac{\Gamma_1 \Rightarrow K\perp}{\Gamma_1 \Rightarrow F} (U) \quad \Gamma_2, F \Rightarrow G}{\Gamma_1, \Gamma_2 \Rightarrow G} (Cut) \rightsquigarrow \frac{\Gamma_1 \Rightarrow K\perp}{\Gamma_1, \Gamma_2 \Rightarrow K\perp} (U) .$$

Case $F = KA$, the last rule in the derivation of the left premise is (KI_1). Then the right premise is also derived by (KI_1):

$$\frac{\frac{\mathcal{D}}{\Gamma, K(\Delta), \Delta \Rightarrow A} (KI_1) \quad \frac{\mathcal{D}'}{\Gamma', K(\Delta'), A, \Delta', A \Rightarrow B} (KI_1)}{\Gamma, K(\Delta) \Rightarrow KA} (Cut) \rightsquigarrow \frac{\Gamma, K(\Delta), \Delta \Rightarrow A \quad \Gamma', K(\Delta'), A, \Delta', A \Rightarrow B}{\Gamma, K(\Delta), \Gamma', K(\Delta') \Rightarrow KB} (Cut)$$

From $\Gamma', K(\Delta'), A, \Delta', A \Rightarrow B$ by K -contraction (Corollary 8) we obtain $\Gamma', K(\Delta'), \Delta', A \Rightarrow B$ and then reduce the rank:

$$\rightsquigarrow \frac{\frac{\mathcal{D}}{\Gamma, K(\Delta), \Delta \Rightarrow A} \quad \frac{\mathcal{D}''}{\Gamma', K(\Delta'), \Delta', A \Rightarrow B}}{\Gamma, K(\Delta), \Delta, \Gamma', K(\Delta'), \Delta' \Rightarrow B} (Cut) (KI_1) \rightsquigarrow \frac{\Gamma, K(\Delta), \Delta, \Gamma', K(\Delta'), \Delta' \Rightarrow B}{\Gamma, K(\Delta), \Gamma', K(\Delta') \Rightarrow KB} (KI_1)$$

In remaining cases (when F has one of the forms $A \wedge B$, $A \vee B$ or $A \rightarrow B$) we follow [5]. \square

Comment. Our formulation of the rule (KI_1) combines K -introduction with contraction. It is done in order to eliminate the contraction rule and to avoid

the case of contraction in the proof of Lemma 9. But the contraction rule remains admissible and can be added as a ground rule too, so we can simplify the formulation of the K -introduction rule. It results in a “light” cut-free version $\text{IEL}_{\mathcal{G}}$:

Axioms

$$\Gamma, A \Rightarrow A, \quad A \text{ is a variable or } \perp.$$

Rules

$$\frac{\Gamma, \Delta, \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow G} (C)$$

$$\begin{array}{l} \frac{\Gamma, F, G \Rightarrow H}{\Gamma, F \wedge G \Rightarrow H} (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G} (\Rightarrow \wedge) \\ \frac{\Gamma, F \Rightarrow H \quad \Gamma, G \Rightarrow H}{\Gamma, F \vee G \Rightarrow H} (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2} (\Rightarrow \vee)_i \quad (i = 1, 2) \\ \frac{\Gamma \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H} (\rightarrow \Rightarrow) \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} (\Rightarrow \rightarrow) \\ \frac{\Gamma_1, \Gamma_2 \Rightarrow F}{\Gamma_1, K(\Gamma_2) \Rightarrow KF} (KI) \qquad \frac{\Gamma \Rightarrow K\perp}{\Gamma \Rightarrow F} (U) \end{array}$$

Lemma 10. $\text{IEL}_{\mathcal{G}} \vdash \Gamma \Rightarrow F$ iff $\text{IEL}_{\mathcal{G}}^- \vdash \Gamma \Rightarrow F$.

Proof. Part “only if”. The rule (KI) is a particular case of (KI_{ext}) , so all rules of $\text{IEL}_{\mathcal{G}}$ are admissible in $\text{IEL}_{\mathcal{G}}^-$ (Lemmas 6, 2 and Corollary 3).

Part “if”. All missing rules are derivable in $\text{IEL}_{\mathcal{G}}$:

$$\frac{\Gamma, F \rightarrow G \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \rightarrow G, F \rightarrow G \Rightarrow H} (\rightarrow \Rightarrow), \quad \frac{\Gamma, K(\Delta), \Delta \Rightarrow F}{\Gamma, K(\Delta), K(\Delta) \Rightarrow KF} (KI) .$$

$$\frac{\Gamma, F \rightarrow G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H} (C), \quad \frac{\Gamma, K(\Delta) \Rightarrow KF}{\Gamma, K(\Delta) \Rightarrow KF} (C)$$

□

Theorem 11. (Cut) is admissible in $\text{IEL}_{\mathcal{G}}$.

Proof. Lemma 9 implies the similar statement for the calculus $\text{IEL}_{\mathcal{G}}$. Indeed, one can convert $\text{IEL}_{\mathcal{G}}$ -derivations into $\text{IEL}_{\mathcal{G}}^-$ -derivations, eliminate a single cut in $\text{IEL}_{\mathcal{G}}^-$, and then convert the cut-free $\text{IEL}_{\mathcal{G}}^-$ -derivation backward (Lemma 10). The statement implies the theorem. □

Theorem 12. *The following are equivalent:*

1. $\text{IEL}_{\mathcal{G}}^0 \vdash \Gamma \Rightarrow F$.
2. $\text{IEL}_{\mathcal{G}}^- \vdash \Gamma \Rightarrow F$.
3. $\text{IEL}_{\mathcal{G}} \vdash \Gamma \Rightarrow F$.
4. $\text{IEL} \vdash \wedge \Gamma \rightarrow F$.

Proof. 1. \Leftrightarrow 2. All rules of IEL_G^0 are admissible in IEL_G^- (Lemmas 2, 6, 7, 9) and vice versa.

The equivalence of 2. and 3. is proved in Lemma 10, the equivalence of 1. and 4. – see Theorem 1. \square

7 Complexity of IEL

We prove that IEL is PSPACE-complete. The lower bound follows from the same lower bound for the intuitionistic propositional logic. To prove the upper bound we show that polynomial space is sufficient for the proof search. Our proof search technique is based on monotone derivations and is similar to one used by S.C. Kleene in his G3 systems (see [6]).

Definition 13. For a multiset Γ let $\text{set}(\Gamma)$ be the set of all its members. An instance of a rule

$$\frac{\Gamma_1 \Rightarrow F_1 \dots \Gamma_n \Rightarrow F_n}{\Gamma \Rightarrow F}$$

is *monotone* if $\text{set}(\Gamma) \subseteq \bigcap_i \text{set}(\Gamma_i)$. A derivation is called monotone if it uses monotone instances of inference rules only.

Consider the extension IEL'_G of the calculus IEL_G^- by the following rules: the contraction rule (C) and

$$\begin{array}{l} \frac{\Gamma, F \wedge G, F, G \Rightarrow H}{\Gamma, F \wedge G, F \Rightarrow H} (\wedge_1^C \Rightarrow), \quad \frac{\Gamma, F \wedge G, F, G \Rightarrow H}{\Gamma, F \wedge G, G \Rightarrow H} (\wedge_2^C \Rightarrow), \\ \frac{\Gamma, F \wedge G, F, G \Rightarrow H}{\Gamma, F \wedge G \Rightarrow H} (\wedge^C \Rightarrow), \quad \frac{\Gamma, F \vee G, F \Rightarrow H \quad \Gamma, F \vee G, G \Rightarrow H}{\Gamma, F \vee G \Rightarrow H} (\vee^C \Rightarrow), \\ \frac{\Gamma, F \Rightarrow G}{\Gamma, F \Rightarrow F \rightarrow G} (\Rightarrow^W), \quad \frac{\Gamma, F \rightarrow G \Rightarrow F \quad \Gamma, F \rightarrow G, G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H} (\rightarrow^C \Rightarrow), \end{array}$$

$$\frac{\Gamma, K(\Delta_1, \Delta_2), \Delta_1, \Delta_2 \Rightarrow F}{\Gamma, \Delta_1, K(\Delta_1, \Delta_2) \Rightarrow KF} (KI_1^W).$$

In (KI_1^W) we require that the multiset Γ, Δ_1 does not contain formulas of the form KG .

Lemma 14. $\text{IEL}'_G \vdash \Gamma \Rightarrow F$ iff $\text{IEL}_G^- \vdash \Gamma \Rightarrow F$.

Proof. All new rules are some combinations of corresponding ground rules with structural rules. The latter are admissible in IEL_G^- (Lemmas 6, 2). \square

Lemma 15. *Any derivation in $\text{IEL}'_{\mathcal{G}}$ can be converted into a monotone derivation of the same sequent.*

Proof. Consider a derivation which is not monotone. Choose the first (top-down) non-monotone instance (R) of a rule in it. (R) introduces a new formula A in the antecedent of its conclusion which is not present in antecedents of some of its premises. Add a copy of A to the antecedent of the conclusion and to antecedents of all sequents above it. When A has the form KB and is added to the antecedent of the conclusion of some instance of rules (KI_1) or (KI_1^W) above (R), add a copy of B to the antecedent of the premise of this rule and to antecedents of all sequents above it. When B has the form KC , do the same further up the derivation with C , etc. Finally, insert the contraction rule after (R):

$$\frac{\mathcal{D}}{A, \Gamma \Rightarrow F} (R) \quad \rightsquigarrow \quad \frac{\mathcal{D}'}{A, A, \Gamma \Rightarrow F} (R) \quad \frac{}{A, \Gamma \Rightarrow F} (C).$$

The result is also a correct derivation with one non-monotone instance eliminated. Repeat the transformation until the derivation becomes monotone. \square

Lemma 16. *A monotone derivation of a sequent $\Gamma \Rightarrow F$ in $\text{IEL}'_{\mathcal{G}}$ can be converted into a monotone derivation of the sequent $\text{set}(\Gamma) \Rightarrow F$ that contains only sequents of the form $\text{set}(\Gamma') \Rightarrow F'$. The transformation does not increase the depth of the proof.*

Proof. Given a monotone derivation replace all sequents $\Gamma' \Rightarrow F'$ in it with $\text{set}(\Gamma') \Rightarrow F'$. This transformation converts axioms into axioms. We claim that an instance of an inference rule will be converted either into some other instance of a rule of $\text{IEL}'_{\mathcal{G}}$ or some premise of the converted instance will coincide with its conclusion, so the rule can be removed from the resulting proof. The depth of the proof does not increase.

Indeed, instances of $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$ and (U) will be converted into some other instances of the same rule. An instance of (C) will be converted into the trivial rule that can be removed:

$$\frac{\Gamma, \Delta, \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow G} (C) \quad \rightsquigarrow \quad \frac{\text{set}(\Gamma, \Delta) \Rightarrow G}{\text{set}(\Gamma, \Delta) \Rightarrow G} \quad \rightsquigarrow \quad \text{remove.}$$

The remaining cases. Let $k, l, m, n, k', l', m', n' \geq 0$ and $F^k = \underbrace{F, \dots, F}_k$.

All monotone instances of $(\wedge \Rightarrow)$, $(\wedge_1^C \Rightarrow)$, $(\wedge_2^C \Rightarrow)$, $(\wedge^C \Rightarrow)$ have the form

$$\frac{\Gamma, (F \wedge G)^{k+1}, F^{l+1}, G^{m+1} \Rightarrow H}{\Gamma, (F \wedge G)^{k'+1}, F^{l'}, G^{m'} \Rightarrow H}.$$

Contractions in antecedents will give

$$\frac{\Gamma', F \wedge G, F, G \Rightarrow H}{\Gamma', F \wedge G \Rightarrow H} (\wedge^C \Rightarrow), l' = m' = 0,$$

$$\frac{\Gamma', F \wedge G, F, G \Rightarrow H}{\Gamma', F \wedge G, F \Rightarrow H} (\wedge_1^C \Rightarrow), l' > 0, m' = 0,$$

$$\frac{\Gamma', F \wedge G, F, G \Rightarrow H}{\Gamma', F \wedge G, G \Rightarrow H} (\wedge_2^C \Rightarrow), l' = 0, m' > 0,$$

trivial rule (removed), $l', m' > 0.$

All monotone instances of $(\vee \Rightarrow)$, $(\vee^C \Rightarrow)$ have the form

$$\frac{\Gamma, (F \vee G)^{k+1}, F^{l+1}, G^m \Rightarrow H \quad \Gamma, (F \vee G)^{k+1}, F^l, G^{m+1} \Rightarrow H}{\Gamma, (F \vee G)^{k'+1}, F^l, G^m \Rightarrow H}.$$

Contractions in antecedents will give

$$\frac{\Gamma', F \vee G, F \Rightarrow H \quad \Gamma, F \vee G, G \Rightarrow H}{\Gamma', F \vee G \Rightarrow H} (\vee^C \Rightarrow), l = m = 0,$$

trivial rule (removed), $l > 0$ or $m > 0.$

All monotone instances of $(\Rightarrow \rightarrow)$, $(\Rightarrow \rightarrow^W)$ have the form

$$\frac{\Gamma, F^{k+1} \Rightarrow G}{\Gamma, F^{k'} \Rightarrow F \rightarrow G}.$$

Contractions in antecedents will give

$$\frac{\Gamma', F \Rightarrow G}{\Gamma', F \Rightarrow F \rightarrow G} (\Rightarrow \rightarrow^W), k' > 0,$$

$$\frac{\Gamma', F \Rightarrow G}{\Gamma' \Rightarrow F \rightarrow G} (\Rightarrow \rightarrow), k' = 0.$$

All monotone instances of $(\rightarrow \Rightarrow)$, $(\rightarrow^C \Rightarrow)$ have the form

$$\frac{\Gamma, (F \rightarrow G)^{k+1}, G^l \Rightarrow F \quad \Gamma, (F \rightarrow G)^{k'+1}, G^{l+1} \Rightarrow H}{\Gamma, (F \rightarrow G)^{k+1}, G^l \Rightarrow H}.$$

Contractions in antecedents will give

$$\frac{\Gamma', (F \rightarrow G) \Rightarrow F \quad \Gamma', (F \rightarrow G), G \Rightarrow H}{\Gamma', (F \rightarrow G) \Rightarrow H} (\rightarrow^C \Rightarrow), l = 0,$$

trivial rule (removed), $l > 0.$

All monotone instances of (KI_1) , (KI_1^W) have the form

$$\frac{\Gamma, G_1^{k_1+1}, (KG_1)^{l_1+1}, \dots, G_n^{k_n+1}, (KG_n)^{l_n+1} \Rightarrow F}{\Gamma, G_1^{k'_1}, (KG_1)^{l'_1+1}, \dots, G_n^{k'_n}, (KG_n)^{l'_n+1} \Rightarrow KF}$$

Contractions in antecedents will give

$$\frac{\Gamma', G_1, KG_1, \dots, G_n, KG_n \Rightarrow F}{\Gamma', KG_1, \dots, KG_n \Rightarrow KF} (KI_1), \text{ when } k'_1 = \dots = k'_n = 0,$$

an instance of (KI_1^W) , when $k'_i = 0, k'_j > 0$ for some i, j ,

trivial rule (removed), when $k'_1, \dots, k'_n > 0$.

□

Lemma 17. (Subformula Property). *Consider a derivation of a sequent $\Gamma \Rightarrow F$ in IEL_G^- , IEL_G or IEL'_G . Any sequent in it is composed of subformulas of some formulas from the multiset $\Gamma, F, K\perp$.*

Proof. For any rule of these calculi, its premises are composed of subformulas of formulas occurring in its conclusion and, possibly, of $K\perp$. □

Definition 18. A monotone IEL'_G -derivation of a sequent $\text{set}(\Gamma) \Rightarrow F$ is called minimal if it contains only sequents of the form $\text{set}(\Gamma') \Rightarrow F'$ and has the minimal depth.

The size of a sequent $F_1, \dots, F_k \Rightarrow F$ is the sum of the lengths of all formulas F_i and F .

Lemma 19. *Let \mathcal{M}_n be the set of all minimal IEL'_G -derivations of sequents of size n . There exist polynomials p and q such that for any derivation $\mathcal{D} \in \mathcal{M}_n$, its depth is bounded by $p(n)$ and the sizes of all sequents in \mathcal{D} do not exceed $q(n)$.*

Proof. Consider a derivation $\mathcal{D} \in \mathcal{M}_n$ and a path from the root to some leaf in it:

$$\Gamma_0 \Rightarrow F_0, \dots, \Gamma_N \Rightarrow F_N.$$

All sequents in it are distinct from each other, all of them composed of subformulas of the first sequent, \perp and $K\perp$ (Lemma 17), and $\Gamma_i \subseteq \Gamma_{i+1}$ holds for $i < N$.

Divide the path into maximal intervals with the same Γ_i inside. The length of such interval is bounded by the number of possible formulas F_i , which is $O(n)$. The number of intervals is $O(n)$ too, because it does not exceed the maximal length of a strictly monotone sequence $\Delta_0 \subset \Delta_1 \subset \dots \subset \Delta_k$ of subsets of S where S is the set of all subformulas of the first sequent extended by \perp and $K\perp$. So, $|S| = O(n)$ and $N = O(n^2)$.

Any sequent $\Gamma_i \Rightarrow F_i$ consists of at most $|S| + 1$ formulas of length $O(n)$, so its size is $O(n^2)$. □

Corollary 20. *The set of all $\text{IEL}'_{\mathcal{G}}$ -derivable sequents belongs to PSPACE.*

Proof. The result follows from the known game characterization $\text{AP} = \text{PSPACE}$ ([7], see also [9] or [10]). We reproduce here the argument from [11] where Kleene's technique is used in a similar way.

Let p, q be the polynomials from Lemma 19. Consider the following two-person game with players (P) and (V). The initial configuration b_0 is a sequent of the form $\text{set}(\Gamma) \Rightarrow F$ of size n . Player (P) moves the first. He writes down one or two sequents of sizes less than $q(n)$ and his opponent (V) chooses one of them, and so on. The game is over after $p(n)$ moves of (V) or when (V) chooses a sequent that is an axiom of $\text{IEL}'_{\mathcal{G}}$.

Let w_i and b_i for $i > 0$ denote the moves of players (P) and (V) respectively, so $b_0, w_1, b_1, w_2, b_2, \dots$ is a run of the game. The winning conditions for (P) are:

1. For every move of (P) the figure $\frac{w_i}{b_{i-1}}$ is a monotone instance of some inference rule of $\text{IEL}'_{\mathcal{G}}$.
2. All sequents written by (P) have the form $\text{set}(\Delta) \Rightarrow G$.
3. At his last move (V) is forced to choose an axiom of $\text{IEL}'_{\mathcal{G}}$.

The number and the sizes of moves are bounded by polynomials and the winning conditions are polynomial-time decidable, so the set M of initial configurations that admit a winning strategy for (P) belongs to PSPACE (see [7]).

By Lemma 19, a sequent belongs to M iff it has a minimal derivation. But it follows from Lemmas 15, 16, 2, that a sequent $\Gamma \Rightarrow F$ is $\text{IEL}'_{\mathcal{G}}$ -derivable iff $\text{set}(\Gamma) \Rightarrow F$ has a minimal derivation. Thus, the general derivability problem for $\text{IEL}'_{\mathcal{G}}$ belongs to PSPACE too. \square

Theorem 21. *The derivability problems for $\text{IEL}_{\mathcal{G}}^0$, $\text{IEL}_{\mathcal{G}}^-$, $\text{IEL}_{\mathcal{G}}$, $\text{IEL}'_{\mathcal{G}}$ and IEL are PSPACE-complete.*

Proof. The lower bound PSPACE follows from the same lower bound for intuitionistic propositional logic [8]. The upper bound PSPACE for $\text{IEL}'_{\mathcal{G}}$ is established in Corollary 20. It can be extended to other calculi by Theorem 12 and Lemma 14. \square

Acknowledgements. We are grateful to Sergey Artemov for inspiring discussions of intuitionistic epistemic logic. We thank the anonymous referees for their comments.

The research described in this paper was partially supported by Russian Foundation for Basic Research (grant 14-01-00127).

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