Chapter 3
The Hidden Geometrical Nature of Spinors

Abstract This chapter reviews the classification of the real and complex Clifford algebras and analyze the relationship between some particular algebras that are important in physical applications, namely the quaternion algebra ($\mathbb{R}_{0,2}$), Pauli algebra ($\mathbb{R}_{3,0}$), the spacetime algebra ($\mathbb{R}_{1,3}$), the Majorana algebra ($\mathbb{R}_{3,1}$) and the Dirac algebra ($\mathbb{R}_{4,1}$). A detailed and original theory disclosing the hidden geometrical meaning of spinors is given through the introduction of the concepts of algebraic, covariant and Dirac-Hestenes spinors. The relationship between these kinds of spinors (that carry the same mathematical information) is elucidated with special emphasis for cases of physical interest. We investigate also how to reconstruct a spinor from their so-called bilinear invariants and present Lounesto's classification of spinors. Also, Majorana, Weyl spinors, the dotted and undotted algebraic spinors are discussed with the Clifford algebra formalism.

3.1 Notes on the Representation Theory of Associative Algebras

To achieve our goal mentioned in Chap. 1 of disclosing the real secret geometrical meaning of Dirac spinors, we shall need to briefly recall some few results of the theory of representations of associative algebras. Propositions are presented without proofs and the interested reader may consult [3, 8, 12, 16, 20, 21] for details.

Let $\mathbf{V}$ be a *finite* dimensional *linear* space over $\mathbb{K}$ (a division ring). Suppose that $\dim_{\mathbb{K}} \mathbf{V} = n$, where $n \in \mathbb{Z}$. We are interested in what follows in the cases where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In this case we also call $\mathbf{V}$ a *vector space* over $\mathbb{K}$. When $\mathbb{K} = \mathbb{H}$ it is necessary to distinguish between right or left $\mathbb{H}$-linear spaces and in this case $\mathbf{V}$ will be called a right or left $\mathbb{H}$-module. Recall that $\mathbb{H}$ is a division ring (sometimes called a noncommutative field or a skew field) and since $\mathbb{H}$ has a natural vector space structure over the real field, then $\mathbb{H}$ is also a division algebra.

Definition 3.1 Let $\mathbf{V}$ be a vector space over $\mathbb{R}$ and $\dim_{\mathbb{R}} \mathbf{V} = 2m = n$. A linear mapping

$$\mathbf{J} : \mathbf{V} \to \mathbf{V} \tag{3.1}$$

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such that

$$\mathbf{J}^2 = -\mathrm{Id}_{\mathbf{V}}, \tag{3.2}$$

s called a complex structure mapping.

Definition 3.2 Let $\mathbf{V}$ be as in the previous definition. The pair $(\mathbf{V}, \mathbf{J})$ is called a complex vector space structure and denote by $\mathbf{V}_{\mathbb{C}}$ if the following product holds. Let $\mathbb{C} \ni z = a + \mathrm{i}b$ ($\mathrm{i} = \sqrt{-1}$) and let $\mathbf{v} \in \mathbf{V}$. Then

$$z\mathbf{v} = (a + \mathrm{i}b)\mathbf{v} = a\mathbf{v} + b\mathbf{J}\mathbf{v}. \tag{3.3}$$

It is obvious that $\dim_{\mathbb{C}} = \frac{m}{2}$.

Definition 3.3 Let $\mathbf{V}$ be a vector space over $\mathbb{R}$. A *complexification* of $\mathbf{V}$ is a complex structure associated with the real vector space $\mathbf{V} \oplus \mathbf{V}$. The resulting complex vector space is denoted by $V^{\mathbb{C}}$. Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}$. Elements of $\mathbf{V}^{\mathbb{C}}$ are usually denoted by $\mathbf{c} = \mathbf{v} + i\mathbf{w}$, and if $\mathbb{C} \ni z = a + \mathrm{i}b$ we have

$$z\mathbf{c} = a\mathbf{v} - b\mathbf{w} + i(a\mathbf{w} + b\boldsymbol{v}). \tag{3.4}$$

Of course, we have that $\dim_{\mathbb{C}} \mathbf{V}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathbf{V}$.

Definition 3.4 A $\mathbb{H}$-module is a real vector space $\mathbf{S}$ carrying three linear transformation, $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ each one of them satisfying

$$\begin{aligned} &\mathbf{I}^2 = \mathbf{J}^2 = -\mathrm{Id}_{\mathbf{S}}, \\ &\mathbf{IJ} = -\mathbf{JI} = \mathbf{K},\ \mathbf{JK} = -\mathbf{KJ} = \mathbf{I},\ \mathbf{KI} = -\mathbf{IK} = \mathbf{J}. \end{aligned} \tag{3.5a}$$

Exercise 3.5 Show that $\mathbf{K}^2 = -\mathrm{Id}_{\mathbf{S}}$

In what follows $\mathcal{A}$ denotes an *associative* algebra on the commutative field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{F} \subseteq \mathcal{A}$.

Definition 3.6 Any subset $I \subseteq \mathcal{A}$ such that

$$\begin{aligned} &a\psi \in I, \forall a \in \mathcal{A},\ \forall \psi \in I, \\ &\psi + \phi \in I, \forall \psi, \phi \in I \end{aligned} \tag{3.6}$$

is called a left ideal of A.

Remark 3.7 An analogous definition holds for right ideals where Eq. (3.6) reads $\psi a \in I, \forall a \in \mathcal{A}, \forall \psi \in I$, for bilateral ideals where in this case Eq. (3.6) reads $a\psi b \in I, \forall a, b \in \mathcal{A}, \forall \psi \in I$.

Definition 3.8 An associative algebra $\mathcal{A}$ is simple if the only bilateral ideals are the zero ideal and $\mathcal{A}$ itself.

Not all algebras are simple and in particular *semi-simple* algebras are important for our considerations. A definition of semi-simple algebras requires the introduction of the concepts of *nilpotent* ideals and radicals. To define these concepts adequately would lead us to a long incursion on the theory of associative algebras, so we avoid to do that here. We only quote that semi-simple algebras are the direct sum of simple algebras and of course simple algebras are semi simple. Then, for our objectives in this chapter the study of semi-simple algebras is reduced to the study of simple algebras.

Definition 3.9 We say that $\mathrm{e} \in \mathcal{A}$ is an *idempotent* element if $\mathrm{e}^2 = \mathrm{e}$. An idempotent is said to be *primitive* if it cannot be written as the sum of two non zero annihilating (or orthogonal) idempotent, i.e., $\mathrm{e} \neq \mathrm{e}_1 + \mathrm{e}_2$, with $\mathrm{e}_1\mathrm{e}_2 = \mathrm{e}_2\mathrm{e}_1 = 0$ and $\mathrm{e}_1^2 = \mathrm{e}_1$, $\mathrm{e}_2^2 = \mathrm{e}_2$.

We give without proofs the following theorems valid for semi-simple (and thus simple) algebras $\mathcal{A}$:

Theorem 3.10 *All minimal left (respectively right) ideals of semi-simple $\mathcal{A}$ are of the form $J = \mathcal{A}\mathrm{e}$ (respectively $\mathrm{e}\mathcal{A}$), where e is a primitive idempotent of $\mathcal{A}$.*

Theorem 3.11 *Two minimal left ideals of a semi-simple algebra $\mathcal{A}$, $J = \mathcal{A}\mathrm{e}$ and $J = \mathcal{A}\mathrm{e}'$ are isomorphic, if and only if, there exist a non null $Y' \in J'$ such that $J' = JY'$.*

Let $\mathcal{A}$ be an associative and simple algebra on the *field* $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$, and let $\mathbf{S}$ be a finite dimensional linear space over a division ring $\mathbb{K} \supseteq \mathbb{F}$ and let $\mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S} = \mathrm{Hom}_{\mathbb{K}}(\mathbf{S}, \mathbf{S})$ be the endomorphism algebra of $\mathbf{S}$.[1]

Definition 3.12 A representation of $\mathcal{A}$ in $\mathbf{S}$ is a $\mathbb{K}$ algebra homomorphism[2] $\rho : \mathcal{A} \to \mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S}$ which maps the unit element of $\mathcal{A}$ to $\mathbf{Id}_{\mathbf{E}}$. The dimension $\mathbb{K}$ of $\mathbf{S}$ is called the degree of the representation.

Definition 3.13 The addition in $\mathbf{S}$ together with the mapping $\mathcal{A} \times \mathbf{S} \to \mathbf{S}$, $(a, x) \mapsto \rho(a)x$ turns $\mathbf{S}$ in a left $\mathcal{A}$-module,[3] called the left representation module.

Remark 3.14 It is important to recall that when $\mathbb{K} = \mathbb{H}$ the usual recipe for $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S}, \mathbf{S})$ to be a linear space over $\mathbb{H}$ fails and in general $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S}, \mathbf{S})$ is considered as a linear space over $\mathbb{R}$, which is the centre of $\mathbb{H}$.

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[1]Recall that $\mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ is the algebra of linear transformations of a finite dimensional vector space $\mathbf{V}$ over $\mathbb{K}$ into a finite vector space $\mathbf{W}$ over $\mathbb{K}$. When $\mathbf{V} = \mathbf{W}$ the set $\mathrm{End}_{\mathbb{K}}\mathbf{V} = \mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{V})$ is called the set of endomorphisms of $\mathbf{V}$.

[2]We recall that a $\mathbb{K}$-algebra homomorphism is a $\mathbb{K}$-linear map ρ such that $\forall X, Y \in \mathcal{A}$, $\rho(XY) = \rho(X)\rho(Y)$.

[3]We recall that there are left and right modules, so we can also define right modular representations of $\mathcal{A}$ by defining the mapping $\mathbf{S} \times \mathcal{A} \to \mathbf{S}$, $(x, a) \mapsto x\rho(a)$. This turns $\mathbf{S}$ in a right $\mathcal{A}$-module, called the right *representation module*.

Remark 3.15 We also have that if $\mathcal{A}$ is an algebra on $\mathbb{F}$ and $\mathbf{S}$ is an $\mathcal{A}$-module, then $\mathbf{S}$ can always be considered as a vector space over $\mathbb{F}$ and if $a \in \mathcal{A}$, the mapping $\chi : a \to \chi_a$ with $\chi_a(\mathbf{s}) = a\mathbf{s}, \mathbf{s} \in \mathbf{S}$, is a homomorphism $\mathcal{A} \to \mathrm{End}_{\mathbb{F}}\mathbf{S}$, and so it is a representation of $\mathcal{A}$ in $\mathbf{S}$. The study of $\mathcal{A}$ modules is then equivalent to the study of the $\mathbb{F}$ representations of $\mathcal{A}$.

Definition 3.16 A representation ρ is faithful if its kernel is zero, i.e., $\rho(a)x = 0, \forall x \in \mathbf{S} \Rightarrow a = 0$. The kernel of ρ is also known as the annihilator of its module.

Definition 3.17 ρ is said to be simple or irreducible if the only invariant subspaces of $\rho(a), \forall a \in \mathcal{A}$, are $\mathbf{S}$ and $\{0\}$.

Then, the representation module is also simple. That means that it has no proper submodules.

Definition 3.18 ρ is said to be semi-simple, if it is the direct sum of simple modules, and in this case $\mathbf{S}$ is the direct sum of subspaces which are globally invariant under $\rho(a), \forall a \in \mathcal{A}$.

When no confusion arises $\rho(a)x$ may be denoted by $a * x$ or ax.

Definition 3.19 Two $\mathcal{A}$-modules $\mathbf{S}$ and $\mathbf{S}'$ (with the "exterior" multiplication being denoted respectively by $\diamondsuit$ and $*$) are isomorphic if there exists a bijection $\varphi : \mathbf{S} \to \mathbf{S}'$ such that,

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y), \ \forall x, y \in \mathbf{S}, \\ \varphi(a \diamondsuit x) &= a * \varphi(x), \ \forall a \in \mathcal{A}, \end{aligned} \tag{3.7}$$

and we say that the representations ρ and ρ' of A are equivalent if their modules are isomorphic.

This implies the existence of a $\mathbb{K}$-linear isomorphism $\varphi : \mathbf{S} \to \mathbf{S}'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi, \forall a \in \mathcal{A}$ or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$. If $\dim \mathbf{S} = n$, then $\dim \mathbf{S}' = n$.

Definition 3.20 A complex representation of $\mathcal{A}$ is simply a real representation $\rho : \mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\mathbf{S}, \mathbf{S})$ for which

$$\rho(Y) \circ \mathbf{J} = \mathbf{J} \circ \rho(Y), \forall Y \in \mathcal{A}. \tag{3.8}$$

This means that the image of ρ commutes with the subalgebra generated by $\{\mathrm{Id}_{\mathbf{S}}, \mathbf{J}\} \sim \mathbb{C}$.

Definition 3.21 A quaternionic representation of $\mathcal{A}$ is a representation $\rho : \mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\mathbf{S}, \mathbf{S})$ such that

$$\rho(Y) \circ \mathbf{I} = \mathbf{I} \circ \rho(Y),\ \rho(Y) \circ \mathbf{J} = \mathbf{J} \circ \rho(Y),\ \rho(Y) \circ \mathbf{K} = \mathbf{K} \circ \rho(Y), \forall Y \in \mathcal{A}. \tag{3.9}$$

This means that the representation ρ has a commuting subalgebra isomorphic to the quaternion ring.

The following theorem is crucial:

Theorem 3.22 *(Wedderburn). If $\mathcal{A}$ is simple algebra over $\mathbb{F}$ then $\mathcal{A}$ is isomorphic to $\mathbb{D}(m)$, where $\mathbb{D}(m)$ is a matrix algebra with entries in $\mathbb{D}$ (a division algebra), and m and $\mathbb{D}$ are unique (modulo isomorphisms).*

3.2 Real and Complex Clifford Algebras and Their Classification

Now, it is time to specialize the previous results to the Clifford algebras on the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We are particularly interested in the case of *real* Clifford algebras. In what follows we take $\mathbf{V} = \mathbb{R}^n$. We denote as in the previous chapter by $\mathbb{R}^{p,q}$ $(n = p + q)$ the real vector space $\mathbb{R}^n$ endowed with a nondegenerate metric $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Let $\{E_i\}, (i = 1, 2, \ldots, n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$\mathbf{g}(E_i, E_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, 2, \ldots p, \\ -1, & i = j = p + 1, \ldots, p + q = n, \\ 0, & i \neq j. \end{cases} \tag{3.10}$$

We recall (Definition 2.37 that the Clifford algebra $\mathbb{R}_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q})$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}, (i = 1, 2, \ldots, n)$ such that $E_i^2 = \mathbf{g}(E_i, E_i)$, $E_iE_j = -E_jE_i (i \neq j)$, and $E_1E_2 \ldots E_n \neq \pm 1$.

$\mathbb{R}_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\bigwedge^k \mathbb{R}^n$ of dimensions $\binom{n}{k}, 0 \leq k \leq n$. The canonical basis of $\bigwedge^k \mathbb{R}^n$ is given by the elements $e_A = E_{\alpha_1} \cdots E_{\alpha_k}, 1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. The element $e_J = E_1 \cdots E_n \in \bigwedge^n \mathbb{R}^n \hookrightarrow \mathbb{R}_{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1, \ldots, E_n \in \bigwedge^1 \mathbb{R}^n \equiv \mathbb{R}^n$. The center $\mathbb{R}_{p,q}$ is $\bigwedge^0 \mathbb{R}^n \equiv \mathbb{R}$ if n is even and it is the direct sum $\bigwedge^0 \mathbb{R}^n \oplus \bigwedge^0 \mathbb{R}^n$ if n is odd.[4]

All Clifford algebras are semi-simple. If $p + q = n$ is even, $\mathbb{R}_{p,q}$ is simple and if $p + q = n$ is odd we have the following possibilities:

(a) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p - q \neq 1 \pmod 4 \leftrightarrow$ center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{C}$;

(b) $\mathbb{R}_{p,q}$ is not simple (but is a direct sum of two simple algebras) $\leftrightarrow e_J^2 = +1 \leftrightarrow p - q = 1 \pmod 4 \leftrightarrow$ center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

[4] For a proof see [20].

Now, for $\mathbb{R}_{p,q}$ the division algebras $\mathbb{D}$ are the division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The explicit isomorphism can be discovered with some hard but not difficult work. It is possible to give a general classification of all real (and also the complex) Clifford algebras and a classification table can be found, e.g., in [20]. One convenient table is the following one (where $\mu = [n/2]$ means the integer part of $n/2$).

We denoted by $\mathbb{R}^0_{p,q}$ the even subalgebra of $\mathbb{R}_{p,q}$ and by $\mathbb{R}^1_{p,q}$ the set of odd elements of $\mathbb{R}_{p,q}$. The following very important result holds true

Proposition 3.23 $\mathbb{R}^0_{p,q} \simeq \mathbb{R}_{p,q-1}$ *and also* $\mathbb{R}^0_{p,q} \simeq \mathbb{R}_{q,p-1}$.

Now, to complete the classification we need the following theorem:

Theorem 3.24 (Periodicity)[5] *We have*

$$\begin{aligned} \mathbb{R}_{n+8} &= \mathbb{R}_{n,0} \otimes \mathbb{R}_{8,0} \quad \mathbb{R}_{0,n+8} = \mathbb{R}_{0,n} \otimes \mathbb{R}_{0,8} \\ \mathbb{R}_{p+8,q} &= \mathbb{R}_{p,q} \otimes \mathbb{R}_{8,0} \quad \mathbb{R}_{p,q+8} = \mathbb{R}_{p,q} \otimes \mathbb{R}_{0,8}. \end{aligned} \tag{3.11}$$

Remark 3.25 We emphasize here that since the general results concerning the representations of simple algebras over a field $\mathbb{F}$ applies to the Clifford algebras $\mathbb{R}_{p,q}$ we can talk about real, complex or quaternionic representation of a given Clifford algebra, even if the natural matrix identification is not a matrix algebra over one of these fields. A case that we shall need is that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$. But it is clear that $\mathbb{R}_{1,3}$ has a complex representation, for any quaternionic representation of $\mathbb{R}_{p,q}$ is automatically *complex*, once we restrict $\mathbb{C} \subset \mathbb{H}$ and of course, the complex dimension of any $\mathbb{H}$-module must be even. Also, any complex representation of $\mathbb{R}_{p,q}$ extends automatically to a representation of $\mathbb{C} \otimes \mathbb{R}_{p,q}$.

Remark 3.26 $\mathbb{C} \otimes \mathbb{R}_{p,q}$ is isomorphic to the complex Clifford algebra $\mathcal{C}\ell_{p+q}$. The algebras $\mathbb{C}$ and $\mathbb{R}_{p,q}$ are subalgebras of $\mathcal{C}\ell_{p+q}$

3.2.1 Pauli, Spacetime, Majorana and Dirac Algebras

For the purposes of our book we shall need to have in mind that:

$\mathbb{R}_{0,1} \simeq \mathbb{C}$,
$\mathbb{R}_{0,2} \simeq \mathbb{H}$,
$\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$,
$\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$,
$\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$,
$\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$.

(3.12)

[5] See [20].

$\mathbb{R}_{3,0}$ is called the Pauli algebra, $\mathbb{R}_{1,3}$ is called the *spacetime* algebra, $\mathbb{R}_{3,1}$ is called *Majorana* algebra and $\mathbb{R}_{4,1}$ is called the *Dirac* algebra. Also, the following particular results, which can be easily proved, will be used many times in what follows:

$$\mathbb{R}^0_{1,3} \simeq \mathbb{R}^0_{3,1} = \mathbb{R}_{3,0}, \quad \mathbb{R}^0_{4,1} \simeq \mathbb{R}_{1,3}, \quad \mathbb{R}^0_{1,4} \simeq \mathbb{R}_{1,3},$$
$$\mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{3,1}, \quad \mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{3,1}. \tag{3.13}$$

In words: the even subalgebras of both the spacetime and Majorana algebras is the Pauli algebra. The even subalgebra of the Dirac algebra is the spacetime algebra and finally the Dirac algebra is the complexification of the spacetime algebra or of the Majorana algebra.

Equation (3.13) show moreover, in view of Remark 3.26 that the spacetime algebra has also a matrix representation in $\mathbb{C}(4)$. Obtaining such a representation is very important for the introduction of the concept of a Dirac-Hestenes spinor, an important ingredient of the present work.

3.3 The Algebraic, Covariant and Dirac-Hestenes Spinors

3.3.1 Minimal Lateral Ideals of $\mathbb{R}_{p,q}$

We now give some results concerning the minimal lateral ideals of $\mathbb{R}_{p,q}$.

Theorem 3.27 *The maximum number of pairwise orthogonal idempotents in* $\mathbb{K}(m)$ *(where* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$*) is* m *.*

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is then characterized by a spectral set $\{\mathrm{e}_{pq,j}\}$ of idempotents elements of $\mathbb{R}_{p,q}$ such that:

(a) $\sum_{i=1}^{n} \mathrm{e}_{pq,j} = 1$;
(b) $\mathrm{e}_{pq,j}\mathrm{e}_{pq,k} = \delta_{jk}\mathrm{e}_{pq,j}$;
(c) the rank of $\mathrm{e}_{pq,j}$ is minimal and non zero, i.e., is primitive.

By rank of $\mathrm{e}_{pq,j}$ we mean the rank of the $\bigwedge \mathbb{R}^{p,q}$ morphism, $\mathrm{e}_{pq,j} : \phi \longmapsto \phi\mathrm{e}_{pq,j}$. Conversely, any $\phi \in I_{pq,j}$ can be characterized by an idempotent $\mathrm{e}_{pq,j}$ of minimal rank $\neq 0$, with $\phi = \phi\mathrm{e}_{pq,j}$.

We now need to know the following theorem [13]:

Theorem 3.28 *A minimal left ideal of* $\mathbb{R}_{p,q}$ *is of the type*

$$I_{pq} = \mathbb{R}_{p,q}\mathrm{e}_{pq}, \tag{3.14}$$

where

$$\mathrm{e}_{pq} = \frac{1}{2}(1 + e_{\alpha_1}) \cdots \frac{1}{2}(1 + e_{\alpha_k}) \tag{3.15}$$

is a primitive idempotent of $R_{p,q}$ *and where* $e_{\alpha_1}, \ldots, e_{\alpha_k}$ *are commuting elements in the canonical basis of* $\mathbb{R}_{p,q}$ *(generated in the standard way through the elements of a basis* $(E_{1,\ldots}, E_p, E_{p+1}, \ldots E_{p+q})$ *of* $\mathbb{R}^{p,q}$*) such that* $(e_{\alpha_i})^2 = 1$, $(i = 1, 2, \ldots, k)$ *generate a group of order* 2^k, $k = q - r_{q-p}$ *and* r_i *are the Radon-Hurwitz numbers, defined by the recurrence formula* $r_{i+8} = r_i + 4$ *and*

i		0	1	2	3	4	5	6	7
r_i		0	1	2	2	3	3	3	3

. (3.16)

Recall that $\mathbb{R}_{p,q}$ is a ring and the minimal lateral ideals are modules over the ring $\mathbb{R}_{p,q}$. They are *representation modules* of $\mathbb{R}_{p,q}$, and indeed we have (recall the above table) the following theorem [13]:

Theorem 3.29 *If* $p + q$ *is even or odd with* $p - q \neq 1 (\mathrm{mod}\, 4)$*, then*

$$\mathbb{R}_{p,q} = \mathrm{Hom}_{\mathbb{K}}(I_{pq}, I_{pq}) \simeq \mathbb{K}(m), \tag{3.17}$$

where (as we already know) $\mathbb{K} = \mathbb{R}$*,* $\mathbb{C}$ *or* $\mathbb{H}$*. Also,*

$$\dim_{\mathbb{K}}(I_{pq}) = m, \tag{3.18}$$

and

$$\mathbb{K} \simeq \mathrm{e}\mathbb{K}(m)\mathrm{e}, \tag{3.19}$$

where e *is the representation of* e_{pq} *in* $\mathbb{K}(m)$*.*

If $p + q = n$ *is odd, with* $p - q = 1\ (\mathrm{mod}\, 4)$*, then*

$$\mathbb{R}_{p,q} = \mathrm{Hom}_{\mathbb{K}}(I_{pq}, I_{pq}) \simeq \mathbb{K}(m) \oplus \mathbb{K}(m), \tag{3.20}$$

with

$$\dim_{\mathbb{K}}(I_{pq}) = m \tag{3.21}$$

and

$$\begin{gathered} \mathrm{e}\mathbb{K}(m)\mathrm{e} \simeq \mathbb{R} \oplus \mathbb{R} \\ \textit{or} \\ \mathrm{e}\mathbb{K}(m)\mathrm{e} \simeq \mathbb{H} \oplus \mathbb{H}. \end{gathered} \tag{3.22}$$

With the above isomorphisms we can immediately identify the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $\mathbb{K}(m)$.

Table 3.1 Representation of the Clifford algebras $\mathbb{R}_{p,q}$ as matrix algebras

$p-q$ mod 8	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^\mu)$	$\mathbb{R}(2^\mu) \oplus \mathbb{R}(2^\mu)$	$\mathbb{R}(2^\mu)$	$\mathbb{C}(2^\mu)$	$\mathbb{H}(2^{\mu-1})$	$\mathbb{H}(2^{\mu-1}) \oplus \mathbb{H}(2^{\mu-1})$	$\mathbb{H}(2^{\mu-1})$	$\mathbb{C}(2^\mu)$

3.3.2 Algorithm for Finding Primitive Idempotents of $\mathbb{R}_{p,q}$

With the ideas introduced above it is now a simple exercise to find primitive idempotents of $\mathbb{R}_{p,q}$. First we look at Table 3.1 and find the matrix algebra to which our particular Clifford algebra $\mathbb{R}_{p,q}$ is isomorphic. Suppose $\mathbb{R}_{p,q}$ is simple.[6] Let $\mathbb{R}_{p,q} \simeq \mathbb{K}(m)$ for a particular $\mathbb{K}$ and m. Next we take an element $\mathbf{e}_{\alpha_1} \in \{\mathbf{e}_A\}$ from the canonical basis $\{\mathbf{e}_A\}$ of $\mathbb{R}_{p,q}$ such that

$$\mathbf{e}_{\alpha_1}^2 = 1. \tag{3.23}$$

Next we construct the idempotent $\mathrm{e}_{pq} = (1 + \mathbf{e}_{\alpha_1})/2$ and the ideal $I_{pq} = \mathbb{R}_{p,q}\mathrm{e}_{pq}$ and calculate $\dim_{\mathbb{K}}(I_{pq})$. If $\dim_{\mathbb{K}}(I_{pq}) = m$, then e_{pq} is primitive. If $\dim_{\mathbb{K}}(I_{pq}) \neq m$, we choose $\mathbf{e}_{\alpha_2} \in \{\mathbf{e}_A\}$ such that $\mathbf{e}_{\alpha_2}$ commutes with $\mathbf{e}_{\alpha_1}$ and $\mathbf{e}_{\alpha_2}^2 = 1$ and construct the idempotent $\mathrm{e}'_{pq} = (1 + \mathbf{e}_{\alpha_1})(1 + \mathbf{e}_{\alpha_2})/4$. If $\dim_{\mathbb{K}}(I'_{pq}) = m$, then e'_{pq} is primitive. Otherwise we repeat the procedure. According to Theorem 3.28 the procedure is finite.

3.3.3 $\mathbb{R}^{\star}_{p,q}$, Clifford, Pinor and Spinor Groups

The set of the invertible elements of $\mathbb{R}_{p,q}$ constitutes a non-abelian group which we denote by $\mathbb{R}^{\star}_{p,q}$. It acts naturally on $\mathbb{R}_{p,q}$ as an algebra homomorphism through its twisted adjoint representation ($\hat{\mathrm{A}}\mathrm{d}$) or adjoint representation (Ad)

$$\hat{\mathrm{A}}\mathrm{d} : \mathbb{R}^{\star}_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q});\ u \mapsto \mathrm{Ad}_u,\ \text{with } \mathrm{Ad}_u(x) = ux\hat{u}^{-1}, \tag{3.24}$$

$$\mathrm{Ad} : \mathbb{R}^{\star}_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q});\ u \mapsto \mathrm{Ad}_u,\ \text{with } \mathrm{Ad}_u(x) = uxu^{-1} \tag{3.25}$$

[6] Once we know the algorithm for a simple Clifford algebra it is straightforward to devise an algorithm for the semi-simple Clifford algebras.

Definition 3.30 The Clifford-Lipschitz group is the set

$$\Gamma_{p,q} = \left\{ u \in \mathbb{R}^{\star}_{p,q} \;\middle|\; \forall x \in \mathbb{R}^{p,q},\ ux\hat{u}^{-1} \in \mathbb{R}^{p,q} \right\}, \tag{3.26a}$$

or

$$\Gamma_{p,q} = \left\{ u \in \mathbb{R}^{\star(0)}_{p,q} \cup \mathbb{R}^{\star(1)}_{p,q} \;\middle|\; \forall x \in \mathbb{R}^{p,q},\ uxu^{-1} \in \mathbb{R}^{p,q} \right\}, \tag{3.26b}$$

Note in Eq. (3.26b) the restriction to the even ($\mathbb{R}^{\star(0)}_{p,q}$) and odd ($\mathbb{R}^{\star(1)}_{p,q}$) parts of $\mathbb{R}^{\star}_{p,q}$.

Definition 3.31 The set $\Gamma^0_{p,q} = \Gamma_{p,q} \cap \mathbb{R}^0_{p,q}$ is called special Clifford-Lipschitz group.

Definition 3.32 The Pinor group $\text{Pin}_{p.q}$ is the subgroup of $\Gamma_{p,q}$ such that

$$\text{Pin}_{p,q} = \left\{ u \in \Gamma_{p,q} \,|N(u) = \pm 1 \right\}, \tag{3.27}$$

where

$$N : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, N(x) = \langle \bar{x}x \rangle_0. \tag{3.28}$$

Definition 3.33 The Spin group $\text{Spin}_{p,q}$ is the set

$$\text{Spin}_{p,q} = \left\{ u \in \Gamma^0_{p,q} | N(u) = \pm 1 \right\}. \tag{3.29}$$

It is easy to see that $\text{Spin}_{p,q}$ is not connected.

Definition 3.34 The Special Spin Group $\text{Spin}^e_{p,q}$ is the set

$$\text{Spin}^e_{p,q} = \left\{ u \in \text{Spin}_{p,q} | N(u) = +1 \right\}. \tag{3.30}$$

The superscript e, means that $\text{Spin}^e_{p,q}$ is the connected component to the identity. We can prove that $\text{Spin}^e_{p,q}$ is connected for all pairs (p, q) with the exception of $\text{Spin}^e(1, 0) \simeq \text{Spin}^e(0, 1)$.

We recall now some classical results [17] associated with the pseudo-orthogonal groups $\text{O}_{p,q}$ of a vector space $\mathbb{R}^{p,q}$ ($n = p+q$) and its subgroups. Let $\mathbf{G}$ be a diagonal $n \times n$ matrix whose elements are G_{ij}

$$\mathbf{G} = [G_{ij}] = \text{diag}(1, 1, \ldots, -1, -1, \ldots - 1), \tag{3.31}$$

with p positive and q negative numbers.

Definition 3.35 $\mathrm{O}_{p,q}$ is the set of $n \times n$ real matrices $\mathbf{L}$ such that

$$\mathbf{LGL}^T = \mathbf{G}, \;\; \det \mathbf{L}^2 = 1. \tag{3.32}$$

Equation (3.32) shows that $\mathrm{O}_{p,q}$ is not connected.

Definition 3.36 $\mathrm{SO}_{p,q}$, the special (proper) pseudo orthogonal group is the set of $n \times n$ real matrices $\mathbf{L}$ such that

$$\mathbf{LGL}^T = \mathbf{G}, \;\; \det \mathbf{L} = 1. \tag{3.33}$$

When $p = 0$ ($q = 0$) $\mathrm{SO}_{p,q}$ is connected. However, $\mathrm{SO}_{p,q}$ (for, $p, q \neq 0$) is not connected and has two connected components for $p, q \geq 1$.

Definition 3.37 The group $\mathrm{SO}^e_{p,q}$, the connected component to the identity of $\mathrm{SO}_{p,q}$ will be called the special orthochronous pseudo-orthogonal group.[7]

Theorem 3.38 *$Ad_{|\mathrm{Pin}_{p,q}} : \mathrm{Pin}_{p,q} \to \mathrm{O}_{p,q}$ is onto with kernel $\mathbb{Z}_2$.*

$Ad_{|\mathrm{Spin}_{p,q}} : \mathrm{Spin}_{p,q} \to \mathrm{SO}_{p,q}$ is onto with kernel $\mathbb{Z}_2$. $Ad_{|\mathrm{Spin}^e_{p,q}} : \mathrm{Spin}^e_{p,q} \to \mathrm{SO}^e_{p,q}$ is onto with kernel $\mathbb{Z}_2$.

We have,

$$\mathrm{O}_{p,q} = \frac{\mathrm{Pin}_{p,q}}{\mathbb{Z}_2}, \;\; \mathrm{SO}_{p,q} = \frac{\mathrm{Spin}_{p,q}}{\mathbb{Z}_2}, \;\; \mathrm{SO}^e_{p,q} = \frac{\mathrm{Spin}^e_{p,q}}{\mathbb{Z}_2}. \tag{3.34}$$

The group homomorphism between $\mathrm{Spin}^e_{p,q}$ and $\mathrm{SO}^e(p, q)$ will be denoted by

$$\mathbf{L} : \mathrm{Spin}^e_{p,q} \to \mathrm{SO}^e_{p,q}. \tag{3.35}$$

The following theorem that first appears in [20] is very important.

Exercise 3.39 (Porteous). Show that for $p + q \leq 4$, $\mathrm{Spin}^e(p, q) = \{u \in \mathbb{R}_{p,q} | u\tilde{u} = 1\}$.

Solution We must show that for any $u \in \mathbb{R}^0_{p,q}$, $N(u) = \pm 1$ and $\boldsymbol{x} \in \mathbb{R}^{p,q}$ we have that $\hat{\mathrm{A}}\mathrm{d}_u(\boldsymbol{x}) \in \mathbb{R}^{p,q}$. But when $u \in \mathbb{R}^0_{p,q}$, $\hat{\mathrm{A}}\mathrm{d}_u(\boldsymbol{x}) = u\boldsymbol{x}u^{-1}$. We must then show that

$$\boldsymbol{y} = u\boldsymbol{x}u^{-1} \in \mathbb{R}^{p,q}.$$

[7] This nomenclature comes from the fact that $\mathrm{SO}^e(1, 3) = \mathcal{L}^{\uparrow}_{+}$ is the special (proper) orthochronous Lorentz group. In this case the set is easily defined by the condition $L^0_0 \geq +1$. For the general case see [17].

Since $u \in \mathbb{R}^0_{p,q}$ we have that $\boldsymbol{y} \in \mathbb{R}^1_{p,q}$. Let $\boldsymbol{e}_i, i = 1, 2, 3, 4$ an orthonormal basis of $\mathbb{R}_{p,q}$, $p + q = 4$. Now, $\bar{\boldsymbol{y}} = (u\boldsymbol{y}u^{-1})^{\hat{}\tilde{}} = -u\boldsymbol{x}u^{-1} = -\boldsymbol{y}$. Writing

$$\boldsymbol{y} = y^i\boldsymbol{e}_i + \frac{1}{3!}y^{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k,$$
$$y^i,\ y^{ijk} \in \mathbb{R},$$

we get

$$\bar{\boldsymbol{y}} = -y^i\boldsymbol{e}_{i,}$$

from which follows that $\boldsymbol{y} \in \mathbb{R}^{p,q}$.

3.3.4 Lie Algebra of $\mathrm{Spin}^e_{1,3}$

It can be shown [14, 16, 23] that for each $u \in \mathrm{Spin}^e_{1,3}$ it holds $u = \pm e^F, F \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ and F can be chosen in such a way to have a positive sign in Eq. (3.33), except in the particular case $F^2 = 0$ when $u = -e^F$. From Eq. (3.33) it follows immediately that the Lie algebra of $\mathrm{Spin}^e_{1,3}$ is generated by the bivectors $F \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ through the commutator product.

Exercise 3.40 Show that when $F^2 = 0$ we must have $u = -e^F$.

3.4 Spinor Representations of $\mathbb{R}_{4,1}$, $\mathbb{R}^0_{4,1}$ and $\mathbb{R}_{1,3}$

We investigate now some spinor representations of $\mathbb{R}_{4,1}$, $\mathbb{R}^0_{4,1}$ and $\mathbb{R}_{1,3}$ which will permit us to introduce the concepts algebraic, Dirac and Dirac-Hestenes spinors in the next section.

Let $b_0 = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, such that $\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu = 2\eta_{\mu\nu}$, with $\eta_{\upsilon\mu} = \mathrm{diag}(+1, -1, -1, -1)$. Now, with the results of the previous section we can verify without difficulties that the elements $\mathrm{e}, \mathrm{e}', \mathrm{e}'' \in \mathbb{R}_{1,3}$

$$\mathrm{e} = \frac{1}{2}(1 + \mathbf{e}_0) \tag{3.36}$$

$$\mathrm{e}' = \frac{1}{2}(1 + \mathbf{e}_3\mathbf{e}_0) \tag{3.37}$$

$$\mathrm{e}'' = \frac{1}{2}(1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \tag{3.38}$$

are primitive idempotents of $\mathbb{R}_{1,3}$. The minimal left ideals,[8] $I = \mathbb{R}_{1,3}\mathrm{e}$, $I' = \mathbb{R}_{1,3}\mathrm{e}'$, $I'' = \mathbb{R}_{1,3}\mathrm{e}''$ are *right* two dimension linear spaces over the quaternion field ($\mathbb{H}\mathrm{e} = \mathrm{e}\mathbb{H} = \mathrm{e}\mathbb{R}_{1,3}\mathrm{e}$).

An elements $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1 + \mathbf{e}_0)$ has been called by Lounesto [15] a *mother* spinor.[9] Let us see the justice of this denomination. First recall from the general result of the previous section that $\frac{\mathrm{Pin}_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{O}_{1,3}$, $\frac{\mathrm{Spin}_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{SO}_{1,3}$, $\frac{\mathrm{Spin}^e_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{SO}^e_{1,3}$, and $\mathrm{Spin}^e_{1,3} \simeq \mathrm{Sl}(2,\mathbb{C})$ is the universal covering group of $\mathcal{L}^{\uparrow}_{+} \equiv \mathrm{SO}^e_{1,3}$, the *special* (proper) *orthochronous* Lorentz group. We can show [10, 11] that the ideal $I = \mathbb{R}_{1,3}\mathrm{e}$ carries the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Sl}(2,\mathbb{C})$. Here we need to know [10, 11] that each Φ can be written as

$$\Phi = \psi_1\mathrm{e} + \psi_2\mathbf{e}_3\mathbf{e}_1\mathrm{e} + \psi_3\mathbf{e}_3\mathbf{e}_0\mathrm{e} + \psi_4\mathbf{e}_1\mathbf{e}_0\mathrm{e} = \sum_i \psi_i s_i, \tag{3.39}$$

$$s_1 = \mathrm{e}, \quad s_2 = \mathbf{e}_3\mathbf{e}_1\mathrm{e}, \quad s_3 = \mathbf{e}_3\mathbf{e}_0\mathrm{e}, \quad s_4 = \mathbf{e}_1\mathbf{e}_0\mathrm{e} \tag{3.40}$$

and where the ψ_i are *formally* complex numbers, i.e., each $\psi_i = (a_i + b_i\mathbf{e}_2\mathbf{e}_1)$ with $a_i, b_i \in \mathbb{R}$ and the set $\{s_i, i = 1,2,3,4\}$ is a basis in the mother spinors space.

Exercise 3.41 Prove Eq. (3.39).

Now we determine an explicit relation between representations of $\mathbb{R}_{4,1}$ and $\mathbb{R}_{3,1}$. Let $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ be an orthonormal basis of $\mathbb{R}_{4,1}$ with

$$-\mathbf{f}_0^2 = \mathbf{f}_1^2 = \mathbf{f}_2^2 = \mathbf{f}_3^2 = \mathbf{f}_4^2 = 1,$$
$$\mathbf{f}_A\mathbf{f}_B = -\mathbf{f}_B v_A, A \neq B \text{ and } A, B = 0,1,2,3,4.$$

Define the pseudo-scalar

$$\mathfrak{i} = \mathbf{f}_0\mathbf{f}_1\mathbf{f}_2\mathbf{f}_3\mathbf{f}_4, \quad \mathfrak{i}^2 = -1, \quad \mathfrak{i}\mathbf{f}_A = \mathbf{f}_A\mathfrak{i}, \quad A = 0,1,2,3,4. \tag{3.41}$$

Put

$$\mathcal{E}_\mu = \mathbf{f}_\mu\mathbf{f}_4, \tag{3.42}$$

we can immediately verify that

$$\mathcal{E}_\mu\mathcal{E}_\nu + \mathcal{E}_\nu\mathcal{E}_\mu = 2\eta_{\mu\nu}. \tag{3.43}$$

[8] According to Definition 3.47 these ideals are algebraically equivalent. For example, $\mathrm{e}' = u\mathrm{e}u^{-1}$, with $u = (1 + \mathbf{e}_3) \notin \Gamma_{1,3}$.

[9] Elements of I' are sometimes called Hestenes ideal spinors.

Taking into account that $\mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1}^0$ we can explicitly exhibit here this isomorphism by considering the map $\mathtt{j}\colon \mathbb{R}_{1,3} \to \mathbb{R}_{4,1}^0$ generated by the linear extension of the map $\mathtt{j}^\# : \mathbb{R}^{1,3} \to \mathbb{R}_{4,1}^0$, $\mathtt{j}^\#(\mathbf{e}_\mu) = \mathcal{E}_\mu = \mathbf{f}_\mu \mathbf{f}_4$, where $\mathcal{E}_\mu$, $(\mu = 0, 1, 2, 3)$ is an orthogonal basis of $\mathbb{R}^{1,3}$. Note that $\mathtt{j}(1_{\mathbb{R}_{1,3}}) = 1_{\mathbb{R}_{4,1}^0}$, where $1_{\mathbb{R}_{1,3}}$ and $1_{\mathbb{R}_{4,1}^0}$ (usually denoted simply by 1) are the identity elements in $\mathbb{R}_{1,3}$ and $\mathbb{R}_{4,1}^0$. Now consider the primitive idempotent of $\mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1}^0$,

$$\mathrm{e}_{41}^0 = \mathtt{j}(\mathrm{e}) = \frac{1}{2}(1 + \mathcal{E}_0) \tag{3.44}$$

and the minimal left ideal $I_{4,1}^0 = \mathbb{R}_{4,1}^0 \mathrm{e}_{41}^0$.

The elements $Z \in I_{4,1}^0$ can be written analogously to $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1+\ \mathbf{e}_0)$ as,

$$Z = \sum z_i \bar{s}_i \tag{3.45}$$

where

$$\bar{s}_1 = \mathrm{e}_{41}^0,\ \bar{s}_2 = \mathcal{E}_1\mathcal{E}_3\mathrm{e}_{41}^0,\ \bar{s}_3 = \mathcal{E}_3\mathcal{E}_0\mathrm{e}_{41}^0,\ \bar{s}_4 = \mathcal{E}_1\mathcal{E}_0\mathrm{e}_{41}^0 \tag{3.46}$$

and where

$$z_i = a_i + \mathcal{E}_2\mathcal{E}_1 b_i,$$

are formally complex numbers, $a_i, b_i \in \mathbb{R}$.

Consider now the element $f \in \mathbb{R}_{4,1}$

$$\begin{aligned} f &= \mathrm{e}_{41}^0 \frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2) \\ &= \frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2), \end{aligned} \tag{3.47}$$

with i defined as in Eq. (3.41).

Since $f\mathbb{R}_{4,1}f = \mathbb{C}f = f\mathbb{C}$ it follows that f is a primitive idempotent of $\mathbb{R}_{4,1}$. We can easily show that each $\Phi \in I = \mathbb{R}_{4,1}f$ can be written

$$\Psi = \sum_i \psi_i f_i,\ \ \psi_i \in \mathbb{C},$$

$$f_1 = f,\ f_2 = -\mathcal{E}_1\mathcal{E}_3 f,\ f_3 = \mathcal{E}_3\mathcal{E}_0 f,\ f_4 = \mathcal{E}_1\mathcal{E}_0 f. \tag{3.48}$$

With the methods described in [10, 11] we find the following representation in $\mathbb{C}(4)$ for the generators $\mathcal{E}_\mu$ of $\mathbb{R}_{4,1} \simeq \mathbb{R}_{1,3}$

$$\mathcal{E}_0 \mapsto \underline{\gamma}_0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \leftrightarrow \mathcal{E}_i \mapsto \underline{\gamma}_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{3.49}$$

where $\mathbf{1}_2$ is the unit 2×2 matrix and σ_i, $(i = 1, 2, 3)$ are the standard *Pauli matrices*. We immediately recognize the $\underline{\gamma}$-matrices in Eq. (3.49) as the standard ones appearing, e.g., in [4].

The matrix representation of $\Psi \in I$ will be denoted by the same letter in boldface, i.e., $\Psi \mapsto \boldsymbol{\Psi} \in \mathbb{C}(4)f$, where

$$f = \frac{1}{2}(1 + \underline{\gamma}_0)\frac{1}{2}(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2), \quad \mathrm{i} = \sqrt{-1}. \tag{3.50}$$

We have

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \quad \psi_i \in \mathbb{C}. \tag{3.51}$$

Equations (3.49)–(3.51) are sufficient to prove that there are bijections between the elements of the ideals $\mathbb{R}_{1,3}\frac{1}{2}(1+\ \mathbf{e}_0)$, $\mathbb{R}_{4,1}^0\frac{1}{2}(1+\mathcal{E}_0)$ and $\mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathrm{i}\mathcal{E}_1\mathcal{E}_2)$.

We can easily find that the following relation exist between $\Psi \in \mathbb{R}_{4,1}f$ and $Z \in \mathbb{R}_{4,1}^0\frac{1}{2}(1 + \mathcal{E}_0)$,

$$\Psi = Z\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2). \tag{3.52}$$

Decomposing Z into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $\mathbb{R}_{4,1}^0 \simeq \mathbb{R}_{1,3}$, $Z = Z^0 + Z^1$ we obtain $Z^0 = Z^1\mathcal{E}_0$ which clearly shows that all information of Z is contained in Z^0. Then,

$$\Psi = Z^0\frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2). \tag{3.53}$$

Now, if we take into account that $\mathbb{R}_{4,1}^0\frac{1}{2}(1 + \mathcal{E}_0) = \mathbb{R}_{4,1}^{00}\frac{1}{2}(1 + \mathcal{E}_0)$ where the symbol $\mathbb{R}_{4,1}^{00}$ means $\mathbb{R}_{4,1}^{00} \simeq \mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$ we see that each $Z \in \mathbb{R}_{4,1}^{00}\frac{1}{2}(1 + \mathcal{E}_0)$ can be written

$$Z = \psi\frac{1}{2}(1 + \mathcal{E}_0) \quad \psi \in \mathbb{R}_{4,1}^{00} \simeq \mathbb{R}_{1,3}^0. \tag{3.54}$$

Then putting $Z^0 = \psi/2$, Eq. (3.54) can be written

$$\begin{aligned}\Psi &= \psi\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathrm{i}\mathcal{E}_1\mathcal{E}_2)\\ &= Z^0\frac{1}{2}(1+\mathrm{i}\mathcal{E}_1\mathcal{E}_2).\end{aligned} \tag{3.55}$$

The matrix representation of ψ and Z in $\mathbb{C}(4)$ (denoted by the same letter in boldface) in the matrix representation generated by the spin basis given by Eq. (3.48) are

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}. \tag{3.56}$$

3.5 Algebraic Spin Frames and Spinors

We introduce now the fundamental concept of *algebraic spin frames*.[10] This is the concept that will permit us to define spinors (steps (i)–(vii)).[11]

(i) In this section $(\mathbf{V}, \boldsymbol{\eta})$ refers always to Minkowski vector space.

(ii) Let $\mathrm{SO}(\mathbf{V}, \boldsymbol{\eta})$ be the group of endomorphisms of $\mathbf{V}$ that preserves $\boldsymbol{\eta}$ and the space orientation. This group is isomorphic to $\mathrm{SO}_{1,3}$ but there is no *natural* isomorphism. We write $\mathrm{SO}(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{SO}_{1,3}$. Also, the connected component to the identity is denoted by $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ and $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{SO}^e_{1,3}$. Note that $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ preserves besides *orientation* also the *time* orientation.

(iii) We denote by $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ the Clifford algebra[12] of $(\mathbf{V}, \boldsymbol{\eta})$ and by $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{Spin}^e_{1,3}$ the connected component of the spin group $\mathrm{Spin}(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{Spin}_{1,3}$. Consider the $2:1$ homomorphism $\mathbf{L} : \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) \to \mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta}), u \mapsto \mathbf{L}(u) \equiv \mathbf{L}_u$. $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ acts on $\mathbf{V}$ identified as the space of 1-vectors of $\mathcal{C}\ell(\mathbf{V}, \eta) \simeq \mathbb{R}_{1,3}$ through its adjoint representation in the Clifford algebra $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ which

[10] The name *spin frame* will be reserved for a section of the spinor bundle structure $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ which will be introduced in Chap. 7.

[11] This section follows the developments given in [22].

[12] We reserve the notation $\mathbb{R}_{p,q}$ for the Clifford algebra of the vector space $\mathbb{R}^n$ equipped with a metric of signature (p, q), $p + q = n$. $\mathcal{C}\ell(\mathbf{V}, \mathbf{g})$ and $\mathbb{R}_{p,q}$ are isomorphic, but there is no canonical isomorphism. Indeed, an isomorphism can be exhibit only after we fix an orthonormal basis of $\mathbf{V}$.

is related with the vector representation of $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ as follows[13]:

$$\mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta}) \ni u \mapsto \mathrm{Ad}_u \in \mathrm{Aut}(\mathcal{C}\ell(\mathbf{V},\eta))$$
$$\mathrm{Ad}_u|_{\mathbf{V}} : \mathbf{V} \to \mathbf{V},\ \mathbf{v} \mapsto u\mathbf{v}u^{-1} = \mathbf{L}_u \odot \mathbf{v}. \tag{3.57}$$

In Eq. (3.57) $\mathbf{L}_u \odot \mathbf{v}$ denotes the standard action $\mathbf{L}_u$ on $\mathbf{v}$ and where we identified $\mathbf{L}_u \in \mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ with $\mathbf{L}_u \in \mathbf{V} \otimes \mathbf{V}^*$ and

$$\boldsymbol{\eta}\,(\mathbf{L}_u \odot \mathbf{v}, \mathbf{L}_u \odot \mathbf{v}) = \boldsymbol{\eta}\,(\mathbf{v},\mathbf{v})\,. \tag{3.58}$$

(iv) Let $\mathcal{B}$ be the set of all oriented and time oriented orthonormal basis[14] of $\mathbf{V}$. Choose among the elements of $\mathcal{B}$ a basis $b_0 = \{\mathbf{b}_0, \ldots, \mathbf{b}_3\}$, hereafter called the *fiducial frame* of $\mathbf{V}$. With this choice, we define a $1-1$ mapping

$$\boldsymbol{\Sigma} : \mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta}) \to \mathcal{B}, \tag{3.59}$$

given by

$$\mathbf{L}_u \mapsto \boldsymbol{\Sigma}\ (\mathbf{L}_u) := \boldsymbol{\Sigma}_{\mathbf{L}_u} = \mathbf{L}b_0 \tag{3.60}$$

where $\boldsymbol{\Sigma}_{\mathbf{Lu}} = \mathbf{L}_u b_0$ is a short for $\{\mathbf{e}_1, \ldots, \mathbf{e}_{3,}\} \in \mathcal{B}$, such that denoting the action of $\mathbf{L}_u$ on $\mathbf{b}_i \in b_0$ by $\mathbf{L}_u \odot \mathbf{b}_i$ we have

$$\mathbf{e}_i = \mathbf{L}_u \odot \mathbf{b}_i := L^{j}_{\cdot i}\mathbf{b}_j, \quad i,j = 0,\ldots,3. \tag{3.61}$$

In this way, we can identify a given vector basis b of $\mathbf{V}$ with the isometry $\mathbf{L}_u$ that takes the fiducial basis b_0 to b. The fiducial basis b_0 will be also denoted by $\boldsymbol{\Sigma}_{\mathbf{L}_0}$, where $\mathbf{L}_0 = e$, is the *identity* element of $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$.

Since the group $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ is *not* simple connected their elements cannot distinguish between frames whose spatial axes are *rotated* in relation to the fiducial vector frame $\boldsymbol{\Sigma}_{\mathbf{L}_0}$ by multiples of 2π or by multiples of 4π. For what follows it is crucial to make such a distinction. This is done by introduction of the concept of *algebraic spin frames*.

Definition 3.42 Let $b_0 \in \mathcal{B}$ be a fiducial frame and choose an arbitrary $u_0 \in \mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta})$. Fix once and for all the pair (u_0, b_0) with $u_0 = 1$ and call it the fiducial algebraic spin frame.

Definition 3.43 The space $\mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta}) \times \mathcal{B} = \{(u,b), ubu^{-1} = u_0 b_0 u_0^{-1}\}$ will be called the space of algebraic spin frames and denoted by $\mathcal{S}$.

---

[13] $\mathrm{Aut}(\mathcal{C}\ell(\mathbf{V},g))$ denotes the (inner) automorphisms of $\mathcal{C}\ell(\mathbf{V},g)$.

[14] We will call the elements of $\mathcal{B}$ (in what follows) simply by orthonormal basis.

Remark 3.44 It is crucial for what follows to observe here that Definition 3.43 implies that a given $b \in \mathcal{B}$ determines two, and only two, algebraic spin frames, namely (u, b) and $(-u, b)$, since $\pm ub(\pm u^{-1}) = u_0 b_0 u_0^{-1}$.

(v) We now parallel the construction in (i**v**) but replacing $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ by its universal covering group $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ and $\mathcal{B}$ by S. Thus, we define the $1-1$ mapping

$$\begin{aligned} \Xi : \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) &\to \qquad \mathcal{S}, \\ u \qquad &\mapsto \Xi(u) := \Xi_u = (u, b), \end{aligned} \tag{3.62}$$

where $ubu^{-1} = b_0$.

The fiducial algebraic spin frame will be denoted in what follows by Ξ_0. It is obvious from Eq. (3.62) that $\Xi(-u) \equiv \Xi_{-u} = (-u, b) \neq \Xi_u$.

Definition 3.45 The natural right action of $a \in \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ denoted by $\odot$ on $\mathcal{S}$ is given by

$$a \odot \Xi_u = a \odot (u, b) = (ua, Ad_{a^{-1}} b) = (ua, a^{-1} ba). \tag{3.63}$$

Observe that if $\Xi_{u'} = (u', b') = u' \odot \Xi_0$ and $\Xi_u = (u, b) = u \odot \Xi_0$ then,

$$\Xi_{\acute{u}} = (u^{-1} u') \odot \Xi_u = (u', u'^{-1} ubu^{-1} u').$$

Note that there is a natural $2-1$ mapping

$$\mathbf{s} : \mathcal{S} \to \mathcal{B}, \qquad \Xi_{\pm u} \mapsto b = (\pm u^{-1}) b_0 (\pm u), \tag{3.64}$$

such that

$$\mathbf{s}((u^{-1} u') \odot \Xi_u)) = \mathrm{Ad}_{(u^{-1} u')^{-1}} (\mathbf{s}(\Xi_u)). \tag{3.65}$$

Indeed,

$$\begin{aligned} \mathbf{s}((u^{-1} u') \odot \Xi_u)) &= \mathbf{s}((u^{-1} u') \odot (u, b)) \\ &= u'^{-1} ub(u'^{-1} u)^{-1} = b' \\ &= \mathrm{Ad}_{(u^{-1} u')^{-1}} b = \mathrm{Ad}_{(u^{-1} u')^{-1}} (\mathbf{s}(\Xi_u)). \end{aligned} \tag{3.66}$$

This means that the natural right actions of $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$, respectively on $\mathcal{S}$ and $\mathcal{B}$, commute. In particular, this implies that the algebraic spin frames $\Xi_u, \Xi_{-u} \in \mathcal{S}$, which are, of course distinct, determine the same vector frame $\Sigma_{\mathbf{L}_u} = \mathbf{s}(\Xi_u) = \mathbf{s}(\Xi_{-u}) = \Sigma_{\mathbf{L}_{-u}}$. We have,

$$\boldsymbol{\Sigma}_{\mathbf{L}_u} = \boldsymbol{\Sigma}_{\mathbf{L}_{-u}} = \mathbf{L}_{u^{-1} u_0} \boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}, \ \mathbf{L}_{u^{-1} u_0} \in \mathrm{SO}^e_{1,3}. \tag{3.67}$$

Also, from Eq. (3.65), we can write explicitly

$$u_0\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}u_0^{-1} = u\boldsymbol{\Sigma}_{\mathbf{L}_u}u^{-1},\; u_0\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}u_0^{-1} = (-u)\boldsymbol{\Sigma}_{\mathbf{L}_{-u}}(-u)^{-1}, u \in \mathrm{Spin}^e(\mathbf{V},\mathbf{g}), \tag{3.68}$$

where the meaning of Eq. (3.68) of course, is that *if* $\boldsymbol{\Sigma}_{\mathbf{L}_u} = \boldsymbol{\Sigma}_{\mathbf{L}_{-u}} = b = \{\mathbf{e}_0, \ldots., \mathbf{e}_{3,}\} \in \mathcal{B}$ and $\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}} = b_0 \in \mathcal{B}$ is the fiducial frame, then

$$u_0\mathbf{b}_j u_0^{-1} = (\pm u)\mathbf{e}_j(\pm u^{-1}). \tag{3.69}$$

In resume, we can say that the space $\mathcal{S}$ of algebraic spin frames can be thought as an *extension* of the space $\mathcal{B}$ of *vector frames*, where even if two vector frames have the *same* ordered vectors, they are considered distinct if the spatial axes of one vector frame is rotated by an odd number of 2π rotations relative to the other vector frame and are considered the same if the spatial axes of one vector frame is rotated by an even number of 2π rotations relative to the other frame. Even if the possibility of such a distinction seems to be impossible at first sight, Aharonov and Susskind [1] claim that it can be implemented physically in a spacetime where the concept of algebraic spin frame is enlarged to the concept of spin frame used for the definition of spinor fields. See Chap. 7 for details.

(vi) Before we proceed an important *digression* on the notation used below is necessary. We recalled above how to construct a minimum left (or right) ideal for a given real Clifford algebra once a vector basis $b \in \mathcal{B}$ for $\mathbf{V} \hookrightarrow \mathcal{C}\ell(\mathbf{V},\mathbf{g})$ is given. That construction suggests to *label* a given primitive idempotent and its corresponding ideal with the subindex b. However, taking into account the above discussion of vector and algebraic spin frames and their relationship we find useful for what follows (specially in view of the Definition 3.46 and the definitions of algebraic and Dirac-Hestenes spinors (see Definitions 3.48 and 3.50 below) to label a given primitive idempotent and its corresponding ideal with a subindex Ξ_u. This notation is also justified by the fact that a given idempotent is according to definition 3.48 *representative* of a particular spinor in a given algebraic spin frame Ξ_u.

(vii) Next we recall Theorem 3.28 which says that a minimal left ideal of $\mathcal{C}\ell(\mathbf{V},\boldsymbol{\eta})$ is of the type

$$I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V},\boldsymbol{\eta})\mathrm{e}_{\Xi_u} \tag{3.70}$$

where e_{Ξ_u} is a primitive idempotent of $\mathcal{C}\ell(\mathbf{V},\boldsymbol{\eta})$.

It is easy to see that all ideals $I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V},\boldsymbol{\eta})\mathrm{e}_{\Xi_u}$ and $I_{\Xi_{u'}} = \mathcal{C}\ell(\mathbf{V},\boldsymbol{\eta})\mathrm{e}_{\Xi_{u'}}$ such that

$$\mathrm{e}_{\Xi_{u'}} = (u'^{-1}u)\mathrm{e}_{\Xi_u}(u'^{-1}u)^{-1} \tag{3.71}$$

$u, u' \in \mathrm{Spin}^e(\mathbf{V},\eta)$ are isomorphic. We have the

Definition 3.46 Any two ideals $I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u}$ and $I_{\Xi_{u'}} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_{u'}}$ such that their generator idempotents are related by Eq. (3.71) are said geometrically equivalent.

Remark 3.47 If u is simply an element of the Clifford group, then the ideals are said to be algebraically equivalent.

But take care, no *equivalence relation* has been defined until now. We observe moreover that we can write

$$I_{\Xi_{u'}} = I_{\Xi_u}(u'^{-1}u)^{-1}, \tag{3.72}$$

an equation that will play a key role in what follows.

3.6 Algebraic Dirac Spinors of Type I_{Ξ_u}

Let $\{I_{\Xi_u}\}$ be the set of all ideals geometrically equivalent to a given minimal $I_{\Xi_{u_o}}$ as defined by Eq. (3.72). Let be

$$\mathfrak{T} = \{(\Xi_u, \Psi_{\Xi_u}) \mid u \in \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}),\ \Xi_u \in S,\ \Psi_{\Xi_u} \in I_{\Xi_u}\}. \tag{3.73}$$

Let $\Xi_u, \Xi_{u'} \in S$, $\Psi_{\Xi_u} \in I_{\Xi_u}$, $\Psi_{\Xi_{u'}} \in I_{\Xi_{u'}}$. We define an equivalence relation $\mathcal{E}$ on $\mathfrak{T}$ by setting

$$(\Xi_u, \Psi_{\Xi_u}) \sim (\Xi_{u'}, \Psi_{\Xi_{u'}}), \tag{3.74}$$

if and only if and

$$\begin{aligned} &\text{(i) } u\mathbf{s}(\Xi_u)u^{-1} = u'\mathbf{s}(\Xi_{u'})u'^{-1}, \\ &\text{(ii) } \Psi_{\Xi_{u'}}u'^{-1} = \Psi_{\Xi_u}u^{-1}. \end{aligned} \tag{3.75}$$

Definition 3.48 An equivalence class

$$\boldsymbol{\Psi}_{\Xi_u} = [(\Xi_u, \Psi_{\Xi_u})] \in \mathfrak{T}/\mathcal{E} \tag{3.76}$$

is called an algebraic spinor of type I_{Ξ_u} for $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$. $\psi_{\Xi_u} \in I_{\Xi_u}$ is said to be a representative of the algebraic spinor $\boldsymbol{\Psi}_{\Xi_u}$ in the algebraic spin frame Ξ_u.

We observe that the pairs (Ξ_u, Ψ_{Ξ_u}) and $(\Xi_{-u}, \Psi_{\Xi_{-u}}) = (\Xi_{-u}, -\Psi_{\Xi_u})$ are equivalent, but the pairs (Ξ_u, Ψ_{Ξ_u}) and $(\Xi_{-u}, -\Psi_{\Xi_{-u}}) = (\Xi_{-u}, \Psi_{\Xi_u})$ are not. This distinction is *essential* in order to give a structure of linear space (over the real field) to the set $\mathfrak{T}$. Indeed, a natural linear structure on $\mathfrak{T}$ is given by

$$a[(\Xi_u, \Psi_{\Xi_u})] + b[(\Xi_u, \Psi'_{\Xi_u})] = [(\Xi_u, a\Psi_{\Xi_u})] + [(\Xi_{u'}, b\Psi'_{\Xi_u})],$$
$$(a+b)[(\Xi_u, \Psi_{\Xi_u})] = a[(\Xi_u, \Psi_{\Xi_u})] + b[(\Xi_u, \Psi_{\Xi_u})]. \tag{3.77}$$

Remark 3.49 The definition just given is not a standard one in the literature [5, 8]. However, the fact is that the standard definition (licit as it is from the mathematical point of view) is *not* adequate for a comprehensive formulation of the Dirac equation using algebraic spinor fields or Dirac-Hestenes spinor fields which will be introduced in Chap. 7.

We end this section recalling that as observed above a given Clifford algebra $\mathbb{R}_{p,q}$ may have minimal ideals that are not geometrically equivalent since they may be generated by primitive idempotents that are related by elements of the group $\mathbb{R}^{\star}_{p,q}$ which are not elements of $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ [see Eqs. (3.36)–(3.38)] where different, non geometrically equivalent primitive ideals for $\mathbb{R}_{1,3}$ are shown). These ideals may be said to be of different *types*. However, from the point of view of the representation theory of the real Clifford algebras all these primitive ideals carry equivalent (i.e., isomorphic) *modular* representations of the Clifford algebra and no preference may be given to any one.[15] In what follows, when no confusion arises and the ideal I_{Ξ_u} is clear from the context, we use the wording algebraic Dirac spinor for any one of the possible types of ideals.

The most important property concerning algebraic Dirac spinors is a coincidence given by Eq. (3.78) below. It permit us to define a *new* kind of spinors.

3.7 Dirac-Hestenes Spinors (DHS)

Let $\Xi_u \in \mathcal{S}$ be an algebraic spin frame for $(\mathbf{V}, \boldsymbol{\eta})$ such that

$$\mathbf{s}(\Xi_u) = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathcal{B}.$$

Then, it follows from Eq. (3.54) that

$$I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u} = \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u}, \tag{3.78}$$

when

$$\mathrm{e}_{\Xi_u} = \frac{1}{2}(1 + \mathbf{e}_0). \tag{3.79}$$

[15] The fact that there are ideals that are algebraically, but not geometrically equivalent seems to contain the seed for new Physics, see [18, 19].

Then, each $\Psi_{\Xi_u} \in I_{\Xi_u}$ can be written as

$$\Psi_{\Xi_u} = \psi_{\Xi_u} \mathrm{e}_{\Xi_u}, \quad \psi_{\Xi_u} \in \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}). \tag{3.80}$$

From Eq. (3.75) we get

$$\psi_{\Xi_{u'}} u'^{-1} u \mathrm{e}_{\Xi_u} = \psi_{\Xi_u} \mathrm{e}_{\Xi_u}, \quad \psi_{\Xi_u}, \psi_{\Xi_{u'}} \in \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}). \tag{3.81}$$

A possible solution for Eq. (3.81) is

$$\psi_{\Xi_{u'}} u'^{-1} = \psi_{\Xi_u} u^{-1}. \tag{3.82}$$

Let $\mathcal{S} \times \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ and consider an equivalence relation $\mathcal{E}$ such that

$$(\Xi_u, \phi_{\Xi_u}) \sim (\Xi_{u'}, \phi_{\Xi_{u'}}) \ (\mathrm{mod}\ \mathcal{E}), \tag{3.83}$$

if and only if $\phi_{\Xi_{u'}}$ and ϕ_{Ξ_u} are related by

$$\phi_{\Xi_{u'}} u'^{-1} = \phi_{\Xi_u} u^{-1}. \tag{3.84}$$

This suggests the following

Definition 3.50 The equivalence classes $[(\Xi_u, \phi_{\Xi_u})] \in \mathcal{S} \times \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})/\mathcal{E}$ are the Hestenes spinors.

Among the Hestenes spinors, an important subset is the one consisted of Dirac-Hestenes spinors where $[(\Xi_u, \psi_{\Xi_u})] \in (\mathcal{S} \times \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}))/\mathcal{E}$.

We say that ϕ_{Ξ_u} (ψ_{Ξ_u}) is a representative of a Hestenes (Dirac-Hestenes) spinor in the algebraic spin frame Ξ_u.

3.7.1 What is a Covariant Dirac Spinor (CDS)

Let $\mathbf{L}' : \mathcal{S} \to \mathcal{B}$ and let $\mathbf{L}'(\Xi_u) = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ and $\mathbf{L}'(\Xi_{u'}) = \{\mathcal{E}'_0, \mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3\}$ with $\mathbf{L}'(\Xi_u) = u\mathbf{L}'(\Xi_o)u^{-1}$, $\mathbf{L}(\Xi_{u'}) = u'\mathbf{L}'(\Xi_o)u'^{-1}$ be two arbitrary basis for $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{4,1}$.

As we already know $f_{\Xi_0} = \frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2)$ [Eq. (3.48)] is a primitive idempotent of $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. If $u \in \mathrm{Spin}\,(1,3) \subset \mathrm{Spin}(4,1)$ then all ideals $I_{\Xi_u} = I_{\Xi_0} u^{-1}$ are geometrically equivalent to I_{Ξ_0}. From Eq. (3.49) we can write

$$I_{\Xi_u} \ni \Psi_{\Xi_u} = \sum \psi_i f_i, \text{ and } I_{\Xi'_u} \ni \Psi_{\Xi_u} = \sum \psi'_i f'_i, \tag{3.85}$$

where

$$f_1 = f_{\Xi_u}, \quad f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Xi_u}, \quad f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Xi_u}, \quad f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Xi_u}$$

and

$$f_1' = f_{\Xi_u}, \quad f_2' = -\mathcal{E}_1'\mathcal{E}_3' f_{\Xi_u}, \quad f_3' = \mathcal{E}_3'\mathcal{E}_0' f_{\Xi_u}, \quad f_4 = \mathcal{E}_1'\mathcal{E}_0' f_{\Xi_u}.$$

Since $\Psi_{\Xi_{u'}} = \Psi_{\Xi_u}(u'^{-1}u)^{-1}$, we get

$$\Psi_{\Xi_u'} = \sum_i \psi_i (u'^{-1}u)^{-1} f_i' = \sum_{i,k} S_{ik}[(u^{-1}u')]\psi_i f_k = \sum_k \psi_k f_k.$$

Then

$$\psi_k = \sum_i S_{ik}(u^{-1}u')\psi_i, \tag{3.86}$$

where $S_{ik}(u^{-1}u')$ are the matrix components of the representation in $\mathbb{C}(4)$ of $(u^{-1}u') \in \mathrm{Spin}^e_{1,3}$. As proved in [10, 11] the matrices $S(u)$ correspond to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $\mathrm{Sl}(2,\mathbb{C}) \simeq \mathrm{Spin}^e_{1,3}$.

Remark 3.51 We remark that all the elements of the set $\{I_{\Xi_u}\}$ of the ideals geometrically equivalent to I_{Ξ_0} under the action of $u \in \mathrm{Spin}^e_{1,3} \subset \mathrm{Spin}^e_{4,1}$ have the same image $I = \mathbb{C}(4)f$ where f is given by Eq. (3.47), i.e.,

$$f = \frac{1}{2}(1 + \underline{\gamma}_0)(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2), \quad \mathrm{i} = \sqrt{-1},$$

where $\underline{\gamma}_\mu$, $\mu = 0, 1, 2, 3$ are the Dirac matrices given by Eq. (3.50). Then, if

$$\begin{aligned} \gamma &: \mathbb{R}_{4,1} \to \mathbb{C}(4) \equiv \mathrm{End}(\mathbb{C}(4)), \\ x &\mapsto \gamma(x) : \mathbb{C}(4)f \to \mathbb{C}(4)f \end{aligned} \tag{3.87}$$

it follows that

$$\gamma(\mathcal{E}_\mu) = \gamma(\mathcal{E}_\mu'), \quad \gamma(f_\mu) = \gamma(f_\mu') \tag{3.88}$$

for all $\{\mathcal{E}_\mu\}$, $\{\mathcal{E}_\mu'\}$ such that $\mathcal{E}_\mu' = (u'^{-1}u)\mathcal{E}_\mu(u'^{-1}u)^{-1}$. Observe that *all information* concerning the geometrical images of the algebraic spin frames $\mathbf{\Xi}_u$, $\mathbf{\Xi}_{u'}, \ldots$, under $\mathbf{L}'$*disappear* in the matrix representation of the ideals $I_{\Xi_u}, I_{\Xi_{u'}}, \ldots$, in $\mathbb{C}(4)$ since all these ideals are mapped in the same ideal $I = \mathbb{C}(4)f$.

Taking into account Remark 3.51 and taking into account the definition of algebraic spinors given above and Eq. (3.86) we are lead to the following

Definition 3.52 A covariant Dirac spinor for $\mathbb{R}^{1,3}$ is an equivalence class of pairs $(\Xi^m_u, \mathbf{\Psi})$, where Ξ^m_u is a matrix algebraic spin frame associated to the algebraic spin frame $\mathbf{\Xi}_u$ through the $S(u^{-1}) \in D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of $\mathrm{Spin}^e_{1,3}$, $u \in \mathrm{Spin}^e_{1,3}$.

We say that $\mathbf{\Psi}, \mathbf{\Psi}' \in \mathbb{C}(4)f$ are equivalent and write

$$(\Xi_u^m, \mathbf{\Psi}) \sim (\Xi_{u'}^m, \mathbf{\Psi}'), \tag{3.89}$$

if and only if,

$$\mathbf{\Psi}' = S(u'^{-1}u)\mathbf{\Psi}, \quad u\mathbf{s}(\mathbf{\Xi}_u)u^{-1} = u'\mathbf{s}(\mathbf{\Xi}_{u'})u'^{-1}. \tag{3.90}$$

Remark 3.53 The definition of *CDS* just given agrees with that given in [6] except for the irrelevant fact that there, as well as in the majority of Physics textbook's, authors use as the space of representatives of a *CDS* a complex four-dimensional space $\mathbb{C}^4$ instead of $I = \mathbb{C}(4)f$.

3.7.2 Canonical Form of a Dirac-Hestenes Spinor

Let $\mathbf{v} \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ be a *non* lightlike vector, i.e., $\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v} \neq 0$ and consider a linear mapping

$$L_\psi : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}, \mathbf{v} \mapsto \mathbf{z} = \psi \mathbf{v} \tilde{\psi}, \quad \mathbf{z}^2 = \rho \mathbf{v}^2 \tag{3.91}$$

with $\psi \in \mathbb{R}_{1,3}$ and $\rho \in \mathbb{R}^+$. Now, recall that if $R \in \mathrm{Spin}_{1,3}^e$ then $\mathbf{w} = R\mathbf{v}\tilde{R}$ is such that $\mathbf{w}^2 = \mathbf{v}^2$. It follows that the most general solution of Eq. (3.91) is

$$\psi = \rho^{1/2} e^{\frac{\beta}{2}\mathbf{e}_5} R, \tag{3.92}$$

where $\beta \in \mathbb{R}$ is called the Takabayasi angle and $\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \in \bigwedge^4 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ is the pseudoscalar of the algebra. Now, Eq. (3.92) shows that $\psi \in \mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$. Moreover, we have that $\psi\tilde{\psi} \neq 0$ since

$$\begin{aligned} \psi\tilde{\psi} &= \rho e^{\beta \mathbf{e}_5} = \sigma + \mathbf{e}_5 \omega, \\ \sigma &= \rho \cos\beta, \quad \omega = \rho \sin\beta. \end{aligned} \tag{3.93}$$

The Secret

Now, let ψ_{Ξ_u} be a representative of a Dirac-Hestenes spinor (Definition 3.50) in a given spin frame $\mathbf{\Xi}_u$. Since $\psi_{\Xi_u} \in \mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$ we have disclosed the *real geometrical meaning* of a Dirac-Hestenes spinor. Indeed, a Dirac-Hestenes spinor such that $\psi_{\Xi_u}\tilde{\psi}_{\Xi_u} \neq 0$ induces the linear mapping given by Eq. (3.91), which *rotates* a vector and *dilate* it. Observe, that even if we started our considerations with $\mathbf{v} \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ and $\mathbf{v}^2 \neq 0$, the linear mapping (3.91) also rotates and 'dilate' a light vector.

3.7.3 Bilinear Invariants and Fierz Identities

Definition 3.54 Given a representative ψ_{Ξ_u} of a *DHS* in the algebraic spin frame field Ξ_u the bilinear invariants[16] associated with it are the objects: $\sigma - \star\omega \in (\bigwedge^0 \mathbb{R}^{1,3} + \bigwedge^4 \mathbb{R}^{1,3}) \hookrightarrow \mathbb{R}_{1,3}$, $J = J_\mu \mathbf{e}^\mu \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, $S = \frac{1}{2} S_{\mu\nu} \mathbf{e}^\mu \mathbf{e}^\nu \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, $K = K_\mu \mathbf{e}^\mu \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ such that

$$\begin{aligned} \psi_{\Xi_u}\tilde{\psi}_{\Xi_u} &= \sigma - \star\omega & \psi_{\Xi_u}\mathbf{e}^0\tilde{\psi}_{\Xi_u} &= J, \\ \psi_{\Xi_u}\mathbf{e}^1\mathbf{e}^2\tilde{\psi}_{\Xi_u} &= S & \psi_{\Xi_u}\mathbf{e}^0\mathbf{e}^3\tilde{\psi}_{\Xi_u} &= \star S, \\ \psi_{\Xi_u}\mathbf{e}^3\tilde{\psi}_{\Xi_u} &= K & \psi_{\Xi_u}\mathbf{e}^0\mathbf{e}^1\mathbf{e}^2\tilde{\psi}_{\Xi_u} &= \star K. \end{aligned} \tag{3.94}$$

and where $\star\omega = -\mathbf{e}_5 \rho \sin\beta$

The bilinear invariants satisfy the so called *Fierz identities*, which are

$$J^2 = \sigma^2 + \omega^2, \quad J \cdot K = 0, \quad J^2 = -K^2, \quad J \wedge K = (\omega - \star\sigma)S \tag{3.95}$$

$$\begin{cases} S \llcorner J = -\omega K & S \llcorner K = -\omega J, \\ (\star S) \llcorner J = -\sigma K & (\star S) \llcorner K = -\sigma J, \\ S \cdot S = \langle S \tilde{S} \rangle_0 = \sigma^2 - \omega^2 & (\star S) \cdot S = -2\sigma\omega. \end{cases} \tag{3.96}$$

$$\begin{cases} JS = (\omega - \star\sigma)K, \\ SJ = -(\omega + \star\sigma)K, \\ SK = (\omega - \star\sigma)J, \\ KS = -(\omega + \star\sigma)J, \\ S^2 = \omega^2 - \sigma^2 - 2\sigma(\star\omega), \\ S^{-1} = KSK/J^4. \end{cases} \tag{3.97}$$

Exercise 3.55 Prove the Fierz identities.

[16] In Physics literature the components of J, S and K when written in terms of covariant Dirac spinors are called *bilinear covariants*.

3.7.4 Reconstruction of a Spinor

The importance of the bilinear invariants is that once we know ω, σ, J, K and F we can recover from them the associate covariant Dirac spinor (and thus the *DHS*) except for a phase. This can be done with an algorithm due to Crawford [7] and presented in a very pedagogical way in [14–16]. Here we only give the result for the case where σ and/or ω are non null. Define the object $\mathfrak{B} \in \mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1}$ called *boomerang* and given by ($\mathrm{i} = \sqrt{-1}$)

$$\mathfrak{B} = \sigma + J + \mathrm{i}S - \mathrm{i}\mathbf{e}_5 K + \mathbf{e}_5\omega \tag{3.98}$$

Then, we can construct $\boldsymbol{\Psi} = \mathfrak{B}f \in \mathbb{R}_{4,1}f$, where f is the idempotent given by Eq. (3.47) which has the following matrix representation in $\mathbb{C}(4)$ (once the standard representation of the Dirac gamma matrices are used)

$$\hat{\boldsymbol{\Psi}} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \tag{3.99}$$

Now, it can be easily verified that $\boldsymbol{\Psi} = \mathfrak{B}f$ determines the same bilinear covariants as the ones determined by ψ_{Ξ_u}. Note however that this spinor is not unique. In fact, $\mathfrak{B}$ determines a class of elements $\mathfrak{B}\xi$ where ξ is an arbitrary element of $\mathbb{R}_{4,1}f$ which differs one from the other by a complex phase factor.

3.7.5 Lounesto Classification of Spinors

A very interesting classification of spinors have been devised by Lounesto [14–16] based on the values of the bilinear invariants. He identified *six* different classes and proved that there are *no* other classes based on distinctions between bilinear covariants. Lounesto classes are:

1. $\sigma \neq 0, \quad \omega \neq 0.$
2. $\sigma \neq 0, \quad \omega = 0.$
3. $\sigma = 0, \quad \omega \neq 0.$
4. $\sigma = 0 = \omega, \quad K \neq 0, \quad S \neq 0.$
5. $\sigma = 0 = \omega, \quad K = 0, \quad S \neq 0.$
6. $\sigma = 0 = \omega, \quad K \neq 0, \quad S = 0.$

The current density J is always non-zero. Type 1, 2 and 3 spinor are denominated *Dirac spinor* for spin-1/2 particles and type 4, 5, and 6 are *singular* spinors respectively called *flag-dipole*, *flagpole* and *Weyl spinor*. Majorana spinor is a particular case of a type 5 spinor. It is worthwhile to point out a peculiar feature of types 4, 5 and 6 spinor: although J is always non-zero, we have due to Fierz identities that $J^2 = -K^2 = 0$.

Spinors belonging to class 4 have not previously been identified in the literature. For the applications we have in mind we are interested (besides Dirac spinors which belong to classes 1 or 2 or 3) in spinors belonging to classes 5 and 6, respectively the Majorana and Weyl spinors.

Remark 3.56 In [2] Ahluwalia-Khalilova and Grumiller introduced from physical considerations a supposedly new kind of spinors representing dark matter that they dubbed *ELKO* spinors. The acronym stands for the German word *Eigenspinoren des Ladungskonjugationsoperators*. It has been proved in [9] that from the algebraic point of view *ELKO* spinors are simply class 5 spinors. In [2] it is claimed that differently from the case of Dirac, Majorana and Weyl spinor fields which have mass dimension $3/2$, *ELKO* spinor fields must have mass dimension 1 and thus instead of satisfying Dirac equation satisfy a Klein-Gordon equation. A thoughtful analysis of this claim is given in Chap. 16.

3.8 Majorana and Weyl Spinors

Recall that for *Majorana* spinors $\sigma = 0, \omega = 0, K = 0, S \neq 0, J, \neq 0$

Given a representative ψ of an arbitrary Dirac-Hestenes spinor we may construct the Majorana spinors

$$\psi_M^{\pm} = \frac{1}{2}(\psi \pm \psi \mathbf{e}_{01}). \tag{3.100}$$

Note that defining an operator $C{:}\psi \mapsto \psi \mathbf{e}_{01}$ (*charge conjugation*) we have

$$C\psi_M^{\pm} = \pm\psi_M^{\pm}, \tag{3.101}$$

i.e., Majorana spinors are eigenvectors of the charge conjugation operator. Majorana spinors satisfy

$$\tilde{\psi}_M^{\pm}\psi_M^{\pm} = \psi_M^{\pm}\tilde{\psi}_M^{\pm} = 0. \tag{3.102}$$

For Weyl spinors $\sigma = \omega = S = 0$ and $K \neq 0, J \neq 0$.

Given a representative ψ of an arbitrary Dirac-Hestenes spinor we may construct the Weyl spinors

$$\psi_W^{\pm} = \frac{1}{2}(\psi \mp \mathbf{e}_5\psi\mathbf{e}_{21}). \tag{3.103}$$

Weyl spinors are 'eigenvectors' of the *chirality operator* $\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, i.e.,

$$\mathbf{e}_5\psi_W^{\pm} = \pm\psi_W^{\pm}\mathbf{e}_{21}. \tag{3.104}$$

We have also,

$$\tilde{\psi}_W^{\pm}\psi_W^{\pm} = \psi_W^{\pm}\tilde{\psi}_W^{\pm} = 0. \tag{3.105}$$

For future reference we introduce the parity operator acting on the space of Dirac-Hestenes spinors. The parity operator P in this formalism [13] is represented in such a way that for $\psi \in \mathbb{R}_{1,3}^0$

$$P\psi = -\mathbf{e}_0\psi\mathbf{e}_0 \ . \tag{3.106}$$

The following Dirac-Hestenes spinors are eigenstates of the parity operator with eigenvalues ± 1:

$$\begin{array}{ll} P\psi^{\uparrow} = +\psi^{\uparrow}\ , & \psi^{\uparrow} = \mathbf{e}_0\psi_-\mathbf{e}_0 - \psi_-\ , \\ P\psi^{\downarrow} = -\psi^{\downarrow}\ , & \psi^{\downarrow} = \mathbf{e}_0\psi_+\mathbf{e}_0 + \psi_+\ , \end{array} \tag{3.107}$$

where $\psi_{\pm} := \psi_W^{\pm}$

3.9 Dotted and Undotted Algebraic Spinors

Dotted and undotted covariant spinor *fields* are very popular subjects in General Relativity. Dotted and undotted algebraic spinor *fields* may be introduced using the methods of Chap. 7 and are briefly discussed in Exercise 7.63. A preliminary to that job is a deep understanding of the algebraic aspects of those concepts, i.e., the dotted and undotted algebraic spinors which we now discuss. Their relation with Weyl spinors will become apparent in a while.

Recall that the spacetime algebra $\mathbb{R}_{1,3}$ is the *real* Clifford algebra associated with Minkowski vector space $\mathbb{R}^{1,3}$, which is a four dimensional real vector space, equipped with a Lorentzian bilinear form

$$\boldsymbol{\eta} : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{R}. \tag{3.108}$$

Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an arbitrary orthonormal basis of $\mathbb{R}^{1,3}$, i.e.,

$$\boldsymbol{\eta}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \eta_{\mu\nu}, \tag{3.109}$$

where the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix $\text{diag}(1, -1, -1, -1)$. Also, $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is the *reciprocal* basis of $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e., $\boldsymbol{\eta}(\mathbf{e}^\mu, \mathbf{e}_\nu) = \delta^\mu_\nu$. We have in obvious notation

$$\boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) = \eta^{\mu\nu},$$

where the matrix with entries $\eta^{\mu\nu}$ is the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$.

The spacetime algebra $\mathbb{R}_{1,3}$ is generate by the following algebraic fundamental relation

$$\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \mathbf{e}^{\nu}\mathbf{e}^{\mu} = 2\eta^{\mu\nu}. \tag{3.110}$$

As we already know (Sect. 3.7.1) the spacetime algebra $\mathbb{R}_{1,3}$ as a vector space over the real field is isomorphic to the exterior algebra $\bigwedge\mathbb{R}^{1,3} = \bigoplus_{j=0}^{4}\bigwedge^{j}\mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$. We code that information writing $\bigwedge\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Also, we make the following identifications: $\bigwedge^{0}\mathbb{R}^{1,3} \equiv \mathbb{R}$ and $\bigwedge^{1}\mathbb{R}^{1,3} \equiv \mathbb{R}^{1,3}$. Moreover, we identify the exterior product of vectors by

$$\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu} = \frac{1}{2}\left(\mathbf{e}^{\mu}\mathbf{e}^{\nu} - \mathbf{e}^{\nu}\mathbf{e}^{\mu}\right), \tag{3.111}$$

and also, we identify the scalar product of vectors by

$$\boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) = \frac{1}{2}\left(\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \mathbf{e}^{\nu}\mathbf{e}^{\mu}\right). \tag{3.112}$$

Then we can write

$$\mathbf{e}^{\mu}\mathbf{e}^{\nu} = \boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) + \mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}. \tag{3.113}$$

Now, an arbitrary element $\mathbf{C} \in \mathbb{R}_{1,3}$ can be written as sum of *nonhomogeneous multivectors*, i.e.,

$$\mathbf{C} = s + c_{\mu}\mathbf{e}^{\mu} + \frac{1}{2}c_{\mu\nu}\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \frac{1}{3!}c_{\mu\nu\rho}\mathbf{e}^{\mu}\mathbf{e}^{\nu}\mathbf{e}^{\rho} + p\mathbf{e}^{5} \tag{3.114}$$

where $s, c_{\mu}, c_{\mu\nu}, c_{\mu\nu\rho}, p \in \mathbb{R}$ and $c_{\mu\nu}, c_{\mu\nu\rho}$ are completely antisymmetric in all indices. Also $\mathbf{e}^{5} = \mathbf{e}^{0}\mathbf{e}^{1}\mathbf{e}^{2}\mathbf{e}^{3}$ is the generator of the pseudoscalars. Recall also that as a matrix algebra we have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, the algebra of the 2×2 quaternionic matrices.

3.9.1 Pauli Algebra

Next, we recall (again) that the Pauli algebra $\mathbb{R}_{3,0}$ is the real Clifford algebra associated with the Euclidean vector space $\mathbb{R}^{3,0}$, equipped as usual, with a positive definite bilinear form. As a matrix algebra we have that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of 2×2 complex matrices. Moreover, we recall that $\mathbb{R}_{3,0}$ is isomorphic to the even subalgebra of the spacetime algebra, i.e., writing $\mathbb{R}_{1,3} = \mathbb{R}_{1,3}^{(0)} \oplus \mathbb{R}_{1,3}^{(1)}$ we have,

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)}. \tag{3.115}$$

The isomorphism is easily exhibited by putting $\boldsymbol{\sigma}^i = \mathbf{e}^i\mathbf{e}^0$, $i = 1, 2, 3$. Indeed, with $\delta^{ij} = \text{diag}(1, 1, 1)$, we have

$$\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + \boldsymbol{\sigma}^j\boldsymbol{\sigma}^i = 2\delta^{ij}, \tag{3.116}$$

which is the fundamental relation defining the algebra $\mathbb{R}_{3,0}$. Elements of the Pauli algebra will be called Pauli numbers.[17] As a vector space over the real field, we have that $\mathbb{R}_{3,0}$ is isomorphic to $\bigwedge\mathbb{R}^{3,0} \hookrightarrow \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$. So, any Pauli number can be written as

$$\mathbf{P} = s + p^i\boldsymbol{\sigma}^i + \frac{1}{2}p^i_{ij}\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + p\mathrm{I}, \tag{3.117}$$

where $s, p_i, p_{ij}, p \in \mathbb{R}$ and $p_{ij} = -p_{ji}$ and also

$$\mathrm{I} = -\mathbf{i} = \boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3 = \mathbf{e}^5. \tag{3.118}$$

Note that $\mathrm{I}^2 = -1$ and that I commutes with any Pauli number. We can trivially verify

$$\begin{aligned} \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j &= \mathrm{I}\varepsilon^{ij}_k\boldsymbol{\sigma}^k + \delta^{ij}, \\ [\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] &= \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\boldsymbol{\sigma}^i = 2\boldsymbol{\sigma}^i\wedge\boldsymbol{\sigma}^j = 2\mathrm{I}\varepsilon^{ij}_k\boldsymbol{\sigma}^k. \end{aligned} \tag{3.119}$$

In that way, writing $\mathbb{R}_{3,0} = \mathbb{R}_{3,0}^{(0)} + \mathbb{R}_{3,0}^{(1)}$, any Pauli number can be written as

$$\mathbf{P} = \mathbf{Q}_1 + \mathrm{I}\mathbf{Q}_2, \quad \mathbf{Q}_1 \in \mathbb{R}_{3,0}^{(0)}, \quad \mathrm{I}\mathbf{Q}_2 \in \mathbb{R}_{3,0}^{(1)}, \tag{3.120}$$

with

$$\begin{aligned} \mathbf{Q}_1 &= a_0 + a_k(\mathrm{I}\boldsymbol{\sigma}^k), \quad a_0 = s, \quad a_k = \frac{1}{2}\varepsilon^{ij}_k p_{ij}, \\ \mathbf{Q}_2 &= \mathrm{I}\left(b_0 + b_k(\mathrm{I}\boldsymbol{\sigma}^k)\right), \quad b_0 = p, \quad b_k = -p_k. \end{aligned} \tag{3.121}$$

[17] Sometimes they are also called 'complex quaternions'. This last terminology will become obvious in a while.

3.9.2 Quaternion Algebra

Equation (3.121) show that the quaternion algebra $\mathbb{R}_{0,2} = \mathbb{H}$ can be identified as the even subalgebra of $\mathbb{R}_{3,0}$, i.e.,

$$\mathbb{R}_{0,2} = \mathbb{H} \simeq \mathbb{R}_{3,0}^{(0)}. \tag{3.122}$$

The statement is obvious once we identify the basis $\{1, \hat{\jmath}, \hat{\imath}, \hat{k}\}$ of $\mathbb{H}$ with

$$\{\mathbf{1}, \mathrm{I}\boldsymbol{\sigma}^1, \mathrm{I}\boldsymbol{\sigma}^2, \mathrm{I}\boldsymbol{\sigma}^3\}, \tag{3.123}$$

which are the generators of $\mathbb{R}_{3,0}^{(0)}$. We observe moreover that the even subalgebra of the quaternions can be identified (in an obvious way) with the complex field, i.e., $\mathbb{R}_{0,2}^{(0)} \simeq \mathbb{C}$. Returning to Eq. (3.117) we see that any $\mathbf{P} \in \mathbb{R}_{3,0}$ can also be written as

$$\mathbf{P} = \mathbf{P}_1 + \mathrm{I}\mathbf{L}_2, \tag{3.124}$$

where

$$\begin{aligned} \mathbf{P}_1 &= (s + p_k\boldsymbol{\sigma}^k) \in \bigwedge^0 \mathbb{R}^{3,0} \oplus \bigwedge^1 \mathbb{R}^{3,0} \equiv \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0}, \\ \mathrm{I}\mathbf{L}_2 &= \mathrm{I}(p + \mathrm{I}l_k\boldsymbol{\sigma}^k) \in \bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0}, \end{aligned} \tag{3.125}$$

with $l_k = -\varepsilon_k^{ij} p_{ij} \in \mathbb{R}$. The important fact that we want to emphasize here is that the subspaces $(\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ and $(\bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0})$ do not close separately any algebra. In general, if $\mathbf{A}, \mathbf{C} \in (\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ then

$$\mathbf{AC} \in \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0} \oplus \bigwedge^2 \mathbb{R}^{3,0}. \tag{3.126}$$

To continue, we introduce

$$\boldsymbol{\sigma}_i = \mathbf{e}_i\mathbf{e}_0 = -\boldsymbol{\sigma}^i, \quad i = 1, 2, 3. \tag{3.127}$$

Then, $\mathrm{I} = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ and the basis $\{1, \hat{\imath}\ , \hat{\jmath}, \hat{k}\}$ of $\mathbb{H}$ can be identified with $\{1, -\mathrm{I}\boldsymbol{\sigma}_1, -\mathrm{I}\boldsymbol{\sigma}_2, -\mathrm{I}\boldsymbol{\sigma}_3\}$.

Now, we know that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. This permit us to represent the Pauli numbers by 2×2 complex matrices, in the usual way ($\mathrm{i} = \sqrt{-1}$). We write $\mathbb{R}_{3,0} \ni \mathbf{P} \mapsto P \in \mathbb{C}(2)$, with

$$\begin{aligned} \boldsymbol{\sigma}^1 &\mapsto \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^2 &\mapsto \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^3 &\mapsto \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{3.128}$$

3.9.3 Minimal Left and Right Ideals in the Pauli Algebra and Spinors

The elements $\mathrm{e}_{\pm} = \frac{1}{2}(1 + \boldsymbol{\sigma}_3) = \frac{1}{2}(1 + \mathbf{e}_3\mathbf{e}_0) \in \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}$, $\mathrm{e}_{\pm}^2 = \mathrm{e}_{\pm}$ are *minimal idempotents* of $\mathbb{R}_{3,0}$. They generate the minimal left and right ideals

$$\mathbf{I}_{\pm} = \mathbb{R}_{1,3}^{(0)}\mathrm{e}_{\pm}, \quad \mathbf{R}_{\pm} = \mathrm{e}_{\pm}\mathbb{R}_{1,3}^{(0)}. \tag{3.129}$$

From now on we write $\mathrm{e} = \mathrm{e}_{+}$. It can be easily shown (see below) that, e.g., $\mathbf{I} = \mathbf{I}_{+}$ has the structure of a 2-dimensional vector space over the complex field [10, 13], i.e., $\mathbf{I} \simeq \mathbb{C}^2$. The elements of the vector space $\mathbf{I}$ are called *representatives* of algebraic *contravariant undotted spinors*[18] and the elements of $\mathbb{C}^2$ are the usual *contravariant undotted spinors* used in physics textbooks. They carry the $D^{(\frac{1}{2},0)}$ representation of $\mathrm{Sl}(2,\mathbb{C})$ [17]. If $\boldsymbol{\varphi} \in \mathbf{I}$ we denote by $\varphi \in \mathbb{C}^2$ the usual matrix representative[19] of $\boldsymbol{\varphi}$ is

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \varphi^1, \varphi^2 \in \mathbb{C}. \tag{3.130}$$

Denoting by $\dot{\mathbf{I}} = \mathrm{e}\mathbb{R}_{1,3}^{(0)}$ the space of the algebraic covariant dotted spinors, we have the isomorphism, $\dot{\mathbf{I}} \simeq (\mathbb{C}^2)^{\dagger} \simeq \mathbb{C}_2$, where $\dagger$ denotes Hermitian conjugation. The elements of $(\mathbb{C}^2)^{\dagger}$ are the usual contravariant spinor fields used in physics textbooks. They carry the $D^{(0,\frac{1}{2})}$ representation of $\mathrm{Sl}(2,\mathbb{C})$ [17]. If $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$, then

[18] We omit in the following the term representative and call the elements of $\mathbf{I}$ simply by algebraic contravariant undotted spinors. However, the reader must always keep in mind that any algebraic spinor is an equivalence class, as defined and discussed in Sect. 4.6.

[19] The matrix representation of the elements of the ideals $\mathbf{I}$, $\dot{\mathbf{I}}$, are of course, 2×2 complex matrices (see, [10], for details). It happens that both columns of that matrices have the *same* information and the representation by column matrices is enough here for our purposes.

its matrix representation in $(\mathbb{C}^2)^\dagger$ is a row matrix usually denoted by

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix}, \quad \xi_{\dot{1}}, \xi_{\dot{2}} \in \mathbb{C}. \tag{3.131}$$

The following representation of $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ in $(\mathbb{C}^2)^\dagger$ is extremely convenient. We say that to a covariant undotted spinor ξ there corresponds a covariant dotted spinor $\dot{\xi}$ given by

$$\dot{\mathbf{I}} \ni \dot{\boldsymbol{\xi}} \mapsto \dot{\xi} = \bar{\xi}\varepsilon \in (\mathbb{C}^2)^\dagger, \quad \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{C}, \tag{3.132}$$

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.133}$$

We can easily find a basis for $\mathbf{I}$ and $\dot{\mathbf{I}}$. Indeed, since $\mathbf{I} = \mathbb{R}_{1,3}^{(0)}\mathrm{e}$ we have that any $\boldsymbol{\varphi} \in \mathbf{I}$ can be written as

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^1 \boldsymbol{\vartheta}_1 + \boldsymbol{\varphi}^2 \boldsymbol{\vartheta}_2$$

where

$$\begin{aligned} &\boldsymbol{\vartheta}_1 = \mathrm{e}, \quad \boldsymbol{\vartheta}_2 = \boldsymbol{\sigma}_1 \mathrm{e}, \\ &\boldsymbol{\varphi}^1 = a + \mathbf{i}b, \quad \boldsymbol{\varphi}^2 = c + \mathbf{i}d, \quad a, b, c, d \in \mathbb{R}. \end{aligned} \tag{3.134}$$

Analogously we find that any $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ can be written as

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \boldsymbol{\xi}_{\dot{1}} \mathbf{s}^{\dot{1}} + \boldsymbol{\xi}_{\dot{2}} \mathbf{s}^{\dot{2}}, \\ \mathbf{s}^{\dot{1}} &= \mathrm{e}, \quad \mathbf{s}^{\dot{2}} = \mathrm{e}\boldsymbol{\sigma}_1. \end{aligned} \tag{3.135}$$

Defining the mapping

$$\begin{aligned} \boldsymbol{\iota} : \mathbf{I} \otimes \dot{\mathbf{I}} &\to \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}, \\ \boldsymbol{\iota}(\boldsymbol{\varphi} \otimes \dot{\boldsymbol{\xi}}) &= \boldsymbol{\varphi}\dot{\boldsymbol{\xi}}, \end{aligned} \tag{3.136}$$

we have

$$\begin{aligned} 1 \equiv \boldsymbol{\sigma}_0 &= \boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{1}} + \mathbf{s}_2 \otimes \mathbf{s}^{\dot{2}}), \\ \boldsymbol{\sigma}_1 &= -\boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{2}} + \mathbf{s}_2 \otimes \mathbf{s}^{\dot{1}}), \end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}_2 &= \boldsymbol{\iota}[\mathbf{i}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{2}} - \mathbf{s}_2 \otimes \mathbf{s}^{\dot{1}})],\\ \boldsymbol{\sigma}_3 &= -\boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{1}} - \mathbf{s}_2 \otimes \mathbf{s}^{\dot{2}}).\end{aligned} \tag{3.137}$$

From this it follows the identification

$$\mathbb{R}_{3,0} \simeq \mathbb{R}^{(0)}_{1,3} \simeq \mathbb{C}(2) = \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}, \tag{3.138}$$

and then, each Pauli number can be written as an appropriate sum of Clifford products of algebraic contravariant undotted spinors and algebraic covariant dotted spinors. And, of course, a representative of a Pauli number in $\mathbb{C}^2$ can be written as an appropriate Kronecker product of a complex column vector by a complex row vector.

Take an arbitrary $\mathbf{P} \in \mathbb{R}_{3,0}$ such that

$$\mathbf{P} = \frac{1}{j!} p^{\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_j} \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_j}, \tag{3.139}$$

where $p^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} \in \mathbb{R}$ and

$$\boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} = \boldsymbol{\sigma}_{\mathbf{k}_1} \cdots \boldsymbol{\sigma}_{\mathbf{k}_j}, \quad \text{and } \boldsymbol{\sigma}_0 \equiv 1 \in \mathbb{R}. \tag{3.140}$$

With the identification $\mathbb{R}_{3,0} \simeq \mathbb{R}^{(0)}_{1,3} \simeq \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}$, we can also write

$$\mathbf{P} = \mathbf{P}^{A}{}_{\dot{B}}\boldsymbol{\iota}(\mathbf{s}_A \otimes \mathbf{s}^{\dot{B}}) = \mathbf{P}^{A}{}_{\dot{B}}\mathbf{s}_A\mathbf{s}^{\dot{B}}, \tag{3.141}$$

where the $\mathbf{P}^{A}{}_{\dot{B}} = \mathbf{X}^{A}{}_{\dot{B}} + \mathbf{i}\mathbf{Y}^{A}{}_{\dot{B}}$, $\mathbf{X}^{A}{}_{\dot{B}}, \mathbf{Y}^{A}{}_{\dot{B}} \in \mathbb{R}$.

Finally, the matrix representative of the Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ is $P \in \mathbb{C}(2)$ given by

$$P = P^{A}{}_{\dot{B}} s_A s^{\dot{B}}, \tag{3.142}$$

with $P^{A}{}_{\dot{B}} \in \mathbb{C}$ and

$$\begin{aligned} s_1 &= \begin{pmatrix}1\\0\end{pmatrix},\ s_2 = \begin{pmatrix}0\\1\end{pmatrix},\\ s^{\dot{1}} &= \begin{pmatrix}1 & 0\end{pmatrix},\ s^{\dot{2}} = \begin{pmatrix}0 & 1\end{pmatrix}.\end{aligned} \tag{3.143}$$

It is convenient for our purposes to introduce also covariant undotted spinors and contravariant dotted spinors. Let $\varphi \in \mathbb{C}^2$ be given as in Eq. (3.130). We define the *covariant* version of undotted spinor $\varphi \in \mathbb{C}^2$ as $\varphi^* \in (\mathbb{C}^2)^t \simeq \mathbb{C}_2$ such that

$$\begin{aligned} \varphi^* &= (\varphi_1, \varphi_2) \equiv \varphi_A s^A, \\ \varphi_A &= \varphi^B \varepsilon_{BA,} \quad \varphi^B = \varepsilon^{BA}\varphi_A, \\ s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \end{aligned} \tag{3.144}$$

where[20] $\varepsilon_{AB} = \varepsilon^{AB} = \text{adiag}(1, -1)$. We can write due to the above identifications that there exists $\varepsilon \in \mathbb{C}(2)$ given by Eq. (3.133) which can be written also as

$$\varepsilon = \varepsilon^{AB} s_A \boxtimes s_B = \varepsilon_{AB} s^A \boxtimes s^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \tag{3.145}$$

where $\boxtimes$ denotes here the *Kronecker* product of matrices. We have, e.g.,

$$\begin{aligned} s_1 \boxtimes s_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ s^1 \boxtimes s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.146}$$

We now introduce the *contravariant* version of the dotted spinor

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix} \in \mathbb{C}_2$$

as being $\dot{\xi}^* \in \mathbb{C}^2$ such that

$$\begin{aligned} \dot{\xi}^* &= \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \end{pmatrix} = \xi^{\dot{A}} s_{\dot{A}}, \\ \xi^{\dot{B}} &= \varepsilon^{\dot{B}\dot{A}} \xi_{\dot{A}}, \quad \xi_{\dot{A}} = \varepsilon_{\dot{B}\dot{A}}\, \xi^{\dot{B}}, \\ s_{\dot{1}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_{\dot{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{3.147}$$

where $\varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \text{adiag}(1, -1)$. Then, due to the above identifications we see that there exists $\dot{\varepsilon} \in \mathbb{C}(2)$ such that

$$\dot{\varepsilon} = \varepsilon^{\dot{A}\dot{B}} s_{\dot{A}} \boxtimes s_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} s^{\dot{A}} \boxtimes \dot{s}^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon. \tag{3.148}$$

[20]The symbol adiag means the antidiagonal matrix.

Also, recall that even if $\{\mathbf{s}_A\}$,$\{\mathbf{s}_{\dot{A}}\}$ and $\{s^{\dot{A}}\}$,$\{s^A\}$ are bases of distinct spaces, we can identify their matrix representations, as it is obvious from the above formulas. So, we have $s_A \equiv s_{\dot{A}}$ and also $s^{\dot{A}} = s^A$. This is the reason for the representation of a dotted covariant spinor as in Eq. (3.132). Moreover, the above identifications permit us to write the *matrix representation* of a Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ as, e.g.,

$$P = P_{AB} s^A \boxtimes s^B \tag{3.149}$$

besides the representation given by Eq. (3.142).

Exercise 3.57 Consider the ideal $I = \mathbb{R}_{1,3}\frac{1}{2}(1 - \mathbf{e}_0\mathbf{e}_3)$. Show that $\phi \in I$ is a representative in a spin frame Ξ_u of a covariant Dirac spinor[21] $\Psi \in \mathbb{C}(4)\frac{1}{2}(1 + \underline{\gamma}_0)(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2)$. Let ψ be the representative (in the same spin frame Ξ_u) of a Dirac-Hestenes spinor, associated to a mother spinor $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1 + \mathbf{e}_0)$ by $\Phi = \psi\frac{1}{2}(1 + \mathbf{e}_0)$. Show that $\phi \in I$ can be written as $\phi = \psi\frac{1}{2}(1 + \mathbf{e}_0)\frac{1}{2}(1 - \mathbf{e}_0\mathbf{e}_3)$.

(a) Show that $\phi\mathbf{e}_5 = \phi\mathbf{e}_{21}$.
(b) Weyl spinors are defined as eigenspinors of the chirality operator, i.e., $\underline{\gamma}_5\Psi_\pm = \pm\mathrm{i}\Psi_\pm$. Show that Weyl spinors corresponds to the even and odd parts of ϕ.
(c) Relate the even and odd parts of ϕ to the algebraic dotted and undotted spinors.[22]

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[21] Recall that $\underline{\gamma}_\mu$ are the Dirac matrices defined by Eq. (3.49).

[22] As a suggestion for solving the above exercise the reader may consult [10].

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