Chapter 4
Some Differential Geometry

Abstract The main objective of this chapter is to present a Clifford bundle formalism for the formulation of the differential geometry of a manifold M, equipped with metric fields $\boldsymbol{g} \in \sec T_2^0 M$ and $\mathtt{g} \in \sec T_0^2 M$ for the tangent and cotangent bundles. We start by first recalling the standard formulation and main concepts of the differential geometry of a differential manifold M. We introduce in M the Cartan bundle of differential forms, define the exterior derivative, Lie derivatives, and also briefly review concepts as chains, homology and cohomology groups, de Rham periods, the integration of form fields and Stokes theorem. Next, after introducing the metric fields $\boldsymbol{g}$ and $\mathtt{g}$ in M we introduce the Hodge bundle presenting the Hodge star and the Hodge coderivative operators acting on sections of this bundle. We moreover recall concepts as the pullback and the differential of maps, connections and covariant derivatives, Cartan's structure equations, the exterior covariant differential of $(p+q)$-indexed r-forms, Bianchi identities and the classification of geometries on M when it is equipped with a metric field and a particular connection. The spacetime concept is rigorously defined. We introduce and scrutinized the structure of the Clifford bundle of differential forms ($\mathcal{C}\ell(M,\mathtt{g})$) of M and introduce the fundamental concept of the Dirac operator (associated to a given particular connection defined in M) acting on Clifford fields (sections of $\mathcal{C}\ell(M,\mathtt{g})$). We show that the square of the Dirac operator (associated to a Levi-Civita connection in M) has two fundamental decompositions, one in terms of the derivative and Hodge codifferential operators and other in terms of the so-called Ricci and D'Alembertian operators. A so-called Einstein operator is also introduced in this context. These decompositions of the square of the Dirac operator are crucial for the formulation of important ideas concerning the construction of gravitational theories as discussed in particular in Chaps. 9, 11, 15. The Dirac operator associated to an arbitrary (metrical compatible) connection defined in M and its relation with the Dirac operator associated to the Levi-Civita connection of the pair $(M, \boldsymbol{g})$ is discussed in details and some important formulas are obtained. The chapter also discuss some applications of the formalism, e.g., the formulation of Maxwell equations in the Hodge and Clifford bundles and formulation of Einstein equation in the Clifford bundle using the concept of the Ricci and Einstein operators. A preliminary account of the crucial difference between the concepts of curvature of a connection in M and the concept of bending of M as a hypersurface embedded in a (pseudo)-Euclidean space of high dimension (a property characterized by

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the concept of the *shape tensor*, discussed in details in Chap. 5) is given by analyzing a specific example, namely the one involving the Levi-Civita and the Nunes connections defined in a punctured 2-dimensional sphere. The chapter ends analyzing a statement referred in most physical textbooks as "tetrad postulate" and shows how not properly defining concepts can produce a lot of misunderstanding and invalid statements.

4.1 Differentiable Manifolds

In this section we briefly recall, in order to fix our notations, some results concerning the theory of differentiable manifolds, that we shall need in the following.

Definition 4.1 A topological space is a pair $(M, \mathcal{U})$ where M is a set and $\mathcal{U}$ a collection of subsets of M such that

(i) $\varnothing, M \in \mathcal{U}$.
(ii) $\mathcal{U}$ contains the union of each one of its subsystems.
(iii) $\mathcal{U}$ contains the intersection of each one of its finite subsystems.

We recall some more terminology.[1] Each $U_\alpha \in \mathcal{U}$ (α belongs to an index set which eventually is infinite) is called an *open* set. Of course we can give many different topologies to a given set by choosing different collections of open sets. Given two topologies for M, i.e., the collections of subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ if $\mathcal{U}_1 \subset \mathcal{U}_2$ we say that $\mathcal{U}_1$ is *coarse* than $\mathcal{U}_2$ and $\mathcal{U}_2$ is *finer* than $\mathcal{U}_1$. Given two coverings $\{U_\alpha\}$ and $\{V_\alpha\}$ of M we say that $\{V_\alpha\}$ is a *refinement* of $\{U_\alpha\}$ if for each V_α there exists an U_α such that $V_\alpha \subset U_\alpha$. A *neighborhood* of a point $x \in M$ is any subset of M containing some (at least one) open set $U_\alpha \in \mathcal{U}$. A subset $X \subset M$ is called closed if its complement is open in the topology $(M, \mathcal{U})$. A family $\{U_\alpha\}, U_\alpha \in \mathcal{U}$ is called a *covering* of M if $\cup_\alpha U_\alpha = M$. A topological space $(M, \mathcal{U})$ is said to be Hausdorff (or *separable*) if for any distinct points $x, x' \in M$ there exists open neighborhoods U and U' of these points such that $U \cap U' = \varnothing$. Moreover, a topological space $(M, \mathcal{U})$ is said to be *compact if for every open covering* $\{U_\alpha\}, U_\alpha \in \mathcal{U}$ *of* M there exists a finite subcovering, i.e., there exists a finite subset of indices, say $\alpha = 1, 2, \ldots m$, such that $\cup_{\alpha=1}^{m} U_\alpha = M$. A Hausdorff space is said *paracompact* if there exists a covering $\{V_\alpha\}$ of M such that every point of M is covered by a finite number of the V_α, i.e., we say that every covering has a locally finite refinement.

Definition 4.2 A *smooth differentiable* manifold M is a set such that

(i) M is a Hausdorff topological space.
(ii) M is provided with a family of pairs $(U_\alpha, \varphi_\alpha)$ called charts, where $\{U_\alpha\}$ is a family of open sets covering M, i.e., $\cup_\alpha U_\alpha = M$ and being $\{V_\alpha\}$ a family

[1] In general we are not going to present proofs of the propositions, except for a few cases, which may considered as exercises. If you need further details, consult e.g., [3, 11, 25].

of open sets covering $\mathbb{R}^n$, i.e., $\cup_\alpha V_\alpha = \mathbb{R}^n$, the $\varphi_\alpha : U_\alpha \to V_\alpha$ are homeomorphisms. We say that any point $x \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$. The integer n is said the *dimension* of M, and we write $\dim M = n$.

(iii) Given any two charts (U, φ) and (U', φ') of the family described in (ii) such that $U \cap U' \neq \varnothing$ the mapping $\Phi = \varphi \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi(U \cap U')$ is differentiable of class C^r.

The word *smooth* means that the integer r is large enough for all statements that we shall done to be valid. For the applications we have in mind we will suppose that M is also paracompact. The whole family of charts $\{(U_\alpha, \varphi_\alpha)\}$ is called an *atlas*.

The *coordinate functions* of a chart (U, φ) are the functions $\mathbf{x}^i = a^i \circ \varphi : U \to \mathbb{R}$, $i = 1, 2, \ldots, n$ where $a^i : \mathbb{R}^n \to \mathbb{R}$ are the usual coordinate functions of $\mathbb{R}^n$ (see Fig. 4.1). We write $\mathbf{x}^i(x) = x^i$ and call the set $(x^1, .., x^n)$ (denoted $\{x^i\}$) the *coordinates* of the points $x \in U$ in the chart (U, φ), or briefly, the coordinates.[2] If (U', φ') is another chart of the maximal atlas of M with coordinate functions $\mathbf{x}'^i$ such that $x \in U \cap U'$ we write $\mathbf{x}'^i(x) = x'^i$ and

$$\mathbf{x}'^j(x) = f^j(\mathbf{x}^1(x), \ldots, \mathbf{x}^n(x)), \tag{4.1}$$

and we use the short notation $x'^j = f^j(x^i)$, $i, j = 1, \ldots, n$. Moreover, we often denote the derivatives $\partial f^j / \partial x^i$ by $\partial x'^j / \partial x^i$.

Let (U, φ) be a chart of the maximal *atlas* of M and $\mathtt{h} : M \to M$, $x \mapsto y = \mathtt{h}(x)$ a diffeomorphism such that $x, y \in U \cap \mathtt{h}(U)$. Putting $\mathbf{x}^i(x) = x^i$ and $y^j = \mathbf{x}^j(\mathtt{h}(x))$

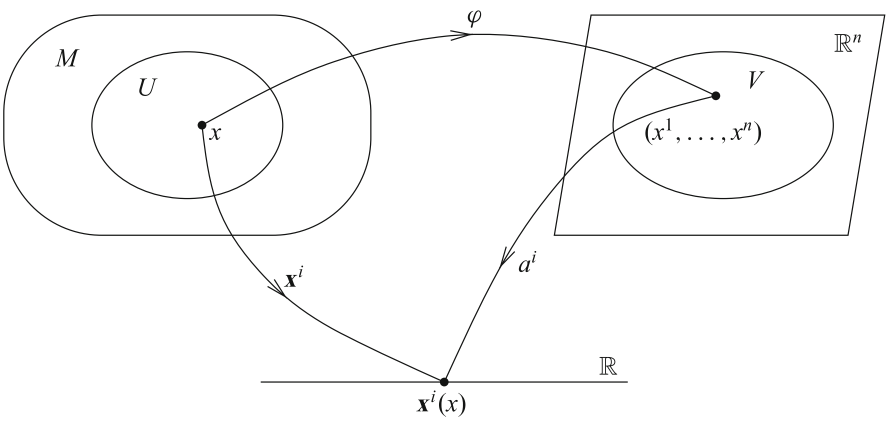

Fig. 4.1 Coordinate chart (U, ϕ), coordinate functions $\mathbf{x} : U \to \mathbb{R}$ and coordinates $\mathbf{x}^i(x) = x^i$

[2]We remark that some authors (see, e.g., [25]) call sometimes the coordinate function $\mathbf{x}^i$ simply by coordinate. Also, some authors (see, e.g., [11]) call sometimes $\{x^i\}$ a *coordinate system* (for $U \subset M$). We eventually also use these terminologies.

we write the mappings $\mathtt{h}^j : (x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$ as

$$y^j = \mathtt{h}^j(x^i), \tag{4.2}$$

and often denote the derivatives $\partial \mathtt{h}^j/\partial x^i$ of the functions $\mathtt{h}^j$ by $\partial y^j/\partial x^i$.

Observe that in the chart $(V, \varkappa)$, $V \subset \mathtt{h}(U)$ with coordinate functions $\{\boldsymbol{y}^j\}$ such that $\boldsymbol{x}^j = \boldsymbol{y}^j \circ \mathtt{h}$, $\boldsymbol{x}^j(x) = x^j = \boldsymbol{y}^j(y) = y^j$ and $\partial y^j/\partial x^i = \delta^i_j$.

4.1.1 Manifold with Boundary

In the definition of a n-dimensional (real) manifold we assumed that each coordinate neighborhoods, $U_\alpha \in M$ is homeomorphic to an open set of $\mathbb{R}^n$. We now give the

Definition 4.3 A n-dimensional (real) manifold M with boundary is a topological space covered by a family of open sets $\{U_\alpha\}$ such that each one is homeomorphic to an open set of $\mathbb{R}^{n+} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$.

Definition 4.4 The boundary of M is the set ∂M of points of M that are mapped to points in $\mathbb{R}^n$ with $x^n = 0$.

Of course, the coordinates of ∂M are given by $(x^1, \ldots, x^{n-1}, 0)$ and thus ∂M is a $(n-1)$-dimensional manifold. of the same class (C^r) as M.

4.1.2 Tangent Vectors

Let $C^r(M, x)$ be the set of all differentiable functions of class C^r (smooth functions) which domain in some neighborhood of $x \in M$. Given a curve in M, $\sigma : \mathbb{R} \supseteq I \to M$, $t \mapsto \sigma(t)$ we can construct a linear function

$$\sigma_*(t) : C^r(M, x) \to \mathbb{R}, \tag{4.3}$$

such that given any $f \in C^r(M, x)$,

$$\sigma_*(t)[f] = \frac{d}{dt}[f \circ \sigma](t). \tag{4.4}$$

Now, $\sigma_*(t)$ is a derivation, i.e., a linear function that satisfy the Leibniz's rule:

$$\sigma_*(t)[fg] = \sigma_*(t)[f]g + f\sigma_*(t)[g], \tag{4.5}$$

for any $f, g \in C^r(M, x)$.

This linear mapping has all the properties that we would like to impose to the *tangent* to σ at $\sigma(t)$ as a generalization of the concept of directional derivative of the calculus on $\mathbb{R}^n$. It can shown that to every linear derivation it is associated a curve (indeed, an infinity of curves) as just described, i.e., curves $\sigma, \gamma : \mathbb{R} \supseteq I \to M$ are equivalent at $x_0 = \sigma(0) = \gamma(0)$ provided $\frac{d}{dt}[f \circ \sigma](t)\big|_{t=0} = \frac{d}{dt}[f \circ \gamma](t)\big|_{t=0}$ for any $f \in C^r(M, x_0)$. This suggests the

Definition 4.5 A tangent to M at the point $x \in M$ is a mapping $\mathbf{v}|_x : C^r(M, x) \to \mathbb{R}$ such that for any $f, g \in C^r(M, x)$, $a, b \in \mathbb{R}$,

$$\begin{array}{ll} \text{(i)} & \mathbf{v}|_x\,[af + bg] = a\mathbf{v}_x[f] + b\,\mathbf{v}|_x\,[g], \\ \text{(ii)} & \mathbf{v}|_x\,[fg] = \mathbf{v}|_x\,[f]g + f\,\mathbf{v}|_x\,[g]. \end{array} . \tag{4.6}$$

As can be easily verified the tangents at x form a linear space over the real field. For that reason a tangent at x is also called a *tangent vector* to M at x.

Definition 4.6 The set of all tangent vectors at x is denoted by T_xM and called the tangent space at x. The dual space of T_xM is denoted by T_x^*M and called the cotangent space at x. Finally $T^r_{sx}M$ is the space of r-contravariant and s-covariant tensors at x.

Definition 4.7 Let $\{\mathbf{x}^i\}$ be the coordinate functions of a chart (U, φ). The partial derivative at x with respect to x^i is the *representative* in the given chart of the tangent vector denoted $\frac{\partial}{\partial x^i}\big|_x \equiv \partial_i|_x$ such that

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|_x f &:= \frac{\partial}{\partial x^i}[f \circ \varphi^{-1}]\bigg|_{\varphi(x)}, \\ &= \frac{\partial \breve{f}}{\partial x^i}(x^i), \end{aligned} \tag{4.7}$$

with

$$f(x) = f \circ \varphi^{-1}(\mathbf{x}^1(x), \ldots, \mathbf{x}^n) = \breve{f}(x^1, \ldots, x^n). \tag{4.8}$$

Remark 4.8 Eventually we should represent the tangent vector $\frac{\partial}{\partial x^i}\big|_x$ by a different symbol, say $\frac{\partial}{\partial \mathbf{x}^i}\big|_x$. This would cause less misunderstandings. However, $\frac{\partial}{\partial x^i}\big|_x$ is almost universal notation and we shall use it. We note moreover that other notations and abuses of notations are widely used, in particular $f \circ \varphi^{-1}$ is many times denoted simply by f and then $\breve{f}(x^i)$ is denoted simply by $f(x^i)$ and also we find $\frac{\partial}{\partial x^i}\big|_x [f] \equiv \frac{\partial \breve{f}}{\partial x^i}(x)$, (or worse) $\frac{\partial}{\partial x^i}\big|_x [f] \equiv \frac{\partial f}{\partial x^i}$. We shall use these (and other) sloppy notations, which are simply to typewrite when no confusion arises, in particular we will use the sloppy notations $\frac{\partial x^j}{\partial x^i}(x)$ or $\frac{\partial x^j}{\partial x^i}$ for $\left.\frac{\partial}{\partial x^i}\right|_x [\mathbf{x}^j]$, i.e.

$$\left.\frac{\partial}{\partial x^i}\right|_x [\mathbf{x}^j] \equiv \frac{\partial x^j}{\partial x^i}(x) \equiv \frac{\partial x^j}{\partial x^i} = \delta^j_i.$$

If $\{\mathbf{x}^i\}$ are the coordinate functions of a chart (U, φ) and $\mathbf{v}|_x \in T_xM$, then we can easily show that

$$\mathbf{v}|_x = \mathbf{v}|_x [\mathbf{x}^i] \left.\frac{\partial}{\partial x^i}\right|_x = v^i \left.\frac{\partial}{\partial x^i}\right|_x , \tag{4.9}$$

with $\mathbf{v}|_x [\mathbf{x}^i] = v^i : U \to \mathbb{R}$.

As a trivial consequence we can verify that the set of tangent vectors $\left\{ \left.\frac{\partial}{\partial x^i}\right|_x , i = 1, 2, \ldots, n \right\}$ is linearly independent and so $\dim T_xM = n$.

Remark 4.9 Have always in mind that $\mathbf{v}|_x = v^i \left.\frac{\partial}{\partial x^i}\right|_x \in T_xU$ and its representative in $T\varphi_{(x)}\mathbb{R}^n$ is the tangent vector $\check{\mathbf{v}}|_{\varphi(x)} =: \check{v}^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(x)}$ such that $v^i \left.\frac{\partial}{\partial x^i}\right|_x f = \check{\mathbf{v}}|_{\varphi(x)} \check{f}$.

Definition 4.10 The tangent vector field to a curve $\sigma : \mathbb{R} \supseteq I \to M$ is denoted by $\sigma_*(t)$ or $\frac{d\sigma}{dt}$.

This means that $\sigma_*(t) = \frac{d\sigma}{dt}(t)$ is the tangent vector to the curve σ at the point $\sigma(t)$. Note that $\sigma_*(t)$ has the expansion

$$\sigma_*(t) = v^i(\sigma(t)) \left.\frac{\partial}{\partial x^i}\right|_{x=\sigma(t)} , \tag{4.10}$$

where, of course,

$$v^i(\sigma(t)) = \sigma_*(t)[\mathbf{x}^i] = \frac{d\mathbf{x}^i \circ \sigma(t)}{dt} = \frac{d\sigma^i(t)}{dt}, \tag{4.11}$$

with $\sigma^i = \mathbf{x}^i \circ \sigma$. We then see, that given any tangent vector $\mathbf{v}|_x \in T_xM$, the solution of the differential equation, Eq. (4.11) permit us to find the components $\sigma^i(t)$ of the curve to which $\mathbf{v}|_x$ is tangent at x. Indeed, the theorem of existence of local solutions of ordinary differential equations warrants the existence of such a curve. More precisely, since the theorem holds only locally, the uniqueness of the solution is warranted only in a neighborhood of the point $x = \sigma(t)$ and in that way, we have in general many curves through x to which $\mathbf{v}|_x$ is tangent to the curve at x.

4.1.3 Tensor Bundles

In what follows we denote respectively by $TM = \bigcup_{x\in M} T_x M$ and $T^*M = \bigcup_{x\in M} T_x^* M$ the tangent and cotangent bundles[3] of M and more generally, we denote by $T_s^r M = \bigcup_{x\in M} T_{sx}^r M$ the bundle of r-contracovariant and s-covariant tensors. A tensor field t of type (r, s) is a section of the $T_s^r M$ bundle and we write[4] $t \in \sec T_s^r M$. Also, $T_0^0 M \equiv M \times \mathbb{R}$ is the module of real functions over M and $T_1^0 M \equiv TM$, $T_s^r M = T^*M$.

4.1.4 Vector Fields and Integral Curves

Let $\sigma : I \to M$ a curve and $\boldsymbol{v} \in \sec TM$ a vector field which is tangent to each one of the points of σ. Then, taking into account Eq. (4.11) we can write that condition as

$$\boldsymbol{v}(\sigma(t)) = \frac{d\sigma(t)}{dt}. \tag{4.12}$$

Definition 4.11 A curve $\sigma : I \to M$ satisfying Eq. (4.12) is called an integral curve of the vector field $\boldsymbol{v}$.

4.1.5 Derivative and Pullback Mappings

Let M and N be two differentiable manifolds, $\dim M = m$, $\dim N = n$ and $\phi : M \to N$ a differentiable mapping of class C^r. ϕ is a diffeomorphism of class C^r if ϕ is a bijection and if ϕ and ϕ^{-1} are of class C^r.

Definition 4.12 The reciprocal image or pullback of a function $f : N \to \mathbb{R}$ is the function $\phi^* f : M \to \mathbb{R}$ given by

$$\phi^* f = f \circ \phi. \tag{4.13}$$

Definition 4.13 Given a mapping $\phi : M \to N$, $\phi(x) = y$ and $\mathbf{v} \in T_x M$, the image of $\mathbf{v}$ under ϕ is the vector $\mathbf{w}$ such that for any $f : N \to \mathbb{R}$

$$\mathbf{w}[f] = \mathbf{v}[f \circ \phi]. \tag{4.14}$$

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[3] In Appendix we list the main concepts concerning fiber bundle theory that we need for the purposes of this book.

[4] See details in Notation A.6 in the Appendix.

The mapping $\phi_* |_x : \sec T_xM \to \sec T_yN$ is called the differential or derivative (or pushforward) mapping of ϕ at x. We write $\boldsymbol{w} = \phi_* |_x \boldsymbol{v}$.

Remark 4.14 When the point $x \in M$ is left unspecified (or is arbitrary), we sometimes write ϕ_* instead of $\phi_* |_x$.

The image a vector field $\boldsymbol{v} \in \sec TM$ at an arbitrary point $x \in M$ is

$$\phi_* \boldsymbol{v}[f](y) = \boldsymbol{v}[f \circ \phi](x). \tag{4.15}$$

Note that if $\phi(x) = y$, and if ϕ is invertible, i.e., $x = \phi^{-1}(y)$ then Eq. (4.15) says that or

$$\phi_* \boldsymbol{v}[f](y) = \boldsymbol{v}[f \circ \phi](x) = \boldsymbol{v}[f \circ \phi](\phi^{-1}(y)). \tag{4.16}$$

This suggests the

Definition 4.15 Let $\phi : M \to N$ be invertible mapping. Let $\boldsymbol{v} \in \sec TM$. The image of $\boldsymbol{v}$ under ϕ is the vector field $\phi_* \boldsymbol{v} \in \sec TN$ such that for any $f : N \to \mathbb{R}$

$$\phi_* \boldsymbol{v}[f] = \boldsymbol{v}[f \circ \phi] = \boldsymbol{v}[f \circ \phi] \circ \phi^{-1}. \tag{4.17}$$

In this case we call

$$\phi_* : \sec TM \to \sec TN, \tag{4.18}$$

the derivative mapping of ϕ.

Remark 4.16 If $\boldsymbol{v} \in \sec TM$ is a differentiable field of class C^r over M and ϕ is a diffeomorphism of class C^{r+1}, then $\phi_* \boldsymbol{v} \in \sec TN$ is a differentiable vector field of class C^r over N. Observe however, that if ϕ is not invertible the image of $\boldsymbol{v}$ under ϕ is not in general a vector field on N [3]. If ϕ is invertible, but not differentiable the image is not differentiable. When the image of a vector field $\boldsymbol{v}$ under some differentiable mapping ϕ is a differentiable vector field, v is said to be projectable. Also, $\boldsymbol{v}$ and $\phi_* \boldsymbol{v}$ are said ϕ-related.

Remark 4.17 We have denoted by $\sigma_*(t)$ the tangent vector to a curve $\sigma : I \to M$. If we look for the definition of that tangent vector and the definition of the derivative mapping we see that the rigorous notation that should be used for that tangent vector is $\sigma_*|_t [\frac{d}{dt}]$, which is really cumbersome, and thus avoided, unless some confusion arises. We will also use sometimes the simplified notation σ_* to refer to the tangent vector field to the curve σ.

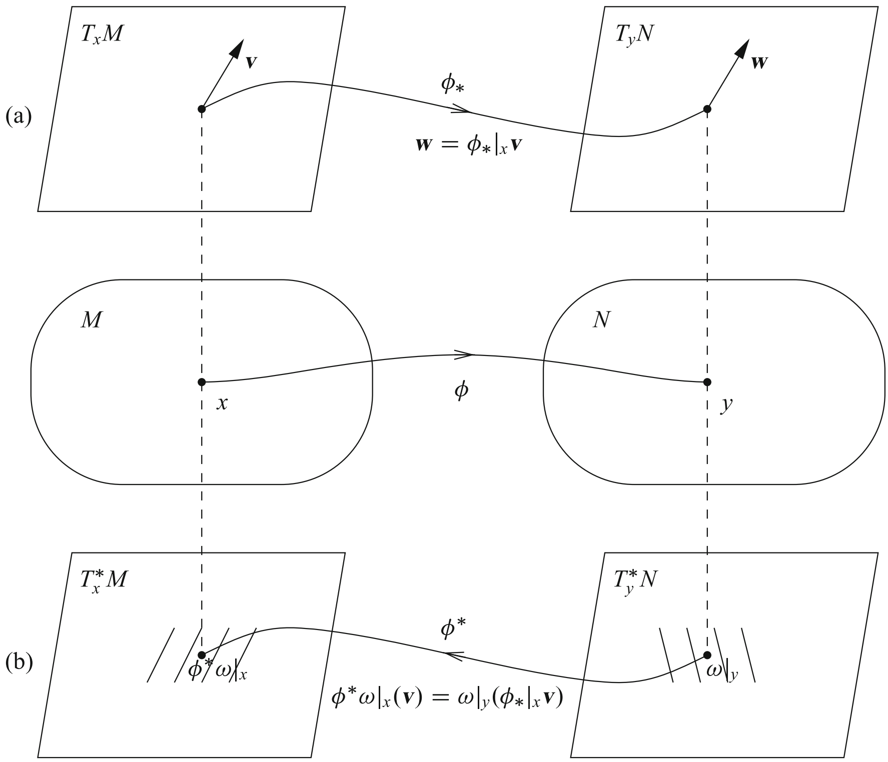

Fig. 4.2 (**a**) The derivative mapping ϕ_*. (**b**) The pullback mapping ϕ^*

Definition 4.18 Given a mapping $\phi : M \to N$, the pullback mapping is the mapping

$$\phi^* : \sec T^*N \to \sec T^*M,$$
$$\phi^\star\omega(\boldsymbol{v}) = \omega(\phi_*\boldsymbol{v}) \circ \phi, \tag{4.19}$$

for any projectable vector field $\boldsymbol{v} \in \sec TM$. Also, $\phi^\star\omega \in \sec T^*N$ is called the pullback of ω (Fig. 4.2).

Remark 4.19 Note that differently from what happens for the image of vector fields, the formula for the reciprocal image of a covector field does not use the inverse mapping ϕ^{-1}. This shows that covector fields are more interesting than vector fields, since $\phi^\star\omega$ is always differentiable if ω and ϕ are differentiable.

Remark 4.20 From now, we assume that $\phi : M \to N$ is a diffeomorphism, *unless explicitly said the contrary* and generalize the concepts of image and reciprocal images defined for vector and covector fields for arbitrary tensor fields.

Definition 4.21 The image of a function $f : M \to \mathbb{R}$ under a diffeomorphism $\phi : M \to N$ is the function $\phi_\star f : N \to \mathbb{R}$ such that

$$\phi_\star f = f \circ \phi^{-1} \tag{4.20}$$

The image of a covector field $\beta \in \sec T^*M$ under a diffeomorphism $\phi : M \to N$ is the covector field $\phi_*\beta$ such that for any projectable vector field $\boldsymbol{v} \in \sec TM$, $\boldsymbol{w} = \phi_*\boldsymbol{v} \in \sec TN$, we have $\phi_*\beta(\boldsymbol{w}) = \beta(\phi_*^{-1}\boldsymbol{w})$, or

$$\phi_*\beta = (\phi^{-1})^*\beta. \tag{4.21}$$

For $\mathbf{S} \in \sec T^r_s M$ we define its image $\phi_*\mathbf{S} \in \sec T^r_s N$ by

$$\phi_*\mathbf{S}(\phi_*\beta_1, \ldots \phi_*\beta_r, \phi_*\boldsymbol{v}_1, \ldots, \phi_*\boldsymbol{v}_s) = \mathbf{S}(\beta_1, \ldots \beta_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_s), \tag{4.22}$$

for any projectable vector fields $\boldsymbol{v}_i \in \sec TM$, $i = 1, 2, \ldots, s$ and covector fields $\beta_j \in \sec T^*M, j = 1, 2, \ldots, r$.

If $\{\mathbf{e}_i\}$ is any basis for TU, $U \subset M$ and $\{\theta^i\}$ is the dual basis for T^*U, then

$$\mathbf{S} = S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \theta^{j_1} \otimes \cdots \otimes \theta^{j_s} \otimes \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_r} \tag{4.23}$$

and

$$\phi_*\mathbf{S} = (S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \circ \phi^{-1})\phi_*\theta^{j_1} \otimes \cdots \otimes \phi_*\theta^{j_s} \otimes \phi_*\boldsymbol{e}_{i_1} \otimes \cdots \otimes \phi_*\boldsymbol{e}_{i_r}. \tag{4.24}$$

Definition 4.22 Let $\mathbf{S} \in \sec T^r_s N$, and $\beta_1, \beta_2, \ldots, \beta_r \in \sec T^*M$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s \in \sec TM$ be projectable vector fields.The reciprocal image (or pullback) of $\mathbf{S}$ is the tensor field $\phi^*\mathbf{S} \in \sec T^r_s M$ such that

$$\phi^*\mathbf{S}(\beta_1, \ldots, \beta_{\mathbf{r}}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\mathbf{s}}) = \mathbf{S}(\phi_*\beta_1, \ldots, \phi_*\beta_{\mathbf{r}}, \phi_*\boldsymbol{v}_1, \ldots, \phi_*\boldsymbol{v}_{\mathbf{s}}), \tag{4.25}$$

and

$$\phi^*\mathbf{S} = (S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \circ \phi)\phi^*\theta^{j_1} \otimes \cdots \otimes \phi^*\theta^{j_s} \otimes \phi_*^{-1}\boldsymbol{e}_{i_1} \otimes \cdots \otimes \phi_*^{-1}\boldsymbol{e}_{i_r}. \tag{4.26}$$

Let $\mathbf{x}^i$ be the coordinate functions of the chart (U, φ) of $U \subset M$ and $\{\partial/\partial x^j\}, \{dx^i\}$, $i, j = 1, \ldots, m$ dual[5] coordinate bases for TU and T^*U, i.e., $dx^i(\partial/\partial x^j) = \delta^i_j$. Let moreover $\mathbf{y}^l$ be the coordinate functions of (V, χ), $V \subset N$ and $\{\partial/\partial y^k\}$,$\{dy^l\}$, $k, l = 1, \ldots, n$ dual bases for TV and T^*V. Let $x \in M, y \in N$ with $y = \phi(x)$ and $\mathbf{x}^i(x) = x^i$, $\mathbf{y}^l(y) = y^l$. If $S^{k_1 \ldots k_r}_{l_1 \ldots l_s}(y^1, \ldots, y^n) \equiv S^{k_1 \ldots k_r}_{l_1 \ldots l_s}(y^j)$ are the

[5]See Remark 4.41 for the reason of the notation dx^i.

components of $\mathbf{S}$ at the point y in the chart (V,χ), then the components $\mathbf{S}' = \phi^*\mathbf{S}$ in the chart (U,φ) at the point x are

$$S'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i) = S^{k_1\ldots k_r}_{l_1\ldots l_s}(y^j(x^i))\frac{\partial y^{l_1}}{\partial x^{j_1}}\cdots\frac{\partial y^{l_s}}{\partial x^{j_s}}\frac{\partial x^{i_1}}{\partial y^{k_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{k_r}},$$
$$S'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i) = (\mathtt{h}^\star S)'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i). \tag{4.27}$$

4.1.6 *Diffeomorphisms, Pushforward and Pullback when M = N*

Definition 4.23 The set of all diffeomorphisms in a differentiable manifold M define a group denoted by $\mathfrak{G}_M$ and called the *manifold mapping group*.

Let $\mathcal{A},\mathcal{B} \subset M$. Let $\mathfrak{G}_M \ni \mathtt{h} : M \to M$ be a diffeomorphism such that $\mathtt{h} : \mathcal{A} \to \mathcal{B}, \mathfrak{e} \mapsto \mathtt{h}\mathfrak{e}$. The diffeomorphism $\mathtt{h}$ induces two important mappings in the tensor bundle $\mathcal{T}M = \bigoplus_{r,s=0} T^r_s M$, the derivative mapping $\mathtt{h}_*$, in this case known as *pushforward*, and the pullback mappings $\mathtt{h}^*$. The definitions of these mappings are the ones given above.

We now recall how to calculate, e.g., the pullback mapping of a tensor field in this case.

Suppose now that $\mathcal{A}$ and $\mathtt{h}(\mathcal{A}) \subset \mathcal{B}$ can be covered by a local charts (U,φ) and (V,χ) of the maximal atlas of M (with $\mathcal{A}, h(\mathcal{A}) \subset U \cap V$) with respective coordinate functions $\{\mathbf{x}^\mu\}$ and $\{\mathbf{y}^\mu\}$ defined by[6]

$$\mathbf{x}^\mu(\mathfrak{e}) = x^\mu, \mathbf{x}^\mu(\mathtt{h}(\mathfrak{e})) = y^\mu, \mathbf{y}^\mu(\mathfrak{e}) = y^\mu. \tag{4.28}$$

We then have the following coordinate transformation

$$y^\mu = \mathbf{x}^\mu(\mathtt{h}(\mathfrak{e})) = \mathtt{h}^\mu(x^\nu). \tag{4.29}$$

Let $\{\partial/\partial x^\mu\}$ and $\{\partial/\partial y^\mu\}$ be a coordinate bases for $T(U\cap V)$ and $\{dx^\mu\}$ and $\{dy^\mu\}$ the corresponding dual basis for $T^*(U\cap V)$.

Then, if the local representation of $\mathbf{S} \in \sec T^r_s M \subset \sec \mathcal{T}M$ in the coordinate chart $\{\mathbf{y}^\mu\}$ at any point of $U\cap V$ is $\check{\mathbf{S}} \in \sec T^r_s\mathbb{R}^n$,

$$\check{\mathbf{S}} = S^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}(y^j)dy^{\nu_1}\otimes\ldots\otimes dy^{\nu_s}\otimes\frac{\partial}{\partial y^{\mu_1}}\otimes\ldots\otimes\frac{\partial}{\partial y^{\mu_r}}, \tag{4.30}$$

[6] Note that in general $\mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) \neq y^\mu$.

we have that the representative of $\mathbf{S}' = \mathtt{h}^*\mathbf{S}$ in $T^r_s\mathbb{R}^n$at any point $\mathfrak{e} \in U \cap V$ is given by

$$\mathtt{h}^*\check{\mathbf{S}} = S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots \rho_s}(x^j) dx^{\rho_1} \otimes \ldots \otimes dx^{\rho_s} \otimes \frac{\partial}{\partial x^{\sigma_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\sigma_r}}$$

$$S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots \rho_s}(x^j) = S^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(y^i(x^j)) \frac{\partial y^{\nu_1}}{\partial x^{\rho_1}} \ldots \frac{\partial y^{\nu_s}}{\partial x^{\rho_s}} \frac{\partial x^{\sigma_1}}{\partial y^{\mu_1}} \ldots \frac{\partial x^{\sigma_r}}{\partial y^{\mu_r}}. \tag{4.31}$$

Remark 4.24 Another important expression for the pullback mapping can be found if we choice charts with the coordinate functions $\{\mathbf{x}^\mu\}$ and $\{\mathbf{y}^\mu\}$ defined by

$$\mathbf{x}^\mu(\mathfrak{e}) = \mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) \tag{4.32}$$

Then writing

$$\mathbf{x}^\mu(\mathfrak{e}) = x^\mu, \; \mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) = y^\mu, \tag{4.33}$$

we have the following coordinate transformation

$$y^\mu = \mathtt{h}^\mu(x^\nu) = x^\mu, \tag{4.34}$$

from where it follows that in this case

$$S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots \rho_s}(x^j) = S^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(y^i(x^j)). \tag{4.35}$$

4.1.7 Lie Derivatives

Definition 4.25 Let M be a differentiable manifold. We say that a mapping $\sigma : M \times \mathbb{R} \to M$ is a one parameter group if

(i) σ is differentiable,
(ii) $\sigma(x, 0) = x$, $\forall x \in M$,
(iii) $\sigma(\sigma(x, s), t) = \sigma(x, s + t)$, $\forall x \in M$, $\forall s, t \in \mathbb{R}$.

These conditions may be expressed in a more convenient way introducing the mappings $\sigma_t : M \to M$ such that

$$\sigma_t(x) = \sigma(x, t). \tag{4.36}$$

For each $t \in \mathbb{R}$, the mapping σ_t is differentiable, since $\sigma_t = \sigma \circ l_t$, where $l_t : M \to M \times \mathbb{R}$ is the differentiable mapping given by $l_t(x) = (x, t)$.

Also, condition (ii) says that $\sigma_0 = \mathrm{id}_M$. Finally, condition (iii) implies, as can be easily verified that

$$\sigma_t \circ \sigma_s = \sigma_{s+t}. \tag{4.37}$$

Observe also that if we take $s = -t$ in Eq. (4.37) we get $\sigma_t \circ \sigma_{-t} = \mathrm{id}_M$. It follows that for each $t \in \mathbb{R}$, the mapping σ_t is a diffeomorphism and $(\sigma_t)^{-1} = \sigma_{-t}$.

Definition 4.26 We say that a family $(\sigma_t, t \in \mathbb{R})$ of mappings $\sigma_t : M \to M$ is a one-parameter group of diffeomorphisms G_1 of M.

Definition 4.27 Given a one-parameter group $\sigma : M \times \mathbb{R} \to M$ for each $x \in M$, we may construct the mapping

$$\begin{aligned} \sigma_x &: \mathbb{R} \to M, \\ \sigma_x(t) &= \sigma(x, t), \end{aligned} \tag{4.38}$$

which in view of condition (ii) is a curve in M, called the orbit (or trajectory) of x generate by the group. Also, the set of all orbits for all points of M are the trajectories of G_1.

It is possible to show, using condition (iii) that for each point $x \in M$ pass one and only one trajectory of the one-parameter group. As a consequence it is uniquely determined by a vector field $\boldsymbol{v} \in \sec TM$ which is constructed by associating to each point $x \in M$ the tangent vector to the orbit of the group in that point, i.e.,

$$\boldsymbol{v}(\sigma_x(t)) = \frac{d}{dt}\sigma_x(t). \tag{4.39}$$

Definition 4.28 The vector field $\boldsymbol{v} \in \sec TM$ determined by Eq. (4.39) is called a Killing vector field relative to the one parameter group of diffeomorphisms $(\sigma_t, t \in \mathbb{R})$.

Remark 4.29 It is important to have in mind that in general, given a vector field $\boldsymbol{v} \in \sec TM$ it does not define a group (even locally) of diffeomorphisms in M. In truth, it will be only possible, in general, to find a local one-parameter pseudo-group that induces $\boldsymbol{v}$. A local one parameter pseudo-group means that σ_t is not defined for all $t \in \mathbb{R}$, but for any $x \in M$, there exists a neighborhood $U(x)$ of x, an interval $I(x) = (-\varepsilon(x), \varepsilon(x)) \subset \mathbb{R}$ and a family $(\sigma_t, t \in I(x))$ of mappings $\sigma_t : M \to M$, such that the properties (i)–(iii) in *Definition 4.27* are valid, when $|t| < \varepsilon(x)$, $|s| < \varepsilon(x)$ and $|t + s| < \varepsilon(x)$.

Definition 4.30 Taking into account the previous remark, the vector field $\boldsymbol{v} \in \sec TM$ is called the infinitesimal generator of the one parameter local pseudo-group $(\sigma_t, t \in I(x))$ and the mapping $\sigma : M \times I(x) \to M$ is called the flow of the vector field ξ.

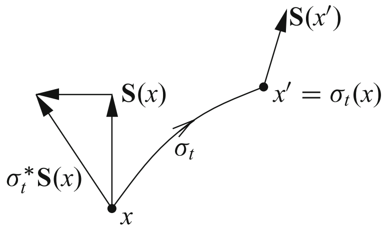

Fig. 4.3 The Lie derivative

Of course, given $\boldsymbol{v} \in \sec TM$ we obtain the one parameter local pseudo-group that induces $\boldsymbol{v}$ by integration of the differential equation Eq. (4.39). From that, we see that the trajectories of the group are *also* the integral lines of the vector field $\boldsymbol{v}$.

Definition 4.31 Let $(\sigma_t, t \in I(x))$ a one-parameter local pseudo group of diffeomorphisms of M that induces the vector field $\boldsymbol{v}$ and let $\mathbf{S} \in \sec T_s^r M$. The Lie derivative of $\mathbf{S}$ in the direction of $\boldsymbol{v}$ is the mapping

$$\pounds_{\boldsymbol{v}} : \sec T_s^r M \to \sec T_s^r M,$$
$$\pounds_{\boldsymbol{v}} \mathbf{S} = \lim_{t \to 0} \frac{\sigma_t^* \mathbf{S} - \mathbf{S}}{t}. \tag{4.40}$$

Remark 4.32 It is possible to define the Lie derivative using the pushforward mapping, the results that follows are the same. In this case we have $\pounds_{\boldsymbol{v}} \mathbf{S} = \lim_{t \to 0} \frac{\mathbf{S} - \sigma_{*t} \mathbf{S}}{t}$ (Fig. 4.3).

4.1.8 Properties of $\pounds_{\boldsymbol{v}}$

(i) $\pounds_{\boldsymbol{v}}$ is a linear mapping and preserve contractions.
(ii) Leibniz's rule. If $S \in \sec T_s^r M$, $S' \in \sec T_{s'}^{r'} M$, we have

$$\pounds_{\boldsymbol{v}} (S \otimes S') = \pounds_v S \otimes S' + S \otimes \pounds_v S'. \tag{4.41}$$

(iii) If $f : M \to \mathbb{R}$, we have

$$\pounds_{\boldsymbol{v}} f = \boldsymbol{v}(f). \tag{4.42}$$

(iv) If $\boldsymbol{v}, \boldsymbol{w} \in \sec TM$, we have

$$\pounds_{\boldsymbol{v}} \boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{v}], \tag{4.43}$$

where $[\boldsymbol{v}, \boldsymbol{w}]$ is the *commutator* of the vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$, such that

$$[\boldsymbol{v}, \boldsymbol{w}](f) = \boldsymbol{v}(\boldsymbol{w}(f)) - \boldsymbol{w}(\boldsymbol{v}(f)). \tag{4.44}$$

(v) If $\omega \in \sec T^*M$, we have

$$\pounds_{\boldsymbol{v}}\omega = \left(\boldsymbol{v}(\omega_k) + \omega_i \boldsymbol{e}_k(v^i) - c_{\cdot jk}^{i\cdot\cdot}\omega_i v^j\right)\theta^k, \tag{4.45}$$

where v^j and ω_i are the components of in the dual basis $\{\boldsymbol{e}_j\}$ and $\{\theta^i\}$ and the $c_{\cdot jk}^{i\cdot\cdot}$ are called the structure coefficients of the frame $\{\boldsymbol{e}_j\}$, and

$$[\boldsymbol{e}_j, \boldsymbol{e}_k] = c_{\cdot jk}^{i\cdot\cdot}\boldsymbol{e}_i. \tag{4.46}$$

Exercise 4.33 Show that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \sec TM$, then they satisfy Jacobi's identity, i.e.,

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0. \tag{4.47}$$

Exercise 4.34 Show that for $\boldsymbol{u}, \boldsymbol{v} \in \sec TM$

$$\pounds_{[\boldsymbol{u},\boldsymbol{v}]} = [\pounds_{\boldsymbol{u}}, \pounds_{\boldsymbol{v}}]. \tag{4.48}$$

4.1.9 Invariance of a Tensor Field

The concept of Lie derivative is intimately associated to the notion of invariance of a tensor field $\mathbf{S} \in \sec T_s^r M$.

Definition 4.35 We say that $\mathbf{S}$ is invariant under a diffeomorphism $h : M \to M$, or h is a symmetry of $\mathbf{S}$, if and only if

$$h^*\mathbf{S}|_x = \mathbf{S}|_x \,. \tag{4.49}$$

We extend naturally this definition for the case in which we have a local one-parameter pseudo-group σ_t of diffeomorphisms. Observe, that in this case, it follows from the definition of Lie derivative, that if $\mathbf{S}$ is invariant under σ_t, then

$$\pounds_{\boldsymbol{v}}\mathbf{S} = 0 \tag{4.50}$$

More properties of Lie derivatives of differential forms that we shall need in future chapters, will be given at the appropriate places.

Remark 4.36 A correct concept for the Lie derivative of spinor fields is as yet a research subject and will not be discussed in this book. A Clifford bundle approach to the subject which we think worth to be known is presented in [22].

4.2 Cartan Bundle, de Rham Periods and Stokes Theorem

In this section, we briefly discuss the processes of differentiation in the Cartan bundle and the concept of de Rham periods and Stokes theorem.

4.2.1 Cartan Bundle

Definition 4.37 The *Cartan bundle* over the cotangent bundle of M is the set

$$\bigwedge T^*M = \bigcup_{x\in M} \bigwedge T^*_x M = \bigcup_{x\in M} \bigoplus_{r=0}^{n} \bigwedge^r T^*_x M, \tag{4.51}$$

where $\bigwedge T^*_x M$, $x \in M$, is the exterior algebra of the vector space $T^*_x M$. The subbundle $\bigwedge^r T^*M \subset \bigwedge T^*M$ given by:

$$\bigwedge^r T^*M = \bigcup_{x\in M} \bigwedge^r T^*_x M \tag{4.52}$$

is called the *r-forms bundle* $(r = 0, \dots, n)$.

Definition 4.38 The *exterior derivative is a mapping*

$$d : \sec \bigwedge T^*M \to \sec \bigwedge T^*M,$$

satisfying:

$$\begin{aligned} &\text{(i)}\ \ d(A + B) = dA + dB; \\ &\text{(ii)}\ \ d(A \wedge B) = dA \wedge B + \hat{A} \wedge dB; \\ &\text{(iii)}\ df(\boldsymbol{v}) = \boldsymbol{v}(f); \\ &\text{(iv)}\ \ d^2 = 0, \end{aligned} \tag{4.53}$$

for every $A, B \in \sec \bigwedge T^*M, f \in \sec \bigwedge^0 T^*M$ and $v \in \sec TM$.

Exercise 4.39 Show that for $A \in \sec \bigwedge^p T^*M$ and $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_p \in \sec TM$,

$$\begin{aligned} dA(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_p) = &\sum_{i=1}^{p} (-1)^i \boldsymbol{v}_i (A(\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \check{\boldsymbol{v}}_i, \dots, \boldsymbol{v}_p)) \\ &+ \sum_{0\le i<j\le p} (-1)^{i+j} A([\boldsymbol{v}_i, \boldsymbol{v}_j]\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \check{\boldsymbol{v}}_i, \dots, \check{\boldsymbol{v}}_j, \dots \boldsymbol{v}_p). \end{aligned} \tag{4.54}$$

Remark 4.40 Note that due to property (ii) the exterior derivative does not satisfy the Leibniz's rule, and as such it is not a derivation. In fact the technical term is antiderivation (see [3]).

Remark 4.41 Let $\mathbf{x}^i$ be coordinate functions of a chart (U, φ) of an atlas of M. A coordinate basis for TU in that chart is denoted $\{\partial/\partial x^i\}$. This means that for each $x \in U$, $\partial/\partial x^i\big|_x$ is a basis of T_xU. As we already know, the dual (coordinate) basis for T^*U is denoted[7] $\{dx^j\}$. This means that $dx^j\big|_x$ is a basis for T_x^*U. We have (indeed) that

$$dx^j(\partial/\partial x^i)\big|_x = \partial x^j/\partial x^i\big|_x = \delta_i^j. \tag{4.55}$$

4.2.2 The Interior Product of Forms and Vector Fields

Another important antiderivation is the so called interior product (sometimes also called inner product).

Definition 4.42 Given a vector field $\boldsymbol{v} \in \sec TM$ we define the interior product extensor of $\boldsymbol{v}$ with $\alpha \in \sec \bigwedge^p T^*M$ as the mapping

$$\begin{aligned} \sec T^*M \times \sec \bigwedge\nolimits^p T^*M &\to \sec \bigwedge\nolimits^{p-1} T^*M, \\ (\boldsymbol{v}, \alpha) &\mapsto \mathbf{i}_{\boldsymbol{v}}\alpha, \end{aligned} \tag{4.56}$$

where $\mathbf{i}_{\boldsymbol{v}} : \sec \bigwedge^p T^*M \to \sec \bigwedge^{p-1} T^*M$ satisfy

(i) For any $\alpha,\beta \in \sec \bigwedge T^*M$ and $a, b \in \mathbb{R}$,

$$\mathbf{i}_{\boldsymbol{v}}(a\alpha + b\beta) = a\mathbf{i}_{\boldsymbol{v}}\alpha + b\mathbf{i}_{\boldsymbol{v}}\beta. \tag{4.57}$$

(ii) if $f \in \sec \bigwedge^0 T^*M$ is a smooth function, then $\mathbf{i}_{\boldsymbol{v}} f = 0$,

(iii) If $\{\boldsymbol{e}_i\}$ is an arbitrary basis for TU, $U \subset M$, and $\{\theta^i\}$ its dual basis,

$$\mathbf{i}_{\boldsymbol{e}_k}\theta^{j_1} \wedge \ldots . \wedge \theta^{j_p} = \sum_{r=1}^{p} (-1)^{r+1} \delta_k^{j_r} \theta^{j_1} \wedge \ldots . \check{\theta}^{j_k} \wedge \ldots \wedge \theta^{j_p}, \tag{4.58}$$

where as usual $\check{\theta}^{j_k}$ means that the term θ^{j_k} is missing in the expression.

[7] Eventually a more rigorously notation for a basis of T^*U should be $\{d\mathbf{x}^i\}$.

From Eq. (4.58) it follows that for $A_p \in \sec \bigwedge^p T^*M$ and $B_q \in \sec \bigwedge^q T^*M$ we have

$$\mathbf{i}_{\boldsymbol{v}}(A_p \wedge B_q) = \mathbf{i}_{\boldsymbol{v}} A_p \wedge B_q + (-1)^{pq} A_p \wedge \mathbf{i}_{\boldsymbol{v}} B_q \tag{4.59}$$

and we usually say that $\mathbf{i}_{\boldsymbol{v}}$ is an antiderivation.

Exercise 4.43 If $\{\mathbf{x}^i\}$ are coordinate functions of a local chart of M, and $\boldsymbol{v} = v^i \frac{\partial}{\partial x^i}$, show that $\mathbf{i}_{\boldsymbol{v}} dx^i = v^i$.

Exercise 4.44 Properties of $\mathbf{i}_v$. Show that

$$\mathbf{i}_{\boldsymbol{v}}^2 = 0, \tag{4.60}$$

$$d\mathbf{i}_{\boldsymbol{v}} + \mathbf{i}_{\boldsymbol{v}} d = \pounds_{\boldsymbol{v}}, \tag{4.61}$$

$$[\pounds_{\boldsymbol{v}}, \mathbf{i}_{\boldsymbol{w}}] = \pounds_{\boldsymbol{v}} \mathbf{i}_{\boldsymbol{w}} - \mathbf{i}_{\boldsymbol{w}} \pounds_{\boldsymbol{v}} = \mathbf{i}[\boldsymbol{v}, \boldsymbol{w}], \tag{4.62}$$

$$\pounds_{\boldsymbol{v}} d = d\pounds_{\boldsymbol{v}}. \tag{4.63}$$

Equation (4.61) is sometimes called Cartan's magical formula. It is really, a very important formula in the formulation of conservation laws, as we shall see in Chap. 9.

4.2.3 *Extensor Fields*

Let $\{\theta^i\}$ be an arbitrary basis for $\sec T^*U$, $U \subset M$. Let $\kappa = \kappa_i \theta^i \in \sec \bigwedge^1 T^*M$ and $\omega = \frac{1}{r!}\omega_{i_1 \ldots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \in \sec \bigwedge^r T^*M$, $r = 1, 2, \ldots, n$.

Definition 4.45 A $(1,1)$-extensor field $t : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$ and its extension $\underline{t} : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$ are the linear operators given by

$$\begin{aligned} t(\kappa) &= t(\kappa_i \theta^i) = \kappa_i t(\theta^i), \\ \underline{t}(\omega) &= \underline{t}(\frac{1}{r!}\omega_{i_1 \ldots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}) = \frac{1}{r!}\omega_{i_1 \ldots i_r} t(\theta^{i_1}) \wedge \cdots \wedge t(\theta^{i_r}) \end{aligned} \tag{4.64}$$

for all κ and ω, $r = 1, 2, \ldots n$. Moreover, if $f \in \sec \bigwedge^0 T^*M$, we put $\underline{t}(f) = f$.

4.2.4 Exact and Closed Forms and Cohomology Groups

Definition 4.46 A r-form $G_r \in \sec \bigwedge^r T^*M$ is called *closed* (or a *cocycle*) if and only if $dG_r = 0$. A r-form $F_r \in \sec \bigwedge^r T^*M$ is called *exact* (or a *coboundary*) if and only if $F_r = dA_{r-1}$, with $A_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

Definition 4.47 The space of closed r-forms is called the r-cocycle group and denoted by $Z^r(M)$. The space of exact r-forms is called the r-coboundary group and denoted by $B^r(M)$.

We recall that the sets $Z^r(M)$ and $B^r(M)$ have the structures of vector spaces over the real field $\mathbb{R}$. Since according to Eq. (4.53iv) $d^2 = 0$ it follows that $B^r(M) \subset Z^r(M)$. Then if $F_r = dA_{r-1} \Rightarrow dF_r = 0$, but in general $dG_r = 0 \nRightarrow G_r = dC_{r-1}$, with $C_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

Definition 4.48 The space $H^r(M) = Z^r(M)/B^r(M)$ is the r-de Rham cohomology group of the manifold M. Obviously, the elements of $H^r(M)$ are equivalent classes of closed forms, i.e., if $F_r, F'_r \in \sec H^r(M)$, then $F_r - F'_r = dW_{r-1}$, $W_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

As a vector space over the real field, $H^r(M)$ is called the r-de Rham vector space group of the manifold M.

Definition 4.49 The dimension of the r-homology[8] (respectively cohomology) group is called the Betti number b_r (respectively b^r) of M.

A very important result is the

Proposition 4.50 (Poincaré Lemma) *If $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ then any closed r-form $F_r \in \sec \bigwedge^r T^*U$ $(r \geq 1)$ which is differentiable on U is also exact.*

Proof For a proof see , e.g., [25].■

Note that if $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ then U is contractible to a point $p \in M$. Also, from Poincaré's lemma it follows that the Betti numbers of U, $b^r = 0, r = 1, 2, \ldots, r$.

Any closed form is exact at least locally and the non triviality of de Rham cohomology group is an obstruction to the global exactness of closed forms.

Remark 4.51 It is very important to observe that Poincaré's lemma does not hold if $F_r \in \sec \bigwedge^r T^*M$ is not differentiable at certain points of $\mathbb{R}^n$, since in that case the manifold where F_r is differentiable is not homeomorphic to $\mathbb{R}^n$. The 'classical' example according to Spivack [43] is $A \in \sec \bigwedge^1 T^*\mathbb{R}^2$,

$$A = \frac{-ydx + xdy}{x^2 + y^2} = d(\arctan \frac{y}{x}). \tag{4.65}$$

[8] See Definition 4.65.

Observe that A is differentiable on $\mathbb{R}^2 - \{0\}$, but despite the third member of Eq. (4.65) A is not exact on $\mathbb{R}^2$, because $\arctan \frac{y}{x}$ is not a differentiable function on $\mathbb{R}^2$.

4.3 Integration of Forms

In what follows we briefly recall some concepts related to the integration of forms on *orientable* manifolds. First we introduce the definition of the integral of a n-form in an n-dimensional manifold M and next the integration of a r-form $A_r \in \sec T^*M$ which is realized over a r-chain.

4.3.1 Orientation

Let M be an n-dimensional connected manifold and $U_\alpha, U_\beta \subset M$, $U_\alpha \cap U_\beta \neq \varnothing$. Let $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be coordinate charts of the maximal atlas of M with coordinate functions $\{\mathbf{x}_\alpha^i\}$ and $\{\mathbf{x}_\beta^j\}$, $i, j = 1, 2 \ldots, n$. Let $e \in U_\alpha \cap U_\beta$. The natural ordered bases $\{\frac{\partial}{\partial x_\alpha^i}\Big|_e\}$ and $\{\frac{\partial}{\partial x_\beta^i}\Big|_e\}$ of T_eM are said to have the same orientation if $J = \det\left[\frac{\partial x_\alpha^i}{\partial x_\beta^i}\Big|_e\right] > 0$. If $J < 0$ the bases are said to have opposite orientations. An orientation at $e \in U_\alpha \cap U_\beta$ is a choice of an ordered basis (not necessarily a coordinate one) for T_eM.

Now, suppose that the basis $\{\frac{\partial}{\partial x_\alpha^i}\Big|_e\}$ is *declared* positive (a right-handed basis). A orientation in T_eM induces naturally an orientation in T_e^*M as follows. Let $\{\theta^i\big|_e\}$ be an ordered basis of T_e^*M. Let $\tau_e = \theta^1\big|_e \wedge \cdots \wedge \theta^n\big|_e$. Then,

$$\boldsymbol{\tau}_e(\frac{\partial}{\partial x_\alpha^1}\Big|_e, \ldots, \frac{\partial}{\partial x_\alpha^n}\Big|_e)$$
$$= \frac{1}{n!}\det\begin{bmatrix} \theta^1\big|_e\left(\frac{\partial}{\partial x_\alpha^1}\Big|_e\right) & \theta^1\big|_e\left(\frac{\partial}{\partial x_\alpha^2}\Big|_e\right) & \ldots & \theta^1\big|_e\left(\frac{\partial}{\partial x_\alpha^n}\Big|_e\right) \\ \theta^2\big|_e\left(\frac{\partial}{\partial x_\alpha^1}\Big|_e\right) & \theta^2\big|_e\left(\frac{\partial}{\partial x_\alpha^2}\Big|_e\right) & \ldots & \theta^2\big|_e\left(\frac{\partial}{\partial x_\alpha^n}\Big|_e\right) \\ \ldots & \ldots & \ldots & \ldots \\ \theta^n\big|_e\left(\frac{\partial}{\partial x_\alpha^1}\Big|_e\right) & \theta^n\big|_e\left(\frac{\partial}{\partial x_\alpha^2}\Big|_e\right) & \ldots & \theta^n\big|_e\left(\frac{\partial}{\partial x_\alpha^n}\Big|_e\right) \end{bmatrix}. \tag{4.66}$$

If $\boldsymbol{\tau}_e(\frac{\partial}{\partial x_\alpha^1}\Big|_e, \ldots, \frac{\partial}{\partial x_\alpha^n}\Big|_e) > 0$ we say that the ordered basis $\{\theta^i\big|_e\}$ of T_e^*M is positive. If $\boldsymbol{\tau}_e(\frac{\partial}{\partial x_\alpha^1}\Big|_e, \ldots, \frac{\partial}{\partial x_\alpha^n}\Big|_e) < 0$ we say that the ordered basis $\{\theta^i\big|_e\}$ of T_e^*M is negative.

Suppose that for all $e \in U_\alpha \cap U_\beta$ we have $J = \det \left[\frac{\partial x^i_\alpha}{\partial x^i_\beta} \right] > 0$. In this case we define that on $U_\alpha \cap U_\beta$ that the bases $\{\frac{\partial}{\partial x^i_\alpha}\}$ and $\{\frac{\partial}{\partial x^i_\beta}\}$ of TU_α and TU_β have the same orientation. If $J = \det \left[\frac{\partial x^i_\alpha}{\partial x^i_\beta} \right] < 0$ we say that the bases have opposite orientation on $U_\alpha \cap U_\beta$.

Definition 4.52 Let $\{U_\alpha\}$ be a covering for M, an n-dimensional connected manifold. We say that M is orientable if for any two overlapping charts U_α and U_β there exist coordinate functions $\{\mathbf{x}^i_\alpha\}$, $\{\mathbf{x}^j_\beta\}$ for U_α and U_β such that $\det \left[\frac{\partial x^i_\alpha}{\partial x^i_\beta} \right] > 0$.

Remark 4.53 From what has been said above it is clear that if M is orientable, there exists an n-form $\tau \in \sec \bigwedge^n T^*M$ called a volume element which is never null.

Thus, we have the alternative (equivalent) definition of an orientable manifold.

Definition 4.54 A connected n-dimensional manifold M is orientable if there exists a non null global section of $\bigwedge^n T^*M$ and $\tau, \tau' \in \sec \bigwedge^n T^*M$ define the same orientation (respectively opposite orientation) if there exists a global function $\lambda \in \sec \bigwedge^0 T^*M$ such that $\lambda > 0$ (respectively $\lambda < 0$) such that $\tau' = \lambda \tau$.

Remark 4.55 Of course, a given orientable manifold M admits two inequivalent orientations, one is declared right-handed, and the other left-handed. It is quite obvious that there are manifolds which are not orientable, the classical example is the Möbius strip, which may be found in almost all books in differential geometry, as, e.g., [3, 25].

4.3.2 *Integration of a n-Form*

In what follows we suppose that M is orientable.[9] Let (U, φ) be a chart of the maximal atlas of M and $\{x^i\}$ the coordinate functions of the chart. Let $h \in \sec \bigwedge^0 T^*M$ be a Lebesgue *integrable* function and[10] $\tau = dx^1 \wedge \cdots \wedge dx^n \in \sec \bigwedge^n T^*M$.

Definition 4.56 The integral of $h\tau \in \sec \bigwedge^n T^*M$ in $\mathfrak{A} \subset U \subset M$ is

$$\int_{\mathfrak{A}} h\tau := \int_{\varphi(\mathfrak{A})} h \circ \varphi^{-1}(x^i) dx^1 \cdots dx^n \tag{4.67}$$

[9] In Chap. 6 we will learn that a spacetime manifold admitting spinor fields must necessarily be orientable.

[10] Of course, we should write $\tau = \varphi^*_\alpha(dx^1 \wedge \cdots \wedge dx^n)$ since dx^i are 1-forms in $T_{\varphi_{\alpha(U)}}\mathbb{R}^n$. So, ours is a sloppy (universally used) notation.

where in the second member of Eq. (4.67) is the ordinary multiple integral of a Lebesgue integrable function $h = h \circ \varphi^{-1}(x^i)$ of n variables.

Let be $\mathfrak{A} \subset U \cap V$ and (V, ψ) another chart of the maximal atlas of M with coordinate functions $\{\mathbf{x}'^j\}$ and suppose that $J = \det\left[\frac{\partial x^i}{\partial x'^j}\right] > 0$ on $U \cap V$. Then we can write that

$$h\tau = h \circ \psi^{-1}(x'^i) J dx'^1 \wedge \cdots \wedge dx'^n = h \circ \psi^{-1}(x'^i)\, |J|\, dx'^1 \wedge \cdots \wedge dx'^n \tag{4.68}$$

and

$$\int_{\mathfrak{A}} h\tau = \int_{\psi(\mathfrak{A})} h \circ \psi^{-1}(x'^i))\, |J|\, dx'^1 \cdots dx'^n, \tag{4.69}$$

which corresponds to the classical formula for a change of variables in a multiple integral.

Now, if M is *paracompact*, i.e., there is an open covering $\{U_\alpha\}$ of M such that each $e \in M$ is covered by a finite number of the U_α a partition of the unity associated to the covering $\{U_\alpha\}$ is a family of differentiable functions $p_\alpha : M \to \mathbb{R}$ such that: (a) $0 \leq p_\alpha \leq 1$; (b) $p_\alpha(e) = 0$ for all $e \notin U_\alpha$; (c) If k is the finite number of U_α covering e then for any $e \in M$ we have that $\sum_{\alpha=1}^{k} p_\alpha(e) = 1$. It is obvious that we can write

$$h(e) = \sum_{\alpha=1}^{k} p_\alpha(e) h(e) = \sum_{\alpha=1}^{k} h_\alpha(e). \tag{4.70}$$

We then have the

Definition 4.57 The integral of $h\tau \in \sec \bigwedge^n T^*M$ in M is

$$\int_M h\tau := \sum_\alpha \int_{U_\alpha} h_\alpha \tau = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} h_\alpha \circ \varphi_\alpha^{-1}(x^i) dx^1 \cdots dx^n \tag{4.71}$$

We may verify that the definition is independent of the choice of atlas used for M (and thus of the partition of the unity used) if the new atlas has the same orientation as the previous one.

4.3.3 *Chains and Homology Groups*

Orientation of Subspaces

Let $(u^1, \ldots, u^n)$ be a right handed coordinate system for $\mathbb{R}^n$. For any $\mathbb{R}^r \subset \mathbb{R}^n$ $(u^1, \ldots, u^r)$ is a naturally right handed coordinate system for $\mathbb{R}^r$, which is supposed to be coherently oriented with $\mathbb{R}^n$.

Definition 4.58 A r-rectangle P^r in $\mathbb{R}^r \subset \mathbb{R}^n$ is a naturally positive oriented subset of $\mathbb{R}^r$ such that $a^i \leq u^i \leq b^i$, $i = 1, \ldots, r$. The boundary of the rectangle P^r is the set $\partial P^r$of $2r$ rectangles $P^{r-1} \in \mathbb{R}^{r-1}$ defined by the faces $u^i = a^i$ and $u^i = b^i$ of P^r. We suppose that the boundary ∂P^r is coherently oriented with P^r. That means that any face has the orientation $(u^1, \ldots \check{u}^i, \ldots u^r)$ if $u^i = a^i, i$ is even and $u^i = b^i$, i is odd and the opposite orientation if $u^i = a^i, i$ is odd and $u^i = b^i$, i is even.

Next we introduce the concept of elementary chain in M.

Definition 4.59 An elementary r-chain or c_r in a n-dimensional connected manifold M is a pair (P^r, f), with $f : \mathbb{R}^r \supset U \to M$ a differentiable mapping. The image of the P^r rectangle is denoted by suppc_r. When f is a diffeomorphism suppc_r is called an elementary r-domain of integration.

Definition 4.60 The boundary of an elementary r-chain is the image of ∂P^r.

Definition 4.61 A r-chain on M is a formal linear combination of elementary r-chains c_{rj} with real coefficients $C_r = \sum_j a_j c_{rj}$. The space of r-chains in M forms a vector space over the real field. It is denoted by $C_r(M)$ and called the r-chain group.

Remark 4.62 We are in general interested in formal locally finite linear combinations with $a_j = \pm 1$, in which case C_r is said a domain of integration on M. More generally, in algebraic topology the coefficients a_j are in many applications elements of a finite group. In that case $C_r(M)$ is a group, but it is not a vector space. That is the reason why $C_r(M)$ has been called the r-chain group.

Definition 4.63 The boundary operator ∂ is a mapping

$$\partial : C_r(M) \to C_{r-1}(M) \tag{4.72}$$

such that for any r-chain $C_r = \sum_j a_j c_{rj}$

$$\partial C_r = \sum_j a_j \partial c_{rj}, \tag{4.73}$$

where ∂c_{rj} is the image under f of an elementary P_j^r-rectangle.

The boundary operator ∂ has the fundamental property

$$\partial^2 = 0, \tag{4.74}$$

a formula that will be proved below.

Definition 4.64 A finite r-chain C_r is said to be a cycle if and only if $\partial C_r = 0$. The space of cycles is denoted $Z_r(M)$. Also, a finite r-chain C_r is said to be a boundary if and only if $C_r = \partial C_{r-1}$and the space of boundaries is denoted by $B_r(M)$.

Since $\partial^2 = 0$ it follows that $B_r(M) \subset Z_r(M)$. We then have

Definition 4.65 The quotient set $H_r(M) = Z_r(M)/B_r(M)$ is called the r-homology group of M.

Remark 4.66 Recall that the dimension of the r-homology group is called the Betti number b_r of M.

In what follows we use the standard convention that $Z^0(M)$ is the space of differentiable functions h such that $dh = 0$. Also, we agree that $B^0(M) = \varnothing$. Finally, we agree that $Z_0(M) = C_0(M)$ and that $B_0(M) = \varnothing$.

4.3.4 Integration of a r-Form

Definition 4.67 The integration of $F_r \in \sec \bigwedge^r T^*M$ over suppC_r is

$$\int_{C_r} F_r = \sum\nolimits_j a_j \int_{c_{rj}} F_r = \sum\nolimits_j a_j \int_{P_j^r} f^* F_r, \tag{4.75}$$

where f^* is the pullback mapping induced by f.

When F_r is continuous and C_r is finite the integral is always defined. The integral is also always defined if F_r has compact support and C_r is locally finite. In what follows we suppose that this is the case. Definition 4.67 shows very clearly that it is bilinear in F_r and C_r and suggests the definition of a *non degenerated* inner product $\langle\rangle : C_r(M) \times \sec \bigwedge^r M \to \mathbb{R}$ given

$$\langle C_r, F_r \rangle = \int_{C_r} F_r. \tag{4.76}$$

With the aid of that definition we can say that two chains C_r and $C_r{}'$ are equal if and only if $\langle C_r, F_r \rangle = \langle C_r', F_r \rangle$. This observation is important because the decomposition of a chain into elementary chains is not unique.

Recall that given a manifold, say M with boundary, its boundary is denoted by ∂M. The manifold M is called triangulable if it can be decomposed as a union of adjacent n-domains of integration with orientation compatible with the *orientation* of M.

4.3.5 Stokes Theorem

Theorem 4.68 (Stokes) *For any $F_r \in \sec \bigwedge^r T^*M$ and $C_r \in C_r(M)$ it holds*

$$\int_{C_r} dF_r = \int_{\partial C_r} F_r. \tag{4.77}$$

Proof For a proof, see, e.g., [25].■

Stokes formula can be written in the suggestive way

$$\langle C_r, dF_r \rangle = \langle \partial C_r, F_r \rangle \tag{4.78a}$$

Proposition 4.69 *The boundary operator ∂ has the fundamental property*

$$\partial^2 = 0. \tag{4.79}$$

Proof It follows directly from the fact that $d^2 = 0$ and Stokes theorem. Indeed,

$$\langle \partial^2 C_r, F_r \rangle = \langle \partial C_r, dF_r \rangle = \langle C_r, d^2 F_r \rangle = 0,$$

which proves the proposition.■

4.3.6 Integration of Closed Forms and de Rham Periods

We now investigate integration in the case when $G_r \in \sec \bigwedge^r T^*M$ is closed. The inner product introduced by Eq. (4.76) permit us to define a mapping from the space of closed (cocycles) forms $Z^r(M)$ into the (dual) space of cycles $Z_r(M)$, by

$$\mathbf{I} : Z^r(M) \to Z_r(M), \tag{4.80}$$

such that for any $G_r \in \sec \bigwedge^r T^*M$ and $z_r \in Z_r(M)$,

$$\mathbf{I}(G_r)(z_r) = \langle z_r, G_r \rangle. \tag{4.81}$$

Note now that

$$\langle z_r + \partial c, G_r \rangle = \langle z_r, G_r \rangle + \langle \partial c, G_r \rangle = \langle z_r, G_r \rangle + \langle c, dG_r \rangle = \langle z_r, G_r \rangle, \tag{4.82}$$

because G_r is closed. This implies that $\mathbf{I}(G_r)$ can be considered as a linear function on the equivalent class of z_r modulus $B_r(M)$, i.e., it defines a mapping

$$\mathbf{I} : Z^r(M) \to H_r(M). \tag{4.83}$$

Also, $\mathbf{I}(G_r + dG_{r-1}) \equiv \mathbf{I}(G_r)$, so it is obvious that $\mathbf{I}$ really defines a linear transformation

$$\mathbf{I} : H^r(M) \to H_r(M). \tag{4.84}$$

Theorem 4.70 (de Rham 1) *The mapping $\mathbf{I} : H^r(M) \to H_r(M)$ is an isomorphism. If $H_r(M)$ is finite dimensional as when M is compact and if $z_r^{(1)}, \ldots z_r^{(b)}$ (with $b =$*

the r-Betti number) is a r-cycle basis of $H_r(M)$ *and if* $\pi_1, \ldots, \pi_r \in \mathbb{R}$ *are arbitrary numbers then there is a closed r-form* $G_r \in Z^r(M)$ *such that*

$$\langle z_r^{(i)}, G_r \rangle = \pi_i,\ i = 1, \ldots, r. \tag{4.85}$$

Proof See, e.g., [25].■

Definition 4.71 The number π_r in Eq. (4.85) is called the period of the form G_r on the cycle $z_r^{(i)}$.

Corollary 4.72 (de Rham 2) *If for a closed form* $G_r \in \sec \bigwedge^r T^*M$ *and for any* $z_r^{(i)} \in H_r(M)$ *we have* $\langle z_r^{(i)}, G_r \rangle = 0$ *then* G_r *is exact, i.e.,* $G_r = dG_{r-1}$ *for some form* $G_{r-1} \in \sec \bigwedge^r T^*M$.

Note also, that when M is compact the spaces $H_r(M)$ and $H^r(M)$ are finite dimensional and $\dim H^r(M) = b^p$. Thus de Rham theorem justifies writing

$$H^r(M) = (H_r(M))^*, \tag{4.86}$$

and the nomenclature: *homology* and *cohomology* groups for $H_r(M)$ and $H^r(M)$.

4.4 Differential Geometry in the Hodge Bundle

4.4.1 Riemannian and Lorentzian Structures on M

Next we introduce on M a smooth metric field $\boldsymbol{g} \in \sec T_2^0 M$ and gives the

Definition 4.73 A pair $(M, \boldsymbol{g})$, $\dim M = n$ is a n-dimensional Riemann structure (or Riemann manifold) if $\boldsymbol{g} \in \sec T_2^0 M$ is a smooth *metric* of signature $(n, 0)$. If $\boldsymbol{g}$ has signature (p, q) with $p + q = n, p \neq n$ or $q \neq n$ then the pair $(M, \boldsymbol{g})$ is called a pseudo Riemannian manifold. When $\boldsymbol{g}$ has signature $(1, n-1)$ the pair $(M, \boldsymbol{g})$ is called an hyperbolic manifold. When $\dim M = 4$ and $\boldsymbol{g}$ has signature $(1, 3)$ the pair $(M, \boldsymbol{g})$ is called a Lorentzian manifold.[11]

We already defined the concept of oriented manifold. Thus, we say that a Riemannian (or pseudo Riemannian or Lorentzian) manifold is orientable if and only if it admits a continuous metric volume element field $\tau_{\boldsymbol{g}} \in \sec \bigwedge^n T^*M$ given in local coordinate functions $\{\boldsymbol{x}^i\}$ covering $U \subset M$ by

$$\tau_{\boldsymbol{g}} = \sqrt{|\det \boldsymbol{g}|} dx^1 \wedge \ldots \wedge dx^n, \tag{4.87}$$

[11]When Lorentzian manifolds serve as models of spacetimes it is also imposed that M is noncompact. See Sect. 4.7.1.

where

$$\det \boldsymbol{g} = \det \left[\boldsymbol{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \right]. \tag{4.88}$$

Proposition 4.74 *Any C^r manifold M,* $\dim M = n$ *admits a C^{r-1} Riemannian metric **g** (signature $(n, 0)$) if and only if it is paracompact.*

Proof For a proof see, e.g., [3].■

Let us consider now a smooth oriented metric manifold $\mathrm{M} = (M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$, $\dim M = n$, where $\boldsymbol{g}$ is a smooth metric field of signature (p, q) and $\tau_{\boldsymbol{g}} \in \sec \bigwedge^n T^*M$. We denote by $\mathtt{g} \in \sec T_0^2 M$ the *metric* tensor of the cotangent bundle. Also we denote the scalar product induced on $\bigwedge T^*M$ by the metric tensor $\mathtt{g} \in \sec T_0^2 M$ by[12] $\underset{\boldsymbol{g}}{\cdot}$: $\sec \bigwedge T^*M \times \sec \bigwedge T^*M \to \sec \bigwedge^0 T^*M$. If $A, B \in \sec \bigwedge^p T^*M$ we have (recall Eq. (2.123))

$$(A \underset{\boldsymbol{g}}{\cdot} B)\tau_{\boldsymbol{g}} = A \wedge \underset{\boldsymbol{g}}{\star} B \tag{4.89}$$

4.4.2 Hodge Bundle

Definition 4.75 The *Hodge bundle* of the structure M is the triple

$$\bigwedge(\mathrm{M}) = (\bigwedge T^*M, \underset{\boldsymbol{g}}{\cdot}, \tau_{\boldsymbol{g}}). \tag{4.90}$$

The importance of the Hodge bundle is that besides the exterior derivative operator, we can now introduce a new differential operator called the Hodge codifferential. Equipped with these two operators we can write, e.g., Maxwell equations (with currents) in a *diffeomorphism* invariant way[13] (see Sect. 4.9.1). This is a very important fact, which is often not well known as it should be.

[12] When there is no chance of confusion we eventually used the symbol $\cdot$ instead of the symbol $\underset{\boldsymbol{g}}{\cdot}$ in order to simplify the notation.

[13] For the exact meaning of the concept of diffeomorphism invariance of a spacetime physical theory (as used in this text) see Sect. 6.6.3.

Definition 4.76 The *Hodge codifferential* operator in the Hodge bundle of $\bigwedge(M)$ is the mapping $\underset{g}{\delta} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$, given, for homogeneous multi-forms, by:

$$\underset{g}{\delta} = (-1)^r \underset{g}{\star}^{-1} d \underset{g}{\star}, \tag{4.91}$$

where $\underset{g}{\star}$ is the Hodge star operator associated to the scalar product $\underset{g}{\cdot}$.

Definition 4.77 The *Hodge Laplacian* operator is the mapping

$$\underset{g}{\diamond} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$$

given by:

$$\underset{g}{\diamond} = -(d\underset{g}{\delta} + \underset{g}{\delta} d). \tag{4.92}$$

The exterior derivative, the Hodge codifferential and the Hodge Laplacian satisfy the relations:

$$\begin{aligned} dd &= \underset{g}{\delta}\underset{g}{\delta} = 0; \quad \underset{g}{\diamond} = (d - \underset{g}{\delta})^2; \\ d\underset{g}{\diamond} &= \underset{g}{\diamond} d; \quad \underset{g}{\delta}\underset{g}{\diamond} = \underset{g}{\diamond}\underset{g}{\delta}; \\ \underset{g}{\delta}\underset{g}{\star} &= (-1)^{r+1}\underset{g}{\star} d; \quad \underset{g}{\star}\underset{g}{\delta} = (-1)^r d\underset{g}{\star}; \\ d\underset{g}{\delta}\underset{g}{\star} &= \underset{g}{\star}\underset{g}{\delta} d; \quad \underset{g}{\star} d\underset{g}{\delta} = \underset{g}{\delta} d\underset{g}{\star}; \quad \underset{g}{\star}\underset{g}{\diamond} = \underset{g}{\diamond}\underset{g}{\star}. \end{aligned} \tag{4.93}$$

Remark 4.78 When it is clear from the context which metric field is involved we use the symbols $\star$, δ and $\diamond$ in place of the symbols $\underset{g}{\star}$, $\underset{g}{\delta}$ and $\underset{g}{\diamond}$ in order to simplify the writing of equations.

4.4.3 *The Global Inner Product of p-Forms*

Definition 4.79 Let $A, B \in \sec \bigwedge^p T^*M$ and suppose that the support of A or B is *compact*. The global inner product of these p-forms is

$$\langle A, B \rangle = \int_M A \wedge \star B. \tag{4.94}$$

Definition 4.80 Let $T : \sec \bigwedge^p T^*M \to \sec \bigwedge^q T^*M$ be a (p, q) extensor field acting on the sections of $\bigwedge^p T^*M$ of compact support. We define the metric transpose of T as the the (q, p) extensor field T^t such that

$$\langle TA, B\rangle = \langle A, T^t B\rangle \tag{4.95}$$

Exercise 4.81 Show that d and δ are metric transposes of each other i.e.,

$$\begin{aligned} \langle dA, B\rangle &= \langle A, \delta B\rangle, \\ \langle \delta A, B\rangle &= \langle A, dB\rangle \end{aligned} \tag{4.96}$$

Are the formulas given in Eq. (4.96) true for a compact manifold with boundary?

4.5 Pullbacks and the Differential

Proposition 4.82 *Let $\phi^* : M \to N$ be a differentiable mapping and let* h* *be the pullback mapping. Let $A, B \in \sec \bigwedge T^*M$. Then*

$$\phi^*(A \wedge B) = \phi^* A \wedge \phi^* B. \tag{4.97}$$

Proof It is a simple algebraic manipulation.■

Proposition 4.83 *Let $\phi : M \to N$ be a differentiable mapping and let ϕ^* be the pullback mapping. Let $A \in \sec \bigwedge T^*M$. Then,*

$$\phi^* dA = d(\phi^* A) \tag{4.98}$$

Proof Since an arbitrary form is a finite sum of exterior products of functions and differential of functions, we see that it is only necessary to prove the theorem for a 0-form and an exact 1-form α. The first case is true because,

$$\begin{aligned} \phi^* dg &= d(g \circ \phi) \\ &= d(\phi^* g) \end{aligned} \tag{4.99}$$

where we used the definition of reciprocal image. Now, if $\alpha = dg$, i.e., α is exact, we have

$$\phi^* d\alpha = \phi^* ddg = 0.$$

Also,

$$d(\phi^*\alpha) = d(\phi^* dg) = d[d(\phi^* g)] = d^2\phi^* g = 0, \tag{4.100}$$

and the proposition is proved.■

Proposition 4.83 is also very much important in proving the invariance of some exterior differential system of equations under diffeomorphisms.

4.6 Structure Equations I

Let us now endow the metric manifold $(M, \boldsymbol{g})$, with an arbitrary linear connection ∇ obtaining the structure $(M, \boldsymbol{g}, \nabla)$.

Definition 4.84 The *torsion and curvature operations* and the torsion and *curvature* tensors of a connection ∇, are respectively the mappings[14]:

$$\boldsymbol{\tau} : \sec(TM \times TM) \to \sec TM,$$
$$\boldsymbol{\rho} : \sec(TM \times TM) \to \mathrm{End}TM$$

$$\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \tag{4.101}$$
$$\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]} \tag{4.102}$$

and

$$\Theta(\alpha, \boldsymbol{u},\boldsymbol{v}) = \alpha\left(\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v})\right), \tag{4.103}$$
$$\mathbf{R}(\alpha, \boldsymbol{w}, \boldsymbol{u},\boldsymbol{v}) = \alpha(\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}), \tag{4.104}$$

for every $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$.

Exercise 4.85 Show that for any differentiable functions f, g and h we have

$$\boldsymbol{\tau}(g\boldsymbol{u},h\boldsymbol{v}) = gh\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}),$$
$$\boldsymbol{\rho}(g\boldsymbol{u},h\boldsymbol{v})f\boldsymbol{w} = ghf\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v}). \tag{4.105}$$

[14]EndTM means the set of endomorphisms $TM \to TM$.

Given an arbitrary moving frame $\{\boldsymbol{e}_\alpha\}$ on *TM*, let $\{\theta^\rho\}$ be the *dual frame of* $\{\boldsymbol{e}_\alpha\}$ (i.e., $\theta^\rho(\boldsymbol{e}_\alpha) = \delta^\rho_\alpha$). We write:

$$\begin{aligned} [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] &= c^{\rho\cdot\cdot}_{\cdot\alpha\beta} \boldsymbol{e}_\rho, \\ \nabla_{\boldsymbol{e}_\alpha} \boldsymbol{e}_\beta &= L^{\rho\cdot\cdot}_{\cdot\alpha\beta} \boldsymbol{e}_\rho, \end{aligned} \tag{4.106}$$

where $c^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the *structure coefficients* of the frame $\{\boldsymbol{e}_\alpha\}$ and $L^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the *connection coefficients* in this frame. Then, the components of the torsion and curvature tensors are given, respectively, by:

$$\begin{aligned} [c]rclT^{\rho\cdot\cdot}_{\cdot\alpha\beta} &:= \Theta(\theta^\rho, e_\alpha, \boldsymbol{e}_\beta) = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} - L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - c^{\rho\cdot\cdot}_{\cdot\alpha\beta}, \\ R^{\rho\cdot\cdot\cdot}_{\cdot\mu\alpha\beta} &:= \mathbf{R}(\theta^\rho, \boldsymbol{e}_\mu, \boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) \\ &= \boldsymbol{e}_\alpha(L^{\rho\cdot\cdot}_{\cdot\beta\mu}) - \boldsymbol{e}_\beta(L^{\rho\cdot\cdot}_{\cdot\alpha\mu}) + L^{\rho\cdot\cdot}_{\cdot\alpha\sigma} L^{\sigma\cdot\cdot}_{\cdot\beta\mu} - L^{\rho\cdot\cdot}_{\beta\sigma} L^{\sigma\cdot\cdot}_{\alpha\mu} - c^{\sigma\cdot\cdot}_{\cdot\alpha\beta} L^{\rho\cdot\cdot}_{\cdot\sigma\mu}. \end{aligned} \tag{4.107}$$

We also have:

$$\begin{aligned} d\theta^\rho &= -\tfrac{1}{2} c^{\rho\cdot\cdot}_{\cdot\alpha\beta} \theta^\alpha \wedge \theta^\beta, \\ \nabla_{e_\alpha} \theta^\rho &= -L^{\rho\cdot\cdot}_{\cdot\alpha\beta} \theta^\beta, \end{aligned} \tag{4.108}$$

where $\omega^{\rho\cdot}_{\cdot\beta} \in \sec \bigwedge^1 T^*M$ are the *connection 1-forms,* $\Theta^\rho \in \sec \bigwedge^2 T^*M$ are the *torsion 2-forms* and $\mathcal{R}^{\rho\cdot}_{\cdot\beta} \in \sec \bigwedge^2 T^*M$are the *curvature 2-forms*, given by:

$$\begin{aligned} \omega^{\rho\cdot}_{\cdot\beta} &:= L^{\rho\cdot\cdot}_{\cdot\alpha\beta} \theta^\alpha, \\ \Theta^\rho &:= \frac{1}{2} T^{\rho\cdot\cdot}_{\cdot\alpha\beta} \theta^\alpha \wedge \theta^\beta, \\ \mathcal{R}^{\rho\cdot}_{\cdot\mu} &:= \frac{1}{2} R^{\rho\cdot\cdot\cdot}_{\cdot\mu\alpha\beta} \theta^\alpha \wedge \theta^\beta. \end{aligned} \tag{4.109}$$

Multiplying Eqs. (4.107) by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ and using Eqs. (4.108) and (4.109), we get the *Cartan's structure equations*:

$$\begin{aligned} d\theta^\rho + \omega^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta &= \Theta^\rho, \\ d\omega^{\rho\cdot}_{\cdot\mu} + \omega^{\rho\cdot}_{\cdot\beta} \wedge \omega^{\beta\cdot}_{\cdot\mu} &= \mathcal{R}^{\rho\cdot}_{\cdot\mu}. \end{aligned} \tag{4.110}$$

Exercise 4.86 Show that the torsion tensor can be written as

$$\Theta = \boldsymbol{e}_\alpha \otimes \Theta^\alpha \tag{4.111}$$

Exercise 4.87 Put $\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \theta^{\mathbf{a}_1} \wedge \cdots \wedge \theta^{\mathbf{a}_r}$ and $\underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \underset{g}{\star}(\theta^{\mathbf{a}_1} \wedge \cdots \wedge \theta^{\mathbf{a}_r})$. Show that when $\Theta^{\mathbf{a}} = 0$ we have

$$d\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega^{\mathbf{a}_1\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}\ldots\mathbf{a}_r} - \cdots - \omega^{\mathbf{a}_r\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{a}_1\ldots\mathbf{b}}, \tag{4.112}$$

$$d\underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega^{\mathbf{a}_1\cdot}_{\cdot\mathbf{b}} \wedge \underset{g}{\star}\theta^{\mathbf{b}\ldots\mathbf{a}_r} - \cdots - \omega^{\mathbf{a}_r\cdot}_{\cdot\mathbf{b}} \wedge \underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{b}}. \tag{4.113}$$

4.6.1 Exterior Covariant Differential of $(p+q)$-Indexed r-Form Fields

Definition 4.88 Suppose that $X \in \sec T^{r+q}_p M$ and let

$$X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(\boldsymbol{v}_1 \ldots, \boldsymbol{v}_r) \in \sec \bigwedge\nolimits^r T^*M, \tag{4.114}$$

such that

$$X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(\boldsymbol{v}_1 \ldots, \boldsymbol{v}_r) = X(\boldsymbol{v}_1 \ldots, \boldsymbol{v}_r, \boldsymbol{e}_{\nu_1}, \ldots .\boldsymbol{e}_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p}). \tag{4.115}$$

for $\boldsymbol{v}_1 \ldots, \boldsymbol{v}_r \in \sec TM$. The $X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ are called $(p+q)$-indexed r-forms.

Definition 4.89 The exterior covariant differential[15] $\mathbf{D}$ of $X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ on a manifold with a general connection ∇ is the mapping:

$$\mathbf{D} : \sec \bigwedge\nolimits^r T^*M \to \sec \bigwedge\nolimits^{r+1} T^*M, 0 \le r \le 4, \tag{4.116}$$

such that[16]

$$\begin{aligned}
&(r+1)\mathbf{D}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(\boldsymbol{v}_0, \boldsymbol{v}_1 \ldots, \boldsymbol{v}_r) \\
&= \sum_{\nu=0}^{r}(-1)^{\nu}\nabla_{\boldsymbol{e}_\nu}X(\boldsymbol{v}_0, \boldsymbol{v}_1 \ldots, \check{\boldsymbol{v}}_\nu, \ldots \boldsymbol{v}_r, \boldsymbol{e}_{\nu_1}, \ldots .\, \boldsymbol{e}_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p}) \\
&\quad - \sum_{0\le\nu,\varsigma\le r}(-1)^{\nu+\varsigma}X(\boldsymbol{\tau}(\boldsymbol{v}_\nu, \boldsymbol{v}_\varsigma), \boldsymbol{v}_0, \boldsymbol{v}_1 \ldots, \check{\boldsymbol{v}}_\nu, \ldots, \boldsymbol{v}_\varsigma, \ldots, \boldsymbol{e}_r, \\
&\qquad \boldsymbol{e}_{\nu_1}, \ldots .\boldsymbol{e}_{\nu_q}, \theta^{\mu_1}, \ldots, \theta^{\mu_p}).
\end{aligned} \tag{4.117}$$

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[15] Sometimes also called exterior covariant derivative.

[16] As usual the inverted hat over a symbol (in Eq. (4.117)) means that the corresponding symbol is missing in the expression.

Then, we may verify that

$$\mathbf{D}X^{\mu_1....\mu_p}_{\nu_1....\nu_q} = dX^{\mu_1....\mu_p}_{\nu_1....\nu_q} + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_s....\mu_p}_{\nu_1....\nu_q} + \cdots + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1....\mu_p}_{\nu_1....\nu_q} \\ - \omega^{\nu_s}_{\nu_1} \wedge X^{\mu_1....\mu_p}_{\nu_s....\nu_q} - \cdots - \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1....\mu_p}_{\nu_1....\nu_s}. \tag{4.118}$$

Remark 4.90 Sometimes, Eqs. (4.110) are written by some authors [45] as:

$$\mathbf{D}\theta^{\rho} := \Theta^{\rho}, \\ \text{“}\mathbf{D}\omega^{\rho\cdot}_{\cdot\mu} := \mathcal{R}^{\rho\cdot}_{\cdot\mu}.\text{”} \tag{4.119}$$

and $\mathbf{D} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$ is said to be the exterior covariant derivative related to the connection ∇. Whereas the equation $\mathbf{D}\theta^{\rho} := \Theta^{\rho}$ is well defined, we see that the equation “$\mathbf{D}\omega^{\rho\cdot}_{\cdot\mu} := \mathcal{R}^{\rho\cdot}_{\cdot\mu}$.” is an equivocated one. Indeed if Eq. (4.118) is applied on the connection 1-forms $\omega^{\mu\cdot}_{\cdot\nu}$ we would get $\mathbf{D}\omega^{\mu\cdot}_{\cdot\nu} = d\omega^{\mu\cdot}_{\cdot\nu} + \omega^{\mu\cdot}_{\cdot\alpha} \wedge \omega^{\alpha\cdot}_{\cdot\nu} - \omega^{\alpha\cdot}_{\cdot\nu} \wedge \omega^{\mu\cdot}_{\cdot\alpha}$. So, we see that the symbol $\mathbf{D}\omega^{\mu\cdot}_{\cdot\nu}$ given by the second formula in Eq. (4.119), supposedly defining the curvature 2-forms is to be avoided. The reason for the failure of Eq. (4.118) in that case is that there do not exist a tensor field $\omega \in \sec T^2_1 M$ which satisfy the corresponding Eq. (4.115). More details on this issue may be found in Appendix A.3.

Exercise 4.91 Show that if $X^J \in \sec \bigwedge^r T^*M$ and $Y^K \in \sec \bigwedge^s T^*M$ are sets of indexed forms,[17] then

$$\mathbf{D}(X^J \wedge Y^K) = \mathbf{D}X^J \wedge Y^K + (-1)^{rs} X^J \wedge \mathbf{D}Y^K. \tag{4.120}$$

Exercise 4.92 Show that if $X^{\mu_1\cdots\mu_p} \in \sec \bigwedge^r T^*M$ then

$$\mathbf{D}\mathbf{D}X^{\mu_1....\mu_p} = dX^{\mu_1....\mu_p} + \mathcal{R}^{\mu_1\cdot}_{\cdot\mu_s} \wedge X^{\mu_s....\mu_p} + \cdots \mathcal{R}^{\mu_p\cdot}_{\cdot\mu_s} \wedge X^{\mu_1....\mu_s}. \tag{4.121}$$

Exercise 4.93 Show that for any metric-compatible connection ∇ if $\boldsymbol{g} = g_{\mu\nu}\theta^{\mu} \otimes \theta^{\nu}$ then,

$$\mathbf{D}g_{\mu\nu} = 0. \tag{4.122}$$

Since we are dealing with a metric manifold, we must complete Cartan's structure equations with the equations stating the relation between the connection and the metric. For this, following the usual nomenclature [1, 40, 47] we give the

[17] Multi indices are here represented by J and K.

Definition 4.94 The *nonmetricity* tensor field of the structures $(M, \boldsymbol{g}, \nabla)$ is the tensor field $\mathbf{Q} \in \sec T_3^0 M$ with components[18] in the basis $\{\theta^\alpha\}$ given by

$$Q_{\mu\alpha\beta} := -\nabla_\mu g_{\alpha\beta} = -\boldsymbol{e}_\mu(g_{\beta\alpha}) + g_{\sigma\alpha} L^{\sigma\cdot\cdot}_{\cdot\mu\beta} + g_{\beta\sigma} L^{\sigma\cdot\cdot}_{\cdot\mu\alpha}. \tag{4.123}$$

Correspondingly, we introduce the *nonmetricity 2-forms*, by:

$$\mathbf{Q}^\rho := \frac{1}{2} Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]} \theta^\alpha \wedge \theta^\beta, \tag{4.124}$$

where $Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]} = g^{\rho\mu}(Q_{\alpha\beta\mu} - Q_{\beta\alpha\mu})$. Multiplying Eq. (4.123) by $\theta^\alpha \wedge \theta^\beta$ and using Eq. (4.110a), we get:

$$\mathbf{D}\theta_\mu \equiv d\theta_\mu - \omega^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta = \boldsymbol{\Phi}_\mu, \tag{4.125}$$

where $\{\theta_\mu\}$ is the reciprocal frame of $\{\theta^\nu\}$ is the (i.e., $\theta_\mu = g_{\mu\nu}\theta^\nu$) and

$$\boldsymbol{\Phi}_\mu = \Theta_\mu - \mathbf{Q}_\mu.$$

Equation (4.125) can be used as the complement of Cartan's structure equations for the case of a *metric* manifold.

4.6.2 Bianchi Identities

Differentiating Eq. (4.110) and Eq. (4.125) we obtain the *Bianchi identities*[19]:

$$\begin{aligned}
&\text{(a)}\quad \mathbf{D}\Theta^\rho = d\Theta^\rho + \omega^{\rho\cdot}_{\cdot\beta} \wedge \Theta^\beta = \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta,\\
&\text{(b)}\quad \mathbf{D}\mathcal{R}^{\rho\cdot}_{\cdot\mu} = d\mathcal{R}^{\rho\cdot}_{\cdot\mu} - \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge \omega^{\beta\cdot}_{\upsilon\mu} + \omega^{\rho\cdot}_{\cdot\beta} \wedge \mathcal{R}^{\beta\cdot}_{\cdot\mu} = 0,\\
&\text{(c)}\quad \mathbf{D}\Phi_\mu = d\boldsymbol{\Phi}_\mu - \omega^{\beta\cdot}_{\cdot\mu} \wedge \boldsymbol{\Phi}_\beta = -\mathcal{R}^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta.
\end{aligned} \tag{4.126}$$

4.6.3 Induced Connections Under Diffeomorphisms

Let M and N be two differentiable manifolds, $\dim M = m$, $\dim N = n$.

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[18]We use the notation $\nabla_\sigma t^{\mu\ldots}_{\nu\ldots} \equiv (\nabla_{e_\sigma} t)^{\mu\ldots}_{\nu\ldots} \equiv (\nabla t)^{\mu\ldots}_{\sigma\nu\ldots}$ for the components of the covariant derivative of a tensor field t. This is not to be confused with $\nabla_{e_\sigma} t^{\mu\ldots}_{\nu\ldots} \equiv e_\sigma(t^{\mu\ldots}_{\nu\ldots})$, the derivative of the components of t in the direction of e_σ.

[19]To our knowledge, Eqs. (4.125) and (4.126c) are not found anywhere in the literature, although they appear to be the most natural extension of the structure equations for metric manifolds.

Definition 4.95 Let ∇ be a connection on N and $\mathbf{X}, \mathbf{Y} \in \sec TN$ and $\mathbf{T} \in \sec T^r_s N$, $f : N \to \mathbb{R}$ and $\mathtt{h}: M \to N$ a diffeomorphism. The induced connection $\mathtt{h}^*\nabla$ on M is defined by

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{T} = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{T}). \tag{4.127}$$

Example 4.96 Let $f : N \to \mathbb{R}$ and $\mathbf{Y} \in \sec TN$. Then,

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{Y} = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{Y}),$$

from where it follows (taking into account that for any vector field $\mathbf{V} \in \sec TN$, $\mathtt{h}^*\mathbf{N} = \mathtt{h}_*^{-1}\mathbf{N}$) that

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{Y}\Big|_{\mathfrak{e}} f\circ h = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{Y})|_{\mathfrak{e}} f\circ h = \nabla_{\mathbf{X}}\mathbf{Y}|_{\mathtt{h}(\mathfrak{e})} f, \ \forall \mathfrak{e} \in M.$$

Remark 4.97 Now, suppose that $M = N$ and $\mathtt{h}$: $M \to M$ a diffeomorphism. Suppose that D is the Levi-Civita connection of $\boldsymbol{g}$, then $\mathtt{h}^*D = D'$ is the Levi-Civita connection of $\mathtt{h}^*\boldsymbol{g} = \boldsymbol{g}'$ since using Eq. (4.127) we infer that

$$\mathtt{h}^*D_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\boldsymbol{g}\Big|_{\mathfrak{e}} = D'_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\boldsymbol{g}\Big|_{\mathfrak{e}} = \mathtt{h}^*(D_{\mathbf{X}}\,\boldsymbol{g})|_{\mathfrak{e}}, \forall \mathfrak{e} \in M. \tag{4.128}$$

Taking into account that[20] $\mathtt{h}^*[\mathbf{X}, \mathbf{Y}] = [\mathtt{h}^*\mathbf{X}, \mathtt{h}^*\mathbf{Y}]$ we have for $\mathbf{X}, \mathbf{Y} \in \sec TM$,

$$\mathtt{h}^*(D_{\mathbf{X}}Y - D_{\mathbf{Y}}X - [\mathbf{X}, \mathbf{Y}]) = 0. \tag{4.129}$$

Remark 4.98 Equation (4.127) applied to the case $M = N$ also implies, as the reader may verify the important fact that the curvature tensor of $\mathtt{h}^*D$ will be null if the curvature tensor of D is null.

4.7 Classification of Geometries on *M* and Spacetimes

Definition 4.99 Given a triple $(M, \boldsymbol{g}, \nabla)$:

(a) it is called a Riemann-*Cartan geometry*[21] *if and only if*

$$\nabla\boldsymbol{g} = 0 \qquad \text{and} \qquad \Theta[\nabla] \neq 0. \tag{4.130}$$

[20] See, e.g., [3, p. 135].

[21] Or Riemann space.

(b) it is called *Weyl geometry if and only if*

$$\nabla \boldsymbol{g} \neq 0 \qquad \text{and} \qquad \Theta[\nabla] = 0. \tag{4.131}$$

(c) it is called a *Riemann geometry* if and only if

$$\nabla \boldsymbol{g} = 0 \qquad \text{and} \qquad \Theta[\nabla] = 0, \tag{4.132}$$

and in that case the pair $(\nabla, \boldsymbol{g})$ is called *Riemannian structure*.

(d) it is called Riemann-*Cartan-Weyl geometry* if and only if

$$\nabla \boldsymbol{g} \neq 0 \qquad \text{and} \qquad \Theta[\nabla] \neq 0. \tag{4.133}$$

(e) it is called a (Riemann) flat geometry if and only if

$$\nabla \boldsymbol{g} = 0 \qquad \text{and} \qquad \mathbf{R}[\nabla] = 0,$$

(f) it is called teleparallel geometry if and only if

$$\nabla \boldsymbol{g} = 0, \;\; \Theta[\nabla] \neq 0 \text{ and } \mathbf{R}[\nabla] = 0. \tag{4.134}$$

For each metric tensor defined on the manifold M there exists one and only one connection in the conditions of Eq. (4.132). It is called *Levi-Civita connection* of the metric considered, and is denoted by D. If in a given context it is necessary to distinguish between the Levi-Civita connections of two different metric tensors $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ on the same manifold, we write $\mathring{D}$, D.

Remark 4.100 When $\dim M = 4$ and the metric $\boldsymbol{g}$ has signature $(1, 3)$ we sometimes substitute the word Riemann by the word Lorentzian in the previous definitions.

4.7.1 Spacetimes

From nowhere besides the constraints already imposed (Hausdorff and paracompact) on M, we suppose also that it is connected and noncompact [14, 38]. We now introduce the concept of *time orientability* on an oriented Lorentzian manifold structure $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$, which plays a key role in physical theories.

Definition 4.101 Let $(M, \boldsymbol{g})$ be a Lorentzian manifold, $TM = \bigcup_{e \in M} T_e M$ its tangent bundle and $\pi : TM \to M$ the canonical projection (see Appendix). The causal character of $(e, \mathbf{v}) \in TM$ is the causal character of $\mathbf{v}$ (Definition 2.62).

Definition 4.102 A line element at $x \in M$ is a one-dimensional subspace of $T_x M$.

Proposition 4.103 *Let M be a C^1paracompact and Hausdorff manifold,* $\dim M = 4$. *Then the existence of a continuous line element field on M is equivalent to the existence of a Lorentzian structure on M.*

Proof For a proof see [3].■

Proposition 4.104 *The set $\mathfrak{T} \subset TM$ of timelike points is an open manifold and it has either one (connected) component or two.*

Proof A proof of this important result can be found in [38].■

Definition 4.105 A connected Lorentzian manifold $(M, \boldsymbol{g})$ is said to be time orientable if and only if $\mathfrak{T}$ has two components and one of the components is labeled the future $\mathfrak{T}^+$ and the other component $\mathfrak{T}^-$ is labelled the past. We denote by $\uparrow$ the time orientability of a Lorentzian manifold.

Definition 4.106 A spacetime is a pentuple $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ where $(M, \boldsymbol{g})$ is a Lorentzian oriented and time oriented manifold and ∇ is an arbitrary covariant derivative operator on M.

Definition 4.107 When $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ is a spacetime and $\nabla = D$ is the Levi-Civita connection of $\boldsymbol{g}$ the spacetime is said to be Lorentzian. When $\nabla \boldsymbol{g} = 0$ and $\Theta(\nabla) \neq 0$ we call the structure $\mathfrak{M} = (M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ a Riemann-Cartan spacetime. The particular Riemann-Cartan spacetime for which $\mathbf{R}(D) = 0$, $\Theta[\nabla] \neq 0$ is called a teleparallel spacetime (also called Weintzenböck spacetime according to [26]).

Definition 4.108 A Lorentzian spacetime structure $\mathcal{M} = (M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ is said to be Minkowski spacetime if and only if $M \simeq \mathbb{R}^4$ and $\mathbf{R}(D) = 0$.

Remark 4.109 We just establish that any Lorentzian manifold admits a continuous element field. If it is also time orientable, we can choose a direction for the continuous element field, and say that it is a *timelike* vector field pointing to the future. This is a nontrivial result and very important for our discussion of the Principle of Relativity (Chap. 6).

4.8 Differential Geometry in the Clifford Bundle

It is well known [28] that the natural operations on metric vector spaces, such as, e.g., direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metric tensors. Then we give the

Definition 4.110 The *Clifford bundleof differential forms* of the metric manifold $(M, \boldsymbol{g})$ is:

$$\mathcal{C}\ell(M, \mathtt{g}) = \frac{\mathcal{T}M}{J_{\mathtt{g}}} = \bigcup_{x \in M} \mathcal{C}\ell(T_x^* M, \mathtt{g}_x), \tag{4.135}$$

where $\mathcal{T}M$ denotes the (covariant) tensor bundle of M, $J_{\mathtt{g}} \subset \mathcal{T}M$ is the bilateral ideal of $\mathcal{T}M$ generated by the elements of the form $\alpha \otimes \beta + \beta \otimes \alpha - 2\mathtt{g}(\alpha, \beta)$, with $\alpha, \beta \in \sec T^*M \subset \mathcal{T}M$ and $\mathcal{C}\ell(T_x^*M, \mathtt{g}_x)$ is the Clifford algebra of the metric vector space structure $(T_x^*M, \mathtt{g}_x)$.

It will be shown in Chap. 7 that the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ (as defined by Eq. (4.135)) is a vector bundle associated to the principal bundle of orthonormal frames $\mathbf{P}_{\mathrm{SO}^e p,q}$, i.e.,

$$\mathcal{C}\ell(M, \mathtt{g}) = \mathbf{P}_{\mathrm{SO}^e p,q} \times_{Ad} \mathbb{R}_{p,q}. \tag{4.136}$$

In Eq. (4.136) Ad is the *adjoint representation* of $\mathrm{Spin}^e_{p,q}$, i.e., $Ad : \mathrm{SO}^e_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q})$, $u \mapsto Ad_u$, with $Ad_u A = \mathrm{A}\, u^{-1}$, $\forall u \in \mathrm{SO}^e_{p,q}$, $\forall A \in \mathbb{R}_{p,q} \simeq \mathcal{C}\ell(T_x^*M, \mathtt{g}_x)$. Details on these groups may be found in Chap. 3. In Chap. 7 we scrutinize the vector bundle structure of the Clifford bundle of differential forms over a general Riemann-Cartan manifold modelling spacetime.

4.8.1 Clifford Fields as Sums of Nonhomogeneous Differential Forms

Definition 4.111 Sections of $\mathcal{C}\ell(M, \mathtt{g})$ are called Clifford fields.

We recall some notations and conventions. By $F(U)$ we denote the frame bundle (see Appendix A.3) of $U \subset M$. A section of $F(U)$ will be denoted by $\{\boldsymbol{e}_\alpha\} \in \sec F(U)$. The dual frame of a frame $\{\boldsymbol{e}_\alpha\}$ will be denoted by $\{\theta^\alpha\}$, where $\theta^\alpha \in \sec T^*U \subset T^*M$. When $\{\boldsymbol{e}_\alpha\}$ is a coordinate frame associated to the coordinate functions $\{\mathbf{x}^\mu\}$ of a local chart covering U we use instead of $\boldsymbol{e}_\alpha$ the notation $e_\alpha = \partial_\alpha$ and in this case $\theta^\alpha = dx^\alpha$. When $\{\boldsymbol{e}_\alpha\}$ refers to an orthonormal frame we use instead of $\boldsymbol{e}_\alpha$ the notation $\boldsymbol{e}_{\mathbf{a}}$ and instead of θ^α the notation $\theta^{\mathbf{a}}$.

Recall that as a vector space over $\mathbb{R}$, $\mathcal{C}\ell(T_x^*M, \mathbf{g}_x)$ is isomorphic to the exterior algebra $\bigwedge T_x^*M$ of the cotangent space and

$$\bigwedge T_x^*M = \bigoplus_{k=0}^{n} \bigwedge{}^k T_x^*M, \tag{4.137}$$

where $\bigwedge^k T_x^*M$ is the $\binom{n}{k}$-dimensional space of k-forms. Then, there is a natural embedding [22] $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ [21] and sections of $\mathcal{C}\ell(M, \mathtt{g})$—Clifford fields (Definition 4.111)—can be represented as a sum of non homogeneous differential forms. Let $\{\boldsymbol{e}_{\mathbf{a}}\}$ be an orthonormal basis for $TU \subset TM$, i.e., $\boldsymbol{g}(\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \eta_{\mathbf{ab}}$, where the matrix with entries $\eta_{\mathbf{ab}}$ is the diagonal matrix, $\mathrm{diag}(1, 1, \ldots .. -1, \ldots, -1)$ and $(\mathbf{a}, \mathbf{b}, \mathbf{i}, \mathbf{j}, \ldots = 1, 2, \ldots, n)$. Moreover, let $\{\theta^{\mathbf{a}}\} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

[22] Recall again that the symbol $A \hookrightarrow B$ means that A is embedded in B and $A \subseteq B$.

such that the set $\{\theta^{\mathbf{a}}\}$ is the dual basis of $\{\mathbf{e}_{\mathbf{a}}\}$. We denote by $\{\theta_{\mathbf{i}}\}$ be the *reciprocal basis* of $\{\theta^{\mathbf{i}}\}$, i.e., $\theta_{\mathbf{i}} \underset{g}{\cdot} \theta^{\mathbf{j}} = \delta_{\mathbf{i}}^{\mathbf{j}}$.

For the particular case of a 4-dimensional spacetime, of course, the range of the bold labels are **a**, **b**, **i**, .. $= 0, 1, 2, 3$. Recall that the fundamental *Clifford product* is generated by

$$\theta^{\mathbf{i}}\theta^{\mathbf{j}} + \theta^{\mathbf{j}}\theta^{\mathbf{i}} = 2\eta^{\mathbf{ij}}. \tag{4.138}$$

If $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ is a Clifford field, we have:

$$\mathcal{C} = s + v_{\mathbf{i}}\theta^{\mathbf{i}} + \frac{1}{2!}b_{\mathbf{ij}}\theta^{\mathbf{i}}\theta^{\mathbf{j}} + \frac{1}{3!}t_{\mathbf{ijk}}\theta^{\mathbf{i}}\theta^{\mathbf{j}}\theta^{\mathbf{k}} + p\theta^{\mathbf{5}}\ , \tag{4.139}$$

where $\theta^{\mathbf{5}} = \theta^{\mathbf{0}}\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{3}$ is the volume element and

$$s, v_{\mathbf{i}}, b_{\mathbf{ij}}, t_{\mathbf{ijk}}, p \in \sec \bigwedge\nolimits^{0} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}). \tag{4.140}$$

4.8.2 Pullbacks and Relation Between Hodge Star Operators

Let M be a n-dimensional manifold and $\mathring{\boldsymbol{g}}$, $\boldsymbol{g} \in \sec T_2^0 M$ two metrics of the same signature with corresponding metrics (for the cotangent bundle) $\mathring{\mathtt{g}}, \mathtt{g} \in \sec T_0^2 M$. Let $\mathring{g}$ and g be the extensor fields associated to $\mathring{\mathtt{g}}$ and $\mathtt{g}$. Let $\mathtt{h}\colon M \to M$ be a diffeomorphism such that

$$\boldsymbol{g} = \mathtt{h}^{*}\mathring{\boldsymbol{g}}. \tag{4.141}$$

From the algebraic results of Sect. 2.8 we easily infer that there exists a metric gauge extensor field h such that

$$g(a) \underset{\mathring{g}}{\cdot} b = h(a) \underset{\mathring{g}}{\cdot} h(b) \tag{4.142}$$

for any $a, b \in \sec \bigwedge^{1} T^{*}M$ and we write $g = h^{\dagger}h$. Then, as in the purely algebraic case discussed in Sect. 2.8 we can also show that we have the following relation between the Hodge star operators associated to $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$

$$\underset{g}{\star} = \underline{h}^{-1} \underset{\mathring{g}}{\star} \underline{h}. \tag{4.143}$$

Remark 4.112 In this case we say that the metric gauge extensor h is related to the pullback mapping $\mathtt{h}^{*}$ and describes an elastic distortion. However, keep in mind that in general given a h it does not implies the existence of $\mathtt{h}^*$ such that Eq. (4.141) holds. In this case h is said to generate a plastic distortion. More details in [9].

We now show the

Proposition 4.113 *Let* $\mathtt{h} : M \to M$ *a diffeomorphism. Let* $\mathring{\boldsymbol{g}}, \boldsymbol{g} \in \sec T_2^0 M$ *two metrics of the same signature. Then for any* $\omega \in \sec \bigwedge^p T^*M$ *we have*

$$\underset{\boldsymbol{g}}{\star} \mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star} \omega \tag{4.144}$$

Proof As in Remark 4.24 take two charts (U, φ) and (V, χ), U, $\mathtt{h}(U), \subset V$ with coordinate functions $\boldsymbol{x}^i$ and $\boldsymbol{y}^i$ such that and $\boldsymbol{x}^i(\mathfrak{e}) = \boldsymbol{y}^i(\mathtt{h}(\mathfrak{e}))$, i.e., calling $\boldsymbol{x}^i(\mathfrak{e}) = x^i$, $\boldsymbol{y}^i(\mathtt{h}(\mathfrak{e})) = y^i$ we have $\partial y^i/\partial x^j = \delta^i_j$, $dx^i = dy^i$. Let also $\mathring{\boldsymbol{g}}(\partial/\partial y^k, \partial/\partial y^k) = \mathring{g}_{kl}(y^j)$. Then it follows that $g_{kl}(x^i) = \mathring{g}_{kl}(y^j(x^i)) = \mathring{g}_{kl}(x^i)$ and $\det \boldsymbol{g} = \det \mathring{\boldsymbol{g}}$. Now, if $\omega = \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(x^i\right) dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, we can write (taking into account that $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and also $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathring{\mathtt{g}})$)

$$\begin{aligned}
\underset{\boldsymbol{g}}{\star}\mathtt{h}^*\omega &= \widetilde{\mathtt{h}^*\omega}\underset{\boldsymbol{g}}{\lrcorner}\tau_{\boldsymbol{g}} = \widetilde{\mathtt{h}^*\omega}\tau_{\boldsymbol{g}} \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(y^i(x^j)\right)\widetilde{dx^{i_1} \wedge \ldots \wedge dx^{i_p}}\sqrt{|\det \boldsymbol{g}|}dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}(y^i(x^j))\widetilde{dy^{i_1} \wedge \ldots \wedge dy^{i_p}}\sqrt{|\det \mathring{\boldsymbol{g}}|}dy^1 \wedge \ldots \wedge dy^n \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(y^i\right)\widetilde{dy^{i_1} \wedge \ldots \wedge dy^{i_p}}\underset{\mathring{\boldsymbol{g}}}{\lrcorner}\sqrt{|\det \mathring{\boldsymbol{g}}|}dy^1 \wedge \ldots \wedge dy^n \\
&= \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star} \omega,
\end{aligned}$$

and the proposition is proved.■

Remark 4.114 When $\boldsymbol{g} = \mathtt{h}^*\mathring{\boldsymbol{g}}$, there exists an associated metric gauge extensor field h such satisfying Eq. (4.142), i.e., $g = h^\dagger h$. The relation $\underset{\boldsymbol{g}}{\star}\mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star} \omega$ and $\underset{\boldsymbol{g}}{\star} = \underline{h}^{-1} \underset{\mathring{\boldsymbol{g}}}{\star} \underline{h}$ permit us to write the suggestive *operator* identity

$$\underset{\mathring{\boldsymbol{g}}}{\star} \underline{h}\mathtt{h}^*\omega = \underline{h}\mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star} \omega. \tag{4.145}$$

Exercise 4.115 Consider any diffeomorphism $\mathtt{h} : M \to M$, and two metrics $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ such that $\boldsymbol{g} = \mathtt{h}^*\mathring{\boldsymbol{g}}$. Show that

$$\underset{\boldsymbol{g}}{\star} d \underset{\boldsymbol{g}}{\star} \mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star} d \underset{\mathring{\boldsymbol{g}}}{\star} \omega, \tag{4.146}$$

for any $\omega \in \bigwedge T^*M$.

Solution The first member of Eq. (4.146) can be writing successively using Eq. (4.144) as

$$\begin{aligned}\underset{g}{\star} d \underset{g}{\star} \mathtt{h}^{*}\omega &= \underset{g}{\star} d\mathtt{h}^{*} \underset{\mathring{g}}{\star}\, \omega \\ &= \underset{g}{\star} \mathtt{h}^{*} d \underset{\mathring{g}}{\star}\, \omega \\ &= \mathtt{h}^{*} \underset{\mathring{g}}{\star}\, d \underset{\mathring{g}}{\star}\, \omega .\end{aligned}$$

4.8.3 Dirac Operators

We now equip the Riemannian (pseudo Riemannian, or Lorentzian) manifold $(M, \mathring{\boldsymbol{g}})$ with a *standard* structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$, where $\mathring{D}$ is the Levi-Civita connection of $\mathring{\boldsymbol{g}}$.

We are going to introduce in the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ a differential operator $\mathring{\boldsymbol{\partial}}$, called the standard Dirac operator,[23] which is associated to the Levi-Civita connection of the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ and we study the properties of that operator. Next we define new Dirac-like operators associated with a connection different from the Levi-Civita one, i.e., to connections ∇ defining a general Riemann-Cartan-Weyl geometry $(M, \mathring{\boldsymbol{g}}, \nabla)$. Moreover, making use of the results developed in Sect. 2.7, we show that it is possible to introduce infinitely many others Dirac-like operators, one for each bilinear form field defined on the manifold M of the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$. These constructions enable us to formulate the geometry of a Riemann-Cartan-Weyl space in the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Some interesting geometrical concepts, like the Dirac *commutator* and *anticommutator*, are introduced. Moreover, we show a new decomposition of a general linear connection, identifying some new relevant tensors which are important for a clear understanding of any formulation of the gravitational theory in flat Minkowski spacetime (Chap. 11) and other related subjects appearing in the literature.

The Standard Dirac Operator

Given $\boldsymbol{u} \in \sec TM$ and $A \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ consider the tensorial mapping $A \mapsto \mathring{D}_{\mathbf{u}} A \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Since $\mathring{D}_{\boldsymbol{u}} J_{\mathring{\mathtt{g}}} \subseteq J_{\mathring{\mathtt{g}}}$, where $J_{\mathring{\mathtt{g}}}$ is the ideal used in the definition of $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$, we see immediately that the notion of covariant derivative (related to the Levi-Civita connection[24]) pass to the quotient

[23] It is crucial to distinguish the Dirac operators introduced in this chapter and which act on sections of Clifford bundles with the spin Dirac operator introduced in Chap. 7 and which act on sections of spin-Clifford bundles.

[24] And more generally, to any metric compatible connection.

bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$, i.e., given $A, B \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ we have taking into account the fact that $\mathring{D}_{\boldsymbol{u}}\mathring{\boldsymbol{g}} = 0 = \mathring{D}_{\boldsymbol{u}}\mathring{\mathtt{g}}$ that

$$\begin{aligned}
\mathring{D}_{\boldsymbol{u}}(AB) &= \mathring{D}_{\boldsymbol{u}}[\frac{1}{2}(A \otimes B - B \otimes A) + \mathring{g}(A, B)] \\
&= \mathring{D}_{\boldsymbol{u}}[\frac{1}{2}(A \otimes B - B \otimes A)] + (\mathring{D}_{\boldsymbol{u}}\mathring{\mathtt{g}})(A, B) + \mathring{\mathtt{g}}(\mathring{D}_{\boldsymbol{u}}A, B) + \mathring{\mathtt{g}}(A, \mathring{D}_{\boldsymbol{u}}B) \\
&= \mathring{D}_{\boldsymbol{u}}(A)B + A\mathring{D}_{\boldsymbol{u}}(B). \qquad (4.147)
\end{aligned}$$

Before continuing we agree that the scalar and contracted products induced by $\mathring{\mathtt{g}}$ will be denoted simply by the symbols $\cdot$ and $\lrcorner$ instead of the symbol $\underset{\mathring{g}}{\cdot}$ and $\underset{\mathring{g}}{\lrcorner}$.

Definition 4.116 The *standard Dirac operatoracting on sections of* $\mathcal{C}\ell(M, \mathring{g})$ is the first order differential operator

$$\boldsymbol{\partial} = \theta^\alpha \mathring{D}_{\boldsymbol{e}_\alpha}. \qquad (4.148)$$

For $A \in \sec \mathcal{C}\ell(M, \mathring{g})$,

$$\boldsymbol{\partial} A = \theta^\alpha (\mathring{D}_{\boldsymbol{e}_\alpha} A) = \theta^\alpha \lrcorner (\mathring{D}_{\boldsymbol{e}_\alpha} A) + \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha} A)$$

and then we define:

$$\begin{aligned}
\boldsymbol{\partial}\lrcorner A &= \theta^\alpha \lrcorner (\mathring{D}_{\boldsymbol{e}_\alpha} A), \\
\boldsymbol{\partial}\wedge A &= \theta^\alpha \wedge (\mathring{D}_{\boldsymbol{e}_\alpha} A),
\end{aligned}$$

in order to have:

$$\boldsymbol{\partial} = \boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge. \qquad (4.149)$$

Remark 4.117 Note moreover that for $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ we can also write

$$\boldsymbol{\partial}\lrcorner A = \boldsymbol{\partial} \cdot A. \qquad (4.150)$$

Exercise 4.118 Verify that the operators $\boldsymbol{\partial}\lrcorner$ and $\boldsymbol{\partial}\wedge$ satisfy the following identities:

$$\begin{aligned}
&(a) \quad \boldsymbol{\partial}\wedge(A \wedge B) = (\boldsymbol{\partial}\wedge A) \wedge B + \hat{A} \wedge (\boldsymbol{\partial}\wedge B), \\
&(b) \quad \boldsymbol{\partial}\lrcorner(A_r \lrcorner B_s) = (\boldsymbol{\partial}\wedge A_r)\lrcorner B_s + \hat{A}_r \lrcorner(\boldsymbol{\partial}\lrcorner B_s); \quad r + 1 \leq s, \\
&(c) \quad \boldsymbol{\partial}\lrcorner \star = (-1)^r \star \boldsymbol{\partial}\wedge\, ; \quad \star\, \boldsymbol{\partial}\lrcorner = (-1)^{r+1}\, \boldsymbol{\partial}\wedge\, .
\end{aligned} \qquad (4.151)$$

In addition to these identities, we have the important result [24, 32].

Proposition 4.119 *The standard Dirac derivative $\boldsymbol{\partial}$ is related to the exterior derivative d and to the Hodge codifferential δ by:*

$$\boldsymbol{\partial} = d - \delta, \tag{4.152}$$

that is, we have $\boldsymbol{\partial}\wedge = d$ and $\boldsymbol{\partial}\lrcorner = -\delta$.

Proof If f is a function, $\boldsymbol{\partial}\wedge f = \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha} f = \boldsymbol{e}_\alpha(f)\theta^\alpha = df$ and $\boldsymbol{\partial}\lrcorner f = \theta^\alpha \lrcorner \mathring{D}_{\boldsymbol{e}_\alpha} f = \theta^\alpha \cdot \mathring{D}_{\boldsymbol{e}_\alpha} f = 0$. For the 1-form fields θ^ρ of a moving frame on T^*M, we have $\boldsymbol{\partial}\wedge\theta^\rho = \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha}\theta^\rho = -\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha \wedge \theta^\beta = -\hat{\omega}^\rho_\beta \wedge \theta^\beta = d\theta^\rho$.

Now, for a r-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, we get, using Eq. (4.151a),

$$\begin{aligned}\boldsymbol{\partial}\wedge\omega &= \frac{1}{r!}(d\omega_{\alpha_1\ldots\alpha_r} \wedge \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} + \omega_{\alpha_1\ldots\alpha_r} d\theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} \\ &\quad + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}} \wedge d\theta^{\alpha_r}) \\ &= d\omega.\end{aligned}$$

Finally, using Eqs. (4.93c) and (4.151c), we get $\boldsymbol{\partial}\lrcorner\omega = -\delta\omega$.■

Note also that given an arbitrary coordinate moving frame $\{\theta^\mu = dx^\mu\}$ on M ($\mathbf{x}^\rho : U \to \mathbb{R}$, $U \subset M$, are coordinate functions), we have the following interesting relations:

$$\begin{aligned}&\text{(a)}\ \boldsymbol{\partial}\lrcorner\theta^\rho \equiv \boldsymbol{\partial}\cdot\theta^\rho = -\mathring{g}^{\alpha\beta}\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \sqrt{|(\det \mathring{\mathtt{g}})|}\partial_\sigma(\sqrt{|(\det \mathring{\mathtt{g}})^{-1}|}\mathring{g}^{\rho\sigma}) \\ &\text{(b)}\ \boldsymbol{\partial}\lrcorner\theta_\sigma \equiv \boldsymbol{\partial}\cdot\theta_\sigma = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\rho\sigma} = \sqrt{|(\det \mathring{\mathtt{g}})|}\partial_\sigma(\sqrt{|(\det \mathring{\mathtt{g}})^{-1}|}),\end{aligned} \tag{4.153}$$

where $\{\partial_\sigma \equiv \partial/\partial x^\sigma\}$ is the dual frame of $\{\theta^\mu\}$. Note that $\det \mathring{\boldsymbol{g}} = (\det \mathring{\mathtt{g}})^{-1}$.

Exercise 4.120 Verify that if $\alpha, \beta \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ then

$$\boldsymbol{\partial}(\alpha \cdot \beta) = (\alpha \cdot \boldsymbol{\partial})\beta + (\beta \cdot \boldsymbol{\partial})\alpha - \alpha\lrcorner(\boldsymbol{\partial}\wedge\beta) - \beta\lrcorner(\boldsymbol{\partial}\wedge\alpha). \tag{4.154}$$

4.8.4 Standard Dirac Commutator and Dirac Anticommutator

Definition 4.121 Given the 1-form fields $\alpha, \beta \in \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ and $\boldsymbol{\partial}$, the standard Dirac operator of the manifold, the operators $[\![,]\!]$ and $\{,\}$ given by

$$\begin{aligned}[\![\alpha, \beta]\!] &= (\alpha \cdot \boldsymbol{\partial})\beta - (\beta \cdot \boldsymbol{\partial})\alpha \\ \{\alpha, \beta\} &= (\alpha \cdot \boldsymbol{\partial})\beta + (\beta \cdot \boldsymbol{\partial})\alpha,\end{aligned} \tag{4.155}$$

are called, respectively, the Standard *Dirac commutator (or Lie bracket)* and the Standard *Dirac anticommutator* of the 1-form fields α and β.

We have the identities:

$$\begin{aligned} [\![\alpha,\beta]\!] &= \partial\!\!\!/\lrcorner(\alpha\wedge\beta) - \left[(\partial\!\!\!/\cdot\alpha)\wedge\beta - \alpha\wedge(\partial\!\!\!/\cdot\beta)\right] \\ \{\alpha,\beta\} &= \partial\!\!\!/\wedge(\alpha\cdot\beta) - \left[(\partial\!\!\!/\wedge\alpha)\llcorner\beta - \alpha\lrcorner(\partial\!\!\!/\wedge\beta)\right]. \end{aligned} \tag{4.156}$$

The algebraic meaning of these equations is clear: they state that the Dirac commutator and the Dirac anticommutator measure the amount by which the operators $\partial\!\!\!/\lrcorner = -\delta$ and $\partial\!\!\!/\wedge = d$ fail to satisfy the Leibniz's rule when applied, respectively, to the exterior and to the dot product of 1-form fields.

Now, let $\{\boldsymbol{e}_\sigma\}$ be an *arbitrary* moving frame on TM, $\{\theta^\sigma\}$ its dual frame on T^*M and $\{\theta_\alpha\}$ the reciprocal frame of $\{\theta^\sigma\}$. From Eqs. (4.155) we obtain, respectively:

$$\begin{aligned} [\![\theta_\alpha,\theta_\beta]\!] &= \mathring{D}_{\boldsymbol{e}_\alpha}\theta_\beta - \mathring{D}_{\boldsymbol{e}_\beta}\theta_\alpha \\ &= (\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} - \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha})\theta_\rho \\ &= c^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta_\rho, \end{aligned} \tag{4.157}$$

and

$$\begin{aligned} \{\theta_\alpha,\theta_\beta\} &= \mathring{D}_{\boldsymbol{e}_\alpha}\theta_\beta + \mathring{D}_{\boldsymbol{e}_\beta}\theta_\alpha, \\ &= (\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha})\theta_\rho \\ &= b^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta_\rho, \end{aligned} \tag{4.158}$$

where $\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the components of the Levi-Civita connection $\mathring{D}$ of $\mathring{g}$, $c^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the structure coefficients of the frame $\{\boldsymbol{e}_\sigma\}$ and where we introduce the notation $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha}$. The meaning of these coefficients will be discussed below.

Clearly, Eq. (4.157) states that the Dirac commutator is the analogous of the Lie bracket of vector fields. These operations have similar properties. In particular, the Dirac commutator satisfies the *Jacobi identity*:

$$[\![\alpha,[\![\beta,\omega]\!]]\!] + [\![\beta,[\![\omega,\alpha]\!]]\!] + [\![\omega,[\![\alpha,\beta]\!]]\!] = 0, \tag{4.159}$$

$\alpha,\beta,\omega \in \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathring{g})$. Therefore it gives to the cotangent bundle of M the structure of a *local* Lie algebra.

4.8.5 Geometrical Meanings of the Commutator and Anticommutator

The geometrical meanings of the Dirac commutator and the Dirac anticommutator are easily discovered from Eqs. (4.157) and (4.158). Indeed, Eq. (4.157) means that

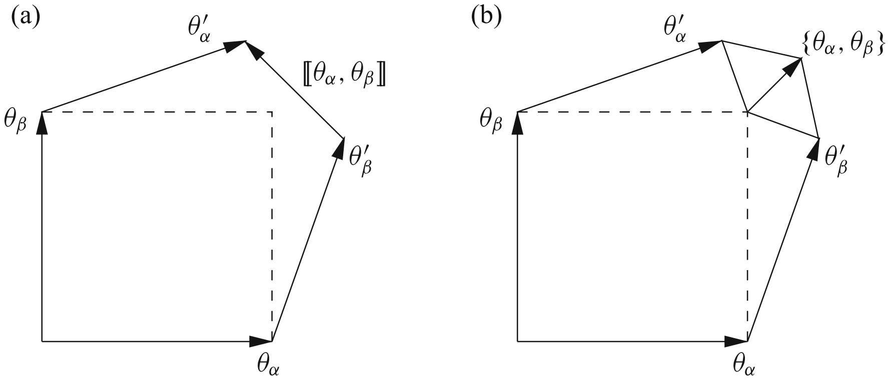

Fig. 4.4 Geometrical interpretation of the: (**a**) Standard commutator $[\![\theta_\alpha, \theta_\beta]\!]$ and (**b**) Standard anticommutator $\{\theta_\alpha, \theta_\beta\}$

the Dirac commutator measures the amount by which the vector fields $\boldsymbol{e}_a = \mathring{\mathtt{g}}(\theta_\alpha,)$ and $\boldsymbol{e}_b = \mathring{\mathtt{g}}(\theta_\beta,)$ and their infinitesimal lifts ($\boldsymbol{e}'_a = \mathring{\mathtt{g}}(\theta'_\alpha,)$, $\boldsymbol{e}'_\beta = \mathring{\mathtt{g}}((\theta'_\beta,))$ along their integral lines fail to form a parallelogram. By its turn, Eq. (4.158) means that the Dirac anticommutator measures the rate of deformation of the frame $\{\theta_\alpha\}$, i.e., $\{\theta_\alpha, \theta_\alpha\}$ gives the rate of dilation of the vector field $\mathring{\mathtt{g}}(\theta_\alpha,)$ under dislocations along its own integral lines, while $\{\theta_\alpha, \theta_\beta\}$, $\alpha \neq \beta$, gives the rate of variation of the angle between $\mathring{\mathtt{g}}(\theta_\alpha,)$ and $\mathring{\mathtt{g}}(\theta_\beta,)$ under dislocations in the direction of each other (Fig. 4.4).

We state now another interesting result:

Proposition 4.122 *The coefficients* $b^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ *of the Dirac anticommutator of a moving frame* $\{\theta_\alpha\}$ *are given by:*

$$b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = -(\pounds_{\boldsymbol{e}^\rho}\boldsymbol{g})_{\alpha\beta}, \tag{4.160}$$

where $\pounds_{\boldsymbol{e}^\rho}$ *denotes the Lie derivative in the direction of the vector field* $\boldsymbol{e}^\rho$ *and* $\{\boldsymbol{e}^\rho\}$ *is the dual frame of* $\{\theta_\alpha\}$.

Proof The coefficients $\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ of the Levi-Civita connection of $\boldsymbol{g}$ are given by: (e.g.,[3])

$$\begin{aligned}\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= \frac{1}{2}\mathring{g}^{\rho\sigma}\left[\boldsymbol{e}_\alpha(\mathring{g}_{\beta\sigma}) + \boldsymbol{e}_\beta(\mathring{g}_{\sigma\alpha}) - \boldsymbol{e}_\sigma(\mathring{g}_{\alpha\beta})\right] \\ &+ \frac{1}{2}\mathring{g}^{\rho\sigma}\left[\mathring{g}_{\mu\alpha}c^{\mu\cdot\cdot}_{\cdot\sigma\beta} + \mathring{g}_{\mu\beta}c^{\mu\cdot\cdot}_{\cdot\sigma\alpha} - \mathring{g}_{\mu\sigma}c^{\mu\cdot\cdot}_{\cdot\alpha\beta}\right].\end{aligned} \tag{4.161}$$

Hence,

$$b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{g}^{\rho\sigma}\left[\boldsymbol{e}_\beta(\mathring{g}_{\alpha\sigma}) + \boldsymbol{e}_\alpha(\mathring{g}_{\sigma\beta}) - \boldsymbol{e}_\sigma(\mathring{g}_{\beta\alpha}) - \mathring{g}_{\mu\alpha}c^{\mu\cdot\cdot}_{\cdot\beta\sigma} - \mathring{g}_{\mu\beta}c^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\right] \tag{4.162}$$

and the r.h.s. of Eq. (4.162) is just the negative of the components of the Lie derivative of the metric tensor in the direction of $\boldsymbol{e}^\rho = \mathring{g}^{\rho\sigma}\boldsymbol{e}_\sigma$.■

Killing Coefficients

In view of the result stated by Eq. (4.160), the attempt to find (if existing) a moving frame for which $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = 0$ is equivalent to solve, locally, the Killing equations for the manifold. Because of this we shall refer to these coefficients as the *Killing coefficients* of the frame. Of course, since the solutions of the Killing equations are restricted by the structure of the metric as well as by the topology of the manifold, it will not be possible, in the more general case, to find any moving frame for which these coefficients are all null.

4.8.6 Associated Dirac Operators

Besides the standard Dirac operator we have just analyzed, we can also introduce in the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ infinitely many other Dirac-like operators, one for each nondegenerate symmetric bilinear form field that can be defined on the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$.

Let $\boldsymbol{g} \in \sec T^0_2 M$ be an arbitrary nondegenerate *positive* symmetric bilinear form field on M. To $\boldsymbol{g}$ corresponds $\mathtt{g} \in \sec T^2_0 M$ as already introduced. We denoted by $g : \sec T^*M \to \sec T^*M$ the associated extensor field to $\mathtt{g}$ and by $h : \sec T^*M \to \sec T^*M$ the field of linear transformations which induces g, i.e., have:

$$\begin{aligned} \mathtt{g}(\alpha, \beta) &= \alpha \cdot g(\beta) = h(\alpha) \cdot h(\beta) \\ &= \mathtt{g}(h(\alpha), h(\beta)) \end{aligned} \tag{4.163}$$

for every $\alpha, \beta \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$.

We also denote by $\vee \equiv \underset{\boldsymbol{g}}{} : \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \times \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \to \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ the "Clifford product" induced on $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ by the bilinear form field $\mathtt{g}$ and by $\bullet \equiv \underset{\boldsymbol{g}}{\cdot} : \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \times \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \to \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ the "dot product" associated to the new Clifford product "$\vee$."

Definition 4.123 Let $\{\theta^\alpha\}$ be a moving frame on T^*M, dual to the moving frame $\{\boldsymbol{e}_\alpha\}$ on TM. We call Dirac operator *associated* to the bilinear form $\mathtt{g} \in \sec T_0^2 M$ the operator:

$$\overset{\vee}{\boldsymbol{\partial}} \equiv \boldsymbol{\partial}\vee = (\theta^\alpha \underset{\boldsymbol{g}}{} \overset{\circ}{D}_{\boldsymbol{e}_\alpha}) \equiv (\theta^\alpha \vee \overset{\circ}{D}_{\boldsymbol{e}_\alpha}). \tag{4.164}$$

We also define

$$\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} = \theta^\alpha \underset{\boldsymbol{g}}{\lrcorner} \overset{\circ}{D}_{\boldsymbol{e}_\alpha}, \tag{4.165}$$

where $\underset{\boldsymbol{g}}{\lrcorner}$ is the contracted product with respect to $\mathtt{g}$. Then,

$$\overset{\vee}{\boldsymbol{\partial}} = \overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} + \overset{\vee}{\boldsymbol{\partial}}\wedge = \overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} + \boldsymbol{\partial}\wedge, \tag{4.166}$$

because the exterior part of the operator $\overset{\vee}{\boldsymbol{\partial}}$ coincides with the exterior part of the operator $\boldsymbol{\partial}$.

Of course, the properties of the operator $\overset{\vee}{\boldsymbol{\partial}}$ differ from those of the standard Dirac operator $\boldsymbol{\partial}$. It is enough to state the properties of the operator $\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}$, which are obtained from the following proposition:

Proposition 4.124 *The operators* $\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}$ *and* $\boldsymbol{\partial}\lrcorner$ *are related by:*

$$\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}\omega = \boldsymbol{\partial}\lrcorner\check{\omega} + s\lrcorner\check{\omega}, \tag{4.167}$$

for every $\omega \in \sec \mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$, *where* $s = g^{\rho\sigma}\overset{\circ}{D}_\rho g_{\sigma\mu}\theta^\mu \in \sec T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$ *is called the dilation 1-form of the bilinear form* $\mathtt{g}$.

Proof Given a r-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} \in \sec \mathcal{C}\ell(M, \overset{\circ}{g})$, we have

$$\overset{\circ}{D}_{\boldsymbol{e}_\rho}\omega = \frac{1}{r!}(D_\rho\omega_{\alpha_1\ldots\alpha_r})\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r},$$

with

$$\overset{\circ}{D}_\rho\omega_{\alpha_1\ldots\alpha_r} = \boldsymbol{e}_\rho(\omega_{\alpha_1\ldots\alpha_r}) - \overset{\circ}{\Gamma}{}^{\mu}_{\rho\alpha_1}\omega_{\mu\alpha_2\ldots\alpha_r} - \cdots - \overset{\circ}{\Gamma}{}^{\mu}_{\rho\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\mu}. \tag{4.168}$$

Then,

$$\begin{aligned}\theta^{\rho} \underset{g}{\lrcorner} \mathring{D}_{e_\rho}\omega &= \frac{1}{r!} D_\rho \omega_{\alpha_1\ldots\alpha_r} \theta^{\rho} \underset{g}{\lrcorner} (\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}) \\ &= \frac{1}{r!} \mathring{D}_\rho \omega_{\alpha_1\ldots\alpha_r} (g^{\rho\alpha_1}\theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} + \cdots \\ &\quad + (-1)^{r+1} g^{\rho\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}}),\end{aligned}$$

or

$$\overset{\vee}{\boldsymbol{\partial}} \underset{g}{\lrcorner} \omega = \frac{1}{(r-1)!} g^{\rho\sigma} \mathring{D}_\rho \omega_{\sigma\alpha_2\ldots\alpha_r} \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r}. \tag{4.169}$$

Now, taking into account that

$$\begin{aligned} g^{\rho\sigma} \mathring{D}_\rho \omega_{\sigma\alpha_2\ldots\alpha_r} &= \mathring{D}_\rho (g^{\rho\sigma} \omega_{\sigma\alpha_2\ldots\alpha_r}) - (\mathring{D}_\rho g^{\rho\sigma}) \omega_{\sigma\alpha_2\ldots\alpha_r}, \\ g_{\sigma\mu} \mathring{D}_\rho g^{\rho\sigma} &= -g^{\rho\sigma} \mathring{D}_\rho g_{\sigma\mu}, \end{aligned}$$

and recalling also that $g^{\rho\sigma} = g^{\rho\mu} \mathring{g}^{\sigma}_{\mu}$, we conclude that

$$\begin{aligned}\overset{\vee}{\boldsymbol{\partial}} \underset{g}{\lrcorner} \omega &= \frac{1}{(r-1)!} \mathring{g}^{\rho\sigma} (\mathring{D}_\rho \mathring{g}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r}) \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} \\ &\quad + \frac{1}{(r-1)!} \mathring{g}^{\rho\sigma} (g^{\alpha\beta} \mathring{D}_\alpha g_{\beta\rho}) \mathring{g}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r} \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r}.\end{aligned}$$

Thus, writing $\check{\omega}_{\sigma\alpha_2\ldots\alpha_r} = \mathring{g}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r}$ and $s_\rho = g^{\alpha\beta} \mathring{D}_\alpha g_{\beta\rho}$, we finally obtain the Eq. (4.167).■

4.8.7 *The Dirac Operator in Riemann-Cartan-Weyl Spaces*

We now consider the structure $(M, \mathring{\boldsymbol{g}}, \nabla)$ where ∇ is an arbitrary linear connection. In this case, the notion of covariant derivative does not pass to the quotient bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ [4]. Despite this fact, it is still a well defined operation and in analogy with the earlier section, we can associate to it, acting on the sections of $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$, the operator:

$$\boldsymbol{\partial} = \theta^\alpha \nabla_{e_\alpha},$$

where $\{\theta^\alpha\}$ is a moving frame on T^*M, dual to the moving frame $\{e_\alpha\}$ on TM.

Definition 4.125 The operator $\boldsymbol{\partial}$ is called the *Dirac operator* (or *Dirac derivative, or sometimes gradient*).

We also define:

$$\begin{aligned} \boldsymbol{\partial} \lrcorner A &= \theta^{\alpha} \lrcorner (\nabla_{\boldsymbol{e}_{\alpha}} A), \\ \boldsymbol{\partial} \wedge A &= \theta^{\alpha} \wedge (\nabla_{\boldsymbol{e}_{\alpha}} A), \end{aligned} \tag{4.170}$$

for every $A \in \sec \mathcal{C}\ell(M, \mathring{g})$, so that:

$$\boldsymbol{\partial} = \boldsymbol{\partial} \lrcorner + \boldsymbol{\partial} \wedge . \tag{4.171}$$

The operator $\boldsymbol{\partial} \wedge$ satisfies, for every $A, B \in \sec \mathcal{C}\ell(M, \mathring{g})$:

$$\boldsymbol{\partial} \wedge (A \wedge B) = (\boldsymbol{\partial} \wedge A) \wedge B + \hat{A} \wedge (\boldsymbol{\partial} \wedge B), \tag{4.172}$$

what generalizes Eq. (4.151a). By its turn, Eq.(4.151c) is generalized according to the following proposition:

Proposition 4.126 *Let* $\mathbf{Q}^{\rho}$ *be the nonmetricity 2-forms associated with the connection* ∇ *in an arbitrary moving frame* $\{\theta^{\rho}\}$ *and* $\nabla_{\boldsymbol{e}_{\alpha}} \boldsymbol{e}_{\beta} = L_{\cdot\alpha\beta}^{\rho\cdot\cdot} \boldsymbol{e}_{\rho}$. *Then we have, for homogeneous multiforms,*

$$\begin{aligned} &(a)\ \ (-1)^{r} \star^{-1} \boldsymbol{\partial} \lrcorner \star &&= \boldsymbol{\partial} \wedge + \mathbf{Q}^{\rho} \wedge \mathbf{i}_{\rho}, \\ &(b)\ \ (-1)^{r+1} \star^{-1} \boldsymbol{\partial} \wedge \star &&= \boldsymbol{\partial} \lrcorner - \mathbf{Q}^{\rho} \lrcorner \mathbf{j}_{\rho}, \end{aligned} \tag{4.173}$$

where $\mathbf{i}_{\rho} A = \theta_{\rho} \lrcorner A$ *and* $\mathbf{j}_{\rho} A = \theta_{\rho} \wedge A$, *for every* $A \in \sec \mathcal{C}\ell(M, \mathring{g})$.

Proof Let $\omega = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{g})$ be a r-form field on M. We have $(\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge * \omega = ((\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \cdot \omega) \tau_{\mathring{g}} = \omega_{\beta_1 \ldots \beta_r} \tau_{\mathring{g}}$ and it follows that $\nabla_{\boldsymbol{e}_{\alpha}} ((\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge * \omega) = \boldsymbol{e}_{\alpha} (\omega_{\beta_1 \ldots \beta_r}) \tau_{\boldsymbol{g}}$. But on the other hand, we also have

$$\begin{aligned} \nabla_{\boldsymbol{e}_{\alpha}} (\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge \star \omega = {} & \theta_{\beta_1} \wedge v \wedge \theta_{\beta_r} \wedge \nabla_{\boldsymbol{e}_{\alpha}} \star \omega \\ & + (L_{\cdot\sigma\beta_1}^{\rho\cdot\cdot} \omega_{\rho\beta_2 \ldots \beta_r} + \cdots + L_{\cdot\sigma\beta_r}^{\rho\cdot\cdot} \omega_{\beta_1 \ldots \beta_{r-1}\rho}) \tau_{\mathring{g}} \\ & - (Q_{\cdot\sigma\beta_1}^{\rho\cdot\cdot} \omega_{\rho\beta_2 \ldots \beta_r} + \cdots + Q_{\cdot\sigma\beta_r}^{\rho\cdot\cdot} \omega_{\beta_1 \ldots \beta_{r-1}\rho}) \tau_{\mathring{g}} \end{aligned}$$

and therefore we get, after some algebraic manipulation:

$$\nabla_{\boldsymbol{e}_{\alpha}} \star \omega = \star \nabla_{\boldsymbol{e}_{\alpha}} \omega + Q_{\sigma\mu\nu} \star (\theta^{\mu} \wedge (\theta^{\nu} \lrcorner \omega)), \tag{4.174}$$

from which Eqs. (4.173) follow immediately. ■

Taking into account the result stated in the above proposition and the definition of the Hodge codifferential (Eq. (4.91)), we are motivated to introduce in the Clifford bundle the *Dirac coderivative* operator, given, for homogeneous multiforms, by:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \star^{-1} \boldsymbol{\partial} \star \ . \tag{4.175}$$

Of course, we have:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \star^{-1} \boldsymbol{\partial} \lrcorner \star \ + (-1)^r \star^{-1} \boldsymbol{\partial} \wedge \star \tag{4.176}$$

and we can, then, define:

$$\begin{aligned} \overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner &:= (-1)^r \star^{-1} \boldsymbol{\partial} \wedge \star = -\boldsymbol{\partial} \lrcorner + \ \mathbf{Q}^{\rho} \lrcorner \mathbf{j}_{\rho} \\ \overset{\blacklozenge}{\boldsymbol{\partial}} \wedge &:= (-1)^r \star^{-1} \boldsymbol{\partial} \lrcorner \star \boldsymbol{\partial} \wedge + \mathbf{Q}^{\rho} \wedge \mathbf{i}_{\rho}, \end{aligned} \tag{4.177}$$

so that:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = \overset{\blacklozenge}{\boldsymbol{\partial}} \wedge + \overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner \ . \tag{4.178}$$

The following identities are trivially established:

$$\begin{aligned} \boldsymbol{\partial} &= (-1)^{r+1} \star^{-1} \overset{\blacklozenge}{\boldsymbol{\partial}} \star \\ \star \boldsymbol{\partial} &= (-1)^{r+1} \overset{\blacklozenge}{\boldsymbol{\partial}} \star; \quad \star \overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \boldsymbol{\partial} \star \\ \boldsymbol{\partial} \overset{\blacklozenge}{\boldsymbol{\partial}} \star &= \star \overset{\blacklozenge}{\boldsymbol{\partial}} \boldsymbol{\partial}; \quad \star \boldsymbol{\partial} \overset{\blacklozenge}{\boldsymbol{\partial}} = \overset{\blacklozenge}{\boldsymbol{\partial}} \boldsymbol{\partial} \star \\ \star \overset{\blacklozenge}{\boldsymbol{\partial}}{}^2 &= -(\overset{\blacklozenge}{\boldsymbol{\partial}})^2 \star; \quad \star (\overset{\blacklozenge}{\boldsymbol{\partial}})^2 = -\boldsymbol{\partial}^2 \star \ . \end{aligned} \tag{4.179}$$

In addition, we note that the Dirac coderivative permit us to generalize Eq. (4.151b) in a very elegant way. In fact, in consequence of Proposition 4.126 we have:

Corollary 4.127 *For* $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec C\ell(M, \mathring{g})$, $B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec C\ell(M, \mathring{g})$, *with* $r + 1 \leq s$, *it holds:*

$$\boldsymbol{\partial} \lrcorner (A_r \lrcorner B_s) = (\overset{\blacklozenge}{\boldsymbol{\partial}} \wedge A_r) \lrcorner B_s + (-1)^r A_r \lrcorner (\overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner B_s). \tag{4.180}$$

Proof Given a 1-form field $\alpha \in \bigwedge^1 T^*M$ and a s-form field $\omega \in \sec \bigwedge^s T^*M$, we have, from Eq. (4.174), that $\nabla_{\boldsymbol{e}_\sigma} \star (\alpha \lrcorner \omega) = \star \nabla_{\boldsymbol{e}_\sigma} (\alpha \lrcorner \omega + Q_{\sigma\mu\nu} \star [\theta^\mu \wedge (\theta^\nu \lrcorner (\alpha \lrcorner \omega))]$.

We also have that

$$\begin{aligned} \nabla_{\boldsymbol{e}_\sigma} \star (\alpha \lrcorner \omega) &= (-1)^{s+1} \nabla_{\boldsymbol{e}_\sigma} (\alpha \wedge \star \omega) \\ &= \star [(\nabla_{\boldsymbol{e}_\sigma} \alpha) \lrcorner \omega + \alpha \lrcorner (\nabla_{\boldsymbol{e}_\sigma} \omega + Q_{\sigma\mu\nu} (\theta^\mu \wedge (\theta^\nu \lrcorner \omega))], \end{aligned}$$

where we have used Eq. (4.174) once again. It follows that:

$$\nabla_{e_\sigma}(\alpha\lrcorner\omega) = (\nabla_{e_\sigma}\alpha)\lrcorner\omega + \alpha\lrcorner(\nabla_{e_\sigma}\omega) + Q_{\sigma\mu\nu}\alpha^\mu\theta^\nu\lrcorner\omega. \tag{4.181}$$

Then, recalling that $(\alpha_1\wedge\ldots\wedge\alpha_r)\lrcorner\omega = \alpha_1\lrcorner\ldots\lrcorner\alpha_r\lrcorner\omega$, with $\alpha_1,\ldots,\alpha_r \in \sec T^*M, \omega \in \sec\bigwedge^s T^*M, r \le s+1$, and applying Eq. (4.181) successively in this expression, we get Eq. (4.180).■

Another very important consequence of Proposition 4.126 states the relation between the operators $\boldsymbol{\partial}$ and $\oint$:

Proposition 4.128 *Let* $\Phi^\rho = \Theta^\rho - \mathbf{Q}^\rho$, *where* Θ^ρ *and* $\mathbf{Q}^\rho$ *denote, respectively, the torsion and the nonmetricity 2-forms of the connection* ∇ *in an arbitrary moving frame* $\{\theta^\alpha\}$. *Then:*

$$\begin{aligned} &(a)\quad \boldsymbol{\partial}\wedge = \oint\wedge - \Theta^\rho\wedge\mathbf{i}_\rho\ ,\\ &(b)\quad \boldsymbol{\partial}\lrcorner\ \ = \oint\lrcorner - \Phi^\rho\lrcorner\mathbf{j}_\rho\quad . \end{aligned} \tag{4.182}$$

Proof If f is a function, $\boldsymbol{\partial}\wedge f = \theta^\alpha\wedge\nabla_{e_\alpha}f = \boldsymbol{e}_\alpha(f)\theta^\alpha = df$ and $\boldsymbol{\partial}\lrcorner f = \theta^\alpha\lrcorner\nabla_{e_\alpha}f = 0$. For the 1-form field θ^ρ of a moving frame on T^*M, we have $\boldsymbol{\partial}\wedge\theta^\rho = \theta^\alpha\wedge\nabla_{e_\alpha}\theta^\rho = -L^\rho_{\alpha\beta}\theta^\alpha\wedge\theta^\beta = -\omega^\rho_\beta\wedge\theta^\beta = d\theta^\rho - \Theta^\rho$.

Now, for a r-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, we get

$$\begin{aligned} \boldsymbol{\partial}\wedge\omega &= \frac{1}{r!}(d\omega_{\alpha_1\ldots\alpha_r}\wedge\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r} + \omega_{\alpha_1\ldots\alpha_r}d\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_r}\\ &\quad+\cdots+(-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_{r-1}}\wedge d\theta^{\alpha_r})\\ &\quad-\frac{1}{r!}(\omega_{\alpha_1\ldots\alpha_r}\Theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_r}+\cdots\\ &\quad+(-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_{r-1}}\wedge\Theta^{\alpha_r})\\ &= d\omega - \frac{1}{r!}\Theta^\rho\wedge(\omega_{\rho\alpha_2\ldots\alpha_r}\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_r}+\cdots\\ &\quad+(-1)^{r+1}\omega_{\alpha_1\ldots\alpha_{r-1}\rho}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_{r-1}})\\ &= d\omega - \Theta^\rho\wedge\mathbf{i}_\rho\omega \end{aligned}$$

and Eq. (4.182a) is proved.

Finally, from Eqs. (4.173b) and (4.182a) we obtain

$$\begin{aligned} \boldsymbol{\partial}\wedge\star\omega &= (-1)^{r+1}\star\boldsymbol{\partial}\lrcorner\omega - (-1)^{r+1}\star\mathbf{Q}^\rho\lrcorner\mathbf{j}_\rho\omega\\ &= \boldsymbol{\partial}\wedge\star\omega - \Theta^\rho\wedge\star\omega\\ &= (-1)^{r+1}\star\oint\lrcorner\omega - (-1)^{r+1}\star\Theta^\rho\lrcorner\mathbf{j}_\rho\omega. \end{aligned}$$

Therefore, $\boldsymbol{\partial}\lrcorner\omega = \oint\lrcorner\omega - \Phi^\rho\lrcorner\mathbf{j}_\rho\omega$, and Eq. (4.182b) is proved.■

From Eqs. (4.182) we obtain the expressions of $\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner$ and $\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge$ in terms of $\boldsymbol{\eth}\lrcorner$ and $\boldsymbol{\eth}\wedge$:

$$\begin{aligned}\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner &= -\boldsymbol{\eth}\lrcorner + \Theta^{\rho}\lrcorner\mathbf{j}_{\rho}\\ \overset{\blacklozenge}{\boldsymbol{\partial}}\wedge &= \boldsymbol{\eth}\wedge - \Phi^{\rho}\wedge\mathbf{i}_{\rho}.\end{aligned} \tag{4.183}$$

Obviously, the Dirac coderivative associated to the standard Dirac operator is given by:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = \boldsymbol{\eth}\wedge - \boldsymbol{\eth}\lrcorner = d + \delta. \tag{4.184}$$

We observe finally that we can still introduce another Dirac operator, obtained by combining the arbitrary affine connection ∇ with the algebraic structure induced by the generic bilinear form field $\mathtt{g} \in \sec T_0^2 M$. With respect to an arbitrary moving frame $\{\theta^{\alpha}\}$ on T^*M, this operator has the expression:

$$\boldsymbol{\partial}\vee = \theta^{\alpha}\vee\nabla_{\boldsymbol{e}_{\sigma}}. \tag{4.185}$$

It is clear that in the particular case where $\nabla = D$ is the Levi-Civita connection of $\boldsymbol{g}$, the operator $\boldsymbol{\partial}$—which in this case *is* the standard Dirac operator associated to $\boldsymbol{g}$—will satisfy the properties of Sect. 4.8.3, with the usual Clifford product exchanged by the new Clifford product "$\vee$." In addition, for a more general connection we can apply the results of Sect. 4.8.6, once again with all the occurrences of $\mathring{\mathtt{g}}$ replaced by $\mathtt{g}$. (In particular, the standard Dirac operator associated to $\mathring{\mathtt{g}}$ is replaced by that associated with $\mathtt{g}$.)

4.8.8 Torsion, Strain, Shear and Dilation of a Connection

In analogy with the introduction of the Dirac commutator and the Dirac anticommutator, let us define the operations:

Definition 4.129 Given $\alpha, \beta \in \sec \bigwedge^1 T^*M$ the Dirac commutator and anticommutator of these 1-form fields are

$$\begin{aligned}&(a)\ [\![\alpha,\beta]\!] = (\alpha\cdot\boldsymbol{\partial})\beta - (\beta\cdot\boldsymbol{\partial})\alpha - [\![\alpha,\beta]\!]\\ &(b)\ \{\!\{\alpha,\beta\}\!\} = (\alpha\cdot\boldsymbol{\partial})\beta + (\beta\cdot\boldsymbol{\partial})\alpha - \{\alpha,\beta\}.\end{aligned} \tag{4.186}$$

We have subtracted the Dirac commutator and the Dirac anticommutator in the r.h.s. of these expressions in order to have objects which are independent of the structure of the fields on which they are applied.

If $\{\theta_\alpha\}$ is the reciprocal of an arbitrary moving frame $\{\theta^\alpha\}$ on T^*M, we get, from Eq. (4.186a):

$$[\![\theta_\alpha, \theta_\beta]\!] = (T^{\rho\cdot\cdot}_{\cdot\alpha\beta} - Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]})\theta_\rho, \tag{4.187}$$

where $T^{\rho}_{\alpha\beta}$ are the components of the usual torsion tensor (Eq. (4.107)). Note from this last equation that the operation defined through Eq. (4.186a) does not satisfy the Jacobi identity. Indeed we have:

$$\sum_{[\alpha\beta\sigma]} [\![[\![\theta_\alpha, \theta_\beta]\!], \theta_\sigma]\!] = \sum_{[\alpha\beta\sigma]} (T^{\rho\cdot\cdot}_{\cdot\alpha\mu} - Q^{\rho\cdot\cdot}_{\cdot[\alpha\mu]})(T^{\mu\cdot\cdot}_{\cdot\beta\sigma} - Q^{\mu\cdot\cdot}_{\cdot[\beta\sigma]})\theta_\rho, \tag{4.188}$$

where the summation in this equation is to be performed on the cyclic permutations of the indices α, β and σ.

From Eq. (4.186b), we get:

$$\{\!\{\theta_\alpha, \theta_\beta\}\!\} = (S^{\rho\cdot\cdot}_{\cdot\alpha\beta} - Q^{\rho\cdot\cdot}_{\cdot(\alpha\beta)})\theta_\rho,$$

where $Q^{\rho\cdot\cdot}_{\cdot(\alpha\beta)} := g^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\alpha\sigma})$ and we have written:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta}. \tag{4.189}$$

It can be easily shown that the object having these components is also a tensor. Using the nomenclature of the theories of continuum media [39, 42] we will call it the *strain tensor* of the connection. Note that it can be further decomposed into:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \breve{S}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{2}{n} s^{\rho} \mathring{g}_{\alpha\beta} \tag{4.190}$$

where $\breve{S}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ is its traceless part, which will be called the *shear* of the connection, and

$$s^{\rho} = \frac{1}{2} \mathring{g}^{\mu\nu} S^{\rho\cdot\cdot}_{\cdot\mu\nu} \tag{4.191}$$

is its trace part, which will be called the *dilation* of the connection.

It is trivially established that:

$$L^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{1}{2} T^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{1}{2} S^{\rho\cdot\cdot}_{\cdot\alpha\beta}. \tag{4.192}$$

where $\overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2}(b^{\rho\cdot\cdot}_{\cdot\alpha\beta} + c^{\rho\cdot\cdot}_{\cdot\alpha\beta})$ are the components of the Levi-Civita connection of $\overset{\circ}{\boldsymbol{g}}$.[25]

Equation (4.192) can be used to relate the covariant derivatives with respect to the connections $\overset{\circ}{D}$ and ∇ of any tensor field on the manifold. In particular, recalling that $\overset{\circ}{D}_\alpha \overset{\circ}{g}_{\beta\sigma} = \boldsymbol{e}_\alpha(\overset{\circ}{g}_{\beta\sigma}) - \overset{\circ}{g}_{\mu\sigma}\overset{\circ}{\Gamma}{}^{\mu\cdot\cdot}_{\cdot\alpha\beta} - \overset{\circ}{g}_{\beta\mu}\overset{\circ}{\Gamma}{}^{\mu\cdot\cdot}_{\cdot\alpha\sigma} = 0$, we get the expression of the nonmetricity tensor of ∇ in terms of the torsion and the strain, namely,

$$Q_{\alpha\beta\sigma} = \frac{1}{2}(\overset{\circ}{g}_{\mu\sigma}T^{\mu\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{g}_{\beta\mu}T^{\mu\cdot\cdot}_{\cdot\alpha\sigma}) + \frac{1}{2}(\overset{\circ}{g}_{\mu\sigma}S^{\mu\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{g}_{\beta\mu}S^{\mu\cdot\cdot}_{\cdot\alpha\sigma}). \tag{4.193}$$

Equation (4.193) can be inverted to yield the expression of the strain in terms of the torsion and the nonmetricity. We get:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \overset{\circ}{g}{}^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) - \overset{\circ}{g}{}^{\rho\sigma}(\overset{\circ}{g}_{\beta\mu}T^{\mu\cdot\cdot}_{\cdot\alpha\sigma} + \overset{\circ}{g}_{\sigma\mu}T^{\mu\cdot\cdot}_{\cdot\beta\alpha}). \tag{4.194}$$

From Eqs. (4.193) and (4.194) it is clear that nonmetricity and strain can be used interchangeably in the description of the geometry of a Riemann-Cartan-Weyl space. In particular, we have the relation:

$$Q_{\alpha\beta\sigma} + Q_{\sigma\alpha\beta} + Q_{\beta\sigma\alpha} = S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha}, \tag{4.195}$$

where $S_{\sigma\alpha\beta} = \overset{\circ}{g}_{\rho\sigma}S^{\rho\cdot\cdot}_{\cdot\alpha\beta}$. Thus, the strain tensor of a Weyl geometry satisfies the relation:

$$S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha} = 0.$$

In order to simplify our next equations, let us introduce the notation:

$$K^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} - \overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2}(T^{\rho\cdot\cdot}_{\cdot\alpha\beta} + S^{\rho\cdot\cdot}_{\cdot\alpha\beta}). \tag{4.196}$$

From Eq. (4.194) it follows that:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= -\frac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}(\nabla_\alpha \overset{\circ}{g}_{\beta\sigma} + \nabla_\beta \overset{\circ}{g}_{\sigma\alpha} - \nabla_\sigma \overset{\circ}{g}_{\alpha\beta}) \\ &\quad - \frac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}(\overset{\circ}{g}_{\mu\alpha}T^{\mu\cdot\cdot}_{\cdot\sigma\beta} + \overset{\circ}{g}_{\mu\beta}T^{\mu\cdot\cdot}_{\cdot\sigma\alpha} - \overset{\circ}{g}_{\mu\sigma}T^{\mu\cdot}_{\cdot\alpha\beta}), \end{aligned} \tag{4.197}$$

[25] We note that the possibility of decomposing the connection coefficients into rotation (torsion), shear and dilation has already been suggested in a Physics paper by Baekler et al. [1] but in their work they do not arrive at the identification of a tensor-like quantity associated to these last two objects. The idea of the decompositions already appeared in [40].

where we have used that $Q_{\alpha\beta\sigma} = -\nabla_\alpha \mathring{g}_{\beta\sigma}$. Note the similarity of this equation with that which gives the coefficients of a Riemannian connection (Eq. (4.161)). Note also that for $\nabla\mathring{\boldsymbol{g}} = 0$, $K^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ is the so-called *contorsion tensor*.[26]

Returning to Eq. (4.192), we obtain now the relation between the curvature tensor $R^{\rho\cdots}_{\cdot\mu\alpha\beta}$ associated with the connection ∇ and the Riemann curvature tensor $\mathring{R}^{\rho\cdots}_{\cdot\mu\alpha\beta}$ of the Levi-Civita connection D associated with the metric $\mathring{\boldsymbol{g}}$. We get, by a simple calculation:

$$R^{\rho\cdots}_{\cdot\mu\alpha\beta} = \mathring{R}^{\rho\cdots}_{\cdot\mu\alpha\beta} + J^{\rho\cdots}_{\cdot\mu[\alpha\beta]}, \tag{4.198}$$

where:

$$J^{\rho\cdots}_{\cdot\mu\alpha\beta} = \mathring{D}_\alpha K^{\rho\cdot\cdot}_{\cdot\beta\mu} - K^{\rho\cdot\cdot}_{\cdot\beta\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu} = \nabla_\alpha K^{\rho\cdot\cdot}_{\cdot\beta\mu} - K^{\rho\cdot\cdot}_{\cdot\alpha\sigma} K^{\sigma\cdot}_{\cdot\beta\mu} + K^{\sigma\cdot\cdot}_{\cdot\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\sigma\mu}\ . \tag{4.199}$$

Multiplying both sides of Eq. (4.198) by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ we get:

$$\mathcal{R}^{\rho\cdot}_{\cdot\mu} = \mathring{\mathcal{R}}^{\rho\cdot}_{\cdot\mu} + \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \tag{4.200}$$

where we have written:

$$\mathfrak{J}^{\rho\cdot}_{\cdot\mu} = \frac{1}{2} J^{\rho\cdots}_{\cdot\mu[\alpha\beta]} \theta^\alpha \wedge \theta^\beta. \tag{4.201}$$

From Eq. (4.198) we also get the relation between the Ricci tensors of the connections ∇ and $\mathring{D}$. We define the *Ricci tensor* by

$$\begin{aligned} Ricci &= R_{\mu\alpha} dx^\mu \otimes dx^\alpha, \\ R_{\mu\alpha} &:= R^{\rho\cdots}_{\cdot\mu\alpha\rho}\ . \end{aligned} \tag{4.202}$$

Then, we have

$$R_{\mu\alpha} = \mathring{R}_{\mu\alpha} + J_{\mu\alpha}, \tag{4.203}$$

with

$$\begin{aligned} J_{\mu\alpha} &= \mathring{D}_\alpha K^{\rho\cdot\cdot}_{\cdot\rho\mu} - \mathring{D}_\rho K^{\rho\cdot\cdot}_{\cdot\alpha\mu} + K^{\rho\cdot\cdot}_{\cdot\alpha\sigma} K^{\sigma\cdot\cdot}_{\cdot\rho\mu} - K^{\rho\cdot\cdot}_{\cdot\rho\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu} \\ &= \nabla_\alpha K^{\rho\cdot\cdot}_{\cdot\rho\mu} - \nabla_\rho K^{\rho\cdot\cdot}_{\cdot\alpha\mu} - K^{\rho\cdot\cdot}_{\cdot\sigma\alpha} K^{\sigma\cdot\cdot}_{\cdot\rho\mu} + K^{\rho\cdot\cdot}_{\cdot\rho\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu}\ . \end{aligned} \tag{4.204}$$

[26] Equations (4.196) and (4.197) have appeared in the literature in two different contexts: with $\nabla\mathring{\boldsymbol{g}} = 0$, they have been used in the formulations of the theory of the spinor fields in Riemann-Cartan spaces [15, 46] and with $\Theta[\nabla] = 0$ they have been used in the formulations of the gravitational theory in a space endowed with a background metric [8, 13, 23, 35, 36].

Observe that since the connection ∇ is arbitrary, its Ricci tensor will be *not* be symmetric in general. Then, since the Ricci tensor $\overset{\circ}{R}_{\mu\alpha}$ of $\overset{\circ}{D}$ is necessarily symmetric, we can split Eq. (4.203) into:

$$\begin{aligned} R_{[\mu\alpha]} &= J_{[\mu\alpha]}, \\ R_{(\mu\alpha)} &= \overset{\circ}{R}_{\mu\alpha} + J_{(\mu\alpha)}. \end{aligned} \tag{4.205}$$

Now we specialize the above results for the case where the general connection $\nabla = D$ is the Levi-Civita connection of a bilinear form field $\boldsymbol{g} \in \sec T_2^0 M$, i.e., $\boldsymbol{\Theta} = 0$ and $\nabla \boldsymbol{g} = 0$. The results that we show next generalize and clear up those found in the formulations of the gravitational theory in a background metric space [13, 23, 35, 36].

First of all, note that the connection $\overset{\circ}{D}$ plays with respect to the tensor field $\overset{\circ}{\boldsymbol{g}}$ a role analogous to that played by the connection ∇ with respect to the metric tensor $\boldsymbol{g}$ and in consequence we shall have similar equations relating these two pairs of objects. In particular, the strain of $\overset{\circ}{D}$ with respect to $\boldsymbol{g}$ equals the negative of the strain of ∇ with respect to $\overset{\circ}{\boldsymbol{g}}$, since we have:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = -(\overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\beta\alpha} - d^{\rho\cdot\cdot}_{\cdot\alpha\beta}) = S^{\rho\cdot\cdot}_{\cdot\beta\alpha},$$

where $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\beta\alpha}$ and $d^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha}$ denote the Killing coefficients of the frame with respect to the tensors $\overset{\circ}{\boldsymbol{g}}$ and $\boldsymbol{g}$ respectively. Furthermore, in view of Eq. (4.197), we can write $K^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2} S^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ as:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= -\frac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}(\nabla_\alpha \overset{\circ}{g}_{\beta\sigma} + \nabla_\beta \overset{\circ}{g}_{\alpha\sigma} - \nabla_\sigma \overset{\circ}{g}_{\alpha\beta}) \\ &= \frac{1}{2} g^{\rho\sigma}(\overset{\circ}{D}_\alpha g_{\beta\sigma} + \overset{\circ}{D}_\beta g_{\alpha\sigma} - \overset{\circ}{D}_\sigma g_{\alpha\beta}). \end{aligned} \tag{4.206}$$

Introducing the notation:

$$\varkappa = \sqrt{\frac{\det \boldsymbol{g}}{\det \overset{\circ}{\boldsymbol{g}}}}, \tag{4.207}$$

we have the following relations:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\rho\sigma} &= -\frac{1}{2}\overset{\circ}{g}{}^{\alpha\beta}\nabla_\sigma \overset{\circ}{g}_{\alpha\beta} = \frac{1}{2} g^{\alpha\beta}\overset{\circ}{D}_\sigma g_{\alpha\beta} = \frac{1}{\varkappa} e_\sigma(\varkappa), \\ g^{\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= -\frac{1}{\varkappa}\overset{\circ}{D}_\sigma(\varkappa g^{\rho\sigma}), \\ \overset{\circ}{g}{}^{\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= \frac{1}{\varkappa^{-1}}\nabla_\sigma(\varkappa^{-1}\overset{\circ}{g}{}^{\rho\sigma}). \end{aligned} \tag{4.208}$$

Another important consequence of the assumption that ∇ is a Levi-Civita connection is that its Ricci tensor will then be symmetric. In view of Eqs. (4.205), this will be achieved, if and only if, the following equivalent conditions hold:

$$\begin{aligned} \mathring{D}_\alpha K^{\rho\cdot\cdot}_{\rho\beta} &= \mathring{D}_\beta K^{\rho\cdot\cdot}_{\cdot\rho\alpha}, \\ \nabla_\alpha K^{\rho}_{\cdot\rho\beta} &= \nabla_\beta K^{\rho\cdot}_{\cdot\rho\alpha}. \end{aligned} \tag{4.209}$$

4.8.9 Structure Equations II

With the results stated above, we can write down the structure equations of the RCWS structure defined by the connection ∇ in terms of the Riemannian structure defined by the metric $\mathring{\boldsymbol{g}}$. For this, let us write Eq. (4.192) in the form:

$$\omega^{\rho\cdot}_{\cdot\beta} = \hat{\omega}^{\rho\cdot}_{\cdot\beta} + w^{\rho\cdot}_{\cdot\beta} = \hat{\omega}^{\rho\cdot}_{\cdot\beta} + \tau^{\rho\cdot}_{\cdot\beta} + \sigma^{\rho\cdot}_{\cdot\beta}, \tag{4.210}$$

with $\omega^{\rho\cdot}_{\cdot\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha$, $\hat{\omega}^{\rho\cdot}_{\cdot\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha$, $w^{\rho\cdot}_{\cdot\beta} = K^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha$, $\tau^{\rho\cdot}_{\cdot\beta} = \frac{1}{2}T^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha$ and $\sigma^{\rho\cdot}_{\cdot\beta} = \frac{1}{2}S^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha$. Then, recalling Eq. (4.200) and the structure equations for both the RCWS and the Riemannian structures, we easily conclude that:

$$\begin{aligned} w^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta &= \Theta^\rho, \\ w^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta &= -\Phi_\mu, \\ \mathring{\mathbf{D}} w^{\rho\cdot}_{\cdot\mu} + w^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta\cdot}_{\cdot\mu} &= \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \end{aligned} \tag{4.211}$$

where $\mathring{\mathbf{D}}$ is the exterior covariant differential (of indexed form fields) associated to the Levi-Civita connection $\mathring{D}$ of $\mathring{\boldsymbol{g}}$. The third of these equations can also be written as:

$$\mathbf{D} w^{\rho\cdot}_{\cdot\mu} - w^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta\cdot}_{\cdot\mu} = \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \tag{4.212}$$

where $\mathbf{D}$ is the exterior covariant differential (of indexed form fields) associated to the connection ∇.

Now, the *Bianchi identities* for the RCWS structure are easily obtained by differentiating the above equations. We get:

$$\begin{aligned} &\text{(a)}\ \ \mathring{\mathbf{D}}\Theta^\rho = \mathfrak{J}^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta - w^{\rho\cdot}_{\cdot\beta} \wedge \Theta^\beta, \\ &\text{(b)}\ \ \mathring{\mathbf{D}}\Phi_\mu = \mathfrak{J}^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta + w^{\beta\cdot}_{\cdot\mu} \wedge \Phi_\beta, \\ &\text{(c)}\ \ \mathring{\mathbf{D}}\mathfrak{J}^{\rho\cdot}_{\cdot\mu} = \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta}_{\cdot\mu} - w^{\rho\cdot}_{\cdot\beta} \wedge \mathcal{R}^{\beta\cdot}_{\cdot\mu}, \end{aligned} \tag{4.213}$$

or equivalently,

$$\begin{aligned}\mathbf{D}\Theta^{\rho} &= \mathfrak{J}_{\cdot\beta}^{\rho\cdot} \wedge \theta^{\beta},\\ \mathbf{D}\Phi_{\mu} &= \mathfrak{J}_{\cdot\mu}^{\beta\cdot} \wedge \theta_{\beta},\\ \mathbf{D}\mathfrak{J}_{\mu}^{\rho} &= \overset{\circ}{\mathcal{R}}{}_{\cdot\beta}^{\rho\cdot} \wedge w_{\cdot\mu}^{\beta\cdot} - w_{\cdot\beta}^{\rho\cdot} \wedge \overset{\circ}{\mathcal{R}}{}_{\cdot\mu}^{\beta\cdot}\ .\end{aligned} \tag{4.214}$$

4.8.10 D'Alembertian, Ricci and Einstein Operators

As we have seen in the Sect. 4.8.3 given the structure $(M, \overset{\circ}{D}, \overset{\circ}{\boldsymbol{g}})$ we can construct the Clifford algebra $\mathcal{C}\ell(M, \overset{\circ}{g})$ and the standard Dirac operator ∂ given by (Eq. (4.152))

$$\partial = d - \delta. \tag{4.215}$$

We investigate now the square of the standard Dirac operator. We shall see that this operator can be separated in some interesting parts that are related to the D'Alembertian, Ricci and Einstein operators of $(M, \overset{\circ}{D}, \overset{\circ}{\boldsymbol{g}})$.

Definition 4.130 The square of standard Dirac operator ∂ is the operator, $\partial^2 = \partial\partial$: $\sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \overset{\circ}{g}) \rightarrow \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \overset{\circ}{g})$ given by:

$$\partial^2 = (d - \delta)(d - \delta) = -(d\delta + \delta d). \tag{4.216}$$

We recognize that $\partial^2 \equiv \Diamond$ is the *Hodge Laplacian* of the manifold introduced by (Eq. (4.92)). On the other hand, remembering also that Eq. (4.148)

$$\partial = \theta^{\alpha} \overset{\circ}{D}_{\boldsymbol{e}_{\alpha}},$$

where $\{\theta^{\alpha}\}$ is an arbitrary reference frame on the manifold and $\overset{\circ}{D}$ is the Levi-Civita connection of the metric $\overset{\circ}{\boldsymbol{g}}$, we have:

$$\begin{aligned}\partial^2 &= (\theta^{\alpha} \overset{\circ}{D}_{\boldsymbol{e}_{\alpha}})(\theta^{\beta} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}}) = \theta^{\alpha}(\theta^{\beta} \overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}} + (\overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \theta^{\beta}) \overset{\circ}{D}_{\boldsymbol{e}_{\beta}})\\ &= \overset{\circ}{g}{}^{\alpha\beta}(\overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}} - \overset{\circ}{\Gamma}{}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \overset{\circ}{D}_{\boldsymbol{e}_{\rho}}) + \theta^{\alpha} \wedge \theta^{\beta}(\overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}} - \overset{\circ}{\Gamma}{}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \overset{\circ}{D}_{\boldsymbol{e}_{\rho}}).\end{aligned}$$

Then defining the operators:

$$\begin{aligned}&\text{(a)}\ \ \partial \cdot \partial = \overset{\circ}{g}{}^{\alpha\beta}(\overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}} - \overset{\circ}{\Gamma}{}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \overset{\circ}{D}_{\boldsymbol{e}_{\rho}}),\\ &\text{(b)}\ \ \partial \wedge \partial = \theta^{\alpha} \wedge \theta^{\beta}(\overset{\circ}{D}_{\boldsymbol{e}_{\alpha}} \overset{\circ}{D}_{\boldsymbol{e}_{\beta}} - \overset{\circ}{\Gamma}{}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \overset{\circ}{D}_{\boldsymbol{e}_{\rho}}),\end{aligned} \tag{4.217}$$

we can write:

$$\Diamond = \boldsymbol{\partial}^2 = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial} \tag{4.218}$$

or,

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)(\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)\\ &= \boldsymbol{\partial}\lrcorner\,\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner\,.\end{aligned} \tag{4.219}$$

Remark 4.131 It is important to observe that the operators $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ and $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ do not have anything analogous in the formulation of the differential geometry in the Cartan and Hodge bundles.

Remark 4.132 Moreover we write for $\omega \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, $\boldsymbol{\partial}\cdot\boldsymbol{\partial}\omega$ and $\boldsymbol{\partial}\wedge\boldsymbol{\partial}\omega$ to mean respectively $(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\omega$ and $(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\omega$. The parenthesis will be included in a formula only if there is a risk of confusion.

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}\mathring{g}^{\alpha\beta}\left[\mathring{D}_{\boldsymbol{e}_\alpha}\mathring{D}_{\boldsymbol{e}_\beta} + \mathring{D}_{\boldsymbol{e}_\beta}\mathring{D}_{\boldsymbol{e}_\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta}\mathring{D}_{\boldsymbol{e}_\rho}\right]. \tag{4.220}$$

Applying this operator to the 1-forms of the frame $\{\theta^\alpha\}$, we get:

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial}\theta^\mu = -\frac{1}{2}\mathring{g}^{\alpha\beta}\mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta}\theta^\rho, \tag{4.221}$$

where:

$$\mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta} = \boldsymbol{e}_\alpha(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) + \boldsymbol{e}_\beta(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\rho} - b^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho}. \tag{4.222}$$

The proof that an object with these components is a tensor is a consequence of the following proposition:

Proposition 4.133 *For every r-form field $\omega \in \sec \bigwedge^r T^*M$, $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, we have:*

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial}\omega = \frac{1}{r!}\mathring{g}^{\alpha\beta}\mathring{D}_\alpha\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}. \tag{4.223}$$

Proof We have $\mathring{D}_{\boldsymbol{e}_\beta}\omega = \frac{1}{r!}\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with

$$\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r} = \boldsymbol{e}_\beta(\omega_{\alpha_1\ldots\alpha_r}) - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\alpha_1}\omega_{\sigma\alpha_2\ldots\alpha_r} - \cdots - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\sigma}.$$

Observe moreover that we have

$$\mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r} = \boldsymbol{e}_\alpha(\mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r}) - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\alpha_1} \mathring{D}_\sigma \omega_{\sigma\alpha_2\ldots\alpha_r} \\ - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\alpha_1} \mathring{D}_\beta \omega_{\sigma\alpha_2\ldots\alpha_r} - \ldots \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\alpha_r} \mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_{r-1}\sigma}$$

but

$$D_{\boldsymbol{e}_\alpha} D_{\boldsymbol{e}_\beta} \omega = D_{\boldsymbol{e}_\alpha}(\frac{1}{r!} \mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}) \\ \frac{1}{r!}(\boldsymbol{e}_\alpha(\mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r}) - \mathring{\Gamma}^{\sigma}_{\alpha\alpha_1} \mathring{D}_\beta \omega_{\sigma\alpha_2\ldots\alpha_r} - \cdots \\ - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\alpha_r} \mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_{r-1}\sigma})\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Thus we conclude that:

$$(\mathring{D}_{\boldsymbol{e}_\alpha} \mathring{D}_{\boldsymbol{e}_\beta} - \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} \mathring{D}_{\boldsymbol{e}_\rho})\omega = \frac{1}{r!} \mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Finally, multiplying this equation by $\mathring{g}^{\alpha\beta}$ and using the Eq. (4.217a), we get the Eq. (4.223).■

In view of Eq. (4.223), we give the

Definition 4.134 The operator $\square = \not\partial \cdot \not\partial$ is called (covariant) *D'Alembertian*.

Note that the D'Alembertian of the 1-forms θ^μ can also be written as:

$$\not\partial \cdot \not\partial \theta^\mu = \mathring{g}^{\alpha\beta} \mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho \theta^\rho = \frac{1}{2} \mathring{g}^{\alpha\beta}(\mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho + \mathring{D}_\beta \mathring{D}_\alpha \delta^\mu_\rho)\theta^\rho$$

and therefore, taking into account the Eq. (4.221), we conclude that:

$$\mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta} = -(\mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho + \mathring{D}_\beta \mathring{D}_\alpha \delta^\mu_\rho), \tag{4.224}$$

what proves our assertion that $\mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta}$ are the components of a tensor.

By its turn, the operator $\not\partial \wedge \not\partial$ can also be written as:

$$\not\partial \wedge \not\partial = \frac{1}{2} \theta^\alpha \wedge \theta^\beta \left[\mathring{D}_{\boldsymbol{e}_\alpha} \mathring{D}_{\boldsymbol{e}_\beta} - \mathring{D}_{\boldsymbol{e}_\beta} \mathring{D}_{\boldsymbol{e}_\alpha} - c^{\rho\cdot\cdot}_{\cdot\alpha\beta} \mathring{D}_{\boldsymbol{e}_\rho} \right]. \tag{4.225}$$

Applying this operator to the 1-forms of the frame $\{\theta^\mu\}$, we get

$$\not\partial \wedge \not\partial \theta^\mu = -\frac{1}{2} \mathring{R}^{\mu\cdots}_{\cdot\rho\alpha\beta}(\theta^\alpha \wedge \theta^\beta)\theta^\rho = -\mathring{\mathcal{R}}^{\rho\mu}\theta_\rho, \tag{4.226}$$

where $\overset{\circ}{R}{}^{\mu\cdots}_{\cdot\rho\alpha\beta}$ are the components of the curvature tensor of the connection $\overset{\circ}{D}$. From Eq. (2.46), we get:

$$\overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho}\theta^{\rho} = \overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho}\llcorner\theta^{\rho} + \overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho} \wedge \theta^{\rho}.$$

The second term in the r.h.s. of this equation is identically null because of the Bianchi identity given by Eq.(4.213a) for the particular case of a symmetric connection ($\Theta^{\mu} = 0$). Using Eqs. (2.35) and (2.37) we can write the first term in the r.h.s. as:

$$\begin{aligned}
\overset{\circ}{\mathcal{R}}{}^{\rho\mu}\llcorner\theta^{\rho} &= \frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}(\theta^{\alpha}\wedge\theta^{\beta})\llcorner\theta_{\rho} \\
&= -\frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta_{\rho}\lrcorner(\theta^{\alpha}\wedge\theta^{\beta}) \\
&= -\frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}(\delta^{\alpha}_{\rho}\theta^{\beta} - \delta^{\alpha}_{\beta}\theta^{\alpha}) \\
&= -\overset{\circ}{R}{}^{\alpha\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta^{\beta} = \overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}\theta^{\beta},
\end{aligned} \tag{4.227}$$

where $\overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}$ are the components of the Ricci tensor of the Levi-Civita connection $\overset{\circ}{D}$ of $\overset{\circ}{\boldsymbol{g}}$. Thus we have:

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial}\theta^{\mu} = \overset{\circ}{\mathcal{R}}{}^{\mu}, \tag{4.228}$$

where $\overset{\circ}{\mathcal{R}}{}^{\mu} = \overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}\theta^{\beta}$ are the Ricci 1-forms of the manifold. Because of this relation, we give the

Definition 4.135 The operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ is called the *Ricci operator* of the manifold associated to the Levi-Civita connection $\overset{\circ}{D}$ of $\overset{\circ}{\boldsymbol{g}}$.

The proposition below shows that the Ricci operator can be written in a purely algebraic way:

Proposition 4.136 *The Ricci operator* $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ *satisfies the relation:*

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial} = \overset{\circ}{\mathcal{R}}{}^{\sigma}\wedge\mathbf{i}_{\sigma} + \overset{\circ}{\mathcal{R}}{}^{\rho\sigma}\wedge\mathbf{i}_{\rho}\mathbf{i}_{\sigma}, \tag{4.229}$$

where (keep in mind) $\overset{\circ}{\mathcal{R}}{}^{\rho\sigma} := \overset{\circ}{g}{}^{\sigma\mu}\overset{\circ}{\mathcal{R}}{}^{\rho\cdot}_{\cdot\mu} = \frac{1}{2}\overset{\circ}{g}{}^{\sigma\mu}\overset{\circ}{R}{}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}$.

Proof The Hodge Laplacian of an arbitrary r-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$ is given by: (e.g., [3]—recall that our definition differs by a sign from that given there) $\Diamond\omega = \eth^2\omega = \frac{1}{r!}(\eth^2\omega)_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with:

$$\begin{aligned}(\Diamond\omega)_{\alpha_1\ldots\alpha_r} &= \mathring{g}^{\alpha\beta}\mathring{D}_\alpha\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r}\\ &\quad -\sum_p(-1)^p\mathring{R}^{\sigma\cdot}_{\cdot\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r}\\ &\quad -2\sum_{\substack{p,q\\p<q}}(-1)^{p+q}\mathring{R}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha_q\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r},\end{aligned}\tag{4.230}$$

where the notation $\check{\alpha}$ means that the index α was exclude of the sequence.

The first term in the r.h.s. of this expression are the components of the D'Alembertian of the field ω.

Now, recalling that $\mathbf{i}_\sigma\omega = \theta_\sigma \lrcorner\omega$, we obtain:

$$\mathring{\mathcal{R}}^\sigma\wedge\mathbf{i}_\sigma\omega = -\frac{1}{r!}\left[\sum_p(-1)^p\mathring{R}^{\sigma\cdot}_{\cdot\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}$$

and also,

$$\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\omega = -\frac{2}{r!}\left[\sum_{\substack{p,q\\p<q}}(-1)^{p+q}\mathring{R}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha_q\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}.$$

Hence, taking into account Eq. (4.218), we conclude that:

$$(\eth\wedge\eth)\omega = \mathring{\mathcal{R}}^\sigma\wedge\mathbf{i}_\sigma\omega + \mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\omega,$$

for every r-form field ω.■

Observe that applying the operator given by the second term in the r.h.s. of Eq. (4.229) to the dual of the 1-forms θ^μ, we get:

$$\begin{aligned}\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\star\theta^\mu &= \mathring{\mathcal{R}}_{\rho\sigma}\star\theta^\rho\lrcorner(\theta^\sigma\lrcorner\theta^\mu))\\ &= -\mathring{\mathcal{R}}_{\rho\sigma}\wedge\star(\theta^\rho\wedge\theta^\sigma\theta^\mu)\\ &= \star(\mathring{\mathcal{R}}_{\rho\sigma}\lrcorner(\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu)),\end{aligned}\tag{4.231}$$

where we have used the Eq. (2.77). Then, recalling the definition of the curvature forms and using the Eq. (2.36), we conclude that:

$$\mathring{\mathcal{R}}^{\rho\sigma}\wedge\theta_\rho\lrcorner\theta_\sigma\lrcorner\star\theta^\mu = -2\star(\mathring{\mathcal{R}}^\mu - \frac{1}{2}\mathring{R}\theta^\mu) = -2\star\mathring{\mathcal{G}}^\mu,\tag{4.232}$$

where $\mathring{R}$ is the scalar curvature of the manifold and the $\mathring{\mathcal{G}}^{\mu}$ may be called the Einstein 1-form fields. That observation motivate us to give the

Definition 4.137 The *Einstein operator* of the Levi-Civita connection $\mathring{D}$ of $\mathring{\boldsymbol{g}}$ on the manifold M is the mapping $\blacksquare$: $\sec \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \to \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ given by:

$$\blacksquare = -\frac{1}{2} \star^{-1} (\mathring{\mathcal{R}}^{\rho\sigma} \wedge \mathbf{i}_{\rho}\mathbf{i}_{\sigma}) \star . \tag{4.233}$$

Obviously, we have:

$$\blacksquare\theta^{\mu} = \mathring{\mathcal{G}}^{\mu} = \mathring{\mathcal{R}}^{\mu} - \frac{1}{2}\mathring{R}\theta^{\mu}. \tag{4.234}$$

In addition, it is easy to verify that $\star^{-1}(\boldsymbol{\partial} \wedge \boldsymbol{\partial})\star = -\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ and $\star^{-1}(\mathring{\mathcal{R}}^{\sigma} \wedge \mathbf{i}_{\sigma})\star = \mathring{\mathcal{R}}^{\sigma} \lrcorner \mathbf{j}_{\sigma}$. Thus we can also write the Einstein operator as:

$$\blacksquare = \frac{1}{2}(\boldsymbol{\partial} \wedge \boldsymbol{\partial} - \mathring{\mathcal{R}}^{\sigma} \lrcorner \mathbf{j}_{\sigma}). \tag{4.235}$$

Another important result is given by the following proposition:

Proposition 4.138 *Let $\mathring{\omega}^{\mu\cdot}_{\cdot\rho}$ be the Levi-Civita connection 1-forms fields in an arbitrary moving frame $\{\theta^{\mu}\}$ on $(M, \mathring{D}, \mathring{g})$. Then:*

$$\begin{aligned} &(a)\ \boldsymbol{\partial} \cdot \boldsymbol{\partial}\theta^{\mu} = -(\boldsymbol{\partial} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma}_{\rho} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho} \\ &(b)\ \boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mu} = -(\boldsymbol{\partial} \wedge \mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \wedge \mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho}, \end{aligned} \tag{4.236}$$

that is,

$$\boldsymbol{\partial}^{2}\theta^{\mu} = -(\boldsymbol{\partial}\mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho}\mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho}. \tag{4.237}$$

Proof We have

$$\begin{aligned} \boldsymbol{\partial} \cdot \mathring{\omega}^{\mu}_{\rho} &= \theta^{\alpha} \cdot \mathring{D}_{e_{\alpha}}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}\theta^{\beta}) \\ &= \theta^{\alpha} \cdot (\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho})\theta^{\beta} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\theta^{\beta}) \\ &= \mathring{g}^{\alpha\beta}(\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu}_{\beta\rho}) - \mathring{\Gamma}^{\mu}_{\sigma\rho}\mathring{\Gamma}^{\sigma}_{\alpha\beta}) \end{aligned}$$

and $\mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\sigma} = (\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho}\theta^{\beta}) \cdot (\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\theta^{\alpha}) = \mathring{g}^{\beta\alpha}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho}$. Then,

$$\begin{aligned} &-(\boldsymbol{\partial} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \cdot \mathring{\omega}^{\mu\cdot}_{v\sigma})\theta^{\rho} \\ &= \mathring{g}^{\alpha\beta}(\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho})\theta^{\rho} \end{aligned}$$

$$= -\frac{1}{2}\mathring{g}^{\alpha\beta}(\boldsymbol{e}_\alpha(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) + \boldsymbol{e}_\beta(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\rho} - b^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho})\theta^\rho$$
$$= \mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}}\theta^\mu.$$

Equation (4.236b) is proved analogously.■

Exercise 4.139 Show that $-(\theta_\rho \wedge \theta_\sigma)\lrcorner\mathring{\mathcal{R}}^{\rho\sigma} = \mathring{R}(\theta_\rho \wedge \theta_\sigma)\cdot\mathring{\mathcal{R}}^{\rho\sigma} = \mathring{R}$, where $\mathring{R}$ is the curvature scalar.

4.8.11 The Square of a General Dirac Operator

Consider the structure $(M, \nabla, \mathring{\boldsymbol{g}})$, where ∇ is an arbitrary Riemann-Cartan-Weyl connection and the Clifford algebra $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Let us now compute the square of the (general) Dirac operator $\boldsymbol{\partial} = \mathrm{tr}(u\nabla_{\boldsymbol{u}})$. As in the earlier section, we have, by one side,

$$\boldsymbol{\partial}^2 = (\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)(\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)$$
$$= \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge$$

and we write $\boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner \equiv \boldsymbol{\partial}^2\lrcorner$, $\boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge \equiv \boldsymbol{\partial}^2\wedge$ and

$$\mathcal{L}_+ = \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner, \tag{4.238}$$

so that:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2\lrcorner\boldsymbol{\partial} + \mathcal{L}_+\boldsymbol{\partial} + \boldsymbol{\partial}^2\wedge\ . \tag{4.239}$$

The operator $\mathcal{L}_+$ when applied to scalar functions corresponds, for the case of a Riemann-Cartan space, to the wave operator introduced in [30]. Obviously, for the case of the standard Dirac operator, $\mathcal{L}_+$ reduces to the usual Hodge Laplacian of the manifold, which preserve graduation of forms.

Now, a similar calculation for the product $\boldsymbol{\partial}\overset{\blacklozenge}{\boldsymbol{\partial}}$ of the Dirac derivative and the Dirac coderivative yields:

$$\boldsymbol{\partial}\overset{\blacklozenge}{\boldsymbol{\partial}} = \boldsymbol{\partial}\lrcorner\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner + \mathcal{L}_- + \boldsymbol{\partial}\wedge\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge, \tag{4.240}$$

with

$$\mathcal{L}_- = \boldsymbol{\partial}\lrcorner\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge + \boldsymbol{\partial}\wedge\cdot\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner\,. \tag{4.241}$$

On the other hand, we have also:

$$\begin{aligned}\blacklozenge &= (\theta^{\alpha}\nabla_{\boldsymbol{e}_{\alpha}})(\theta^{\beta}\nabla_{\boldsymbol{e}_{\beta}}) = \theta^{\alpha}(\theta^{\beta}\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} + (\nabla_{\boldsymbol{e}_{\alpha}}\theta^{\beta})\nabla_{\boldsymbol{e}_{\beta}}) \\ &= \mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) + \theta^{\alpha}\wedge\theta^{\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}})\end{aligned}$$

and we can then define:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \theta^{\alpha}\wedge\theta^{\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}})\end{aligned} \tag{4.242}$$

in order to have:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial} = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\ . \tag{4.243}$$

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \frac{1}{2}\theta^{\alpha}\cdot\theta^{\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) + \frac{1}{2}\theta^{\beta}\cdot\theta^{\alpha}(\nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - L_{\cdot\beta\alpha}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) \\ &= \frac{1}{2}\mathring{g}^{\alpha\beta}[\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} + \nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - (L_{\cdot\alpha\beta}^{\rho\cdot\cdot} + L_{\cdot\beta\alpha}^{\rho\cdot\cdot})\nabla_{\boldsymbol{e}_{\rho}}]\end{aligned}$$

or,

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}\mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} + \nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - b_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) - s^{\rho}\nabla_{\boldsymbol{e}_{\rho}}, \tag{4.244}$$

where s^{ρ} has been defined in Eq. (4.191).

By its turn, the operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \frac{1}{2}\theta^{\alpha}\wedge\theta^{\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - L_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) + \frac{1}{2}\theta^{\beta}\wedge\theta^{\alpha}(\nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - L_{\cdot\beta\alpha}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) \\ &= \frac{1}{2}\theta^{\alpha}\wedge\theta^{\beta}[\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - \nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - (L_{\cdot\alpha\beta}^{\rho\cdot\cdot} - L_{\cdot\beta\alpha}^{\rho\cdot\cdot})\nabla_{\boldsymbol{e}_{\rho}}]\end{aligned} \tag{4.245}$$

or,

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial} = \frac{1}{2}\theta^{\alpha}\wedge\theta^{\beta}(\nabla_{\boldsymbol{e}_{\alpha}}\nabla_{\boldsymbol{e}_{\beta}} - \nabla_{\boldsymbol{e}_{\beta}}\nabla_{\boldsymbol{e}_{\alpha}} - c_{\cdot\alpha\beta}^{\rho\cdot\cdot}\nabla_{\boldsymbol{e}_{\rho}}) - \Theta^{\rho}\nabla_{\boldsymbol{e}_{\rho}}. \tag{4.246}$$

Exercise 4.140 Prove that the Ricci and Einstein operators are $(1, 1)$-extensor fields on a Lorentzian spacetime, i.e., for any $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned}\boldsymbol{\partial}\wedge\boldsymbol{\partial}\, A &= \boldsymbol{\partial}\wedge\boldsymbol{\partial}\,(A_{\mu}\theta^{\mu}) = A_{\mu}\boldsymbol{\partial}\wedge\boldsymbol{\partial}\theta^{\mu}, \\ \blacksquare A &= \blacksquare(A_{\mu}\theta^{\mu}) = A_{\mu}\blacksquare\theta^{\mu}.\end{aligned} \tag{4.247}$$

Solution We prove the first formula, since after proving it the second one is obvious. We choose for simplicity an orthonormal cobasis $\{\theta^{\mathbf{a}}\}$ for T^*M dual to the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$ for TM, such that $[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}] = c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}$. Let ∇ be a connection on a Riemann-Cartan-Weyl spacetime, such that $\nabla_{\boldsymbol{e}_{\mathbf{a}}}\boldsymbol{e}_{\mathbf{b}} = L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}$. Recalling (Eq. (4.245)) we have

$$\begin{aligned}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A &= \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\{[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}](A_{\mathbf{k}}) - L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}(A_{\mathbf{k}}) - L_{\cdot\mathbf{ba}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}(A_{\mathbf{k}})\}\theta^{\mathbf{k}}\} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}}\\ &= \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\{c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot} - L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot} - L_{\cdot\mathbf{ba}}^{\mathbf{d}\cdot\cdot}\}\theta^{\mathbf{k}} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}}\\ &= \frac{1}{2}T_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}} = A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}},\end{aligned}$$

since for a Lorentzian spacetime the torsion tensor (with components $T_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}$) is null.

Exercise 4.141 Show that for any $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A = \mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}}\, A + \mathbf{J}^{\alpha} \cdot \theta_{\alpha}\check{A}, \tag{4.248}$$

where $\check{A} := \check{A}_{\sigma}\theta^{\sigma}$, $\check{A}_{\kappa} := \mathring{g}_{\beta\kappa}g^{\beta\sigma}A_{\sigma}$ and $\mathbf{J}^{\alpha} := \mathring{g}^{\alpha\beta}J_{\beta\sigma}\theta^{\sigma}$, where $J_{\beta\sigma}$ is given by

4.9 Some Applications

4.9.1 Maxwell Equations in the Hodge Bundle

The system of Maxwell equations has many faces.[27] Here we show how to express that system of equations in the Hodge bundle and then in the Clifford bundle. To start, let $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$ be an oriented Lorentzian manifold.

Maxwell equations on $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$ refers to an exterior system of differential equations given by a closed 2-form $F \in \sec \bigwedge^2 T^*M$ and a exact 3-form $\boldsymbol{J}_e \in \sec \bigwedge^3 T^*M$. Then there exists $G \in \sec \bigwedge^2 T^*M$ such that

$$dF = 0 \text{ and } dG = -\boldsymbol{J}_e. \tag{4.249}$$

It is postulated that in vacuum there is a relation between G and F (said constitutive relation) given by

$$G = \star F. \tag{4.250}$$

[27] Besides the ones presented in this chapter, others will be exhibited in Chap. 13.

In that case putting $\boldsymbol{J}_e = \star J_e$, $J_e \in \sec \bigwedge^1 T^*M$ and taking into account Eq. (4.91) we can write the system (4.249) as[28]

$$dF = 0 \text{ and } \delta F = -J_e. \tag{4.251}$$

F is called the Faraday field and J_e is called the electric current.

4.9.2 Charge Conservation

Of course, $\delta J_e = 0$, which means that charge is conserved. Indeed, let C_3 be a three dimensional volume contained in a space slice, i.e., in a spacelike surface. Then the electric flux contained in $C_2 = \partial C_3$ is

$$Q = \int_{C_3} \star J_e = -\int_{C_3} dG = -\int_{\partial C_3} \star F. \tag{4.252}$$

It is an empirical fact that all observable *free* charges are integer multiple of the electron charge. This phenomenon is called *charge quantization*. On the other hand consider a 4-volume C_4 with boundary given by $\partial C_4 = C_3^{(2)} - C_3^{(1)} + S$ where with the condition $J_e|_S = 0$ and where $C_3^{(2)}$ and $C_3^{(1)}$are three dimensional volumes contained in two different space slices. Then,

$$\int_{\partial C_4} \star J_e = \int_{\partial C_4} dG = \int_{C_4} d^2 G = 0, \tag{4.253}$$

from where it follows that

$$\int_{C_3^{(1)}} \star J_e = \int_{C_3^{(2)}} \star J_e. \tag{4.254}$$

We postulate that F is closed but it may be (eventually) not exact. In that case it may have period integrals according to de Rham theorem, i.e.,

$$\int_{z_2^{(i)}} F = g_{(i)}, \tag{4.255}$$

where $z_2^{(i)} \in H_2(M)$ are cycles. It seems to be an empirical fact that all $g_{(i)} = 0$, at least for cycles in the region of the universe where men already did experiments. This means that F is exact, i.e., it is possible to define globally a differentiable

[28] Thirring [44] said that the two equations in Eq. (4.251) is the twentieth Century presentation of Maxwell equations.

potential $A \in \sec \bigwedge^1 T^*M$ such that $F = dA$. This also means that there are no magnetic monopoles in nature.[29] Indeed, if z_2 is a cycle (a closed surface) then we have

$$\int_{z_2} F = \int_{z_2} dA = \langle \partial z_2, A \rangle = \langle 0, A \rangle = 0. \tag{4.256}$$

4.9.3 Flux Conservation

Of course, A is only defined modulus a gauge, i.e., $A + A'$, with $A' \in \sec \bigwedge^1 T^*M$ a closed form. The period integrals of A' according to de Rham theorem are

$$\int_{z_i^{(i)}} A' = \Phi_{(i)}. \tag{4.257}$$

Now, it is an empirical fact that $\Phi_{(i)}$ is quantized in some (*but not all*) physical systems, like, e.g., in superconductors [16]. The phenomenon is then called flux quantization. In appropriate units

$$\int_{z_1} A' = nh/2e, \tag{4.258}$$

where n is an integer and h is Planck constant and e is the electron charge.

Note also that from $\boldsymbol{J}_e = -dG$ in Eq. (4.249) it follows that G is defined also only modulus a closed form G'. The period integrals of G' may eventually correspond to topological charges. Another possibility of having 'charge without charge' coming from statistical distributions of quantized flux loops has been investigated in [18, 19]. We shall not discuss these interesting issues in this book.

4.9.4 Quantization of Action

Finally we mention the following. As we shall see in Chap. 7 the Lagrangian *density* of the electromagnetic field in *free space* is given by

$$\mathcal{L}(A) = -\frac{1}{2} F \wedge \star F. \tag{4.259}$$

[29]See however the news in [31] where it is claimed that magnetic monopoles have been observed in a synthetic magnetic field.

Calling $\boldsymbol{K} = A \wedge \star F$, we can write

$$\mathcal{L}(A) = -\frac{1}{2} d\boldsymbol{K}. \tag{4.260}$$

Now, it seems an empirical fact that action is quantized, i.e., we have

$$\begin{aligned} a &= \int_{C_4} \mathcal{L}(A) \\ &= \int_{C_3 = \partial C_4} \boldsymbol{K} = nh. \end{aligned} \tag{4.261}$$

Remark 4.142 We observe that $\int_{C_3=\partial C_4} \boldsymbol{K}$ has been introduced by Kiehn (see [20]). However he called $A \wedge \star F$ the topological spin, which is not a good name (and identification of observable) in our opinion. The reason is that according to the Lagrangian formalism (see Chap. 8, Eq. (8.124))[30] the spin density is proportional to $A \wedge F$. This result and the other period integrals discussed above suggests that quantization may be linked to topology in a way not suspected by contemporary physicists. On this issue, see also [29].

4.9.5 A Comment on the Use of de Rham Pseudo-Forms and Electromagnetism

Besides the forms we have been working until now, in a famous book, de Rham [6] introduces also the concept of *impair* forms[31] in a n-dimensional manifold M, which is essential for the formulation of a theory of integration in a non orientable manifold.

Definition 4.143 An impair p-form in M is a pair of p-forms such that if its representative in a given $\mathfrak{A} \subset M$ in a cobasis $\{\theta^i\}$ for T^*U $(U \supset \mathfrak{A})$ is declared as being

$$\omega|_U = \frac{1}{p!} \omega_{i_1 \ldots i_p} \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \in \sec \bigwedge\nolimits^p T^*M$$

[30] See also [7].

[31] Also called by some authors pseudo forms.

then its representative $\omega|_V$ in $\mathfrak{A} \subset V \subset M$ in a cobasis $\{\bar{\theta}^i\}$, $\bar{\theta}^i = \Lambda^i_j \theta^j$, for T^*V ($V \cap U \supset \mathfrak{A}$) is

$$\omega|_V = \frac{1}{p!} \bar{\omega}_{j_1 \ldots j_p} \bar{\theta}^{i_1} \wedge \cdots \wedge \bar{\theta}^{i_p} \in \sec \bigwedge^p T^*M, \tag{4.262}$$

with

$$\bar{\omega}_{j_1 \ldots j_p} = \frac{\det\left[\Lambda^i_j\right]}{\left|\det\left[\Lambda^i_j\right]\right|} \omega_{i_1 \ldots i_p} \Lambda^{i_1}_{j_1} \cdots \Lambda^{i_1}_{j_1}. \tag{4.263}$$

The introduction of impair forms leads to the question of exterior (and interior) multiplication of forms of different parities (i.e., even and odd). The rule introduced by de Rham [6] is that the product of two forms of the same parity is a form, whereas the product of two forms of different parities is an impair form. Also de Rham introduces the rule that application of the differential operator d to a form preserves its parity.

We can verify that if we denote by $\bigwedge_{impair} T^*M = \sum_{p=0}^{n} \bigwedge^p_{impair} T^*M$ the real vector space of the pseudo forms we can give a structure of associative algebra to the (exterior) direct sum $\bigwedge T^*M \oplus \bigwedge_{impair} T^*M$ equipped with the exterior product satisfying the de Rham rules mentioned above.

Having introduced the concept of de Rham pseudo forms we call the reader's attention to the following remarks.

Remark 4.144 In our brief presentation above of Maxwell equations we introduced the electromagnetic current as $J_e = \star^{-1} \boldsymbol{J}_e$, $J_e \in \sec \bigwedge^1 T^*M$. Since until that point we have not introduced the concept of impair forms its is clear that we supposed that $\boldsymbol{J}_e$ is 3-form. This certainly means that the theory as presented presupposes that we use always bases with the same orientation in order to calculate the charge in a certain three dimensional volume contained in a given space slice (Eq. (4.253)). The use of bases with the same orientation presupposes that spacetime is an orientable manifold. As will be discussed in Chap. 7 orientability of a spacetime manifold is a necessary condition for the existence of spinor fields. Since these objects seems to be an essential tool for the understanding of the world we live in, we restrict all our considerations to orientable manifolds. Eventually, if is discovered some of these days that our universe cannot be represented by an orientable manifold, then it will be necessary to study deeply the theory of impair forms.

Remark 4.145 If the spacetime manifold is orientable we do not need to consider, as some authors claim (e.g., [20, 29]) that $\boldsymbol{J}_e$ and G must be considered as pseudo forms. A thoughtful discussion of this issue may be found in [5].

4.9.6 Maxwell Equation in the Clifford Bundle

Let now, $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ be a Lorentzian spacetime and let $\mathcal{C}\ell(M, \mathtt{g})$ be the Clifford bundle of differential forms. Since D is the Levi-Civita connection of $\boldsymbol{g}$ we know (Eq. (4.152)) that the action of the Dirac operator $\boldsymbol{\partial}$ on any $P \in \sec \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ is $\boldsymbol{\partial} P = (d - \delta)P$. So, let us suppose that the Faraday field and the electric current are sections of the Clifford bundle, i.e., $F \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, $J_e \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$. In that case, it is licit two sum the equations $dF = 0$ and $\delta F = -J_e$, which according to Eq. (4.251) represent the system of Maxwell equations in the Hodge bundle. We get, of course, the single equation

$$\boldsymbol{\partial} F = J_e, \tag{4.264}$$

which we be call *Maxwell equation*. Parodying Thirring [44] we may say that Eq. (4.264) the twenty-first century representation of Maxwell system of equations.

Exercise 4.146 Show that in Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ (Definition 4.108) Eq. (4.264) is equivalent to the standard vector form of Maxwell equations, that appears in elementary electrodynamics textbooks.

Solution We recall (see Table 3.1 in Chap. 3) that for any $x \in M$, $\mathcal{C}\ell(T_x^*M, \eta_x) \simeq \mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, is the so called spacetime algebra. The even elements of $\mathbb{R}_{1,3}$ close a subalgebra called the Pauli algebra. That subalgebra is denoted by $\mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. Also, $\mathbb{H}(2)$ is the algebra of the 2×2 quaternionic matrices and $\mathbb{C}(2)$ is the algebra of the 2×2 complex matrices. As in Sect. 3.9.1 a convenient isomorphism $\mathbb{R}^0_{1,3} \approx \mathbb{R}_{3,0}$ is easily exhibited. Choose a global orthonormal tetrad coframe $\{\gamma^\mu\}$, $\gamma^\mu = dx^\mu$, $\mu = 0, 1, 2, 3$, and let $\{\gamma_\mu\}$ be the reciprocal tetrad of $\{\gamma^\mu\}$, i.e., $\gamma_\nu \cdot \gamma^\mu = \delta^\mu_\nu$. Now, put

$$\sigma_i = \gamma_i \gamma_0, \; \mathbf{i} = -\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^5. \tag{4.265}$$

Observe that $\mathbf{i}$ commutes with bivectors and thus *acts* like the imaginary unity $\mathrm{i} = \sqrt{-1}$ in the subbundle $\mathcal{C}\ell^0(M, \eta) = \bigcup_{x \in M} \mathcal{C}\ell^0(T_x^*M, \eta_x) \hookrightarrow \mathcal{C}\ell(M, \eta)$, which we call *Pauli bundle*. Now, the electromagnetic field is represented in $\mathcal{C}\ell(M, \eta)$ by $F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta))$ with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{4.266}$$

where (E_1, E_2, E_3) and (B_1, B_2, B_3) are the *usual* Cartesian components of the electric and magnetic fields. Then, as it is easy to verify we can write

$$F = \vec{E} + \mathbf{i}\vec{B}, \tag{4.267}$$

with , $\vec{E} = \sum_{i=1}^3 E_i \sigma_i$, $\vec{B} = \sum_{i=1}^3 B_i \sigma_i$.

For the electric current density $J_e = \rho\gamma^0 + J^i\gamma_i$ we can write

$$\gamma_0 J_e = \rho - \vec{j} = \rho - J^i\sigma_i. \tag{4.268}$$

For the Dirac operator we have

$$\gamma_0 \boldsymbol{\partial} = \frac{\partial}{\partial x^0} + \sum_{i=1}^{3} \sigma_i \partial_i = \frac{\partial}{\partial t} + \nabla. \tag{4.269}$$

Multiplying both members of Eq. (4.264) on the left by γ_0 we obtain

$$\begin{gathered} \gamma_0 \boldsymbol{\partial} F = \gamma_0 J_e, \\ (\frac{\partial}{\partial t} + \nabla)(\vec{E} + \mathbf{i}\vec{B}) = \rho - \vec{j} \end{gathered} \tag{4.270}$$

From Eq. (4.270) we obtain

$$\partial_0 \vec{E} + \mathbf{i}\partial_0 \vec{B} + \nabla \cdot \vec{E} + \nabla \wedge \vec{E} + \mathbf{i}\nabla \cdot \vec{B} + \mathbf{i}\nabla \wedge \vec{B} = \rho - \vec{j}. \tag{4.271}$$

For any 'vector field' $\vec{A} \in \mathcal{C}\ell^0(M,\eta) \hookrightarrow \mathcal{C}\ell(M,\eta)$ we define the *rotational operator* $\nabla\times$ by

$$\nabla \times \vec{A} = -\mathbf{i}\nabla \wedge \vec{A}. \tag{4.272}$$

This relation follows once we realize that the usual *vector* product of two vectors $\vec{a} = \sum_{i=1}^{3} a_i\sigma_i$ and $\vec{b} = \sum_{i=1}^{3} b_i\sigma_i$ can be identified with the dual of the bivector $\vec{a} \wedge \vec{b}$ through the formula $\vec{a} \times \vec{b} = -\mathbf{i}\vec{a} \wedge \vec{b}$. Finally we obtain from Eq. (4.271) by equating terms with the same grades (in the Pauli subbundle)

$$\begin{array}{ll} \text{(a)} \quad \nabla \cdot \vec{E} = \rho, & \text{(b)}\ \nabla \times \vec{B} - \partial_0 \vec{E} = \vec{j}, \\ \text{(c)}\ \nabla \times \vec{E} + \partial_0 \vec{B} = 0, & \text{(d)} \quad \nabla \cdot \vec{B} = 0, \end{array} \tag{4.273}$$

which we recognize as the system of Maxwell equations in the usual vector notation.

We just exhibit three equivalent presentations of Maxwell systems of equations, namely Eqs. (4.251), (4.264), and (4.273). They are some of the many faces of Maxwell equations. Other faces exist as we shall see in Chap. 11.

4.9.7 Einstein Equations and the Field Equations for the $\theta^{\mathbf{a}}$

As, it is the case of Maxwell equations, also Einstein equations have many faces. Here we exhibit an interesting one which is possible once we have at our disposal the Clifford bundle formalism. So, let now $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ be a Lorentzian spacetime (Definition 4.107) modelling a gravitational field in the general theory of Relativity [38]. Let $\{\boldsymbol{e}_{\mathbf{a}}\}$ be an arbitrary orthonormal basis of TU (a tetrad[32]) and $\{\theta^{\mathbf{b}}\}$ of T^*M its dual basis (a cotetrad), with $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$. We recall that Einstein's equations relating the distribution of matter energy represented by the energy-momentum tensor $\boldsymbol{T} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \otimes \boldsymbol{e}_{\mathbf{a}} \in \sec T^1_1 U \subset \sec T^1_1 M$ can be written (in appropriated units)

$$R^{\mathbf{a}}_{\mathbf{b}} - \frac{1}{2}\delta^{\mathbf{a}}_{\mathbf{b}} R = -T^{\mathbf{a}}_{\mathbf{b}}, \tag{4.274}$$

where $R^{\mathbf{a}}_{\mathbf{b}}$ is the Ricci tensor and R is the scalar curvature. Multiplying both members of Eq. (4.274) by $\theta^{\mathbf{b}}$ and taking into account Eq. (4.228) defining the Ricci 1-forms in terms of the Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ (with $\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\boldsymbol{e}_{\mathbf{a}}}$) we can write after some trivial algebra

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ \theta^{\mathbf{a}} + \frac{T}{2}\theta^{\mathbf{a}} = -T^{\mathbf{a}}, \tag{4.275}$$

where[33] $T^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g})$ are the energy-momentum 1-form fields and $T = T^{\mathbf{a}}_{\mathbf{a}}$.

Now, taking into account Eqs. (4.218) and (4.219) we can write

$$-\boldsymbol{\partial} \cdot \boldsymbol{\partial}\ \theta^{\mathbf{a}} + \boldsymbol{\partial} \wedge (\boldsymbol{\partial} \cdot \theta^{\mathbf{a}}) + \boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \theta^{\mathbf{a}}) + \frac{1}{2}T\theta^{\mathbf{a}} = -T^{\mathbf{a}}. \tag{4.276}$$

Now, let $\{x^\mu\}$ be the coordinate functions of a local chart of the maximal atlas of M covering $U \subset M$. When $\theta^{\mathbf{a}}$ is an exact differential, and in that case we write $\theta^{\mathbf{a}} \mapsto \theta^\mu = dx^\mu$ and if the coordinate functions are *harmonic* [10], i.e., $\delta\theta^\mu = -\boldsymbol{\partial} \lrcorner \theta^\mu = g^{\alpha\beta}\Gamma^{\mu\cdot\cdot}_{\cdot\alpha\beta} = 0$, Eq. (4.276) becomes

$$\Box\theta^\mu - \frac{1}{2}R\theta^\mu = T^\mu, \tag{4.277}$$

where $\Box$ is the covariant D'Alembertian operator (Definition 4.134).

[32] We shall see in Chap. 6 that any Lorentzian spacetime admitting spinor fields must have a global tetrad.

[33] Sometimes in the written of some formulas in the next chapters it is convenient to use the notation $\mathcal{T}^{\mathbf{a}} = -T^{\mathbf{a}}$.

4.9.8 Curvature of a Connection and Bending. The Nunes Connection of $\mathring{S}^2$

Consider the manifold $\mathring{S}^2 = \{S^2\backslash\text{north pole} + \text{south pole}\} \subset \mathbb{R}^3$, it is an sphere of radius $\mathfrak{R} = 1$ excluding the north and south poles. Let $\boldsymbol{g} \in \sec T^0_2\mathring{S}^2$ be a metric field for $\mathring{S}^2$, which is the pullback on it of the metric of the ambient space $\mathbb{R}^3$. Now, consider two different connections on $\mathring{S}^2$, D—the Levi-Civita connection— and $\boldsymbol{\nabla}^c$, a connection—here called the Nunes[34] (or navigator) connection[35]— defined by the following parallel transport rule: a vector is parallel transported along a curve, if at any $x \in S^2$ the angle between the vector and the vector tangent to the latitude line passing through that point is constant during the transport (see Fig. 4.5).

Exercise 4.147 (i) Show that the structure $(\mathring{S}^2, \boldsymbol{g}, D)$ is a Riemann geometry of constant curvature and;

(ii) that the structure $(\mathring{S}^2, \boldsymbol{g}, \boldsymbol{\nabla}^c)$ is a teleparallel geometry, with zero Riemann curvature tensor, but non zero tensor.

Solution The first part of the exercise is a standard one and can be found in many good textbooks on differential geometry. Here, we only show (ii). We clearly see from Fig. 4.5a that if we transport a vector along the infinitesimal quadrilateral *pqrs* composed of latitudes and longitudes, first starting from p along *pqr* and then starting from p along *psr* the parallel transported vectors that result in both cases will coincide. Using the definition of the Riemann curvature tensor, we see that it is null. So, we see that $\mathring{S}^2$ considered as part of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla^c)$ is flat!

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[34]Pedro Salacience Nunes (1502–1578) was one of the leading mathematicians and cosmographers of Portugal during the Age of Discoveries. He is well known for his studies in Cosmography, Spherical Geometry, Astronomic Navigation, and Algebra, and particularly known for his discovery of loxodromic curves and the nonius. Loxodromic curves, also called rhumb lines, are spirals that converge to the poles. They are lines that maintain a fixed angle with the meridians. In other words, loxodromic curves directly related to the construction of the Nunes connection. A ship following a fixed compass direction travels along a loxodromic, this being the reason why Nunes connection is also known as navigator connection. Nunes discovered the loxodromic lines and advocated the drawing of maps in which loxodromic spirals would appear as straight lines. This led to the celebrated Mercator projection, constructed along these recommendations. Nunes invented also the Nonius scales which allow a more precise reading of the height of stars on a quadrant. The device was used and perfected at the time by several people, including Tycho Brahe, Jacob Kurtz, Christopher Clavius and further by Pierre Vernier who in 1630 constructed a practical device for navigation. For some centuries, this device was called nonius. During the nineteenth century, many countries, most notably France, started to call it vernier. More details in http://www.mlahanas.de/Stamps/Data/Mathematician/N.htm.

[35]Some authors call the Columbus connection the Nunes connection. Such name is clearly unappropriated.

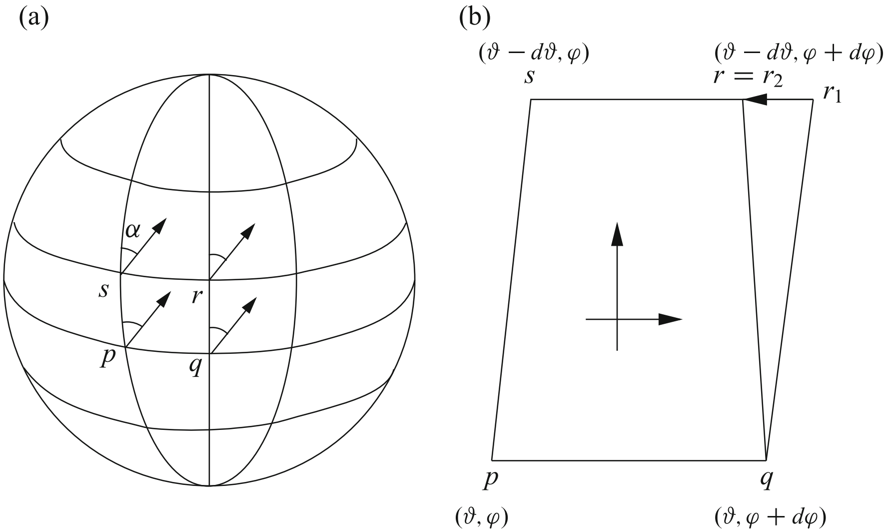

Fig. 4.5 Characterization of the Nunes connection

Let $(x^1, x^2) = (\vartheta, \varphi)$ $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, be the standard spherical coordinates of a $\mathring{S}^2$ or unitary radius, which covers all the open set U which is $\mathring{S}^2$ with the exclusion of a semi-circle uniting the north and south poles.

Introduce first the *coordinate bases*

$$\{\partial_\mu = \partial/\partial x^\mu\}, \{\theta^\mu = dx^\mu\} \tag{4.278}$$

for TU and T^*U.

Introduce next the *orthonormal bases* $\{\boldsymbol{e}_\mathbf{a}\}, \{\theta^\mathbf{a}\}$ for TU and T^*U with

$$\boldsymbol{e}_\mathbf{1} = \partial_1,\ \boldsymbol{e}_\mathbf{2} = \frac{1}{\sin x^1}\partial_2, \tag{4.279}$$

$$\theta^\mathbf{1} = dx^1, \theta^\mathbf{2} = \sin x^1 dx^2. \tag{4.280}$$

Then,

$$[\boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}] = c_{\cdot\mathbf{ij}}^{\mathbf{k}\cdot\cdot}\boldsymbol{e}_\mathbf{k}, \tag{4.281}$$
$$c_{\cdot\mathbf{12}}^{\mathbf{2}\cdot\cdot} = -c_{\cdot\mathbf{21}}^{\mathbf{2}\cdot\cdot} = -\cot x^\mathbf{1},$$

and

$$\begin{aligned} \boldsymbol{g} &= dx^1 \otimes dx^1 + \sin^2 x^1 dx^2 \otimes dx^2 \\ &= \theta^\mathbf{1} \otimes \theta^\mathbf{1} + \theta^\mathbf{2} \otimes \theta^\mathbf{2}. \end{aligned} \tag{4.282}$$

Now, it is obvious from what has been said above that our teleparallel connection is characterized by

$$\nabla^{c}_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{e}_{\mathbf{i}} = 0. \tag{4.283}$$

Then taking into account the definition of the curvature operator (definition (4.104)), we have

$$\mathbf{R}(\theta^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{k}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{a}}\left(\left[\nabla^{c}_{\boldsymbol{e}_{\mathbf{i}}}\nabla^{c}_{\boldsymbol{e}_{\mathbf{j}}} - \nabla^{c}_{\boldsymbol{e}_{\mathbf{j}}}\nabla^{c}_{\boldsymbol{e}_{\mathbf{i}}} - \nabla^{c}_{[\boldsymbol{e}_{\mathbf{i}},\boldsymbol{e}_{\mathbf{j}}]}\right]\boldsymbol{e}_{\mathbf{k}}\right) = 0. \tag{4.284}$$

Also, taking into account the definition of the torsion operation (definition (4.103)) we have

$$\begin{aligned}\boldsymbol{\tau}(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) &= \nabla^{c}_{\boldsymbol{e}_{\mathbf{j}}\boldsymbol{e}_{\mathbf{i}}} - \nabla^{c}_{\boldsymbol{e}_{\mathbf{i}}\boldsymbol{e}_{\mathbf{j}}} - [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}] \\ &= [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}],\end{aligned} \tag{4.285}$$

and $T^{\mathbf{2}\cdot\cdot}_{\cdot\mathbf{21}} = -T^{\mathbf{2}\cdot\cdot}_{\cdot\mathbf{12}} = \cot\vartheta$.

If you still need more details, concerning this last result, consider Fig. 4.5b which shows the standard parametrization of the points p, q, r, s in terms of the spherical coordinates introduced above. According to the geometrical meaning of torsion, we determine its value at a given point by calculating the difference between the (infinitesimal)[36] segments (vectors) $pr_1$and pr_2 determined as follows. If we transport the vector pq along ps we get (recalling that $\mathfrak{R} = 1$) the vector $\vec{v} = sr_1$ such that $\left|\boldsymbol{g}(\vec{v}, \vec{v})\right|^{\frac{1}{2}} = \sin\vartheta\,\triangle\varphi$. On the other hand, if we transport the vector ps along pr we get the vector $qr_2 = qr$. Let $\vec{w} = sr$. Then,

$$\left|\boldsymbol{g}(\vec{w}, \vec{w})\right| = \sin(\vartheta - \triangle\vartheta)\triangle\varphi \simeq \sin\vartheta\,\triangle\varphi - \cos\vartheta\,\triangle\vartheta\,\triangle\varphi, \tag{4.286}$$

Also,

$$\vec{u} = r_1 r_2 = -u(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}), u = \left|\boldsymbol{g}(\vec{u}, \vec{u})\right| = \cos\vartheta\,\triangle\vartheta\,\triangle\varphi \tag{4.287}$$

Then, the (Riemann-Cartan) connection ∇^{c} of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla^{c}, \tau_{\boldsymbol{g}})$ has a non null torsion tensor Θ. Indeed, the component of $\vec{u} = r_1 r_2$ in the direction $\partial/\partial\varphi$ is precisely $T^{\varphi}_{\vartheta\varphi}\triangle\vartheta\,\triangle\varphi$. So, we get (recalling that $\nabla^{c}{}_{\partial j}\partial_i = \Gamma^{k\cdot\cdot}_{\cdot ji}\partial_k$)

$$T^{\varphi\cdot\cdot}_{\cdot\vartheta\varphi} = \left(\Gamma^{\varphi\cdot\cdot}_{\cdot\vartheta\varphi} - \Gamma^{\varphi\cdot\cdot}_{\cdot\varphi\vartheta}\right) = -\cot\theta. \tag{4.288}$$

[36]This wording, of course, means that this vectors are identified as elements of the appropriate tangent spaces.

To complete the exercise we must show that $\boldsymbol{\nabla}^c \boldsymbol{g} = 0$. We have,

$$\begin{aligned} 0 &= \boldsymbol{\nabla}^c_{\boldsymbol{e}_{\mathbf{c}}} \boldsymbol{g}(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = (\boldsymbol{\nabla}^c_{\boldsymbol{e}_{\mathbf{c}}} \boldsymbol{g})(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) + \boldsymbol{g}(\boldsymbol{\nabla}^c_{\boldsymbol{e}_{\mathbf{c}}} \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) + \boldsymbol{g}(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{\nabla}^c_{\boldsymbol{e}_{\mathbf{c}}} \boldsymbol{e}_{\mathbf{j}}) \\ &= (\boldsymbol{\nabla}^c_{\boldsymbol{e}_{\mathbf{c}}} \boldsymbol{g})(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}). \end{aligned} \tag{4.289}$$

Remark 4.148 This exercise, shows clearly that we cannot *mislead* the Riemann curvature tensor of a connection with the fact that the manifold where that connection is defined may be bend as a surface in an Euclidean manifold where it is embedded. Bending is characterized by the shape operator[37] (a fundamental concept in differential geometry that will be presented in Chap. 5 using the Clifford bundle formalism). Neglecting this fact may generate a lot of wishful thinking. Taking it into account may suggest new formulations of the gravitational field theory as we will show in Chap. 11.

4.9.9 "Tetrad" Postulate? On the Necessity of Precise Notations

Given a differentiable manifold M, let $\boldsymbol{X}, \boldsymbol{Y} \in \sec TM$ be vector fields and $C \in \sec T^*M$ a covector field. Let $\mathcal{T}M = \bigoplus_{r,s=0}^{\infty} T^r_s M$ be the tensor bundle of M and $\mathbf{P} \in \sec \mathcal{T}M$ a general tensor field. We already introduced in M a rule for differentiation of tensor fields, namely the Lie derivative. Taking into account Appendix A.4 we introduce three covariant derivatives operators, $\boldsymbol{\nabla}^+$, $\boldsymbol{\nabla}^-$ and $\boldsymbol{\nabla}$, defined as follows:

$$\begin{aligned} \boldsymbol{\nabla}^+ &: \sec TM \times \sec TM \to \sec TM, \\ (\boldsymbol{X}, \boldsymbol{Y}) &\mapsto \boldsymbol{\nabla}^+_{\boldsymbol{X}} \boldsymbol{Y}, \end{aligned} \tag{4.290}$$

$$\begin{aligned} \boldsymbol{\nabla}^- &: \sec TM \times \sec T^*M \to \sec TM, \\ (\boldsymbol{X}, C) &\mapsto \boldsymbol{\nabla}^-_{\boldsymbol{X}} C, \end{aligned} \tag{4.291}$$

$$\begin{aligned} \boldsymbol{\nabla} &: \sec TM \times \sec \boldsymbol{\tau} M \to \sec TM, \\ (X, \mathbf{P}) &\mapsto \boldsymbol{\nabla}_X \mathbf{P}, \end{aligned} \tag{4.292}$$

Each one of the covariant derivative operators introduced above satisfy the following properties: Given, differentiable functions $f, g : M \to \mathbb{R}$, vector fields

[37] See, e.g., [17, 27, 34, 41] for details.

$X, Y \in \sec TM$ and $\mathbf{P}, \mathbf{Q} \in \sec \mathcal{T}M$ we have

$$\begin{aligned}
\boldsymbol{\nabla}_{fX+gY}\mathbf{P} &= f\boldsymbol{\nabla}_X\mathbf{P} + g\boldsymbol{\nabla}_Y\mathbf{P},\\
\boldsymbol{\nabla}_X(\mathbf{P}+\mathbf{Q}) &= \boldsymbol{\nabla}_X\mathbf{P} + \boldsymbol{\nabla}_X\mathbf{Q},\\
\boldsymbol{\nabla}_X(f\mathbf{P}) &= f\boldsymbol{\nabla}_X(\mathbf{P}) + X(f)\mathbf{P},\\
\boldsymbol{\nabla}_X(\mathbf{P}\otimes\mathbf{Q}) &= \boldsymbol{\nabla}_X\mathbf{P}\otimes\mathbf{Q} + \mathbf{P}\otimes\boldsymbol{\nabla}_X\mathbf{Q}.
\end{aligned} \tag{4.293}$$

The *absolute differential* of $\mathbf{P} \in \sec T_s^r M$ is given by the mapping

$$\begin{aligned}
\boldsymbol{\nabla} : \sec T_s^r M &\to \sec T_{s+1}^r M,\\
\boldsymbol{\nabla}\mathbf{P}(\boldsymbol{X}, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s, \alpha_1, \ldots, \alpha_r) &= \boldsymbol{\nabla}_X\mathbf{P}(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s, \alpha_1, \ldots, \alpha_r),\\
\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s &\in \sec TM, \alpha_1, \ldots \alpha_r \in \sec T^*M.
\end{aligned} \tag{4.294}$$

To continue we must give the relationship between $\boldsymbol{\nabla}^+$, $\boldsymbol{\nabla}^-$ and $\boldsymbol{\nabla}$. Let $U \subset M$ and consider a chart of the maximal atlas of M covering U coordinate functions $\{\mathbf{x}^\mu\}$. Let $\boldsymbol{g} \in \sec T_2^0 M$ be a metric field for TM and $g \in \sec T_0^2 M$ the corresponding metric for TM (as introduced previously). Let $\{\partial_\mu\}$ be a basis for TU, $U \subset M$ and let $\{\theta^\mu = dx^\mu\}$ be the dual basis of $\{\partial_\mu\}$. The reciprocal basis of $\{\theta^\mu\}$ is denoted $\{\theta_\mu\}$, and we have $\mathtt{g}(\theta^\mu, \theta_\nu) = \delta_\nu^\mu$. Introduce next a set of differentiable functions $h_\mu^{\mathbf{a}}, h_{\mathbf{b}}^\nu : U \to \mathbb{R}$ such that:

$$h_{\mathbf{a}}^\mu q_\mu^{\mathbf{b}} = \boldsymbol{\delta}_{\mathbf{a}}^{\mathbf{b}}, \qquad h_{\mathbf{a}}^\mu h_\nu^{\mathbf{a}} = \delta_\nu^\mu. \tag{4.295}$$

Define

$$\boldsymbol{e}_{\mathbf{b}} = h_{\mathbf{b}}^\nu \partial_\nu$$

where the set $\{\boldsymbol{e}_{\mathbf{a}}\}$ is an orthonormal basis[38] for TU, i.e., $\boldsymbol{g}(\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \eta^{\mathbf{ab}}$. The reciprocal basis of $\{\boldsymbol{e}_{\mathbf{a}}\}$ is $\{\boldsymbol{e}^a\}$ and $\boldsymbol{g}(\boldsymbol{e}^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \delta_{\mathbf{b}}^{\mathbf{a}}$ The dual basis of TU is $\{\boldsymbol{\theta}^{\mathbf{a}}\}$, with $\boldsymbol{\theta}^{\mathbf{a}} = h_\mu^{\mathbf{a}} dx^\mu$ and $\mathtt{g}(\boldsymbol{\theta}^{\mathbf{a}}, \boldsymbol{\theta}^{\mathbf{b}}) = \eta^{\mathbf{ab}}$. Also, $\{\boldsymbol{\theta}_{\mathbf{b}}\}$ is the reciprocal basis of $\{\boldsymbol{\theta}^{\mathbf{a}}\}$, i.e. $\mathtt{g}(\boldsymbol{\theta}^{\mathbf{a}}, \boldsymbol{\theta}_{\mathbf{b}}) = \delta_{\mathbf{b}}^{\mathbf{a}}$. It is trivial to verify the formulas

$$\begin{aligned}
g_{\mu\nu} &= h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} \eta_{\mathbf{ab}}, \qquad & g^{\mu\nu} &= h_{\mathbf{a}}^\mu h_{\mathbf{b}}^\nu \eta^{\mathbf{ab}},\\
\eta_{\mathbf{ab}} &= h_{\mathbf{a}}^\mu h_{\mathbf{b}}^\nu g_{\mu\nu}, \qquad & \eta^{\mathbf{ab}} &= h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} g^{\mu\nu}.
\end{aligned} \tag{4.296}$$

[38] $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ is the orthonormal frame bundle (see Appendix A.1.2).

The connection coefficients associated to the respective covariant derivatives in the respective bases are denoted as:

$$\boldsymbol{\nabla}^{+}_{\partial_\mu}\partial_\nu = \Gamma^{\rho\cdot\cdot}_{\cdot\mu\nu}\partial_\rho, \quad \boldsymbol{\nabla}^{-}_{\partial_\sigma}\partial^\mu = -\Gamma^{\mu\cdot\cdot}_{\cdot\sigma\alpha}\partial^\alpha, \tag{4.297}$$

$$\boldsymbol{\nabla}^{+}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{e}_\mathbf{b} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\boldsymbol{e}_\mathbf{c}, \qquad \boldsymbol{\nabla}^{+}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{e}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\,\boldsymbol{e}^\mathbf{c}, \quad \boldsymbol{\nabla}^{+}_{\partial_\mu}\,\boldsymbol{e}_\mathbf{b} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mu\mathbf{b}}\boldsymbol{e}_\mathbf{c}, \tag{4.298}$$

$$\boldsymbol{\nabla}^{-}_{\partial_\mu}dx^\nu = -\Gamma^{\nu\cdot\cdot}_{\cdot\mu\alpha}dx^\alpha, \quad \boldsymbol{\nabla}^{-}_{\partial_\mu}\theta_\nu = \Gamma^{\rho\cdot\cdot}_{\cdot\mu\nu}\theta_\rho, \tag{4.299}$$

$$\boldsymbol{\nabla}^{-}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{\theta}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\boldsymbol{\theta}^\mathbf{c}, \quad \boldsymbol{\nabla}^{-}_{\partial_\mu}\boldsymbol{\theta}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mu\mathbf{a}}\boldsymbol{\theta}^\mathbf{a}, \tag{4.300}$$

$$\boldsymbol{\nabla}^{-}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{\theta}^\mathbf{b} = -\omega_{\mathbf{cab}}\boldsymbol{\theta}^\mathbf{c}, \tag{4.301}$$

$$\omega_{\mathbf{abc}} = \eta_{\mathbf{ad}}\omega^{\mathbf{d}\cdot\cdot}_{\cdot\mathbf{bc}} = -\omega_{\mathbf{cba}},\ \omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot} = \eta^{\mathbf{bk}}\omega_{\mathbf{kal}}\eta^{\mathbf{cl}},\ \omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot} = -\omega^{\mathbf{c}\cdot\mathbf{b}}_{\cdot\mathbf{a}\cdot} \tag{4.302}$$

$$\text{etc.}\ldots \tag{4.303}$$

To understood how $\boldsymbol{\nabla}$ works, consider its action, e.g., on the sections of $T^1_1M = TM \otimes T^*M$. For that case, if $\boldsymbol{X} \in \sec TM$, $C \in \sec T^*M$, we have that

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}^{+} \otimes \mathrm{Id}_{T^*M} + \mathrm{Id}_{TM} \otimes \boldsymbol{\nabla}^{-}, \tag{4.304}$$

and

$$\boldsymbol{\nabla}(\boldsymbol{X} \otimes C) = (\boldsymbol{\nabla}^{+}\boldsymbol{X}) \otimes C + X \otimes \boldsymbol{\nabla}^{-}C. \tag{4.305}$$

The general case, of $\boldsymbol{\nabla}$ acting on sections of $\mathcal{T}M$ is an obvious generalization of the previous one, and details are left to the reader.

For every vector field $\boldsymbol{V} \in \sec TU$ and a covector field $C \in \sec T^*U$ we have

$$\boldsymbol{\nabla}^{+}_{\partial_\mu}\boldsymbol{V} = \boldsymbol{\nabla}^{+}_{\partial_\mu}(V^\alpha\partial_\alpha), \quad \boldsymbol{\nabla}^{-}_{\partial_\mu}C = \boldsymbol{\nabla}^{-}_{\partial_\mu}(C_\alpha\theta^\alpha) \tag{4.306}$$

and using the properties of a covariant derivative operator introduced above, $\boldsymbol{\nabla}^{+}_{\partial_\mu}V$ can be written as:

$$\begin{aligned}\boldsymbol{\nabla}^{+}_{\partial_\mu}\boldsymbol{V} &= \boldsymbol{\nabla}^{+}_{\partial_\mu}(V^\alpha\partial_\alpha) = (\boldsymbol{\nabla}^{+}_{\partial_\mu}V)^\alpha\partial_\alpha \\ &= (\partial_\mu V^\alpha)\partial_\alpha + V^\alpha\boldsymbol{\nabla}^{+}_{\partial_\mu}\partial_\alpha \\ &= \left(\frac{\partial V^\alpha}{\partial x^\mu} + V^\rho\Gamma^{\alpha\cdot\cdot}_{\cdot\mu\rho}\right)\partial_\alpha := (\boldsymbol{\nabla}^{+}_\mu V^\alpha)\partial_\alpha,\end{aligned} \tag{4.307}$$

where it is to be kept in mind that the symbol $\boldsymbol{\nabla}^{+}_\mu V^\alpha$ is a short notation for

$$\boldsymbol{\nabla}^{+}_\mu V^\alpha := (\boldsymbol{\nabla}^{+}_{\partial_\mu}\boldsymbol{V})^\alpha. \tag{4.308}$$

Also, we have

$$\begin{aligned}\nabla^{-}_{\partial_\mu} C &= \nabla^{-}_{\partial_\mu}(C_\alpha \theta^\alpha) = (\nabla^{-}_{\partial_\mu} C)_\alpha \theta^\alpha \\ &= \left(\frac{\partial C_\alpha}{\partial x^\mu} - C_\beta \Gamma^{\beta\cdot\cdot}_{\cdot\mu\alpha}\right)\theta^\alpha, \\ &:= (\nabla^{-}_{\mu} C_\alpha)\theta^\alpha, \end{aligned} \tag{4.309}$$

where it is to be kept in mind that[39] that the symbol $\nabla^{-}_{\mu} C_\alpha$ is a short notation for

$$\nabla^{-}_{\mu} C_\alpha := (\nabla^{-}_{\partial_\mu} C)_\alpha. \tag{4.310}$$

Remark 4.149 When there is no possibility of confusion, we shall use only the symbol $\boldsymbol{\nabla}$ to denote any one of the covariant derivatives introduced above. However, the standard practice of many Physics textbooks of representing, $\nabla^{+}_{\mu} V^\alpha$ and $\nabla^{+}_{\mu} V^\alpha$ by $\nabla_\mu V^\alpha$ should be avoided whenever possible in order to not produce misunderstandings (see Exercise below).

Exercise 4.150 Calculate $\nabla^{-}_{\mu} h^{\mathbf{a}}_{\nu} := (\nabla^{-}_{\partial_\mu} \theta^{\mathbf{a}})_\nu = (\nabla^{-}_{\partial_\mu} h^{\mathbf{a}}_{\alpha} \partial^\alpha)_\nu$ and $\nabla^{+}_{\mu} h^{\mathbf{a}}_{\nu} := (\nabla^{+}_{\partial_\mu} \partial_\nu)^{\mathbf{a}} = (\nabla^{+}_{\partial_\mu} h^{\mathbf{b}}_{\nu} \boldsymbol{e}_{\mathbf{b}})^{\mathbf{a}}$. Show that in general $\nabla^{-}_{\mu} h^{\mathbf{a}}_{\nu} \neq \nabla^{+}_{\mu} h^{\mathbf{a}}_{\nu} \neq 0$ and that

$$\partial_\mu h^{\mathbf{a}}_{\nu} + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_{\nu} - \Gamma^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_{\nu} = 0. \tag{4.311}$$

Exercise 4.151 Define the object

$$\mathbf{e} = \boldsymbol{e}_{\mathbf{a}} \otimes \theta^{\mathbf{a}} = h^{\mathbf{a}}_{\mu} \partial_\mu \otimes dx^\mu \in \sec T^1_1 M, \tag{4.312}$$

which is clearly the identity endomorphism acting on sections of TU. Show that

$$\nabla_\mu h^{\mathbf{a}}_{\nu} := (\nabla_{\partial_\mu} \mathbf{e})^{\mathbf{a}}_{\nu} = \partial_\mu h^{\mathbf{a}}_{\nu} + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_{\nu} - \Gamma^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_{\nu} = 0. \tag{4.313}$$

Remark 4.152 Equation (4.313) is presented in many textbooks (see., e.g., [2, 12, 37]) under the name 'tetrad postulate'. In that books, since authors do not distinguish clearly the derivative operators ∇^{+}, ∇^{-} and ∇, Eq. (4.313) becomes sometimes misunderstood as meaning $\nabla^{-}_{\mu} h^{\mathbf{a}}_{\nu}$ or $\nabla^{+}_{\mu} h^{\mathbf{a}}_{\nu}$, thus generating a big confusion. For a discussion of this issue see [33].

[39] Recall that other authors prefer the notations $(\nabla_{\partial_\mu} V)^\alpha := V^\alpha_{:\mu}$ and $(\nabla_{\partial_\mu} C)_\alpha := C_{\alpha:\mu}$. What is important is always to have in mind the meaning of the symbols.

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