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The Many Faces of Maxwell, Dirac and Einstein Equations

A Clifford Bundle Approach

Second Edition

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Preface

In this second edition of our book, we rewrote some sections and added new ones to some chapters for better intelligibility, e.g., the new Sect. 6.9 titled "Schwarzschild Original Solution and the Existence of Black Holes." Three new chapters (in this edition, Chaps. 5, 15, and 16) are included. Chapter 5 gives a Clifford bundle approach to the Riemannian or semi-Riemannian differential geometry of branes understood as submanifolds of a Euclidean or pseudo-Euclidean space of large dimension. We introduce the important concept of the projection operator, study its properties, and introduce several other operators associated to it, as the shape operator and the shape biform, crucial objects to define the bending of a submanifold equipped with a Euclidean or pseudo-Euclidean embedded in a Euclidean or pseudo-Euclidean space of large dimension. We give several different expressions for the curvature biform operator in terms of derivatives of the projection operator and of the shape operator and prove the remarkable formula $\mathbf{S}^2(v) = -\boldsymbol{\partial} \wedge \boldsymbol{\partial}(v)$, which says that the square of the shape operator applied to a 1-form field v is equal to the negative of the Ricci operator (introduced in Chap. 4) applied to a 1-form field v. Such result is used in Chap. 11 to show how to transform in " marble" the "wood" part of Einstein equation. By this we mean that we can express its second member containing a phenomenological energy-momentum tensor in a purely geometrical term involving the square of the shape operator. More, Chap. 11 besides including the results just mentioned has been completely rewritten and is now titled "On the Nature of the Gravitational Field." We hope that this chapter leaves clear that the interpretation of the gravitational field as the geometry of a Lorentzian spacetime structure is only one among different possible choices, not a necessary one. We added to this second edition Chap. 15 titled "Maxwell, Einstein, Dirac, and Navier–Stokes Equations," which, besides reveling some surprising relations concerning the many faces of Maxwell, Dirac, Einstein, and the Navier–Stokes equations also, clarifies the meaning of the so-called Komar currents and finds their explicit form in General Relativity theory. There is now also Chap. 16 that analyzes the similarities and main differences between Dirac, Majorana, and ELKO spinor fields, a subject that is receiving a lot of attention in the last few years. We present an alternative theory for ELKO spinor fields of mass dimension $3/2$ (instead of mass dimension

1 as originally proposed in [1]) and show that our ELKO spinor fields can be used to describe electric neutral particles carrying “magnetic-like” charges with short-range interaction mediated by a $su(2)$-valued gauge potential. We also change (in relation to the first edition) some symbols for better clarity on the typing of some formulas. Of course, in a book of this size, it is almost impossible not to use the same symbol to represent (at different places) different objects. We tried to minimize such occurrences, and a list of the principal symbols is given at the end of the book. There, the reader will find also a list of acronyms and an index. References are given at the end of each chapter. A detailed description of the contents of the chapters is given in Chap. 1.

We are particularly grateful to the many helpful discussions on the subjects presented in this book that we had for years with P. Anglès, G. Bruhn, F. W. Hehl, R.F. Leão, (the late) P. Lounesto, (the late) Ian Porteous, R. A. Mosna, A.M. Moya, E.A. Notte Cuello, R. da Rocha, J.E. Maiorino, Z. Oziewicz, F. Grangeiro Rodrigues, Q.A.G. Souza, J. Vaz Jr., and S.A. Wainer. We are also grateful to the many readers of the first edition who pointed to us misprints and some errors that (we hope) have been corrected in this edition. Of course, they are not guilty for any remaining error or misconception that the reader may eventually find. Moreover, the authors will be grateful to anyone who inform them of any additional correction that should be made to the text.

Campinas, Brazil
August 2015

Waldyr Alves Rodrigues Jr.
Edmundo Capelas de Oliveira

Reference

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Chapter 1
Introduction

Abstract In this chapter we describe the contents of all chapters of the book and their interrelationship.

Maxwell, Dirac and Einstein's equations are certainly among the most important equations of twentieth century Physics and it is our intention in this book to investigate some of the many faces[1] of these equations and their relationship and to discuss some foundational issues involving some of the theories where they appear. To do that, let us briefly recall some facts.

Maxwell equations which date back to the nineteenth century encodes all classical electromagnetism, i.e., they describe the electromagnetic fields generated by charge distributions in arbitrary motion. Of course, when Maxwell formulated his theory the arena where physical phenomena were supposed to occur was a Newtonian spacetime, a structure containing a manifold which is diffeomorphic to $\mathbb{R} \times \mathbb{R}^3$, the first factor describing Newtonian absolute time [30] and the second factor the Euclidean space of our immediate perception.[2] In his original approach Maxwell presented his equations as a system of eight linear first order partial differential equations involving the components of the electric and magnetic fields [22] generated by charge and currents distributions with prescribed motions in vacuum.[3] It was only after Heaviside [16], Hertz and Gibbs that those equations were presented using vector calculus, which by the way, is the form they appear until today in elementary textbooks on Electrodynamics and Engineering Sciences. In the vector calculus formalism Maxwell equations are encoded in four equations involving the well known divergent and rotational operators. The motion of charged particles under the action of prescribed electric and magnetic fields was supposed in Maxwell's time to be given by Newton's second law of motion, with the so-called Lorentz force acting on the charged particles. It is a well known story that

---

[1] By many faces of Maxwell, Dirac and Einstein's equations, we mean the many different ways in which those equations can be presented using different mathematical theories.

[2] For details on the Newtonian spacetime structure see, e.g., [30].

[3] Inside a material the equations involve also other fields, the so-called polarization fields. See details in [22].

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at the beginning of the twentieth century, due to the works of Lorentz, Poincaré and Einstein [35], it has been established that the system of Maxwell equations was compatible with a new version of spacetime structure serving as arena for physical phenomena, namely Minkowski spacetime (to be discussed in details in the following chapters). It became also obvious that the classical Lorentz force law needed to be modified in order to leave the theory of classical charged particles and their electromagnetic fields invariant under spacetime transformations in Minkowski spacetime defining a representation of the Poincaré group. Such a condition was a necessary one for the theory to satisfy the Principle of Relativity.[4]

Modern presentations of Maxwell equations make use of the theory of differential forms and succeed in writing the original system of Maxwell equations as two equations involving the exterior derivative operator and the so-called Hodge star operator.[5]

Now, one of the most important constructions of human mind in the twentieth century has been Quantum Theory. Here, we need to recall that in a version of that theory which is also in accordance with the Principle of Relativity, the motion of a charged particle under the action of a prescribed electromagnetic field is not described by the trajectory of the particle in Minkowski spacetime predicted by the Lorentz force law once prescribed initial conditions are given. Instead (as it is supposed well known by any reader of this book) the state of motion of the particle is described by a (covariant) Dirac spinor field (Chap. 7), which is a section of a particular spinor bundle. In elementary presentations[6] of the subject a Dirac spinor field is simply a mapping from Minkowski spacetime (expressed in global coordinates in the Einstein-Lorentz-Poincaré gauge) to $\mathbb{C}^4$. At any given spacetime point, the space of Dirac spinor fields is said to carry[7] the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $Sl(2,\mathbb{C})$, which is the universal covering group of the proper orthochronous Lorentz group. Moreover the Dirac spinor field in interaction with a prescribed electromagnetic potential is supposed to satisfy a linear differential equation called the Dirac equation. Such theory was called relativistic quantum mechanics (or first quantized relativistic quantum mechanics).[8]

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[4]The reader is invited to study Chap. 6 in detail to know the exact meaning of this statement.

[5]These mathematical tools are introduced in Chap. 4.

[6]The elementary approach is related with a choice of a global spin coframe (Chap. 7) in spacetime.

[7]The precise mathematical meaning of this statement can be given only within the theory of spin-Clifford bundles, as described in Chap. 6.

[8]Soon it became clear that the interaction of the electromagnetic field with the Dirac spinor field could produce pairs (electrons and positrons). Besides that it was known since 1905 that the classical concept of the electromagnetic field was not in accord with experience and that the concept of photons as quanta of the said field needed to be introduced. The theory that deals with the interaction of photons and electrons (and positrons) is a particular case of a second quantized renormalizable quantum field theory and is called quantum electrodynamics. In that theory the electromagnetic and the Dirac spinor fields are interpreted as operator valued distributions [3] acting on the Hilbert space of the state vectors of the system. We shall not discuss further this theory in this book, but will return to some of its issues in a sequel volume [6].

Now, the covariant Dirac spinor field used in quantum electrodynamics is, at first sight, an object of a mathematical nature very different from that of the electromagnetic field, which is described by a 2-form field (or the electromagnetic potential that is described by a 1-form field). As a consequence we cannot see any relationship between these fields or between Maxwell and Dirac equations.

It would be nice if those fields which are the dependent variables in Maxwell and Dirac equations, could be represented by objects of the same mathematical nature. As we are going to see, this is indeed possible. It turns out that Maxwell fields can be represented by appropriate homogeneous sections of the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathtt{g})$ and Dirac fields can be represented (once we fix a spin coframe) by a sum of even homogeneous sections of $\mathcal{C}\ell(M, \mathtt{g})$. These objects are called representatives of Dirac-Hestenes spinor fields.[9] Once we arrive at this formulation, which requires of course the introduction of several mathematical tools, we can see relationships between those equations that are not apparent in the standard formalism (Chap. 13). Finally, we can also easily see the meaning of the many Dirac-like presentations of Maxwell equations that appeared in the literature during the last century. Moreover, we will see that our formalism is related in an intriguing way to formalisms used in modern theories of Physics, like the theories of superparticles and superfields (Chap. 14).

Besides providing mathematical unity to the theory of Maxwell and Dirac fields, we give also a Clifford bundle approach to the differential geometry of a Riemann-Cartan-Weyl spacetime. This is done with the objective of finding a description of the gravitational field as a set of sections of an appropriate Clifford bundle over Minkowski spacetime, which moreover satisfy equations equivalent to Einstein's equations on an effective Lorentzian spacetime. That enterprise is not just a mathematical game. There are serious reasons for formulating such a theory. To understand the most important reason (in our opinion), let us recall some facts. First, keep in mind that in Einstein's General Relativity theory (GRT) a gravitational field is modeled by a Lorentzian spacetime[10] (which is a particular Riemann-Cartan-Weyl spacetime). This means that in GRT the gravitational field is an object of a different physical nature from the electromagnetic and Dirac-Hestenes fields, which are fields living in a spacetime. This distinct nature of the gravitational field implies (as will be proved in Chap. 10) the *lack* of conservation laws of energy-momentum and angular momentum in that theory. But how to formulate a more comprehensible theory?

A possible answer is provided by the theory of extensors and symmetric automorphisms of Clifford algebras developed in Chap. 2, which, together with the Clifford bundle formalism of Chap. 4, suggests naturally to interpret Einstein theory as a field theory for cotetrad fields (defining a coframe) in Minkowski spacetime.

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[9]The details are given in Chap. 7 whose intelligibility presupposes that the reader has studied Chap. 3.

[10]Indeed, by an equivalence class of diffeomorphic Lorentzian spacetimes.

How this is done is described in the appropriate chapters that follow and which are now summarized.

Chapter 2, called *Multiform and Extensor Calculus*, gives a detailed introduction to exterior and Grassmann algebras and to the Clifford algebra of multiforms. The presentation has been designed in order for any serious student to acquire quickly the necessary skill which will permit him to reproduce the calculations done in the text. Thus, several exercises are proposed and many of them are solved in detail. Also in Chap. 2 we introduce the concept of *extensor*, which is a natural generalization of the concept of tensor and which plays an important role in our developments. In particular extensors appear in the formulation of the theory of symmetric automorphisms and orthogonal Clifford products, which permit to see different Clifford algebras associated with vector spaces of the same dimension, but equipped with metrics of different signatures, as deformations of each other. A theory of multiform functions and of the several different derivatives operators acting on those multiform functions is also presented in Chap. 2. Such a theory is the basis for the presentation of Lagrangian field theory, a subject discussed in Chaps. 8 and 9. There, we see also the crucial role of *extensor fields* when representing energy-momentum and angular momentum in field theory.

Chapter 3 describes the hidden geometrical nature of *spinors,* and it is our hope to have presented a fresh view on the subject. So, that chapter starts recalling some fundamental results from the representation theory of associative algebras and then gives the classification of real and complex Clifford algebras, and in particular discloses the relationship between the spacetime algebra and the Majorana, Dirac and Pauli algebras, which are the most important Clifford algebras in our study of Maxwell, Dirac and Einstein's equations. Next, the concepts of left, right and bilateral ideals on Clifford algebras are introduced, and the notion of *algebraically* and *geometrically* equivalent ideals is given. Equipped with these notions we give original definitions of algebraic and covariant spinors. For the case of a vector space equipped with a metric of signature $(1,3)$ we show that it is very useful to introduce the concept of Dirac-Hestenes spinors. The hidden geometrical nature of these objects is then disclosed. We think that the concept of Dirac-Hestenes spinor and of Dirac-Hestenes spinor fields (introduced in Chap. 6) are very important and worth to be known by every physicist and mathematician. Indeed, we shall see in Chap. 12 how the concept of Dirac-Hestenes spinor fields permits us to find unsuspected mathematical relations between Maxwell and Dirac equations and also between those equations and the Seiberg-Witten equations (in Minkowski spacetime). Chapter 3 discuss also Majorana, Weyl and dotted and undotted algebraic spinors. Bilinear invariants, Fierz identities and the notion of boomerangs are also introduced since, in particular, the Fierz identities play an important role in the interpretation of Dirac theory.

Chapter 4 discusses aspects of differential geometry that are essential for a reasonable understanding of spacetime theories and in our opinion necessary to avoid wishful thinking concerning mathematical possibilities and physical reality. Our main purpose is to present a Clifford bundle approach to the geometry of a general Riemann-Cartan-Weyl *space M* (a differentiable manifold) carrying a *metric* $\boldsymbol{g} \in \sec T_2^0 M$ of signature (p, q), a *connection* ∇, an *orientation* $\tau_{\boldsymbol{g}}$. Such a structure is denoted by $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}})$. The pair $(\nabla, \boldsymbol{g})$ defines a *geometry* for M. When M is 4-dimensional (and satisfies some other requirements to be discussed later) and $\boldsymbol{g}$ has signature $(1, 3)$, the pair $(M, \boldsymbol{g})$ is called a Lorentzian manifold. Moreover, endow M with a general connection ∇, with a spacetime orientation $\tau_{\boldsymbol{g}}$ and with a *time orientation* $\uparrow$ (see Definition 4.105). Such a general structure will be denoted by $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ and is said to be a Riemann-Cartan-Weyl *spacetime. Lorentzian* spacetimes are structures $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ restricted by the condition that the connection D is metric compatible, i.e., $D\boldsymbol{g} = 0$ and that the torsion tensor of that connection is zero, i.e., $\Theta[D] = 0$. A connection satisfying these two requirements is called a Levi-Civita connection[11] (and it is unique). Moreover, if in addition to the previous requirements, $M \simeq \mathbb{R}^4$ and the Riemann curvature tensor of the connection is null (i.e., $\mathbf{R}[D] = 0$) the Lorentzian spacetime is called Minkowski spacetime. That structure is the 'arena' for what is called special relativistic theories.

It is necessary that the reader realize very soon that a given n-dimensional manifold M may *eventually* admit many different metrics[12] and many different connections. Thus, there is no meaning in saying that a given manifold has torsion and/or curvature. What is meaningful is to say that a given connection on a given manifold has torsion and/or curvature. A classical example [23], is the *Nunes connection* on the punctured sphere $M = \mathring{S}^2 = \{S^2 \setminus$ north + south poles$\}$, discussed in Sect. 4.9.8. Indeed, let $(\mathring{S}^2, \boldsymbol{g}, D)$ be the usual Riemann structure for a punctured sphere. That means that D is the Levi-Civita connection of $\boldsymbol{g}$, the metric on $\mathring{S}^2$ which it inherits (by pullback) from the ambient three dimensional Euclidean space. According to that structure, as it is well known, the Riemann curvature tensor of D on $\mathring{S}^2$ is not null and the torsion of D on $\mathring{S}^2$ is null. However, we can give to the punctured sphere the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla^c)$ where ∇^c is the *Nunes connection*. For that connection the Riemann curvature tensor $\mathbf{R}[\nabla^c] = 0$, but its torsion $\Theta[\nabla^c] \neq 0$. So, according to the connection D the punctured sphere has (Riemann) curvature different from zero, but according to the Nunes connection it has zero (Riemann) curvature.

If the above statements looks odd to you, it is because you always thought of the sphere as being a curved surface (bent) living in Euclidean space. Notice, however, that there may exist a surface $\mathfrak{S}$ that is also bent as a surface in the three dimensional Euclidean space, but is such that the structure $(\mathfrak{S}, \boldsymbol{g}, D)$ is *flat*. As an example, take the cylinder $\mathfrak{S} = S^1 \times \mathbb{R}$ with $\boldsymbol{g}$ the usual metric on $S^1 \times \mathbb{R}$ that it inherits from the ambient three-dimensional Euclidean space, and where D is the Levi-Civita connection of $\boldsymbol{g}$, then $\mathbf{R}[D] = 0$ and $\Theta[D] = 0$.

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[11] Of course that denomination holds for any manifold M, $\dim M = n$ equipped with a metric of signature (p, q).

[12] The possible types of different metrics depend on some topological restrictions. This will be discussed at the appropriate place.

The above discussion makes clear that we cannot confuse the Riemann curvature of a connection on a manifold M, with its *bending*,[13] i.e., with the topological fact that M may be realized as a (hyper)surface *embedded* on an Euclidean (or pseudo-Euclidean) space of sufficiently high dimension.[14] Keeping in mind these (simple) ideas is important for appreciating the theory (and the nature) of the gravitational field, to be discussed in Chap. 11.

As we already said the main objective of Chap. 4 is to introduce a *Clifford bundle formalism*, which can efficiently be used in the study of the differential geometry of manifolds an also to give an unified mathematical description of the Maxwell, Dirac and gravitational fields.

So, we introduce the *Cartan*, *Hodge* and *Clifford* bundles[15] and present the relationship between them and we also recall Cartan's formulation of differential geometry, extending it to a general Riemann-Cartan-Weyl space or spacetime (hereafter denoted RCWS).

We study the geometry of a RCWS in the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ of the cotangent bundle.[16]

First we introduce, for a given manifold M, a structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$, where $\mathring{D}$ is the Levi-Civita connection of an *arbitrary* fiducial metric $\mathring{\boldsymbol{g}}$ (which is supposed to be compatible with the structure of M) and call such a structure the *standard structure*. Next we introduce the concept of the *standard Dirac operator* $\mathring{\boldsymbol{\partial}}$ acting on sections of the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ of differential forms. Using $\mathring{\boldsymbol{\partial}}$, we define the concepts of standard Dirac *commutator* and *anticommutator* and we discuss the geometrical meaning of those operators. With the theory of symmetric automorphisms of a Clifford algebra (discussed in Chap. 2) we introduce *infinitely* many other Dirac-like operators given $(M, \mathring{D})$, one for each non-degenerated bilinear form field $\boldsymbol{g} \in \sec T_2^0 M$ that can be defined on the standard structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$. Such new

[13]Bending of a manifold viewed as submanifold of a Euclidean or pseudo-Euclidean space of large dimension is characterized by the shape operator, a concept introduced in Chap. 5.

[14]Any manifold M, $\dim M = n$, according to Whitney's theorem, can be realized as a submanifold of $\mathbb{R}^m$, with $m = 2n$. However, if M carries additional structure the number m in general must be greater than $2n$. Indeed, it has been shown by Eddington [7] that if dim $M = 4$ and if M carries a Lorentzian metric $\boldsymbol{g}$, which moreover satisfies Einstein's equations, then M can be locally embedded in a (pseudo)Euclidean space $\mathbb{R}^{1,9}$. Also, isometric embeddings of general Lorentzian spacetimes would require a lot of extra dimensions [4]. Indeed, a compact Lorentzian manifold can be embedded isometrically in $\mathbb{R}^{2,46}$ and a non-compact one can be embedded isometrically in $\mathbb{R}^{2,87}$!

[15]Spin-Clifford bundles are introduced in Chap. 7.

[16]In this book, the metric of the tangent bundle is always denoted by a boldsymbol letter, e.g., $\boldsymbol{g} \in \sec T_2^0 M$. The corresponding metric of the cotangent bundle is always represented by a typewriter symbol, in this case, $\mathtt{g} \in \sec T_0^2 M$. Moreover, we represent by $\underline{\mathtt{g}} : TM \to TM$ the endomorphism associated with $\boldsymbol{g}$. We have $\boldsymbol{g}(\mathbf{u}, \mathbf{v}) = \underline{\mathtt{g}}(\mathbf{u}) \underset{\boldsymbol{g}_{\mathrm{E}}}{\cdot} \mathbf{v}$, for any $\mathbf{u}, \mathbf{v} \in \sec TM$, where $\boldsymbol{g}_{\mathrm{E}}$ is an appropriate Euclidean metric on TM. The inverse of the endomorphism $\underline{\mathtt{g}}$ is denoted $\underline{\mathtt{g}}^{-1}$. We represent by $\underline{g} : T^*M \to T^*M$ the endomorphism corresponding to $\mathtt{g}$. Finally, the inverse of $\underline{g}$ is denoted by $\underline{g}^{-1}$. See details in Sect. 2.8.

Dirac-like operators are denoted by $\overset{\vee}{\boldsymbol{\partial}}$. Using $\overset{\vee}{\boldsymbol{\partial}}$ we introduce the analogous of the concepts of Dirac commutator and anticommutator, which are first introduced using $\boldsymbol{\partial}$ and explore their geometrical meaning. In particular, the standard Dirac commutator permits us to give the structure of a *local* Lie algebra to the cotangent bundle, in analogy with the way in which the bracket of vector fields defines a *local* Lie algebra structure for the tangent bundle. The coefficients of the standard Dirac anticommutator—called the Killing coefficients—are related to the Lie derivative of the metric by a very interesting relation [see Eq. (4.160)].

Subsequently, we introduce, besides the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ on M, also a general RCWS structure $(M, \mathring{\boldsymbol{g}}, \nabla)$, $\nabla \neq \mathring{D}$, on the *same* manifold M and study its geometry with the help of $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Associated with that structure we introduce a Dirac operator $\boldsymbol{\partial}$ and again using the theory of symmetric automorphisms of a Clifford algebra we introduce infinitely many other Dirac-like operators $\overset{\vee}{\boldsymbol{\partial}}$, one for each non-degenerated bilinear form field $\boldsymbol{g} \in \sec T_2^0 M$ that can be defined on the structure $(M, \mathring{\boldsymbol{g}}, \nabla)$.

With respect to the structures $(M, \mathring{\boldsymbol{g}}, \nabla)$ and $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ we also obtain new decompositions of a general connection ∇. It is then possible to exhibit some tensor quantities which are not well known, and have been first introduced (for the best of our knowledge) in the literature in [32] and exhibits interesting relations between the geometries of the structures $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ and $(M, \boldsymbol{g}, \nabla)$. These results are used latter to shed a new light on the *flat* space[17] formulations of the theory of the gravitational field (Chap. 11) and on the theory of spinor fields in RCWS, as discussed in Chap. 10.

We also show that the square of the standard Dirac operator is (up to a signal, which depends on convention) equal to the *Hodge Laplacian* $\diamond$ of the standard structure. The Hodge Laplacian $\diamond$ maps p-forms on p-forms. However, the *square* of the Dirac operator $\boldsymbol{\partial}$ in a general Riemann-Cartan space does not maps p-forms on p-forms and as such cannot play the role of a wave operator in such spaces. The role of such an operator must be played by an appropriate generalization of the Hodge Laplacian in such spaces. We have identified such a wave operator $\mathcal{L}_+$, which (apart from a constant factor) is the relativistic Hamiltonian operator that describes the theory of Markov processes, as used, e.g., in [24].

Chapter 4 presents also some applications of the formalism, namely Maxwell equations in the Hodge and Clifford bundles, flux and action quantization. The concepts of *Ricci* and *Einstein operators* acting on a set of cotetrad fields $\theta^{\mathbf{a}}$ defining a coframe are also introduced. Such operators are used to write 'wave' equations for the cotetrad fields which for Lorentzian spacetimes are equivalent to Einstein's equations.

[17]The word flat here refers to formulations of the gravitational field, in which this field is a physical field, in the sense of Faraday, living on Minkowski spacetime.

Chapter 5 gives a Clifford bundle approach to the Riemannian or semi-Riemannian differential geometry of branes understood as submanifolds of a Euclidean or pseudo-Euclidean space of large dimension. We introduce the important concept of the projection operator and define some other operators associated to it, as the shape operator and the shape biform. The shape operator is essential to define the concept of bending of a submanifold (as introduced above) and to leave it clear that a surface can be bended and yet the Riemann curvature of a connection defined in it may be null (as already mentioned for the case of the Nunes connection).

We give several different expression for the curvature biform operator in terms of derivatives of the projection operator and the shape operator and prove the remarkable formula $\mathbf{S}^2(v) = -\boldsymbol{\partial} \wedge \boldsymbol{\partial}(v)$, which says that the square of the shape operator applied to a 1-form field v is equal to the negative of the Ricci operator (introduced in Chap. 4) applied to a 1-form field v. Such result is used in Chap. 11 to show how to transform in marble the "wood" part of Einstein equation. By this we mean that we can express its second member containing a phenomenological energy-momentum tensor in a purely geometrical term involving the square of the shape operator.

Chapter 6 introduces concepts and discusses issues that are in our opinion crucial for a perfect understanding of the Physics behind the theories of Special and General Relativity.[18] To fulfill our goals it is necessary to give a mathematically well formulated statement of the *Principle of Relativity*.[19] This requires a precise formulation (using the mathematical tools introduced in previous chapters and some new ones) of an ensemble of essential concepts as, e.g., observers, reference frames,[20] physical equivalence of reference frames, naturally adapted coordinate chart to a given reference frame (among others) which are rarely discussed in textbooks or research articles. Using the Clifford bundle formalism developed earlier, Chap. 6 presents a detailed discussion of the concept of local rotation as detected by an observer and the Fermi-Walker transport. After that, a mathematical definition of *reference frames* (which are modeled by timelike vector fields in the spacetime manifold) and their *classifications* according to two complementary schemes are presented. In particular, one classification refers to the concept of synchronizability. Some simple examples of the formalism, including a discussion of the *Sagnac effect*, are given. We define the concepts of *covariance* and *invariance* of theories based on the spacetime concept, discussing in details and with examples (including Maxwell and Einstein's equations) what we think must be understood by

[18]We presuppose that the reader of our book knows Relativity Theory at least at the level presented at the classical book [18].

[19]A perfect understanding of the Principle of Relativity is also crucial in our forthcoming book [6] which discusses 'superluminal wave phenomena'.

[20]It is important to distinguish between the concept of a *frame* (which are sections of the frame bundle) introduced in Appendix A.1.1 with the concept of a *reference frame* to be defined in Chap. 6.

the concept of *diffeomorphism invariance* of a phyisical theory. Next, the concept of the *indistinguishable group* of a class of reference frames is introduced and the *Principle of Relativity* for theories which have *Minkowski* spacetime as part of their structures is rigorously formulated. We briefly discuss the empirical status of that principle, which is also associated with what is known as Poincaré invariance of physical laws. We show also the not well known fact that in a general Lorentzian spacetime, modeling a gravitational field according to the GRT, inertial reference frames as the ones that exist in Minkowski spacetime do *not* exist in general. Ignorance of this and other facts, e.g., that distinct reference frames are in general not physically equivalent in GRT generated a lot of confusion for decades, in particular lead many people to believe in the validity of a "General Principle of Relativity". The reference frames in a general Lorentzian spacetime which more closely resembles inertial reference frames are the *pseudo-inertial reference frames* (PIRFs)[21] and the *local Lorentz reference frame associated with* γ (LLRFγ).[22] With the help of these concepts we prove that the so-called 'Principle of Local Lorentz Invariance'[23] of GRT is *not* a true law of nature, despite statements in contrary by many physicists. In general a single PIRF is selected as preferred in reasonable cosmological models in a precise sense discussed in Sect. 6.8. Such a selected PIRF $\mathbf{V}$ is usually identified with the reference frame where the *cosmic background radiation* is isotropic (or the comoving frame of the galaxies). It is important to keep this point in mind, for the following reason. Suppose that physical phenomena occur in Minkowski spacetime, and that there is some phenomenon breaking Lorentz invariance, as, e.g., would be the case if *genuine* superluminal motion existed. In that case, the phenomenon breaking Lorentz invariance could be used to identify a preferred *inertial* reference frame $\mathbf{I}_0$, as has been shown by many authors.[24] However, the identification (as done, e.g., in [25–27]) of $\mathbf{I}_0$ with $\mathbf{V}$, the reference frame where the cosmic background radiation is isotropic and which is supposed to exist in a Lorentzian spacetime (solution of Einstein's equations for some distribution of energy-momentum) cannot be done, because $\mathbf{I}_0$ is an inertial reference frame and $\mathbf{V}$ is a PIRF and these are concepts belonging to different theories. Chapter 6 gives also a short account of the Schwarzschild original solution to Einstein's equation and the notion of black holes, emphasizing that the global topology of a given solution to Einstein equations obtained, of course in an open set U of a not yet defined manifold M is most the time added by hand in the process of obtaining the maximal extension of that solution.

In Chap. 7 a thoughtful presentation of the theory of a Dirac-Hestenes spinor field (DHSF) on a general RCST $(M, \boldsymbol{g}, \nabla, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$ is given, together with a clarification

[21] See Definition 6.59.

[22] See Definition 6.61. γ is a timelike geodesic in the Lorentzian manifold representing spacetime.

[23] Here, this principle is a statement about indistinguishable of LLRFγ. It is not to be confused with the imposition of (active) local Lorentz invariance of Lagrangians and field equations discussed in Sect. 10.2.

[24] This issue is discussed in details in [6].

of its *ontology*.[25] A DHSF is a section of a particular spinor bundle (to be described below), but the important fact that we shall explore is that any DHSF has *representatives* on the even subbundle of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ of differential forms. Each representative is relative to a given spin coframe.

In order to present the details of our theory we scrutinize the vector bundle structure of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$, define spinor structures, spin frame bundles, spinor bundles, spin manifolds, and introduce Geroch's theorem. Particularly important for our purposes are the *left* ($\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) and *right* ($\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) spin-Clifford bundles on a *spin* manifold $(M, \boldsymbol{g}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$. We study in details how these bundles are related with $\mathcal{C}\ell(M, \mathtt{g})$. Left algebraic spinor fields and Dirac-Hestenes spinor fields (both fields are sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) are defined and the relation between them is established. Then, we show that to each DHSF $\boldsymbol{\Psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and to each *spin coframe* $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ there is a well defined sum of even multiform fields (EMFS) $\psi_\Xi \in \sec \mathcal{C}\ell(M, \mathtt{g})$ associated with $\boldsymbol{\Psi}$. Such an EMFS is called a *representative* of the DHSF on the given spin coframe. Of course, such an EMFS (the representative of the DHSF) is *not* a spinor field, but it plays a very important role in calculations. Indeed, with this crucial distinction between a DHSF and their EMFS representatives, we find useful formulas for calculating the derivatives of both Clifford fields and representatives of DHSF on $\mathcal{C}\ell(M, \mathtt{g})$ using the general theory of covariant derivatives of sections of a vector bundle, briefly recalled in Appendix A.5. This is done by introducing an *effective* spinorial connection[26] for the derivation of representatives of a DHSF on $\mathcal{C}\ell(M, \mathtt{g})$. We thus provide a consistent theory for the covariant derivatives of Clifford and spinor fields of all kinds. Besides that, we introduce the concepts of curvature and torsion extensors of a (spin) connection and clarify some misunderstandings appearing in the literature.

Next, we introduce the Dirac equation for a DHSF $\boldsymbol{\Psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ (denoted DE$\mathcal{C}\ell^l$) on a Lorentzian spacetime.[27] Then, we obtain a *representation* of the DE$\mathcal{C}\ell^l$ in the Clifford bundle. It is that equation that we call the Dirac-Hestenes equation (DHE) and which is satisfied by even Clifford fields $\psi_\Xi \in \sec \mathcal{C}\ell(M, \mathtt{g})$.

We study also the concepts of local Lorentz invariance and the electromagnetic gauge invariance. We show that for the DHE such transformations are of the same mathematical nature, thus suggesting a possible link between them. Chapter 7 also discusses the concept of amorphous spinor fields, which are ideal sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ and which have been some times confused with true spinor fields (see also Chap. 12).

Chapter 8 deals with the Lagrangian formalism of classical field theory in Minkowski spacetime. Recall that in Chaps. 4 and 7 we show how to represent

[25]For the genesis of these objects we quote [28].

[26]The same as that used in [28].

[27]The case of Dirac-Hestenes equation on a Riemann-Cartan manifold is discussed in Sect. 10.1.

Maxwell, Dirac and Einstein equations in several different formalisms. We hope to have convinced the reader that studied those chapters(and will study Chaps. 9 and 11) of the elegance and conciseness of the representation of the electromagnetic and gravitational fields as *appropriate* homogeneous sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ (Clifford fields).[28] Also, we hope to have convinced the reader that the Dirac-Hestenes spinor fields—which are sections of the left spin-Clifford bundle, $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$—can be represented, once we fix a spin coframe, as even sections of $\mathcal{C}\ell(M, \mathtt{g})$, i.e., a sum of non homogeneous differential forms.

Taking into account these observations Chap. 8 is dedicated to the formulation of the Lagrangian formalism for (interacting) Clifford fields and representatives of Dirac-Hestenes fields living on *Minkowski* spacetime using the theory of multiform functions and extensors developed in Chap. 2. We show that it is possible to exhibit trustful conservation laws of energy-momentum and angular momentum for such fields. Several exercises are proposed and many solved in details, in order to help the reader to achieve a complete domain on the mathematical methods employed. A thoughtful discussion is given of non-symmetric energy-momentum extensors, since in any case the non-symmetric part is responsible for the spin of the field (see Sect. 8.7). The cases of the electromagnetic and DHSF are studied in details.

Chapter 9 is dedicated to the study of conservation laws on Riemann-Cartan and Lorentzian spacetimes. We already observed that the nature of the gravitational field and the nature of other fields (e.g., the electromagnetic and Dirac fields) in GRT are very distinct. The former, according to the orthodox interpretation, is to be identified with some aspects of the geometry of the world manifold (spacetime) while the latter are physical fields, in the sense of Faraday living in a background spacetime. This crucial distinction implies that there are no genuine conservation laws of energy, momentum and angular momentum in GRT. A proof of this statement is one of the main objectives of Chap. 9. To motivate the reader for the importance of the issue we quote page 98 of Sachs and Wu [31]:

> As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage.

The problem of the conservations laws in GRT is a particular case of the problem of the conservation laws of energy-momentum and angular momentum for fields living in a general Riemann-Cartan spacetime $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$. This latter problem is also relevant in view of the fact that recently a geometric alternative formulation of the theory of gravitational field (called the teleparallel equivalent of GRT [21]) is being presented (see, e.g., [2, 5]) as one that solves the issue of the conservation laws. This statement must be qualified and it is discussed in Sect. 11.6, after we prove in Chap. 9 that for any field theory describing a set of interacting fields living in a *background* Riemann-Cartan spacetime there are conservation laws involving

[28]For a description of the gravitational field by a set of 1-forms $\mathfrak{g}^{\mathbf{a}} \in \sec \bigwedge^{1} T^{*}M$, $\mathbf{a} = 0, 1, 2, 3$ see Chap. 11.

only the energy-momentum and angular momentum tensors of the *matter* fields, *if and only if*, the Riemann-Cartan spacetime time has special global vector fields that besides being Killing vectors satisfy also *additional* constraints, a fact that unfortunately is not well known as it should be. In the teleparallel equivalent of GRT (with null or non null cosmological constant) it is possible to find a true tensor corresponding to the energy of the gravitational field and thus to have an energy-momentum conservation law for the coupled system made of the matter and gravitational fields. However, this result as we shall see is a triviality once we know how to formulate a gravitational field in Minkowski spacetime, an issue also discussed in Chap. 11. In Chap. 9 we treat in details only the case where each one of the fields ϕ^A, $A = 1, 2, \ldots n$, is a homogeneous Clifford field $\phi^A \in \sec \bigwedge^r TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Moreover, we restrict ourselves to the case where the Lagrangian density is of the form, $\mathcal{L}_\wedge(\phi) \equiv \mathcal{L}\circ\mathbf{j}_\wedge(\phi) = \mathcal{L}_\wedge(x, \phi, d\phi)$. This case is enough for our purposes, which refers to matters of principles. However, the results are general and can be easily extended for nonhomogeneous Clifford fields, and thus includes the case of the representatives in the Clifford bundle of DHSF on a general Riemann-Cartan spacetime. We remind also that Chap. 9 ends with a series of non trivial exercises (with detailed solutions) including one that gives the detailed derivation of Einstein's equation from a Lagrangian density written directly for the cotetrad fields $\theta^{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$.

Chapter 10 gives a presentation of the theory of DHSFs on a general Riemann-Cartan spacetime. Such a theory reveals a hidden problem, the one of knowing the exact meaning of *active local Lorentz invariance* of the DHSF Lagrangian and of the DHE. We show that a rigorous mathematical meaning to that concept can be implemented with the concept of generalized gauge connections introduced in Appendix A.5.2 and surprisingly implies in a "gauge equivalence" between spacetimes with different connections which have different torsion and curvature tensors.

Chapter 11[29] in this second edition has been completely rewritten and is now titled *On the Nature of the Gravitational Field.* It first presents a theory of the gravitational field where this field is described by global gravitational potentials $\{\mathfrak{g}^{\mathbf{a}}\}$, $\mathfrak{g}^{\mathbf{a}} \in \sec \bigwedge^1 TM$, $\mathbf{a} = 0, 1, 2, 3$, living on a *parallelizable* manifold M. Using the $\mathfrak{g}^{\mathbf{a}}$ to introduce a metric like field on M, namely $\boldsymbol{g} = \boldsymbol{\eta}_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ makes the Lorentzian structure $(M, \boldsymbol{g})$ a spin manifold. With the field $\boldsymbol{g}$ we introduce a Hodge star operator $\underset{\boldsymbol{g}}{\star}$ which is then used in the writing of a postulated Lagrangian density for the gravitational potentials $\mathfrak{g}^{\mathbf{a}}$ in interaction with matter fields. Such a Lagrangian does not involves *any* connection defined in M, which thus permit us to present convincing arguments that the geometrical interpretation of the gravitational field modelled by a Lorentzian spacetime structure (LSTS) $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ according to GRT, is no more than a possible choice.

[29]Chapter 10 in the first edition.

This geometrical interpretation becomes almost obvious once we prove that the field equations for $\mathfrak{g}^{\mathbf{a}}$ are equivalent to Einstein equations describing the gravitational field by the LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$. However , our approach makes clear that there are other geometrical structures such as, e.g., a teleparallel spacetime structure $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ (and yet other geometrical structures) that can equally well represent the gravitational field.

Recalling that we learned in Chap. 9 that in GRT there are no genuine energy-momentum and angular momentum for the gravitational field and no genuine conservations laws for the matter and the gravitational fields,[30] we mention yet that:

(a) Einstein's gravitational equation has as second member the energy-momentum tensor of matter fields, and this tensor is symmetric. However, we learned in Chap. 8 that the canonical energy-momentum tensors of the electromagnetic field and of the Dirac field are not symmetric, this fact being associated with the *existence* of spin. If this fact is taken into account (even if we leave out quantum theory from our considerations) it immediately becomes clear that Einstein's gravitational theory, must be an approximation to a more complete theory. Science does not yet know, which is the correct theory of the gravitational field, and indeed many alternatives have been and continues to be investigated. We are particularly sympathetic with the view that gravitation is a low energy manifestation of the quantum vacuum, as described, e.g., in Volovik[31] [33]. This sympathy comes from the fact that, as we shall see, it is possible to formulate theories of the gravitational field in Minkowski spacetime in such a way that a gravitational field results as a kind of plastic distortion of the *physical quantum vacuum*. However we do not discuss such a theory here, the interested reader may consult[32] [15].

(b) The crucial distinction between the gravitational field and the other physical fields, mentioned above, has made it impossible so far to formulate a well-defined and satisfactory quantum theory for the gravitational field, despite the efforts of a legion of physicists and mathematicians. Eventually, as a first step in arriving at such a quantum theory we should promote the gravitational field described by the potentials $\mathfrak{g}^{\mathbf{a}}$ to the same status of all other physical fields, i.e., a physical field living in Minkowski spacetime which satisfies field equations such that genuine conservation laws of energy-momentum and

[30]There are, of course, other serious problems with the formulation of a quantum theory of Einstein's gravitational field, that we are not going to discuss in this book. The interested reader should consult on this issue, e.g., [19, 20].

[31]Honestly, we think that gravitation is an emergent macroscopic phenomenon which need not to be quantized and which will eventually find its correct description in a theory about the real structure of the physical vacuum as suggested, e.g., in [33]. However, we are not going to discuss such a possibility in this book.

[32]On this issue, see also the book by Kleinert [17], which however describes plastic distortions by means of multivalued functions.

angular momentum hold. Theories of this type are indeed possible and it seems, at least to the authors, that they are more satisfactory than GRT, and indeed many presentations have been devised in a form or another, by several eminent physicists in the past. In Chap. 11 our version of such a possible theory is given. In it is possible to formulate a genuine energy-momentum tensor for the gravitational field. The formula [Eq. (11.9)] derived by standard methods directly from the postulated Lagrangian density is a very complicated one involving many terms. However, using some results of the Clifford bundle formalism and taking into account that we can use D, the Levi-Civita connection of the metric like field $\boldsymbol{g}$ as no more than a mathematical device to simplify eventual calculations, we show that it is possible to represent the energy momentum 1-form fields for the gravitational field by a very nice and short formula [29] Eq. (11.35) worth to be registered.

We mention again that in our theory the field $\boldsymbol{g}$ obeys Einstein's gravitational equation in an *effective* Lorentzian spacetime. However, $\boldsymbol{g}$ is considered a physical field (in the sense of Faraday) living in Minkowski spacetime (the true arena where physical phenomena takes place), thus being an object of analogous nature as the electromagnetic field and the other physical fields we are aware of. The geometrical interpretation, i.e., the orthodox view that the gravitational field is the geometry described by a LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ is a simple coincidence, as emphasized by Weinberg [34], valid as an approximation. Such view is a useful one, because the motion of *probe* particles and photons can be described with a very good approximation by geodetic motions in the effective Lorentzian spacetime generated by a 'big' source of energy-momentum. However, keep in mind that as mentioned above other possible representations of the gravitational field are also possible. Chapter 11 has also a section on Einstein's most happy thought, i.e., the equivalence principle (EP). We remark (see also the discussions in Chap. 6) that the interpretations of the equivalence principle in GRT (as originally suggested by Einstein) is subject to criticisms. It has been recently suggested that the original Einstein suggestion for the meaning of the EP seems more reasonable described in the teleparallel interpretation of GRT. However, even that interpretation, as will be shown, is subject to criticism. We mention also that Chap. 11 discuss also the Hamiltonian formalism for our theory and how it relates to the ADM energy concept in GRT. Finally, despite our view that gravitation is well described by a field theory in Minkowski spacetime we present (using results proved in Chap. 5) in Sects. 11.7 and 11.8 a mathematical formulation of Clifford's idea of matter as curvature in a brane world, in particular showing how the "wood" part of Einstein equation (i.e., the second member with the phenomenological energy-momentum tensor) can be transformed in "marble", i.e., can be given a geometrical formulation in terms of the square of the shape tensor.

In Chap. 12, *On the Many Faces of Einstein's Equations*, we show how the orthodox Einstein theory can be formulated in a way that resembles the gauge theories of particle physics, in particular a gauge theory with gauge group $Sl(2, \mathbb{C})$. This exercise will reveal yet another face of Einstein's equations, besides the ones already discussed in previous chapters. For our presentation we introduce new mathematical objects, namely, the Clifford valued differential forms (cliforms) and a new operator, $\mathcal{D}$, which we called the *fake exterior covariant differential* (FECD) and an associated operator $\mathcal{D}_{\mathbf{e_r}}$ acting on them. Moreover, with our formalism we show that Einstein's equations can be put in a form that *apparently* resembles Maxwell equations. Chapter 12 also clarifies some misunderstandings appearing in the literature concerning that Maxwell-like form of Einstein's equations.

In Chap. 13, called *Maxwell, Dirac and Seiberg-Witten Equations*, we first discuss how $\mathrm{i} = \sqrt{-1}$ enters Dirac theory since complex numbers do not appear in the equivalent Dirac-Hestenes description of fermionic fields. Next we discuss how $\mathrm{i} = \sqrt{-1}$ enters classical Maxwell theory and give (using Clifford bundle methods) a detailed presentation of the theory of polarization and Stokes parameters. Our approach leaves the reader equipped to appreciate the *nonsense* that sometimes even serious publishing houses leave to appear, as e.g., [8–14]. We also present several Dirac-like representations of Maxwell equations. These Dirac-like forms of Maxwell equations (which are trivial within the Clifford bundle formalism) really use amorphous spinor fields and do not seem to have any real importance until now. A three dimensional Majorana-like representation of Maxwell equations is also easily derived and it looks like Schrödinger equation. After that, we exhibit *mathematical* equivalences of the first and second kinds between Maxwell and Dirac equations. We think that the results presented are really nice and worth to be more studied. The chapter ends showing how the Maxwell-Dirac equivalence of the first kind plus a reasonable *ansatz* can provide an interpretation for the Seiberg-Witten equations in Minkowski spacetime.

In Chap. 14 we explore the potential of the mathematical methods developed previously. We show that we can describe the motion of *classical* charged spinning particles (CCSP) when free or in interaction with the electromagnetic field using DHSF. We show that in the free case there is a unitary DHSF describing the motion of the CCSP which satisfies a linear Dirac-Hestenes equation. When the CCSP interacts with an electromagnetic field, a non linear equation that we called the classical Dirac-Hestenes equation is satisfied by a DHSF describing the motion of the particle. We study the meaning of the nonlinear terms and suggests a possible conjecture: that this nonlinear term may compensate the term due to radiation reaction, thus providing the linear Dirac-Hestenes equation introduced in Chap. 7. Moreover, we show that our approach suggests by itself an interpretation for the Dirac-Hestenes wave function, namely, as a probability distribution. This may be important as it concerns the interpretation of quantum theory, but that issue will not be discussed in this book. After that we introduce (using the multiform calculus developed in Chap. 2 and some generalizations of the Lagrangian formalism developed in Chap. 8) the dynamics of the *superparticle*. This consists in showing that it is possible to give a Lagrangian formulation to the Frenet equations describing a CCSP and that the resulting equations are in one to one correspondence with the famous Berezin-Marinov equations for the superparticle. We recall moreover how the Clifford-algebraic methods previously developed suggest a very simple interpretation for Berezin calculus and provides alternative geometric intelligibility for superfields, a concept appearing in modern physics theories such as supergravity and string theory. We show moreover that superfields may be correctly interpreted as non homogeneous sections of a particular Clifford bundle over spacetime, and we observe that we already had contact with objects of this kind in previous chapters, namely the representatives of DHSF introduced in Chap. 7 and the generalized potential (Sect. 13.4.5) appearing in the theory of the Hertz potential. Our main intention in presenting these issues is to induce some readers to think about the ontology of abstract objects being used in advanced theories in alternative ways.

We exhibit in Chaps. 4–14 several different, mathematical faces of Maxwell, Einstein and Dirac equations. In Chap. 15 titled—"*Maxwell, Einstein, Dirac and Navier-Stokes Equations*"—we show that given certain conditions we can encode the contents of Einstein equations in Maxwell like equations for a field $\mathring{F} = d\mathring{A} \in \sec \bigwedge^2 T^*M$ (or $F = dA \in \sec \bigwedge^2 T^*M$),[33] whose contents can be also encoded in a Navier-Stokes equation. For the particular cases when it happens that $F^2 \neq 0$ we can also using the Maxwell-Dirac equivalence of the first kind discussed in Chap. 13 to encode the contents of the previous quoted equations in a Dirac-Hestenes like equation for $\psi \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M)$ such that $F = \psi\gamma^{21}\tilde{\psi}$.

Specifically, Sect. 15.1 shows how each LSTS $(M = \mathbb{R}^4, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ which, which as we already mentioned, is a model of a gravitational field generated by $\mathbf{T} \in \sec T_2^0 M$ (the matter plus non gravitational fields energy-momentum tensor) in Einstein GRT is such that for any $\boldsymbol{K} \in \sec TM$—which is a vector field generating a one parameter group of diffeomorphisms of M—we can encode Einstein equations in Maxwell like equations satisfied by $F = dK$ where $K = \boldsymbol{g}(\boldsymbol{K},)$ with a well determined current term named the *Komar current* $J_{\boldsymbol{K}} = -\underset{\boldsymbol{g}}{\delta}\boldsymbol{K}$, whose explicit form is given.

Next we show in Sect. 15.2 that when $\boldsymbol{K} = \boldsymbol{A}$ is a Killing vector field, due to some noticeable results [Eqs. (15.28) and (15.29)] the Komar current acquires a very simple form and is then denoted $J_{\boldsymbol{A}}$. Then, interpreting, as in Chap. 11 the Lorentzian spacetime structure $(M = \mathbb{R}^4, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ as no more than an useful representation for the gravitational field represented by the gravitational potentials $\{\mathfrak{g}^a\}$ which lives in Minkowski spacetime we show in Sect. 15.3 that we can find a Navier-Stokes equation which encodes the contents of the Maxwell like equations (encoding Einstein equations) once a proper identification is made between the variables entering the Navier-Stokes equations and the ones defining $\mathring{A}$ and $\mathring{F}$, objects clearly related to A and $F = dA$. We also explicitly determine also the constraints imposed by the nonhomogeneous Maxwell like equation $\underset{\boldsymbol{g}}{\delta}F = -J_{\boldsymbol{A}}$ on the variables entering the Navier-Stokes equations and the ones defining A (or $\mathring{A}$).

[33] $\mathring{A} = \mathring{\boldsymbol{g}}(\boldsymbol{A},)$ and $A = \boldsymbol{g}(\boldsymbol{A},)$ with $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ the metrics of Minkowski spacetime denoted in Chap. 15 by $(M = \mathbb{R}^4, \mathring{\boldsymbol{g}}, \mathring{D}, \tau_{\mathring{\boldsymbol{g}}}, \uparrow)$ and of the structure $(M = \mathbb{R}^4, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ describing an effective Lorentzian spacetime.

Finally we comment on relations between Einstein and the Navier-Stokes equations living in spacetimes of different dimensions found by other authors.

Chapter 16 analyzes the similarities and main differences between Dirac, Majorana and elko spinor fields and the equations satisfied by these fields, a subject that is receiving a lot of attention in the last few years. We present an alternative theory for elko spinor fields of mass dimension $3/2$ (instead of mass dimension 1 as originally proposed in [1]) and show that our elko spinor fields can be used to describe electric neutral particles carrying "magnetic like" charges with short range interaction mediated by a su(2)-valued gauge potential $\mathcal{A} = A^{\mathbf{i}} \otimes \boldsymbol{\tau}_{\mathbf{i}} \in \sec \bigwedge^1 T^*M \otimes \mathrm{spin}_{3,0} \hookrightarrow \sec \mathcal{C}\ell(M, \eta) \otimes \mathbb{R}_{1,3}$. Some crucial criticisms to the mass dimension 1 elko spinor field theory are also given, since that theory breaks Lorentz and rotational symmetries in a very odd way as shown in Sect. 16.7.

The book contains an appendix where we review some of the main definitions and concepts of the theory of principal bundles and their associated vector bundles, including the theory of connections in principal and vector bundles, exterior covariant derivatives, etc., which are needed in order to introduce the Clifford and spin-Clifford bundles and to discuss some other issues in the main text. Jet bundles are also defined. We believe that the material presented in the appendix is enough to guide our reader permitting him to follow the most difficult passages of the text, and in particular to see the reason for our use of many eventually sloppy notations.

Having resumed the contents of our book, the following observations are necessary. First, it is not a Mathematics book, despite the format of the presentation in some sections, a format that has been used simply because it is in our opinion the most efficient one, for quotations. Even though it is not a Mathematics book, several mathematical theories (some sophisticated, indeed) have been introduced and we hope that they do not scare a potential reader. We are sure that any reader (be him a student, a physicist or even a mathematician) who will spend the appropriate time studying our book will really benefit from its reading, since he (or she) will start to see under a different point of view some of the foundational issues associated with the theories discussed. Hopefully this will give to some readers new insights on several subjects, a necessary condition to advance knowledge.

Second, we recall that we mentioned in the introduction of the first edition that it was our intention to discuss the question of the arbitrary velocities solutions of the relativistic wave equations and that due to the size attained by that first edition with its thirteen chapters plus an Appendix, it has been decided to discuss that subject in a sequel volume entitled *Subluminal, Luminal and Superluminal Wave Motion.* Well, unfortunately the planned new book is not ready yet as the beginning of 2014, being still being written.

We tried to quote all papers and books that we have studied and that influenced our work, and we offer our apologies to any author not cited who feels that some of his writings should be quoted.

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Chapter 2
Multivector and Extensor Calculus

Abstract This chapter is dedicated to a thoughtful exposition of the multiform and extensor calculus. Starting from the tensor algebra of a real n-dimensional vector space $\boldsymbol{V}$ we construct the exterior algebra $\bigwedge \boldsymbol{V}$ of $\boldsymbol{V}$. Equipping $\boldsymbol{V}$ with a metric tensor $\overset{\circ}{g}$ we introduce the Grassmann algebra and next the Clifford algebra $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ associated to the pair $(\boldsymbol{V}, \overset{\circ}{g})$. The concept of Hodge dual of elements of $\bigwedge \boldsymbol{V}$ (called nonhomogeneous multiforms) and of $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ (also called nonhomogeneous multiforms or Clifford numbers) is introduced, and the scalar product and operations of left and right contractions in these structures are defined. Several important formulas and "tricks of the trade" are presented. Next we introduce the concept of *extensors* which are multilinear maps from p subspaces of $\bigwedge \boldsymbol{V}$ to q subspaces of $\bigwedge \boldsymbol{V}$ and study their properties. Equipped with such concept we study some properties of symmetric automorphisms and the orthogonal Clifford algebras introducing the gauge metric extensor (an essential ingredient for theories presented in other chapters). Also, we define the concepts of strain, shear and dilation associated with endomorphisms. A preliminary exposition of the Minkoswski *vector space* is given and the Lorentz and Poincaré groups are introduced. In the remaining of the chapter we give an original presentation of the theory of *multiform functions* of multiform variables. For these objects we define the concepts of limit, continuity and differentiability. We study in details the concept of directional derivatives of multiform functions and solve several nontrivial exercises to clarify how to work with these notions, which in particular are crucial for the formulation of Chap. 8 which deals with a Clifford algebra Lagrangian formalism of field theory in Minkowski spacetime.

2.1 Tensor Algebra

We recall here some basic facts of tensor algebra. Let $\mathbf{V}$ be a vector space over the real field $\mathbb{R}$ of finite dimension, i.e., $\dim \mathbf{V} = n$, $n \in \mathbb{N}$. By $\boldsymbol{V} = \mathbf{V}^*$ we denote the dual space of $\mathbf{V}$. Recall that $\dim \mathbf{V} = \dim \boldsymbol{V}$. The elements of $\mathbf{V}$ are called vectors and the elements of $\boldsymbol{V}$ are called covectors or 1-forms.

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2.1.1 Cotensors

Definition 2.1 We call space of k-cotensors (denoted $T_k\boldsymbol{V}$) the set of all k-linear mappings τ_k such that

$$\tau_k : \underbrace{\mathbf{V} \times \mathbf{V} \ldots \times \mathbf{V}}_{k\text{-copies}} \to \mathbb{R}. \tag{2.1}$$

Remark 2.2 In what follows we identify $T_0\boldsymbol{V} \equiv \mathbb{R}$, and $T_1\boldsymbol{V} \equiv \boldsymbol{V}$.

2.1.2 Multicotensors

Definition 2.3 Consider the (exterior) direct sum $T\boldsymbol{V} := \sum_{k=0}^{\infty} \oplus T_k\boldsymbol{V} \equiv \bigoplus_{k=0}^{\infty} T_k\boldsymbol{V}$. A multicotensor τ of *order* $M_\tau \in \mathbb{N}$ is an element of $T\boldsymbol{V}$ of the form $\tau = \sum_{k=0}^{M_\tau} \oplus \tau_k$, $\tau_k \in T_k\boldsymbol{V}$, such that all the components $\tau_k \in T_k\boldsymbol{V}$ of τ are null for $k > M$. $T\boldsymbol{V}$ is called the space of multicotensors.

We can easily show that $T\boldsymbol{V}$ is a vector space over $\mathbb{R}$. We have that the *order* of $\tau + \sigma$, $\tau, \sigma \in T\boldsymbol{V}$ is the greatest of the orders of τ or σ, and of course, the order of $a\tau$, $a \in \mathbb{R}$, $\tau \in T\boldsymbol{V}$ is equal to the order of τ. The set $T_M\boldsymbol{V} = \sum_{k=0}^{M} \oplus T_k\boldsymbol{V}$ is clearly a subspace of $T\boldsymbol{V}$. Sometimes it is convenient to denote an element of $T_M\boldsymbol{V}$ by $\tau = (\tau_0, \tau_1, \ldots, \tau_k, \ldots, \tau_M)$.

Definition 2.4 The k-part operator is a mapping $\langle\rangle_k : T\boldsymbol{V} \to T_k\boldsymbol{V}$ such that for all $j \in \mathbb{N}, j \neq k$ we have that

$$\langle\langle\tau\rangle_k\rangle_j = 0, \tag{2.2}$$

where $\langle\langle\tau\rangle_k\rangle_j \in T_j\boldsymbol{V}$.

Then, if $\tau = (\tau_0, \tau_1, \ldots, \tau_k, \ldots, \tau_M) \in T_M\boldsymbol{V}$ we have that

$$\langle\tau\rangle_k = (0, \ldots, \tau_k, \ldots, 0).$$

Definition 2.5 A multicotensor $\tau \in T\boldsymbol{V}$ is said to be homogeneous of grade k if and only if $\tau = \langle\tau\rangle_k$.

Of course, the set of multicotensors of grade k is a subspace of $T\boldsymbol{V}$ which is isomorphic to $T_k\boldsymbol{V}$ and, we can write any $\tau \in T_M\boldsymbol{V}$ as

$$\tau = \sum_{k=0}^{M} \langle\tau\rangle_k. \tag{2.3}$$

2.1.3 Tensor Product of Multicotensors

Definition 2.6 The tensor product of multicotensors is a mapping $\otimes : T\boldsymbol{V} \times T\boldsymbol{V} \to T\boldsymbol{V}$ such that:

(i) if $a, b \in T_0\boldsymbol{V} = \mathbb{R}$, $a \otimes b = ab$,
(ii) if $a \in \mathbb{R}$ and $\tau \in T_p\boldsymbol{V}$, $p \geq 1$, then $a \otimes \tau = \tau \otimes a = a\tau$,
(iii) if $\sigma \in T_j\boldsymbol{V}$ and $\tau \in T_k\boldsymbol{V}$, with $j, k \geq 1$ then $\sigma \otimes \tau \in T_{j+k}\boldsymbol{V}$ and is such that for $\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{j+k} \in \boldsymbol{V}$ we have

$$\sigma \otimes \tau(\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{j+k}) = \sigma(\mathbf{v}_1, \ldots, \mathbf{v}_j)\tau(\mathbf{v}_{j+1}, \ldots, \mathbf{v}_{j+k}), \tag{2.4}$$

(iv) the tensor product satisfies the distributive laws on the right and on the left and it is associative, i.e., for $a, b \in \mathbb{R}$, $\sigma \in T_j\boldsymbol{V}$, $\tau \in T_k\boldsymbol{V}$, $\phi \in T_l\boldsymbol{V}$

$$\begin{aligned}(\sigma + \tau) \otimes \phi &= \sigma \otimes \phi + \tau \otimes \phi,\\ \phi \otimes (\sigma + \tau) &= \phi \otimes \sigma + \phi \otimes \tau,\\ (\sigma \otimes \tau) \otimes \phi &= \sigma \otimes (\tau \otimes \phi).\end{aligned} \tag{2.5}$$

(v) If $\sigma, \tau \in T\boldsymbol{V}$ then

$$(\sigma \otimes \tau)_k = \sum\nolimits_{J=0}^{K} \sigma_j \otimes \tau_{k-j}, \tag{2.6}$$

where $\sigma_j \otimes \tau_{k-j}$ is the tensor product of the j component of σ by the $(k-j)$ component of τ.

Of course, from (iii) we may verify that the tensor product of cotensors is *not* commutative in general.

Remark 2.7 We recall that the tensor product of multitensors is defined in complete analogy to the previous one. The reader may easily fill in the details. We recall also that it is possible to extend the definition of tensor product by allowing tensor products of a r-tensor by a s-cotensor. We then denote, as usual, by

$$T_s^r\mathbf{V} = \mathbf{V} \otimes \cdots \otimes \mathbf{V} \otimes \boldsymbol{V} \otimes \cdots \otimes \boldsymbol{V} \equiv \boldsymbol{V} \otimes \cdots \otimes \boldsymbol{V} \otimes \mathbf{V} \otimes \cdots \otimes \mathbf{V} \tag{2.7}$$

the space of the r-contravariant and s-covariant tensors. $\mathbf{P} \in T_s^r\mathbf{V}$ is a $(r+s)$-multilinear map

$$\mathbf{P} : \underbrace{\mathbf{V} \times \mathbf{V} \times \ldots \times \mathbf{V}}_{s\text{-copies}} \times \underbrace{\boldsymbol{V} \times \boldsymbol{V} \times \ldots \times \boldsymbol{V}}_{r\text{-copies}} \to \mathbb{R}. \tag{2.8}$$

Remark 2.8 If $\mathbf{P} \in T_s^r\mathbf{V}$ and $\mathbf{S} \in T_q^p\mathbf{V}$ we define the tensor product of $\mathbf{P}$ by $\mathbf{S}$ as the multilinear mapping $\mathbf{R} \otimes \mathbf{S} \in T_{s+q}^{r+p}\mathbf{V}$. Note that in general, $\mathbf{P} \otimes \mathbf{S} \neq \mathbf{S} \otimes \mathbf{P}$. The (exterior) direct sum $\mathcal{T}\mathbf{V} = \sum_{r=0,s=0}^{M} \oplus T_s{}^r\mathbf{V}$ equipped with the tensor product is a real vector space over $\mathbb{R}$, called the general tensor algebra of $\mathbf{V}$. Note also that we can define $TV = \bigoplus_{k=0}^{\infty} T^k\mathbf{V}$, the space of multitensors in complete analogy to Definition 2.3. Note moreover that $T_s{}^0\mathbf{V} = \mathbf{T}_s V$.

2.1.4 Involutions

Definition 2.9 The main involution or *grade* involution is an automorphism $\wedge : TV \to TV$ such that:

(i) if $\alpha \in \mathbb{R}$, $\hat{\alpha} = \alpha$;
(ii) if $a_1 \otimes \cdots \otimes a_k \in T_kV$, $k \geq 1$, $(a_1 \otimes \cdots \otimes a_k)^\wedge = (-1)^k a_1 \otimes \cdots \otimes a_k$;
(iii) if $a, b \in \mathbb{R}$ and $\sigma, \tau \in TV$ then $(a\sigma + b\tau)^\wedge = a\hat{\sigma} + b\hat{\tau}$;
(iv) if $\tau = \sum \tau_k$, $\tau_k \in T_kV$ then

$$\hat{\tau} = \sum_{k=0}^{n} \hat{\tau}_k, \tag{2.9}$$

Definition 2.10 The reversion operator is the anti-automorphism $\sim : TV \ni \tau \mapsto \tilde{\tau} \in TV$ such that if $\tau = \sum_k \tau_k$, $\tau_k \in T_kV$ then

(i) if $\alpha \in \mathbb{R}$, $\tilde{\alpha} = \alpha$;
(ii) if $a_1 \otimes \cdots \otimes a_k \in T_kV$, $k \geq 1$, $(a_1 \otimes \cdots \otimes a_k)^\sim = a_k \otimes \cdots \otimes a_1$;
(iii) if $a, b \in \mathbb{R}$ and $\sigma, \tau \in TV$ then $(a\sigma + b\tau)^\sim = a\tilde{\sigma} + b\tilde{\tau}$;
(iv) if $\tau = \sum_k \tau_k$, $\tau_k \in T_kV$ then

$$\tilde{\tau} = \sum_{k=0}^{n} \tilde{\tau}_k, \tag{2.10}$$

where $\tilde{\tau}$ is called the reverse of τ.

Definition 2.11 The composition of the grade evolution with the reversion operator, denoted by the symbol $-$ is called by some authors the *conjugation* and, $\bar{\tau}$ is said to be the *conjugate* of τ. We have $\bar{\tau} = (\tilde{\tau})^\wedge = (\hat{\tau})^\sim$.

2.2 Scalar Products in V and *V*

Let $\mathbf{V}$ be real vector space, $\dim \mathbf{V} = n$.

Definition 2.12 A metric tensor is a 2-cotensor $\mathbf{g} \in T_2V$ which is symmetric and non degenerated. A basis $\{\mathbf{e}_k\}$ of $\mathbf{V}$ is said to be orthonormal if $\mathbf{g}(\mathbf{e}_k, \mathbf{e}_k)$ is equal to $+1$ or equal to -1 and $\mathbf{g}(\mathbf{e}_k, \mathbf{e}_j) = 0$ for $j \neq k$. We have

$$\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{g}(\mathbf{e}_j, \mathbf{e}_i) = g_{ij} = \begin{cases} 1 \text{ if } & i, j = 1, 2, \ldots p \\ -1 \text{ if } & i, j = p+1, p+2, \ldots, p+q \\ 0 \text{ if } & i \neq j \end{cases}, \tag{2.11}$$

with $p+q = n$. The signature of $\mathbf{g}$ is defined by the difference $(p-q)$ and it is usual to say that the metric has signature (p, q). We denote $\mathbf{g}(\mathbf{v}, \mathbf{w}) \equiv \mathbf{v} \cdot \mathbf{w}$ and called the dot $\cdot$ a scalar product in $\mathbf{V}$.

Definition 2.13 A metric $\mathbf{g}$ induces a fundamental isomorphism between $\mathbf{V}$ and V given by $\mathbf{V} \ni \mathbf{v} \mapsto \sharp\mathbf{v} = \mathbf{g}(\mathbf{v}, \,) \in V$ such that given any $\mathbf{w} \in \mathbf{V}$, we have

$$\sharp\mathbf{v}(\mathbf{w}) = \mathbf{g}(\mathbf{v}, \mathbf{w}). \tag{2.12}$$

Definition 2.14 A general metric in V is a 2-tensor $g \in T^2\mathbf{V}$, i.e., a mapping $g : V \times V \to \mathbb{R}$ which is symmetric and non degenerated.

Let $\{\varepsilon^k\}$ be the basis of V dual to a basis $\{\mathbf{e}_k\}$ of $\mathbf{V}$, i.e., $\varepsilon^k(\mathbf{e}_j) = \delta^k_j$. We are particularly interested in a metric $g \in T^2\mathbf{V}$ such that if $\{\varepsilon^k\}$ is the basis of V dual to an arbitrary basis $\{\mathbf{e}_k\}$ of $\mathbf{V}$ then

$$g(\varepsilon^i, \varepsilon^j) = g(\varepsilon^j, \varepsilon^i) = g^{ij}, \tag{2.13}$$

and $g^{ij} g_{ik} = \delta^i_k$, i.e., the matrix with elements g^{ij} is the inverse of the matrix with elements g_{ij} in Eq. (2.11). Recall also that the inverse of the isomorphism $\sharp$ is the mapping $\sharp^{-1} : V \ni \alpha \mapsto \sharp^{-1}\alpha = g(\alpha, \,) \in \mathbf{V}$.

Definition 2.15 The scalar product of $\alpha, \beta \in V$ equipped with the metric g given by Eq. (2.13) is denoted (to emphasize the relation between the components of g and $\mathbf{g}$)

$$\alpha \underset{\mathbf{g}}{\cdot} \beta = g(\alpha, \beta) = \mathbf{g}(\sharp^{-1}\alpha, \sharp^{-1}\beta). \tag{2.14}$$

Remark 2.16 Take notice that $\mathbf{v} \cdot \mathbf{w} = \mathbf{g}(\mathbf{v}, \mathbf{w}) = g(\sharp\mathbf{v}, \sharp\mathbf{w})$ and that $\alpha \underset{\mathbf{g}}{\cdot} \beta$ will be denoted simply by $\alpha \cdot \beta$ when the context is clear.

Definition 2.17 Let $\{\varepsilon^k\}$ be the basis of V dual to the basis $\{\mathbf{e}_k\}$ of $\mathbf{V}$. A basis $\{\mathbf{e}^k\}$ of $\mathbf{V}$ is said to be the reciprocal basis of $\{\mathbf{e}_k\}$ if and only if $\mathbf{e}^k = \sharp^{-1}\varepsilon^k$, for all $k = 1, 2, .., n$. Also, a basis $\{\varepsilon_k\}$ of V is called the reciprocal basis of $\{\varepsilon^k\}$ if and only if $\varepsilon_k = \sharp\mathbf{e}_k$.

The reader may verify that $\mathbf{e}^k \cdot \mathbf{e}_j = \delta^k_j$ and $\varepsilon^k \cdot \varepsilon_j = \delta^k_j$.

Exercise 2.18 Let $\{\mathbf{e}_k\}$ and $\{\varepsilon^k\}$ be bases of $\mathbf{V}$ and $\boldsymbol{V}$, such that $\varepsilon^k(\mathbf{e}_j) = \delta^k_j$. Show that the set $\{(1, \boldsymbol{\varepsilon}^1, \ldots, \varepsilon^k, \ldots, \varepsilon^{p_1} \otimes \cdots \otimes \varepsilon^{p_k}, \ldots., \varepsilon^{p_1} \otimes \cdots \otimes \varepsilon^{p_M})\}_{1 \leq k \leq M}$ is a basis of $T_M\boldsymbol{V}$ and $\dim T_M\boldsymbol{V} = \frac{n^{M+1}-1}{n-1}$.

2.3 Exterior and Grassmann Algebras

Definition 2.19 The *exterior algebra*of $\boldsymbol{V}$ is the quotient algebra

$$\bigwedge \boldsymbol{V} = \frac{T(\boldsymbol{V})}{J}, \tag{2.15}$$

where $J \subset T\boldsymbol{V}$ is the bilateral ideal[1] in $T\boldsymbol{V}$ generated by the elements of the form $u \otimes v + v \otimes u$, with $u, v \in \boldsymbol{V}$. The elements of $\bigwedge \boldsymbol{V}$ will be called *multiforms*[2]

Let $\rho : T\boldsymbol{V} \to \bigwedge \boldsymbol{V}$ be the canonical projection of $T\boldsymbol{V}$ onto $\bigwedge \boldsymbol{V}$. Multiplication in $\bigwedge \boldsymbol{V}$ will be denoted as usually by $\wedge : \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V}$ and called *exterior product*. We have

Definition 2.20 For every $A, B \in \bigwedge \boldsymbol{V}$,

$$A \wedge B = \rho(A \otimes B), \tag{2.16}$$

where $\otimes : T\boldsymbol{V} \to T\boldsymbol{V}$ is the usual tensor product.

Remark 2.21 Note that if $u, v \in \boldsymbol{V}$ we can write

$$u \otimes v = \frac{1}{2}(u \otimes v - v \otimes u) + \frac{1}{2}(u \otimes v + v \otimes u), \tag{2.17}$$

and then,

$$\rho(u \otimes v) = u \wedge v = \frac{1}{2}(u \otimes v - v \otimes u). \tag{2.18}$$

We can easily show that $\bigwedge \boldsymbol{V}$ is a 2^n-dimensional associative algebra with unity. In addition, it is a $\mathbb{Z}$-grade algebra, i.e.,

$$\bigwedge \boldsymbol{V} = \bigoplus_{r=0}^{n} \bigwedge^r \boldsymbol{V},$$

[1]Given an associative algebra $\mathfrak{A}$, a bilateral ideal I is a subalgebra of $\mathfrak{A}$ such that for any $a, b \in \mathfrak{A}$ and for $y \in I$, $ay \in I$, $yb \in I$ and $ayb \in I$. More on ideals on Chap. 3.

[2]If we do the analogous construction of the exterior algebra using $\mathbf{V}$ instead of $\boldsymbol{V} \equiv \mathbf{V}^*$, then the elements of the resulting space are called *multivectors*.

and

$$\bigwedge^r \boldsymbol{V} \wedge \bigwedge^s \boldsymbol{V} \subset \bigwedge^{r+s} \boldsymbol{V},$$

$r, s \geq 0$, where $\bigwedge^r \boldsymbol{V} = \rho(T_r \boldsymbol{V})$ is the $\binom{n}{r}$-dimensional subspace of the *r-forms*on $\boldsymbol{V}$. ($\bigwedge^0 \boldsymbol{V} = \mathbb{R}$; $\bigwedge^1 \boldsymbol{V} = \boldsymbol{V}$; $\bigwedge^r \boldsymbol{V} = \{0\}$ if $r > n$). If $A \in \bigwedge^r \boldsymbol{V}$ for some fixed r ($r = 0, \ldots, n$), then A is said to be *homogeneous*. For any such multivectors we have:

$$A \wedge B = (-1)^{rs} B \wedge A, \tag{2.19}$$

$A \in \bigwedge^r \boldsymbol{V}, B \in \bigwedge^s \boldsymbol{V}$.

The exterior algebra (as can easily be verified) inherits the associativity of the tensor algebra, a very important property. It satisfies also, of course, the distributive laws (on the left and on the right), i.e.,

$$\begin{aligned} (A + B) \wedge C &= A \wedge C + B \wedge C, \\ A \wedge (B + C) &= A \wedge B + A \wedge C. \end{aligned} \tag{2.20}$$

Definition 2.22 The *antisymmetrization operator* A is the linear mapping $\mathsf{A} : T_k \boldsymbol{V} \to \bigwedge^k \boldsymbol{V}$ such that (i) for all $\alpha \in \mathbb{R} : \mathsf{A}\alpha = \alpha$, (ii) for all $v \in \boldsymbol{V} : \mathsf{A}v = v$, (iii) for all $X_1 \otimes X_2 \otimes \cdots \otimes X_k \in T_k \boldsymbol{V}$, with $k \geq 2$,

$$\mathsf{A}(X_1 \otimes X_2 \otimes \cdots \otimes X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(k)}, \tag{2.21}$$

where $\sigma : \{1, 2, \ldots k\} \to \{\sigma(1), \sigma(2), \ldots \sigma(k)\}$ is a permutation of k elements $\{1, 2, \ldots k\}$. Of course, the composition of permutations is a permutation and the set of all permutations is S_k, the symmetric group.

Exercise 2.23

(a) For $\tau \in T_k \boldsymbol{V}$ a general k-cotensor and $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \boldsymbol{V}$ show that

$$\mathsf{A}\tau(\mathbf{v}^1, \ldots, \mathbf{v}^k) = \frac{1}{k!} \epsilon_{i_1 \cdots i_k} \tau(\mathbf{v}^{i_1}, \ldots, \mathbf{v}^{i_k}), \tag{2.22}$$

where $\epsilon_{i_1 \ldots i_k}$ is the permutation symbol of order k,

$$\epsilon_{i_1 \cdots i_k} = \begin{cases} 1, & \text{if } i_1 \cdots i_k \text{ is an even permutation of } 1 \ldots k \\ -1, & \text{if } i_1 \cdots i_k \text{ is an odd permutation of } 1 \ldots k \\ 0, & \text{otherwise} \end{cases} \tag{2.23}$$

(b) Show that if $A_p \in \bigwedge^p \boldsymbol{V}$ and $B_q \in \bigwedge^q \boldsymbol{V}$ then

$$A_p \wedge B_q = \mathsf{A}(A_p \otimes B_q). \tag{2.24}$$

Remark 2.24 Some authors define the exterior product $A_p \dot{\wedge} B_q$ by

$$A_p \dot{\wedge} B_q = \frac{(p+q)!}{p!q!}\mathsf{A}(A_p \otimes B_q). \tag{2.25}$$

This definition is more used by differential geometers, whereas the definition given by Eq. (2.24) is more used by algebraists. Additional material on this issue may be found in [5]. See also Exercise *2.31*

2.3.1 Scalar Product in $\bigwedge V$ and Hodge Star Operator

Now let us suppose that $\mathbf{V}$ and $\boldsymbol{V}$ are metric vector spaces that is, they are endowed with nondegenerate metric tensors which we denote conveniently in what follows by $\mathring{\mathbf{g}} \in T_2\mathbf{V}$ and $\mathring{g} \in T^2\boldsymbol{V}$ of signature (p,q) and such that $\mathring{g}_{ij}\mathring{g}^{jk} = \delta_i^k$, with $\mathring{g}_{ij} = \mathring{\mathbf{g}}(\mathbf{e}_i, \mathbf{e}_j)$ and $\mathring{g}^{jk} = \mathring{g}(\varepsilon^j, \varepsilon^j)$, where $\{\mathbf{e}_i\}$ is a basis for $\mathbf{V}$ and $\{\varepsilon^j\}$ a basis for $\boldsymbol{V}$ with $\varepsilon^j(\mathbf{e}_i) = \delta_i^j$. We can use those metric tensors to induce scalar products on $\bigwedge \mathbf{V}$ and $\bigwedge \boldsymbol{V}$. We give the construction for $\bigwedge \boldsymbol{V}$.

Definition 2.25 The (fiducial) scalar product in $\bigwedge \boldsymbol{V}$ is the linear mapping $\underset{\mathring{\mathbf{g}}}{\cdot} : \bigwedge \boldsymbol{V} \times \bigwedge \boldsymbol{V} \to \mathbb{R}$ given by

$$A \underset{\mathring{\mathbf{g}}}{\cdot} B = \det(\mathring{g}(u_i, v_j)), \tag{2.26}$$

for homogeneous multivectors $A = u_1 \wedge \cdots \wedge u_r \in \bigwedge^r \boldsymbol{V}$ and $B = v_1 \wedge \cdots \wedge v_r \in \bigwedge^r \boldsymbol{V}$, $u_i, v_i \in \boldsymbol{V}$, $i = 1, \ldots, r$. This scalar product is extended to all of $\bigwedge \boldsymbol{V}$ due to linearity and orthogonality and $A \underset{\mathring{\mathbf{g}}}{\cdot} B = 0$ if $A \in \bigwedge^r \boldsymbol{V}$, $B \in \bigwedge^s \boldsymbol{V}$, $r \neq s$. We shall agree that if $a, b \in \bigwedge^0 \boldsymbol{V} \equiv \mathbb{R}$, then $a \cdot b = ab$.

If the metric vector space $(\boldsymbol{V}, \mathring{g})$ is also endowed with an *orientation*, i.e., a *metric volume n-covector denoted by* $\tau_{\mathring{\mathbf{g}}} \in \bigwedge^n \boldsymbol{V}$ such that

$$\tau_{\mathring{\mathbf{g}}} = \sqrt{\left|\det \mathring{\mathbf{g}}\right|}\, \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \frac{1}{\sqrt{\left|\det \mathring{g}\right|}}\varepsilon^1 \wedge \cdots \wedge \varepsilon^n,$$

$$\tilde{\tau}_{\mathring{\mathbf{g}}} \underset{\mathring{\mathbf{g}}}{\cdot} \tau_{\mathring{\mathbf{g}}} = (-1)^q, \tag{2.27}$$

then we can introduce (see Definition 2.27) a natural isomorphism between the spaces $\bigwedge^r \boldsymbol{V}$ and $\bigwedge^{n-r} \boldsymbol{V}$.

Remark 2.26 When the metric $\overset{\circ}{\mathbf{g}}$ used in the definition of the scalar product is obvious we use from now on only the symbol $\cdot$ in order to simplify the formulas.

Definition 2.27 The *Hodge star operator* (or *Hodge dual*) is the linear mapping $\underset{\overset{\circ}{\mathbf{g}}}{\star} : \bigwedge^r \boldsymbol{V} \to \bigwedge^{n-r} \boldsymbol{V}$ such that

$$A \wedge \underset{\overset{\circ}{\mathbf{g}}}{\star} B = (A \cdot B)\tau_{\overset{\circ}{\mathbf{g}}}, \tag{2.28}$$

for every $A, B \in \bigwedge^r \boldsymbol{V}$. Of course, this operator is naturally extended to an isomorphism $\underset{\overset{\circ}{\mathbf{g}}}{\star} : \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V}$ by linearity. The inverse $\underset{\overset{\circ}{\mathbf{g}}}{\star}^{-1} : \bigwedge^{n-r} \boldsymbol{V} \to \bigwedge^r \boldsymbol{V}$ of the Hodge star operator is given by:

$$\underset{\overset{\circ}{\mathbf{g}}}{\star}^{-1} A = (-1)^{r(n-r)} \operatorname{sgn} \overset{\circ}{\mathbf{g}} \underset{\overset{\circ}{\mathbf{g}}}{\star} A, \tag{2.29}$$

for $A \in \bigwedge^{n-r} \boldsymbol{V}$ and where $\operatorname{sgn} \overset{\circ}{\mathbf{g}} = \det \overset{\circ}{\mathbf{g}} / |\det \overset{\circ}{\mathbf{g}}|$ denotes the sign of the determinant of the matrix with entries $\overset{\circ}{g}_{ij}$ where $\{\mathbf{e}_i\}$ is an arbitrary basis of $\mathbf{V}$.

Note that $\underset{\overset{\circ}{\mathbf{g}}}{\star}$ is a linear isomorphism but is not an algebra isomorphism.

Remark 2.28 When the metric $\overset{\circ}{\mathbf{g}}$ used in the definition of the Hodge star operator is obvious we use only the symbol $\star$ in order to simplify the formulas.

Exercise 2.29 Show that for any $X, Y \in \bigwedge \boldsymbol{V}$

$$X \cdot Y = \langle \tilde{X} Y \rangle_0 = \langle X \tilde{Y} \rangle_0 = Y \cdot X \tag{2.30}$$

Exercise 2.30 Let $\{\varepsilon_i\}$ and orthonormal basis of $\boldsymbol{V}$ and $\{\varepsilon^j\}$ is reciprocal basis, i.e., $\varepsilon_i \cdot \varepsilon^k = \delta_i^k$. Then, any $Y \in \bigwedge \boldsymbol{V}$ can be written as

$$\begin{aligned} Y &= \frac{1}{p!} Y^{j_1 \cdots j_p} \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p} \\ &= \frac{1}{p!} Y_{j_1 \cdots j_p} \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}. \end{aligned} \tag{2.31}$$

Show that:

$$\begin{aligned} &\text{(a)} \quad Y^{j_1 \cdots j_p} = Y \cdot (\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}), \\ &\text{(b)} \quad Y_{j_1 \cdots j_p} = Y \cdot (\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}). \end{aligned} \tag{2.32}$$

Exercise 2.31 Verify that, e.g., that if $Y = \frac{1}{2!}Y^{ij}\varepsilon_i \wedge \varepsilon_j \in \bigwedge^2 \boldsymbol{V}$ ($Y^{ij} = -Y^{ji}$, of course) then $Y^{ij} \neq Y(\mathbf{e}^i, \mathbf{e}^j)$. Find[3] $Y(\mathbf{e}^i, \mathbf{e}^j,)$ in terms of Y^{ij}. On the other hand show that if $\mathcal{Y} = \frac{1}{2!}\mathcal{Y}^{ij}\varepsilon_i \dot{\wedge} \varepsilon_j \in \bigwedge^2 \boldsymbol{V}$ then $\mathcal{Y}(\varepsilon^i, \varepsilon^j) = \mathcal{Y}^{ij}$.

The algebra $\bigwedge \boldsymbol{V}$ inherits the operators ^ (main automorphism) , ~ (reversion) and $^-$ (conjugation) of the tensor algebra $\boldsymbol{TV}$, and we have

$$\begin{aligned}(AB)^\wedge &= \hat{A}\hat{B},\\ (AB)^\sim &= \tilde{B}\tilde{A},\\ (A)^- = (\tilde{A})^\wedge &= (\hat{A})^\sim\end{aligned} \tag{2.33}$$

for all $A, B \in \bigwedge \boldsymbol{V}$, with $\hat{A} = A$ if $A \in \mathbb{R}$, $\hat{A} = -A$ if $A \in \boldsymbol{V}$ and $\tilde{A} = A$ if $A \in \mathbb{R}$ or $A \in \boldsymbol{V}$.

2.3.2 *Contractions*

Definition 2.32 For arbitrary multiforms $X, Y, Z \in \bigwedge \boldsymbol{V}$ the left ($\lrcorner$) and right ($\llcorner$) contractions of X and Y are the mappings $\underset{\mathring{\mathbf{g}}}{\lrcorner} : \bigwedge \boldsymbol{V} \times \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V}$, $\underset{\mathring{\mathbf{g}}}{\llcorner} : \bigwedge \boldsymbol{V} \times \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V}$such that

$$\begin{aligned}(X \underset{\mathring{\mathbf{g}}}{\lrcorner} Y) \cdot Z &= Y \underset{\mathring{\mathbf{g}}}{\cdot} (\tilde{X} \wedge Z),\\ (X \underset{\mathring{\mathbf{g}}}{\llcorner} Y) \cdot Z &= X \underset{\mathring{\mathbf{g}}}{\cdot} (Z \wedge \tilde{Y}).\end{aligned} \tag{2.34}$$

These contracted products $\underset{\mathring{\mathbf{g}}}{\lrcorner}$ and $\underset{\mathring{\mathbf{g}}}{\llcorner}$ are internal laws on $\bigwedge \boldsymbol{V}$. Sometimes the contractions are called interior products. Both contract products satisfy the distributive laws (on the left and on the right) but they are *not* associative.

Remark 2.33 When the metric $\mathring{\mathbf{g}}$ used in the definitions of the left and right contractions is obvious from the context we use the symbols $\lrcorner$ and $\llcorner$.

Definition 2.34 The vector space $\bigwedge \boldsymbol{V}$ endowed with each one of these contracted products (either $\lrcorner$ or $\llcorner$) is a non-associative algebra. We call Grassmann algebra of multiforms the algebraic structure $(\bigwedge \boldsymbol{V}, \wedge, \lrcorner, \llcorner)$, which will simply be denoted by $(\bigwedge \boldsymbol{V}, \mathring{g})$.

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[3]Recall again that when the exterior product is defined (as, e.g., in [5, 9]) by Eq. (2.25), then $X_{j_1} \ldots {}_{j_p}$ in Eq. (2.32) really means $X(\varepsilon_{j_1}, \ldots, \varepsilon_{j_p})$. So, care is need when reading textbooks or articles in order to avoid errors.

We present now some of the most *important* properties of the contractions:

(i) For any $a, b \in \mathbb{R}$, and $Y \in \bigwedge \boldsymbol{V}$

$$a \lrcorner b = a \llcorner b = ab \text{ (product in } \mathbb{R}\text{)},$$
$$a \lrcorner Y = Y \llcorner a = aY \text{ (multiplication by scalars).} \tag{2.35}$$

(ii) If $a, b_1, \ldots, b_k \in \boldsymbol{V}$ then

$$a \lrcorner (b_1 \wedge \cdots \wedge b_k) = \sum_{j=1}^{k} (-1)^{j+1} (a \cdot b_j) b_1 \wedge \cdots \wedge \check{b}_j \wedge \cdots \wedge b_k, \tag{2.36}$$

where the symbol $\check{b}_j$ means that the b_j factor did not appear in the j-term of the sum.

(iii) For any $Y_j \in \bigwedge^j \boldsymbol{V}$ and $Y_k \in \bigwedge^k \boldsymbol{V}$ with $j \leq k$

$$Y_j \lrcorner Y_k = (-1)^{j(k-j)} Y_k \llcorner Y_j. \tag{2.37}$$

(iv) For any $Y_j \in \bigwedge^j \boldsymbol{V}$ and $Y_k \in \bigwedge^k \boldsymbol{V}$

$$Y_j \lrcorner Y_k = 0, \text{ if } j > k,$$
$$Y_j \llcorner Y_k = 0, \text{ if } j < k. \tag{2.38}$$

(v) For any $X_k, Y_k \in \bigwedge^k \boldsymbol{V}$

$$X_k \lrcorner Y_k = Y_k \llcorner Y_k = \tilde{X}_k \cdot Y_k = X_k \cdot \tilde{Y}_k, \tag{2.39}$$

where $\tilde{Y}_k$ the *reverse* of $Y_k \in \bigwedge^k \boldsymbol{V}$ is the antiautomorphism given by

$$\tilde{Y}_k = (-1)^{\frac{k}{2}(k-1)} Y_k. \tag{2.40}$$

(vi) For any $v \in \boldsymbol{V}$ and $X, Y \in \bigwedge \boldsymbol{V}$

$$v \lrcorner (X \wedge Y) = (v \lrcorner X) \wedge Y + \hat{X} \wedge (v \lrcorner Y). \tag{2.41}$$

2.4 Clifford Algebras

Definition 2.35 The *Clifford algebra* $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ of a metric vector space $(\boldsymbol{V}, \overset{\circ}{g})$ is defined as the quotient algebra[4]

$$\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g}) = \frac{T(\boldsymbol{V})}{J_{\overset{\circ}{g}}} ,$$

where $J_{\overset{\circ}{g}} \subset T\boldsymbol{V}$ is the bilateral ideal of $T\boldsymbol{V}$ generated by the elements of the form $u \otimes v + v \otimes u - 2\overset{\circ}{g}(u, v)$, with $u, v \in \boldsymbol{V} \subset T\boldsymbol{V}$.

Clifford algebras generated by symmetric bilinear forms are sometimes referred as *orthogonal*, in order to be distinguished from the *symplectic* Clifford algebras, which are generated by skew-symmetric bilinear forms (see, e.g., [3]).

Let $\rho_{\overset{\circ}{g}} : T\boldsymbol{V} \to \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ be the natural projection of $T\boldsymbol{V}$ onto the quotient algebra $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$. Multiplication in $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ will be denoted as usually by juxtaposition and called *Clifford product*. We have:

$$AB := \rho_{\overset{\circ}{g}}(A \otimes B), \tag{2.42}$$

$A, B \in \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$. The subspaces $\mathbb{R}, \boldsymbol{V} \subset T\boldsymbol{V}$ are identified with their images in $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$. In particular, for $u, v \in \boldsymbol{V} \subset \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{\mathrm{g}})$, we have:

$$u \otimes v = \frac{1}{2}(u \otimes v - v \otimes u) + \overset{\circ}{g}(u, v) + \frac{1}{2}\{(u \otimes v + v \otimes u - 2\overset{\circ}{g}(u, v)\}, \tag{2.43}$$

and then

$$\rho_\gamma(u \otimes v) \equiv uv = \frac{1}{2}(u \otimes v - v \otimes u) + \overset{\circ}{g}(u, v) = u \wedge v + \overset{\circ}{g}(u, v). \tag{2.44}$$

From here we get the *standard* relation characterizing the Clifford algebra $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$,

$$uv + vu = 2\overset{\circ}{g}(u, v). \tag{2.45}$$

[4]For other possible definitions of Clifford algebras see, e.g., [1, 12, 13]. In this chapter we shall be concerned only with Clifford algebras over *real* vector spaces, induced by *nondegenerate* bilinear forms.

2.4.1 Properties of the Clifford Product

With some work the reader can prove [5] the following *rules* satisfied by the Clifford product of multiforms:

(i) For all $a \in \mathbb{R}$ and $Y \in \bigwedge \boldsymbol{V} : aY = Ya$ equals multiplication of multiform Y by scalar a.
(ii) For all $v \in \boldsymbol{V}$ and $Y \in \bigwedge \boldsymbol{V}$:

$$vY = v \lrcorner Y + v \wedge Y \text{ and } Yv = Y \llcorner v + Y \wedge v. \tag{2.46}$$

(iii) For all $X, Y, Z \in \bigwedge \boldsymbol{V} : X(YZ) = (XY)Z$.

The Clifford product is an internal law on $\bigwedge \boldsymbol{V}$. It is associative (by (iii)) and satisfies the distributive laws (on the left and on the right). The distributive laws follow from the corresponding distributive laws of the contracted and exterior products.

Recall that the Clifford product is associative but it is not commutative (as follows from (ii)) .

Note that since the ideal $J_{\mathring{g}} \subset T\boldsymbol{V}$ is nonhomogeneous, of even grade, it induces a parity grading in the algebra $\bigwedge \boldsymbol{V}$, i.e.,

$$\mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) = \mathcal{C}\ell^0(\boldsymbol{V}, \mathring{g}) \oplus \mathcal{C}\ell^1(\boldsymbol{V}, \mathring{g}), \tag{2.47}$$

with

$$\begin{aligned} \mathcal{C}\ell^0(\boldsymbol{V}, \mathring{g}) &= \rho_{\mathring{g}}(\bigoplus\nolimits_{r=0}^{\infty} T_{2r}\boldsymbol{V}) \\ \mathcal{C}\ell^1(\boldsymbol{V}, \mathring{g}) &= \rho_{\mathring{g}}(\bigoplus\nolimits_{r=0}^{\infty} T_{2r+1}\boldsymbol{V}). \end{aligned} \tag{2.48}$$

We say that $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ is $\mathbb{Z}_2$ graded algebra. The elements of $\mathcal{C}\ell^0(\boldsymbol{V}, \mathring{g})$ form a subalgebra of $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, called *even subalgebra* of $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$. Note that $\mathcal{C}\ell^1(\boldsymbol{V}, \mathring{g})$ is not a Clifford algebra.

2.4.2 Universality of $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$

We now quote a standard theorem concerning real Clifford algebras [3]:

Theorem 2.36 *If $\mathcal{A}$ is a real associative algebra with unity, then each linear mapping $\phi : \boldsymbol{V} \to \mathcal{A}$ such that:*

$$(\phi(u))^2 = \mathring{g}(u, u) \tag{2.49}$$

for every $u \in \boldsymbol{V}$, *can be extended in an unique way to a homomorphism* $C_\phi : \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g}) \to \mathcal{A}$, *satisfying the relation:*

$$\phi = C_\phi \circ \rho_{\overset{\circ}{g}}. \tag{2.50}$$

Let $(\boldsymbol{V}, \overset{\circ}{g})$ and $(\boldsymbol{V}', \overset{\circ}{g}')$ be two metric vector spaces and $\psi : \boldsymbol{V} \to \boldsymbol{V}'$ a linear mapping satisfying:

$$\overset{\circ}{g}'(\psi(u), \psi(v)) = \overset{\circ}{g}(u, v) \tag{2.51}$$

for every $u, v \in \boldsymbol{V}$.

We denote by $\Lambda\psi$ the natural extension of ψ, i.e., the *linear* mapping $\Lambda\psi : \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V}'$ called *exterior power extension* (see more details in Sect. 2.7) such that:

(i) for $s \in \bigwedge^0 \boldsymbol{V} \subset \bigwedge \boldsymbol{V}$,

$$\Lambda\psi(s) = s, \tag{2.52}$$

(ii) for any homogenous multiform $\alpha_1 \wedge \cdots \wedge \alpha_r \in \bigwedge \boldsymbol{V} \subset \bigwedge \boldsymbol{V}$, $\alpha_i \in \bigwedge \boldsymbol{V}$ we have

$$\Lambda\psi(\alpha_1 \wedge \cdots \wedge \alpha_r) = \psi(\alpha_1) \wedge \cdots \wedge \psi(\alpha_r) \in \bigwedge^r \boldsymbol{V}' \subset \bigwedge \boldsymbol{V}', \tag{2.53}$$

(iii) If $A = \bigoplus_{j=0}^n A_j \in \bigwedge^j V$, then

$$\Lambda\psi(A) = \bigoplus_{j=0}^n \Lambda\psi(A_j) \in \bigwedge^j \boldsymbol{V}' \subset \bigwedge \boldsymbol{V}'. \tag{2.54}$$

Then, using Theorem 2.36 we can show that there exists a homomorphism $C_\psi : \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g}) \to \mathcal{C}\ell(\boldsymbol{V}', \overset{\circ}{g}')$ between their Clifford algebras such that:

$$C_\psi \circ \rho_{\overset{\circ}{g}} = \rho_{\overset{\circ}{g}'} \circ \Lambda\psi. \tag{2.55}$$

Moreover, if $\boldsymbol{V}$ and $\boldsymbol{V}'$ are metrically isomorphic[5] vector spaces, then their Clifford algebras are *isomorphic*. In particular, two Clifford algebras $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ and $\mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g}')$ with the same underlying vector space $\boldsymbol{V}$ are isomorphic if and only if the bilinear forms $\overset{\circ}{g}$ and $\overset{\circ}{g}'$ have the same signature. Therefore, there is essentially one Clifford algebra for each signature on a given vector space $\boldsymbol{V}$.[6]

[5]We call *metric isomorphism* a vector space isomorphism satisfying Eq. (2.51). The term *isometry* will be reserved to designate a metric isomorphism from a space onto itself.

[6]However, take into account that Clifford algebras $\mathcal{C}\ell(\mathbf{V}, \overset{\circ}{\mathrm{g}})$ and $\mathcal{C}\ell(\mathbf{V}, \overset{\circ}{\mathrm{g}}')$ over the same vector space may be isomorphic as *algebras* (but not as graded algebras) even if $\overset{\circ}{\mathrm{g}}$ and $\overset{\circ}{\mathrm{g}}'$ do not have

Definition 2.37 Let $\mathbb{R}^{p,q}$ $(p+q=n)$ denote the vector space $\mathbb{R}^n$ endowed with a metric tensor of signature (p,q). We denote by $\mathbb{R}_{p,q}$ the Clifford algebra of $\mathbb{R}^{p,q}$.

Natural Embedding $\bigwedge V \hookrightarrow \mathcal{C}\ell(V, \mathring{g})$

Another important (indirect) consequence of the universality is that $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ is isomorphic, as a vector space over $\mathbb{R}$, to the Grassmann algebra $(\bigwedge \boldsymbol{V}, \mathring{g})$. Let the symbol $A \hookrightarrow B$ means that A is *embedded* in B and $A \subseteq B$.There is a natural embedding [11] $\bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$. Then $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ is a 2^n-dimensional vector space and given $A \in \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ we can write:

$$A = \sum_{r=0}^{n} \langle A \rangle_r, \tag{2.56}$$

with $\langle A \rangle_r \in \bigwedge^r \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ the projection of A (see Definition 2.4) in the $\bigwedge^r \boldsymbol{V}$ subspace of $\bigwedge \boldsymbol{V}$.

Definition 2.38 The elements of $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ will also be called *multiforms and sometimes also called "Clifford numbers"emph.* Furthermore, if $A = \langle A \rangle_r$ for some fixed r, we say that A is *homogeneous* of *grade* r. In that case, we also write $A = A_r \in \bigwedge^r \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$.

Since $\bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ inherits the main antiautomorphism, the reversion and the conjugation operators that we defined (see Eq. (2.33) in $\bigwedge \boldsymbol{V}$. Note moreover that $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ also inherits from $(\bigwedge \boldsymbol{V}, \mathring{g})$ the scalar and contracted products of multiforms.

Exercise 2.39 Show that:

(i) If $A \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ then

$$\langle \bar{A} \rangle_r = (-1)^r \langle A \rangle_r, \quad \langle \tilde{A} \rangle_r = (-1)^{\frac{1}{2}r(r-1)} \langle A \rangle_r. \tag{2.57}$$

(ii) If $a \in \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, $A_r \in \bigwedge^r \boldsymbol{V}$, $B_s \in \bigwedge^s \boldsymbol{V}$, $r, s \geq 0$: (see [10])

$$\begin{aligned} a \lrcorner (A_r B_s) &= (a \lrcorner A_r) B_s + (-1)^r A_r (a \lrcorner B_s) \\ &= (a \wedge A_r) B_s - (-1)^r A_r (a \wedge B_s), \\ a \wedge (A_r B_s) &= (a \wedge A_r) B_s - (-1)^r A_r (a \lrcorner B_s) \\ &= (a \lrcorner A_r) B_s + (-1)^r A_r (a \wedge B_s). \end{aligned} \tag{2.58}$$

the *same* signature. The reader may verify the validity of this statement by inspecting Table 3.1 in Chap. 3 and finding examples.

(iii) if $a \in \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ and $A \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$a \lrcorner A = -\bar{A} \llcorner a, \; a \wedge A = \bar{A} \wedge a. \tag{2.59}$$

(iv) if $a \in \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ and $A \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$a \lrcorner (A \wedge B) = (a \lrcorner A) \wedge B + \hat{A} \wedge (a \lrcorner B). \tag{2.60}$$

(v) if $A, B \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$\frac{1}{2} \langle A\tilde{B} - \tilde{B}\bar{A} \rangle_1 = \left\langle \tilde{\bar{B}} \lrcorner A \right\rangle_1, \tag{2.61}$$

$$\frac{1}{2} \langle A\tilde{B} + \tilde{B}\bar{A} \rangle_1 = \left\langle \tilde{\bar{A}} \lrcorner B \right\rangle_1. \tag{2.62}$$

(vi) if $a \in \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ and $A, B \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$(a \lrcorner A) \cdot B = A \cdot (a \wedge B). \tag{2.63}$$

(vii) if $A, B, C \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$A \lrcorner (B \lrcorner C) = (A \wedge B) \lrcorner C, \tag{2.64}$$

$$(A \llcorner B) \llcorner C = A \llcorner (B \wedge C). \tag{2.65}$$

(viii) if $A, B, C \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$(A \lrcorner B) \cdot C = B \cdot (\tilde{A} \wedge C), \tag{2.66}$$

$$(B \llcorner A) \cdot C = B \cdot (C \wedge \tilde{A}). \tag{2.67}$$

(ix) if $A, B, C \in \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then

$$(AB) \cdot C = B \cdot (\tilde{A}C), \tag{2.68}$$

$$(BA) \cdot C = B \cdot (C\tilde{A}). \tag{2.69}$$

(x) if $A_r \in \bigwedge^r \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g}), B_s \in \bigwedge^s \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ then [10] :

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s} = \sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}, \tag{2.70}$$

with $m = \frac{1}{2}(r + s - |r - s|)$.

(xi) if $A_r \in \bigwedge^r \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}), B_s \in \bigwedge^s \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ and $r \leq s$ show that

$$A_r \lrcorner B_s = \langle A_r B_s \rangle_{|r-s|} = (-1)^{r(s-r)} \langle B_s A_r \rangle_{|r-s|} = (-1)^{r(s-r)} B_s \llcorner A_r. \tag{2.71}$$

(xii) Show that for $\varepsilon^{\mathbf{a}}, \varepsilon^{\mathbf{b}} \in \bigwedge^1 \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ and $B \in \bigwedge^2 \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$

$$\langle (\varepsilon^{\mathbf{a}} \wedge \varepsilon^{\mathbf{b}}) B \rangle_2 = \varepsilon^{\mathbf{a}} \wedge (\varepsilon^{\mathbf{b}} \lrcorner B) + \varepsilon^{\mathbf{a}} \lrcorner (\varepsilon^{\mathbf{b}} \wedge B). \tag{2.72}$$

Exercise 2.40 Define the commutator product of $A, B \in \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ by

$$A \times B = \frac{1}{2}[A, B]. \tag{2.73}$$

Show that the commutator product satisfy the Jacobi identity, i.e. for $A, B, C \in \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0 \tag{2.74}$$

Exercise 2.41 Show that if $F \in \bigwedge^2 \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, then

$$F^2 = -F \cdot F + F \wedge F. \tag{2.75}$$

If the metric vector space $(\boldsymbol{V}, \mathring{g})$ is oriented by $\tau_{\mathring{\mathbf{g}}}$, then we can also extend the Hodge star operator defined in the Grassmann algebra to the Clifford algebra $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, by letting $\star : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ be given by:

$$\star A = \sum_r \star \langle A \rangle_r \,. \tag{2.76}$$

Exercise 2.42 Show that for any $A_r \in \bigwedge^r \boldsymbol{V}$ and $B_s \in \bigwedge^s \boldsymbol{V}$, $r, s \geq 0$:

$$\begin{aligned} A_r \wedge \star B_s &= B_s \wedge \star A_r; \quad r = s \\ A_r \cdot \star B_s &= B_s \cdot \star A_r; \quad r + s = n \\ A_r \wedge \star B_s &= (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s \\ A_r \lrcorner \star B_s &= (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq n \\ \star A_r &= \tilde{A}_r \lrcorner \tau_{\mathring{\mathbf{g}}} = \tilde{A}_r \tau_{\mathring{\mathbf{g}}} \\ \star \tau_{\mathring{\mathbf{g}}} &= \operatorname{sgn} \mathring{\mathbf{g}}; \quad \star 1 = \tau_{\mathring{\mathbf{g}}}. \end{aligned} \tag{2.77}$$

Exercise 2.43 Let $\{\varepsilon^\mu\}$ be a basis of $\boldsymbol{V}$ (a n-dimensional real vector space) dual to a basis $\{\mathbf{e}_\mu\}$ of $\mathbf{V}$. Let $\mathring{\mathbf{g}}$ be the metric of $\mathbf{V}$ such that $\mathring{\mathbf{g}}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \mathring{g}_{\mu\nu}$ and let $\mathring{g}$ be a metric for $\boldsymbol{V}$ such that $\mathring{g}(\varepsilon^\mu, \varepsilon^\nu) = \mathring{g}^{\mu\nu}$ and $\mathring{g}^{\mu\nu} \mathring{g}_{\nu\beta} = \delta^\mu_\beta$. Show that writing

$\varepsilon^{\mu_1\cdots\mu_p} = \varepsilon^{\mu_1} \wedge \cdots \wedge \varepsilon^{\mu_p}$, $\varepsilon^{\nu_{p+1}\cdots\nu_n} = \varepsilon^{\nu_{p+1}} \wedge \cdots \wedge \varepsilon^{\nu_n}$ we have

$$\star\, \varepsilon^{\mu_1\cdots\mu_p} = \frac{1}{(n-p)!}\sqrt{\left|\det \mathring{\mathbf{g}}\right|}\mathring{g}^{\mu_1\nu_1}\cdots\mathring{g}^{\mu_p\nu_p}\epsilon_{\nu_1\cdots\nu_n}\varepsilon^{\nu_{p+1}\cdots\nu_n}. \tag{2.78}$$

All identities in the above exercises are very important for calculations. They are part of the tricks of the trade.

2.5 Extensors

In what follows $\bigwedge^{\diamond} \boldsymbol{V}$ denotes an arbitrary subspace of $\bigwedge \boldsymbol{V}$, called a *subspace part* of $\bigwedge \boldsymbol{V}$. Consider the arbitrary subspace parts $\bigwedge_1^{\diamond} \boldsymbol{V}, \ldots, \bigwedge_m^{\diamond} \boldsymbol{V}$ and $\bigwedge^{\diamond} \boldsymbol{V}$ of $\bigwedge \boldsymbol{V}$.

Definition 2.44 Any linear mapping $t : \bigwedge_1^{\diamond} \boldsymbol{V} \times \ldots \times \bigwedge_m^{\diamond} \boldsymbol{V} \to \bigwedge^{\diamond} \boldsymbol{V}$ is called an extensor [6] over $\boldsymbol{V}$**.**

The set of extensors over $\boldsymbol{V}$, with domain $\bigwedge_1^{\diamond} \boldsymbol{V} \times \ldots \times \bigwedge_m^{\diamond} \boldsymbol{V}$ and codomain $\bigwedge^{\diamond} \boldsymbol{V}$ has a natural structure of a vector space over $\mathbb{R}$ and will be called **EXT**-($\boldsymbol{V}$). In what follows we shall need to study mainly the properties of extensors where the domain is a single subspace part, say $\bigwedge_1^{\diamond} \boldsymbol{V}$. The space of that extensors where $\bigwedge_1^{\diamond} \boldsymbol{V} = \bigwedge^{\diamond} \boldsymbol{V} = \bigwedge \boldsymbol{V}$ will be denoted by *ext*-($\boldsymbol{V}$).

2.5.1 (p, q)-Extensors

Definition 2.45 Let $p, q \in \mathbb{N}$ with $0 \leq p, q \leq n$. A extensor over $\boldsymbol{V}$ with domain $\bigwedge^p \boldsymbol{V}$ and codomain $\bigwedge^q \boldsymbol{V}$ is called a (p, q)-extensor over $\boldsymbol{V}$. The real vector space of the (p, q)-extensors over $\boldsymbol{V}$ is denoted by $ext - (\bigwedge^p \boldsymbol{V}, \bigwedge^q \boldsymbol{V})$.

Note that if $\dim \boldsymbol{V} = n$ then $\dim ext(\bigwedge^p \boldsymbol{V}, \bigwedge^q \boldsymbol{V}) = \binom{n}{p}\binom{n}{q}$.

2.5.2 Adjoint Operator

Definition 2.46 Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be a pair of arbitrary reciprocal basis for $\boldsymbol{V}$ (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). The linear operator

$$^{\dagger} : ext\Big(\bigwedge^p \boldsymbol{V}, \bigwedge^q \boldsymbol{V}\Big) \ni t \mapsto t^{\dagger} \in ext\Big(\bigwedge^q \boldsymbol{V}, \bigwedge^p \boldsymbol{V}\Big)$$

such that

$$t^\dagger(Y) = \frac{1}{p!}(t(\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}) \cdot Y)\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}$$
$$= \frac{1}{p!}(t(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}) \cdot Y)\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}, \tag{2.79}$$

is called the adjoint operator relative to the scalar product defined by $\overset{\circ}{g}$. $t^\dagger$ is called the adjoint of t.

The adjoint operator is well defined since the sums in the second members of the above equations are independent of the chosen pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$.

2.5.3 *Properties of the Adjoint Operator*

(i) The operator † is involutive, i.e.,

$$(t^\dagger)^\dagger = t. \tag{2.80}$$

(ii) Let $t \in ext(\bigwedge^q \boldsymbol{V}, \bigwedge^p \boldsymbol{V})$. Then, for any $X \in \bigwedge^p \boldsymbol{V}$ and $Y \in \bigwedge^q \boldsymbol{V}$

$$t(X) \cdot Y = X \cdot t^\dagger(Y). \tag{2.81}$$

(iii) Let $t \in ext(\bigwedge^q \boldsymbol{V}, \bigwedge^r \boldsymbol{V})$ and $u \in ext(\bigwedge^p \boldsymbol{V}, \bigwedge^q \boldsymbol{V})$. Then, composition of u with t denoted $t \circ u \in ext(\bigwedge^p \boldsymbol{V}, \bigwedge^r \boldsymbol{V})$ and we have

$$(t \circ u)^\dagger = u^\dagger \circ t^\dagger. \tag{2.82}$$

2.6 (1, 1)-Extensors

2.6.1 *Symmetric and Antisymmetric Parts of* (1, 1)*-Extensors*

Definition 2.47 An extensor $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ is said adjoint symmetric relative to the scalar product defined by $\overset{\circ}{g}$ (adjoint antisymmetric) if, and only if $t = t^\dagger$ $(t = -t^\dagger)$.

It is easy to see that for any $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$, there are two (1, 1)-extensors over $\boldsymbol{V}$, say t_+ and t_-, such that t_+ is adjoint symmetric (i.e., $t_+ = t_+^\dagger$) and t_- is adjoint antisymmetric (i.e., $t_- = -t_-^\dagger$) and

$$t(a) = t_+(a) + t_-(a). \tag{2.83}$$

Moreover,

$$t_\pm(a) = \frac{1}{2}(t(a) \pm t^\dagger(a)). \tag{2.84}$$

Definition 2.48 t_+ and t_- are called the adjoint symmetric and adjoint antisymmetric parts of t.

2.6.2 *Exterior Power Extension of* $(1,1)$*-Extensors*

Definition 2.49 Let $t \in ext(\bigwedge^1 V, \bigwedge^1 V))$ and $Y \in \bigwedge V$. The linear operator

$$\underline{\ } : ext(\bigwedge^1 V, \bigwedge^1 V) \to ext\text{-}(V), \quad t \mapsto \underline{t}, \tag{2.85}$$

such that for any

$$\underline{t}(Y) = 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!}((\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}) \cdot Y)t(\varepsilon_{j_1}) \wedge \cdots \wedge t(\varepsilon_{j_k}) \tag{2.86}$$

$$= 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!}((\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k}) \cdot Y)t(\varepsilon^{j_1}) \wedge \cdots \wedge t(\varepsilon^{j_k}), \tag{2.87}$$

is called the (exterior power) extension operator relative to the scalar product defined by $\overset{\circ}{g}$. We read $\underline{t}$ as the extended of t.

The extension operator is well defined since the sums in the second members of the above equations is independent of the chosen pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$. Take into account also that the extension operator preserve grade, i.e., if $Y \in \bigwedge^k V$ then $\underline{t}(Y) \in \bigwedge^k V$.

2.6.3 *Properties of* $\underline{t}$

(i) For any $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$, and any $\alpha \in \mathbb{R}, v, v_1, \ldots, v_k \in \bigwedge^1 V$ we have

$$\underline{t}(\alpha) = \alpha, \tag{2.88}$$

$$\underline{t}(v) = t(v), \tag{2.89}$$

$$\underline{t}(v_1 \wedge \cdots \wedge v_k) = t(v_1) \wedge \cdots \wedge t(v_k). \tag{2.90}$$

An obvious corollary of the last property is

$$\underline{t}(X \wedge Y) = \underline{t}(X) \wedge \underline{t}(Y), \tag{2.91}$$

for any $X, Y \in \bigwedge \boldsymbol{V}$.

(ii) For all $t, u \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ it holds

$$\underline{t \circ u} = \underline{t} \circ \underline{u}. \tag{2.92}$$

(iii) Let $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ with inverse $t^{-1} \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ (i.e., $t \circ t^{-1} = t^{-1} \circ t = i_d$, where $i_d \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ is the identity extensor). Then

$$(\underline{t})^{-1} = \underline{(t^{-1})} \equiv \underline{t}^{-1}. \tag{2.93}$$

(iv) For any $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$

$$\underline{(t^{\dagger})} = (\underline{t})^{\dagger} \equiv \underline{t}^{\dagger}. \tag{2.94}$$

(v) Let $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$, for any $X, Y \in \bigwedge \boldsymbol{V}$

$$X \lrcorner \underline{t}(Y) = \underline{t}(\underline{t}^{\dagger}(X) \lrcorner Y). \tag{2.95}$$

2.6.4 Characteristic Scalars of a (1, 1)*-Extensor*

Definition 2.50 The trace of $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$ relative to the scalar product defined by $\overset{\circ}{g}$ is a mapping

$$tr : ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V}) \to \mathbb{R}$$

such that

$$tr(t) = t(\varepsilon^j) \cdot \varepsilon_j = t(\varepsilon_j) \cdot \varepsilon^j. \tag{2.96}$$

Note that the definition is independent of the chosen pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$. Also, for any $t \in ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V})$,

$$tr(t^{\dagger}) = tr(t). \tag{2.97}$$

Definition 2.51 The determinant of $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ relative to the scalar product defined by $\overset{\circ}{g}$ is the mapping $\mathfrak{det} : ext(\bigwedge^1 V, \bigwedge^1 V) \to \mathbb{R}$ such that

$$\mathfrak{det}\, t = \frac{1}{n!}\underline{t}(\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_n}) \cdot (\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_n}) \tag{2.98}$$

$$= \frac{1}{n!}\underline{t}(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_n}) \cdot (\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_n}). \tag{2.99}$$

Remark 2.52 Note that the definition is *independent* of the chosen pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$. When it becomes necessary to explicitly specify the metric $\overset{\circ}{g}$ relative to which the determinant of t is defined we will write $\underset{\overset{\circ}{\mathfrak{g}}}{\mathfrak{det}}\, t$ instead of $\mathfrak{det}\, t$.

Remark 2.53 For the relation between the definition of $\mathfrak{det}\, t$ and the classical determinant of a square matrix representing t in a given basis see Exercise 2.58.

Using the combinatorial formulas $v^{j_1} \wedge \cdots \wedge v^{j_n} = \epsilon^{j_1 \cdots j_n} v^1 \wedge \cdots \wedge v^n$ and $v_{j_1} \wedge \cdots \wedge v_{j_n} = \epsilon_{j_1 \cdots j_n} v_1 \wedge \cdots \wedge v_n$, where $\epsilon^{j_1 \cdots j_n}$ and $\epsilon_{j_1 \cdots j_n}$are the permutation symbols of order n and $v^1, \ldots, v^n$ and $v_1, \ldots, v_n$ are linearly independent covectors, we can also write

$$\mathfrak{det}\, t = \underline{t}(\varepsilon^1 \wedge \cdots \wedge \varepsilon^n) \cdot (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) = \underline{t}(\overset{\triangle}{\varepsilon}) \cdot \underset{\triangle}{\varepsilon} \tag{2.100}$$

$$= \underline{t}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \cdot (\varepsilon^1 \wedge \cdots \wedge \varepsilon^n) = \underline{t}(\underset{\triangle}{\varepsilon}) \cdot \overset{\triangle}{\varepsilon}, \tag{2.101}$$

where we used: $\overset{\triangle}{\varepsilon} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ and $\underset{\triangle}{\varepsilon} = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$.

2.6.5 Properties of $\mathfrak{det}\, t$

(i) For any $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$,

$$\mathfrak{det}\, t^\dagger = \mathfrak{det}\, t. \tag{2.102}$$

(ii) For any $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$, and $I \in \bigwedge^n V$ we have

$$\underline{t}(I) = I\, \mathfrak{det}\, t. \tag{2.103}$$

(iii) For any $t, u \in ext(\bigwedge^1 V, \bigwedge^1 V)$,

$$\mathfrak{det}(t \circ u) = \mathfrak{det}\, t\; \mathfrak{det}\, u. \tag{2.104}$$

(iv) Let $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ with inverse $t^{-1} \in (1,1)\text{-}ext(V)$ (i.e., $t \circ t^{-1} = t^{-1} \circ t = i_d$, where $i_d \in (1,1)\text{-}ext(V)$ is the identity extensor), then

$$\mathfrak{det}\, t^{-1} = (\mathfrak{det}\, t)^{-1}. \tag{2.105}$$

In what follows we use the notation $\mathfrak{det}^{-1} t$ meaning $\mathfrak{det}\, t^{-1}$ or $(\mathfrak{det}\, t)^{-1}$.

(v) If $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ is non degenerated (i.e., $\mathfrak{det}\, t \neq 0$), then it has an inverse $t^{-1} \in ext(\bigwedge^1 V, \bigwedge^1 V)$, given by

$$t^{-1}(a) = \mathfrak{det}^{-1}\, t \underline{t}^{\dagger}(aI)I^{-1}, \tag{2.106}$$

where $I \in \bigwedge^n V$ is any non null pseudoscalar.

2.6.6 Characteristic Biform of a (1, 1)-Extensor

Definition 2.54 The 2-form [7] of $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ is

$$bif(t) = t(\varepsilon^j) \wedge \varepsilon_j = t(\varepsilon_j) \wedge \varepsilon^j \in \bigwedge^2 V. \tag{2.107}$$

Note that $bif(t)$ is indeed a characteristic of t since the definition is independent of the chosen pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$.

Properties of *bif*(*t*)

(i) Let $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$, then

$$bif(t^{\dagger}) = -bif(t). \tag{2.108}$$

(ii) The adjoint antisymmetric part of any $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ can be 'factored' by the formula [6]

$$t_{-}(a) = \frac{1}{2} bif(t) \times a, \tag{2.109}$$

where $\times$ means the commutator product [see Eq. (2.73)].

2.6.7 Generalization of $(1, 1)$*-Extensors*

Definition 2.55 Let $\mathcal{G} : ext(\bigwedge^1 V, \bigwedge^1 V) \ni t \mapsto T \in ext\text{-}(V)$ such that for any $Y \in \bigwedge V$

$$\mathcal{G}t\,(Y) = T(Y) = t(\varepsilon^j) \wedge (\varepsilon_j \lrcorner Y) = t(\varepsilon_j) \wedge (\varepsilon^j \lrcorner Y), \tag{2.110}$$

The linear operator $\mathcal{G}$ is called the generalization operator of t relative to the scalar product defined by $\overset{\circ}{g}$, T is read as the generalized t.

Note that $\mathcal{G}$ is well defined since it does not depend on the choice of the pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$. Note also that $\mathcal{G}$ preserves grade, i.e., if $Y \in \bigwedge^k V$, then $T(Y) \in \bigwedge^k V$.

Properties of $\mathcal{G}$

(i) For any $\alpha \in \mathbb{R}$ and $v \in \bigwedge^1 V$ we have

$$T(\alpha) = 0, \tag{2.111}$$

$$T(v) = t(v). \tag{2.112}$$

(ii) For any $X, Y \in \bigwedge V$ we have

$$T(X \wedge Y) = T(X) \wedge Y + X \wedge T(Y). \tag{2.113}$$

(iii) $\mathcal{G}$ commutes with the adjoint operator. Thus, $T^\dagger$ means either the adjoint of the generalized as well as the generalized of the adjoint.

(iv) The adjoint antisymmetric part of the generalized of t is equal to the generalized of the antisymmetric adjoint part of t, and can be factored as

$$T_-(Y) = \frac{1}{2} bif(t) \times Y, \tag{2.114}$$

for any $Y \in \bigwedge V$.

2.7 Symmetric Automorphisms and Orthogonal Clifford Products

Besides the "natural" Clifford product of $\mathcal{C}\ell(V, \overset{\circ}{g})$, we can introduce infinitely many other Clifford-like products on this same algebra, one for each symmetric automorphism of its underlying vector space. In what follows we are going to construct such new Clifford products.[7]

[7]The possibility of introducing different Clifford products in the same Clifford algebra was already established by Arcuri [2]. A complete study of that issue is given in [8]. The relation between

There is a one-to-one correspondence between the endomorphisms of $(\boldsymbol{V}, \overset{\circ}{g})$ and the bilinear forms over $\boldsymbol{V}$. Indeed, to each endomorphism $\boldsymbol{g} : \boldsymbol{V} \to \boldsymbol{V}$ we can associate a bilinear form $g : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, by the relation:

$$g(u, v) = \overset{\circ}{g}(\boldsymbol{g}(u), v) \tag{2.115}$$

for every $u, v \in \boldsymbol{V}$.

As we know [recall Eq. (2.79)] the adjoint of the extensor $\boldsymbol{g} : \boldsymbol{V} \to \boldsymbol{V}$ is the extensor (linear mapping) $\boldsymbol{g}^{\dagger} : \boldsymbol{V} \to \boldsymbol{V}$ such that for any $u, v \in \boldsymbol{V}$

$$\overset{\circ}{g}(\boldsymbol{g}(u), v) = \boldsymbol{g}(u) \cdot v = u \cdot \boldsymbol{g}^{\dagger}(u).$$

Definition 2.56 An endomorphism $G : \boldsymbol{V} \to \boldsymbol{V}$ is said to be *symmetric*or *skew-symmetric* if its associated bilinear form $\mathtt{G} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is, respectively, symmetric or skew-symmetric. In the more general case we can write a bilinear form $\mathtt{G}$ as:

$$\mathtt{G} = \mathtt{G}_{+} + \mathtt{G}_{-}, \tag{2.116}$$

with $\mathtt{G}_{\pm}(u, v) = \frac{1}{2}[\mathtt{G}(u, v) \pm \mathtt{G}(v, u)]$, for every $u, v \in \boldsymbol{V}$.

Then, correspondingly, its associated endomorphism G will be written as the sum of a symmetric and a skew-symmetric endomorphism, i.e.,

$$G = G_{+} + G_{-}, \tag{2.117}$$

with $G_{+}, G_{-} : \boldsymbol{V} \to \boldsymbol{V}$ standing for the endomorphisms associated to the bilinear forms $\mathtt{G}_{+}$ and $\mathtt{G}_{-}$, respectively. We see immediately that for a symmetric automorphism, i.e., $G = G_{+}$ we have $G_{+} \equiv G^{\dagger}$.

If $\boldsymbol{g} \equiv \boldsymbol{g}_{+} = \boldsymbol{g}^{\dagger}$ is a symmetric automorphism of $(\boldsymbol{V}, \overset{\circ}{g})$, the bilinear form $g \in T^2\mathbf{V}$ associated to it has all the properties of a metric tensor on $\boldsymbol{V}$ and in that case g can be used to define a new Clifford algebra $\mathcal{C}\ell(\boldsymbol{V}, g)$ associated to the pair $(\boldsymbol{V}, g)$.

This is done by associating to the bilinear form g [10] a *new* scalar product $\bullet \equiv \underset{\mathbf{g}}{\cdot}$ of vectors in the algebra in the space $\boldsymbol{V} \equiv \bigwedge^1 V$ related to the old scalar product by

$$u \bullet v := g(u, v) = \boldsymbol{g}(u) \cdot v \tag{2.118}$$

for every $u, v \in \boldsymbol{V}$.

Also, given $A_p = u_1 \wedge \ldots \wedge u_p, B_p = v_1 \wedge \ldots \wedge v_p \in \bigwedge^p V \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \overset{\circ}{g})$ we define in analogy with Definition 2.25

$$A_p \bullet B_p = \det(g(u_i, v_j)) = \det(\boldsymbol{g}(u_i) \cdot v_j) \tag{2.119}$$

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different Clifford products is an essential tool in the theory of the gravitational field as presented in [4], where this field is represented by the gauge metric extensor h (Sect. 2.8.1).

This new scalar product is extended to all of $\bigwedge V$ due to linearity and orthogonality and $A \bullet B = 0$ if $A \in \bigwedge^r V$ and $B \in \bigwedge^s V$, $r \neq s$. Also, we agree that if $a, b \in \mathbb{R} \equiv \bigwedge^0 V$ then $a \bullet b = ab$.

2.7.1 The Gauge Metric Extensor h

If it can be easily proved that $\mathcal{C}\ell(\boldsymbol{V}, g)$ will be *isomorphic* to the original Clifford algebra $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ if and only if there exists an automorphism $h : \boldsymbol{V} \to \boldsymbol{V}$ (called the gauge metric extensor) such that:

$$\boldsymbol{g}(u) \cdot v = h(u) \cdot h(v), \tag{2.120}$$

for every $u, v \in \boldsymbol{V}$. We will say that h is the square root of $\boldsymbol{g}$ (or by abuse of language of g) even when $h \neq h^{\dagger}$,

Eq. (2.120) implies that

$$\boldsymbol{g} = h^{\dagger} h. \tag{2.121}$$

We can prove (see below) that every *positive* symmetric transformation possesses at most $2n$ square roots, all of them being symmetric transformations, but only one being itself positive (see, e.g., [8]).

If Eq. (2.120) is satisfied, we can reproduce the Clifford product of $\mathcal{C}\ell(\boldsymbol{V}, g)$ into the algebra $\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ defining an operation

$$\vee \equiv \underset{\mathbf{g}}{} : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \times \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}),$$

$$A \vee B \equiv A \underset{\mathbf{g}}{} B = \underline{h}^{-1}((\underline{h}(A)\underline{h}(B)), \tag{2.122}$$

for every $A, B \in \mathcal{C}\ell(\boldsymbol{V}, g)$, where h^{-1} is the inverse of the automorphism of h, and $\underline{h} : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \hookleftarrow \bigwedge \boldsymbol{V} \to \bigwedge \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ is the extended of h defined according to Eq. (2.85). In particular, if $u, v \in \boldsymbol{V} \hookrightarrow \mathcal{C}\ell(\boldsymbol{V}, \mathsf{g})$ are covectors, then

$$u \underset{\mathbf{g}}{} v = u \bullet v + u \wedge v.$$

In addition, the product $\vee : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \times \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ satisfies all the properties of a Clifford product which we have stated previously.

We introduce also the contractions $\underset{\mathbf{g}}{\lrcorner} : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \times \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ and $\underset{\mathbf{g}}{\llcorner} : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \times \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$ induced by $\vee : \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \times \mathcal{C}\ell(\boldsymbol{V}, \mathring{g}) \to \mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$, in complete analogy to Definition 2.32, and we left to the reader to fill in the details.

Relations analogous to those given in the earlier section will be obtained, with the usual contraction product "⌟" replaced by this new one.

Furthermore, if we perform a change in the volume scale by introducing another volume n-vector $\tau_{\mathbf{g}} \in \bigwedge^n V$ such that

$$\tilde{\tau}_{\mathbf{g}} \bullet \tau_{\mathbf{g}} = (-1)^q,$$

then we can also define the analogous of the Hodge duality operation for this new Clifford product, by letting $\underset{\mathbf{g}}{\star} : \mathcal{C}\ell(V, \overset{\circ}{g}) \to \mathcal{C}\ell(V, \overset{\circ}{g})$ be given by:

$$A_r \wedge (\underset{\mathbf{g}}{\star} B_r) = (\tilde{A}_r \bullet B_r)\tau_{\mathbf{g}}, \tag{2.123}$$

for every $A_r, B_r \in \bigwedge^r V \hookrightarrow \mathcal{C}\ell(V, \overset{\circ}{g})$, $r = 0, \ldots, n$. Of course, the operator $\underset{\mathbf{g}}{\star}$ just defined satisfies relations that are analogous to those satisfied by the operator $\star (= \underset{\mathbf{g}}{\star})$ [see Eq. (2.28)], and we have:

$$\begin{aligned} \underset{\mathbf{g}}{\star} &= \operatorname{sgn}(\mathfrak{det} h)\underline{h}^{-1} \circ \underset{\overset{\circ}{g}}{\star} \circ \underline{h} \\ &\equiv \operatorname{sgn}(\mathfrak{det} h)\underline{h}^{-1} \star \underline{h} \end{aligned} \tag{2.124}$$

as can be verified (see Exercise 2.60)

Remark 2.57 For the case of two metrics, say $\mathbf{g}$ and $\boldsymbol{\eta}$ of the same Lorentz signature $(1, 3)$ such that $\mathbf{g}$ may be continuously deformed into $\boldsymbol{\eta}$ we have $\det h > 0$. In this case, we can write

$$\underset{\mathbf{g}}{\star} = \underline{h}^{-1} \circ \underset{\eta}{\star} \circ \underline{h}. \tag{2.125}$$

which we abbreviate as $\underset{\mathbf{g}}{\star} = \underline{h}^{-1} \underset{\eta}{\star} \underline{h}$.

2.7.2 *Relation Between* $\mathfrak{det}$ *and the Classical Determinant* $\det[T_{ij}]$

Let $\mathbf{V}$ be a real vector space of finite dimension n and V its dual. Using notations introduced previously, let $\overset{\circ}{\mathbf{g}}_E$ and $\overset{\circ}{g}_E$ be metrics of Euclidean signature and $\overset{\circ}{\mathbf{g}}$ and $\overset{\circ}{\overset{\circ}{g}}$ be metrics of signature (p, q) in $\mathbf{V}$ and V which are related as explained in Sect. 2.2. Let moreover $\mathbf{g} : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ be an arbitrary nondegenerate symmetric bilinear form on $\mathbf{V}$ and $g : V \times V \to \mathbb{R}$ a bilinear form on V such that if $\{\varepsilon^j\}$ is a basis of V dual to $\{\mathbf{e}_j\}$ (an arbitrary basis of $\mathbf{V}$), then $g_{ij}g^{jk} = \delta^k_j$, where $g_{ij} = \mathbf{g}(\mathbf{e}_i, \mathbf{e}_j)$

and $g^{jk} = g(\varepsilon^j, \varepsilon^k)$. Let also $\{\varepsilon_j\}$ be the reciprocal basis of $\{\varepsilon^j\}$ with respect to the Euclidean metric $\overset{\circ}{g}_E$. Let also $\mathfrak{t} \in ext(\bigwedge^1 V, \bigwedge^1 V)$ be the unique extensor that corresponds to $T \in T^2\mathbf{V}$, such that for any $a, b \in \bigwedge^1 V$ we have $T(a,b) = t(a) \underset{\mathbf{g}_E}{\cdot} b$ and $t \in ext(\bigwedge^1 V, \bigwedge^1 V)$ be the unique extensor that corresponds to $T \in T^2\mathbf{V}$, such that for any $a, b \in \bigwedge^1 V$ we have $T(a,b) = \mathfrak{t}(a) \underset{\mathbf{g}}{\cdot} b$. Define $T_{ij} = T(\varepsilon_i, \varepsilon_j)$.

Exercise 2.58 Show that the classical determinant of the matrix $[T_{ij}]$ denoted $\det[T_{ij}]$ and $\underset{\mathbf{g}_E}{\mathfrak{det}}\,\mathfrak{t}$ and $\underset{\mathbf{g}}{\mathfrak{det}}\,t$ are related by:

$$\begin{aligned}\det[T_{ij}] &= \underset{\mathbf{g}_E}{\mathfrak{det}}\,t(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \underset{\mathbf{g}_E}{\cdot} (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \\ &= \underset{\mathbf{g}}{\mathfrak{det}}\,\mathfrak{t}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \underset{\mathbf{g}}{\cdot} (\varepsilon_1 \wedge \cdots . \wedge \varepsilon_n).\end{aligned} \tag{2.126}$$

Solution We show the first line of Eq. (2.126). Recall that for any $v_1, v_2, \ldots, w_1, \ldots, w_n \in V$ we have

$$(v_1 \wedge \cdots \wedge v_k) \underset{\mathbf{g}_E}{\cdot} (w_1 \wedge \cdots \wedge w_k) = \epsilon^{s_1 \ldots s_k} v_1 \underset{\mathbf{g}_E}{\cdot} w_{s_1} \cdots v_k \underset{\mathbf{g}_E}{\cdot} w_{s_k} \tag{2.127}$$

and the property $t(v_1 \wedge \cdots \wedge v_k) = t(v_1) \wedge \cdots \wedge t(v_k)$. By definition of classical determinant of $n \times n$ real matrix we have

$$\begin{aligned}\det[T_{jk}] &= \epsilon^{s_1 \cdots s_n} T_{1s_1} \cdots T_{ns_n} = \epsilon^{s_1 \ldots s_n} T(\varepsilon_1, \varepsilon_{s_1}) \cdots T(\varepsilon_n, \varepsilon_{s_n}) \\ &= \epsilon^{s_1 \ldots s_n} t(\varepsilon_1) \underset{\mathbf{g}_E}{\cdot} \varepsilon_{s_1} \cdots t(\varepsilon_n) \underset{\mathbf{g}_E}{\cdot} \varepsilon_{s_n} = (t(\varepsilon_1) \wedge \cdots \wedge t(\varepsilon_n)) \underset{\mathbf{g}_E}{\cdot} (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \\ &= \underline{t}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \underset{\mathbf{g}_E}{\cdot} (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n),\end{aligned} \tag{2.128}$$

hence, by definition of $\underset{\mathbf{g}_E}{\mathfrak{det}}\,t$, i.e., $\underline{t}(I) = I \underset{\mathbf{g}_E}{\mathfrak{det}}\,t$ for all non-null pseudoscalar I, Eq. (2.126) follows.

Remark 2.59 Note, in particular, for future reference that writing $\det \mathbf{g} = \det[\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j)]$, $\det g = \det[g(\varepsilon^j, \varepsilon^k)]$ we have,

$$\begin{aligned}(\det \mathbf{g})^{-1} = \det g &= \underset{\mathbf{g}_E}{\mathfrak{det}}\,\boldsymbol{g}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \underset{\mathbf{g}_E}{\cdot} (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \\ &= \underset{\mathbf{g}}{\mathfrak{det}}\,\mathfrak{g}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n) \underset{\mathbf{g}}{\cdot} (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n).\end{aligned} \tag{2.129}$$

where $\boldsymbol{g} \in ext(\bigwedge^1 V, \bigwedge^1 V)$ is the unique extensor that corresponds to $g \in T^2\mathbf{V}$ such that $g(a,b) = \boldsymbol{g}(a) \underset{\mathbf{g}_E}{\cdot} b$ and $\mathfrak{g} \in ext(\bigwedge^1 V, \bigwedge^1 V)$ is the unique extensor that corresponds to $g \in T^2\mathbf{V}$ such that $g(a,b) = \mathfrak{g}(a) \underset{\mathbf{g}}{\cdot} b$.

Exercise 2.60 Prove Eq. (2.124).

Solution To prove Eq. (2.124) we need to take into account that for each invertible (1, 1)-extensor h such that $\boldsymbol{g} = h^\dagger h$ and for any $X, Y \in \bigwedge V$ the following identity holds

$$\underline{h}(X \underset{\mathring{\mathbf{g}}}{\lrcorner} Y) = \underline{h}(X) \underset{\mathring{\mathbf{g}}}{\lrcorner} \underline{h}(Y), \tag{2.130}$$

By definition

$$\underset{\mathbf{g}}{\star} X = \tilde{X} \underset{\mathbf{g}}{\lrcorner} \tau_{\mathbf{g}}, \quad \underset{\mathring{\mathbf{g}}}{\star} X = \tilde{X} \underset{\mathring{\mathbf{g}}}{\lrcorner} \tau_{\mathring{\mathbf{g}}}. \tag{2.131}$$

Also, since $\left|\det \mathring{\mathbf{g}}\right| = 1$, we have $\tau_{\mathbf{g}} = \sqrt{|\det \mathbf{g}|}\tau_{\mathring{\mathbf{g}}}$, $\det \mathbf{g} = (\mathfrak{det} h)^{-2}$ and

$$\underline{h}(\tau_{\mathbf{g}}) = (\mathfrak{det} h)\tau_{\mathring{\mathbf{g}}}. \tag{2.132}$$

Then

$$\begin{aligned}
\tilde{X} \underset{\mathbf{g}}{\lrcorner} \tau_{\mathbf{g}} &= \sqrt{|\det \mathbf{g}|}\tilde{X} \underset{\mathbf{g}}{\lrcorner} \tau_{\mathring{\mathbf{g}}} \\
&= \sqrt{|\det \mathbf{g}|}\underline{h}^{-1}\left(\underline{h}(\tilde{X}) \underset{\mathring{\mathbf{g}}}{\lrcorner} \underline{h}(\tau_{\mathring{\mathbf{g}}})\right) \\
&= \sqrt{|\det \mathbf{g}|}(\mathfrak{det} h)\underline{h}^{-1}\left(\underline{h}(\tilde{X}) \underset{\mathring{\mathbf{g}}}{\lrcorner} (\tau_{\mathring{\mathbf{g}}})\right) \\
&= \operatorname{sgn}(\mathfrak{det} h)\underline{h}^{-1} \star \underline{h}(X)
\end{aligned} \tag{2.133}$$

2.7.3 *Strain, Shear and Dilation Associated with Endomorphisms*

Recall that every linear transformation can be expressed as composition of elementary transformations of the types $R_a : \boldsymbol{V} \to \boldsymbol{V}$ and $S_{ab} : \boldsymbol{V} \to \boldsymbol{V}$, defined by: (see, e.g., [10]).

$$\begin{aligned}
R_a(u) &= -aua^{-1} \\
S_{ab}(u) &= u + (u \cdot a)b,
\end{aligned} \tag{2.134}$$

for every $u \in \boldsymbol{V}$, where $a, b \in \boldsymbol{V}$ are non-zero (co)vectors parametrizing the transformation and $a^{-1} = a/a^2 = a/(a \cdot a)$. Transformations of the type R_a are called elementary *reflections*. Recall also that it is a classical result proved in any good book of linear algebra that any isometry of $(\boldsymbol{V}, \overset{\circ}{g})$ can always be written as the composite of at most n such transformations. The skew-symmetric part of a transformation of the type "S_{ab}" will be denoted by $S_{[ab]}$. We have:

$$S_{[ab]}(u) = \frac{1}{2}\left(S_{ab}(u) - S_{ba}(u)\right) = \frac{1}{2} u \lrcorner (a \wedge b), \tag{2.135}$$

for every $u \in \boldsymbol{V}$. It is again an example of an extensor, this time mapping biforms in forms.

The symmetric part of a transformation of the type "S_{ab}" is called a *strain*; it is a *shear* in the $a \wedge b$-plane if $a \cdot b = 0$, or a *dilation* along a, if $a \wedge b = 0$. Obviously, a dilation along a direction a can be written more simply as:

$$S_a(u) = u + \kappa(u \cdot a)\frac{a}{a^2}, \tag{2.136}$$

for every $u \in \boldsymbol{V}$, where $\kappa \in \mathbb{R}$, $\kappa > -1$, is a scalar parameter. If $\kappa = 0$, then S_a is the identity map of $\boldsymbol{V}$, for any $a \in \boldsymbol{V}$. If $\kappa \neq 0$, then S_a is a contraction $(-1 < \kappa < 0)$ or a dilation $(\kappa > 0)$, in the direction of a, by a factor $1 + \kappa$.

Now, we can show that every *positive* symmetric transformation can always be written as the composite of dilations along at most n orthogonal directions. To see this, it is sufficient to remember that for any symmetric transformation g associated to g we can find an orthonormal basis $\{\varepsilon_\mu\}$ of $\boldsymbol{V}$ for which (aligned indices are not to be summed over)

$$\boldsymbol{g}(\varepsilon_\mu) = \lambda_{(\mu)}\varepsilon_\mu,$$

where $\lambda_{(\mu)} \in \mathbb{R}$ is the *eigenvalue* of $\boldsymbol{g}$ associated to the *eigenvector* ε_μ $(\mu = 1, \ldots, n)$. Then, defining

$$S_{\varepsilon_\mu}(u) = u + \kappa_{(\mu)}(u \cdot \varepsilon_\mu)\frac{\varepsilon_\mu}{\varepsilon_\mu^2}, \tag{2.137}$$

for every $u \in \boldsymbol{V}$, with $\kappa_{(\mu)} = \lambda_{(\mu)} - 1$, we get:

$$\boldsymbol{g} = S_1 \circ \cdots \circ S_n. \tag{2.138}$$

If the symmetric transformation $\boldsymbol{g}$ is in addition positive, i.e., $\boldsymbol{g} = hh^\dagger$, we have $\varepsilon_\mu \cdot \boldsymbol{g}(\varepsilon_\mu) = h(\varepsilon_\mu) \cdot h(\varepsilon_\mu) = \lambda_{(\mu)}\varepsilon_\mu \cdot \varepsilon_\mu$. Then, since the signature of a bilinear form is preserved by linear transformations (Sylvester's law of inertia), we conclude that:

$$\lambda_{(\mu)} = \frac{h(\varepsilon_\mu) \cdot h(\varepsilon_\mu)}{\varepsilon_\mu \cdot \varepsilon_\mu} > 0. \tag{2.139}$$

This means that in Eq. (2.137), $\kappa_{(\mu)} = \lambda_{(\mu)} - 1 > -1$ and therefore it satisfies the definition of dilation given by Eq. (2.136). Note also that the positive "square root" of $\boldsymbol{g}$ is given by $h = S_1^{1/2} \circ \cdots \circ S_n^{1/2}$, with

$$S_{\varepsilon_\mu}^{1/2}(u) = u + \zeta_{(\mu)}(u \cdot \varepsilon_\mu)\frac{\varepsilon_\mu}{\varepsilon_\mu^2}, \tag{2.140}$$

for every $u \in \boldsymbol{V}$, where $\zeta_{(\mu)} = -1 + \sqrt{\lambda_{(\mu)}}$.

With these results it is trivial to give an operational form to the product defined through Eq. (2.122), although eventually this may demand a great deal of algebraic manipulation.

We emphasize that although we have considered only the positive symmetric transformations in the developments above, the formalism can be adapted to more general transformations (see, e.g. [8]).

We give as example of the above formalism Table 2.1, where we listed the g's relating all metrics of signature (p, q) in a vector space $\boldsymbol{V}$, with $\dim \boldsymbol{V} = p + q = 4$, relative to a standard metric of Lorentzian signature. Note that if $\{\varepsilon^\mu\}$ is a basis of $\boldsymbol{V}$ with $\varepsilon^\mu \cdot \varepsilon^\nu = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ then

$$\boldsymbol{g}(v) = R_{(\varepsilon^{\mu_1} \ldots . \varepsilon^{\mu_r})} v, \tag{2.141}$$

where $R_{(\varepsilon^{\mu_1} \ldots . \varepsilon^{\mu_r})}$ is a product of r-reflections, each one in relation to the hyperplane orthogonal to ε^{μ_i}. Then, we have, e.g., $R_{\varepsilon^0}(v) = -\varepsilon^0 v \varepsilon^0, R_{\varepsilon^i}(v) = \varepsilon^i v \varepsilon^i, i = 1, 2, 3$. Also, the last column of Table 2.1 list the matrix algebra to which the corresponding Clifford algebra $\mathbb{R}_{p,q}$ is isomorphic (see Chap. 3 for details).

Table 2.1 Endomorphisms generating all Clifford algebras in 4-dimensions

$\boldsymbol{g}$	$\boldsymbol{g}(v)$	g^{00}	g^{11}	g^{22}	g^{33}	(p, q)	$\mathbb{R}_{p,q}$
$R_{(0)}$	$-\varepsilon^0 v \varepsilon^0$	-1	-1	-1	-1	$(0, 4)$	$\mathbb{H}(2)$
$R_{(1)}$	$\varepsilon^1 v \varepsilon^1$	$+1$	$+1$	-1	-1	$(2, 2)$	$\mathbb{R}(4)$
$R_{(01)}$	$-\varepsilon^0 \varepsilon^1 v \varepsilon^1 \varepsilon^0$	-1	$+1$	-1	-1	$(1, 3)$	$\mathbb{H}(2)$
$R_{(12)}$	$\varepsilon^1 \varepsilon^2 v \varepsilon^2 \varepsilon^1$	$+1$	$+1$	$+1$	-1	$(3, 1)$	$\mathbb{R}(4)$
$R_{(012)}$	$-\varepsilon^0 \varepsilon^1 \varepsilon^2 v \varepsilon^2 \varepsilon^1 \varepsilon^1$	-1	$+1$	$+1$	-1	$(2, 2)$	$\mathbb{R}(4)$
$R_{(123)}$	$\varepsilon^1 \varepsilon^2 \varepsilon^3 v \varepsilon^3 \varepsilon^2 \varepsilon^1$	$+1$	$+1$	$+1$	$+1$	$(4, 0)$	$\mathbb{H}(2)$
$R_{(0123)}$	$-\varepsilon^0 \varepsilon^1 \varepsilon^2 \varepsilon^3 v \varepsilon^3 \varepsilon^2 \varepsilon^1 \varepsilon^0$	-1	$+1$	$+1$	$+1$	$(3, 1)$	$\mathbb{R}(4)$

2.8 Minkowski Vector Space

We give now some details concerning the structure of Minkowski *vector* space which plays a fundamental role in the theories presented in this book.[8]

Definition 2.61 Let $\mathbf{V}$ be a real four dimensional space over the real field $\mathbb{R}$ and $\boldsymbol{\eta} : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ a metric of signature[9] -2. The pair $(\mathbf{V}, \boldsymbol{\eta})$ is called Minkowski vector space and is denoted by $\mathbb{R}^{1,3}$.

We recall that the above definition implies that there exists an orthonormal basis $b = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^{1,3}$ such that

$$\boldsymbol{\eta}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \boldsymbol{\eta}(e_\nu, \mathbf{e}_\mu) = \eta_{\mu\nu} = \begin{cases} 1 \text{ if } & \mu = \nu = 0, \\ -1 \text{ if } & \mu = \nu = 1, 2, 3, \\ 0 \text{ if } & \mu \neq \nu. \end{cases} \tag{2.142}$$

We use in what follows the short notation $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

The notation

$$\boldsymbol{\eta}(\mathbf{u}, \mathbf{w}) = \mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = \boldsymbol{\eta}(\mathbf{w}, \mathbf{u}), \tag{2.143}$$

where the symbol $\cdot$ as usual is called a scalar product will also be used. Also, for any $\mathbf{u} \in \mathbb{R}^{1,3}$, we write $\mathbf{u} \cdot \mathbf{u} = \mathbf{u}^2$.

We recall that given a basis $b = \{\mathbf{e}_\mu\}$ of $\mathbf{V}$ the basis $\check{b} = \{\mathbf{e}^\mu\}$ of $\mathbf{V}$ such that

$$\mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu = \begin{cases} 1 \text{ if } \mu = \nu, \\ 0 \text{ if } \mu \neq \nu. \end{cases} \tag{2.144}$$

is called the *reciprocal basis* of b.

Note that $\mathbf{e}^\mu \cdot \mathbf{e}^\nu = \eta^{\mu\nu}$, with the matrix with entries $\eta^{\mu\nu}$ being the matrix $\text{diag}(1, -1, -1, -1)$.

The *dual space* of $\mathbf{V}$ in the definition of the Minkowski vector space is as usual denoted by $\boldsymbol{V}$. The *dual basis* of a basis $b = \{\mathbf{e}_\mu\}$ of $\mathbb{R}^{1,3}$ is the set $b^* = \{\boldsymbol{\varepsilon}^\mu\}$ such that $\varepsilon^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu$. We introduce in $\boldsymbol{V}$ a metric of signature -2, $\eta : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, such that

$$\eta(\varepsilon^\mu, \varepsilon^\nu) = \eta^{\mu\nu}. \tag{2.145}$$

The pair $(\boldsymbol{V}, \eta)$ will be denoted by $\mathbb{R}^{*1,3}$. We write $\eta(u, v) = u \cdot v$. Recall also that the reciprocal basis of the basis $b^* = \{\varepsilon^\mu\}$ of $\boldsymbol{V}$ is the basis $\check{b}^* = \{\varepsilon_\mu\}$ of $\boldsymbol{V}$

[8]The proofs of the propositions of this section are left to the reader.

[9]The signature of a metric tensor is sometimes defined as the number $(p - q)$.

such that

$$\varepsilon_\mu \cdot \varepsilon^\nu = \delta^\mu_\nu. \tag{2.146}$$

The metric $\boldsymbol{\eta}$ induces an isomorphism between $\mathbf{V}$ and $\boldsymbol{V}$ given by

$$\mathbf{V} \ni \mathbf{a} \mapsto a = \boldsymbol{\eta}(\mathbf{a},) \in \boldsymbol{V}, \tag{2.147}$$

such that for any $\mathbf{w} \in \mathbf{V}$ we have

$$a(\mathbf{w}) = \boldsymbol{\eta}(\mathbf{a}, \mathbf{w}). \tag{2.148}$$

Of course the structures $\mathbb{R}^{1,3} = (\mathbf{V}, \boldsymbol{\eta})$ and $\mathbb{R}^{*1,3} = (\boldsymbol{V}, \eta)$ are also isomorphic. To see this it is enough to verify that if $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ and $v = \boldsymbol{\eta}\ (\mathbf{v},), w = \boldsymbol{\eta}\ (\mathbf{w},) \in \boldsymbol{V}$ then, $\eta(u, v) = \boldsymbol{\eta}\ (\mathbf{v}, \mathbf{w})$.

Definition 2.62 Let $\mathbf{v} \in \mathbf{V}$, then we say that $\mathbf{v}$ is spacelike if $\mathbf{v}^2 < 0$ or $\mathbf{v} = 0$, that $\mathbf{v}$ is lightlike if $\mathbf{v}^2 = 0$ and $\mathbf{v} \neq 0$, and that $\mathbf{v}$ is timelike if $\mathbf{v}^2 > 0$. This classification gives the causal character of $\mathbf{v}$. An analogous terminology are used to classify the elements of $\boldsymbol{V}$.

When $\mathbf{v} \in V$ is timelike, we denote by $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ the *norm* of $\mathbf{v}$.

Remark 2.63 From nowhere when no confusion arises we write $\mathbf{u} \in \mathbf{V}$ as $\mathbf{u} \in \mathbb{R}^{1,3}$. That notation emphasize that $\mathbf{u}$ is an element of $\mathbf{V}$ and it can be classified as a spacelike or lightlike or timelike vector. Also, within the same spirit we write when no confusion arises $u \in \boldsymbol{V}$ as $u \in \mathbb{R}^{*1,3}$. The same notation will be used also for subspaces $\boldsymbol{S}$ of $\mathbf{V}$ (S of $\boldsymbol{V}$), i.e., we write $\boldsymbol{S} \subset \mathbf{V}$ ($S \subset \boldsymbol{V}$) as $\boldsymbol{S} \subset \mathbb{R}^{1,3}$ ($S \subset \mathbb{R}^{*1,3}$).

Definition 2.64 Let $S \subset \mathbb{R}^{*1,3}$ be a subspace. We say that S is spacelike if all its covectors are spacelike, that S is lightlike if it contains a lightlike covector but no timelike vector, and S is timelike if it contains a timelike covector.

Definition 2.64 establish that a given subspace $S \subset \mathbb{R}^{*1,3}$ is necessarily spacelike or lightlike or timelike. We now present without proofs some propositions that will give us some insight on the linear algebra of $\mathbb{R}^{*1,3}$.

Proposition 2.65 *A subspace $S \subset \mathbb{R}^{*1,3}$ is timelike if and only if its orthogonal complement is spacelike.*

Proposition 2.66 *Let $S \subset \mathbb{R}^{*1,3}$ be a lightlike subspace. Then its orthogonal complement $S^\perp$ is lightlike and $S \cap S^\perp \neq \{0\}$.*

Proposition 2.67 *Two lightlike covectors, $n_1, n_2 \in \mathbb{R}^{*1,3}$ are orthogonal if and only if they are proportional.*

Proposition 2.68 *There are only two orthogonal spacelike covectors that are orthogonal to a lightlike covector.*

Proposition 2.69 *The unique way to construct an orthonormal basis for* $\mathbb{R}^{*1,3}$ *is with one timelike covector and three spacelike covectors.*

The next proposition shows how to divide the set $\mathfrak{T} \subset \mathbb{R}^{*1,3}$ of all timelike covectors in two disjoint subsets $\mathfrak{T}^+$ and $\mathfrak{T}^-$, which are called respectively *future* pointing and *past* point timelike vectors.

Proposition 2.70 *Let* $u, v \in \mathfrak{T}$. *The relation* $\uparrow$ *on the set* $\mathfrak{T} \times \mathfrak{T}$ *such that* $u \uparrow v$ *if and only if* $u \cdot v > 0$ *is an equivalence relation and it divides* $\mathfrak{T}$ *in two disjoint equivalence classes* $\mathfrak{T}^+$ *and* $\mathfrak{T}^-$.

Remark 2.71 Such equivalence relation is used to define a time orientation in spacetime structures (see Chap. 4).

Proposition 2.72 $\mathfrak{T}^+$ *and* $\mathfrak{T}^-$ *are convex sets.*

Recall that $\mathfrak{T}^+$ to be convex means that given any $u, w \in \mathfrak{T}^+$ and $a \in (0, \infty)$ and $b \in [0, \infty)$ then $(au + bw) \in \mathfrak{T}^+$.

Proposition 2.73 *Let* $u, v \in \mathfrak{T}^+$. *Then they satisfy the anti-Schwarz inequality, i.e.,* $|u \cdot v| \geq ||u||\ ||v||$ *and the equality only occurs if* u *and* v *are proportional.*

We end this section presenting the anti-Minkowski inequality, which is the basis for a trivial solution [16] of the "twin paradox", once we introduce a postulate (see Axiom 6.1, Chap. 6) concerning the behavior of ideal clocks.

Proposition 2.74 *Let* $u, v \in \mathfrak{T}^+$. *Then we have*

$$||u + v|| \geq ||u|| + ||v||, \tag{2.149}$$

which will be called anti-Minkowski inequality.

Proposition 2.75 *Let* u, v *be spacelike covectors such that the* span$[u, v]$ *is spacelike. Then the usual Schwarz and Minkowski inequalities* $|u \cdot v| \leq ||u||\ ||v||$ *and* $||u + v|| \leq ||u|| + ||v||$ *hold. If the equalities hold then* u *and* v *are proportional.*

Proposition 2.76 *Let* u, v *be spacelike covectors such that the* span$[u, v]$ *is timelike. Then the anti-Schwarz inequality* $|u \cdot v| \geq ||u||\ ||v||$ *holds. If the equality holds then* u *and* v *are proportional. If* $u \cdot v \leq 0$ *and* $u + v$ *is spacelike then the anti-Minkowski inequality* $||u + v|| \geq ||u|| + ||v||$ *holds, the equality holding if* u, v *are proportional.*

Exercise 2.77 Prove Propositions 2.65–2.76.

2.8.1 Lorentz and Poincaré Groups

Definition 2.78 A Lorentz transformation $\mathbf{L} : \boldsymbol{V} \to \boldsymbol{V}$ acting on the Minkowski (co)vector space $\mathbb{R}^{*1,3} = (\boldsymbol{V}, \eta)$ is a Lorentz extensor, i.e., an element of

$ext(\bigwedge^1 \boldsymbol{V}, \bigwedge^1 \boldsymbol{V}) \simeq \boldsymbol{V} \otimes \mathbf{V}$ such that for any $u, v \in \boldsymbol{V}$ we have

$$\eta(\mathbf{L}u, \mathbf{L}v) = \eta(u, v). \tag{2.150}$$

The set of all Lorentz transformations has a group structure under composition of mappings. We denote such group by $\mathrm{O}_{1,3}$. As it is well known these transformations can be represented by 4×4 real matrices, once a basis in $\boldsymbol{V}$ is chosen. Equation (2.150) then implies that

$$\det \mathbf{L} = \pm \mathbf{1}. \tag{2.151}$$

Transformations such that $\det \mathbf{L} = -1$ are called *improper* and the ones with $\det \mathbf{L} = 1$ are called *proper*. Improper transformations, of course do not close in a group. The subset of proper Lorentz transformations define a group denoted $\mathrm{SO}_{1,3}$. Finally, the subset of $\mathrm{SO}_{1,3}$ connected with the identity is also a group. We denote it by $\mathrm{SO}^e_{1,3}$. This group also denoted by $\mathcal{L}^{\uparrow}_{+}$ is called the proper orthochronous Lorentz group. The reason for the adjective orthochronous becomes obvious once you solve the next exercise.

Exercise 2.79 Show that if $\mathbf{L} \in \mathrm{SO}^e_{1,3}$ and $u \in \mathfrak{T}^+ \subset \boldsymbol{V}$, then $\mathbf{L}u {\uparrow} u$. Also show that if $v {\in} \mathfrak{T}^-$, then $\mathbf{L}v {\uparrow} v$.[10]

Definition 2.80 Let $\mathbf{L}$ be a Lorentz transformation and let $a \in \boldsymbol{V}$. A Poincaré transformation acting on Minkowski vector space $\mathbb{R}^{1,3}$ is a mapping $\mathbf{P} : \boldsymbol{V} \to \boldsymbol{V}$, $v \mapsto \mathbf{P}v = \mathbf{L}v + a$.

A Poincaré transformation is denoted by $\mathbf{P} = (\mathbf{L}, a)$. The set of all Poincaré transformations define a group denoted by $\mathcal{P}$—called the Poincaré group—under the multiplication rule (group composition law) given by

$$(\mathbf{L}_1, a)(\mathbf{L}_2, b)v = (\mathbf{L}_2\mathbf{L}_1, \mathbf{L}_1 b + a)\mathbf{v} = \mathbf{L}_2\mathbf{L}_1 v + \mathbf{L}_1 b + a. \tag{2.152}$$

This composition law says that the Poincaré group is the *semi direct product* ($\boxtimes$) of the Lorentz group $\mathrm{O}_{1,3}$ by the translation group. We have

$$\mathcal{T} = \{(\mathbf{1}, a), \mathbf{1} \in \mathrm{O}_{1,3}, a {\in} \boldsymbol{V}\}, \tag{2.153}$$

and we write $\mathcal{P} = \mathrm{O}_{1,3} \boxtimes \mathcal{T}$. We also denote by $\mathcal{P}^{\uparrow} = \mathrm{SO}_{1,3} \boxtimes \mathcal{T}$ and by $\mathcal{P}^{\uparrow}_{+} = \mathrm{SO}^e_{1,3} \boxtimes \mathcal{T}$.

The theory of representations of the Lorentz and Poincaré groups is an essential ingredient of relativistic quantum field theory. We shall need only some few results of these theories in this book and that results are present at the places where they are needed.

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[10] If the reader has any difficulty in solving that exercise he must consult, e.g., [17].

Exercise 2.81 Show that if $u, v \in \boldsymbol{V}$, and $\mathbf{P} = (\mathbf{L}, a)$ is a Poincaré transformation then $\mathbf{P}(u - v) \cdot \mathbf{P}(u - v) = (u - v) \cdot (u - v)$.

2.9 Multiform Functions

2.9.1 Multiform Functions of a Real Variable

Definition 2.82 A mapping $Y : I \to \bigwedge \boldsymbol{V}$ (with $I \subseteq \mathbb{R}$) is called a multiform function of real variable [14].

For simplicity when the image of Y is a scalar (or 1-form, or 2-form , ..., etc.) the multiform function $\lambda \mapsto Y(\lambda)$ is said respectively scalar, (or 1-form, or 2-form , ..., etc.) function of real variable.

Limit and Continuity

The notions of limit and continuity for multiform functions can be easily introduced, following a path analogous to the one used in the case of ordinary real functions.

Definition 2.83 A multiform $L \in \bigwedge \boldsymbol{V}$ is said to be the limit of $Y(\lambda)$ for $\lambda \in I$ approaching $\lambda_0 \in I$, if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that[11] $\|Y(\lambda) - L\| < \varepsilon$, if $0 < |\lambda - \lambda_0| < \delta$. We write $\lim\limits_{\lambda \to \lambda_0} Y(\lambda) = L$.

We prove without any difficulty that:

$$\lim_{\lambda \to \lambda_0} (Y(\lambda) + Y(\lambda)) = \lim_{\lambda \to \lambda_0} Y(\lambda) + \lim_{\lambda \to \lambda_0} Y(\lambda). \tag{2.154}$$

$$\lim_{\lambda \to \lambda_0} (Y(\lambda) \circledast Y(\lambda)) = \lim_{\lambda \to \lambda_0} Y(\lambda) \circledast \lim_{\lambda \to \lambda_0} Y(\lambda), \tag{2.155}$$

where $\circledast$ is any one of the product of multiforms, i.e., $(\wedge), (\cdot), (\lrcorner, \llcorner)$ or (*Clifford product*).

$Y(\lambda)$ is said *continuous at* $\lambda_0 \in I$ if and only if $\lim\limits_{\lambda \to \lambda_0} Y(\lambda) = Y(\lambda_0)$. Then, sum and products of any continuous multiform functions are also continuous.

Definition 2.84 The derivative of Y *in* λ_0 is

$$Y'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{Y(\lambda) - Y(\lambda_0)}{\lambda - \lambda_0}. \tag{2.156}$$

[11] Recall that the norm of a multiform $X \in \bigwedge V$ is defined by $\|X\| = \sqrt{X \cdot X}$, where the symbol $\cdot$ refers to the Euclidean scalar product.

The following rules are valid:

$$(Y + Y)' = Y' + Y' \tag{2.157}$$

$$(Y \circledast Y)' = Y' \circledast Y + Y \circledast Y' \ (\textit{Leibniz's rule}) \tag{2.158}$$

$$(Y \circ \phi)' = (Y' \circ \phi)\phi' \ (\textit{chain rule}), \tag{2.159}$$

where $\circledast$ is any one of the product of multiforms, i.e., $(\wedge), (\cdot), (\lrcorner, \llcorner)$ or (*Clifford product*) and ϕ is an ordinary real function (ϕ' is the derivative of ϕ). We also write sometimes $\dfrac{d}{d\lambda}Y(\lambda) = Y'(\lambda)$.

We can generalize easily all ideas above for the case of multiform functions of several real variables, and when need such properties will be used.

2.10 Multiform Functions of Multiform Variables

Definition 2.85 A mapping $F : \bigwedge^{\diamond} \boldsymbol{V} \to \bigwedge \boldsymbol{V}$, is said a multiform function of multiform variable.

If $F(Y)$ is a scalar (or 1-form, or 2-form, ...) then F is said scalar (or 1-form, or 2-form,...) function of multiform variable.

2.10.1 Limit and Continuity

Definition 2.86 $M \in \bigwedge \boldsymbol{V}$ is said the limit of F for $Y \in \bigwedge^{\diamond} \boldsymbol{V}$ *approaching* $Y_0 \in \bigwedge^{\diamond} \boldsymbol{V}$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|F(Y) - M\| < \varepsilon$, if $0 < \|Y - Y_0\| < \delta$. We write $\lim\limits_{Y \to Y_0} F(Y) = M$.

The following properties are easily proved:

$$\lim_{Y \to Y_0} (F(Y) + G(Y)) = \lim_{Y \to Y_0} F(Y) + \lim_{Y \to Y_0} G(Y). \tag{2.160}$$

$$\lim_{Y \to Y_0} (F(Y) \circledast G(Y)) = \lim_{Y \to Y_0} F(Y) \circledast \lim_{Y \to Y_0} G(Y), \tag{2.161}$$

where $\circledast$ is any one of the product of multiforms, i.e., $(\wedge), (\cdot), (\lrcorner, \llcorner)$ or (*Clifford product*).

Definition 2.87 Let $F : \bigwedge^{\diamond} \boldsymbol{V} \to \bigwedge \boldsymbol{V}$. F is said to be continuous at $Y_0 \in \bigwedge^{\diamond} \boldsymbol{V}$ if and only if $\lim\limits_{Y \to Y_0} F(Y) = F(Y_0)$.

Of course, the sum $Y \mapsto (F+G)(Y) = F(Y)+G(Y)$ or *any* product $Y \mapsto (F \circledast G)(Y) = F(Y) \circledast G(Y)$ of continuous multiform functions of multiform variable are continuous.

2.10.2 Differentiability

Definition 2.88 $F : \bigwedge^{\diamond} \boldsymbol{V} \to \bigwedge \boldsymbol{V}$ is said to be differentiable at $Y_0 \in \bigwedge^{\diamond} \boldsymbol{V}$ if and only if there exists an extensor over $\boldsymbol{V}$, say, $f_{Y_0} \in ext\text{-}(\boldsymbol{V})$, such that

$$\lim_{Y \to Y_0} \frac{F(Y) - F(Y_0) - f_{Y_0}(Y - Y_0)}{\|Y - Y_0\|} = 0. \tag{2.162}$$

Remark 2.89 It is possible to show [15] that if such f_{Y_0} *exists*, it must be unique.

Definition 2.90 f_{Y_0} is called the differential (extensor) of F *at* Y_0.

2.11 Directional Derivatives

Let F be any multiform function of multiform variable which is *continuous* at Y_0. Let $\lambda \neq 0$ be any real number and A an arbitrary multiform. Then, there exists the multiform

$$\lim_{\lambda \to 0} \frac{F(Y_0 + \lambda \langle A \rangle_{Y_0}) - F(Y_0)}{\lambda}. \tag{2.163}$$

Definition 2.91 The limit in Eq. (2.163) denoted by $F'_A(Y_0)$ or $(A \cdot \partial_Y)F(Y_0)$ is called the directional derivative of F at Y_0 in the direction of the multivector A [15]. The operator $A \cdot \partial_Y$ is said the directional derivative with respect to A and this notation, i.e., $A \cdot \partial_Y$ is justified in Remark 2.102. We write

$$F'_A(Y_0) = (A \cdot \partial_Y)F(Y_0) = \lim_{\lambda \to 0} \frac{F(Y_0 + \lambda \langle A \rangle_{Y_0}) - F(Y_0)}{\lambda}, \tag{2.164}$$

or more conveniently

$$F'_A(Y_0) = (A \cdot \partial_Y)F(Y_0) = \frac{d}{d\lambda} F(Y_0 + \lambda \langle A \rangle_{Y_0})\bigg|_{\lambda=0}. \tag{2.165}$$

The directional derivative of a multiform function of multifunction variable is linear with respect to the multiform direction, i.e., for all $\alpha, \beta \in \mathbb{R}$ and $A, B \in \bigwedge \boldsymbol{V}$

$$F'_{\alpha A + \beta B}(Y) = \alpha F'_A(Y) + \beta F'_B(Y), \tag{2.166}$$

or

$$(\alpha A + \beta B) \cdot \partial_Y F(Y) = \alpha A \cdot \partial_Y F(Y) + \beta B \cdot \partial_Y F(Y). \tag{2.167}$$

The following propositions are true and their proofs may be found in [15].

Remark 2.92 When there is no risk of confusion we write $(A \cdot \partial_Y)F(Y)$ simply as $A \cdot \partial_Y F$.

Proposition 2.93 *Let $Y \mapsto F(Y)$ and $Y \mapsto G(Y)$ be any two differentiable multiform functions. Then, the sum $Y \mapsto (F + G)(Y) = F(Y) + G(Y)$ and the products $Y \mapsto (F \circledast G)(Y) = F(Y) \circledast G(Y)$, where $\circledast$ means $(\wedge), (\cdot), (\lrcorner, \llcorner)$ or (Clifford product)* of *differentiable multiform functions. Also,*

$$(F + G)'_A(Y) = F'_A(Y) + G'_A(Y). \tag{2.168}$$

$$(F \circledast G)'_A(Y) = F'_A(Y) \circledast G(Y) + F(Y) \circledast G'_A(Y) \text{ (Leibniz's rule)}. \tag{2.169}$$

2.11.1 Chain Rules

Proposition 2.94 *Let $Y \mapsto F(Y)$ and $Y \mapsto G(Y)$ be any two differentiable multiform functions. Then, the composite multiform function $Y \mapsto (F \circ G)(Y) = F(\langle G(Y) \rangle_Y)$, with $Y \in domF$, is a differentiable multiform function and*

$$(F \circ G)'_A(Y) = F'_{G'_A(Y)}(\langle G(Y) \rangle_Y). \tag{2.170}$$

Proposition 2.95 *Let $Y \mapsto F(Y)$ and $\lambda \mapsto Y(\lambda)$ be two differentiable multiform functions, the first one a multiform function of multivector variable and the second one a multiform function of real variable. Then, the composite function $\lambda \mapsto (F \circ Y)(\lambda) = F(\langle Y(\lambda) \rangle_Y)$, with $Y \in domF$, is a multiform function of real variable and*

$$(F \circ Y)'(\lambda) = F'_{Y'(\lambda)}(\langle Y(\lambda) \rangle_Y). \tag{2.171}$$

Proposition 2.96 *Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \bigwedge^\diamond \boldsymbol{V} \to \mathbb{R}$ be respectively an ordinary differentiable real function and a differentiable multiform function. The composition $\phi \circ \Psi : \bigwedge^\diamond \boldsymbol{V} \to \mathbb{R}$, $(\phi \circ \Psi)(Y) = \phi(\Psi(Y))$ is a differentiable scalar function of multiform variable and*

$$(\phi \circ \Psi)'_A(Y) = \phi'(\Psi(Y))\Psi'_A(Y). \tag{2.172}$$

The above formulas can also be written

$$A \cdot \partial_Y (F + G)(Y) = A \cdot \partial_Y F(Y) + A \cdot \partial_Y G(Y). \tag{2.173}$$

$$A \cdot \partial_Y (F \circledast G)(Y) = A \cdot \partial_Y F(Y) \circledast G(Y) + F(Y) \circledast A \cdot \partial_Y G(Y). \tag{2.174}$$

$$A \cdot \partial_Y (F \circ G)(Y) = A \cdot \partial_Y G(Y) \cdot \partial_Y F(\langle G(Y) \rangle_Y), \quad \text{with } Y \in domF. \tag{2.175}$$

$$\frac{d}{d\lambda}(F \circ Y)(\lambda) = \frac{d}{d\lambda} Y(\lambda) \cdot \partial_Y F(\langle Y(\lambda) \rangle_Y), \text{ with } Y \in domF. \tag{2.176}$$

$$A \cdot \partial_Y (\phi \circ \Psi)(Y) = \frac{d}{d\mu} \phi(\Psi(Y)) A \cdot \partial_Y \Psi(Y). \tag{2.177}$$

2.11.2 Derivatives of Multiform Functions

Let F be any multiform function of multiform variable, which is differentiable at Y. Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be an arbitrary pair of reciprocal basis for $\boldsymbol{V}$ (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). Then, there exists a well defined multiform (i.e., independent of the pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$ and depending only on F)

$$\sum_J \frac{1}{\nu(J)!} \varepsilon^J F'_{\varepsilon_J}(Y) \equiv \sum_J \frac{1}{\nu(J)!} \varepsilon_J F'_{\varepsilon^J}(Y). \tag{2.178}$$

The symbol J denotes collective indices. Recall, e.g., that

$$\begin{aligned} \varepsilon_J &= 1, \varepsilon_{j_1}, \ldots, \varepsilon_{j_1 j_2 \cdots j_n} = \varepsilon_{j_1} \wedge \varepsilon_{j_2} \wedge \cdots \varepsilon_{j_n}, \\ \varepsilon^J &= 1, \varepsilon^{j_1}, \ldots, \varepsilon^{j_1 j_2 \cdots j_n} = \varepsilon^{j_1} \wedge \varepsilon^{j_2} \wedge \cdots \wedge \varepsilon^{j_n}. \end{aligned} \tag{2.179}$$

and $\nu(J) = 0, 1, 2, \ldots$ for $J = \varnothing, j_1, j_1 j_2 \cdots j_n, \ldots$ where all indices $j_1, j_2, \ldots, j_n$ run from 1 to n.

Definition 2.97 The multiform given by Eq. (2.178) will be denoted by $F'(Y)$ or $\partial_Y F(Y)$ and is called [8, 14] the derivative of F at Y. We write

$$F'(Y) = \partial_Y F(Y) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J F'_{\varepsilon_J}(Y). \tag{2.180}$$

2.11.3 Properties of $\partial_Y F$

Proposition 2.98

(a) *Let $Y \mapsto \Phi(Y)$ be a scalar function of multiform variable. Then,*

$$A \cdot \partial_Y \Phi(Y) = A \cdot (\partial_Y \Phi(Y)). \tag{2.181}$$

(b) *Let* $Y \mapsto F(Y)$ *and* $Y \mapsto G(Y)$ *be differentiable multiform functions. Then,*

$$\partial_Y(F+G)(Y) = \partial_Y F(Y) + \partial_Y G(Y). \tag{2.182}$$

$$\partial_Y(FG)(Y) = \partial_Y F(Y)G(Y) + F(Y)\partial_Y G(Y) \text{ (Leibniz's rule).} \tag{2.183}$$

Remark 2.99 In field theories formulated in terms of differential forms *(see Chap. 9)* the directional derivative of a multiform function, say $F : \bigwedge^r \ni X \mapsto F(X) \in \bigwedge^n$ in the direction of $W = \boldsymbol{\delta} X \in \bigwedge^r V$ is written as

$$\boldsymbol{\delta} F := \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} \tag{2.184}$$

called the *variational derivative* of F and $\frac{\partial F}{\partial X}$ is called the algebraic derivative of F. Now, since $\boldsymbol{\delta} F = W \cdot \partial_X$ we can, e.g., verify that for $F = X \wedge \star X$ we have the identification

$$\boldsymbol{\delta} X \cdot \partial_X F = \boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} = (-1)^{\frac{r}{2}(r-1)} \boldsymbol{\delta} X \wedge \partial_X F. \tag{2.185}$$

More details may be found in [4].

2.11.4 *The Differential Operator* $\boldsymbol{\partial_Y \circledast}$

Definition 2.100 Given a multiform function of multiform variable, we define the differential operator [15] $\partial_Y \circledast$ by

$$\partial_Y \circledast F(Y_0) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \circledast \varepsilon_J \cdot \partial_Y F(Y_0), \tag{2.186}$$

where $\circledast$ is any one of the products $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (Clifford product).

∂_Y (the derivative operator) is also called the generalized gradient operator and $\partial_Y F$ is said to be the generalized gradient of F. Also $\partial_Y \wedge$ is called the generalized rotational operator and $\partial_Y \wedge F$ is said the generalized rotational of F. Finally the operators $\partial_Y \cdot$ and $\partial_Y \lrcorner$ are called generalized divergent operators and $\partial_Y \cdot F$ is said the generalized scalar divergent of F and $\partial_Y \lrcorner F$ is said the generalized contracted divergent of F.

Note that these differential operators are well defined since the right side of Eq. (2.186) depends only on F and are independent of the pair of reciprocal bases used.

2.11.5 The $A \circledast \partial_Y$ Directional Derivative

Definition 2.101 The $A \circledast \partial_Y$ directional derivative of a differentiable multiform function $F(Y)$ is

$$A \circledast \partial_Y F(Y) = \sum_J \frac{1}{\nu(J)!}(A \circledast \varepsilon^J) F'_{\varepsilon_J}(Y) \tag{2.187}$$

where $\circledast$ is any one of the four products ($\wedge$), ($\cdot$), ($\lrcorner$), ($\llcorner$) or the Clifford product. The symbol J denotes collective indices.

Remark 2.102 We already introduced above the $A \cdot \partial_Y$ directional derivative. The other derivatives will appear in our formulation of the Lagrangian formalism in Chaps. 7 and 8. We see moreover that the symbol $A \cdot \partial_Y$ for the directional derivative is well justified. Indeed, from Eq. (2.187) using $\cdot$ in the place of $\circledast$ we have

$$\begin{aligned} A \cdot \partial_Y F(Y) &= \sum_J \frac{1}{\nu(J)!}(A \cdot \varepsilon^J) F'_{\varepsilon_J}(Y) \\ &= \sum_J \frac{1}{\nu(J)!} A^J F'_{\varepsilon_J}(Y) \\ &= F'_{\sum_J \frac{1}{\nu(J)!} A^J \varepsilon_J}(Y) \\ &= F'_A(Y). \end{aligned} \tag{2.188}$$

We now solve some exercises dealing with calculations of directional derivatives and derivatives of some multiform functions that will be used in the main text, in particular in the Lagrangian formulation of multiform and extensor fields over a Lorentzian spacetime (see Chaps. 8 and 9).

2.12 Solved Exercises

Exercise 2.103 Let $\bigwedge^{\diamond} V \ni Y \mapsto Y \cdot Y \in R$, where $\bigwedge^{\diamond} V$ is some subspace of $\bigwedge V$. Find $A \cdot \partial_Y (Y \cdot Y)$ and $\partial_Y (Y \cdot Y)$.

Solution

$$\begin{aligned} A \cdot \partial_Y (Y \cdot Y) &= \frac{d}{d\lambda}(Y + \lambda \langle A \rangle_Y) \cdot (Y + \lambda \langle A \rangle_Y)\bigg|_{\lambda=0} \\ &= \frac{d}{d\lambda}(Y \cdot Y + 2\lambda \langle A \rangle_Y \cdot Y + \lambda^2 \langle A \rangle_Y \cdot \langle A \rangle_Y)\bigg|_{\lambda=0}, \end{aligned} \tag{2.189}$$

$A \cdot \partial_Y(Y \cdot Y) = 2 \langle A \rangle_Y \cdot Y = 2A \cdot Y.$

$$\partial_Y(Y \cdot Y) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \varepsilon_J \cdot \partial_Y(Y \cdot Y) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J 2(\varepsilon_J \cdot Y) = 2Y. \tag{2.190}$$

Exercise 2.104 Let $\bigwedge^\diamond \boldsymbol{V} \ni Y \mapsto B \cdot Y \in \mathbb{R}$, with $B \in \bigwedge \boldsymbol{V}$. Find $A \cdot \partial_Y(B \cdot Y)$ and $\partial_Y(B \cdot Y)$.

Solution

$$A \cdot \partial_Y(B \cdot Y) = \left. \frac{d}{d\lambda} B \cdot (Y + \lambda \langle A \rangle_Y) \right|_{\lambda=0} = B \cdot \langle A \rangle_Y = A \cdot \langle B \rangle_Y , \tag{2.191}$$

$$\partial_Y(B \cdot Y) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \varepsilon_J \cdot \partial_Y(B \cdot Y) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J (\varepsilon_J \cdot \langle B \rangle_Y) = \langle B \rangle_Y . \tag{2.192}$$

Exercise 2.105 Let $\bigwedge^\diamond \boldsymbol{V} \ni Y \mapsto (BYC) \cdot Y \in \mathbb{R}$, with $B, C \in \bigwedge \boldsymbol{V}$. Find $A \cdot \partial_Y((BYC) \cdot Y)$ and $\partial_Y((BYC) \cdot Y)$.

Solution

$$\begin{aligned} A \cdot \partial_Y((BYC) \cdot Y) &= \left. \frac{d}{d\lambda} (B(Y + \lambda \langle A \rangle_Y)C) \cdot (Y + \lambda \langle A \rangle_Y) \right|_{\lambda=0} \\ &= \frac{d}{d\lambda} ((BYC) \cdot Y + \lambda (B \langle A \rangle_Y C) \cdot Y + \lambda (BYC) \cdot \langle A \rangle_Y) \\ &\qquad + \lambda^2 (B \langle A \rangle_Y C) \cdot \langle A \rangle_Y) \big|_{\lambda=0} \\ &= (B \langle A \rangle_Y C) \cdot Y + (BYC) \cdot \langle A \rangle_Y \\ &= \langle A \rangle_Y \cdot (BYC + \tilde{B} Y \tilde{C}), \\ A \cdot \partial_Y((BYC) \cdot Y) &= A \cdot \left\langle BYC + \tilde{B} Y \tilde{C} \right\rangle_Y . \end{aligned} \tag{2.193}$$

where we used that $(AYB) \cdot Y = Y \cdot (\tilde{A} Y \tilde{B})$. Then,

$$\begin{aligned} \partial_Y((BYC) \cdot Y) &= \sum_J \frac{1}{\nu(J)!} \varepsilon^J \varepsilon_J \cdot \partial_Y(BYC \cdot Y) \\ &= \sum_J \frac{1}{\nu(J)!} \varepsilon^J (\varepsilon_J \cdot \left\langle BYC + \tilde{B} Y \tilde{C} \right\rangle_Y), \end{aligned}$$

and finally

$$\partial_Y((BYC) \cdot Y) = \left\langle BYC + \tilde{B} Y \tilde{C} \right\rangle_Y . \tag{2.194}$$

Exercise 2.106 Let $x \wedge B$, with $x, a \in \bigwedge^1 V$ and $B \in \bigwedge^2 V$. Let also $a \in \bigwedge^1 V$. Find $a \cdot \partial_x (x \wedge B)$, the divergence $\partial_x \lrcorner (x \wedge B)$, the rotational $\partial_x \wedge (x \wedge B)$ and the gradient $\partial_x (x \wedge B)$.

Solution Taking into account that $x \wedge B \in \bigwedge^3 V$ we have

$$a \cdot \partial_x (x \wedge B) = \frac{d}{d\lambda}(x + \lambda a) \wedge B\bigg|_{\lambda=0} = a \wedge B, \tag{2.195}$$

$$\partial_x \lrcorner (x \wedge B) = \sum_{j=1}^{n} \varepsilon^j \lrcorner \varepsilon_j \cdot \partial_x (x \wedge B) = \sum_{j=1}^{n} \varepsilon^j \lrcorner (\varepsilon_j \wedge B) = (n-2)B. \tag{2.196}$$

$$\partial_x \wedge (x \wedge B) = \sum_{j=1}^{n} \varepsilon^j \wedge \varepsilon_j \cdot \partial_x (x \wedge B) = \sum_{j=1}^{n} \varepsilon^j \wedge (\varepsilon_j \wedge B) = 0. \tag{2.197}$$

$$\begin{aligned} \partial_x (x \wedge B) &= \sum_{j=1}^{n} \varepsilon^j \varepsilon_j \cdot \partial_x (x \wedge B) = \sum_{j=1}^{n} \varepsilon^j (\varepsilon_j \wedge B) \\ &= \sum_{j=1}^{n} (\varepsilon^j \lrcorner (\varepsilon_j \wedge B) + \varepsilon^j \wedge (\varepsilon_j \wedge B)), \end{aligned} \tag{2.198}$$

$$\partial_x (x \wedge B) = (n-2)B. \qquad (n = \dim V) \tag{2.199}$$

Exercise 2.107 Let $x \in \bigwedge^1 V$ and $B \in \bigwedge^2 V$ and consider $x \lrcorner B \in \bigwedge^1 V$. Find $a \cdot \partial_x (x \lrcorner B)$, $\partial_x \lrcorner (x \lrcorner B)$, $\partial_x \wedge (x \lrcorner B)$ and $\partial_x (x \lrcorner B)$.

Solution

$$a \cdot \partial_x (x \lrcorner B) = \frac{d}{d\lambda}(x + \lambda a) \lrcorner B\bigg|_{\lambda=0} = a \lrcorner B. \tag{2.200}$$

$$\partial_x \lrcorner (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j \lrcorner \varepsilon_j \cdot \partial_x (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j \lrcorner (\varepsilon_j \lrcorner B) = \sum_{j=1}^{n} (\varepsilon^j \wedge \varepsilon_j) \lrcorner B_k = 0, \tag{2.201}$$

$$\partial_x \wedge (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j \wedge \varepsilon_j \cdot \partial_x (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j \wedge (\varepsilon_j \lrcorner B) = 2B. \tag{2.202}$$

$$\partial_x (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j \varepsilon_j \cdot \partial_x (x \lrcorner B) = \sum_{j=1}^{n} \varepsilon^j (\varepsilon_j \lrcorner B) \tag{2.203}$$

$$\partial_x (x \lrcorner B) = \sum_{j=1}^{n} (\varepsilon^j \lrcorner (\varepsilon_j \lrcorner B) + \varepsilon^j \wedge (\varepsilon_j \lrcorner B)) = 2B. \qquad (n = \dim V) \tag{2.204}$$

Exercise 2.108 Let $a \in \bigwedge^1 V$ and $B_k \in \bigwedge^k V$ and $A \in \bigwedge V$. Then, $a \wedge B_k \in \bigwedge^{k+1} V$ and we can define a $(k+1)$-multiform function of vector variable

$$\bigwedge^1 V \ni a \mapsto a \wedge B_k \in \bigwedge^{k+1} V. \tag{2.205}$$

Calculate $A \cdot \partial_a a \wedge B_k$, the divergence $\partial_a \lrcorner (a \wedge B_k)$, the rotational $\partial_a \wedge (a \wedge B_k)$ and the gradient $\partial_a (a \wedge B_k)$.

Solution

$$A \cdot \partial_a a \wedge B_k = \frac{d}{d\lambda}(a + \lambda \langle A \rangle_a) \wedge B_k \Big|_{\lambda=0} = \langle A \rangle_a \wedge B_k = \langle A \rangle_1 \wedge B_k, \tag{2.206}$$

$$\partial_a \lrcorner (a \wedge B_k) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \lrcorner \varepsilon_J \cdot \partial_a \lrcorner (a \wedge B_k) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \lrcorner (\langle \varepsilon_J \rangle_1 \wedge B_k), \tag{2.207}$$

$$\partial_a \lrcorner (a \wedge B_k) = \sum_{j=1}^n \varepsilon^j \lrcorner (\varepsilon_j \wedge B_k) = (n-k) B_k, \tag{2.208}$$

$$\partial_a \wedge (a \wedge B_k) = \sum_{j=1}^n \varepsilon^j \wedge (\varepsilon_j \wedge B_k) = 0, \tag{2.209}$$

$$\begin{aligned} \partial_a (a \wedge B_k) &= \sum_{j=1}^n \varepsilon^j (\varepsilon_j \wedge B_k) = \sum_{j=1}^n [\varepsilon^j \lrcorner (\varepsilon_j \wedge B_k) + \varepsilon^j \wedge (\varepsilon_j \wedge B_k)] \\ &= \partial_a \lrcorner (a \wedge B_k) = (n-k) B_k. \end{aligned} \tag{2.210}$$

Exercise 2.109 Let $a \in \bigwedge^1 V$ and $B_k \in \bigwedge^k V$, $1 \leq k \leq n$, and $A \in \bigwedge V$. Then, $a \lrcorner B_k \in \bigwedge^{k-1} V$ and we can define a $(k-1)$-multiform function of vector variable

$$\bigwedge^1 V \ni a \mapsto a \lrcorner B_k \in \bigwedge^{k-1} V. \tag{2.211}$$

Calculate $A \cdot \partial_a a \lrcorner B_k$, the divergence $\partial_a \lrcorner (a \lrcorner B_k)$, the rotational $\partial_a \wedge (a \lrcorner B_k)$ and the gradient $\partial_a (a \lrcorner B_k)$.

Solution

$$A \cdot \partial_a a \lrcorner B_k = \frac{d}{d\lambda}(a + \lambda \langle A \rangle_a) \lrcorner B_k \Big|_{\lambda=0} = \langle A \rangle_a \lrcorner B_k = \langle A \rangle_1 \lrcorner B_k, \tag{2.212}$$

$$\partial_a \lrcorner (a \lrcorner B_k) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \lrcorner \varepsilon_J \cdot \partial_a \lrcorner (a \lrcorner B_k) = \sum_J \frac{1}{\nu(J)!} \varepsilon^J \lrcorner (\langle \varepsilon_J \rangle_1 \lrcorner B_k), \tag{2.213}$$

$$\partial_a \lrcorner (a \lrcorner B_k) = \sum_{j=1}^n \varepsilon^j \lrcorner (\varepsilon_j \lrcorner B_k) = \sum_{j=1}^n (\varepsilon^j \wedge \varepsilon_j) \lrcorner B_k = 0, \tag{2.214}$$

$$\partial_a \wedge (a \lrcorner B_k) = \sum_{j=1}^{n} \varepsilon^j \wedge (\varepsilon_j \lrcorner B_k) = kB_k \tag{2.215}$$

$$\partial_a (a \lrcorner B_k) = \sum_{j=1}^{n} \varepsilon^j (\varepsilon_j \lrcorner B_k) \tag{2.216}$$

$$= \sum_{j=1}^{n} [\varepsilon^j \lrcorner (\varepsilon_j \lrcorner B_k) + \varepsilon^j \wedge (\varepsilon_j \lrcorner B_k)]$$

$$= \partial_a \wedge (a \lrcorner B_k) = kB_k. \tag{2.217}$$

Exercise 2.110 Given two multiform functions $F : \bigwedge^r \boldsymbol{V} \to \bigwedge^p \boldsymbol{V}$, $G : \bigwedge^r \boldsymbol{V} \to \bigwedge^q \boldsymbol{V}$ abbreviated $F(Y)$, $G(Y)$ show that

$$\partial_Y [F(Y) \wedge G(Y)] = \partial_Y F(Y) \wedge G(Y) + (-1)^{pq} F(Y) \wedge \partial_Y G(Y). \tag{2.218}$$

Exercise 2.111 Let $a \in \bigwedge^1 \boldsymbol{V}$ and $Y, Y : \bigwedge^1 \boldsymbol{V} \to \bigwedge \boldsymbol{V}$ be differentiable functions of the position form $x = x^k \varepsilon_k \in \bigwedge^1 \boldsymbol{V}$. Prove the following identities

$$\begin{aligned} (a) \partial_x \cdot [\partial_a (a \lrcorner X) \cdot Y] &= (\partial_x \lrcorner X) \cdot Y + X \cdot (\partial_x \wedge Y) \\ (b) \partial_x \cdot [\partial_a (a \wedge X) \cdot Y] &= (\partial_x \wedge X) \cdot Y + X \cdot (\partial_x \lrcorner Y) \\ (c) \partial_x \cdot [\partial_a (aX) \cdot Y] &= (\partial_x X) \cdot Y + X \cdot (\partial_x Y) \end{aligned} \tag{2.219}$$

Solution Note that if we prove (a) and (b) then (c) follows by summing the identities (a) and (b).

(a) Using the algebraic identity $(b \lrcorner B) \cdot C = B \cdot (b \wedge C)$ valid for $b \in \bigwedge^1 \boldsymbol{V}$ and $B, C \in \bigwedge \boldsymbol{V}$ we can write the second member of (a) as

$$\begin{aligned} (\partial_x \lrcorner X) \cdot Y + X \cdot (\partial_x \wedge Y) &= (\varepsilon^k \lrcorner \varepsilon_k \cdot \partial_x X) \cdot Y + X \cdot (\varepsilon^k \wedge \varepsilon_k \cdot \partial_x Y) \\ &= \varepsilon_k \cdot \partial_x (\varepsilon^k \lrcorner X) \cdot Y + \varepsilon^k \lrcorner X \cdot (\varepsilon_k \cdot \partial_x Y). \end{aligned} \tag{2.220a}$$

Now, taking into account Eq. (2.216) we can write the first member of identity (a) in of Eq. (2.111) as follows:

$$\begin{aligned} \partial_x \cdot [\partial_a (a \lrcorner X) \cdot Y] &= \partial_x \cdot \{\varepsilon^k [(\varepsilon_k \lrcorner X) \cdot Y]\} \\ &= \varepsilon^l \cdot \varepsilon^k \partial_l [(\varepsilon_k \lrcorner X) \cdot Y] \\ &= \eta^{kl} [(\varepsilon_k \lrcorner \partial_l X) \cdot Y + (\varepsilon_k \lrcorner X) \cdot \partial_l Y] \\ &= (\varepsilon^k \partial_k X) \cdot Y + (\varepsilon^k \lrcorner X) \cdot \partial_k Y \\ &= \varepsilon_k \cdot \partial_x (\varepsilon^k \lrcorner X \cdot Y) + (\varepsilon^k \lrcorner X) \cdot (\varepsilon_k \cdot \partial_x Y). \end{aligned} \tag{2.221}$$

Comparing Eqs. (2.220a) and (2.221) identity (a) is proved.

(b) The second member of identity (b) in Eq. (2.111) can be written as

$$(\partial_x \wedge X) \cdot Y + X \cdot (\partial_x \wedge Y) = Y \cdot (\partial_x \wedge X) + (\partial_x \wedge Y) \cdot X$$
$$= \partial_x \cdot [\partial_a (a \lrcorner X) \cdot Y]. \tag{2.222}$$

Taken into account the algebraic identity $(b \wedge B) \cdot C = B \cdot (b \lrcorner C)$ valid for $b \in \bigwedge^1 V$ and $B, C \in \bigwedge V$ we can write

$$\partial_x \cdot [\partial_a (a \lrcorner X) \cdot Y] = \partial_x \cdot [\partial_a X \cdot (a \wedge Y)]$$
$$= \partial_x \cdot [\partial_a (a \wedge X) \cdot Y], \tag{2.223}$$

and identity (b) in Eq. (2.111) is proved.

(c) As we already said summing up identities (a) and (b), identity (c) in Eq. (2.111) follows. Nevertheless we give another simple proof of that identity. Indeed, we can work the first member of identity (c) as

$$\begin{aligned}
\partial_x \cdot [\partial_a (aX) \cdot Y] &= \varepsilon^k \cdot \varepsilon^l \partial_k [\varepsilon_l X \cdot Y] \\
&= \eta^{lk} [(\varepsilon_l \partial_k X) \cdot Y + (\varepsilon_l X) \cdot (\partial_k Y)] \\
&= (\varepsilon^k \partial_k X) \cdot Y + \langle \widetilde{(\varepsilon_l X)} (\partial_k Y) \rangle_0 \\
&= (\partial Y) \cdot Y + \langle \tilde{X} \varepsilon_l \partial_k Y) \rangle_0 \\
&= (\partial Y) \cdot Y + \langle \tilde{X} \partial Y) \rangle_0 \\
&= (\partial X) \cdot Y + X \cdot (\partial Y).
\end{aligned} \tag{2.224}$$

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Chapter 3
The Hidden Geometrical Nature of Spinors

Abstract This chapter reviews the classification of the real and complex Clifford algebras and analyze the relationship between some particular algebras that are important in physical applications, namely the quaternion algebra ($\mathbb{R}_{0,2}$), Pauli algebra ($\mathbb{R}_{3,0}$), the spacetime algebra ($\mathbb{R}_{1,3}$), the Majorana algebra ($\mathbb{R}_{3,1}$) and the Dirac algebra ($\mathbb{R}_{4,1}$). A detailed and original theory disclosing the hidden geometrical meaning of spinors is given through the introduction of the concepts of algebraic, covariant and Dirac-Hestenes spinors. The relationship between these kinds of spinors (that carry the same mathematical information) is elucidated with special emphasis for cases of physical interest. We investigate also how to reconstruct a spinor from their so-called bilinear invariants and present Lounesto's classification of spinors. Also, Majorana, Weyl spinors, the dotted and undotted algebraic spinors are discussed with the Clifford algebra formalism.

3.1 Notes on the Representation Theory of Associative Algebras

To achieve our goal mentioned in Chap. 1 of disclosing the real secret geometrical meaning of Dirac spinors, we shall need to briefly recall some few results of the theory of representations of associative algebras. Propositions are presented without proofs and the interested reader may consult [3, 8, 12, 16, 20, 21] for details.

Let $\mathbf{V}$ be a *finite* dimensional *linear* space over $\mathbb{K}$ (a division ring). Suppose that $\dim_{\mathbb{K}} \mathbf{V} = n$, where $n \in \mathbb{Z}$. We are interested in what follows in the cases where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In this case we also call $\mathbf{V}$ a *vector space* over $\mathbb{K}$. When $\mathbb{K} = \mathbb{H}$ it is necessary to distinguish between right or left $\mathbb{H}$-linear spaces and in this case $\mathbf{V}$ will be called a right or left $\mathbb{H}$-module. Recall that $\mathbb{H}$ is a division ring (sometimes called a noncommutative field or a skew field) and since $\mathbb{H}$ has a natural vector space structure over the real field, then $\mathbb{H}$ is also a division algebra.

Definition 3.1 Let $\mathbf{V}$ be a vector space over $\mathbb{R}$ and $\dim_{\mathbb{R}} \mathbf{V} = 2m = n$. A linear mapping

$$\mathbf{J} : \mathbf{V} \to \mathbf{V} \tag{3.1}$$

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such that

$$\mathbf{J}^2 = -\mathrm{Id}_{\mathbf{V}}, \tag{3.2}$$

s called a complex structure mapping.

Definition 3.2 Let $\mathbf{V}$ be as in the previous definition. The pair $(\mathbf{V}, \mathbf{J})$ is called a complex vector space structure and denote by $\mathbf{V}_{\mathbb{C}}$ if the following product holds. Let $\mathbb{C} \ni z = a + \mathrm{i}b$ ($\mathrm{i} = \sqrt{-1}$) and let $\mathbf{v} \in \mathbf{V}$. Then

$$z\mathbf{v} = (a + \mathrm{i}b)\mathbf{v} = a\mathbf{v} + b\mathbf{J}\mathbf{v}. \tag{3.3}$$

It is obvious that $\dim_{\mathbb{C}} = \frac{m}{2}$.

Definition 3.3 Let $\mathbf{V}$ be a vector space over $\mathbb{R}$. A *complexification* of $\mathbf{V}$ is a complex structure associated with the real vector space $\mathbf{V} \oplus \mathbf{V}$. The resulting complex vector space is denoted by $V^{\mathbb{C}}$. Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}$. Elements of $\mathbf{V}^{\mathbb{C}}$ are usually denoted by $\mathbf{c} = \mathbf{v} + i\mathbf{w}$, and if $\mathbb{C} \ni z = a + \mathrm{i}b$ we have

$$z\mathbf{c} = a\mathbf{v} - b\mathbf{w} + i(a\mathbf{w} + b\boldsymbol{v}). \tag{3.4}$$

Of course, we have that $\dim_{\mathbb{C}} \mathbf{V}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathbf{V}$.

Definition 3.4 A $\mathbb{H}$-module is a real vector space $\mathbf{S}$ carrying three linear transformation, $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ each one of them satisfying

$$\begin{aligned} &\mathbf{I}^2 = \mathbf{J}^2 = -\mathrm{Id}_{\mathbf{S}}, \\ &\mathbf{IJ} = -\mathbf{JI} = \mathbf{K},\ \mathbf{JK} = -\mathbf{KJ} = \mathbf{I},\ \mathbf{KI} = -\mathbf{IK} = \mathbf{J}. \end{aligned} \tag{3.5a}$$

Exercise 3.5 Show that $\mathbf{K}^2 = -\mathrm{Id}_{\mathbf{S}}$

In what follows $\mathcal{A}$ denotes an *associative* algebra on the commutative field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{F} \subseteq \mathcal{A}$.

Definition 3.6 Any subset $I \subseteq \mathcal{A}$ such that

$$a\psi \in I, \forall a \in \mathcal{A},\ \forall \psi \in I,$$

$$\psi + \phi \in I, \forall \psi, \phi \in I \tag{3.6}$$

is called a left ideal of A.

Remark 3.7 An analogous definition holds for right ideals where Eq. (3.6) reads $\psi a \in I, \forall a \in \mathcal{A}, \forall \psi \in I$, for bilateral ideals where in this case Eq. (3.6) reads $a\psi b \in I, \forall a, b \in \mathcal{A}, \forall \psi \in I$.

Definition 3.8 An associative algebra $\mathcal{A}$ is simple if the only bilateral ideals are the zero ideal and $\mathcal{A}$ itself.

Not all algebras are simple and in particular *semi-simple* algebras are important for our considerations. A definition of semi-simple algebras requires the introduction of the concepts of *nilpotent* ideals and radicals. To define these concepts adequately would lead us to a long incursion on the theory of associative algebras, so we avoid to do that here. We only quote that semi-simple algebras are the direct sum of simple algebras and of course simple algebras are semi simple. Then, for our objectives in this chapter the study of semi-simple algebras is reduced to the study of simple algebras.

Definition 3.9 We say that $\mathrm{e} \in \mathcal{A}$ is an *idempotent* element if $\mathrm{e}^2 = \mathrm{e}$. An idempotent is said to be *primitive* if it cannot be written as the sum of two non zero annihilating (or orthogonal) idempotent, i.e., $\mathrm{e} \neq \mathrm{e}_1 + \mathrm{e}_2$, with $\mathrm{e}_1\mathrm{e}_2 = \mathrm{e}_2\mathrm{e}_1 = 0$ and $\mathrm{e}_1^2 = \mathrm{e}_1$, $\mathrm{e}_2^2 = \mathrm{e}_2$.

We give without proofs the following theorems valid for semi-simple (and thus simple) algebras $\mathcal{A}$:

Theorem 3.10 *All minimal left (respectively right) ideals of semi-simple $\mathcal{A}$ are of the form $J = \mathcal{A}\mathrm{e}$ (respectively $\mathrm{e}\mathcal{A}$), where e is a primitive idempotent of $\mathcal{A}$.*

Theorem 3.11 *Two minimal left ideals of a semi-simple algebra $\mathcal{A}$, $J = \mathcal{A}\mathrm{e}$ and $J = \mathcal{A}\mathrm{e}'$ are isomorphic, if and only if, there exist a non null $Y' \in J'$ such that $J' = JY'$.*

Let $\mathcal{A}$ be an associative and simple algebra on the *field* $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$, and let $\mathbf{S}$ be a finite dimensional linear space over a division ring $\mathbb{K} \supseteq \mathbb{F}$ and let $\mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S} = \mathrm{Hom}_{\mathbb{K}}(\mathbf{S}, \mathbf{S})$ be the endomorphism algebra of $\mathbf{S}$.[1]

Definition 3.12 A representation of $\mathcal{A}$ in $\mathbf{S}$ is a $\mathbb{K}$ algebra homomorphism[2] $\rho : \mathcal{A} \to \mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S}$ which maps the unit element of $\mathcal{A}$ to $\mathbf{Id}_{\mathbf{E}}$. The dimension $\mathbb{K}$ of $\mathbf{S}$ is called the degree of the representation.

Definition 3.13 The addition in $\mathbf{S}$ together with the mapping $\mathcal{A} \times \mathbf{S} \to \mathbf{S}$, $(a, x) \mapsto \rho(a)x$ turns $\mathbf{S}$ in a left $\mathcal{A}$-module,[3] called the left representation module.

Remark 3.14 It is important to recall that when $\mathbb{K} = \mathbb{H}$ the usual recipe for $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S}, \mathbf{S})$ to be a linear space over $\mathbb{H}$ fails and in general $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S}, \mathbf{S})$ is considered as a linear space over $\mathbb{R}$, which is the centre of $\mathbb{H}$.

[1] Recall that $\mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ is the algebra of linear transformations of a finite dimensional vector space $\mathbf{V}$ over $\mathbb{K}$ into a finite vector space $\mathbf{W}$ over $\mathbb{K}$. When $\mathbf{V} = \mathbf{W}$ the set $\mathrm{End}_{\mathbb{K}}\mathbf{V} = \mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{V})$ is called the set of endomorphisms of $\mathbf{V}$.

[2] We recall that a $\mathbb{K}$-algebra homomorphism is a $\mathbb{K}$-linear map ρ such that $\forall X, Y \in \mathcal{A}, \rho(XY) = \rho(X)\rho(Y)$.

[3] We recall that there are left and right modules, so we can also define right modular representations of $\mathcal{A}$ by defining the mapping $\mathbf{S} \times \mathcal{A} \to \mathbf{S}$, $(x, a) \mapsto x\rho(a)$. This turns $\mathbf{S}$ in a right $\mathcal{A}$-module, called the right *representation module*.

Remark 3.15 We also have that if $\mathcal{A}$ is an algebra on $\mathbb{F}$ and $\mathbf{S}$ is an $\mathcal{A}$-module, then $\mathbf{S}$ can always be considered as a vector space over $\mathbb{F}$ and if $a \in \mathcal{A}$, the mapping $\chi : a \to \chi_a$ with $\chi_a(\mathbf{s}) = a\mathbf{s}, \mathbf{s} \in \mathbf{S}$, is a homomorphism $\mathcal{A} \to \mathrm{End}_{\mathbb{F}}\mathbf{S}$, and so it is a representation of $\mathcal{A}$ in $\mathbf{S}$. The study of $\mathcal{A}$ modules is then equivalent to the study of the $\mathbb{F}$ representations of $\mathcal{A}$.

Definition 3.16 A representation ρ is faithful if its kernel is zero, i.e., $\rho(a)x = 0, \forall x \in \mathbf{S} \Rightarrow a = 0$. The kernel of ρ is also known as the annihilator of its module.

Definition 3.17 ρ is said to be simple or irreducible if the only invariant subspaces of $\rho(a), \forall a \in \mathcal{A}$, are $\mathbf{S}$ and $\{0\}$.

Then, the representation module is also simple. That means that it has no proper submodules.

Definition 3.18 ρ is said to be semi-simple, if it is the direct sum of simple modules, and in this case $\mathbf{S}$ is the direct sum of subspaces which are globally invariant under $\rho(a), \forall a \in \mathcal{A}$.

When no confusion arises $\rho(a)x$ may be denoted by $a * x$ or ax.

Definition 3.19 Two $\mathcal{A}$-modules $\mathbf{S}$ and $\mathbf{S}'$ (with the "exterior" multiplication being denoted respectively by $\diamond$ and $*$) are isomorphic if there exists a bijection $\varphi : \mathbf{S} \to \mathbf{S}'$ such that,

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y),\ \forall x, y \in \mathbf{S}, \\ \varphi(a \diamond x) &= a * \varphi(x),\ \forall a \in \mathcal{A}, \end{aligned} \tag{3.7}$$

and we say that the representations ρ and ρ' of A are equivalent if their modules are isomorphic.

This implies the existence of a $\mathbb{K}$-linear isomorphism $\varphi : \mathbf{S} \to \mathbf{S}'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi, \forall a \in \mathcal{A}$ or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$. If $\dim \mathbf{S} = n$, then $\dim \mathbf{S}' = n$.

Definition 3.20 A complex representation of $\mathcal{A}$ is simply a real representation $\rho : \mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\mathbf{S}, \mathbf{S})$ for which

$$\rho(Y) \circ \mathbf{J} = \mathbf{J} \circ \rho(Y), \forall Y \in \mathcal{A}. \tag{3.8}$$

This means that the image of ρ commutes with the subalgebra generated by $\{\mathrm{Id}_{\mathbf{S}}, \mathbf{J}\} \sim \mathbb{C}$.

Definition 3.21 A quaternionic representation of $\mathcal{A}$ is a representation $\rho : \mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\mathbf{S}, \mathbf{S})$ such that

$$\rho(Y) \circ \mathbf{I} = \mathbf{I} \circ \rho(Y),\ \rho(Y) \circ \mathbf{J} = \mathbf{J} \circ \rho(Y),\ \rho(Y) \circ \mathbf{K} = \mathbf{K} \circ \rho(Y), \forall Y \in \mathcal{A}. \tag{3.9}$$

This means that the representation ρ has a commuting subalgebra isomorphic to the quaternion ring.

The following theorem is crucial:

Theorem 3.22 *(Wedderburn). If $\mathcal{A}$ is simple algebra over $\mathbb{F}$ then $\mathcal{A}$ is isomorphic to $\mathbb{D}(m)$, where $\mathbb{D}(m)$ is a matrix algebra with entries in $\mathbb{D}$ (a division algebra), and m and $\mathbb{D}$ are unique (modulo isomorphisms).*

3.2 Real and Complex Clifford Algebras and Their Classification

Now, it is time to specialize the previous results to the Clifford algebras on the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We are particularly interested in the case of *real* Clifford algebras. In what follows we take $\mathbf{V} = \mathbb{R}^n$. We denote as in the previous chapter by $\mathbb{R}^{p,q}$ $(n = p + q)$ the real vector space $\mathbb{R}^n$ endowed with a nondegenerate metric $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Let $\{E_i\}, (i = 1, 2, \ldots, n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$\mathbf{g}(E_i, E_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, 2, \ldots p, \\ -1, & i = j = p + 1, \ldots, p + q = n, \\ 0, & i \neq j. \end{cases} \tag{3.10}$$

We recall (Definition 2.37 that the Clifford algebra $\mathbb{R}_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q})$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}, (i = 1, 2, \ldots, n)$ such that $E_i^2 = \mathbf{g}(E_i, E_i)$, $E_iE_j = -E_jE_i (i \neq j)$, and $E_1E_2 \ldots E_n \neq \pm 1$.

$\mathbb{R}_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\bigwedge^k \mathbb{R}^n$ of dimensions $\binom{n}{k}, 0 \leq k \leq n$. The canonical basis of $\bigwedge^k \mathbb{R}^n$ is given by the elements $e_A = E_{\alpha_1} \cdots E_{\alpha_k}, 1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. The element $e_J = E_1 \cdots E_n \in \bigwedge^n \mathbb{R}^n \hookrightarrow \mathbb{R}_{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1, \ldots, E_n \in \bigwedge^1 \mathbb{R}^n \equiv \mathbb{R}^n$. The center $\mathbb{R}_{p,q}$ is $\bigwedge^0 \mathbb{R}^n \equiv \mathbb{R}$ if n is even and it is the direct sum $\bigwedge^0 \mathbb{R}^n \oplus \bigwedge^0 \mathbb{R}^n$ if n is odd.[4]

All Clifford algebras are semi-simple. If $p + q = n$ is even, $\mathbb{R}_{p,q}$ is simple and if $p + q = n$ is odd we have the following possibilities:

(a) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p - q \neq 1 \pmod 4 \leftrightarrow$ center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{C}$;
(b) $\mathbb{R}_{p,q}$ is not simple (but is a direct sum of two simple algebras) $\leftrightarrow e_J^2 = +1 \leftrightarrow p - q = 1 \pmod 4 \leftrightarrow$ center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

[4]For a proof see [20].

Now, for $\mathbb{R}_{p,q}$ the division algebras $\mathbb{D}$ are the division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The explicit isomorphism can be discovered with some hard but not difficult work. It is possible to give a general classification of all real (and also the complex) Clifford algebras and a classification table can be found, e.g., in [20]. One convenient table is the following one (where $\mu = [n/2]$ means the integer part of $n/2$).

We denoted by $\mathbb{R}^0_{p,q}$ the even subalgebra of $\mathbb{R}_{p,q}$ and by $\mathbb{R}^1_{p,q}$ the set of odd elements of $\mathbb{R}_{p,q}$. The following very important result holds true

Proposition 3.23 $\mathbb{R}^0_{p,q} \simeq \mathbb{R}_{p,q-1}$ *and also* $\mathbb{R}^0_{p,q} \simeq \mathbb{R}_{q,p-1}$.

Now, to complete the classification we need the following theorem:

Theorem 3.24 (Periodicity)[5] *We have*

$$\begin{array}{llll} \mathbb{R}_{n+8} & = \mathbb{R}_{n,0} \otimes \mathbb{R}_{8,0} & \mathbb{R}_{0,n+8} & = \mathbb{R}_{0,n} \otimes \mathbb{R}_{0,8} \\ \mathbb{R}_{p+8,q} & = \mathbb{R}_{p,q} \otimes \mathbb{R}_{8,0} & \mathbb{R}_{p,q+8} & = \mathbb{R}_{p,q} \otimes \mathbb{R}_{0,8}. \end{array} \tag{3.11}$$

Remark 3.25 We emphasize here that since the general results concerning the representations of simple algebras over a field $\mathbb{F}$ applies to the Clifford algebras $\mathbb{R}_{p,q}$ we can talk about real, complex or quaternionic representation of a given Clifford algebra, even if the natural matrix identification is not a matrix algebra over one of these fields. A case that we shall need is that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$. But it is clear that $\mathbb{R}_{1,3}$ has a complex representation, for any quaternionic representation of $\mathbb{R}_{p,q}$ is automatically *complex*, once we restrict $\mathbb{C} \subset \mathbb{H}$ and of course, the complex dimension of any $\mathbb{H}$-module must be even. Also, any complex representation of $\mathbb{R}_{p,q}$ extends automatically to a representation of $\mathbb{C} \otimes \mathbb{R}_{p,q}$.

Remark 3.26 $\mathbb{C} \otimes \mathbb{R}_{p,q}$ is isomorphic to the complex Clifford algebra $\mathcal{C}\ell_{p+q}$. The algebras $\mathbb{C}$ and $\mathbb{R}_{p,q}$ are subalgebras of $\mathcal{C}\ell_{p+q}$

3.2.1 Pauli, Spacetime, Majorana and Dirac Algebras

For the purposes of our book we shall need to have in mind that:

$\mathbb{R}_{0,1} \simeq \mathbb{C}$,
$\mathbb{R}_{0,2} \simeq \mathbb{H}$,
$\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$,
$\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$,
$\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$,
$\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$.

(3.12)

[5] See [20].

$\mathbb{R}_{3,0}$ is called the Pauli algebra, $\mathbb{R}_{1,3}$ is called the *spacetime* algebra, $\mathbb{R}_{3,1}$ is called *Majorana* algebra and $\mathbb{R}_{4,1}$ is called the *Dirac* algebra. Also, the following particular results, which can be easily proved, will be used many times in what follows:

$$\begin{aligned}&\mathbb{R}^0_{1,3} \simeq \mathbb{R}^0_{3,1} = \mathbb{R}_{3,0}, \quad \mathbb{R}^0_{4,1} \simeq \mathbb{R}_{1,3}, \quad \mathbb{R}^0_{1,4} \simeq \mathbb{R}_{1,3},\\ &\mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{3,1}, \quad \mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{3,1}.\end{aligned} \tag{3.13}$$

In words: the even subalgebras of both the spacetime and Majorana algebras is the Pauli algebra. The even subalgebra of the Dirac algebra is the spacetime algebra and finally the Dirac algebra is the complexification of the spacetime algebra or of the Majorana algebra.

Equation (3.13) show moreover, in view of Remark 3.26 that the spacetime algebra has also a matrix representation in $\mathbb{C}(4)$. Obtaining such a representation is very important for the introduction of the concept of a Dirac-Hestenes spinor, an important ingredient of the present work.

3.3 The Algebraic, Covariant and Dirac-Hestenes Spinors

3.3.1 Minimal Lateral Ideals of $\mathbb{R}_{p,q}$

We now give some results concerning the minimal lateral ideals of $\mathbb{R}_{p,q}$.

Theorem 3.27 *The maximum number of pairwise orthogonal idempotents in* $\mathbb{K}(m)$ *(where* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$*) is* m *.*

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is then characterized by a spectral set $\{\mathrm{e}_{pq,j}\}$ of idempotents elements of $\mathbb{R}_{p,q}$ such that:

(a) $\sum_{i=1}^{n} \mathrm{e}_{pq,j} = 1$;
(b) $\mathrm{e}_{pq,j}\mathrm{e}_{pq,k} = \delta_{jk}\mathrm{e}_{pq,j}$;
(c) the rank of $\mathrm{e}_{pq,j}$ is minimal and non zero, i.e., is primitive.

By rank of $\mathrm{e}_{pq,j}$ we mean the rank of the $\bigwedge \mathbb{R}^{p,q}$ morphism, $\mathrm{e}_{pq,j} : \phi \longmapsto \phi\mathrm{e}_{pq,j}$. Conversely, any $\phi \in I_{pq,j}$ can be characterized by an idempotent $\mathrm{e}_{pq,j}$ of minimal rank $\neq 0$, with $\phi = \phi\mathrm{e}_{pq,j}$.

We now need to know the following theorem [13]:

Theorem 3.28 *A minimal left ideal of* $\mathbb{R}_{p,q}$ *is of the type*

$$I_{pq} = \mathbb{R}_{p,q}\mathrm{e}_{pq}, \tag{3.14}$$

where

$$\mathrm{e}_{pq} = \frac{1}{2}(1 + e_{\alpha_1}) \cdots \frac{1}{2}(1 + e_{\alpha_k}) \tag{3.15}$$

is a primitive idempotent of $R_{p,q}$ *and where* $e_{\alpha_1}, \ldots, e_{\alpha_k}$ *are commuting elements in the canonical basis of* $\mathbb{R}_{p,q}$ *(generated in the standard way through the elements of a basis* $(E_{1,\ldots}, E_p, E_{p+1}, \ldots E_{p+q})$ *of* $\mathbb{R}^{p,q}$*) such that* $(e_{\alpha_i})^2 = 1$, $(i = 1, 2, \ldots, k)$ *generate a group of order* 2^k, $k = q - r_{q-p}$ *and* r_i *are the Radon-Hurwitz numbers, defined by the recurrence formula* $r_{i+8} = r_i + 4$ *and*

i		0	1	2	3	4	5	6	7
r_i		0	1	2	2	3	3	3	3

. (3.16)

Recall that $\mathbb{R}_{p,q}$ is a ring and the minimal lateral ideals are modules over the ring $\mathbb{R}_{p,q}$. They are *representation modules* of $\mathbb{R}_{p,q}$, and indeed we have (recall the above table) the following theorem [13]:

Theorem 3.29 *If* $p + q$ *is even or odd with* $p - q \neq 1 (\text{mod}\, 4)$*, then*

$$\mathbb{R}_{p,q} = \text{Hom}_{\mathbb{K}}(I_{pq}, I_{pq}) \simeq \mathbb{K}(m), \tag{3.17}$$

where (as we already know) $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *Also,*

$$\dim_{\mathbb{K}}(I_{pq}) = m, \tag{3.18}$$

and

$$\mathbb{K} \simeq \text{e}\mathbb{K}(m)\text{e}, \tag{3.19}$$

where e *is the representation of* e_{pq} *in* $\mathbb{K}(m)$.

If $p + q = n$ *is odd, with* $p - q = 1\ (\text{mod}\, 4)$*, then*

$$\mathbb{R}_{p,q} = \text{Hom}_{\mathbb{K}}(I_{pq}, I_{pq}) \simeq \mathbb{K}(m) \oplus \mathbb{K}(m), \tag{3.20}$$

with

$$\dim_{\mathbb{K}}(I_{pq}) = m \tag{3.21}$$

and

$$\begin{gathered} \text{e}\mathbb{K}(m)\text{e} \simeq \mathbb{R} \oplus \mathbb{R} \\ \textit{or} \\ \text{e}\mathbb{K}(m)\text{e} \simeq \mathbb{H} \oplus \mathbb{H}. \end{gathered} \tag{3.22}$$

With the above isomorphisms we can immediately identify the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $\mathbb{K}(m)$.

Table 3.1 Representation of the Clifford algebras $\mathbb{R}_{p,q}$ as matrix algebras

$p-q$ mod 8	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{\mu})$	$\mathbb{R}(2^{\mu}) \oplus \mathbb{R}(2^{\mu})$	$\mathbb{R}(2^{\mu})$	$\mathbb{C}(2^{\mu})$	$\mathbb{H}(2^{\mu-1})$	$\mathbb{H}(2^{\mu-1}) \oplus \mathbb{H}(2^{\mu-1})$	$\mathbb{H}(2^{\mu-1})$	$\mathbb{C}(2^{\mu})$

3.3.2 Algorithm for Finding Primitive Idempotents of $\mathbb{R}_{p,q}$

With the ideas introduced above it is now a simple exercise to find primitive idempotents of $\mathbb{R}_{p,q}$. First we look at Table 3.1 and find the matrix algebra to which our particular Clifford algebra $\mathbb{R}_{p,q}$ is isomorphic. Suppose $\mathbb{R}_{p,q}$ is simple.[6] Let $\mathbb{R}_{p,q} \simeq \mathbb{K}(m)$ for a particular $\mathbb{K}$ and m. Next we take an element $\mathbf{e}_{\alpha_1} \in \{\mathbf{e}_A\}$ from the canonical basis $\{\mathbf{e}_A\}$ of $\mathbb{R}_{p,q}$ such that

$$\mathbf{e}_{\alpha_1}^2 = 1. \tag{3.23}$$

Next we construct the idempotent $\mathrm{e}_{pq} = (1 + \mathbf{e}_{\alpha_1})/2$ and the ideal $I_{pq} = \mathbb{R}_{p,q}\mathrm{e}_{pq}$ and calculate $\dim_{\mathbb{K}}(I_{pq})$. If $\dim_{\mathbb{K}}(I_{pq}) = m$, then e_{pq} is primitive. If $\dim_{\mathbb{K}}(I_{pq}) \neq m$, we choose $\mathbf{e}_{\alpha_2} \in \{\mathbf{e}_A\}$ such that $\mathbf{e}_{\alpha_2}$ commutes with $\mathbf{e}_{\alpha_1}$ and $\mathbf{e}_{\alpha_2}^2 = 1$ and construct the idempotent $\mathrm{e}'_{pq} = (1 + \mathbf{e}_{\alpha_1})(1 + \mathbf{e}_{\alpha_2})/4$. If $\dim_{\mathbb{K}}(I'_{pq}) = m$, then e'_{pq} is primitive. Otherwise we repeat the procedure. According to Theorem 3.28 the procedure is finite.

3.3.3 $\mathbb{R}^{\star}_{p,q}$, Clifford, Pinor and Spinor Groups

The set of the invertible elements of $\mathbb{R}_{p,q}$ constitutes a non-abelian group which we denote by $\mathbb{R}^{\star}_{p,q}$. It acts naturally on $\mathbb{R}_{p,q}$ as an algebra homomorphism through its twisted adjoint representation ($\hat{\mathrm{A}}\mathrm{d}$) or adjoint representation (Ad)

$$\hat{\mathrm{A}}\mathrm{d} : \mathbb{R}^{\star}_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q});\; u \mapsto \mathrm{Ad}_u,\; \text{with } \mathrm{Ad}_u(x) = ux\hat{u}^{-1}, \tag{3.24}$$

$$\mathrm{Ad} : \mathbb{R}^{\star}_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q});\; u \mapsto \mathrm{Ad}_u,\; \text{with } \mathrm{Ad}_u(x) = uxu^{-1} \tag{3.25}$$

[6] Once we know the algorithm for a simple Clifford algebra it is straightforward to devise an algorithm for the semi-simple Clifford algebras.

Definition 3.30 The Clifford-Lipschitz group is the set

$$\Gamma_{p,q} = \left\{u \in \mathbb{R}^{\star}_{p,q} \;\middle|\; \forall x \in \mathbb{R}^{p,q},\; ux\hat{u}^{-1} \in \mathbb{R}^{p,q}\right\}, \tag{3.26a}$$

or

$$\Gamma_{p,q} = \left\{u \in \mathbb{R}^{\star(0)}_{p,q} \cup \mathbb{R}^{\star(1)}_{p,q} \;\middle|\; \forall x \in \mathbb{R}^{p,q},\; uxu^{-1} \in \mathbb{R}^{p,q}\right\}, \tag{3.26b}$$

Note in Eq. (3.26b) the restriction to the even ($\mathbb{R}^{\star(0)}_{p,q}$) and odd ($\mathbb{R}^{\star(1)}_{p,q}$) parts of $\mathbb{R}^{\star}_{p,q}$.

Definition 3.31 The set $\Gamma^0_{p,q} = \Gamma_{p,q} \cap \mathbb{R}^0_{p,q}$ is called special Clifford-Lipschitz group.

Definition 3.32 The Pinor group $\text{Pin}_{p.q}$ is the subgroup of $\Gamma_{p,q}$ such that

$$\text{Pin}_{p,q} = \left\{u \in \Gamma_{p,q} \,| N(u) = \pm 1\right\}, \tag{3.27}$$

where

$$N : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, N(x) = \langle \bar{x}x \rangle_0. \tag{3.28}$$

Definition 3.33 The Spin group $\text{Spin}_{p,q}$ is the set

$$\text{Spin}_{p,q} = \left\{u \in \Gamma^0_{p,q} | N(u) = \pm 1\right\}. \tag{3.29}$$

It is easy to see that $\text{Spin}_{p,q}$ is not connected.

Definition 3.34 The Special Spin Group $\text{Spin}^e_{p,q}$ is the set

$$\text{Spin}^e_{p,q} = \left\{u \in \text{Spin}_{p,q} | N(u) = +1\right\}. \tag{3.30}$$

The superscript e, means that $\text{Spin}^e_{p,q}$ is the connected component to the identity. We can prove that $\text{Spin}^e_{p,q}$ is connected for all pairs (p, q) with the exception of $\text{Spin}^e(1, 0) \simeq \text{Spin}^e(0, 1)$.

We recall now some classical results [17] associated with the pseudo-orthogonal groups $\text{O}_{p,q}$ of a vector space $\mathbb{R}^{p,q}$ $(n = p+q)$ and its subgroups. Let $\mathbf{G}$ be a diagonal $n \times n$ matrix whose elements are G_{ij}

$$\mathbf{G} = [G_{ij}] = \text{diag}(1, 1, \ldots, -1, -1, \ldots -1), \tag{3.31}$$

with p positive and q negative numbers.

Definition 3.35 $\mathrm{O}_{p,q}$ is the set of $n \times n$ real matrices $\mathbf{L}$ such that

$$\mathbf{LGL}^T = \mathbf{G}, \ \det \mathbf{L}^2 = 1. \tag{3.32}$$

Equation (3.32) shows that $\mathrm{O}_{p,q}$ is not connected.

Definition 3.36 $\mathrm{SO}_{p,q}$, the special (proper) pseudo orthogonal group is the set of $n \times n$ real matrices $\mathbf{L}$ such that

$$\mathbf{LGL}^T = \mathbf{G}, \ \det \mathbf{L} = 1. \tag{3.33}$$

When $p = 0$ ($q = 0$) $\mathrm{SO}_{p,q}$ is connected. However, $\mathrm{SO}_{p,q}$ (for, $p, q \neq 0$) is not connected and has two connected components for $p, q \geq 1$.

Definition 3.37 The group $\mathrm{SO}^e_{p,q}$, the connected component to the identity of $\mathrm{SO}_{p,q}$ will be called the special orthochronous pseudo-orthogonal group.[7]

Theorem 3.38 *$Ad_{|\mathrm{Pin}_{p,q}} : \mathrm{Pin}_{p,q} \to \mathrm{O}_{p,q}$ is onto with kernel $\mathbb{Z}_2$.*

$Ad_{|\mathrm{Spin}_{p,q}} : \mathrm{Spin}_{p,q} \to \mathrm{SO}_{p,q}$ is onto with kernel $\mathbb{Z}_2$. $Ad_{|\mathrm{Spin}^e_{p,q}} : \mathrm{Spin}^e_{p,q} \to \mathrm{SO}^e_{p,q}$ is onto with kernel $\mathbb{Z}_2$.

We have,

$$\mathrm{O}_{p,q} = \frac{\mathrm{Pin}_{p,q}}{\mathbb{Z}_2}, \ \mathrm{SO}_{p,q} = \frac{\mathrm{Spin}_{p,q}}{\mathbb{Z}_2}, \ \mathrm{SO}^e_{p,q} = \frac{\mathrm{Spin}^e_{p,q}}{\mathbb{Z}_2}. \tag{3.34}$$

The group homomorphism between $\mathrm{Spin}^e_{p,q}$ and $\mathrm{SO}^e(p,q)$ will be denoted by

$$\mathbf{L} : \mathrm{Spin}^e_{p,q} \to \mathrm{SO}^e_{p,q}. \tag{3.35}$$

The following theorem that first appears in [20] is very important.

Exercise 3.39 (Porteous). Show that for $p + q \leq 4$, $\mathrm{Spin}^e(p,q) = \{u \in \mathbb{R}_{p,q} | u\tilde{u} = 1\}$.

Solution We must show that for any $u \in \mathbb{R}^0_{p,q}$, $N(u) = \pm 1$ and $\boldsymbol{x} \in \mathbb{R}^{p,q}$ we have that $\hat{\mathrm{A}}\mathrm{d}_u(\boldsymbol{x}) \in \mathbb{R}^{p,q}$. But when $u \in \mathbb{R}^0_{p,q}$, $\hat{\mathrm{A}}\mathrm{d}_u(\boldsymbol{x}) = u\boldsymbol{x}u^{-1}$. We must then show that

$$\boldsymbol{y} = u\boldsymbol{x}u^{-1} \in \mathbb{R}^{p,q}.$$

[7] This nomenclature comes from the fact that $\mathrm{SO}^e(1,3) = \mathcal{L}^{\uparrow}_{+}$ is the special (proper) orthochronous Lorentz group. In this case the set is easily defined by the condition $L^0_0 \geq +1$. For the general case see [17].

Since $u \in \mathbb{R}^0_{p,q}$ we have that $\boldsymbol{y} \in \mathbb{R}^1_{p,q}$. Let $\boldsymbol{e}_i, i = 1, 2, 3, 4$ an orthonormal basis of $\mathbb{R}_{p,q}$, $p + q = 4$. Now, $\bar{\boldsymbol{y}} = (u\boldsymbol{y}u^{-1})^{\hat{}\tilde{}} = -u\boldsymbol{x}u^{-1} = -\boldsymbol{y}$. Writing

$$\boldsymbol{y} = y^i\boldsymbol{e}_i + \frac{1}{3!}y^{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k,$$
$$y^i,\ y^{ijk} \in \mathbb{R},$$

we get

$$\bar{\boldsymbol{y}} = -y^i\boldsymbol{e}_{i,}$$

from which follows that $\boldsymbol{y} \in \mathbb{R}^{p,q}$.

3.3.4 Lie Algebra of $Spin^e_{1,3}$

It can be shown [14, 16, 23] that for each $u \in \mathrm{Spin}^e_{1,3}$ it holds $u = \pm e^F, F \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ and F can be chosen in such a way to have a positive sign in Eq. (3.33), except in the particular case $F^2 = 0$ when $u = -e^F$. From Eq. (3.33) it follows immediately that the Lie algebra of $\mathrm{Spin}^e_{1,3}$ is generated by the bivectors $F \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ through the commutator product.

Exercise 3.40 Show that when $F^2 = 0$ we must have $u = -e^F$.

3.4 Spinor Representations of $\mathbb{R}_{4,1}$, $\mathbb{R}^0_{4,1}$ and $\mathbb{R}_{1,3}$

We investigate now some spinor representations of $\mathbb{R}_{4,1}$, $\mathbb{R}^0_{4,1}$ and $\mathbb{R}_{1,3}$ which will permit us to introduce the concepts algebraic, Dirac and Dirac-Hestenes spinors in the next section.

Let $b_0 = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, such that $\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu = 2\eta_{\mu\nu}$, with $\eta_{\upsilon\mu} = \mathrm{diag}(+1, -1, -1, -1)$. Now, with the results of the previous section we can verify without difficulties that the elements $\mathrm{e}, \mathrm{e}', \mathrm{e}'' \in \mathbb{R}_{1,3}$

$$\mathrm{e} = \frac{1}{2}(1 + \mathbf{e}_0) \tag{3.36}$$

$$\mathrm{e}' = \frac{1}{2}(1 + \mathbf{e}_3\mathbf{e}_0) \tag{3.37}$$

$$\mathrm{e}'' = \frac{1}{2}(1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \tag{3.38}$$

are primitive idempotents of $\mathbb{R}_{1,3}$. The minimal left ideals,[8] $I = \mathbb{R}_{1,3}\mathrm{e}$, $I' = \mathbb{R}_{1,3}\mathrm{e}'$, $I'' = \mathbb{R}_{1,3}\mathrm{e}''$ are *right* two dimension linear spaces over the quaternion field ($\mathbb{H}\mathrm{e} = \mathrm{e}\mathbb{H} = \mathrm{e}\mathbb{R}_{1,3}\mathrm{e}$).

An elements $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1 + \mathbf{e}_0)$ has been called by Lounesto [15] a *mother* spinor.[9] Let us see the justice of this denomination. First recall from the general result of the previous section that $\frac{\mathrm{Pin}_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{O}_{1,3}$, $\frac{\mathrm{Spin}_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{SO}_{1,3}$, $\frac{\mathrm{Spin}^e_{1,3}}{\mathbb{Z}_2} \simeq \mathrm{SO}^e_{1,3}$, and $\mathrm{Spin}^e_{1,3} \simeq \mathrm{Sl}(2, \mathbb{C})$ is the universal covering group of $\mathcal{L}^{\uparrow}_{+} \equiv \mathrm{SO}^e_{1,3}$, the *special* (proper) *orthochronous* Lorentz group. We can show [10, 11] that the ideal $I = \mathbb{R}_{1,3}\mathrm{e}$ carries the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Sl}(2, \mathbb{C})$. Here we need to know [10, 11] that each Φ can be written as

$$\Phi = \psi_1 \mathrm{e} + \psi_2 \mathbf{e}_3\mathbf{e}_1\mathrm{e} + \psi_3 \mathbf{e}_3\mathbf{e}_0\mathrm{e} + \psi_4 \mathbf{e}_1\mathbf{e}_0\mathrm{e} = \sum_i \psi_i s_i, \tag{3.39}$$

$$s_1 = \mathrm{e}, \quad s_2 = \mathbf{e}_3\mathbf{e}_1\mathrm{e}, \quad s_3 = \mathbf{e}_3\mathbf{e}_0\mathrm{e}, \quad s_4 = \mathbf{e}_1\mathbf{e}_0\mathrm{e} \tag{3.40}$$

and where the ψ_i are *formally* complex numbers, i.e., each $\psi_i = (a_i + b_i\mathbf{e}_2\mathbf{e}_1)$ with $a_i, b_i \in \mathbb{R}$ and the set $\{s_i, i = 1, 2, 3, 4\}$ is a basis in the mother spinors space.

Exercise 3.41 Prove Eq. (3.39).

Now we determine an explicit relation between representations of $\mathbb{R}_{4,1}$ and $\mathbb{R}_{3,1}$. Let $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ be an orthonormal basis of $\mathbb{R}_{4,1}$ with

$$-\mathbf{f}_0^2 = \mathbf{f}_1^2 = \mathbf{f}_2^2 = \mathbf{f}_3^2 = \mathbf{f}_4^2 = 1,$$
$$\mathbf{f}_A\mathbf{f}_B = -\mathbf{f}_B v_A, A \neq B \text{ and } A, B = 0, 1, 2, 3, 4.$$

Define the pseudo-scalar

$$\mathfrak{i} = \mathbf{f}_0\mathbf{f}_1\mathbf{f}_2\mathbf{f}_3\mathbf{f}_4, \quad \mathfrak{i}^2 = -1, \quad \mathfrak{i}\mathbf{f}_A = \mathbf{f}_A\mathfrak{i}, \quad A = 0, 1, 2, 3, 4. \tag{3.41}$$

Put

$$\mathcal{E}_\mu = \mathbf{f}_\mu\mathbf{f}_4, \tag{3.42}$$

we can immediately verify that

$$\mathcal{E}_\mu\mathcal{E}_\nu + \mathcal{E}_\nu\mathcal{E}_\mu = 2\eta_{\mu\upsilon}. \tag{3.43}$$

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[8] According to Definition 3.47 these ideals are algebraically equivalent. For example, $\mathrm{e}' = u\mathrm{e}u^{-1}$, with $u = (1 + \mathbf{e}_3) \notin \Gamma_{1,3}$.

[9] Elements of I' are sometimes called Hestenes ideal spinors.

Taking into account that $\mathbb{R}_{1,3} \simeq \mathbb{R}^0_{4,1}$ we can explicitly exhibit here this isomorphism by considering the map $\mathtt{j}\colon \mathbb{R}_{1,3} \to \mathbb{R}^0_{4,1}$ generated by the linear extension of the map $\mathtt{j}^\# : \mathbb{R}^{1,3} \to \mathbb{R}^0_{4,1}$, $\mathtt{j}^\#(\mathbf{e}_\mu) = \mathcal{E}_\mu = \mathbf{f}_\mu \mathbf{f}_4$, where $\mathcal{E}_\mu$, $(\mu = 0, 1, 2, 3)$ is an orthogonal basis of $\mathbb{R}^{1,3}$. Note that $\mathtt{j}(1_{\mathbb{R}_{1,3}}) = 1_{\mathbb{R}^0_{4,1}}$, where $1_{\mathbb{R}_{1,3}}$ and $1_{\mathbb{R}^0_{4,1}}$ (usually denoted simply by 1) are the identity elements in $\mathbb{R}_{1,3}$ and $\mathbb{R}^0_{4,1}$. Now consider the primitive idempotent of $\mathbb{R}_{1,3} \simeq \mathbb{R}^0_{4,1}$,

$$\mathrm{e}^0_{41} = \mathtt{j}(\mathrm{e}) = \frac{1}{2}(1 + \mathcal{E}_0) \tag{3.44}$$

and the minimal left ideal $I^0_{4,1} = \mathbb{R}^0_{4,1}\mathrm{e}^0_{41}$.

The elements $Z \in I^0_{4,1}$ can be written analogously to $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1+\ \mathbf{e}_0)$ as,

$$Z = \sum z_i \bar{s}_i \tag{3.45}$$

where

$$\bar{s}_1 = \mathrm{e}^0_{41},\ \bar{s}_2 = \mathcal{E}_1\mathcal{E}_3\mathrm{e}^0_{41},\ \bar{s}_3 = \mathcal{E}_3\mathcal{E}_0\mathrm{e}^0_{41},\ \bar{s}_4 = \mathcal{E}_1\mathcal{E}_0\mathrm{e}^0_{41} \tag{3.46}$$

and where

$$z_i = a_i + \mathcal{E}_2\mathcal{E}_1 b_i,$$

are formally complex numbers, $a_i, b_i \in \mathbb{R}$.

Consider now the element $f \in \mathbb{R}_{4,1}$

$$\begin{aligned} f &= \mathrm{e}^0_{41}\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2) \\ &= \frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2), \end{aligned} \tag{3.47}$$

with i defined as in Eq. (3.41).

Since $f\mathbb{R}_{4,1}f = \mathbb{C}f = f\mathbb{C}$ it follows that f is a primitive idempotent of $\mathbb{R}_{4,1}$. We can easily show that each $\Phi \in I = \mathbb{R}_{4,1}f$ can be written

$$\Psi = \sum_i \psi_i f_i,\ \ \psi_i \in \mathbb{C},$$

$$f_1 = f,\ f_2 = -\mathcal{E}_1\mathcal{E}_3 f,\ f_3 = \mathcal{E}_3\mathcal{E}_0 f,\ f_4 = \mathcal{E}_1\mathcal{E}_0 f. \tag{3.48}$$

With the methods described in [10, 11] we find the following representation in $\mathbb{C}(4)$ for the generators $\mathcal{E}_\mu$ of $\mathbb{R}_{4,1} \simeq \mathbb{R}_{1,3}$

$$\mathcal{E}_0 \mapsto \underline{\gamma}_0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \leftrightarrow \mathcal{E}_i \mapsto \underline{\gamma}_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{3.49}$$

where $\mathbf{1}_2$ is the unit 2×2 matrix and σ_i, $(i = 1, 2, 3)$ are the standard *Pauli matrices*. We immediately recognize the $\underline{\gamma}$-matrices in Eq. (3.49) as the standard ones appearing, e.g., in [4].

The matrix representation of $\Psi \in I$ will be denoted by the same letter in boldface, i.e., $\Psi \mapsto \boldsymbol{\Psi} \in \mathbb{C}(4)f$, where

$$f = \frac{1}{2}(1 + \underline{\gamma}_0)\frac{1}{2}(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2), \quad \mathrm{i} = \sqrt{-1}. \tag{3.50}$$

We have

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \quad \psi_i \in \mathbb{C}. \tag{3.51}$$

Equations (3.49)–(3.51) are sufficient to prove that there are bijections between the elements of the ideals $\mathbb{R}_{1,3}\frac{1}{2}(1+\ \mathbf{e}_0)$, $\mathbb{R}^0_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ and $\mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathrm{i}\mathcal{E}_1\mathcal{E}_2)$.

We can easily find that the following relation exist between $\Psi \in \mathbb{R}_{4,1}f$ and $Z \in \mathbb{R}^0_{4,1}\frac{1}{2}(1 + \mathcal{E}_0)$,

$$\Psi = Z\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2). \tag{3.52}$$

Decomposing Z into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $\mathbb{R}^0_{4,1} \simeq \mathbb{R}_{1,3}$, $Z = Z^0 + Z^1$ we obtain $Z^0 = Z^1\mathcal{E}_0$ which clearly shows that all information of Z is contained in Z^0. Then,

$$\Psi = Z^0\frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2). \tag{3.53}$$

Now, if we take into account that $\mathbb{R}^0_{4,1}\frac{1}{2}(1 + \mathcal{E}_0) = \mathbb{R}^{00}_{4,1}\frac{1}{2}(1 + \mathcal{E}_0)$ where the symbol $\mathbb{R}^{00}_{4,1}$ means $\mathbb{R}^{00}_{4,1} \simeq \mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0}$ we see that each $Z \in \mathbb{R}^{00}_{4,1}\frac{1}{2}(1 + \mathcal{E}_0)$ can be written

$$Z = \psi\frac{1}{2}(1 + \mathcal{E}_0) \quad \psi \in \mathbb{R}^{00}_{4,1} \simeq \mathbb{R}^0_{1,3}. \tag{3.54}$$

Then putting $Z^0 = \psi/2$, Eq. (3.54) can be written

$$\Psi = \psi \frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2)$$
$$= Z^0 \frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2). \quad (3.55)$$

The matrix representation of ψ and Z in $\mathbb{C}(4)$ (denoted by the same letter in boldface) in the matrix representation generated by the spin basis given by Eq. (3.48) are

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}. \quad (3.56)$$

3.5 Algebraic Spin Frames and Spinors

We introduce now the fundamental concept of *algebraic spin frames*.[10] This is the concept that will permit us to define spinors (steps (i)–(vii)).[11]

(i) In this section $(\mathbf{V}, \boldsymbol{\eta})$ refers always to Minkowski vector space.
(ii) Let $\mathrm{SO}(\mathbf{V}, \boldsymbol{\eta})$ be the group of endomorphisms of $\mathbf{V}$ that preserves $\boldsymbol{\eta}$ and the space orientation. This group is isomorphic to $\mathrm{SO}_{1,3}$ but there is no *natural* isomorphism. We write $\mathrm{SO}(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{SO}_{1,3}$. Also, the connected component to the identity is denoted by $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ and $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{SO}^e_{1,3}$. Note that $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ preserves besides *orientation* also the *time* orientation.
(iii) We denote by $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ the Clifford algebra[12] of $(\mathbf{V}, \boldsymbol{\eta})$ and by $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{Spin}^e_{1,3}$ the connected component of the spin group $\mathrm{Spin}(\mathbf{V}, \boldsymbol{\eta}) \simeq \mathrm{Spin}_{1,3}$. Consider the $2:1$ homomorphism $\mathbf{L} : \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) \to \mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta}), u \mapsto \mathbf{L}(u) \equiv \mathbf{L}_u$. $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ acts on $\mathbf{V}$ identified as the space of 1-vectors of $\mathcal{C}\ell(\mathbf{V}, \eta) \simeq \mathbb{R}_{1,3}$ through its adjoint representation in the Clifford algebra $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ which

[10] The name *spin frame* will be reserved for a section of the spinor bundle structure $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ which will be introduced in Chap. 7.

[11] This section follows the developments given in [22].

[12] We reserve the notation $\mathbb{R}_{p,q}$ for the Clifford algebra of the vector space $\mathbb{R}^n$ equipped with a metric of signature (p, q), $p + q = n$. $\mathcal{C}\ell(\mathbf{V}, \mathbf{g})$ and $\mathbb{R}_{p,q}$ are isomorphic, but there is no canonical isomorphism. Indeed, an isomorphism can be exhibit only after we fix an orthonormal basis of $\mathbf{V}$.

is related with the vector representation of $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ as follows[13]:

$$\mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta}) \ni u \mapsto \mathrm{Ad}_u \in \mathrm{Aut}(\mathcal{C}\ell(\mathbf{V},\eta))$$
$$\mathrm{Ad}_u|_{\mathbf{V}} : \mathbf{V} \to \mathbf{V},\ \mathbf{v} \mapsto u\mathbf{v}u^{-1} = \mathbf{L}_u \odot \mathbf{v}. \tag{3.57}$$

In Eq. (3.57) $\mathbf{L}_u \odot \mathbf{v}$ denotes the standard action $\mathbf{L}_u$ on $\mathbf{v}$ and where we identified $\mathbf{L}_u \in \mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ with $\mathbf{L}_u \in \mathbf{V}\otimes\mathbf{V}^*$ and

$$\boldsymbol{\eta}\,(\mathbf{L}_u \odot \mathbf{v}, \mathbf{L}_u \odot \mathbf{v}) = \boldsymbol{\eta}\,(\mathbf{v},\mathbf{v})\,. \tag{3.58}$$

(iv) Let $\mathcal{B}$ be the set of all oriented and time oriented orthonormal basis[14] of $\mathbf{V}$. Choose among the elements of $\mathcal{B}$ a basis $b_0 = \{\mathbf{b}_0,\ldots.,\mathbf{b}_3\}$, hereafter called the *fiducial frame* of $\mathbf{V}$. With this choice, we define a $1-1$ mapping

$$\boldsymbol{\Sigma} : \mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta}) \to \mathcal{B}, \tag{3.59}$$

given by

$$\mathbf{L}_u \mapsto \boldsymbol{\Sigma}\ (\mathbf{L}_u) := \boldsymbol{\Sigma}_{\mathbf{L}_u} = \mathbf{L}b_0 \tag{3.60}$$

where $\boldsymbol{\Sigma}_{\mathbf{Lu}} = \mathbf{L}_u b_0$ is a short for $\{\mathbf{e}_1,\ldots.,\mathbf{e}_{3,}\} \in \mathcal{B}$, such that denoting the action of $\mathbf{L}_u$ on $\mathbf{b}_i \in b_0$ by $\mathbf{L}_u \odot \mathbf{b}_i$ we have

$$\mathbf{e}_i = \mathbf{L}_u \odot \mathbf{b}_i := L^{j}_{\cdot i}\mathbf{b}_j, \quad i,j = 0,\ldots,3. \tag{3.61}$$

In this way, we can identify a given vector basis b of $\mathbf{V}$ with the isometry $\mathbf{L}_u$ that takes the fiducial basis b_0 to b. The fiducial basis b_0 will be also denoted by $\boldsymbol{\Sigma}_{\mathbf{L}_0}$, where $\mathbf{L}_0 = e$, is the *identity* element of $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$.

Since the group $\mathrm{SO}^e(\mathbf{V},\boldsymbol{\eta})$ is *not* simple connected their elements cannot distinguish between frames whose spatial axes are *rotated* in relation to the fiducial vector frame $\boldsymbol{\Sigma}_{\mathbf{L}_0}$ by multiples of 2π or by multiples of 4π. For what follows it is crucial to make such a distinction. This is done by introduction of the concept of *algebraic spin frames*.

Definition 3.42 Let $b_0 \in \mathcal{B}$ be a fiducial frame and choose an arbitrary $u_0 \in \mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta})$. Fix once and for all the pair (u_0, b_0) with $u_0 = 1$ and call it the fiducial algebraic spin frame.

Definition 3.43 The space $\mathrm{Spin}^e(\mathbf{V},\boldsymbol{\eta}) \times \mathcal{B} = \{(u,b), ubu^{-1} = u_0 b_0 u_0^{-1}\}$ will be called the space of algebraic spin frames and denoted by $\mathcal{S}$.

[13] $\mathrm{Aut}(\mathcal{C}\ell(\mathbf{V},g))$ denotes the (inner) automorphisms of $\mathcal{C}\ell(\mathbf{V},g)$.

[14] We will call the elements of $\mathcal{B}$ (in what follows) simply by orthonormal basis.

Remark 3.44 It is crucial for what follows to observe here that Definition 3.43 implies that a given $b \in \mathcal{B}$ determines two, and only two, algebraic spin frames, namely (u, b) and $(-u, b)$, since $\pm ub(\pm u^{-1}) = u_0 b_0 u_0^{-1}$.

(v) We now parallel the construction in (iv) but replacing $\mathrm{SO}^e(\mathbf{V}, \boldsymbol{\eta})$ by its universal covering group $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ and $\mathcal{B}$ by S. Thus, we define the $1-1$ mapping

$$\begin{aligned} \Xi : \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}) &\to \quad\quad \mathcal{S}, \\ u \quad\quad &\mapsto \Xi(u) := \Xi_u = (u, b), \end{aligned} \tag{3.62}$$

where $ubu^{-1} = b_0$.

The fiducial algebraic spin frame will be denoted in what follows by Ξ_0. It is obvious from Eq. (3.62) that $\Xi(-u) \equiv \Xi_{-u} = (-u, b) \neq \Xi_u$.

Definition 3.45 The natural right action of $a \in \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ denoted by $\odot$ on $\mathcal{S}$ is given by

$$a \odot \Xi_u = a \odot (u, b) = (ua, Ad_{a^{-1}} b) = (ua, a^{-1} ba). \tag{3.63}$$

Observe that if $\Xi_{u'} = (u', b') = u' \odot \Xi_0$ and $\Xi_u = (u, b) = u \odot \Xi_0$ then,

$$\Xi_{\acute{u}} = (u^{-1} u') \odot \Xi_u = (u', u'^{-1} ubu^{-1} u').$$

Note that there is a natural $2-1$ mapping

$$\mathbf{s} : \mathcal{S} \to \mathcal{B}, \qquad \Xi_{\pm u} \mapsto b = (\pm u^{-1}) b_0 (\pm u), \tag{3.64}$$

such that

$$\mathbf{s}((u^{-1} u') \odot \Xi_u)) = \mathrm{Ad}_{(u^{-1} u')^{-1}} (\mathbf{s}(\Xi_u)). \tag{3.65}$$

Indeed,

$$\begin{aligned} \mathbf{s}((u^{-1} u') \odot \Xi_u)) &= \mathbf{s}((u^{-1} u') \odot (u, b)) \\ &= u'^{-1} ub (u'^{-1} u)^{-1} = b' \\ &= \mathrm{Ad}_{(u^{-1} u')^{-1}} b = \mathrm{Ad}_{(u^{-1} u')^{-1}} (\mathbf{s}(\Xi_u)). \end{aligned} \tag{3.66}$$

This means that the natural right actions of $\mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$, respectively on $\mathcal{S}$ and $\mathcal{B}$, commute. In particular, this implies that the algebraic spin frames $\Xi_u, \Xi_{-u} \in \mathcal{S}$, which are, of course distinct, determine the same vector frame $\Sigma_{\mathbf{L}_u} = \mathbf{s}(\Xi_u) = \mathbf{s}(\Xi_{-u}) = \Sigma_{\mathbf{L}_{-u}}$. We have,

$$\boldsymbol{\Sigma}_{\mathbf{L}_u} = \boldsymbol{\Sigma}_{\mathbf{L}_{-u}} = \mathbf{L}_{u^{-1} u_0} \boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}, \ \mathbf{L}_{u^{-1} u_0} \in \mathrm{SO}^e_{1,3}. \tag{3.67}$$

Also, from Eq. (3.65), we can write explicitly

$$u_0\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}u_0^{-1} = u\boldsymbol{\Sigma}_{\mathbf{L}_u}u^{-1},\ u_0\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}}u_0^{-1} = (-u)\boldsymbol{\Sigma}_{\mathbf{L}_{-u}}(-u)^{-1}, u \in \mathrm{Spin}^e(\mathbf{V}, \mathbf{g}), \tag{3.68}$$

where the meaning of Eq. (3.68) of course, is that *if* $\boldsymbol{\Sigma}_{\mathbf{L}_u} = \boldsymbol{\Sigma}_{\mathbf{L}_{-u}} = b = \{\mathbf{e}_0, \ldots., \mathbf{e}_{3,}\} \in \mathcal{B}$ and $\boldsymbol{\Sigma}_{\mathbf{L}_{u_0}} = b_0 \in \mathcal{B}$ is the fiducial frame, then

$$u_0\mathbf{b}_j u_0^{-1} = (\pm u)\mathbf{e}_j(\pm u^{-1}). \tag{3.69}$$

In resume, we can say that the space $\mathcal{S}$ of algebraic spin frames can be thought as an *extension* of the space $\mathcal{B}$ of *vector frames*, where even if two vector frames have the *same* ordered vectors, they are considered distinct if the spatial axes of one vector frame is rotated by an odd number of 2π rotations relative to the other vector frame and are considered the same if the spatial axes of one vector frame is rotated by an even number of 2π rotations relative to the other frame. Even if the possibility of such a distinction seems to be impossible at first sight, Aharonov and Susskind [1] claim that it can be implemented physically in a spacetime where the concept of algebraic spin frame is enlarged to the concept of spin frame used for the definition of spinor fields. See Chap. 7 for details.

(vi) Before we proceed an important *digression* on the notation used below is necessary. We recalled above how to construct a minimum left (or right) ideal for a given real Clifford algebra once a vector basis $b \in \mathcal{B}$ for $\mathbf{V} \hookrightarrow \mathcal{C}\ell(\mathbf{V}, \mathbf{g})$ is given. That construction suggests to *label* a given primitive idempotent and its corresponding ideal with the subindex b. However, taking into account the above discussion of vector and algebraic spin frames and their relationship we find useful for what follows (specially in view of the Definition 3.46 and the definitions of algebraic and Dirac-Hestenes spinors (see Definitions 3.48 and 3.50 below) to label a given primitive idempotent and its corresponding ideal with a subindex $\boldsymbol{\Xi}_u$. This notation is also justified by the fact that a given idempotent is according to definition 3.48 *representative* of a particular spinor in a given algebraic spin frame $\boldsymbol{\Xi}_u$.

(vii) Next we recall Theorem 3.28 which says that a minimal left ideal of $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ is of the type

$$I_{\boldsymbol{\Xi}_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\boldsymbol{\Xi}_u} \tag{3.70}$$

where $\mathrm{e}_{\boldsymbol{\Xi}_u}$ is a primitive idempotent of $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$.

It is easy to see that all ideals $I_{\boldsymbol{\Xi}_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\boldsymbol{\Xi}_u}$ and $I_{\boldsymbol{\Xi}_{u'}} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\boldsymbol{\Xi}_{u'}}$ such that

$$\mathrm{e}_{\boldsymbol{\Xi}_{u'}} = (u'^{-1}u)\mathrm{e}_{\boldsymbol{\Xi}_u}(u'^{-1}u)^{-1} \tag{3.71}$$

$u, u' \in \mathrm{Spin}^e(\mathbf{V}, \eta)$ are isomorphic. We have the

Definition 3.46 Any two ideals $I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u}$ and $I_{\Xi_{u'}} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_{u'}}$ such that their generator idempotents are related by Eq. (3.71) are said geometrically equivalent.

Remark 3.47 If u is simply an element of the Clifford group, then the ideals are said to be algebraically equivalent.

But take care, no *equivalence relation* has been defined until now. We observe moreover that we can write

$$I_{\Xi_{u'}} = I_{\Xi_u}(u'^{-1}u)^{-1}, \tag{3.72}$$

an equation that will play a key role in what follows.

3.6 Algebraic Dirac Spinors of Type I_{Ξ_u}

Let $\{I_{\Xi_u}\}$ be the set of all ideals geometrically equivalent to a given minimal $I_{\Xi_{u_o}}$ as defined by Eq. (3.72). Let be

$$\mathfrak{T} = \{(\Xi_u, \Psi_{\Xi_u}) \mid u \in \mathrm{Spin}^e(\mathbf{V}, \boldsymbol{\eta}),\ \Xi_u \in S,\ \Psi_{\Xi_u} \in I_{\Xi_u}\}. \tag{3.73}$$

Let $\Xi_u, \Xi_{u'} \in S$, $\Psi_{\Xi_u} \in I_{\Xi_u}$, $\Psi_{\Xi_{u'}} \in I_{\Xi_{u'}}$. We define an equivalence relation $\mathcal{E}$ on $\mathfrak{T}$ by setting

$$(\Xi_u, \Psi_{\Xi_u}) \sim (\Xi_{u'}, \Psi_{\Xi_{u'}}), \tag{3.74}$$

if and only if and

$$\begin{aligned} &\text{(i) } u\mathbf{s}(\Xi_u)u^{-1} = u'\mathbf{s}(\Xi_{u'})u'^{-1}, \\ &\text{(ii) } \Psi_{\Xi_{u'}}u'^{-1} = \Psi_{\Xi_u}u^{-1}. \end{aligned} \tag{3.75}$$

Definition 3.48 An equivalence class

$$\boldsymbol{\Psi}_{\Xi_u} = [(\Xi_u, \Psi_{\Xi_u})] \in \mathfrak{T}/\mathcal{E} \tag{3.76}$$

is called an algebraic spinor of type I_{Ξ_u} for $\mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$. $\psi_{\Xi_u} \in I_{\Xi_u}$ is said to be a representative of the algebraic spinor $\boldsymbol{\Psi}_{\Xi_u}$ in the algebraic spin frame Ξ_u.

We observe that the pairs (Ξ_u, Ψ_{Ξ_u}) and $(\Xi_{-u}, \Psi_{\Xi_{-u}}) = (\Xi_{-u}, -\Psi_{\Xi_u})$ are equivalent, but the pairs (Ξ_u, Ψ_{Ξ_u}) and $(\Xi_{-u}, -\Psi_{\Xi_{-u}}) = (\Xi_{-u}, \Psi_{\Xi_u})$ are not. This distinction is *essential* in order to give a structure of linear space (over the real field) to the set $\mathfrak{T}$. Indeed, a natural linear structure on $\mathfrak{T}$ is given by

$$a[(\Xi_u, \Psi_{\Xi_u})] + b[(\Xi_u, \Psi'_{\Xi_u})] = [(\Xi_u, a\Psi_{\Xi_u})] + [(\Xi_{u'}, b\Psi'_{\Xi_u})],$$
$$(a+b)[(\Xi_u, \Psi_{\Xi_u})] = a[(\Xi_u, \Psi_{\Xi_u})] + b[(\Xi_u, \Psi_{\Xi_u})]. \tag{3.77}$$

Remark 3.49 The definition just given is not a standard one in the literature [5, 8]. However, the fact is that the standard definition (licit as it is from the mathematical point of view) is *not* adequate for a comprehensive formulation of the Dirac equation using algebraic spinor fields or Dirac-Hestenes spinor fields which will be introduced in Chap. 7.

We end this section recalling that as observed above a given Clifford algebra $\mathbb{R}_{p,q}$ may have minimal ideals that are not geometrically equivalent since they may be generated by primitive idempotents that are related by elements of the group $\mathbb{R}^{\star}_{p,q}$ which are not elements of $\text{Spin}^e(\mathbf{V}, \boldsymbol{\eta})$ [see Eqs. (3.36)–(3.38)] where different, non geometrically equivalent primitive ideals for $\mathbb{R}_{1,3}$ are shown). These ideals may be said to be of different *types*. However, from the point of view of the representation theory of the real Clifford algebras all these primitive ideals carry equivalent (i.e., isomorphic) *modular* representations of the Clifford algebra and no preference may be given to any one.[15] In what follows, when no confusion arises and the ideal I_{Ξ_u} is clear from the context, we use the wording algebraic Dirac spinor for any one of the possible types of ideals.

The most important property concerning algebraic Dirac spinors is a coincidence given by Eq. (3.78) below. It permit us to define a *new* kind of spinors.

3.7 Dirac-Hestenes Spinors (DHS)

Let $\Xi_u \in \mathcal{S}$ be an algebraic spin frame for $(\mathbf{V}, \boldsymbol{\eta})$ such that

$$\mathbf{s}(\Xi_u) = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathcal{B}.$$

Then, it follows from Eq. (3.54) that

$$I_{\Xi_u} = \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u} = \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta})\mathrm{e}_{\Xi_u}, \tag{3.78}$$

when

$$\mathrm{e}_{\Xi_u} = \frac{1}{2}(1 + \mathbf{e}_0). \tag{3.79}$$

[15]The fact that there are ideals that are algebraically, but not geometrically equivalent seems to contain the seed for new Physics, see [18, 19].

Then, each $\Psi_{\Xi_u} \in I_{\Xi_u}$ can be written as

$$\Psi_{\Xi_u} = \psi_{\Xi_u} \mathrm{e}_{\Xi_u}, \quad \psi_{\Xi_u} \in \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}). \tag{3.80}$$

From Eq. (3.75) we get

$$\psi_{\Xi_{u'}} u'^{-1} u \mathrm{e}_{\Xi_u} = \psi_{\Xi_u} \mathrm{e}_{\Xi_u}, \quad \psi_{\Xi_u}, \psi_{\Xi_{u'}} \in \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}). \tag{3.81}$$

A possible solution for Eq. (3.81) is

$$\psi_{\Xi_{u'}} u'^{-1} = \psi_{\Xi_u} u^{-1}. \tag{3.82}$$

Let $\mathcal{S} \times \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})$ and consider an equivalence relation $\mathcal{E}$ such that

$$(\Xi_u, \phi_{\Xi_u}) \sim (\Xi_{u'}, \phi_{\Xi_{u'}}) \ (\mathrm{mod}\ \mathcal{E}), \tag{3.83}$$

if and only if $\phi_{\Xi_{u'}}$ and ϕ_{Ξ_u} are related by

$$\phi_{\Xi_{u'}} u'^{-1} = \phi_{\Xi_u} u^{-1}. \tag{3.84}$$

This suggests the following

Definition 3.50 The equivalence classes $[(\Xi_u, \phi_{\Xi_u})] \in \mathcal{S} \times \mathcal{C}\ell(\mathbf{V}, \boldsymbol{\eta})/\mathcal{E}$ are the Hestenes spinors.

Among the Hestenes spinors, an important subset is the one consisted of Dirac-Hestenes spinors where $[(\Xi_u, \psi_{\Xi_u})] \in (\mathcal{S} \times \mathcal{C}\ell^0(\mathbf{V}, \boldsymbol{\eta}))/\mathcal{E}$.

We say that ϕ_{Ξ_u} (ψ_{Ξ_u}) is a representative of a Hestenes (Dirac-Hestenes) spinor in the algebraic spin frame Ξ_u.

3.7.1 What is a Covariant Dirac Spinor (CDS)

Let $\mathbf{L}' : \mathcal{S} \to \mathcal{B}$ and let $\mathbf{L}'(\Xi_u) = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ and $\mathbf{L}'(\Xi_{u'}) = \{\mathcal{E}'_0, \mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3\}$ with $\mathbf{L}'(\Xi_u) = u\mathbf{L}'(\Xi_o)u^{-1}$, $\mathbf{L}(\Xi_{u'}) = u'\mathbf{L}'(\Xi_o)u'^{-1}$ be two arbitrary basis for $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{4,1}$.

As we already know $f_{\Xi_0} = \frac{1}{2}(1 + \mathcal{E}_0)\frac{1}{2}(1 + \mathrm{i}\mathcal{E}_1\mathcal{E}_2)$ [Eq. (3.48)] is a primitive idempotent of $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. If $u \in \mathrm{Spin}\,(1,3) \subset \mathrm{Spin}(4,1)$ then all ideals $I_{\Xi_u} = I_{\Xi_0} u^{-1}$ are geometrically equivalent to I_{Ξ_0}. From Eq. (3.49) we can write

$$I_{\Xi_u} \ni \Psi_{\Xi_u} = \sum \psi_i f_i, \text{ and } I_{\Xi'_u} \ni \Psi_{\Xi_u} = \sum \psi'_i f'_i, \tag{3.85}$$

where

$$f_1 = f_{\Xi_u}, \quad f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Xi_u}, \quad f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Xi_u}, \quad f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Xi_u}$$

and

$$f_1' = f_{\Xi_u}, \quad f_2' = -\mathcal{E}_1'\mathcal{E}_3' f_{\Xi_u}, \quad f_3' = \mathcal{E}_3'\mathcal{E}_0' f_{\Xi_u}, \quad f_4 = \mathcal{E}_1'\mathcal{E}_0' f_{\Xi_u}.$$

Since $\Psi_{\Xi_{u'}} = \Psi_{\Xi_u}(u'^{-1}u)^{-1}$, we get

$$\Psi_{\Xi_u'} = \sum_i \psi_i (u'^{-1}u)^{-1} f_i' = \sum_{i,k} S_{ik}[(u^{-1}u')]\psi_i f_k = \sum_k \psi_k f_k.$$

Then

$$\psi_k = \sum_i S_{ik}(u^{-1}u')\psi_i, \tag{3.86}$$

where $S_{ik}(u^{-1}u')$ are the matrix components of the representation in $\mathbb{C}(4)$ of $(u^{-1}u')$ $\in \mathrm{Spin}^e_{1,3}$. As proved in [10, 11] the matrices $S(u)$ correspond to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $\mathrm{Sl}(2,\mathbb{C}) \simeq \mathrm{Spin}^e_{1,3}$.

Remark 3.51 We remark that all the elements of the set $\{I_{\Xi_u}\}$ of the ideals geometrically equivalent to I_{Ξ_0} under the action of $u \in \mathrm{Spin}^e_{1,3} \subset \mathrm{Spin}^e_{4,1}$ have the same image $I = \mathbb{C}(4)f$ where f is given by Eq. (3.47), i.e.,

$$f = \frac{1}{2}(1 + \underline{\gamma}_0)(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2), \quad \mathrm{i} = \sqrt{-1},$$

where $\underline{\gamma}_\mu$, $\mu = 0, 1, 2, 3$ are the Dirac matrices given by Eq. (3.50). Then, if

$$\begin{aligned} \gamma &: \mathbb{R}_{4,1} \to \mathbb{C}(4) \equiv \mathrm{End}(\mathbb{C}(4)), \\ x &\mapsto \gamma(x) : \mathbb{C}(4)f \to \mathbb{C}(4)f \end{aligned} \tag{3.87}$$

it follows that

$$\gamma(\mathcal{E}_\mu) = \gamma(\mathcal{E}_\mu'), \quad \gamma(f_\mu) = \gamma(f_\mu') \tag{3.88}$$

for all $\{\mathcal{E}_\mu\}$, $\{\mathcal{E}_\mu'\}$ such that $\mathcal{E}_\mu' = (u'^{-1}u)\mathcal{E}_\mu(u'^{-1}u)^{-1}$. Observe that *all information* concerning the geometrical images of the algebraic spin frames Ξ_u, $\Xi_{u'}, \ldots,$ under $\mathbf{L}'$*disappear* in the matrix representation of the ideals $I_{\Xi_u}, I_{\Xi_{u'}}, \ldots,$ in $\mathbb{C}(4)$ since all these ideals are mapped in the same ideal $I = \mathbb{C}(4)f$.

Taking into account Remark 3.51 and taking into account the definition of algebraic spinors given above and Eq. (3.86) we are lead to the following

Definition 3.52 A covariant Dirac spinor for $\mathbb{R}^{1,3}$ is an equivalence class of pairs $(\Xi_u^m, \boldsymbol{\Psi})$, where Ξ_u^m is a matrix algebraic spin frame associated to the algebraic spin frame Ξ_u through the $S(u^{-1}) \in D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of $\mathrm{Spin}^e_{1,3}$, $u \in \mathrm{Spin}^e_{1,3}$.

We say that $\boldsymbol{\Psi}, \boldsymbol{\Psi}' \in \mathbb{C}(4)f$ are equivalent and write

$$(\Xi_u^m, \boldsymbol{\Psi}) \sim (\Xi_{u'}^m, \boldsymbol{\Psi}'), \tag{3.89}$$

if and only if,

$$\boldsymbol{\Psi}' = S(u'^{-1}u)\boldsymbol{\Psi}, \quad u\mathbf{s}(\boldsymbol{\Xi}_u)u^{-1} = u'\mathbf{s}(\boldsymbol{\Xi}_{u'})u'^{-1}. \tag{3.90}$$

Remark 3.53 The definition of *CDS* just given agrees with that given in [6] except for the irrelevant fact that there, as well as in the majority of Physics textbook's, authors use as the space of representatives of a *CDS* a complex four-dimensional space $\mathbb{C}^4$ instead of $I = \mathbb{C}(4)f$.

3.7.2 Canonical Form of a Dirac-Hestenes Spinor

Let $\mathbf{v} \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ be a *non* lightlike vector, i.e., $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v} \neq 0$ and consider a linear mapping

$$L_\psi : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}, \mathbf{v} \mapsto \mathbf{z} = \psi\mathbf{v}\tilde{\psi}, \quad \mathbf{z}^2 = \rho\mathbf{v}^2 \tag{3.91}$$

with $\psi \in \mathbb{R}_{1,3}$ and $\rho \in \mathbb{R}^+$. Now, recall that if $R \in \mathrm{Spin}_{1,3}^e$ then $\mathbf{w} = R\mathbf{v}\tilde{R}$ is such that $\mathbf{w}^2 = \mathbf{v}^2$. It follows that the most general solution of Eq. (3.91) is

$$\psi = \rho^{1/2} e^{\frac{\beta}{2}\mathbf{e}_5} R, \tag{3.92}$$

where $\beta \in \mathbb{R}$ is called the Takabayasi angle and $\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \in \bigwedge^4 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ is the pseudoscalar of the algebra. Now, Eq. (3.92) shows that $\psi \in \mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$. Moreover, we have that $\psi\tilde{\psi} \neq 0$ since

$$\begin{aligned} \psi\tilde{\psi} &= \rho e^{\beta\mathbf{e}_5} = \sigma + \mathbf{e}_5\omega, \\ \sigma &= \rho\cos\beta, \quad \omega = \rho\sin\beta. \end{aligned} \tag{3.93}$$

The Secret

Now, let $\psi_{\boldsymbol{\Xi}_u}$ be a representative of a Dirac-Hestenes spinor (Definition 3.50) in a given spin frame $\boldsymbol{\Xi}_u$. Since $\psi_{\boldsymbol{\Xi}_u} \in \mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0}$ we have disclosed the *real geometrical meaning* of a Dirac-Hestenes spinor. Indeed, a Dirac-Hestenes spinor such that $\psi_{\boldsymbol{\Xi}_u}\tilde{\psi}_{\boldsymbol{\Xi}_u} \neq 0$ induces the linear mapping given by Eq. (3.91), which *rotates* a vector and *dilate* it. Observe, that even if we started our considerations with $\mathbf{v} \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ and $\mathbf{v}^2 \neq 0$, the linear mapping (3.91) also rotates and 'dilate' a light vector.

3.7.3 Bilinear Invariants and Fierz Identities

Definition 3.54 Given a representative ψ_{Ξ_u} of a *DHS* in the algebraic spin frame field Ξ_u the bilinear invariants[16] associated with it are the objects: $\sigma - \star\omega \in (\bigwedge^0 \mathbb{R}^{1,3} + \bigwedge^4 \mathbb{R}^{1,3}) \hookrightarrow \mathbb{R}_{1,3}$, $J = J_\mu \mathbf{e}^\mu \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, $S = \frac{1}{2} S_{\mu\nu} \mathbf{e}^\mu \mathbf{e}^\nu \in \bigwedge^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$, $K = K_\mu \mathbf{e}^\mu \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ such that

$$
\begin{aligned}
\psi_{\Xi_u} \tilde{\psi}_{\Xi_u} &= \sigma - \star\omega & \psi_{\Xi_u} \mathbf{e}^0 \tilde{\psi}_{\Xi_u} &= J, \\
\psi_{\Xi_u} \mathbf{e}^1 \mathbf{e}^2 \tilde{\psi}_{\Xi_u} &= S & \psi_{\Xi_u} \mathbf{e}^0 \mathbf{e}^3 \tilde{\psi}_{\Xi_u} &= \star S, \\
\psi_{\Xi_u} \mathbf{e}^3 \tilde{\psi}_{\Xi_u} &= K & \psi_{\Xi_u} \mathbf{e}^0 \mathbf{e}^1 \mathbf{e}^2 \tilde{\psi}_{\Xi_u} &= \star K.
\end{aligned}
\tag{3.94}
$$

and where $\star\omega = -\mathbf{e}_5 \rho \sin\beta$

The bilinear invariants satisfy the so called *Fierz identities*, which are

$$
J^2 = \sigma^2 + \omega^2, \qquad J \cdot K = 0, \qquad J^2 = -K^2, \qquad J \wedge K = (\omega - \star\sigma) S \tag{3.95}
$$

$$
\begin{cases}
S \llcorner J = -\omega K & S \llcorner K = -\omega J, \\
(\star S) \llcorner J = -\sigma K & (\star S) \llcorner K = -\sigma J, \\
S \cdot S = \langle S \tilde{S} \rangle_0 = \sigma^2 - \omega^2 & (\star S) \cdot S = -2\sigma\omega.
\end{cases}
\tag{3.96}
$$

$$
\begin{cases}
JS = (\omega - \star\sigma) K, \\
SJ = -(\omega + \star\sigma) K, \\
SK = (\omega - \star\sigma) J, \\
KS = -(\omega + \star\sigma) J, \\
S^2 = \omega^2 - \sigma^2 - 2\sigma(\star\omega), \\
S^{-1} = KSK/J^4.
\end{cases}
\tag{3.97}
$$

Exercise 3.55 Prove the Fierz identities.

---

[16] In Physics literature the components of J, S and K when written in terms of covariant Dirac spinors are called *bilinear covariants*.

3.7.4 Reconstruction of a Spinor

The importance of the bilinear invariants is that once we know ω, σ, J, K and F we can recover from them the associate covariant Dirac spinor (and thus the *DHS*) except for a phase. This can be done with an algorithm due to Crawford [7] and presented in a very pedagogical way in [14–16]. Here we only give the result for the case where σ and/or ω are non null. Define the object $\mathfrak{B} \in \mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1}$ called *boomerang* and given by ($\mathrm{i} = \sqrt{-1}$)

$$\mathfrak{B} = \sigma + J + \mathrm{i}S - \mathrm{i}\mathbf{e}_5 K + \mathbf{e}_5 \omega \tag{3.98}$$

Then, we can construct $\mathbf{\Psi} = \mathfrak{B} f \in \mathbb{R}_{4,1} f$, where f is the idempotent given by Eq. (3.47) which has the following matrix representation in $\mathbb{C}(4)$ (once the standard representation of the Dirac gamma matrices are used)

$$\hat{\mathbf{\Psi}} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \tag{3.99}$$

Now, it can be easily verified that $\mathbf{\Psi} = \mathfrak{B} f$ determines the same bilinear covariants as the ones determined by ψ_{Ξ_u}. Note however that this spinor is not unique. In fact, $\mathfrak{B}$ determines a class of elements $\mathfrak{B}\xi$ where ξ is an arbitrary element of $\mathbb{R}_{4,1} f$ which differs one from the other by a complex phase factor.

3.7.5 Lounesto Classification of Spinors

A very interesting classification of spinors have been devised by Lounesto [14–16] based on the values of the bilinear invariants. He identified *six* different classes and proved that there are *no* other classes based on distinctions between bilinear covariants. Lounesto classes are:

1. $\sigma \neq 0, \quad \omega \neq 0.$
2. $\sigma \neq 0, \quad \omega = 0.$
3. $\sigma = 0, \quad \omega \neq 0.$
4. $\sigma = 0 = \omega, \quad K \neq 0, \quad S \neq 0.$
5. $\sigma = 0 = \omega, \quad K = 0, \quad S \neq 0.$
6. $\sigma = 0 = \omega, \quad K \neq 0, \quad S = 0.$

The current density J is always non-zero. Type 1, 2 and 3 spinor are denominated *Dirac spinor* for spin-1/2 particles and type 4, 5, and 6 are *singular* spinors respectively called *flag-dipole*, *flagpole* and *Weyl spinor*. Majorana spinor is a particular case of a type 5 spinor. It is worthwhile to point out a peculiar feature of types 4, 5 and 6 spinor: although J is always non-zero, we have due to Fierz identities that $J^2 = -K^2 = 0$.

Spinors belonging to class 4 have not previously been identified in the literature. For the applications we have in mind we are interested (besides Dirac spinors which belong to classes 1 or 2 or 3) in spinors belonging to classes 5 and 6, respectively the Majorana and Weyl spinors.

Remark 3.56 In [2] Ahluwalia-Khalilova and Grumiller introduced from physical considerations a supposedly new kind of spinors representing dark matter that they dubbed *ELKO* spinors. The acronym stands for the German word *Eigenspinoren des Ladungskonjugationsoperators*. It has been proved in [9] that from the algebraic point of view *ELKO* spinors are simply class 5 spinors. In [2] it is claimed that differently from the case of Dirac, Majorana and Weyl spinor fields which have mass dimension $3/2$, *ELKO* spinor fields must have mass dimension 1 and thus instead of satisfying Dirac equation satisfy a Klein-Gordon equation. A thoughtful analysis of this claim is given in Chap. 16.

3.8 Majorana and Weyl Spinors

Recall that for *Majorana* spinors $\sigma = 0, \omega = 0, K = 0, S \neq 0, J, \neq 0$

Given a representative ψ of an arbitrary Dirac-Hestenes spinor we may construct the Majorana spinors

$$\psi_M^{\pm} = \frac{1}{2}\left(\psi \pm \psi \mathbf{e}_{01}\right). \tag{3.100}$$

Note that defining an operator $C{:}\psi \mapsto \psi\mathbf{e}_{01}$ (*charge conjugation*) we have

$$C\psi_M^{\pm} = \pm\psi_M^{\pm}, \tag{3.101}$$

i.e., Majorana spinors are eigenvectors of the charge conjugation operator. Majorana spinors satisfy

$$\tilde{\psi}_M^{\pm}\psi_M^{\pm} = \psi_M^{\pm}\tilde{\psi}_M^{\pm} = 0. \tag{3.102}$$

For Weyl spinors $\sigma = \omega = S = 0$ and $K \neq 0, J \neq 0$.

Given a representative ψ of an arbitrary Dirac-Hestenes spinor we may construct the Weyl spinors

$$\psi_W^{\pm} = \frac{1}{2}\left(\psi \mp \mathbf{e}_5\psi\mathbf{e}_{21}\right). \tag{3.103}$$

Weyl spinors are 'eigenvectors' of the *chirality operator* $\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, i.e.,

$$\mathbf{e}_5\psi_W^{\pm} = \pm\psi_W^{\pm}\mathbf{e}_{21}. \tag{3.104}$$

We have also,

$$\tilde{\psi}_W^{\pm}\psi_W^{\pm} = \psi_W^{\pm}\tilde{\psi}_W^{\pm} = 0. \tag{3.105}$$

For future reference we introduce the parity operator acting on the space of Dirac-Hestenes spinors. The parity operator P in this formalism [13] is represented in such a way that for $\psi \in \mathbb{R}_{1,3}^0$

$$P\psi = -\mathbf{e}_0\psi\mathbf{e}_0\ . \tag{3.106}$$

The following Dirac-Hestenes spinors are eigenstates of the parity operator with eigenvalues ± 1:

$$\begin{aligned} P\psi^{\uparrow} &= +\psi^{\uparrow}\ , \quad \psi^{\uparrow} = \mathbf{e}_0\psi_{-}\mathbf{e}_0 - \psi_{-}\ , \\ P\psi^{\downarrow} &= -\psi^{\downarrow}\ , \quad \psi^{\downarrow} = \mathbf{e}_0\psi_{+}\mathbf{e}_0 + \psi_{+}\ , \end{aligned} \tag{3.107}$$

where $\psi_{\pm} := \psi_W^{\pm}$

3.9 Dotted and Undotted Algebraic Spinors

Dotted and undotted covariant spinor *fields* are very popular subjects in General Relativity. Dotted and undotted algebraic spinor *fields* may be introduced using the methods of Chap. 7 and are briefly discussed in Exercise 7.63. A preliminary to that job is a deep understanding of the algebraic aspects of those concepts, i.e., the dotted and undotted algebraic spinors which we now discuss. Their relation with Weyl spinors will become apparent in a while.

Recall that the spacetime algebra $\mathbb{R}_{1,3}$ is the *real* Clifford algebra associated with Minkowski vector space $\mathbb{R}^{1,3}$, which is a four dimensional real vector space, equipped with a Lorentzian bilinear form

$$\boldsymbol{\eta} : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{R}. \tag{3.108}$$

Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an arbitrary orthonormal basis of $\mathbb{R}^{1,3}$, i.e.,

$$\boldsymbol{\eta}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \eta_{\mu\nu}, \tag{3.109}$$

where the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix diag$(1, -1, -1, -1)$. Also, $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is the *reciprocal* basis of $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e., $\boldsymbol{\eta}(\mathbf{e}^\mu, \mathbf{e}_\nu) = \delta^\mu_\nu$. We have in obvious notation

$$\boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) = \eta^{\mu\nu},$$

where the matrix with entries $\eta^{\mu\nu}$ is the diagonal matrix diag$(1, -1, -1, -1)$.

The spacetime algebra $\mathbb{R}_{1,3}$ is generate by the following algebraic fundamental relation

$$\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \mathbf{e}^{\nu}\mathbf{e}^{\mu} = 2\eta^{\mu\nu}. \tag{3.110}$$

As we already know (Sect. 3.7.1) the spacetime algebra $\mathbb{R}_{1,3}$ as a vector space over the real field is isomorphic to the exterior algebra $\bigwedge\mathbb{R}^{1,3} = \bigoplus_{j=0}^{4}\bigwedge^{j}\mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$. We code that information writing $\bigwedge\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Also, we make the following identifications: $\bigwedge^{0}\mathbb{R}^{1,3} \equiv \mathbb{R}$ and $\bigwedge^{1}\mathbb{R}^{1,3} \equiv \mathbb{R}^{1,3}$. Moreover, we identify the exterior product of vectors by

$$\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu} = \frac{1}{2}\left(\mathbf{e}^{\mu}\mathbf{e}^{\nu} - \mathbf{e}^{\nu}\mathbf{e}^{\mu}\right), \tag{3.111}$$

and also, we identify the scalar product of vectors by

$$\boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) = \frac{1}{2}\left(\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \mathbf{e}^{\nu}\mathbf{e}^{\mu}\right). \tag{3.112}$$

Then we can write

$$\mathbf{e}^{\mu}\mathbf{e}^{\nu} = \boldsymbol{\eta}(\mathbf{e}^{\mu}, \mathbf{e}^{\nu}) + \mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}. \tag{3.113}$$

Now, an arbitrary element $\mathbf{C} \in \mathbb{R}_{1,3}$ can be written as sum of *nonhomogeneous multivectors*, i.e.,

$$\mathbf{C} = s + c_{\mu}\mathbf{e}^{\mu} + \frac{1}{2}c_{\mu\nu}\mathbf{e}^{\mu}\mathbf{e}^{\nu} + \frac{1}{3!}c_{\mu\nu\rho}\mathbf{e}^{\mu}\mathbf{e}^{\nu}\mathbf{e}^{\rho} + p\mathbf{e}^{5} \tag{3.114}$$

where $s, c_{\mu}, c_{\mu\nu}, c_{\mu\nu\rho}, p \in \mathbb{R}$ and $c_{\mu\nu}, c_{\mu\nu\rho}$ are completely antisymmetric in all indices. Also $\mathbf{e}^{5} = \mathbf{e}^{0}\mathbf{e}^{1}\mathbf{e}^{2}\mathbf{e}^{3}$ is the generator of the pseudoscalars. Recall also that as a matrix algebra we have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, the algebra of the 2×2 quaternionic matrices.

3.9.1 Pauli Algebra

Next, we recall (again) that the Pauli algebra $\mathbb{R}_{3,0}$ is the real Clifford algebra associated with the Euclidean vector space $\mathbb{R}^{3,0}$, equipped as usual, with a positive definite bilinear form. As a matrix algebra we have that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of 2×2 complex matrices. Moreover, we recall that $\mathbb{R}_{3,0}$ is isomorphic to the even subalgebra of the spacetime algebra, i.e., writing $\mathbb{R}_{1,3} = \mathbb{R}_{1,3}^{(0)} \oplus \mathbb{R}_{1,3}^{(1)}$ we have,

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)}. \tag{3.115}$$

The isomorphism is easily exhibited by putting $\boldsymbol{\sigma}^i = \mathbf{e}^i\mathbf{e}^0$, $i = 1, 2, 3$. Indeed, with $\delta^{ij} = \mathrm{diag}(1, 1, 1)$, we have

$$\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + \boldsymbol{\sigma}^j\boldsymbol{\sigma}^i = 2\delta^{ij}, \tag{3.116}$$

which is the fundamental relation defining the algebra $\mathbb{R}_{3,0}$. Elements of the Pauli algebra will be called Pauli numbers.[17] As a vector space over the real field, we have that $\mathbb{R}_{3,0}$ is isomorphic to $\bigwedge\mathbb{R}^{3,0} \hookrightarrow \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$. So, any Pauli number can be written as

$$\mathbf{P} = s + p^i\boldsymbol{\sigma}^i + \frac{1}{2}p^i_{ij}\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + p\mathrm{I}, \tag{3.117}$$

where $s, p_i, p_{ij}, p \in \mathbb{R}$ and $p_{ij} = -p_{ji}$ and also

$$\mathrm{I} = -\mathbf{i} = \boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3 = \mathbf{e}^5. \tag{3.118}$$

Note that $\mathrm{I}^2 = -1$ and that I commutes with any Pauli number. We can trivially verify

$$\begin{aligned} \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j &= \mathrm{I}\varepsilon^{ij}_{k}\boldsymbol{\sigma}^k + \delta^{ij}, \\ [\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] &= \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\boldsymbol{\sigma}^i = 2\boldsymbol{\sigma}^i \wedge \boldsymbol{\sigma}^j = 2\mathrm{I}\varepsilon^{ij}_{k}\boldsymbol{\sigma}^k. \end{aligned} \tag{3.119}$$

In that way, writing $\mathbb{R}_{3,0} = \mathbb{R}_{3,0}^{(0)} + \mathbb{R}_{3,0}^{(1)}$, any Pauli number can be written as

$$\mathbf{P} = \mathbf{Q}_1 + \mathrm{I}\mathbf{Q}_2, \quad \mathbf{Q}_1 \in \mathbb{R}_{3,0}^{(0)}, \quad \mathrm{I}\mathbf{Q}_2 \in \mathbb{R}_{3,0}^{(1)}, \tag{3.120}$$

with

$$\begin{aligned} \mathbf{Q}_1 &= a_0 + a_k(\mathrm{I}\boldsymbol{\sigma}^k), \quad a_0 = s, \quad a_k = \frac{1}{2}\varepsilon^{ij}_{k}p_{ij}, \\ \mathbf{Q}_2 &= \mathrm{I}\left(b_0 + b_k(\mathrm{I}\boldsymbol{\sigma}^k)\right), \quad b_0 = p, \quad b_k = -p_k. \end{aligned} \tag{3.121}$$

[17]Sometimes they are also called 'complex quaternions'. This last terminology will become obvious in a while.

3.9.2 Quaternion Algebra

Equation (3.121) show that the quaternion algebra $\mathbb{R}_{0,2} = \mathbb{H}$ can be identified as the even subalgebra of $\mathbb{R}_{3,0}$, i.e.,

$$\mathbb{R}_{0,2} = \mathbb{H} \simeq \mathbb{R}_{3,0}^{(0)}. \tag{3.122}$$

The statement is obvious once we identify the basis $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$ of $\mathbb{H}$ with

$$\{\mathbf{1}, \mathrm{I}\boldsymbol{\sigma}^1, \mathrm{I}\boldsymbol{\sigma}^2, \mathrm{I}\boldsymbol{\sigma}^3\}, \tag{3.123}$$

which are the generators of $\mathbb{R}_{3,0}^{(0)}$. We observe moreover that the even subalgebra of the quaternions can be identified (in an obvious way) with the complex field, i.e., $\mathbb{R}_{0,2}^{(0)} \simeq \mathbb{C}$. Returning to Eq. (3.117) we see that any $\mathbf{P} \in \mathbb{R}_{3,0}$ can also be written as

$$\mathbf{P} = \mathbf{P}_1 + \mathrm{I}\mathbf{L}_2, \tag{3.124}$$

where

$$\begin{aligned} \mathbf{P}_1 &= (s + p_k\boldsymbol{\sigma}^k) \in \bigwedge^0 \mathbb{R}^{3,0} \oplus \bigwedge^1 \mathbb{R}^{3,0} \equiv \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0}, \\ \mathrm{I}\mathbf{L}_2 &= \mathrm{I}(p + \mathrm{I}l_k\boldsymbol{\sigma}^k) \in \bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0}, \end{aligned} \tag{3.125}$$

with $l_k = -\varepsilon_k^{ij} p_{ij} \in \mathbb{R}$. The important fact that we want to emphasize here is that the subspaces $(\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ and $(\bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0})$ do not close separately any algebra. In general, if $\mathbf{A}, \mathbf{C} \in (\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ then

$$\mathbf{AC} \in \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0} \oplus \bigwedge^2 \mathbb{R}^{3,0}. \tag{3.126}$$

To continue, we introduce

$$\boldsymbol{\sigma}_i = \mathbf{e}_i\mathbf{e}_0 = -\boldsymbol{\sigma}^i, \quad i = 1, 2, 3. \tag{3.127}$$

Then, $\mathrm{I} = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ and the basis $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$ of $\mathbb{H}$ can be identified with $\{1, -\mathrm{I}\boldsymbol{\sigma}_1, -\mathrm{I}\boldsymbol{\sigma}_2, -\mathrm{I}\boldsymbol{\sigma}_3\}$.

Now, we know that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. This permit us to represent the Pauli numbers by 2×2 complex matrices, in the usual way ($\mathrm{i} = \sqrt{-1}$). We write $\mathbb{R}_{3,0} \ni \mathbf{P} \mapsto P \in \mathbb{C}(2)$, with

$$\begin{aligned} \boldsymbol{\sigma}^1 &\mapsto \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^2 &\mapsto \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\ \boldsymbol{\sigma}^3 &\mapsto \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{3.128}$$

3.9.3 Minimal Left and Right Ideals in the Pauli Algebra and Spinors

The elements $\mathrm{e}_\pm = \frac{1}{2}(1 + \boldsymbol{\sigma}_3) = \frac{1}{2}(1 + \mathbf{e}_3\mathbf{e}_0) \in \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}$, $\mathrm{e}_\pm^2 = \mathrm{e}_\pm$ are *minimal idempotents* of $\mathbb{R}_{3,0}$. They generate the minimal left and right ideals

$$\mathbf{I}_\pm = \mathbb{R}_{1,3}^{(0)}\mathrm{e}_\pm, \quad \mathbf{R}_\pm = \mathrm{e}_\pm\mathbb{R}_{1,3}^{(0)}. \tag{3.129}$$

From now on we write $\mathrm{e} = \mathrm{e}_+$. It can be easily shown (see below) that, e.g., $\mathbf{I} = \mathbf{I}_+$ has the structure of a 2-dimensional vector space over the complex field [10, 13], i.e., $\mathbf{I} \simeq \mathbb{C}^2$. The elements of the vector space $\mathbf{I}$ are called *representatives* of algebraic *contravariant undotted spinors*[18] and the elements of $\mathbb{C}^2$ are the usual *contravariant undotted spinors* used in physics textbooks. They carry the $D^{(\frac{1}{2},0)}$ representation of $\mathrm{Sl}(2,\mathbb{C})$ [17]. If $\boldsymbol{\varphi} \in \mathbf{I}$ we denote by $\varphi \in \mathbb{C}^2$ the usual matrix representative[19] of $\boldsymbol{\varphi}$ is

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \varphi^1, \varphi^2 \in \mathbb{C}. \tag{3.130}$$

Denoting by $\dot{\mathbf{I}} = \mathrm{e}\mathbb{R}_{1,3}^{(0)}$ the space of the algebraic covariant dotted spinors, we have the isomorphism, $\dot{\mathbf{I}} \simeq (\mathbb{C}^2)^\dagger \simeq \mathbb{C}_2$, where $\dagger$ denotes Hermitian conjugation. The elements of $(\mathbb{C}^2)^\dagger$ are the usual contravariant spinor fields used in physics textbooks. They carry the $D^{(0,\frac{1}{2})}$ representation of $\mathrm{Sl}(2,\mathbb{C})$ [17]. If $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$, then

[18]We omit in the following the term representative and call the elements of $\mathbf{I}$ simply by algebraic contravariant undotted spinors. However, the reader must always keep in mind that any algebraic spinor is an equivalence class, as defined and discussed in Sect. 4.6.

[19]The matrix representation of the elements of the ideals $\mathbf{I}$, $\dot{\mathbf{I}}$, are of course, 2×2 complex matrices (see, [10], for details). It happens that both columns of that matrices have the *same* information and the representation by column matrices is enough here for our purposes.

its matrix representation in $(\mathbb{C}^2)^\dagger$ is a row matrix usually denoted by

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix}, \quad \xi_{\dot{1}}, \xi_{\dot{2}} \in \mathbb{C}. \tag{3.131}$$

The following representation of $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ in $(\mathbb{C}^2)^\dagger$ is extremely convenient. We say that to a covariant undotted spinor ξ there corresponds a covariant dotted spinor $\dot{\xi}$ given by

$$\dot{\mathbf{I}} \ni \dot{\boldsymbol{\xi}} \mapsto \dot{\xi} = \bar{\xi}\varepsilon \in (\mathbb{C}^2)^\dagger, \quad \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{C}, \tag{3.132}$$

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.133}$$

We can easily find a basis for $\mathbf{I}$ and $\dot{\mathbf{I}}$. Indeed, since $\mathbf{I} = \mathbb{R}_{1,3}^{(0)}\mathrm{e}$ we have that any $\boldsymbol{\varphi} \in \mathbf{I}$ can be written as

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^1 \boldsymbol{\vartheta}_1 + \boldsymbol{\varphi}^2 \boldsymbol{\vartheta}_2$$

where

$$\begin{aligned} &\boldsymbol{\vartheta}_1 = \mathrm{e}, \quad \boldsymbol{\vartheta}_2 = \boldsymbol{\sigma}_1 \mathrm{e}, \\ &\boldsymbol{\varphi}^1 = a + \mathbf{i}b, \quad \boldsymbol{\varphi}^2 = c + \mathbf{i}d, \quad a, b, c, d \in \mathbb{R}. \end{aligned} \tag{3.134}$$

Analogously we find that any $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ can be written as

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \boldsymbol{\xi}_{\dot{1}} \mathbf{s}^{\dot{1}} + \boldsymbol{\xi}_{\dot{2}} \mathbf{s}^{\dot{2}}, \\ \mathbf{s}^{\dot{1}} &= \mathrm{e}, \quad \mathbf{s}^{\dot{2}} = \mathrm{e}\boldsymbol{\sigma}_1. \end{aligned} \tag{3.135}$$

Defining the mapping

$$\begin{aligned} \boldsymbol{\iota} : \mathbf{I} \otimes \dot{\mathbf{I}} &\to \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}, \\ \boldsymbol{\iota}(\boldsymbol{\varphi} \otimes \dot{\boldsymbol{\xi}}) &= \boldsymbol{\varphi}\dot{\boldsymbol{\xi}}, \end{aligned} \tag{3.136}$$

we have

$$\begin{aligned} 1 \equiv \boldsymbol{\sigma}_0 &= \boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{1}} + \mathbf{s}_2 \otimes \mathbf{s}^{\dot{2}}), \\ \boldsymbol{\sigma}_1 &= -\boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{2}} + \mathbf{s}_2 \otimes \mathbf{s}^{\dot{1}}), \end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}_2 &= \boldsymbol{\iota}[\mathbf{i}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{2}} - \mathbf{s}_2 \otimes \mathbf{s}^{\dot{1}})],\\ \boldsymbol{\sigma}_3 &= -\boldsymbol{\iota}(\mathbf{s}_1 \otimes \mathbf{s}^{\dot{1}} - \mathbf{s}_2 \otimes \mathbf{s}^{\dot{2}}).\end{aligned} \tag{3.137}$$

From this it follows the identification

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{C}(2) = \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}, \tag{3.138}$$

and then, each Pauli number can be written as an appropriate sum of Clifford products of algebraic contravariant undotted spinors and algebraic covariant dotted spinors. And, of course, a representative of a Pauli number in $\mathbb{C}^2$ can be written as an appropriate Kronecker product of a complex column vector by a complex row vector.

Take an arbitrary $\mathbf{P} \in \mathbb{R}_{3,0}$ such that

$$\mathbf{P} = \frac{1}{j!} p^{\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_j} \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_j}, \tag{3.139}$$

where $p^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} \in \mathbb{R}$ and

$$\boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} = \boldsymbol{\sigma}_{\mathbf{k}_1} \cdots \boldsymbol{\sigma}_{\mathbf{k}_j}, \quad \text{and } \boldsymbol{\sigma}_0 \equiv 1 \in \mathbb{R}. \tag{3.140}$$

With the identification $\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}$, we can also write

$$\mathbf{P} = \mathbf{P}^A{}_{\dot{B}}\boldsymbol{\iota}(\mathbf{s}_A \otimes \mathbf{s}^{\dot{B}}) = \mathbf{P}^A{}_{\dot{B}}\mathbf{s}_A\mathbf{s}^{\dot{B}}, \tag{3.141}$$

where the $\mathbf{P}^A{}_{\dot{B}} = \mathbf{X}^A{}_{\dot{B}} + \mathbf{i}\mathbf{Y}^A{}_{\dot{B}}$, $\mathbf{X}^A{}_{\dot{B}}, \mathbf{Y}^A{}_{\dot{B}} \in \mathbb{R}$.

Finally, the matrix representative of the Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ is $P \in \mathbb{C}(2)$ given by

$$P = P^A{}_{\dot{B}} s_A s^{\dot{B}}, \tag{3.142}$$

with $P^A{}_{\dot{B}} \in \mathbb{C}$ and

$$\begin{aligned} s_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\\ s^{\dot{1}} &= \begin{pmatrix} 1 & 0 \end{pmatrix}, s^{\dot{2}} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.143}$$

It is convenient for our purposes to introduce also covariant undotted spinors and contravariant dotted spinors. Let $\varphi \in \mathbb{C}^2$ be given as in Eq. (3.130). We define the *covariant* version of undotted spinor $\varphi \in \mathbb{C}^2$ as $\varphi^* \in (\mathbb{C}^2)^t \simeq \mathbb{C}_2$ such that

$$\begin{aligned}
\varphi^* &= (\varphi_1, \varphi_2) \equiv \varphi_A s^A, \\
\varphi_A &= \varphi^B \varepsilon_{BA,} \quad \varphi^B = \varepsilon^{BA}\varphi_A, \\
s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & 1 \end{pmatrix},
\end{aligned} \tag{3.144}$$

where[20] $\varepsilon_{AB} = \varepsilon^{AB} = \operatorname{adiag}(1, -1)$. We can write due to the above identifications that there exists $\varepsilon \in \mathbb{C}(2)$ given by Eq. (3.133) which can be written also as

$$\varepsilon = \varepsilon^{AB} s_A \boxtimes s_B = \varepsilon_{AB} s^A \boxtimes s^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \tag{3.145}$$

where $\boxtimes$ denotes here the *Kronecker* product of matrices. We have, e.g.,

$$\begin{aligned}
s_1 \boxtimes s_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
s^1 \boxtimes s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.146}$$

We now introduce the *contravariant* version of the dotted spinor

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix} \in \mathbb{C}_2$$

as being $\dot{\xi}^* \in \mathbb{C}^2$ such that

$$\begin{aligned}
\dot{\xi}^* &= \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \end{pmatrix} = \xi^{\dot{A}} s_{\dot{A}}, \\
\xi^{\dot{B}} &= \varepsilon^{\dot{B}\dot{A}} \xi_{\dot{A}}, \quad \xi_{\dot{A}} = \varepsilon_{\dot{B}\dot{A}}\, \xi^{\dot{B}}, \\
s_{\dot{1}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_{\dot{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned} \tag{3.147}$$

where $\varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \operatorname{adiag}(1, -1)$. Then, due to the above identifications we see that there exists $\dot{\varepsilon} \in \mathbb{C}(2)$ such that

$$\dot{\varepsilon} = \varepsilon^{\dot{A}\dot{B}} s_{\dot{A}} \boxtimes s_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} s^{\dot{A}} \boxtimes \dot{s}^{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon. \tag{3.148}$$

[20]The symbol adiag means the antidiagonal matrix.

Also, recall that even if $\{\mathbf{s}_A\}$,$\{\mathbf{s}_{\dot{A}}\}$ and $\{s^{\dot{A}}\}$,$\{s^A\}$ are bases of distinct spaces, we can identify their matrix representations, as it is obvious from the above formulas. So, we have $s_A \equiv s_{\dot{A}}$ and also $s^{\dot{A}} = s^A$. This is the reason for the representation of a dotted covariant spinor as in Eq. (3.132). Moreover, the above identifications permit us to write the *matrix representation* of a Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ as, e.g.,

$$P = P_{AB} s^A \boxtimes s^B \tag{3.149}$$

besides the representation given by Eq. (3.142).

Exercise 3.57 Consider the ideal $I = \mathbb{R}_{1,3}\frac{1}{2}(1 - \mathbf{e}_0\mathbf{e}_3)$. Show that $\phi \in I$ is a representative in a spin frame Ξ_u of a covariant Dirac spinor[21] $\Psi \in \mathbb{C}(4)\frac{1}{2}(1 + \underline{\gamma}_0)(1 + \mathrm{i}\underline{\gamma}_1\underline{\gamma}_2)$. Let ψ be the representative (in the same spin frame Ξ_u) of a Dirac-Hestenes spinor, associated to a mother spinor $\Phi \in \mathbb{R}_{1,3}\frac{1}{2}(1 + \mathbf{e}_0)$ by $\Phi = \psi\frac{1}{2}(1 + \mathbf{e}_0)$. Show that $\phi \in I$ can be written as $\phi = \psi\frac{1}{2}(1 + \mathbf{e}_0)\frac{1}{2}(1 - \mathbf{e}_0\mathbf{e}_3)$.

(a) Show that $\phi\mathbf{e}_5 = \phi\mathbf{e}_{21}$.
(b) Weyl spinors are defined as eigenspinors of the chirality operator, i.e., $\underline{\gamma}_5\Psi_\pm = \pm\mathrm{i}\Psi_\pm$. Show that Weyl spinors corresponds to the even and odd parts of ϕ.
(c) Relate the even and odd parts of ϕ to the algebraic dotted and undotted spinors.[22]

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[21] Recall that $\underline{\gamma}_\mu$ are the Dirac matrices defined by Eq. (3.49).

[22] As a suggestion for solving the above exercise the reader may consult [10].

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Chapter 4
Some Differential Geometry

Abstract The main objective of this chapter is to present a Clifford bundle formalism for the formulation of the differential geometry of a manifold M, equipped with metric fields $\boldsymbol{g} \in \sec T_2^0 M$ and $\mathtt{g} \in \sec T_0^2 M$ for the tangent and cotangent bundles. We start by first recalling the standard formulation and main concepts of the differential geometry of a differential manifold M. We introduce in M the Cartan bundle of differential forms, define the exterior derivative, Lie derivatives, and also briefly review concepts as chains, homology and cohomology groups, de Rham periods, the integration of form fields and Stokes theorem. Next, after introducing the metric fields $\boldsymbol{g}$ and $\mathtt{g}$ in M we introduce the Hodge bundle presenting the Hodge star and the Hodge coderivative operators acting on sections of this bundle. We moreover recall concepts as the pullback and the differential of maps, connections and covariant derivatives, Cartan's structure equations, the exterior covariant differential of $(p+q)$-indexed r-forms, Bianchi identities and the classification of geometries on M when it is equipped with a metric field and a particular connection. The spacetime concept is rigorously defined. We introduce and scrutinized the structure of the Clifford bundle of differential forms ($\mathcal{C}\ell(M,\mathtt{g})$) of M and introduce the fundamental concept of the Dirac operator (associated to a given particular connection defined in M) acting on Clifford fields (sections of $\mathcal{C}\ell(M,\mathtt{g})$). We show that the square of the Dirac operator (associated to a Levi-Civita connection in M) has two fundamental decompositions, one in terms of the derivative and Hodge codifferential operators and other in terms of the so-called Ricci and D'Alembertian operators. A so-called Einstein operator is also introduced in this context. These decompositions of the square of the Dirac operator are crucial for the formulation of important ideas concerning the construction of gravitational theories as discussed in particular in Chaps. 9, 11, 15. The Dirac operator associated to an arbitrary (metrical compatible) connection defined in M and its relation with the Dirac operator associated to the Levi-Civita connection of the pair $(M,\boldsymbol{g})$ is discussed in details and some important formulas are obtained. The chapter also discuss some applications of the formalism, e.g., the formulation of Maxwell equations in the Hodge and Clifford bundles and formulation of Einstein equation in the Clifford bundle using the concept of the Ricci and Einstein operators. A preliminary account of the crucial difference between the concepts of curvature of a connection in M and the concept of bending of M as a hypersurface embedded in a (pseudo)-Euclidean space of high dimension (a property characterized by

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the concept of the *shape tensor*, discussed in details in Chap. 5) is given by analyzing a specific example, namely the one involving the Levi-Civita and the Nunes connections defined in a punctured 2-dimensional sphere. The chapter ends analyzing a statement referred in most physical textbooks as "tetrad postulate" and shows how not properly defining concepts can produce a lot of misunderstanding and invalid statements.

4.1 Differentiable Manifolds

In this section we briefly recall, in order to fix our notations, some results concerning the theory of differentiable manifolds, that we shall need in the following.

Definition 4.1 A topological space is a pair $(M, \mathcal{U})$ where M is a set and $\mathcal{U}$ a collection of subsets of M such that

(i) $\varnothing, M \in \mathcal{U}$.
(ii) $\mathcal{U}$ contains the union of each one of its subsystems.
(iii) $\mathcal{U}$ contains the intersection of each one of its finite subsystems.

We recall some more terminology.[1] Each $U_\alpha \in \mathcal{U}$ (α belongs to an index set which eventually is infinite) is called an *open* set. Of course we can give many different topologies to a given set by choosing different collections of open sets. Given two topologies for M, i.e., the collections of subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ if $\mathcal{U}_1 \subset \mathcal{U}_2$ we say that $\mathcal{U}_1$ is *coarse* than $\mathcal{U}_2$ and $\mathcal{U}_2$ is *finer* than $\mathcal{U}_1$. Given two coverings $\{U_\alpha\}$ and $\{V_\alpha\}$ of M we say that $\{V_\alpha\}$ is a *refinement* of $\{U_\alpha\}$ if for each V_α there exists an U_α such that $V_\alpha \subset U_\alpha$. A *neighborhood* of a point $x \in M$ is any subset of M containing some (at least one) open set $U_\alpha \in \mathcal{U}$. A subset $X \subset M$ is called closed if its complement is open in the topology $(M, \mathcal{U})$. A family $\{U_\alpha\}, U_\alpha \in \mathcal{U}$ is called a *covering* of M if $\cup_\alpha U_\alpha = M$. A topological space $(M, \mathcal{U})$ is said to be Hausdorff (or *separable*) if for any distinct points $x, x' \in M$ there exists open neighborhoods U and U' of these points such that $U \cap U' = \varnothing$. Moreover, a topological space $(M, \mathcal{U})$ is said to be *compact if for every open covering* $\{U_\alpha\}, U_\alpha \in \mathcal{U}$ *of* M there exists a finite subcovering, i.e., there exists a finite subset of indices, say $\alpha = 1, 2, \ldots m$, such that $\cup_{\alpha=1}^m U_\alpha = M$. A Hausdorff space is said *paracompact* if there exists a covering $\{V_\alpha\}$ of M such that every point of M is covered by a finite number of the V_α, i.e., we say that every covering has a locally finite refinement.

Definition 4.2 A *smooth differentiable* manifold M is a set such that

(i) M is a Hausdorff topological space.
(ii) M is provided with a family of pairs $(U_\alpha, \varphi_\alpha)$ called charts, where $\{U_\alpha\}$ is a family of open sets covering M, i.e., $\cup_\alpha U_\alpha = M$ and being $\{V_\alpha\}$ a family

[1] In general we are not going to present proofs of the propositions, except for a few cases, which may considered as exercises. If you need further details, consult e.g., [3, 11, 25].

of open sets covering $\mathbb{R}^n$, i.e., $\cup_\alpha V_\alpha = \mathbb{R}^n$, the $\varphi_\alpha : U_\alpha \to V_\alpha$ are homeomorphisms. We say that any point $x \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$. The integer n is said the *dimension* of M, and we write $\dim M = n$.

(iii) Given any two charts (U, φ) and (U', φ') of the family described in (ii) such that $U \cap U' \neq \varnothing$ the mapping $\Phi = \varphi \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi(U \cap U')$ is differentiable of class C^r.

The word *smooth* means that the integer r is large enough for all statements that we shall done to be valid. For the applications we have in mind we will suppose that M is also paracompact. The whole family of charts $\{(U_\alpha, \varphi_\alpha)\}$ is called an *atlas*.

The *coordinate functions* of a chart (U, φ) are the functions $\mathbf{x}^i = a^i \circ \varphi : U \to \mathbb{R}$, $i = 1, 2, \ldots, n$ where $a^i : \mathbb{R}^n \to \mathbb{R}$ are the usual coordinate functions of $\mathbb{R}^n$ (see Fig. 4.1). We write $\mathbf{x}^i(x) = x^i$ and call the set $(x^1, .., x^n)$ (denoted $\{x^i\}$) the *coordinates* of the points $x \in U$ in the chart (U, φ), or briefly, the coordinates.[2] If (U', φ') is another chart of the maximal atlas of M with coordinate functions $\mathbf{x}'^i$ such that $x \in U \cap U'$ we write $\mathbf{x}'^i(x) = x'^i$ and

$$\mathbf{x}'^j(x) = f^j(\mathbf{x}^1(x), \ldots, \mathbf{x}^n(x)), \tag{4.1}$$

and we use the short notation $x'^j = f^j(x^i)$, $i, j = 1, \ldots, n$. Moreover, we often denote the derivatives $\partial f^j / \partial x^i$ by $\partial x'^j / \partial x^i$.

Let (U, φ) be a chart of the maximal *atlas* of M and $\mathtt{h} : M \to M$, $x \mapsto y = \mathtt{h}(x)$ a diffeomorphism such that $x, y \in U \cap \mathtt{h}(U)$. Putting $\mathbf{x}^i(x) = x^i$ and $y^j = \mathbf{x}^j(\mathtt{h}(x))$

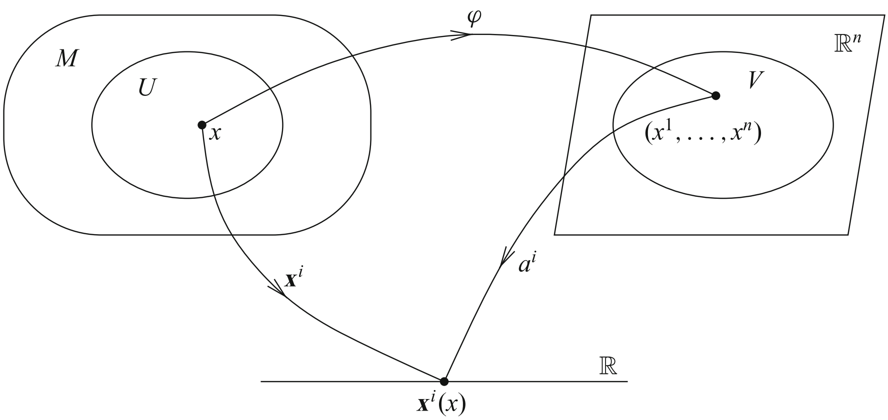

Fig. 4.1 Coordinate chart (U, ϕ), coordinate functions $\mathbf{x} : U \to \mathbb{R}$ and coordinates $\mathbf{x}^i(x) = x^i$

[2]We remark that some authors (see, e.g., [25]) call sometimes the coordinate function $\mathbf{x}^i$ simply by coordinate. Also, some authors (see, e.g., [11]) call sometimes $\{x^i\}$ a *coordinate system* (for $U \subset M$). We eventually also use these terminologies.

we write the mappings $\mathtt{h}^j : (x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$ as

$$y^j = \mathtt{h}^j(x^i), \tag{4.2}$$

and often denote the derivatives $\partial \mathtt{h}^j / \partial x^i$ of the functions $\mathtt{h}^j$ by $\partial y^j / \partial x^i$.

Observe that in the chart $(V, \varkappa)$, $V \subset \mathtt{h}(U)$ with coordinate functions $\{\boldsymbol{y}^j\}$ such that $\boldsymbol{x}^j = \boldsymbol{y}^j \circ \mathtt{h}$, $\boldsymbol{x}^j(x) = x^j = \boldsymbol{y}^j(y) = y^j$ and $\partial y^j / \partial x^i = \delta^i_j$.

4.1.1 Manifold with Boundary

In the definition of a n-dimensional (real) manifold we assumed that each coordinate neighborhoods, $U_\alpha \in M$ is homeomorphic to an open set of $\mathbb{R}^n$. We now give the

Definition 4.3 A n-dimensional (real) manifold M with boundary is a topological space covered by a family of open sets $\{U_\alpha\}$ such that each one is homeomorphic to an open set of $\mathbb{R}^{n+} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$.

Definition 4.4 The boundary of M is the set ∂M of points of M that are mapped to points in $\mathbb{R}^n$ with $x^n = 0$.

Of course, the coordinates of ∂M are given by $(x^1, \ldots, x^{n-1}, 0)$ and thus ∂M is a $(n-1)$-dimensional manifold. of the same class (C^r) as M.

4.1.2 Tangent Vectors

Let $C^r(M, x)$ be the set of all differentiable functions of class C^r (smooth functions) which domain in some neighborhood of $x \in M$. Given a curve in M, $\sigma : \mathbb{R} \supseteq I \to M$, $t \mapsto \sigma(t)$ we can construct a linear function

$$\sigma_*(t) : C^r(M, x) \to \mathbb{R}, \tag{4.3}$$

such that given any $f \in C^r(M, x)$,

$$\sigma_*(t)[f] = \frac{d}{dt}[f \circ \sigma](t). \tag{4.4}$$

Now, $\sigma_*(t)$ is a derivation, i.e., a linear function that satisfy the Leibniz's rule:

$$\sigma_*(t)[fg] = \sigma_*(t)[f]g + f\sigma_*(t)[g], \tag{4.5}$$

for any $f, g \in C^r(M, x)$.

This linear mapping has all the properties that we would like to impose to the *tangent* to σ at $\sigma(t)$ as a generalization of the concept of directional derivative of the calculus on $\mathbb{R}^n$. It can shown that to every linear derivation it is associated a curve (indeed, an infinity of curves) as just described, i.e., curves $\sigma, \gamma : \mathbb{R} \supseteq I \to M$ are equivalent at $x_0 = \sigma(0) = \gamma(0)$ provided $\frac{d}{dt}[f \circ \sigma](t)\big|_{t=0} = \frac{d}{dt}[f \circ \gamma](t)\big|_{t=0}$ for any $f \in C^r(M, x_0)$. This suggests the

Definition 4.5 A tangent to M at the point $x \in M$ is a mapping $\mathbf{v}|_x : C^r(M, x) \to \mathbb{R}$ such that for any $f, g \in C^r(M, x)$, $a, b \in \mathbb{R}$,

$$\begin{array}{ll} \text{(i)} & \mathbf{v}|_x\,[af + bg] = a\mathbf{v}_x[f] + b\,\mathbf{v}|_x\,[g], \\ \text{(ii)} & \mathbf{v}|_x\,[fg] = \mathbf{v}|_x\,[f]g + f\,\mathbf{v}|_x\,[g]. \end{array} \quad . \tag{4.6}$$

As can be easily verified the tangents at x form a linear space over the real field. For that reason a tangent at x is also called a *tangent vector* to M at x.

Definition 4.6 The set of all tangent vectors at x is denoted by T_xM and called the tangent space at x. The dual space of T_xM is denoted by T_x^*M and called the cotangent space at x. Finally $T^r_{sx}M$ is the space of r-contravariant and s-covariant tensors at x.

Definition 4.7 Let $\{\mathbf{x}^i\}$ be the coordinate functions of a chart (U, φ). The partial derivative at x with respect to x^i is the *representative* in the given chart of the tangent vector denoted $\frac{\partial}{\partial x^i}\big|_x \equiv \partial_i|_x$ such that

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|_x f &:= \frac{\partial}{\partial x^i}[f \circ \varphi^{-1}]\bigg|_{\varphi(x)}, \\ &= \frac{\partial \breve{f}}{\partial x^i}(x^i), \end{aligned} \tag{4.7}$$

with

$$f(x) = f \circ \varphi^{-1}(\mathbf{x}^1(x), \ldots, \mathbf{x}^n) = \breve{f}(x^1, \ldots, x^n). \tag{4.8}$$

Remark 4.8 Eventually we should represent the tangent vector $\frac{\partial}{\partial x^i}\big|_x$ by a different symbol, say $\frac{\partial}{\partial \mathbf{x}^i}\big|_x$. This would cause less misunderstandings. However, $\frac{\partial}{\partial x^i}\big|_x$ is almost universal notation and we shall use it. We note moreover that other notations and abuses of notations are widely used, in particular $f \circ \varphi^{-1}$ is many times denoted simply by f and then $\breve{f}(x^i)$ is denoted simply by $f(x^i)$ and also we find $\frac{\partial}{\partial x^i}\big|_x [f] \equiv \frac{\partial f}{\partial x^i}(x)$, (or worse) $\frac{\partial}{\partial x^i}\big|_x [f] \equiv \frac{\partial f}{\partial x^i}$. We shall use these (and other) sloppy notations, which are simply to typewrite when no confusion arises, in particular we will use the sloppy notations $\frac{\partial x^j}{\partial x^i}(x)$ or $\frac{\partial x^j}{\partial x^i}$ for $\left.\frac{\partial}{\partial x^i}\right|_x [\mathbf{x}^j]$, i.e.

$$\left.\frac{\partial}{\partial x^i}\right|_x [\mathbf{x}^j] \equiv \frac{\partial x^j}{\partial x^i}(x) \equiv \frac{\partial x^j}{\partial x^i} = \delta^j_i.$$

If $\{\mathbf{x}^i\}$ are the coordinate functions of a chart (U, φ) and $\mathbf{v}|_x \in T_x M$, then we can easily show that

$$\mathbf{v}|_x = \mathbf{v}|_x [\mathbf{x}^i] \left.\frac{\partial}{\partial x^i}\right|_x = v^i \left.\frac{\partial}{\partial x^i}\right|_x, \tag{4.9}$$

with $\mathbf{v}|_x [\mathbf{x}^i] = v^i : U \to \mathbb{R}$.

As a trivial consequence we can verify that the set of tangent vectors $\left\{\left.\frac{\partial}{\partial x^i}\right|_x, i = 1, 2, \ldots, n\right\}$ is linearly independent and so $\dim T_x M = n$.

Remark 4.9 Have always in mind that $\mathbf{v}|_x = v^i \left.\frac{\partial}{\partial x^i}\right|_x \in T_x U$ and its representative in $T\varphi_{(x)}\mathbb{R}^n$ is the tangent vector $\check{\mathbf{v}}|_{\varphi(x)} =: \check{v}^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(x)}$ such that $v^i \left.\frac{\partial}{\partial x^i}\right|_x f = \check{\mathbf{v}}|_{\varphi(x)} \check{f}$.

Definition 4.10 The tangent vector field to a curve $\sigma : \mathbb{R} \supseteq I \to M$ is denoted by $\sigma_*(t)$ or $\frac{d\sigma}{dt}$.

This means that $\sigma_*(t) = \frac{d\sigma}{dt}(t)$ is the tangent vector to the curve σ at the point $\sigma(t)$. Note that $\sigma_*(t)$ has the expansion

$$\sigma_*(t) = v^i(\sigma(t)) \left.\frac{\partial}{\partial x^i}\right|_{x=\sigma(t)}, \tag{4.10}$$

where, of course,

$$v^i(\sigma(t)) = \sigma_*(t)[\mathbf{x}^i] = \frac{d\mathbf{x}^i \circ \sigma(t)}{dt} = \frac{d\sigma^i(t)}{dt}, \tag{4.11}$$

with $\sigma^i = \mathbf{x}^i \circ \sigma$. We then see, that given any tangent vector $\mathbf{v}|_x \in T_x M$, the solution of the differential equation, Eq. (4.11) permit us to find the components $\sigma^i(t)$ of the curve to which $\mathbf{v}|_x$ is tangent at x. Indeed, the theorem of existence of local solutions of ordinary differential equations warrants the existence of such a curve. More precisely, since the theorem holds only locally, the uniqueness of the solution is warranted only in a neighborhood of the point $x = \sigma(t)$ and in that way, we have in general many curves through x to which $\mathbf{v}|_x$ is tangent to the curve at x.

4.1.3 Tensor Bundles

In what follows we denote respectively by $TM = \bigcup_{x\in M} T_xM$ and $T^*M = \bigcup_{x\in M} T_x^*M$ the tangent and cotangent bundles[3] of M and more generally, we denote by $T_s^rM = \bigcup_{x\in M} T_{sx}^rM$ the bundle of r-contracovariant and s-covariant tensors. A tensor field t of type (r, s) is a section of the T_s^rM bundle and we write[4] $t \in \sec T_s^rM$. Also, $T_0^0M \equiv M \times \mathbb{R}$ is the module of real functions over M and $T_1^0M \equiv TM$, $T_s^rM = T^*M$.

4.1.4 Vector Fields and Integral Curves

Let $\sigma : I \to M$ a curve and $\boldsymbol{v} \in \sec TM$ a vector field which is tangent to each one of the points of σ. Then, taking into account Eq. (4.11) we can write that condition as

$$\boldsymbol{v}(\sigma(t)) = \frac{d\sigma(t)}{dt}. \tag{4.12}$$

Definition 4.11 A curve $\sigma : I \to M$ satisfying Eq. (4.12) is called an integral curve of the vector field $\boldsymbol{v}$.

4.1.5 Derivative and Pullback Mappings

Let M and N be two differentiable manifolds, $\dim M = m$, $\dim N = n$ and $\phi : M \to N$ a differentiable mapping of class C^r. ϕ is a diffeomorphism of class C^r if ϕ is a bijection and if ϕ and ϕ^{-1} are of class C^r.

Definition 4.12 The reciprocal image or pullback of a function $f : N \to \mathbb{R}$ is the function $\phi^*f : M \to \mathbb{R}$ given by

$$\phi^*f = f \circ \phi. \tag{4.13}$$

Definition 4.13 Given a mapping $\phi : M \to N$, $\phi(x) = y$ and $\mathbf{v} \in T_xM$, the image of $\mathbf{v}$ under ϕ is the vector $\mathbf{w}$ such that for any $f : N \to \mathbb{R}$

$$\mathbf{w}[f] = \mathbf{v}[f \circ \phi]. \tag{4.14}$$

[3] In Appendix we list the main concepts concerning fiber bundle theory that we need for the purposes of this book.

[4] See details in Notation A.6 in the Appendix.

The mapping $\phi_* \mid_x : \sec T_xM \to \sec T_yN$ is called the differential or derivative (or pushforward) mapping of ϕ at x. We write $\boldsymbol{w} = \phi_* \mid_x \boldsymbol{v}$.

Remark 4.14 When the point $x \in M$ is left unspecified (or is arbitrary), we sometimes write ϕ_* instead of $\phi_* \mid_x$.

The image a vector field $\boldsymbol{v} \in \sec TM$ at an arbitrary point $x \in M$ is

$$\phi_* \, \boldsymbol{v}[f](y) = \boldsymbol{v}[f \circ \phi](x). \tag{4.15}$$

Note that if $\phi(x) = y$, and if ϕ is invertible, i.e., $x = \phi^{-1}(y)$ then Eq. (4.15) says that or

$$\phi_* \, \boldsymbol{v}[f](y) = \boldsymbol{v}[f \circ \phi](x) = \boldsymbol{v}[f \circ \phi](\phi^{-1}(y)). \tag{4.16}$$

This suggests the

Definition 4.15 Let $\phi : M \to N$ be invertible mapping. Let $\boldsymbol{v} \in \sec TM$. The image of $\boldsymbol{v}$ under ϕ is the vector field $\phi_* \boldsymbol{v} \in \sec TN$ such that for any $f : N \to \mathbb{R}$

$$\phi_* \boldsymbol{v}[f] = \boldsymbol{v}[f \circ \phi] = \boldsymbol{v}[f \circ \phi] \circ \phi^{-1}. \tag{4.17}$$

In this case we call

$$\phi_* : \sec TM \to \sec TN, \tag{4.18}$$

the derivative mapping of ϕ.

Remark 4.16 If $\boldsymbol{v} \in \sec TM$ is a differentiable field of class C^r over M and ϕ is a diffeomorphism of class C^{r+1}, then $\phi_* \, \boldsymbol{v} \in \sec TN$ is a differentiable vector field of class C^r over N. Observe however, that if ϕ is not invertible the image of $\boldsymbol{v}$ under ϕ is not in general a vector field on N [3]. If ϕ is invertible, but not differentiable the image is not differentiable. When the image of a vector field $\boldsymbol{v}$ under some differentiable mapping ϕ is a differentiable vector field, v is said to be projectable. Also, $\boldsymbol{v}$ and $\phi_* \boldsymbol{v}$ are said ϕ-related.

Remark 4.17 We have denoted by $\sigma_*(t)$ the tangent vector to a curve $\sigma : I \to M$. If we look for the definition of that tangent vector and the definition of the derivative mapping we see that the rigorous notation that should be used for that tangent vector is $\sigma_*|_t \, [\frac{d}{dt}]$, which is really cumbersome, and thus avoided, unless some confusion arises. We will also use sometimes the simplified notation σ_* to refer to the tangent vector field to the curve σ.

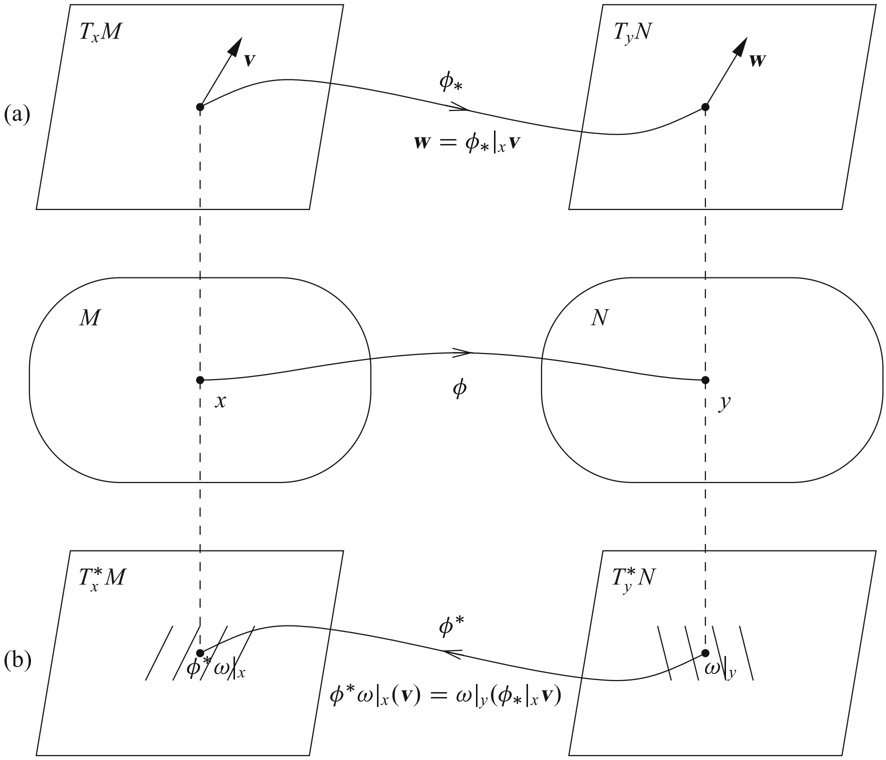

Fig. 4.2 (**a**) The derivative mapping ϕ_*. (**b**) The pullback mapping ϕ^*

Definition 4.18 Given a mapping $\phi : M \to N$, the pullback mapping is the mapping

$$\phi^* : \sec T^*N \to \sec T^*M,$$
$$\phi^\star\omega(\boldsymbol{v}) = \omega(\phi_*\boldsymbol{v}) \circ \phi, \tag{4.19}$$

for any projectable vector field $\boldsymbol{v} \in \sec TM$. Also, $\phi^\star\omega \in \sec T^*N$ is called the pullback of ω (Fig. 4.2).

Remark 4.19 Note that differently from what happens for the image of vector fields, the formula for the reciprocal image of a covector field does not use the inverse mapping ϕ^{-1}. This shows that covector fields are more interesting than vector fields, since $\phi^\star\omega$ is always differentiable if ω and ϕ are differentiable.

Remark 4.20 From now, we assume that $\phi : M \to N$ is a diffeomorphism, *unless explicitly said the contrary* and generalize the concepts of image and reciprocal images defined for vector and covector fields for arbitrary tensor fields.

Definition 4.21 The image of a function $f : M \to \mathbb{R}$ under a diffeomorphism $\phi : M \to N$ is the function $\phi_{\star}f : N \to \mathbb{R}$ such that

$$\phi_{\star}f = f \circ \phi^{-1} \tag{4.20}$$

The image of a covector field $\beta \in \sec T^*M$ under a diffeomorphism $\phi : M \to N$ is the covector field $\phi_*\beta$ such that for any projectable vector field $\boldsymbol{v} \in \sec TM$, $\boldsymbol{w} = \phi_*\boldsymbol{v} \in \sec TN$, we have $\phi_*\beta(\boldsymbol{w}) = \beta(\phi_*^{-1}\boldsymbol{w})$, or

$$\phi_*\beta = (\phi^{-1})^*\beta. \tag{4.21}$$

For $\mathbf{S} \in \sec T^r_sM$ we define its image $\phi_*\mathbf{S} \in \sec T^r_sN$ by

$$\phi_*\mathbf{S}(\phi_*\beta_1, \ldots \phi_*\beta_r, \phi_*\boldsymbol{v}_1, \ldots, \phi_*\boldsymbol{v}_s) = \mathbf{S}(\beta_1, \ldots \beta_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_s), \tag{4.22}$$

for any projectable vector fields $\boldsymbol{v}_i \in \sec TM$, $i = 1, 2, \ldots, s$ and covector fields $\beta_j \in \sec T^*M, j = 1, 2, \ldots, r$.

If $\{\mathbf{e}_{\ i}\}$ is any basis for TU, $U \subset M$ and $\{\theta^i\}$ is the dual basis for T^*U, then

$$\mathbf{S} = S^{i_1 \ldots i_r}_{j_1 \ldots j_s}\theta^{j_1} \otimes \cdots \otimes \theta^{j_s} \otimes \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_r} \tag{4.23}$$

and

$$\phi_*\mathbf{S} = (S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \circ \phi^{-1})\phi_*\theta^{j_1} \otimes \cdots \otimes \phi_*\theta^{j_s} \otimes \phi_*\boldsymbol{e}_{i_1} \otimes \cdots \otimes \phi_*\boldsymbol{e}_{i_r}. \tag{4.24}$$

Definition 4.22 Let $\mathbf{S} \in \sec T^r_sN$, and $\beta_1, \beta_2, \ldots, \beta_r \in \sec T^*M$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s \in \sec TM$ be projectable vector fields.The reciprocal image (or pullback) of $\mathbf{S}$ is the tensor field $\phi^*\mathbf{S} \in \sec T^r_sM$ such that

$$\phi^*\mathbf{S}(\beta_1, \ldots, \beta_{\mathbf{r}}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\mathbf{s}}) = \mathbf{S}(\phi_*\beta_1, \ldots, \phi_*\beta_{\mathbf{r}}, \phi_*\boldsymbol{v}_1, \ldots, \phi_*\boldsymbol{v}_{\mathbf{s}}), \tag{4.25}$$

and

$$\phi^*\mathbf{S} = (S^{i_1 \ldots i_r}_{j_1 \ldots j_s} \circ \phi)\phi^*\theta^{j_1} \otimes \cdots \otimes \phi^*\theta^{j_s} \otimes \phi_*^{-1}\boldsymbol{e}_{i_1} \otimes \cdots \otimes \phi_*^{-1}\boldsymbol{e}_{i_r}. \tag{4.26}$$

Let $\mathbf{x}^i$ be the coordinate functions of the chart (U, φ) of $U \subset M$ and $\{\partial/\partial x^j\}, \{dx^i\}$, $i, j = 1, \ldots, m$ dual[5] coordinate bases for TU and T^*U, i.e., $dx^i(\partial/\partial x^j) = \delta^i_j$. Let moreover $\boldsymbol{y}^l$ be the coordinate functions of (V, χ), $V \subset N$ and $\{\partial/\partial y^k\}$,$\{dy^l\}$, $k, l = 1, \ldots, n$ dual bases for TV and T^*V. Let $x \in M, y \in N$ with $y = \phi(x)$ and $\mathbf{x}^i(x) = x^i$, $\boldsymbol{y}^l(y) = y^l$. If $S^{k_1 \ldots k_r}_{l_1 \ldots l_s}(y^1, \ldots, y^n) \equiv S^{k_1 \ldots k_r}_{l_1 \ldots l_s}(y^j)$ are the

[5]See Remark 4.41 for the reason of the notation dx^i.

components of $\mathbf{S}$ at the point y in the chart (V, χ), then the components $\mathbf{S}' = \phi^*\mathbf{S}$ in the chart (U, φ) at the point x are

$$S'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i) = S^{k_1\ldots k_r}_{l_1\ldots l_s}(y^j(x^i))\frac{\partial y^{l_1}}{\partial x^{j_1}}\cdots\frac{\partial y^{l_s}}{\partial x^{j_s}}\frac{\partial x^{i_1}}{\partial y^{k_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{k_r}},$$
$$S'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i) = (\mathtt{h}^\star S)'^{i_1\ldots i_r}_{j_1\ldots j_s}(x^i). \tag{4.27}$$

4.1.6 Diffeomorphisms, Pushforward and Pullback when $M = N$

Definition 4.23 The set of all diffeomorphisms in a differentiable manifold M define a group denoted by $\mathfrak{G}_M$ and called the *manifold mapping group*.

Let $\mathcal{A}, \mathcal{B} \subset M$. Let $\mathfrak{G}_M \ni \mathtt{h} : M \to M$ be a diffeomorphism such that $\mathtt{h} : \mathcal{A} \to \mathcal{B}, \mathfrak{e} \mapsto \mathtt{h}\mathfrak{e}$. The diffeomorphism $\mathtt{h}$ induces two important mappings in the tensor bundle $\mathcal{T}M = \bigoplus_{r,s=0} T^r_s M$, the derivative mapping $\mathtt{h}_*$, in this case known as *pushforward*, and the pullback mappings $\mathtt{h}^*$. The definitions of these mappings are the ones given above.

We now recall how to calculate, e.g., the pullback mapping of a tensor field in this case.

Suppose now that $\mathcal{A}$ and $\mathtt{h}(\mathcal{A}) \subset \mathcal{B}$ can be covered by a local charts (U, φ) and (V, χ) of the maximal atlas of M (with $\mathcal{A}, h(\mathcal{A}) \subset U \cap V$) with respective coordinate functions $\{\mathbf{x}^\mu\}$ and $\{\mathbf{y}^\mu\}$ defined by[6]

$$\mathbf{x}^\mu(\mathfrak{e}) = x^\mu, \mathbf{x}^\mu(\mathtt{h}(\mathfrak{e})) = y^\mu, \mathbf{y}^\mu(\mathfrak{e}) = y^\mu. \tag{4.28}$$

We then have the following coordinate transformation

$$y^\mu = \mathbf{x}^\mu(\mathtt{h}(\mathfrak{e})) = \mathtt{h}^\mu(x^\nu). \tag{4.29}$$

Let $\{\partial/\partial x^\mu\}$ and $\{\partial/\partial y^\mu\}$ be a coordinate bases for $T(U \cap V)$ and $\{dx^\mu\}$ and $\{dy^\mu\}$ the corresponding dual basis for $T^*(U \cap V)$.

Then, if the local representation of $\mathbf{S} \in \sec T^r_s M \subset \sec \mathcal{T}M$ in the coordinate chart $\{\mathbf{y}^\mu\}$ at any point of $U \cap V$ is $\check{\mathbf{S}} \in \sec T^r_s \mathbb{R}^n$,

$$\check{\mathbf{S}} = S^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}(y^j)dy^{\nu_1} \otimes \ldots \otimes dy^{\nu_s} \otimes \frac{\partial}{\partial y^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{\mu_r}}, \tag{4.30}$$

[6]Note that in general $\mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) \neq y^\mu$.

we have that the representative of $\mathbf{S}' = \mathtt{h}^*\mathbf{S}$ in $T^r_s\mathbb{R}^n$at any point $\mathfrak{e} \in U \cap V$ is given by

$$\mathtt{h}^*\check{\mathbf{S}} = S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots . \rho_s}(x^j) dx^{\rho_1} \otimes \ldots \otimes dx^{\rho_s} \otimes \frac{\partial}{\partial x^{\sigma_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\sigma_r}}$$

$$S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots . \rho_s}(x^j) = S^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots . \nu_s}(y^i(x^j)) \frac{\partial y^{\nu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial y^{\nu_s}}{\partial x^{\rho_s}} \frac{\partial x^{\sigma_1}}{\partial y^{\mu_1}} \cdots \frac{\partial x^{\sigma_r}}{\partial y^{\mu_r}}. \tag{4.31}$$

Remark 4.24 Another important expression for the pullback mapping can be found if we choice charts with the coordinate functions $\{\mathbf{x}^\mu\}$ and $\{\mathbf{y}^\mu\}$ defined by

$$\mathbf{x}^\mu(\mathfrak{e}) = \mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) \tag{4.32}$$

Then writing

$$\mathbf{x}^\mu(\mathfrak{e}) = x^\mu, \ \mathbf{y}^\mu(\mathtt{h}(\mathfrak{e})) = y^\mu, \tag{4.33}$$

we have the following coordinate transformation

$$y^\mu = \mathtt{h}^\mu(x^\nu) = x^\mu, \tag{4.34}$$

from where it follows that in this case

$$S'^{\sigma_1 \ldots \sigma_r}_{\rho_1 \ldots . \rho_s}(x^j) = S^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots . \nu_s}(y^i(x^j)). \tag{4.35}$$

4.1.7 Lie Derivatives

Definition 4.25 Let M be a differentiable manifold. We say that a mapping $\sigma : M \times \mathbb{R} \to M$ is a one parameter group if

(i) σ is differentiable,
(ii) $\sigma(x, 0) = x$, $\forall x \in M$,
(iii) $\sigma(\sigma(x, s), t) = \sigma(x, s + t)$, $\forall x \in M$, $\forall s, t \in \mathbb{R}$.

These conditions may be expressed in a more convenient way introducing the mappings $\sigma_t : M \to M$ such that

$$\sigma_t(x) = \sigma(x, t). \tag{4.36}$$

For each $t \in \mathbb{R}$, the mapping σ_t is differentiable, since $\sigma_t = \sigma \circ l_t$, where $l_t : M \to M \times \mathbb{R}$ is the differentiable mapping given by $l_t(x) = (x, t)$.

Also, condition (ii) says that $\sigma_0 = \mathrm{id}_M$. Finally, condition (iii) implies, as can be easily verified that

$$\sigma_t \circ \sigma_s = \sigma_{s+t}. \tag{4.37}$$

Observe also that if we take $s = -t$ in Eq. (4.37) we get $\sigma_t \circ \sigma_{-t} = \mathrm{id}_M$. It follows that for each $t \in \mathbb{R}$, the mapping σ_t is a diffeomorphism and $(\sigma_t)^{-1} = \sigma_{-t}$.

Definition 4.26 We say that a family $(\sigma_t, t \in \mathbb{R})$ of mappings $\sigma_t : M \to M$ is a one-parameter group of diffeomorphisms G_1 of M.

Definition 4.27 Given a one-parameter group $\sigma : M \times \mathbb{R} \to M$ for each $x \in M$, we may construct the mapping

$$\begin{aligned} \sigma_x &: \mathbb{R} \to M, \\ \sigma_x(t) &= \sigma(x, t), \end{aligned} \tag{4.38}$$

which in view of condition (ii) is a curve in M, called the orbit (or trajectory) of x generate by the group. Also, the set of all orbits for all points of M are the trajectories of G_1.

It is possible to show, using condition (iii) that for each point $x \in M$ pass one and only one trajectory of the one-parameter group. As a consequence it is uniquely determined by a vector field $\boldsymbol{v} \in \sec TM$ which is constructed by associating to each point $x \in M$ the tangent vector to the orbit of the group in that point, i.e.,

$$\boldsymbol{v}(\sigma_x(t)) = \frac{d}{dt}\sigma_x(t). \tag{4.39}$$

Definition 4.28 The vector field $\boldsymbol{v} \in \sec TM$ determined by Eq. (4.39) is called a Killing vector field relative to the one parameter group of diffeomorphisms $(\sigma_t, t \in \mathbb{R})$.

Remark 4.29 It is important to have in mind that in general, given a vector field $\boldsymbol{v} \in \sec TM$ it does not define a group (even locally) of diffeomorphisms in M. In truth, it will be only possible, in general, to find a local one-parameter pseudo-group that induces $\boldsymbol{v}$. A local one parameter pseudo-group means that σ_t is not defined for all $t \in \mathbb{R}$, but for any $x \in M$, there exists a neighborhood $U(x)$ of x, an interval $I(x) = (-\varepsilon(x), \varepsilon(x)) \subset \mathbb{R}$ and a family $(\sigma_t, t \in I(x))$ of mappings $\sigma_t : M \to M$, such that the properties (i)–(iii) in *Definition 4.27* are valid, when $|t| < \varepsilon(x)$, $|s| < \varepsilon(x)$ and $|t + s| < \varepsilon(x)$.

Definition 4.30 Taking into account the previous remark, the vector field $\boldsymbol{v} \in \sec TM$ is called the infinitesimal generator of the one parameter local pseudo-group $(\sigma_t, t \in I(x))$ and the mapping $\sigma : M \times I(x) \to M$ is called the flow of the vector field ξ.

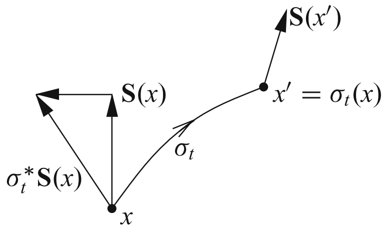

Fig. 4.3 The Lie derivative

Of course, given $\boldsymbol{v} \in \sec TM$ we obtain the one parameter local pseudo-group that induces $\boldsymbol{v}$ by integration of the differential equation Eq. (4.39). From that, we see that the trajectories of the group are *also* the integral lines of the vector field $\boldsymbol{v}$.

Definition 4.31 Let $(\sigma_t, t \in I(x))$ a one-parameter local pseudo group of diffeomorphisms of M that induces the vector field $\boldsymbol{v}$ and let $\mathbf{S} \in \sec T^r_s M$. The Lie derivative of $\mathbf{S}$ in the direction of $\boldsymbol{v}$ is the mapping

$$\pounds_{\boldsymbol{v}} : \sec T^r_s M \to \sec T^r_s M,$$
$$\pounds_{\boldsymbol{v}} \mathbf{S} = \lim_{t \to 0} \frac{\sigma_t^* \mathbf{S} - \mathbf{S}}{t}. \tag{4.40}$$

Remark 4.32 It is possible to define the Lie derivative using the pushforward mapping, the results that follows are the same. In this case we have $\pounds_{\boldsymbol{v}} \mathbf{S} = \lim_{t \to 0} \frac{\mathbf{S} - \sigma_{*t} \mathbf{S}}{t}$ (Fig. 4.3).

4.1.8 Properties of $\pounds_{\boldsymbol{v}}$

(i) $\pounds_{\boldsymbol{v}}$ is a linear mapping and preserve contractions.
(ii) Leibniz's rule. If $S \in \sec T^r_s M$, $S' \in \sec T^{r'}_{s'} M$, we have

$$\pounds_{\boldsymbol{v}} (S \otimes S') = \pounds_v S \otimes S' + S \otimes \pounds_v S'. \tag{4.41}$$

(iii) If $f : M \to \mathbb{R}$, we have

$$\pounds_{\boldsymbol{v}} f = \boldsymbol{v}(f). \tag{4.42}$$

(iv) If $\boldsymbol{v}, \boldsymbol{w} \in \sec TM$, we have

$$\pounds_{\boldsymbol{v}} \boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{v}], \tag{4.43}$$

where $[\boldsymbol{v}, \boldsymbol{w}]$ is the *commutator* of the vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$, such that

$$[\boldsymbol{v}, \boldsymbol{w}](f) = \boldsymbol{v}(\boldsymbol{w}(f)) - \boldsymbol{w}(\boldsymbol{v}(f)). \tag{4.44}$$

(v) If $\omega \in \sec T^*M$, we have

$$\pounds_{\boldsymbol{v}}\omega = \left(\boldsymbol{v}(\omega_k) + \omega_i \boldsymbol{e}_k(v^i) - c_{\cdot jk}^{i\cdot\cdot}\omega_i v^j\right)\theta^k, \tag{4.45}$$

where v^j and ω_i are the components of in the dual basis $\{\boldsymbol{e}_j\}$ and $\{\theta^i\}$ and the $c_{\cdot jk}^{i\cdot\cdot}$ are called the structure coefficients of the frame $\{\boldsymbol{e}_j\}$, and

$$[\boldsymbol{e}_j, \boldsymbol{e}_k] = c_{\cdot jk}^{i\cdot\cdot}\boldsymbol{e}_i. \tag{4.46}$$

Exercise 4.33 Show that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \sec TM$, then they satisfy Jacobi's identity, i.e.,

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0. \tag{4.47}$$

Exercise 4.34 Show that for $\boldsymbol{u}, \boldsymbol{v} \in \sec TM$

$$\pounds_{[\boldsymbol{u},\boldsymbol{v}]} = [\pounds_{\boldsymbol{u}}, \pounds_{\boldsymbol{v}}]. \tag{4.48}$$

4.1.9 Invariance of a Tensor Field

The concept of Lie derivative is intimately associated to the notion of invariance of a tensor field $\mathbf{S} \in \sec T_s^r M$.

Definition 4.35 We say that $\mathbf{S}$ is invariant under a diffeomorphism $h : M \to M$, or h is a symmetry of $\mathbf{S}$, if and only if

$$h^*\mathbf{S}|_x = \mathbf{S}|_x\,. \tag{4.49}$$

We extend naturally this definition for the case in which we have a local one-parameter pseudo-group σ_t of diffeomorphisms. Observe, that in this case, it follows from the definition of Lie derivative, that if $\mathbf{S}$ is invariant under σ_t, then

$$\pounds_{\boldsymbol{v}}\mathbf{S} = 0 \tag{4.50}$$

More properties of Lie derivatives of differential forms that we shall need in future chapters, will be given at the appropriate places.

Remark 4.36 A correct concept for the Lie derivative of spinor fields is as yet a research subject and will not be discussed in this book. A Clifford bundle approach to the subject which we think worth to be known is presented in [22].

4.2 Cartan Bundle, de Rham Periods and Stokes Theorem

In this section, we briefly discuss the processes of differentiation in the Cartan bundle and the concept of de Rham periods and Stokes theorem.

4.2.1 Cartan Bundle

Definition 4.37 The *Cartan bundle* over the cotangent bundle of M is the set

$$\bigwedge T^*M = \bigcup_{x \in M} \bigwedge T_x^*M = \bigcup_{x \in M} \bigoplus_{r=0}^{n} \bigwedge^r T_x^*M, \tag{4.51}$$

where $\bigwedge T_x^*M$, $x \in M$, is the exterior algebra of the vector space T_x^*M. The subbundle $\bigwedge^r T^*M \subset \bigwedge T^*M$ given by:

$$\bigwedge^r T^*M = \bigcup_{x \in M} \bigwedge^r T_x^*M \tag{4.52}$$

is called the *r-forms bundle* $(r = 0, \ldots, n)$.

Definition 4.38 The *exterior derivative is a mapping*

$$d : \sec \bigwedge T^*M \to \sec \bigwedge T^*M,$$

satisfying:

$$\begin{aligned} &\text{(i)}\ \ d(A + B) = dA + dB; \\ &\text{(ii)}\ \ d(A \wedge B) = dA \wedge B + \hat{A} \wedge dB; \\ &\text{(iii)}\ df(\boldsymbol{v}) = \boldsymbol{v}(f); \\ &\text{(iv)}\ d^2 = 0, \end{aligned} \tag{4.53}$$

for every $A, B \in \sec \bigwedge T^*M, f \in \sec \bigwedge^0 T^*M$ and $v \in \sec TM$.

Exercise 4.39 Show that for $A \in \sec \bigwedge^p T^*M$ and $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_p \in \sec TM$,

$$\begin{aligned} dA(\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_p) = &\sum_{i=1}^{p} (-1)^i \boldsymbol{v}_i (A(\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \check{\boldsymbol{v}}_i, \ldots, \boldsymbol{v}_p)) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} A([\boldsymbol{v}_i, \boldsymbol{v}_j]\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \check{\boldsymbol{v}}_i, \ldots, \check{\boldsymbol{v}}_j, \ldots \boldsymbol{v}_p). \end{aligned} \tag{4.54}$$

Remark 4.40 Note that due to property (ii) the exterior derivative does not satisfy the Leibniz's rule, and as such it is not a derivation. In fact the technical term is antiderivation (see [3]).

Remark 4.41 Let $\mathbf{x}^i$ be coordinate functions of a chart (U, φ) of an atlas of M. A coordinate basis for TU in that chart is denoted $\{\partial/\partial x^i\}$. This means that for each $x \in U$, $\partial/\partial x^i\big|_x$ is a basis of T_xU. As we already know, the dual (coordinate) basis for T^*U is denoted[7] $\{dx^j\}$. This means that $dx^j\big|_x$ is a basis for T^*_xU. We have (indeed) that

$$dx^j(\partial/\partial x^i)\big|_x = \partial x^j/\partial x^i\big|_x = \delta^j_i. \tag{4.55}$$

4.2.2 The Interior Product of Forms and Vector Fields

Another important antiderivation is the so called interior product (sometimes also called inner product).

Definition 4.42 Given a vector field $\boldsymbol{v} \in \sec TM$ we define the interior product extensor of $\boldsymbol{v}$ with $\alpha \in \sec \bigwedge^p T^*M$ as the mapping

$$\begin{aligned} \sec T^*M \times \sec \bigwedge{}^p T^*M &\to \sec \bigwedge{}^{p-1} T^*M, \\ (\boldsymbol{v}, \alpha) &\mapsto \mathbf{i}_{\boldsymbol{v}}\alpha, \end{aligned} \tag{4.56}$$

where $\mathbf{i}_{\boldsymbol{v}} : \sec \bigwedge^p T^*M \to \sec \bigwedge^{p-1} T^*M$ satisfy

(i) For any $\alpha, \beta \in \sec \bigwedge T^*M$ and $a, b \in \mathbb{R}$,

$$\mathbf{i}_{\boldsymbol{v}}(a\alpha + b\beta) = a\mathbf{i}_{\boldsymbol{v}}\alpha + b\mathbf{i}_{\boldsymbol{v}}\beta. \tag{4.57}$$

(ii) if $f \in \sec \bigwedge^0 T^*M$ is a smooth function, then $\mathbf{i}_{\boldsymbol{v}} f = 0$,

(iii) If $\{\boldsymbol{e}_i\}$ is an arbitrary basis for TU, $U \subset M$, and $\{\theta^i\}$ its dual basis,

$$\mathbf{i}_{\boldsymbol{e}_k}\theta^{j_1} \wedge \ldots . \wedge \theta^{j_p} = \sum_{r=1}^{p}(-1)^{r+1}\delta^{j_r}_k \theta^{j_1} \wedge \ldots . \check{\theta}^{j_k} \wedge \ldots \wedge \theta^{j_p}, \tag{4.58}$$

where as usual $\check{\theta}^{j_k}$means that the term θ^{j_k} is missing in the expression.

[7] Eventually a more rigorously notation for a basis of T^*U should be $\{d\mathbf{x}^i\}$.

From Eq. (4.58) it follows that for $A_p \in \sec \bigwedge^p T^*M$ and $B_q \in \sec \bigwedge^q T^*M$ we have

$$\mathbf{i}_{\boldsymbol{v}}(A_p \wedge B_q) = \mathbf{i}_{\boldsymbol{v}} A_p \wedge B_q + (-1)^{pq} A_p \wedge \mathbf{i}_{\boldsymbol{v}} B_q \tag{4.59}$$

and we usually say that $\mathbf{i}_{\boldsymbol{v}}$ is an antiderivation.

Exercise 4.43 If $\{\mathbf{x}^i\}$ are coordinate functions of a local chart of M, and $\boldsymbol{v} = v^i \frac{\partial}{\partial x^i}$, show that $\mathbf{i}_{\boldsymbol{v}} dx^i = v^i$.

Exercise 4.44 Properties of $\mathbf{i}_v$. Show that

$$\mathbf{i}_{\boldsymbol{v}}^2 = 0, \tag{4.60}$$

$$d\mathbf{i}_{\boldsymbol{v}} + \mathbf{i}_{\boldsymbol{v}} d = \pounds_{\boldsymbol{v}}, \tag{4.61}$$

$$[\pounds_{\boldsymbol{v}}, \mathbf{i}_{\boldsymbol{w}}] = \pounds_{\boldsymbol{v}} \mathbf{i}_{\boldsymbol{w}} - \mathbf{i}_{\boldsymbol{w}} \pounds_{\boldsymbol{v}} = \mathbf{i}[\boldsymbol{v}, \boldsymbol{w}], \tag{4.62}$$

$$\pounds_{\boldsymbol{v}} d = d\pounds_{\boldsymbol{v}}. \tag{4.63}$$

Equation (4.61) is sometimes called Cartan's magical formula. It is really, a very important formula in the formulation of conservation laws, as we shall see in Chap. 9.

4.2.3 Extensor Fields

Let $\{\theta^i\}$ be an arbitrary basis for $\sec T^*U$, $U \subset M$. Let $\kappa = \kappa_i \theta^i \in \sec \bigwedge^1 T^*M$ and $\omega = \frac{1}{r!} \omega_{i_1 \ldots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \in \sec \bigwedge^r T^*M$, $r = 1, 2, \ldots, n$.

Definition 4.45 A $(1,1)$-extensor field $t : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$ and its extension $\underline{t} : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$ are the linear operators given by

$$t(\kappa) = t(\kappa_i \theta^i) = \kappa_i t(\theta^i),$$
$$\underline{t}(\omega) = \underline{t}(\frac{1}{r!} \omega_{i_1 \ldots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}) = \frac{1}{r!} \omega_{i_1 \ldots i_r} t(\theta^{i_1}) \wedge \cdots \wedge t(\theta^{i_r}) \tag{4.64}$$

for all κ and ω, $r = 1, 2, \ldots n$. Moreover, if $f \in \sec \bigwedge^0 T^*M$, we put $\underline{t}(f) = f$.

4.2.4 Exact and Closed Forms and Cohomology Groups

Definition 4.46 A r-form $G_r \in \sec \bigwedge^r T^*M$ is called *closed* (or a *cocycle*) if and only if $dG_r = 0$. A r-form $F_r \in \sec \bigwedge^r T^*M$ is called *exact* (or a *coboundary*) if and only if $F_r = dA_{r-1}$, with $A_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

Definition 4.47 The space of closed r-forms is called the r-cocycle group and denoted by $Z^r(M)$. The space of exact r-forms is called the r-coboundary group and denoted by $B^r(M)$.

We recall that the sets $Z^r(M)$ and $B^r(M)$ have the structures of vector spaces over the real field $\mathbb{R}$. Since according to Eq. (4.53iv) $d^2 = 0$ it follows that $B^r(M) \subset Z^r(M)$. Then if $F_r = dA_{r-1} \Rightarrow dF_r = 0$, but in general $dG_r = 0 \nRightarrow G_r = dC_{r-1}$, with $C_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

Definition 4.48 The space $H^r(M) = Z^r(M)/B^r(M)$ is the r-de Rham cohomology group of the manifold M. Obviously, the elements of $H^r(M)$ are equivalent classes of closed forms, i.e., if $F_r, F_r' \in \sec H^r(M)$, then $F_r - F_r' = dW_{r-1}$, $W_{r-1} \in \sec \bigwedge^{r-1} T^*M$.

As a vector space over the real field, $H^r(M)$ is called the r-de Rham vector space group of the manifold M.

Definition 4.49 The dimension of the r-homology[8] (respectively cohomology) group is called the Betti number b_r (respectively b^r) of M.

A very important result is the

Proposition 4.50 (Poincaré Lemma) *If $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ then any closed r-form $F_r \in \sec \bigwedge^r T^*U$ $(r \geq 1)$ which is differentiable on U is also exact.*

Proof For a proof see , e.g., [25].■

Note that if $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ then U is contractible to a point $p \in M$. Also, from Poincaré's lemma it follows that the Betti numbers of U, $b^r = 0, r = 1, 2, \ldots, r$.

Any closed form is exact at least locally and the non triviality of de Rham cohomology group is an obstruction to the global exactness of closed forms.

Remark 4.51 It is very important to observe that Poincaré's lemma does not hold if $F_r \in \sec \bigwedge^r T^*M$ is not differentiable at certain points of $\mathbb{R}^n$, since in that case the manifold where F_r is differentiable is not homeomorphic to $\mathbb{R}^n$. The 'classical' example according to Spivack [43] is $A \in \sec \bigwedge^1 T^*\mathbb{R}^2$,

$$A = \frac{-ydx + xdy}{x^2 + y^2} = d(\arctan \frac{y}{x}). \tag{4.65}$$

[8] See Definition 4.65.

Observe that A is differentiable on $\mathbb{R}^2 - \{0\}$, but despite the third member of Eq. (4.65) A is not exact on $\mathbb{R}^2$, because $\arctan \frac{y}{x}$ is not a differentiable function on $\mathbb{R}^2$.

4.3 Integration of Forms

In what follows we briefly recall some concepts related to the integration of forms on *orientable* manifolds. First we introduce the definition of the integral of a n-form in an n-dimensional manifold M and next the integration of a r-form $A_r \in \sec T^*M$ which is realized over a r-chain.

4.3.1 Orientation

Let M be an n-dimensional connected manifold and $U_\alpha, U_\beta \subset M$, $U_\alpha \cap U_\beta \neq \varnothing$. Let $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be coordinate charts of the maximal atlas of M with coordinate functions $\{\mathbf{x}^i_\alpha\}$ and $\{\mathbf{x}^j_\beta\}$, $i, j = 1, 2 \ldots, n$. Let $e \in U_\alpha \cap U_\beta$. The natural ordered bases $\{\frac{\partial}{\partial x^i_\alpha}\Big|_e\}$ and $\{\frac{\partial}{\partial x^i_\beta}\Big|_e\}$ of T_eM are said to have the same orientation if $J = \det\left[\frac{\partial x^i_\alpha}{\partial x^i_\beta}\Big|_e\right] > 0$. If $J < 0$ the bases are said to have opposite orientations. An orientation at $e \in U_\alpha \cap U_\beta$ is a choice of an ordered basis (not necessarily a coordinate one) for T_eM.

Now, suppose that the basis $\{\frac{\partial}{\partial x^i_\alpha}\Big|_e\}$ is *declared* positive (a right-handed basis). A orientation in T_eM induces naturally an orientation in T^*_eM as follows. Let $\{\theta^i\big|_e\}$ be an ordered basis of T^*_eM. Let $\tau_e = \theta^1\big|_e \wedge \cdots \wedge \theta^n\big|_e$. Then,

$$\begin{aligned}
&\boldsymbol{\tau}_e\Big(\frac{\partial}{\partial x^1_\alpha}\Big|_e, \ldots, \frac{\partial}{\partial x^n_\alpha}\Big|_e\Big) \\
&= \frac{1}{n!}\det\begin{bmatrix}
\theta^1\big|_e\left(\frac{\partial}{\partial x^1_\alpha}\Big|_e\right) & \theta^1\big|_e\left(\frac{\partial}{\partial x^2_\alpha}\Big|_e\right) & \cdots & \theta^1\big|_e\left(\frac{\partial}{\partial x^n_\alpha}\Big|_e\right) \\
\theta^2\big|_e\left(\frac{\partial}{\partial x^1_\alpha}\Big|_e\right) & \theta^2\big|_e\left(\frac{\partial}{\partial x^2_\alpha}\Big|_e\right) & \cdots & \theta^2\big|_e\left(\frac{\partial}{\partial x^n_\alpha}\Big|_e\right) \\
\cdots & \cdots & \cdots & \cdots \\
\theta^n\big|_e\left(\frac{\partial}{\partial x^1_\alpha}\Big|_e\right) & \theta^n\big|_e\left(\frac{\partial}{\partial x^2_\alpha}\Big|_e\right) & \cdots & \theta^n\big|_e\left(\frac{\partial}{\partial x^n_\alpha}\Big|_e\right)
\end{bmatrix}.
\end{aligned} \tag{4.66}$$

If $\boldsymbol{\tau}_e(\frac{\partial}{\partial x^1_\alpha}\Big|_e, \ldots, \frac{\partial}{\partial x^n_\alpha}\Big|_e) > 0$ we say that the ordered basis $\{\theta^i\big|_e\}$ of T^*_eM is positive. If $\boldsymbol{\tau}_e(\frac{\partial}{\partial x^1_\alpha}\Big|_e, \ldots, \frac{\partial}{\partial x^n_\alpha}\Big|_e) < 0$ we say that the ordered basis $\{\theta^i\big|_e\}$ of T^*_eM is negative.

Suppose that for all $e \in U_\alpha \cap U_\beta$ we have $J = \det\left[\frac{\partial x^i_\alpha}{\partial x^i_\beta}\right] > 0$. In this case we define that on $U_\alpha \cap U_\beta$ that the bases $\{\frac{\partial}{\partial x^i_\alpha}\}$ and $\{\frac{\partial}{\partial x^i_\beta}\}$ of TU_α and TU_β have the same orientation. If $J = \det\left[\frac{\partial x^i_\alpha}{\partial x^i_\beta}\right] < 0$ we say that the bases have opposite orientation on $U_\alpha \cap U_\beta$.

Definition 4.52 Let $\{U_\alpha\}$ be a covering for M, an n-dimensional connected manifold. We say that M is orientable if for any two overlapping charts U_α and U_β there exist coordinate functions $\{\mathbf{x}^i_\alpha\}$, $\{\mathbf{x}^j_\beta\}$ for U_α and U_β such that $\det\left[\frac{\partial x^i_\alpha}{\partial x^i_\beta}\right] > 0$.

Remark 4.53 From what has been said above it is clear that if M is orientable, there exists an n-form $\tau \in \sec \bigwedge^n T^*M$ called a volume element which is never null.

Thus, we have the alternative (equivalent) definition of an orientable manifold.

Definition 4.54 A connected n-dimensional manifold M is orientable if there exists a non null global section of $\bigwedge^n T^*M$ and $\tau, \tau' \in \sec \bigwedge^n T^*M$ define the same orientation (respectively opposite orientation) if there exists a global function $\lambda \in \sec \bigwedge^0 T^*M$ such that $\lambda > 0$ (respectively $\lambda < 0$) such that $\tau' = \lambda\tau$.

Remark 4.55 Of course, a given orientable manifold M admits two inequivalent orientations, one is declared right-handed, and the other left-handed. It is quite obvious that there are manifolds which are not orientable, the classical example is the Möbius strip, which may be found in almost all books in differential geometry, as, e.g., [3, 25].

4.3.2 Integration of a n-Form

In what follows we suppose that M is orientable.[9] Let (U, φ) be a chart of the maximal atlas of M and $\{x^i\}$ the coordinate functions of the chart. Let $h \in \sec \bigwedge^0 T^*M$ be a Lebesgue *integrable* function and[10] $\tau = dx^1 \wedge \cdots \wedge dx^n \in \sec \bigwedge^n T^*M$.

Definition 4.56 The integral of $h\tau \in \sec \bigwedge^n T^*M$ in $\mathfrak{A} \subset U \subset M$ is

$$\int_{\mathfrak{A}} h\tau := \int_{\varphi(\mathfrak{A})} h \circ \varphi^{-1}(x^i) dx^1 \cdots dx^n \tag{4.67}$$

[9] In Chap. 6 we will learn that a spacetime manifold admitting spinor fields must necessarily be orientable.

[10] Of course, we should write $\tau = \varphi^*_\alpha(dx^1 \wedge \cdots \wedge dx^n)$ since dx^i are 1-forms in $T_{\varphi_{\alpha(U)}}\mathbb{R}^n$. So, ours is a sloppy (universally used) notation.

where in the second member of Eq. (4.67) is the ordinary multiple integral of a Lebesgue integrable function $h = h \circ \varphi^{-1}(x^i)$ of n variables.

Let be $\mathfrak{A} \subset U \cap V$ and (V, ψ) another chart of the maximal atlas of M with coordinate functions $\{\mathbf{x}'^j\}$ and suppose that $J = \det\left[\frac{\partial x^i}{\partial x'^j}\right] > 0$ on $U \cap V$. Then we can write that

$$h\tau = h \circ \psi^{-1}(x'^i) J dx'^1 \wedge \cdots \wedge dx'^n = h \circ \psi^{-1}(x'^i)\, |J|\, dx'^1 \wedge \cdots \wedge dx'^n \tag{4.68}$$

and

$$\int_{\mathfrak{A}} h\tau = \int_{\psi(\mathfrak{A})} h \circ \psi^{-1}(x'^i))\, |J|\, dx'^1 \cdots dx'^n, \tag{4.69}$$

which corresponds to the classical formula for a change of variables in a multiple integral.

Now, if M is *paracompact*, i.e., there is an open covering $\{U_\alpha\}$ of M such that each $e \in M$ is covered by a finite number of the U_α a partition of the unity associated to the covering $\{U_\alpha\}$ is a family of differentiable functions $p_\alpha : M \to \mathbb{R}$ such that: (a) $0 \leq p_\alpha \leq 1$; (b) $p_\alpha(e) = 0$ for all $e \notin U_\alpha$; (c) If k is the finite number of U_α covering e then for any $e \in M$ we have that $\sum_{\alpha=1}^{k} p_\alpha(e) = 1$. It is obvious that we can write

$$h(e) = \sum_{\alpha=1}^{k} p_\alpha(e) h(e) = \sum_{\alpha=1}^{k} h_\alpha(e). \tag{4.70}$$

We then have the

Definition 4.57 The integral of $h\tau \in \sec \bigwedge^n T^*M$ in M is

$$\int_M h\tau := \sum_\alpha \int_{U_\alpha} h_\alpha \tau = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} h_\alpha \circ \varphi_\alpha^{-1}(x^i) dx^1 \cdots dx^n \tag{4.71}$$

We may verify that the definition is independent of the choice of atlas used for M (and thus of the partition of the unity used) if the new atlas has the same orientation as the previous one.

4.3.3 Chains and Homology Groups

Orientation of Subspaces

Let $(u^1, \ldots, u^n)$ be a right handed coordinate system for $\mathbb{R}^n$. For any $\mathbb{R}^r \subset \mathbb{R}^n$ $(u^1, \ldots, u^r)$ is a naturally right handed coordinate system for $\mathbb{R}^r$, which is supposed to be coherently oriented with $\mathbb{R}^n$.

Definition 4.58 A r-rectangle P^r in $\mathbb{R}^r \subset \mathbb{R}^n$ is a naturally positive oriented subset of $\mathbb{R}^r$ such that $a^i \leq u^i \leq b^i$, $i = 1, \ldots, r$. The boundary of the rectangle P^r is the set $\partial P^r$of $2r$ rectangles $P^{r-1} \in \mathbb{R}^{r-1}$ defined by the faces $u^i = a^i$ and $u^i = b^i$ of P^r. We suppose that the boundary ∂P^r is coherently oriented with P^r. That means that any face has the orientation $(u^1, \ldots \check{u}^i, \ldots u^r)$ if $u^i = a^i, i$ is even and $u^i = b^i,\ i$ is odd and the opposite orientation if $u^i = a^i, i$ is odd and $u^i = b^i,\ i$ is even.

Next we introduce the concept of elementary chain in M.

Definition 4.59 An elementary r-chain or c_r in a n-dimensional connected manifold M is a pair (P^r, f), with $f:\ \mathbb{R}^r \supset U \to M$ a differentiable mapping. The image of the P^r rectangle is denoted by suppc_r. When f is a diffeomorphism suppc_r is called an elementary r-domain of integration.

Definition 4.60 The boundary of an elementary r-chain is the image of ∂P^r.

Definition 4.61 A r-chain on M is a formal linear combination of elementary r-chains c_{rj} with real coefficients $C_r = \sum_j a_j c_{rj}$. The space of r-chains in M forms a vector space over the real field. It is denoted by $C_r(M)$ and called the r-chain group.

Remark 4.62 We are in general interested in formal locally finite linear combinations with $a_j = \pm 1$, in which case C_r is said a domain of integration on M. More generally, in algebraic topology the coefficients a_j are in many applications elements of a finite group. In that case $C_r(M)$ is a group, but it is not a vector space. That is the reason why $C_r(M)$ has been called the r-chain group.

Definition 4.63 The boundary operator ∂ is a mapping

$$\partial : C_r(M) \to C_{r-1}(M) \tag{4.72}$$

such that for any r-chain $C_r = \sum_j a_j c_{rj}$

$$\partial C_r = \sum_j a_j \partial c_{rj}, \tag{4.73}$$

where ∂c_{rj} is the image under f of an elementary P^r_j-rectangle.

The boundary operator ∂ has the fundamental property

$$\partial^2 = 0, \tag{4.74}$$

a formula that will be proved below.

Definition 4.64 A finite r-chain C_r is said to be a cycle if and only if $\partial C_r = 0$. The space of cycles is denoted $Z_r(M)$. Also, a finite r-chain C_r is said to be a boundary if and only if $C_r = \partial C_{r-1}$and the space of boundaries is denoted by $B_r(M)$.

Since $\partial^2 = 0$ it follows that $B_r(M) \subset Z_r(M)$. We then have

Definition 4.65 The quotient set $H_r(M) = Z_r(M)/B_r(M)$ is called the r-homology group of M.

Remark 4.66 Recall that the dimension of the r-homology group is called the Betti number b_r of M.

In what follows we use the standard convention that $Z^0(M)$ is the space of differentiable functions h such that $dh = 0$. Also, we agree that $B^0(M) = \varnothing$. Finally, we agree that $Z_0(M) = C_0(M)$ and that $B_0(M) = \varnothing$.

4.3.4 Integration of a r-Form

Definition 4.67 The integration of $F_r \in \sec \bigwedge^r T^*M$ over suppC_r is

$$\int_{C_r} F_r = \sum\nolimits_j a_j \int_{c_{rj}} F_r = \sum\nolimits_j a_j \int_{P_j^r} f^* F_r, \tag{4.75}$$

where f^* is the pullback mapping induced by f.

When F_r is continuous and C_r is finite the integral is always defined. The integral is also always defined if F_r has compact support and C_r is locally finite. In what follows we suppose that this is the case. Definition 4.67 shows very clearly that it is bilinear in F_r and C_r and suggests the definition of a *non degenerated* inner product $\langle\rangle : C_r(M) \times \sec \bigwedge^r M \to \mathbb{R}$ given

$$\langle C_r, F_r \rangle = \int_{C_r} F_r. \tag{4.76}$$

With the aid of that definition we can say that two chains C_r and $C_r\,'$ are equal if and only if $\langle C_r, F_r \rangle = \langle C_r', F_r \rangle$. This observation is important because the decomposition of a chain into elementary chains is not unique.

Recall that given a manifold, say M with boundary, its boundary is denoted by ∂M. The manifold M is called triangulable if it can be decomposed as a union of adjacent n-domains of integration with orientation compatible with the *orientation* of M.

4.3.5 Stokes Theorem

Theorem 4.68 (Stokes) *For any $F_r \in \sec \bigwedge^r T^*M$ and $C_r \in C_r(M)$ it holds*

$$\int_{C_r} dF_r = \int_{\partial C_r} F_r. \tag{4.77}$$

Proof For a proof, see, e.g., [25].■

Stokes formula can be written in the suggestive way

$$\langle C_r, dF_r \rangle = \langle \partial C_r, F_r \rangle \tag{4.78a}$$

Proposition 4.69 *The boundary operator* ∂ *has the fundamental property*

$$\partial^2 = 0. \tag{4.79}$$

Proof It follows directly from the fact that $d^2 = 0$ and Stokes theorem. Indeed,

$$\langle \partial^2 C_r, F_r \rangle = \langle \partial C_r, dF_r \rangle = \langle C_r, d^2 F_r \rangle = 0,$$

which proves the proposition.■

4.3.6 Integration of Closed Forms and de Rham Periods

We now investigate integration in the case when $G_r \in \sec \bigwedge^r T^*M$ is closed. The inner product introduced by Eq. (4.76) permit us to define a mapping from the space of closed (cocycles) forms $Z^r(M)$ into the (dual) space of cycles $Z_r(M)$, by

$$\mathbf{I} : Z^r(M) \to Z_r(M), \tag{4.80}$$

such that for any $G_r \in \sec \bigwedge^r T^*M$ and $z_r \in Z_r(M)$,

$$\mathbf{I}(G_r)(z_r) = \langle z_r, G_r \rangle. \tag{4.81}$$

Note now that

$$\langle z_r + \partial c, G_r \rangle = \langle z_r, G_r \rangle + \langle \partial c, G_r \rangle = \langle z_r, G_r \rangle + \langle c, dG_r \rangle = \langle z_r, G_r \rangle, \tag{4.82}$$

because G_r is closed. This implies that $\mathbf{I}(G_r)$ can be considered as a linear function on the equivalent class of z_r modulus $B_r(M)$, i.e., it defines a mapping

$$\mathbf{I} : Z^r(M) \to H_r(M). \tag{4.83}$$

Also, $\mathbf{I}(G_r + dG_{r-1}) \equiv \mathbf{I}(G_r)$, so it is obvious that $\mathbf{I}$ really defines a linear transformation

$$\mathbf{I} : H^r(M) \to H_r(M). \tag{4.84}$$

Theorem 4.70 (de Rham 1) *The mapping* $\mathbf{I} : H^r(M) \to H_r(M)$ *is an isomorphism. If* $H_r(M)$ *is finite dimensional as when* M *is compact and if* $z_r^{(1)}, \ldots z_r^{(b)}$ *(with* $b =$

the r-Betti number) is a r-cycle basis of $H_r(M)$ *and if* $\pi_1, \ldots, \pi_r \in \mathbb{R}$ *are arbitrary numbers then there is a closed r-form* $G_r \in Z^r(M)$ *such that*

$$\langle z_r^{(i)}, G_r \rangle = \pi_i,\ i = 1, \ldots, r. \tag{4.85}$$

Proof See, e.g., [25].■

Definition 4.71 The number π_r in Eq. (4.85) is called the period of the form G_r on the cycle $z_r^{(i)}$.

Corollary 4.72 (de Rham 2) *If for a closed form* $G_r \in \sec \bigwedge^r T^*M$ *and for any* $z_r^{(i)} \in H_r(M)$ *we have* $\langle z_r^{(i)}, G_r \rangle = 0$ *then* G_r *is exact, i.e.,* $G_r = dG_{r-1}$ *for some form* $G_{r-1} \in \sec \bigwedge^r T^*M$.

Note also, that when M is compact the spaces $H_r(M)$ and $H^r(M)$ are finite dimensional and $\dim H^r(M) = b^p$. Thus de Rham theorem justifies writing

$$H^r(M) = (H_r(M))^*, \tag{4.86}$$

and the nomenclature: *homology* and *cohomology* groups for $H_r(M)$ and $H^r(M)$.

4.4 Differential Geometry in the Hodge Bundle

4.4.1 Riemannian and Lorentzian Structures on M

Next we introduce on M a smooth metric field $\boldsymbol{g} \in \sec T_2^0 M$ and gives the

Definition 4.73 A pair $(M, \boldsymbol{g})$, $\dim M = n$ is a n-dimensional Riemann structure (or Riemann manifold) if $\boldsymbol{g} \in \sec T_2^0 M$ is a smooth *metric* of signature $(n, 0)$. If $\boldsymbol{g}$ has signature (p, q) with $p + q = n$, $p \neq n$ or $q \neq n$ then the pair $(M, \boldsymbol{g})$ is called a pseudo Riemannian manifold. When $\boldsymbol{g}$ has signature $(1, n-1)$ the pair $(M, \boldsymbol{g})$ is called an hyperbolic manifold. When $\dim M = 4$ and $\boldsymbol{g}$ has signature $(1, 3)$ the pair $(M, \boldsymbol{g})$ is called a Lorentzian manifold.[11]

We already defined the concept of oriented manifold. Thus, we say that a Riemannian (or pseudo Riemannian or Lorentzian) manifold is orientable if and only if it admits a continuous metric volume element field $\tau_{\boldsymbol{g}} \in \sec \bigwedge^n T^*M$ given in local coordinate functions $\{\boldsymbol{x}^i\}$ covering $U \subset M$ by

$$\tau_{\boldsymbol{g}} = \sqrt{|\det \boldsymbol{g}|}\, dx^1 \wedge \ldots \wedge dx^n, \tag{4.87}$$

[11]When Lorentzian manifolds serve as models of spacetimes it is also imposed that M is noncompact. See Sect. 4.7.1.

where

$$\det \boldsymbol{g} = \det \left[\boldsymbol{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \right]. \tag{4.88}$$

Proposition 4.74 *Any C^r manifold M, $\dim M = n$ admits a C^{r-1} Riemannian metric $\boldsymbol{g}$ (signature $(n, 0)$) if and only if it is paracompact.*

Proof For a proof see, e.g., [3].■

Let us consider now a smooth oriented metric manifold $\mathrm{M} = (M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$, $\dim M = n$, where $\boldsymbol{g}$ is a smooth metric field of signature (p, q) and $\tau_{\boldsymbol{g}} \in \sec \bigwedge^n T^*M$. We denote by $\mathtt{g} \in \sec T_0^2 M$ the *metric* tensor of the cotangent bundle. Also we denote the scalar product induced on $\bigwedge T^*M$ by the metric tensor $\mathtt{g} \in \sec T_0^2 M$ by[12] $\underset{\boldsymbol{g}}{\cdot}$: $\sec \bigwedge T^*M \times \sec \bigwedge T^*M \to \sec \bigwedge^0 T^*M$. If $A, B \in \sec \bigwedge^p T^*M$ we have (recall Eq. (2.123))

$$(A \underset{\boldsymbol{g}}{\cdot} B)\tau_{\boldsymbol{g}} = A \wedge \underset{\boldsymbol{g}}{\star} B \tag{4.89}$$

4.4.2 Hodge Bundle

Definition 4.75 The *Hodge bundle* of the structure M is the triple

$$\bigwedge(\mathrm{M}) = (\bigwedge T^*M, \underset{\boldsymbol{g}}{\cdot}, \tau_{\boldsymbol{g}}). \tag{4.90}$$

The importance of the Hodge bundle is that besides the exterior derivative operator, we can now introduce a new differential operator called the Hodge codifferential. Equipped with these two operators we can write, e.g., Maxwell equations (with currents) in a *diffeomorphism* invariant way[13] (see Sect. 4.9.1). This is a very important fact, which is often not well known as it should be.

[12]When there is no chance of confusion we eventually used the symbol $\cdot$ instead of the symbol $\underset{\boldsymbol{g}}{\cdot}$ in order to simplify the notation.

[13]For the exact meaning of the concept of diffeomorphism invariance of a spacetime physical theory (as used in this text) see Sect. 6.6.3.

Definition 4.76 The *Hodge codifferential* operator in the Hodge bundle of $\bigwedge(M)$ is the mapping $\underset{g}{\delta} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$, given, for homogeneous multiforms, by:

$$\underset{g}{\delta} = (-1)^r \underset{g}{\star}^{-1} d \underset{g}{\star}, \tag{4.91}$$

where $\underset{g}{\star}$ is the Hodge star operator associated to the scalar product $\underset{g}{\cdot}$.

Definition 4.77 The *Hodge Laplacian* operator is the mapping

$$\underset{g}{\diamond} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$$

given by:

$$\underset{g}{\diamond} = -(d\underset{g}{\delta} + \underset{g}{\delta} d). \tag{4.92}$$

The exterior derivative, the Hodge codifferential and the Hodge Laplacian satisfy the relations:

$$\begin{aligned} dd &= \underset{g}{\delta}\underset{g}{\delta} = 0; \quad \underset{g}{\diamond} = (d - \underset{g}{\delta})^2; \\ d\underset{g}{\diamond} &= \underset{g}{\diamond} d; \quad \underset{g}{\delta}\underset{g}{\diamond} = \underset{g}{\diamond}\underset{g}{\delta}; \\ \underset{g}{\delta}\underset{g}{\star} &= (-1)^{r+1} \underset{g}{\star} d; \quad \underset{g}{\star}\underset{g}{\delta} = (-1)^r d \underset{g}{\star}; \\ d\underset{g}{\delta}\underset{g}{\star} &= \underset{g}{\star}\underset{g}{\delta} d; \quad \underset{g}{\star} d \underset{g}{\delta} = \underset{g}{\delta} d \underset{g}{\star}; \quad \underset{g}{\star}\underset{g}{\diamond} = \underset{g}{\diamond}\underset{g}{\star}. \end{aligned} \tag{4.93}$$

Remark 4.78 When it is clear from the context which metric field is involved we use the symbols $\star$, δ and $\diamond$ in place of the symbols $\underset{g}{\star}$, $\underset{g}{\delta}$ and $\underset{g}{\diamond}$ in order to simplify the writing of equations.

4.4.3 The Global Inner Product of p-Forms

Definition 4.79 Let $A, B \in \sec \bigwedge^p T^*M$ and suppose that the support of A or B is *compact*. The global inner product of these p-forms is

$$\langle A, B \rangle = \int_M A \wedge \star B. \tag{4.94}$$

Definition 4.80 Let $T : \sec \bigwedge^p T^*M \to \sec \bigwedge^q T^*M$ be a (p, q) extensor field acting on the sections of $\bigwedge^p T^*M$ of compact support. We define the metric transpose of T as the the (q, p) extensor field T^t such that

$$\langle TA, B\rangle = \langle A, T^t B\rangle \tag{4.95}$$

Exercise 4.81 Show that d and δ are metric transposes of each other i.e.,

$$\begin{aligned} \langle dA, B\rangle &= \langle A, \delta B\rangle, \\ \langle \delta A, B\rangle &= \langle A, dB\rangle \end{aligned} \tag{4.96}$$

Are the formulas given in Eq. (4.96) true for a compact manifold with boundary?

4.5 Pullbacks and the Differential

Proposition 4.82 *Let $\phi^* : M \to N$ be a differentiable mapping and let* h* *be the pullback mapping. Let $A, B \in \sec \bigwedge T^*M$. Then*

$$\phi^*(A \wedge B) = \phi^* A \wedge \phi^* B. \tag{4.97}$$

Proof It is a simple algebraic manipulation.■

Proposition 4.83 *Let $\phi : M \to N$ be a differentiable mapping and let ϕ^* be the pullback mapping. Let $A \in \sec \bigwedge T^*M$. Then,*

$$\phi^* dA = d(\phi^* A) \tag{4.98}$$

Proof Since an arbitrary form is a finite sum of exterior products of functions and differential of functions, we see that it is only necessary to prove the theorem for a 0-form and an exact 1-form α. The first case is true because,

$$\begin{aligned} \phi^* dg &= d(g \circ \phi) \\ &= d(\phi^* g) \end{aligned} \tag{4.99}$$

where we used the definition of reciprocal image. Now, if $\alpha = dg$, i.e., α is exact, we have

$$\phi^* d\alpha = \phi^* ddg = 0.$$

Also,

$$d(\phi^*\alpha) = d(\phi^*dg) = d[d(\phi^*g)] = d^2\phi^*g = 0, \tag{4.100}$$

and the proposition is proved.■

Proposition 4.83 is also very much important in proving the invariance of some exterior differential system of equations under diffeomorphisms.

4.6 Structure Equations I

Let us now endow the metric manifold $(M, \boldsymbol{g})$, with an arbitrary linear connection ∇ obtaining the structure $(M, \boldsymbol{g}, \nabla)$.

Definition 4.84 The *torsion and curvature operations* and the torsion and *curvature* tensors of a connection ∇, are respectively the mappings[14]:

$$\begin{aligned} \boldsymbol{\tau} &: \sec(TM \times TM) \to \sec TM, \\ \boldsymbol{\rho} &: \sec(TM \times TM) \to \mathrm{End}TM \end{aligned}$$

$$\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \tag{4.101}$$

$$\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]} \tag{4.102}$$

and

$$\Theta(\alpha, \boldsymbol{u},\boldsymbol{v}) = \alpha\left(\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v})\right), \tag{4.103}$$

$$\mathbf{R}(\alpha, \boldsymbol{w}, \boldsymbol{u},\boldsymbol{v}) = \alpha(\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}), \tag{4.104}$$

for every $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$.

Exercise 4.85 Show that for any differentiable functions f, g and h we have

$$\begin{aligned} \boldsymbol{\tau}(g\boldsymbol{u},h\boldsymbol{v}) &= gh\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}), \\ \boldsymbol{\rho}(g\boldsymbol{u},h\boldsymbol{v})f\boldsymbol{w} &= ghf\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v}). \end{aligned} \tag{4.105}$$

[14]EndTM means the set of endomorphisms $TM \to TM$.

Given an arbitrary moving frame $\{\boldsymbol{e}_\alpha\}$ on *TM*, let $\{\theta^\rho\}$ be the *dual frame of* $\{\boldsymbol{e}_\alpha\}$ (i.e., $\theta^\rho(\boldsymbol{e}_\alpha) = \delta^\rho_\alpha$). We write:

$$\begin{aligned} [\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta] &= c^{\rho\cdot\cdot}_{\cdot\alpha\beta}\boldsymbol{e}_\rho, \\ \nabla_{\boldsymbol{e}_\alpha}\boldsymbol{e}_\beta &= L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\boldsymbol{e}_\rho, \end{aligned} \tag{4.106}$$

where $c^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the *structure coefficients* of the frame $\{\boldsymbol{e}_\alpha\}$ and $L^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the *connection coefficients* in this frame. Then, the components of the torsion and curvature tensors are given, respectively, by:

$$\begin{aligned} [c]rclT^{\rho\cdot\cdot}_{\cdot\alpha\beta} &:= \Theta(\theta^\rho, e_\alpha,\boldsymbol{e}_\beta) = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} - L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - c^{\rho\cdot\cdot}_{\cdot\alpha\beta}, \\ R^{\rho\cdot\cdot\cdot}_{\cdot\mu\alpha\beta} &:= \mathbf{R}(\theta^\rho, \boldsymbol{e}_\mu, \boldsymbol{e}_\alpha,\boldsymbol{e}_\beta) \\ &= \boldsymbol{e}_\alpha(L^{\rho\cdot\cdot}_{\cdot\beta\mu}) - \boldsymbol{e}_\beta(L^{\rho\cdot\cdot}_{\cdot\alpha\mu}) + L^{\rho\cdot\cdot}_{\cdot\alpha\sigma}L^{\sigma\cdot\cdot}_{\cdot\beta\mu} - L^{\rho\cdot\cdot}_{\beta\sigma}L^{\sigma\cdot\cdot}_{\alpha\mu} - c^{\sigma\cdot\cdot}_{\cdot\alpha\beta}L^{\rho\cdot\cdot}_{\cdot\sigma\mu}. \end{aligned} \tag{4.107}$$

We also have:

$$\begin{aligned} d\theta^\rho &= -\tfrac{1}{2}c^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha \wedge \theta^\beta, \\ \nabla_{e_\alpha}\theta^\rho &= -L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\beta, \end{aligned} \tag{4.108}$$

where $\omega^{\rho\cdot}_{\cdot\beta} \in \sec \bigwedge^1 T^*M$ are the *connection 1-forms,* $\Theta^\rho \in \sec \bigwedge^2 T^*M$ are the *torsion 2-forms* and $\mathcal{R}^{\rho\cdot}_{\cdot\beta} \in \sec \bigwedge^2 T^*M$are the *curvature 2-forms*, given by:

$$\begin{aligned} \omega^{\rho\cdot}_{\cdot\beta} &:= L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha, \\ \Theta^\rho &:= \frac{1}{2}T^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha \wedge \theta^\beta, \\ \mathcal{R}^{\rho\cdot}_{\cdot\mu} &:= \frac{1}{2}R^{\rho\cdot\cdot\cdot}_{\cdot\mu\alpha\beta}\theta^\alpha \wedge \theta^\beta. \end{aligned} \tag{4.109}$$

Multiplying Eqs. (4.107) by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ and using Eqs. (4.108) and (4.109), we get the *Cartan's structure equations*:

$$\begin{aligned} d\theta^\rho + \omega^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta &= \Theta^\rho, \\ d\omega^{\rho\cdot}_{\cdot\mu} + \omega^{\rho\cdot}_{\cdot\beta} \wedge \omega^{\beta\cdot}_{\cdot\mu} &= \mathcal{R}^{\rho\cdot}_{\cdot\mu}. \end{aligned} \tag{4.110}$$

Exercise 4.86 Show that the torsion tensor can be written as

$$\Theta = \boldsymbol{e}_\alpha \otimes \Theta^\alpha \tag{4.111}$$

Exercise 4.87 Put $\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \theta^{\mathbf{a}_1} \wedge \cdots \wedge \theta^{\mathbf{a}_r}$ and $\underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \underset{g}{\star}(\theta^{\mathbf{a}_1} \wedge \cdots \wedge \theta^{\mathbf{a}_r})$. Show that when $\Theta^{\mathbf{a}} = 0$ we have

$$d\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega_{\cdot\mathbf{b}}^{\mathbf{a}_1\cdot} \wedge \theta^{\mathbf{b}\ldots\mathbf{a}_r} - \cdots - \omega_{\cdot\mathbf{b}}^{\mathbf{a}_r\cdot} \wedge \theta^{\mathbf{a}_1\ldots\mathbf{b}}, \tag{4.112}$$

$$d\underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega_{\cdot\mathbf{b}}^{\mathbf{a}_1\cdot} \wedge \underset{g}{\star}\theta^{\mathbf{b}\ldots\mathbf{a}_r} - \cdots - \omega_{\cdot\mathbf{b}}^{\mathbf{a}_r\cdot} \wedge \underset{g}{\star}\theta^{\mathbf{a}_1\ldots\mathbf{b}}. \tag{4.113}$$

4.6.1 *Exterior Covariant Differential of* $(p+q)$*-Indexed r-Form Fields*

Definition 4.88 Suppose that $X \in \sec T_p^{r+q}M$ and let

$$X_{\nu_1\ldots.\nu_q}^{\mu_1\ldots.\mu_p}(\boldsymbol{v}_1\ldots,\boldsymbol{v}_r) \in \sec \bigwedge\nolimits^r T^*M, \tag{4.114}$$

such that

$$X_{\nu_1\ldots.\nu_q}^{\mu_1\ldots.\mu_p}(\boldsymbol{v}_1\ldots,\boldsymbol{v}_r) = X(\boldsymbol{v}_1\ldots,\boldsymbol{v}_r,\boldsymbol{e}_{\nu_1},\ldots.\boldsymbol{e}_{\nu_q},\theta^{\mu_1},\ldots,\theta^{\mu_p}). \tag{4.115}$$

for $\boldsymbol{v}_1\ldots,\boldsymbol{v}_r \in \sec TM$. The $X_{\nu_1\ldots.\nu_q}^{\mu_1\ldots.\mu_p}$ are called $(p+q)$-indexed r-forms.

Definition 4.89 The exterior covariant differential[15] **D** of $X_{\nu_1\ldots.\nu_q}^{\mu_1\ldots.\mu_p}$ on a manifold with a general connection ∇ is the mapping:

$$\mathbf{D} : \sec \bigwedge\nolimits^r T^*M \to \sec \bigwedge\nolimits^{r+1} T^*M, 0 \le r \le 4, \tag{4.116}$$

such that[16]

$$\begin{aligned}
&(r+1)\mathbf{D}X_{\nu_1\ldots.\nu_q}^{\mu_1\ldots.\mu_p}(\boldsymbol{v}_0,\boldsymbol{v}_1\ldots,\boldsymbol{v}_r) \\
&\quad= \sum_{\nu=0}^{r}(-1)^{\nu}\nabla_{\boldsymbol{e}_\nu}X(\boldsymbol{v}_0,\boldsymbol{v}_1\ldots,\check{\boldsymbol{v}}_\nu,\ldots\boldsymbol{v}_r,\boldsymbol{e}_{\nu_1},\ldots.\,\boldsymbol{e}_{\nu_q},\theta^{\mu_1},\ldots,\theta^{\mu_p}) \\
&\qquad- \sum_{0\le\nu,\varsigma\le r}(-1)^{\nu+\varsigma}X(\boldsymbol{\tau}(\boldsymbol{v}_\nu,\boldsymbol{v}_\varsigma),\boldsymbol{v}_0,\boldsymbol{v}_1\ldots,\check{\boldsymbol{v}}_\nu,\ldots,\boldsymbol{v}_\varsigma,\ldots,\boldsymbol{e}_r, \\
&\qquad\quad \boldsymbol{e}_{\nu_1},\ldots.\boldsymbol{e}_{\nu_q},\theta^{\mu_1},\ldots,\theta^{\mu_p}).
\end{aligned} \tag{4.117}$$

[15] Sometimes also called exterior covariant derivative.

[16] As usual the inverted hat over a symbol (in Eq. (4.117)) means that the corresponding symbol is missing in the expression.

Then, we may verify that

$$\begin{aligned}\mathbf{D}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} &= dX^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_s\ldots\mu_p}_{\nu_1\ldots\nu_q} + \cdots + \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \\ &\quad - \omega^{\nu_s}_{\nu_1} \wedge X^{\mu_1\ldots\mu_p}_{\nu_s\ldots\nu_q} - \cdots - \omega^{\mu_1}_{\mu_s} \wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_s}.\end{aligned} \tag{4.118}$$

Remark 4.90 Sometimes, Eqs. (4.110) are written by some authors [45] as:

$$\begin{aligned}\mathbf{D}\theta^\rho &:= \Theta^\rho, \\ \text{“}\mathbf{D}\omega^{\rho\cdot}_{\cdot\mu} &:= \mathcal{R}^{\rho\cdot}_{\cdot\mu}.\text{”}\end{aligned} \tag{4.119}$$

and $\mathbf{D} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$ is said to be the exterior covariant derivative related to the connection ∇. Whereas the equation $\mathbf{D}\theta^\rho := \Theta^\rho$ is well defined, we see that the equation "$\mathbf{D}\omega^{\rho\cdot}_{\cdot\mu} := \mathcal{R}^{\rho\cdot}_{\cdot\mu}$." is an equivocated one. Indeed if Eq. (4.118) is applied on the connection 1-forms $\omega^{\mu\cdot}_{\cdot\nu}$ we would get $\mathbf{D}\omega^{\mu\cdot}_{\cdot\nu} = d\omega^{\mu\cdot}_{\cdot\nu} + \omega^{\mu\cdot}_{\cdot\alpha} \wedge \omega^{\alpha\cdot}_{\cdot\nu} - \omega^{\alpha\cdot}_{\cdot\nu} \wedge \omega^{\mu\cdot}_{\cdot\alpha}$. So, we see that the symbol $\mathbf{D}\omega^{\mu\cdot}_{\cdot\nu}$ given by the second formula in Eq. (4.119), supposedly defining the curvature 2-forms is to be avoided. The reason for the failure of Eq. (4.118) in that case is that there do not exist a tensor field $\omega \in \sec T^2_1 M$ which satisfy the corresponding Eq. (4.115). More details on this issue may be found in Appendix A.3.

Exercise 4.91 Show that if $X^J \in \sec \bigwedge^r T^*M$ and $Y^K \in \sec \bigwedge^s T^*M$ are sets of indexed forms,[17] then

$$\mathbf{D}(X^J \wedge Y^K) = \mathbf{D}X^J \wedge Y^K + (-1)^{rs} X^J \wedge \mathbf{D}Y^K. \tag{4.120}$$

Exercise 4.92 Show that if $X^{\mu_1\ldots\mu_p} \in \sec \bigwedge^r T^*M$ then

$$\mathbf{D}\mathbf{D}X^{\mu_1\ldots\mu_p} = dX^{\mu_1\ldots\mu_p} + \mathcal{R}^{\mu_1\cdot}_{\cdot\mu_s} \wedge X^{\mu_s\ldots\mu_p} + \cdots \mathcal{R}^{\mu_p\cdot}_{\cdot\mu_s} \wedge X^{\mu_1\ldots\mu_s}. \tag{4.121}$$

Exercise 4.93 Show that for any metric-compatible connection ∇ if $\boldsymbol{g} = g_{\mu\nu}\theta^\mu \otimes \theta^\nu$ then,

$$\mathbf{D}g_{\mu\nu} = 0. \tag{4.122}$$

Since we are dealing with a metric manifold, we must complete Cartan's structure equations with the equations stating the relation between the connection and the metric. For this, following the usual nomenclature [1, 40, 47] we give the

[17] Multi indices are here represented by J and K.

Definition 4.94 The *nonmetricity* tensor field of the structures $(M, \boldsymbol{g}, \nabla)$ is the tensor field $\mathbf{Q} \in \sec T_3^0 M$ with components[18] in the basis $\{\theta^\alpha\}$ given by

$$Q_{\mu\alpha\beta} := -\nabla_\mu g_{\alpha\beta} = -\boldsymbol{e}_\mu(g_{\beta\alpha}) + g_{\sigma\alpha} L^{\sigma\cdot\cdot}_{\cdot\mu\beta} + g_{\beta\sigma} L^{\sigma\cdot\cdot}_{\cdot\mu\alpha}. \tag{4.123}$$

Correspondingly, we introduce the *nonmetricity 2-forms*, by:

$$\mathbf{Q}^\rho := \frac{1}{2} Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]} \theta^\alpha \wedge \theta^\beta, \tag{4.124}$$

where $Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]} = g^{\rho\mu}(Q_{\alpha\beta\mu} - Q_{\beta\alpha\mu})$. Multiplying Eq. (4.123) by $\theta^\alpha \wedge \theta^\beta$ and using Eq. (4.110a), we get:

$$\mathbf{D}\theta_\mu \equiv d\theta_\mu - \omega^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta = \boldsymbol{\Phi}_\mu, \tag{4.125}$$

where $\{\theta_\mu\}$ is the reciprocal frame of $\{\theta^\nu\}$ is the (i.e., $\theta_\mu = g_{\mu\nu}\theta^\nu$) and

$$\boldsymbol{\Phi}_\mu = \Theta_\mu - \mathbf{Q}_\mu.$$

Equation (4.125) can be used as the complement of Cartan's structure equations for the case of a *metric* manifold.

4.6.2 Bianchi Identities

Differentiating Eq. (4.110) and Eq. (4.125) we obtain the *Bianchi identities*[19]:

$$\begin{aligned}
&\text{(a)} \quad \mathbf{D}\Theta^\rho = d\Theta^\rho + \omega^{\rho\cdot}_{\cdot\beta} \wedge \Theta^\beta = \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge \theta^\beta, \\
&\text{(b)} \quad \mathbf{D}\mathcal{R}^{\rho\cdot}_{\cdot\mu} = d\mathcal{R}^{\rho\cdot}_{\cdot\mu} - \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge \omega^{\beta\cdot}_{\upsilon\mu} + \omega^{\rho\cdot}_{\cdot\beta} \wedge \mathcal{R}^{\beta\cdot}_{\cdot\mu} = 0, \\
&\text{(c)} \quad \mathbf{D}\Phi_\mu = d\boldsymbol{\Phi}_\mu - \omega^{\beta\cdot}_{\cdot\mu} \wedge \boldsymbol{\Phi}_\beta = -\mathcal{R}^{\beta\cdot}_{\cdot\mu} \wedge \theta_\beta.
\end{aligned} \tag{4.126}$$

4.6.3 Induced Connections Under Diffeomorphisms

Let M and N be two differentiable manifolds, $\dim M = m$, $\dim N = n$.

[18] We use the notation $\nabla_\sigma t^{\mu\ldots}_{\nu\ldots} \equiv (\nabla_{e_\sigma} t)^{\mu\ldots}_{\nu\ldots} \equiv (\nabla t)^{\mu\ldots}_{\sigma\nu\ldots}$ for the components of the covariant derivative of a tensor field t. This is not to be confused with $\nabla_{e_\sigma} t^{\mu\ldots}_{\nu\ldots} \equiv e_\sigma(t^{\mu\ldots}_{\nu\ldots})$, the derivative of the components of t in the direction of e_σ.

[19] To our knowledge, Eqs. (4.125) and (4.126c) are not found anywhere in the literature, although they appear to be the most natural extension of the structure equations for metric manifolds.

Definition 4.95 Let ∇ be a connection on N and $\mathbf{X}, \mathbf{Y} \in \sec TN$ and $\mathbf{T} \in \sec T_s^r N$, $f : N \to \mathbb{R}$ and $\mathtt{h} : M \to N$ a diffeomorphism. The induced connection $\mathtt{h}^*\nabla$ on M is defined by

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{T} = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{T}). \tag{4.127}$$

Example 4.96 Let $f : N \to \mathbb{R}$ and $\mathbf{Y} \in \sec TN$. Then,

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{Y} = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{Y}),$$

from where it follows (taking into account that for any vector field $\mathbf{V} \in \sec TN$, $\mathtt{h}^*\mathbf{N} = \mathtt{h}_*^{-1}\mathbf{N}$) that

$$\mathtt{h}^*\nabla_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\mathbf{Y}\Big|_{\mathfrak{e}} f \circ h = \mathtt{h}^*(\nabla_{\mathbf{X}}\mathbf{Y})|_{\mathfrak{e}} f \circ h = \nabla_{\mathbf{X}}\mathbf{Y}|_{\mathtt{h}(\mathfrak{e})} f, \ \forall \mathfrak{e} \in M.$$

Remark 4.97 Now, suppose that $M = N$ and $\mathtt{h}$: $M \to M$ a diffeomorphism. Suppose that D is the Levi-Civita connection of $\boldsymbol{g}$, then $\mathtt{h}^*D = D'$ is the Levi-Civita connection of $\mathtt{h}^*\boldsymbol{g} = \boldsymbol{g}'$ since using Eq. (4.127) we infer that

$$\mathtt{h}^*D_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\boldsymbol{g}\Big|_{\mathfrak{e}} = D'_{\mathtt{h}_*^{-1}\mathbf{X}}\mathtt{h}^*\boldsymbol{g}\Big|_{\mathfrak{e}} = \mathtt{h}^*(D_{\mathbf{X}}\,\boldsymbol{g})|_{\mathfrak{e}}\,, \forall \mathfrak{e} \in M. \tag{4.128}$$

Taking into account that[20] $\mathtt{h}^*[\mathbf{X}, \mathbf{Y}] = [\mathtt{h}^*\mathbf{X}, \mathtt{h}^*\mathbf{Y}]$ we have for $\mathbf{X}, \mathbf{Y} \in \sec TM$,

$$\mathtt{h}^*(D_{\mathbf{X}}Y - D_{\mathbf{Y}}X - [\mathbf{X}, \mathbf{Y}]) = 0. \tag{4.129}$$

Remark 4.98 Equation (4.127) applied to the case $M = N$ also implies, as the reader may verify the important fact that the curvature tensor of $\mathtt{h}^*D$ will be null if the curvature tensor of D is null.

4.7 Classification of Geometries on M and Spacetimes

Definition 4.99 Given a triple $(M, \boldsymbol{g}, \nabla)$:

(a) it is called a Riemann-*Cartan geometry*[21] *if and only if*

$$\nabla\boldsymbol{g} = 0 \qquad \text{and} \qquad \Theta[\nabla] \neq 0. \tag{4.130}$$

[20]See, e.g., [3, p. 135].

[21]Or Riemann space.

(b) it is called *Weyl geometry if and only if*

$$\nabla \boldsymbol{g} \neq 0 \qquad \text{and} \qquad \Theta[\nabla] = 0. \tag{4.131}$$

(c) it is called a *Riemann geometry* if and only if

$$\nabla \boldsymbol{g} = 0 \qquad \text{and} \qquad \Theta[\nabla] = 0, \tag{4.132}$$

and in that case the pair $(\nabla, \boldsymbol{g})$ is called *Riemannian structure*.

(d) it is called Riemann-*Cartan-Weyl geometry* if and only if

$$\nabla \boldsymbol{g} \neq 0 \qquad \text{and} \qquad \Theta[\nabla] \neq 0. \tag{4.133}$$

(e) it is called a (Riemann) flat geometry if and only if

$$\nabla \boldsymbol{g} = 0 \qquad \text{and} \qquad \mathbf{R}[\nabla] = 0,$$

(f) it is called teleparallel geometry if and only if

$$\nabla \boldsymbol{g} = 0, \ \Theta[\nabla] \neq 0 \text{ and } \mathbf{R}[\nabla] = 0. \tag{4.134}$$

For each metric tensor defined on the manifold M there exists one and only one connection in the conditions of Eq. (4.132). It is called *Levi-Civita connection* of the metric considered, and is denoted by D. If in a given context it is necessary to distinguish between the Levi-Civita connections of two different metric tensors $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ on the same manifold, we write $\mathring{D}$, D.

Remark 4.100 When $\dim M = 4$ and the metric $\boldsymbol{g}$ has signature $(1, 3)$ we sometimes substitute the word Riemann by the word Lorentzian in the previous definitions.

4.7.1 Spacetimes

From nowhere besides the constraints already imposed (Hausdorff and paracompact) on M, we suppose also that it is connected and noncompact [14, 38]. We now introduce the concept of *time orientability* on an oriented Lorentzian manifold structure $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$, which plays a key role in physical theories.

Definition 4.101 Let $(M, \boldsymbol{g})$ be a Lorentzian manifold, $TM = \bigcup_{e \in M} T_e M$ its tangent bundle and $\pi : TM \to M$ the canonical projection (see Appendix). The causal character of $(e, \mathbf{v}) \in TM$ is the causal character of $\mathbf{v}$ (Definition 2.62).

Definition 4.102 A line element at $x \in M$ is a one-dimensional subspace of $T_x M$.

Proposition 4.103 *Let M be a C^1paracompact and Hausdorff manifold,* $\dim M = 4$. *Then the existence of a continuous line element field on M is equivalent to the existence of a Lorentzian structure on M.*

Proof For a proof see [3].■

Proposition 4.104 *The set $\mathfrak{T} \subset TM$ of timelike points is an open manifold and it has either one (connected) component or two.*

Proof A proof of this important result can be found in [38].■

Definition 4.105 A connected Lorentzian manifold $(M, \boldsymbol{g})$ is said to be time orientable if and only if $\mathfrak{T}$ has two components and one of the components is labeled the future $\mathfrak{T}^+$ and the other component $\mathfrak{T}^-$ is labelled the past. We denote by $\uparrow$ the time orientability of a Lorentzian manifold.

Definition 4.106 A spacetime is a pentuple $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ where $(M, \boldsymbol{g})$ is a Lorentzian oriented and time oriented manifold and ∇ is an arbitrary covariant derivative operator on M.

Definition 4.107 When $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ is a spacetime and $\nabla = D$ is the Levi-Civita connection of $\boldsymbol{g}$ the spacetime is said to be Lorentzian. When $\nabla \boldsymbol{g} = 0$ and $\Theta(\nabla) \neq 0$ we call the structure $\mathfrak{M} = (M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ a Riemann-Cartan spacetime. The particular Riemann-Cartan spacetime for which $\mathbf{R}(D) = 0$, $\Theta[\nabla] \neq 0$ is called a teleparallel spacetime (also called Weintzenböck spacetime according to [26]).

Definition 4.108 A Lorentzian spacetime structure $\mathcal{M} = (M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ is said to be Minkowski spacetime if and only if $M \simeq \mathbb{R}^4$ and $\mathbf{R}(D) = 0$.

Remark 4.109 We just establish that any Lorentzian manifold admits a continuous element field. If it is also time orientable, we can choose a direction for the continuous element field, and say that it is a *timelike* vector field pointing to the future. This is a nontrivial result and very important for our discussion of the Principle of Relativity (Chap. 6).

4.8 Differential Geometry in the Clifford Bundle

It is well known [28] that the natural operations on metric vector spaces, such as, e.g., direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metric tensors. Then we give the

Definition 4.110 The *Clifford bundleof differential forms* of the metric manifold $(M, \boldsymbol{g})$ is:

$$\mathcal{C}\ell(M, \mathtt{g}) = \frac{\mathcal{T}M}{J_{\mathtt{g}}} = \bigcup_{x \in M} \mathcal{C}\ell(T_x^* M, \mathtt{g}_x), \tag{4.135}$$

where $\mathcal{T}M$ denotes the (covariant) tensor bundle of M, $J_{\mathtt{g}} \subset \mathcal{T}M$ is the bilateral ideal of $\mathcal{T}M$ generated by the elements of the form $\alpha \otimes \beta + \beta \otimes \alpha - 2\mathtt{g}(\alpha, \beta)$, with $\alpha, \beta \in \sec T^*M \subset \mathcal{T}M$ and $\mathcal{C}\ell(T_x^*M, \mathtt{g}_x)$ is the Clifford algebra of the metric vector space structure $(T_x^*M, \mathtt{g}_x)$.

It will be shown in Chap. 7 that the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ (as defined by Eq. (4.135)) is a vector bundle associated to the principal bundle of orthonormal frames $\mathbf{P}_{\mathrm{SO}^e p,q}$, i.e.,

$$\mathcal{C}\ell(M, \mathtt{g}) = \mathbf{P}_{\mathrm{SO}^e p,q} \times_{Ad} \mathbb{R}_{p,q}. \tag{4.136}$$

In Eq. (4.136) Ad is the *adjoint representation* of $\mathrm{Spin}^e_{p,q}$, i.e., $Ad : \mathrm{SO}^e_{p,q} \to \mathrm{Aut}(\mathbb{R}_{p,q})$, $u \mapsto Ad_u$, with $Ad_u A = \mathrm{A}\, u^{-1}$, $\forall u \in \mathrm{SO}^e_{p,q}$, $\forall A \in \mathbb{R}_{p,q} \simeq \mathcal{C}\ell(T_x^*M, \mathtt{g}_x)$. Details on these groups may be found in Chap. 3. In Chap. 7 we scrutinize the vector bundle structure of the Clifford bundle of differential forms over a general Riemann-Cartan manifold modelling spacetime.

4.8.1 Clifford Fields as Sums of Nonhomogeneous Differential Forms

Definition 4.111 Sections of $\mathcal{C}\ell(M, \mathtt{g})$ are called Clifford fields.

We recall some notations and conventions. By $F(U)$ we denote the frame bundle (see Appendix A.3) of $U \subset M$. A section of $F(U)$ will be denoted by $\{\boldsymbol{e}_\alpha\} \in \sec F(U)$. The dual frame of a frame $\{\boldsymbol{e}_\alpha\}$ will be denoted by $\{\theta^\alpha\}$, where $\theta^\alpha \in \sec T^*U \subset T^*M$. When $\{\boldsymbol{e}_\alpha\}$ is a coordinate frame associated to the coordinate functions $\{\mathbf{x}^\mu\}$ of a local chart covering U we use instead of $\boldsymbol{e}_\alpha$ the notation $e_\alpha = \partial_\alpha$ and in this case $\theta^\alpha = dx^\alpha$. When $\{\boldsymbol{e}_\alpha\}$ refers to an orthonormal frame we use instead of $\boldsymbol{e}_\alpha$ the notation $\boldsymbol{e}_\mathbf{a}$ and instead of θ^α the notation $\theta^\mathbf{a}$.

Recall that as a vector space over $\mathbb{R}$, $\mathcal{C}\ell(T_x^*M, \mathbf{g}_x)$ is isomorphic to the exterior algebra $\bigwedge T_x^*M$ of the cotangent space and

$$\bigwedge T_x^*M = \bigoplus\nolimits_{k=0}^{n} \bigwedge{}^k T_x^*M, \tag{4.137}$$

where $\bigwedge^k T_x^*M$ is the $\binom{n}{k}$-dimensional space of k-forms. Then, there is a natural embedding [22] $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ [21] and sections of $\mathcal{C}\ell(M, \mathtt{g})$—Clifford fields (Definition 4.111)—can be represented as a sum of non homogeneous differential forms. Let $\{\boldsymbol{e}_\mathbf{a}\}$ be an orthonormal basis for $TU \subset TM$, i.e., $\boldsymbol{g}(\boldsymbol{e}_\mathbf{a}, \boldsymbol{e}_\mathbf{b}) = \eta_{\mathbf{ab}}$, where the matrix with entries $\eta_{\mathbf{ab}}$ is the diagonal matrix, $\mathrm{diag}(1, 1, \ldots.. -1, \ldots, -1)$ and $(\mathbf{a}, \mathbf{b}, \mathbf{i}, \mathbf{j}, \ldots = 1, 2, \ldots, n)$. Moreover, let $\{\theta^\mathbf{a}\} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

[22] Recall again that the symbol $A \hookrightarrow B$ means that A is embedded in B and $A \subseteq B$.

such that the set $\{\theta^{\mathbf{a}}\}$ is the dual basis of $\{\mathbf{e}_{\mathbf{a}}\}$. We denote by $\{\theta_{\mathbf{i}}\}$ be the *reciprocal basis* of $\{\theta^{\mathbf{i}}\}$, i.e., $\theta_{\mathbf{i}} \underset{g}{\cdot} \theta^{\mathbf{j}} = \delta_{\mathbf{i}}^{\mathbf{j}}$.

For the particular case of a 4-dimensional spacetime, of course, the range of the bold labels are $\mathbf{a}, \mathbf{b}, \mathbf{i}, .. = 0, 1, 2, 3$. Recall that the fundamental *Clifford product* is generated by

$$\theta^{\mathbf{i}}\theta^{\mathbf{j}} + \theta^{\mathbf{j}}\theta^{\mathbf{i}} = 2\eta^{\mathbf{ij}}. \tag{4.138}$$

If $\mathcal{C} \in \sec \mathcal{C\ell}(M, \mathtt{g})$ is a Clifford field, we have:

$$\mathcal{C} = s + v_{\mathbf{i}}\theta^{\mathbf{i}} + \frac{1}{2!}b_{\mathbf{ij}}\theta^{\mathbf{i}}\theta^{\mathbf{j}} + \frac{1}{3!}t_{\mathbf{ijk}}\theta^{\mathbf{i}}\theta^{\mathbf{j}}\theta^{\mathbf{k}} + p\theta^{\mathbf{5}} , \tag{4.139}$$

where $\theta^{\mathbf{5}} = \theta^{\mathbf{0}}\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{3}$ is the volume element and

$$s, v_{\mathbf{i}}, b_{\mathbf{ij}}, t_{\mathbf{ijk}}, p \in \sec \bigwedge^{0} T^{*}M \hookrightarrow \sec \mathcal{C\ell}(M, \mathtt{g}). \tag{4.140}$$

4.8.2 Pullbacks and Relation Between Hodge Star Operators

Let M be a n-dimensional manifold and $\mathring{\boldsymbol{g}}, \boldsymbol{g} \in \sec T_{2}^{0}M$ two metrics of the same signature with corresponding metrics (for the cotangent bundle) $\mathring{\mathtt{g}}, \mathtt{g} \in \sec T_{0}^{2}M$. Let $\mathring{g}$ and g be the extensor fields associated to $\mathring{\mathtt{g}}$ and $\mathtt{g}$. Let $\mathtt{h}$: $M \to M$ be a diffeomorphism such that

$$\boldsymbol{g} = \mathtt{h}^{*}\mathring{\boldsymbol{g}}. \tag{4.141}$$

From the algebraic results of Sect. 2.8 we easily infer that there exists a metric gauge extensor field h such that

$$g(a) \underset{\mathring{g}}{\cdot} b = h(a) \underset{\mathring{g}}{\cdot} h(b) \tag{4.142}$$

for any $a, b \in \sec \bigwedge^{1} T^{*}M$ and we write $g = h^{\dagger}h$. Then, as in the purely algebraic case discussed in Sect. 2.8 we can also show that we have the following relation between the Hodge star operators associated to $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$

$$\underset{g}{\star} = \underline{h}^{-1} \underset{\mathring{g}}{\star} \underline{h}. \tag{4.143}$$

Remark 4.112 In this case we say that the metric gauge extensor h is related to the pullback mapping $\mathtt{h}^{*}$ and describes an elastic distortion. However, keep in mind that in general given a h it does not implies the existence of $\mathtt{h}^*$ such that Eq. (4.141) holds. In this case h is said to generate a plastic distortion. More details in [9].

We now show the

Proposition 4.113 *Let* $\mathtt{h}: M \to M$ *a diffeomorphism. Let* $\mathring{\boldsymbol{g}}, \boldsymbol{g} \in \sec T_2^0 M$ *two metrics of the same signature. Then for any* $\omega \in \sec \bigwedge^p T^*M$ *we have*

$$\underset{\boldsymbol{g}}{\star}\, \mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star}\, \omega \tag{4.144}$$

Proof As in Remark 4.24 take two charts (U, φ) and (V, χ), U, $\mathtt{h}(U), \subset V$ with coordinate functions $\boldsymbol{x}^i$ and $\boldsymbol{y}^i$ such that and $\boldsymbol{x}^i(\mathfrak{e}) = \boldsymbol{y}^i(\mathtt{h}(\mathfrak{e}))$, i.e., calling $\boldsymbol{x}^i(\mathfrak{e}) = x^i$, $\boldsymbol{y}^i(\mathtt{h}(\mathfrak{e})) = y^i$ we have $\partial y^i/\partial x^j = \delta^i_j$, $dx^i = dy^i$. Let also $\mathring{\boldsymbol{g}}(\partial/\partial y^k, \partial/\partial y^k) = \mathring{g}_{kl}(y^j)$. Then it follows that $g_{kl}(x^i) = \mathring{g}_{kl}(y^j(x^i)) = \mathring{g}_{kl}(x^i)$ and $\det \boldsymbol{g} = \det \mathring{\boldsymbol{g}}$. Now, if $\omega = \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(x^i\right) dx^{i_1} \wedge \ldots \wedge dx^{i_p}$, we can write (taking into account that $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and also $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathring{\mathtt{g}})$)

$$\begin{aligned}
\underset{\boldsymbol{g}}{\star}\mathtt{h}^*\omega &= \widetilde{\mathtt{h}^*\omega} \underset{\boldsymbol{g}}{\lrcorner} \tau_{\boldsymbol{g}} = \widetilde{\mathtt{h}^*\omega}\tau_{\boldsymbol{g}} \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(y^i(x^j)\right)\widetilde{dx^{i_1} \wedge \ldots \wedge dx^{i_p}} \sqrt{|\det \boldsymbol{g}|} dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}(y^i(x^j))\widetilde{dy^{i_1} \wedge \ldots \wedge dy^{i_p}} \sqrt{|\det \mathring{\boldsymbol{g}}|} dy^1 \wedge \ldots \wedge dy^n \\
&= \frac{1}{p!}\omega_{i_1 \ldots i_p}\left(y^i\right) \widetilde{dy^{i_1} \wedge \ldots \wedge dy^{i_p}} \underset{\mathring{\boldsymbol{g}}}{\lrcorner} \sqrt{|\det \mathring{\boldsymbol{g}}|} dy^1 \wedge \ldots \wedge dy^n \\
&= \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star}\, \omega,
\end{aligned}$$

and the proposition is proved.■

Remark 4.114 When $\boldsymbol{g} = \mathtt{h}^*\mathring{\boldsymbol{g}}$, there exists an associated metric gauge extensor field h such satisfying Eq. (4.142), i.e., $g = h^\dagger h$. The relation $\underset{\boldsymbol{g}}{\star}\mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star}\, \omega$ and $\underset{\boldsymbol{g}}{\star} = \underline{h}^{-1} \underset{\mathring{\boldsymbol{g}}}{\star}\, \underline{h}$ permit us to write the suggestive *operator* identity

$$\underset{\mathring{\boldsymbol{g}}}{\star}\, \underline{h}\mathtt{h}^*\omega = \underline{h}\mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star}\, \omega. \tag{4.145}$$

Exercise 4.115 Consider any diffeomorphism $\mathtt{h}: M \to M$, and two metrics $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ such that $\boldsymbol{g} = \mathtt{h}^*\mathring{\boldsymbol{g}}$. Show that

$$\underset{\boldsymbol{g}}{\star}\, d \underset{\boldsymbol{g}}{\star}\, \mathtt{h}^*\omega = \mathtt{h}^* \underset{\mathring{\boldsymbol{g}}}{\star}\, d \underset{\mathring{\boldsymbol{g}}}{\star}\, \omega, \tag{4.146}$$

for any $\omega \in \bigwedge T^*M$.

Solution The first member of Eq. (4.146) can be writing successively using Eq. (4.144) as

$$\begin{aligned}\underset{g}{\star}d\underset{g}{\star}\,\mathtt{h}^{*}\omega &= \underset{g}{\star}d\mathtt{h}^{*}\underset{\mathring{g}}{\star}\,\omega\\ &= \underset{g}{\star}\mathtt{h}^{*}d\underset{\mathring{g}}{\star}\,\omega\\ &= \mathtt{h}^{*}\underset{\mathring{g}}{\star}\,d\underset{\mathring{g}}{\star}\,\omega.\end{aligned}$$

4.8.3 Dirac Operators

We now equip the Riemannian (pseudo Riemannian, or Lorentzian) manifold $(M, \mathring{\boldsymbol{g}})$ with a *standard* structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$, where $\mathring{D}$ is the Levi-Civita connection of $\mathring{\boldsymbol{g}}$.

We are going to introduce in the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ a differential operator $\boldsymbol{\partial}$, called the standard Dirac operator,[23] which is associated to the Levi-Civita connection of the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$ and we study the properties of that operator. Next we define new Dirac-like operators associated with a connection different from the Levi-Civita one, i.e., to connections ∇ defining a general Riemann-Cartan-Weyl geometry $(M, \mathring{\boldsymbol{g}}, \nabla)$. Moreover, making use of the results developed in Sect. 2.7, we show that it is possible to introduce infinitely many others Dirac-like operators, one for each bilinear form field defined on the manifold M of the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$. These constructions enable us to formulate the geometry of a Riemann-Cartan-Weyl space in the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Some interesting geometrical concepts, like the Dirac *commutator* and *anticommutator*, are introduced. Moreover, we show a new decomposition of a general linear connection, identifying some new relevant tensors which are important for a clear understanding of any formulation of the gravitational theory in flat Minkowski spacetime (Chap. 11) and other related subjects appearing in the literature.

The Standard Dirac Operator

Given $\boldsymbol{u} \in \sec TM$ and $A \in \sec \bigwedge^{r} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ consider the tensorial mapping $A \mapsto \mathring{D}_{\mathbf{u}}A \in \sec \bigwedge^{r} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Since $\mathring{D}_{\boldsymbol{u}}J_{\mathring{\mathtt{g}}} \subseteq J_{\mathring{\mathtt{g}}}$, where $J_{\mathring{\mathtt{g}}}$ is the ideal used in the definition of $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$, we see immediately that the notion of covariant derivative (related to the Levi-Civita connection[24]) pass to the quotient

[23] It is crucial to distinguish the Dirac operators introduced in this chapter and which act on sections of Clifford bundles with the spin Dirac operator introduced in Chap. 7 and which act on sections of spin-Clifford bundles.

[24] And more generally, to any metric compatible connection.

bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$, i.e., given $A, B \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ we have taking into account the fact that $\mathring{D}_{\boldsymbol{u}}\mathring{\boldsymbol{g}} = 0 = \mathring{D}_{\boldsymbol{u}}\mathring{\mathtt{g}}$ that

$$\begin{aligned}\mathring{D}_{\boldsymbol{u}}(AB) &= \mathring{D}_{\boldsymbol{u}}[\frac{1}{2}(A \otimes B - B \otimes A) + \mathring{g}(A, B)] \\ &= \mathring{D}_{\boldsymbol{u}}[\frac{1}{2}(A \otimes B - B \otimes A)] + (\mathring{D}_{\boldsymbol{u}}\mathring{\mathtt{g}})(A, B) + \mathring{\mathtt{g}}(\mathring{D}_{\boldsymbol{u}}A, B) + \mathring{\mathtt{g}}(A, \mathring{D}_{\boldsymbol{u}}B) \\ &= \mathring{D}_{\boldsymbol{u}}(A)B + A\mathring{D}_{\boldsymbol{u}}(B). \end{aligned} \tag{4.147}$$

Before continuing we agree that the scalar and contracted products induced by $\mathring{\mathtt{g}}$ will be denoted simply by the symbols $\cdot$ and $\lrcorner$ instead of the symbol $\underset{\mathring{g}}{\cdot}$ and $\underset{\mathring{g}}{\lrcorner}$.

Definition 4.116 The *standard Dirac operatoracting on sections of* $\mathcal{C}\ell(M, \mathring{g})$ is the first order differential operator

$$\boldsymbol{\partial} = \theta^\alpha \mathring{D}_{\boldsymbol{e}_\alpha}. \tag{4.148}$$

For $A \in \sec \mathcal{C}\ell(M, \mathring{g})$,

$$\boldsymbol{\partial} A = \theta^\alpha (\mathring{D}_{\boldsymbol{e}_\alpha} A) = \theta^\alpha \lrcorner (\mathring{D}_{\boldsymbol{e}_\alpha} A) + \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha} A)$$

and then we define:

$$\begin{aligned}\boldsymbol{\partial}\lrcorner A &= \theta^\alpha \lrcorner (\mathring{D}_{\boldsymbol{e}_\alpha} A), \\ \boldsymbol{\partial}\wedge A &= \theta^\alpha \wedge (\mathring{D}_{\boldsymbol{e}_\alpha} A),\end{aligned}$$

in order to have:

$$\boldsymbol{\partial} = \boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge. \tag{4.149}$$

Remark 4.117 Note moreover that for $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ we can also write

$$\boldsymbol{\partial}\lrcorner A = \boldsymbol{\partial}\cdot A. \tag{4.150}$$

Exercise 4.118 Verify that the operators $\boldsymbol{\partial}\lrcorner$ and $\boldsymbol{\partial}\wedge$ satisfy the following identities:

$$\begin{aligned}&\text{(a)} \quad \boldsymbol{\partial}\wedge(A \wedge B) = (\boldsymbol{\partial}\wedge A) \wedge B + \hat{A} \wedge (\boldsymbol{\partial}\wedge B), \\ &\text{(b)} \quad \boldsymbol{\partial}\lrcorner(A_r \lrcorner B_s) = (\boldsymbol{\partial}\wedge A_r)\lrcorner B_s + \hat{A}_r \lrcorner(\boldsymbol{\partial}\lrcorner B_s); \quad r + 1 \leq s, \\ &\text{(c)} \quad \boldsymbol{\partial}\lrcorner\star = (-1)^r \star \boldsymbol{\partial}\wedge\,; \quad \star\,\boldsymbol{\partial}\lrcorner = (-1)^{r+1}\,\boldsymbol{\partial}\wedge\,.\end{aligned} \tag{4.151}$$

In addition to these identities, we have the important result [24, 32].

Proposition 4.119 *The standard Dirac derivative* $\boldsymbol{\partial}$ *is related to the exterior derivative* d *and to the Hodge codifferential* δ *by:*

$$\boldsymbol{\partial} = d - \delta, \tag{4.152}$$

that is, we have $\boldsymbol{\partial}\wedge = d$ *and* $\boldsymbol{\partial}\lrcorner = -\delta$.

Proof If f is a function, $\boldsymbol{\partial}\wedge f = \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha} f = \boldsymbol{e}_\alpha(f)\theta^\alpha = df$ and $\boldsymbol{\partial}\lrcorner f = \theta^\alpha \lrcorner \mathring{D}_{\boldsymbol{e}_\alpha} f = \theta^\alpha \cdot \mathring{D}_{\boldsymbol{e}_\alpha} f = 0$. For the 1-form fields θ^ρ of a moving frame on T^*M, we have $\boldsymbol{\partial}\wedge\theta^\rho = \theta^\alpha \wedge \mathring{D}_{\boldsymbol{e}_\alpha}\theta^\rho = -\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^\alpha \wedge \theta^\beta = -\hat{\omega}^\rho_\beta \wedge \theta^\beta = d\theta^\rho$.

Now, for a r-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, we get, using Eq. (4.151a),

$$\begin{aligned}\boldsymbol{\partial}\wedge\omega &= \frac{1}{r!}(d\omega_{\alpha_1\ldots\alpha_r} \wedge \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} + \omega_{\alpha_1\ldots\alpha_r} d\theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} \\ &\quad + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}} \wedge d\theta^{\alpha_r}) \\ &= d\omega.\end{aligned}$$

Finally, using Eqs. (4.93c) and (4.151c), we get $\boldsymbol{\partial}\lrcorner\omega = -\delta\omega$.■

Note also that given an arbitrary coordinate moving frame $\{\theta^\mu = dx^\mu\}$ on M ($\mathbf{x}^\rho : U \to \mathbb{R}$, $U \subset M$, are coordinate functions), we have the following interesting relations:

$$\begin{aligned}&(a)\ \boldsymbol{\partial}\lrcorner\theta^\rho \equiv \boldsymbol{\partial}\cdot\theta^\rho = -\mathring{g}^{\alpha\beta}\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \sqrt{|(\det \mathring{\mathtt{g}})|}\partial_\sigma(\sqrt{|(\det \mathring{\mathtt{g}})^{-1}|}\mathring{g}^{\rho\sigma}) \\ &(b)\ \boldsymbol{\partial}\lrcorner\theta_\sigma \equiv \boldsymbol{\partial}\cdot\theta_\sigma = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\rho\sigma} = \sqrt{|(\det \mathring{\mathtt{g}})|}\partial_\sigma(\sqrt{|(\det \mathring{\mathtt{g}})^{-1}|}),\end{aligned} \tag{4.153}$$

where $\{\partial_\sigma \equiv \partial/\partial x^\sigma\}$ is the dual frame of $\{\theta^\mu\}$. Note that $\det \mathring{\boldsymbol{g}} = (\det \mathring{\mathtt{g}})^{-1}$.

Exercise 4.120 Verify that if $\alpha, \beta \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ then

$$\boldsymbol{\partial}(\alpha \cdot \beta) = (\alpha \cdot \boldsymbol{\partial})\beta + (\beta \cdot \boldsymbol{\partial})\alpha - \alpha\lrcorner(\boldsymbol{\partial}\wedge\beta) - \beta\lrcorner(\boldsymbol{\partial}\wedge\alpha). \tag{4.154}$$

4.8.4 Standard Dirac Commutator and Dirac Anticommutator

Definition 4.121 Given the 1-form fields $\alpha, \beta \in \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ and $\boldsymbol{\partial}$, the standard Dirac operator of the manifold, the operators $[\![,]\!]$ and $\{,\}$ given by

$$\begin{aligned}[\![\alpha, \beta]\!] &= (\alpha \cdot \boldsymbol{\partial})\beta - (\beta \cdot \boldsymbol{\partial})\alpha \\ \{\alpha, \beta\} &= (\alpha \cdot \boldsymbol{\partial})\beta + (\beta \cdot \boldsymbol{\partial})\alpha,\end{aligned} \tag{4.155}$$

are called, respectively, the Standard *Dirac commutator (or Lie bracket)* and the Standard *Dirac anticommutator* of the 1-form fields α and β.

We have the identities:

$$\begin{aligned}[\![\alpha,\beta]\!] &= \partial\!\!\!/\lrcorner(\alpha\wedge\beta) - \left[(\partial\!\!\!/\cdot\alpha)\wedge\beta - \alpha\wedge(\partial\!\!\!/\cdot\beta)\right]\\ \{\alpha,\beta\} &= \partial\!\!\!/\wedge(\alpha\cdot\beta) - \left[(\partial\!\!\!/\wedge\alpha)\llcorner\beta - \alpha\lrcorner(\partial\!\!\!/\wedge\beta)\right].\end{aligned} \tag{4.156}$$

The algebraic meaning of these equations is clear: they state that the Dirac commutator and the Dirac anticommutator measure the amount by which the operators $\partial\!\!\!/\lrcorner = -\delta$ and $\partial\!\!\!/\wedge = d$ fail to satisfy the Leibniz's rule when applied, respectively, to the exterior and to the dot product of 1-form fields.

Now, let $\{\boldsymbol{e}_\sigma\}$ be an *arbitrary* moving frame on TM, $\{\theta^\sigma\}$ its dual frame on T^*M and $\{\theta_\alpha\}$ the reciprocal frame of $\{\theta^\sigma\}$. From Eqs. (4.155) we obtain, respectively:

$$\begin{aligned}[\![\theta_\alpha,\theta_\beta]\!] &= \mathring{D}_{\boldsymbol{e}_\alpha}\theta_\beta - \mathring{D}_{\boldsymbol{e}_\beta}\theta_\alpha\\ &= (\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} - \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha})\theta_\rho\\ &= c^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta_\rho,\end{aligned} \tag{4.157}$$

and

$$\begin{aligned}\{\theta_\alpha,\theta_\beta\} &= \mathring{D}_{\boldsymbol{e}_\alpha}\theta_\beta + \mathring{D}_{\boldsymbol{e}_\beta}\theta_\alpha,\\ &= (\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha})\theta_\rho\\ &= b^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta_\rho,\end{aligned} \tag{4.158}$$

where $\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the components of the Levi-Civita connection $\mathring{D}$ of $\mathring{g}$, $c^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ are the structure coefficients of the frame $\{\boldsymbol{e}_\sigma\}$ and where we introduce the notation $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha}$. The meaning of these coefficients will be discussed below.

Clearly, Eq. (4.157) states that the Dirac commutator is the analogous of the Lie bracket of vector fields. These operations have similar properties. In particular, the Dirac commutator satisfies the *Jacobi identity*:

$$[\![\alpha,[\![\beta,\omega]\!]]\!] + [\![\beta,[\![\omega,\alpha]\!]]\!] + [\![\omega,[\![\alpha,\beta]\!]]\!] = 0, \tag{4.159}$$

$\alpha,\beta,\omega \in \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathring{g})$. Therefore it gives to the cotangent bundle of M the structure of a *local* Lie algebra.

4.8.5 Geometrical Meanings of the Commutator and Anticommutator

The geometrical meanings of the Dirac commutator and the Dirac anticommutator are easily discovered from Eqs. (4.157) and (4.158). Indeed, Eq. (4.157) means that

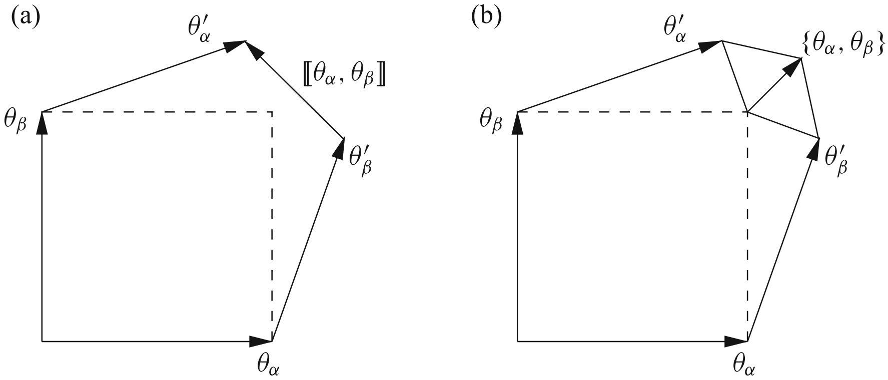

Fig. 4.4 Geometrical interpretation of the: (**a**) Standard commutator $[\![\theta_\alpha, \theta_\beta]\!]$ and (**b**) Standard anticommutator $\{\theta_\alpha, \theta_\beta\}$

the Dirac commutator measures the amount by which the vector fields $\boldsymbol{e}_a = \mathring{g}(\theta_\alpha,)$ and $\boldsymbol{e}_b = \mathring{g}(\theta_\beta,)$ and their infinitesimal lifts ($\boldsymbol{e}'_a = \mathring{g}(\theta'_\alpha,)$, $\boldsymbol{e}'_\beta = \mathring{g}((\theta'_\beta,))$ along their integral lines fail to form a parallelogram. By its turn, Eq. (4.158) means that the Dirac anticommutator measures the rate of deformation of the frame $\{\theta_\alpha\}$, i.e., $\{\theta_\alpha, \theta_\alpha\}$ gives the rate of dilation of the vector field $\mathring{g}(\theta_\alpha,)$ under dislocations along its own integral lines, while $\{\theta_\alpha, \theta_\beta\}$, $\alpha \neq \beta$, gives the rate of variation of the angle between $\mathring{g}(\theta_\alpha,)$ and $\mathring{g}(\theta_\beta,)$ under dislocations in the direction of each other (Fig. 4.4).

We state now another interesting result:

Proposition 4.122 *The coefficients $b^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ of the Dirac anticommutator of a moving frame $\{\theta_\alpha\}$ are given by:*

$$b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = -(\pounds_{\boldsymbol{e}^\rho} \boldsymbol{g})_{\alpha\beta}, \tag{4.160}$$

where $\pounds_{\boldsymbol{e}^\rho}$ denotes the Lie derivative in the direction of the vector field $\boldsymbol{e}^\rho$ and $\{\boldsymbol{e}^\rho\}$ is the dual frame of $\{\theta_\alpha\}$.

Proof The coefficients $\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ of the Levi-Civita connection of $\boldsymbol{g}$ are given by: (e.g.,[3])

$$\begin{aligned} \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= \frac{1}{2}\mathring{g}^{\rho\sigma}\left[\boldsymbol{e}_\alpha(\mathring{g}_{\beta\sigma}) + \boldsymbol{e}_\beta(\mathring{g}_{\sigma\alpha}) - \boldsymbol{e}_\sigma(\mathring{g}_{\alpha\beta})\right] \\ &\quad + \frac{1}{2}\mathring{g}^{\rho\sigma}\left[\mathring{g}_{\mu\alpha}c^{\mu\cdot\cdot}_{\cdot\sigma\beta} + \mathring{g}_{\mu\beta}c^{\mu\cdot\cdot}_{\cdot\sigma\alpha} - \mathring{g}_{\mu\sigma}c^{\mu\cdot\cdot}_{\cdot\alpha\beta}\right]. \end{aligned} \tag{4.161}$$

Hence,

$$b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{g}^{\rho\sigma}\left[\boldsymbol{e}_\beta(\mathring{g}_{\alpha\sigma}) + \boldsymbol{e}_\alpha(\mathring{g}_{\sigma\beta}) - \boldsymbol{e}_\sigma(\mathring{g}_{\beta\alpha}) - \mathring{g}_{\mu\alpha}c^{\mu\cdot\cdot}_{\cdot\beta\sigma} - \mathring{g}_{\mu\beta}c^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\right] \tag{4.162}$$

and the r.h.s. of Eq. (4.162) is just the negative of the components of the Lie derivative of the metric tensor in the direction of $\boldsymbol{e}^\rho = \mathring{g}^{\rho\sigma}\boldsymbol{e}_\sigma$.■

Killing Coefficients

In view of the result stated by Eq. (4.160), the attempt to find (if existing) a moving frame for which $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = 0$ is equivalent to solve, locally, the Killing equations for the manifold. Because of this we shall refer to these coefficients as the *Killing coefficients* of the frame. Of course, since the solutions of the Killing equations are restricted by the structure of the metric as well as by the topology of the manifold, it will not be possible, in the more general case, to find any moving frame for which these coefficients are all null.

4.8.6 Associated Dirac Operators

Besides the standard Dirac operator we have just analyzed, we can also introduce in the Clifford bundle $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ infinitely many other Dirac-like operators, one for each nondegenerate symmetric bilinear form field that can be defined on the structure $(M, \mathring{\boldsymbol{g}}, \mathring{D})$.

Let $\boldsymbol{g} \in \sec T^0_2 M$ be an arbitrary nondegenerate *positive* symmetric bilinear form field on M. To $\boldsymbol{g}$ corresponds $\mathtt{g} \in \sec T^2_0 M$ as already introduced. We denoted by $g : \sec T^*M \to \sec T^*M$ the associated extensor field to $\mathtt{g}$ and by $h : \sec T^*M \to \sec T^*M$ the field of linear transformations which induces g, i.e., have:

$$\begin{aligned} \mathtt{g}(\alpha, \beta) &= \alpha \cdot g(\beta) = h(\alpha) \cdot h(\beta) \\ &= \mathtt{g}(h(\alpha), h(\beta)) \end{aligned} \tag{4.163}$$

for every $\alpha, \beta \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$.

We also denote by $\vee \equiv \underset{\boldsymbol{g}}{} : \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \times \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \to \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ the "Clifford product" induced on $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ by the bilinear form field $\mathtt{g}$ and by $\bullet \equiv \underset{\boldsymbol{g}}{\cdot} : \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \times \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \to \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ the "dot product" associated to the new Clifford product "$\vee$."

Definition 4.123 Let $\{\theta^\alpha\}$ be a moving frame on T^*M, dual to the moving frame $\{\boldsymbol{e}_\alpha\}$ on TM. We call Dirac operator *associated* to the bilinear form $\mathtt{g} \in \sec T_0^2 M$ the operator:

$$\overset{\vee}{\boldsymbol{\partial}} \equiv \boldsymbol{\partial}\vee = (\theta^\alpha \underset{\boldsymbol{g}}{} \overset{\circ}{D}_{\boldsymbol{e}_\alpha}) \equiv (\theta^\alpha \vee \overset{\circ}{D}_{\boldsymbol{e}_\alpha}). \tag{4.164}$$

We also define

$$\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} = \theta^\alpha \underset{\boldsymbol{g}}{\lrcorner} \overset{\circ}{D}_{\boldsymbol{e}_\alpha}, \tag{4.165}$$

where $\underset{\boldsymbol{g}}{\lrcorner}$ is the contracted product with respect to $\mathtt{g}$. Then,

$$\overset{\vee}{\boldsymbol{\partial}} = \overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} + \overset{\vee}{\boldsymbol{\partial}}\wedge = \overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner} + \boldsymbol{\partial}\wedge, \tag{4.166}$$

because the exterior part of the operator $\overset{\vee}{\boldsymbol{\partial}}$ coincides with the exterior part of the operator $\boldsymbol{\partial}$.

Of course, the properties of the operator $\overset{\vee}{\boldsymbol{\partial}}$ differ from those of the standard Dirac operator $\boldsymbol{\partial}$. It is enough to state the properties of the operator $\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}$, which are obtained from the following proposition:

Proposition 4.124 *The operators* $\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}$ *and* $\boldsymbol{\partial}\lrcorner$ *are related by:*

$$\overset{\vee}{\boldsymbol{\partial}}\underset{\boldsymbol{g}}{\lrcorner}\omega = \boldsymbol{\partial}\lrcorner\check{\omega} + s\lrcorner\check{\omega}, \tag{4.167}$$

for every $\omega \in \sec \mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$, *where* $s = g^{\rho\sigma}\overset{\circ}{D}_\rho g_{\sigma\mu}\theta^\mu \in \sec T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$ *is called the dilation 1-form of the bilinear form* $\mathtt{g}$.

Proof Given a r-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} \in \sec \mathcal{C}\ell(M, \overset{\circ}{g})$, we have

$$\overset{\circ}{D}_{\boldsymbol{e}_\rho}\omega = \frac{1}{r!}(D_\rho\omega_{\alpha_1\ldots\alpha_r})\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r},$$

with

$$\overset{\circ}{D}_\rho\omega_{\alpha_1\ldots\alpha_r} = \boldsymbol{e}_\rho(\omega_{\alpha_1\ldots\alpha_r}) - \overset{\circ}{\Gamma}{}^{\mu}_{\rho\alpha_1}\omega_{\mu\alpha_2\ldots\alpha_r} - \cdots - \overset{\circ}{\Gamma}{}^{\mu}_{\rho\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\mu}. \tag{4.168}$$

Then,

$$\theta^\rho \underset{g}{\lrcorner} \overset{\circ}{D}_{e_\rho}\omega = \frac{1}{r!} D_\rho \omega_{\alpha_1\ldots\alpha_r} \theta^\rho \underset{g}{\lrcorner} (\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r})$$
$$= \frac{1}{r!} \overset{\circ}{D}_\rho \omega_{\alpha_1\ldots\alpha_r} (g^{\rho\alpha_1}\theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} + \cdots$$
$$+ (-1)^{r+1} g^{\rho\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}}),$$

or

$$\overset{\vee}{\boldsymbol{\partial}} \underset{g}{\lrcorner} \omega = \frac{1}{(r-1)!} g^{\rho\sigma} \overset{\circ}{D}_\rho \omega_{\sigma\alpha_2\ldots\alpha_r} \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r}. \tag{4.169}$$

Now, taking into account that

$$g^{\rho\sigma} \overset{\circ}{D}_\rho \omega_{\sigma\alpha_2\ldots\alpha_r} = \overset{\circ}{D}_\rho (g^{\rho\sigma} \omega_{\sigma\alpha_2\ldots\alpha_r}) - (\overset{\circ}{D}_\rho g^{\rho\sigma}) \omega_{\sigma\alpha_2\ldots\alpha_r},$$
$$g_{\sigma\mu} \overset{\circ}{D}_\rho g^{\rho\sigma} = -g^{\rho\sigma} \overset{\circ}{D}_\rho g_{\sigma\mu},$$

and recalling also that $g^{\rho\sigma} = g^{\rho\mu} \overset{\circ}{g}{}^{\sigma}_{\mu}$, we conclude that

$$\overset{\vee}{\boldsymbol{\partial}} \underset{g}{\lrcorner} \omega = \frac{1}{(r-1)!} \overset{\circ}{g}{}^{\rho\sigma} (\overset{\circ}{D}_\rho \overset{\circ}{g}{}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r}) \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r}$$
$$+ \frac{1}{(r-1)!} \overset{\circ}{g}{}^{\rho\sigma} (g^{\alpha\beta} \overset{\circ}{D}_\alpha g_{\beta\rho}) \overset{\circ}{g}{}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r} \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Thus, writing $\check{\omega}_{\sigma\alpha_2\ldots\alpha_r} = \overset{\circ}{g}{}^{\mu}_{\sigma} \omega_{\mu\alpha_2\ldots\alpha_r}$ and $s_\rho = g^{\alpha\beta} \overset{\circ}{D}_\alpha g_{\beta\rho}$, we finally obtain the Eq. (4.167).■

4.8.7 The Dirac Operator in Riemann-Cartan-Weyl Spaces

We now consider the structure $(M, \overset{\circ}{\boldsymbol{g}}, \nabla)$ where ∇ is an arbitrary linear connection. In this case, the notion of covariant derivative does not pass to the quotient bundle $\mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$ [4]. Despite this fact, it is still a well defined operation and in analogy with the earlier section, we can associate to it, acting on the sections of $\mathcal{C}\ell(M, \overset{\circ}{\mathtt{g}})$, the operator:

$$\boldsymbol{\partial} = \theta^\alpha \nabla_{e_\alpha},$$

where $\{\theta^\alpha\}$ is a moving frame on T^*M, dual to the moving frame $\{\boldsymbol{e}_\alpha\}$ on TM.

Definition 4.125 The operator $\boldsymbol{\partial}$ is called the *Dirac operator* (or *Dirac derivative, or sometimes gradient*).

We also define:

$$\begin{aligned} \boldsymbol{\partial} \lrcorner A &= \theta^\alpha \lrcorner (\nabla_{\boldsymbol{e}_\alpha} A), \\ \boldsymbol{\partial} \wedge A &= \theta^\alpha \wedge (\nabla_{\boldsymbol{e}_\alpha} A), \end{aligned} \tag{4.170}$$

for every $A \in \sec \mathcal{C}\ell(M, \mathring{g})$, so that:

$$\boldsymbol{\partial} = \boldsymbol{\partial} \lrcorner + \boldsymbol{\partial} \wedge . \tag{4.171}$$

The operator $\boldsymbol{\partial} \wedge$ satisfies, for every $A, B \in \sec \mathcal{C}\ell(M, \mathring{g})$:

$$\boldsymbol{\partial} \wedge (A \wedge B) = (\boldsymbol{\partial} \wedge A) \wedge B + \hat{A} \wedge (\boldsymbol{\partial} \wedge B), \tag{4.172}$$

what generalizes Eq. (4.151a). By its turn, Eq.(4.151c) is generalized according to the following proposition:

Proposition 4.126 *Let* $\mathbf{Q}^\rho$ *be the nonmetricity 2-forms associated with the connection* ∇ *in an arbitrary moving frame* $\{\theta^\rho\}$ *and* $\nabla_{\boldsymbol{e}_\alpha} \boldsymbol{e}_\beta = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} \boldsymbol{e}_\rho$. *Then we have, for homogeneous multiforms,*

$$\begin{aligned} (a)\ \ (-1)^r \star^{-1} \boldsymbol{\partial} \lrcorner \star &= \boldsymbol{\partial} \wedge + \mathbf{Q}^\rho \wedge \mathbf{i}_\rho, \\ (b)\ \ (-1)^{r+1} \star^{-1} \boldsymbol{\partial} \wedge \star &= \boldsymbol{\partial} \lrcorner - \mathbf{Q}^\rho \lrcorner \mathbf{j}_\rho, \end{aligned} \tag{4.173}$$

where $\mathbf{i}_\rho A = \theta_\rho \lrcorner A$ *and* $\mathbf{j}_\rho A = \theta_\rho \wedge A$, *for every* $A \in \sec \mathcal{C}\ell(M, \mathring{g})$.

Proof Let $\omega = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{g})$ be a r-form field on M. We have $(\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge *\omega = ((\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \cdot \omega) \tau_{\mathring{g}} = \omega_{\beta_1 \ldots \beta_r} \tau_{\mathring{g}}$ and it follows that $\nabla_{\boldsymbol{e}_\alpha}((\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge *\omega) = \boldsymbol{e}_\alpha(\omega_{\beta_1 \ldots \beta_r}) \tau_{\boldsymbol{g}}$. But on the other hand, we also have

$$\begin{aligned} \nabla_{\boldsymbol{e}_\alpha}(\theta_{\beta_1} \wedge \cdots \wedge \theta_{\beta_r}) \wedge \star\omega = {} & \theta_{\beta_1} \wedge v \wedge \theta_{\beta_r} \wedge \nabla_{\boldsymbol{e}_\alpha} \star \omega \\ & + (L^{\rho\cdot\cdot}_{\cdot\sigma\beta_1} \omega_{\rho\beta_2 \ldots \beta_r} + \cdots + L^{\rho\cdot\cdot}_{\cdot\sigma\beta_r} \omega_{\beta_1 \ldots \beta_{r-1}\rho}) \tau_{\mathring{g}} \\ & - (Q^{\rho\cdot\cdot}_{\cdot\sigma\beta_1} \omega_{\rho\beta_2 \ldots \beta_r} + \cdots + Q^{\rho\cdot\cdot}_{\cdot\sigma\beta_r} \omega_{\beta_1 \ldots \beta_{r-1}\rho}) \tau_{\mathring{g}} \end{aligned}$$

and therefore we get, after some algebraic manipulation:

$$\nabla_{\boldsymbol{e}_\alpha} \star \omega = \star \nabla_{\boldsymbol{e}_\alpha} \omega + Q_{\sigma\mu\nu} \star (\theta^\mu \wedge (\theta^\nu \lrcorner \omega)), \tag{4.174}$$

from which Eqs. (4.173) follow immediately. ■

Taking into account the result stated in the above proposition and the definition of the Hodge codifferential (Eq. (4.91)), we are motivated to introduce in the Clifford bundle the *Dirac coderivative* operator, given, for homogeneous multiforms, by:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \star^{-1} \boldsymbol{\partial} \star \ . \tag{4.175}$$

Of course, we have:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \star^{-1} \boldsymbol{\partial} \lrcorner \star \ + (-1)^r \star^{-1} \boldsymbol{\partial} \wedge \star \tag{4.176}$$

and we can, then, define:

$$\begin{aligned} \overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner &:= (-1)^r \star^{-1} \boldsymbol{\partial} \wedge \star = -\boldsymbol{\partial} \lrcorner + \ \mathbf{Q}^{\rho} \lrcorner \mathbf{j}_{\rho} \\ \overset{\blacklozenge}{\boldsymbol{\partial}} \wedge &:= (-1)^r \star^{-1} \boldsymbol{\partial} \lrcorner \star \boldsymbol{\partial} \wedge + \mathbf{Q}^{\rho} \wedge \mathbf{i}_{\rho}, \end{aligned} \tag{4.177}$$

so that:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = \overset{\blacklozenge}{\boldsymbol{\partial}} \wedge + \overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner \ . \tag{4.178}$$

The following identities are trivially established:

$$\begin{aligned} \boldsymbol{\partial} &= (-1)^{r+1} \star^{-1} \overset{\blacklozenge}{\boldsymbol{\partial}} \star \\ \star \boldsymbol{\partial} &= (-1)^{r+1} \overset{\blacklozenge}{\boldsymbol{\partial}} \star; \quad \star \overset{\blacklozenge}{\boldsymbol{\partial}} = (-1)^r \boldsymbol{\partial} \star \\ \boldsymbol{\partial} \overset{\blacklozenge}{\boldsymbol{\partial}} \star &= \star \overset{\blacklozenge}{\boldsymbol{\partial}} \boldsymbol{\partial}; \quad \star \boldsymbol{\partial} \overset{\blacklozenge}{\boldsymbol{\partial}} = \overset{\blacklozenge}{\boldsymbol{\partial}} \boldsymbol{\partial} \star \\ \star \overset{\blacklozenge}{\boldsymbol{\partial}}{}^2 &= -(\overset{\blacklozenge}{\boldsymbol{\partial}})^2 \star; \quad \star (\overset{\blacklozenge}{\boldsymbol{\partial}})^2 = -\boldsymbol{\partial}^2 \star \ . \end{aligned} \tag{4.179}$$

In addition, we note that the Dirac coderivative permit us to generalize Eq. (4.151b) in a very elegant way. In fact, in consequence of Proposition 4.126 we have:

Corollary 4.127 *For* $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec C\ell(M, \mathring{g})$*,* $B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec C\ell(M, \mathring{g})$*, with* $r + 1 \leq s$*, it holds:*

$$\boldsymbol{\partial} \lrcorner (A_r \lrcorner B_s) = (\overset{\blacklozenge}{\boldsymbol{\partial}} \wedge A_r) \lrcorner B_s + (-1)^r A_r \lrcorner (\overset{\blacklozenge}{\boldsymbol{\partial}} \lrcorner B_s). \tag{4.180}$$

Proof Given a 1-form field $\alpha \in \bigwedge^1 T^*M$ and a s-form field $\omega \in \sec \bigwedge^s T^*M$, we have, from Eq. (4.174), that $\nabla_{e_\sigma} \star (\alpha \lrcorner \omega) = \star \nabla_{e_\sigma} (\alpha \lrcorner \omega + Q_{\sigma\mu\nu} \star [\theta^\mu \wedge (\theta^\nu \lrcorner (\alpha \lrcorner \omega))]$.

We also have that

$$\begin{aligned} \nabla_{e_\sigma} \star (\alpha \lrcorner \omega) &= (-1)^{s+1} \nabla_{e_\sigma} (\alpha \wedge \star \omega) \\ &= \star[(\nabla_{e_\sigma} \alpha) \lrcorner \omega + \alpha \lrcorner (\nabla_{e_\sigma} \omega + Q_{\sigma\mu\nu} (\theta^\mu \wedge (\theta^\nu \lrcorner \omega))], \end{aligned}$$

where we have used Eq. (4.174) once again. It follows that:

$$\nabla_{e_\sigma}(\alpha \lrcorner \omega) = (\nabla_{e_\sigma}\alpha) \lrcorner \omega + \alpha \lrcorner (\nabla_{e_\sigma}\omega) + Q_{\sigma\mu\nu}\alpha^\mu \theta^\nu \lrcorner \omega. \tag{4.181}$$

Then, recalling that $(\alpha_1 \wedge \ldots \wedge \alpha_r) \lrcorner \omega = \alpha_1 \lrcorner \ldots \lrcorner \alpha_r \lrcorner \omega$, with $\alpha_1, \ldots, \alpha_r \in \sec T^*M, \omega \in \sec \bigwedge^s T^*M, r \leq s+1$, and applying Eq. (4.181) successively in this expression, we get Eq. (4.180).■

Another very important consequence of Proposition 4.126 states the relation between the operators $\boldsymbol{\partial}$ and $\oint$:

Proposition 4.128 *Let $\Phi^\rho = \Theta^\rho - \mathbf{Q}^\rho$, where Θ^ρ and $\mathbf{Q}^\rho$ denote, respectively, the torsion and the nonmetricity 2-forms of the connection ∇ in an arbitrary moving frame $\{\theta^\alpha\}$. Then:*

$$\begin{array}{ll} (a) & \boldsymbol{\partial}\wedge = \oint\wedge - \Theta^\rho \wedge \mathbf{i}_\rho\ , \\ (b) & \boldsymbol{\partial}\lrcorner\ = \oint\lrcorner - \Phi^\rho \lrcorner \mathbf{j}_\rho\ \ . \end{array} \tag{4.182}$$

Proof If f is a function, $\boldsymbol{\partial}\wedge f = \theta^\alpha \wedge \nabla_{e_\alpha} f = e_\alpha(f)\theta^\alpha = df$ and $\boldsymbol{\partial}\lrcorner f = \theta^\alpha \lrcorner \nabla_{e_\alpha} f = 0$. For the 1-form field θ^ρ of a moving frame on T^*M, we have $\boldsymbol{\partial}\wedge\theta^\rho = \theta^\alpha \wedge \nabla_{e_\alpha}\theta^\rho = -L^\rho_{\alpha\beta}\theta^\alpha \wedge \theta^\beta = -\omega^\rho_\beta \wedge \theta^\beta = d\theta^\rho - \Theta^\rho$.

Now, for a r-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, we get

$$\begin{aligned}
\boldsymbol{\partial}\wedge\omega &= \frac{1}{r!}(d\omega_{\alpha_1\ldots\alpha_r} \wedge \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r} + \omega_{\alpha_1\ldots\alpha_r} d\theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} \\
&\quad + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}} \wedge d\theta^{\alpha_r}) \\
&\quad - \frac{1}{r!}(\omega_{\alpha_1\ldots\alpha_r}\Theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} + \cdots \\
&\quad + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}} \wedge \Theta^{\alpha_r}) \\
&= d\omega - \frac{1}{r!}\Theta^\rho \wedge (\omega_{\rho\alpha_2\ldots\alpha_r}\theta^{\alpha_2} \wedge \cdots \wedge \theta^{\alpha_r} + \cdots \\
&\quad + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_{r-1}\rho}\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_{r-1}}) \\
&= d\omega - \Theta^\rho \wedge \mathbf{i}_\rho\omega
\end{aligned}$$

and Eq. (4.182a) is proved.

Finally, from Eqs. (4.173b) and (4.182a) we obtain

$$\begin{aligned}
\boldsymbol{\partial}\wedge\star\omega &= (-1)^{r+1}\star\boldsymbol{\partial}\lrcorner\omega - (-1)^{r+1}\star\mathbf{Q}^\rho \lrcorner \mathbf{j}_\rho\omega \\
&= \boldsymbol{\partial}\wedge\star\omega - \Theta^\rho \wedge \star\omega \\
&= (-1)^{r+1}\star\oint\lrcorner\omega - (-1)^{r+1}\star\Theta^\rho \lrcorner \mathbf{j}_\rho\omega.
\end{aligned}$$

Therefore, $\boldsymbol{\partial}\lrcorner\omega = \oint\lrcorner\omega - \Phi^\rho \lrcorner \mathbf{j}_\rho\omega$, and Eq. (4.182b) is proved.■

From Eqs. (4.182) we obtain the expressions of $\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner$ and $\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge$ in terms of $\boldsymbol{\eth}\lrcorner$ and $\boldsymbol{\eth}\wedge$:

$$\begin{aligned} \overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner &= -\boldsymbol{\eth}\lrcorner + \Theta^{\rho}\lrcorner\mathbf{j}_{\rho} \\ \overset{\blacklozenge}{\boldsymbol{\partial}}\wedge &= \boldsymbol{\eth}\wedge - \Phi^{\rho}\wedge\mathbf{i}_{\rho}. \end{aligned} \tag{4.183}$$

Obviously, the Dirac coderivative associated to the standard Dirac operator is given by:

$$\overset{\blacklozenge}{\boldsymbol{\partial}} = \boldsymbol{\eth}\wedge - \boldsymbol{\eth}\lrcorner = d + \delta. \tag{4.184}$$

We observe finally that we can still introduce another Dirac operator, obtained by combining the arbitrary affine connection ∇ with the algebraic structure induced by the generic bilinear form field $\mathtt{g} \in \sec T_0^2 M$. With respect to an arbitrary moving frame $\{\theta^{\alpha}\}$ on T^*M, this operator has the expression:

$$\boldsymbol{\partial}\vee = \theta^{\alpha} \vee \nabla_{\boldsymbol{e}_{\sigma}}. \tag{4.185}$$

It is clear that in the particular case where $\nabla = D$ is the Levi-Civita connection of $\boldsymbol{g}$, the operator $\boldsymbol{\partial}$—which in this case *is* the standard Dirac operator associated to $\boldsymbol{g}$—will satisfy the properties of Sect. 4.8.3, with the usual Clifford product exchanged by the new Clifford product "$\vee$." In addition, for a more general connection we can apply the results of Sect. 4.8.6, once again with all the occurrences of $\mathring{\mathtt{g}}$ replaced by $\mathtt{g}$. (In particular, the standard Dirac operator associated to $\mathring{\mathtt{g}}$ is replaced by that associated with $\mathtt{g}$.)

4.8.8 Torsion, Strain, Shear and Dilation of a Connection

In analogy with the introduction of the Dirac commutator and the Dirac anticommutator, let us define the operations:

Definition 4.129 Given $\alpha, \beta \in \sec \bigwedge^1 T^*M$ the Dirac commutator and anticommutator of these 1-form fields are

$$\begin{aligned} &\text{(a)}\ [\![\alpha,\beta]\!] = (\alpha\cdot\boldsymbol{\partial})\beta - (\beta\cdot\boldsymbol{\partial})\alpha - [\![\alpha,\beta]\!] \\ &\text{(b)}\ \{\!\{\alpha,\beta\}\!\} = (\alpha\cdot\boldsymbol{\partial})\beta + (\beta\cdot\boldsymbol{\partial})\alpha - \{\alpha,\beta\}. \end{aligned} \tag{4.186}$$

We have subtracted the Dirac commutator and the Dirac anticommutator in the r.h.s. of these expressions in order to have objects which are independent of the structure of the fields on which they are applied.

If $\{\theta_\alpha\}$ is the reciprocal of an arbitrary moving frame $\{\theta^\alpha\}$ on T^*M, we get, from Eq. (4.186a):

$$[\![\theta_\alpha, \theta_\beta]\!] = (T^{\rho\cdot\cdot}_{\cdot\alpha\beta} - Q^{\rho\cdot\cdot}_{\cdot[\alpha\beta]})\theta_\rho, \tag{4.187}$$

where $T^\rho_{\alpha\beta}$ are the components of the usual torsion tensor (Eq. (4.107)). Note from this last equation that the operation defined through Eq. (4.186a) does not satisfy the Jacobi identity. Indeed we have:

$$\sum_{[\alpha\beta\sigma]} [\![[\![\theta_\alpha, \theta_\beta]\!], \theta_\sigma]\!] = \sum_{[\alpha\beta\sigma]} (T^{\rho\cdot\cdot}_{\cdot\alpha\mu} - Q^{\rho\cdot\cdot}_{\cdot[\alpha\mu]})(T^{\mu\cdot\cdot}_{\cdot\beta\sigma} - Q^{\mu\cdot\cdot}_{\cdot[\beta\sigma]})\theta_\rho, \tag{4.188}$$

where the summation in this equation is to be performed on the cyclic permutations of the indices α, β and σ.

From Eq. (4.186b), we get:

$$\{\!\{\theta_\alpha, \theta_\beta\}\!\} = (S^{\rho\cdot\cdot}_{\cdot\alpha\beta} - Q^{\rho\cdot\cdot}_{\cdot(\alpha\beta)})\theta_\rho,$$

where $Q^{\rho\cdot\cdot}_{\cdot(\alpha\beta)} := g^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\alpha\sigma})$ and we have written:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta}. \tag{4.189}$$

It can be easily shown that the object having these components is also a tensor. Using the nomenclature of the theories of continuum media [39, 42] we will call it the *strain tensor* of the connection. Note that it can be further decomposed into:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \breve{S}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{2}{n} s^\rho \mathring{g}_{\alpha\beta} \tag{4.190}$$

where $\breve{S}^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ is its traceless part, which will be called the *shear* of the connection, and

$$s^\rho = \frac{1}{2} \mathring{g}^{\mu\nu} S^{\rho\cdot\cdot}_{\cdot\mu\nu} \tag{4.191}$$

is its trace part, which will be called the *dilation* of the connection.

It is trivially established that:

$$L^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{1}{2} T^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \frac{1}{2} S^{\rho\cdot\cdot}_{\cdot\alpha\beta}. \tag{4.192}$$

where $\overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2}(b^{\rho\cdot\cdot}_{\cdot\alpha\beta} + c^{\rho\cdot\cdot}_{\cdot\alpha\beta})$ are the components of the Levi-Civita connection of $\overset{\circ}{\boldsymbol{g}}$.[25]

Equation (4.192) can be used to relate the covariant derivatives with respect to the connections $\overset{\circ}{D}$ and ∇ of any tensor field on the manifold. In particular, recalling that $\overset{\circ}{D}_\alpha \overset{\circ}{g}_{\beta\sigma} = \boldsymbol{e}_\alpha(\overset{\circ}{g}_{\beta\sigma}) - \overset{\circ}{g}_{\mu\sigma}\overset{\circ}{\Gamma}{}^{\mu\cdot\cdot}_{\cdot\alpha\beta} - \overset{\circ}{g}_{\beta\mu}\overset{\circ}{\Gamma}{}^{\mu\cdot\cdot}_{\cdot\alpha\sigma} = 0$, we get the expression of the nonmetricity tensor of ∇ in terms of the torsion and the strain, namely,

$$Q_{\alpha\beta\sigma} = \frac{1}{2}(\overset{\circ}{g}_{\mu\sigma}T^{\mu\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{g}_{\beta\mu}T^{\mu\cdot\cdot}_{\cdot\alpha\sigma}) + \frac{1}{2}(\overset{\circ}{g}_{\mu\sigma}S^{\mu\cdot\cdot}_{\cdot\alpha\beta} + \overset{\circ}{g}_{\beta\mu}S^{\mu\cdot\cdot}_{\cdot\alpha\sigma}). \tag{4.193}$$

Equation (4.193) can be inverted to yield the expression of the strain in terms of the torsion and the nonmetricity. We get:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \overset{\circ}{g}{}^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) - \overset{\circ}{g}{}^{\rho\sigma}(\overset{\circ}{g}_{\beta\mu}T^{\mu\cdot\cdot}_{\cdot\alpha\sigma} + \overset{\circ}{g}_{\sigma\mu}T^{\mu\cdot\cdot}_{\cdot\beta\alpha}). \tag{4.194}$$

From Eqs. (4.193) and (4.194) it is clear that nonmetricity and strain can be used interchangeably in the description of the geometry of a Riemann-Cartan-Weyl space. In particular, we have the relation:

$$Q_{\alpha\beta\sigma} + Q_{\sigma\alpha\beta} + Q_{\beta\sigma\alpha} = S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha}, \tag{4.195}$$

where $S_{\sigma\alpha\beta} = \overset{\circ}{g}_{\rho\sigma}S^{\rho\cdot\cdot}_{\cdot\alpha\beta}$. Thus, the strain tensor of a Weyl geometry satisfies the relation:

$$S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha} = 0.$$

In order to simplify our next equations, let us introduce the notation:

$$K^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} - \overset{\circ}{\Gamma}{}^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2}(T^{\rho\cdot\cdot}_{\cdot\alpha\beta} + S^{\rho\cdot\cdot}_{\cdot\alpha\beta}). \tag{4.196}$$

From Eq. (4.194) it follows that:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} = &-\frac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}(\nabla_\alpha\overset{\circ}{g}_{\beta\sigma} + \nabla_\beta\overset{\circ}{g}_{\sigma\alpha} - \nabla_\sigma\overset{\circ}{g}_{\alpha\beta}) \\ &- \frac{1}{2}\overset{\circ}{g}{}^{\rho\sigma}(\overset{\circ}{g}_{\mu\alpha}T^{\mu\cdot\cdot}_{\cdot\sigma\beta} + \overset{\circ}{g}_{\mu\beta}T^{\mu\cdot\cdot}_{\cdot\sigma\alpha} - \overset{\circ}{g}_{\mu\sigma}T^{\mu\cdot}_{\cdot\alpha\beta}), \end{aligned} \tag{4.197}$$

[25] We note that the possibility of decomposing the connection coefficients into rotation (torsion), shear and dilation has already been suggested in a Physics paper by Baekler et al. [1] but in their work they do not arrive at the identification of a tensor-like quantity associated to these last two objects. The idea of the decompositions already appeared in [40].

where we have used that $Q_{\alpha\beta\sigma} = -\nabla_\alpha \mathring{g}_{\beta\sigma}$. Note the similarity of this equation with that which gives the coefficients of a Riemannian connection (Eq. (4.161)). Note also that for $\nabla\mathring{\boldsymbol{g}} = 0$, $K^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ is the so-called *contorsion tensor*.[26]

Returning to Eq. (4.192), we obtain now the relation between the curvature tensor $R^{\rho\cdots}_{\cdot\mu\alpha\beta}$ associated with the connection ∇ and the Riemann curvature tensor $\mathring{R}^{\rho\cdots}_{\cdot\mu\alpha\beta}$ of the Levi-Civita connection D associated with the metric $\mathring{\boldsymbol{g}}$. We get, by a simple calculation:

$$R^{\rho\cdots}_{\cdot\mu\alpha\beta} = \mathring{R}^{\rho\cdots}_{\cdot\mu\alpha\beta} + J^{\rho\cdots}_{\cdot\mu[\alpha\beta]}, \tag{4.198}$$

where:

$$J^{\rho\cdots}_{\cdot\mu\alpha\beta} = \mathring{D}_\alpha K^{\rho\cdot\cdot}_{\cdot\beta\mu} - K^{\rho\cdot\cdot}_{\cdot\beta\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu} = \nabla_\alpha K^{\rho\cdot\cdot}_{\cdot\beta\mu} - K^{\rho\cdot\cdot}_{\cdot\alpha\sigma} K^{\sigma\cdot}_{\cdot\beta\mu} + K^{\sigma\cdot\cdot}_{\cdot\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\sigma\mu}\ . \tag{4.199}$$

Multiplying both sides of Eq. (4.198) by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ we get:

$$\mathcal{R}^{\rho\cdot}_{\cdot\mu} = \mathring{\mathcal{R}}^{\rho\cdot}_{\cdot\mu} + \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \tag{4.200}$$

where we have written:

$$\mathfrak{J}^{\rho\cdot}_{\cdot\mu} = \frac{1}{2} J^{\rho\cdots}_{\cdot\mu[\alpha\beta]} \theta^\alpha \wedge \theta^\beta. \tag{4.201}$$

From Eq. (4.198) we also get the relation between the Ricci tensors of the connections ∇ and $\mathring{D}$. We define the *Ricci tensor* by

$$\begin{aligned} Ricci &= R_{\mu\alpha} dx^\mu \otimes dx^\alpha, \\ R_{\mu\alpha} &:= R^{\rho\cdots}_{\cdot\mu\alpha\rho}\ . \end{aligned} \tag{4.202}$$

Then, we have

$$R_{\mu\alpha} = \mathring{R}_{\mu\alpha} + J_{\mu\alpha}, \tag{4.203}$$

with

$$\begin{aligned} J_{\mu\alpha} &= \mathring{D}_\alpha K^{\rho\cdot\cdot}_{\cdot\rho\mu} - \mathring{D}_\rho K^{\rho\cdot\cdot}_{\cdot\alpha\mu} + K^{\rho\cdot\cdot}_{\cdot\alpha\sigma} K^{\sigma\cdot\cdot}_{\cdot\rho\mu} - K^{\rho\cdot\cdot}_{\cdot\rho\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu} \\ &= \nabla_\alpha K^{\rho\cdot\cdot}_{\cdot\rho\mu} - \nabla_\rho K^{\rho\cdot\cdot}_{\cdot\alpha\mu} - K^{\rho\cdot\cdot}_{\cdot\sigma\alpha} K^{\sigma\cdot\cdot}_{\cdot\rho\mu} + K^{\rho\cdot\cdot}_{\cdot\rho\sigma} K^{\sigma\cdot\cdot}_{\cdot\alpha\mu}\ . \end{aligned} \tag{4.204}$$

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[26] Equations (4.196) and (4.197) have appeared in the literature in two different contexts: with $\nabla\mathring{\boldsymbol{g}} = 0$, they have been used in the formulations of the theory of the spinor fields in Riemann-Cartan spaces [15, 46] and with $\Theta[\nabla] = 0$ they have been used in the formulations of the gravitational theory in a space endowed with a background metric [8, 13, 23, 35, 36].

Observe that since the connection ∇ is arbitrary, its Ricci tensor will be *not* be symmetric in general. Then, since the Ricci tensor $\mathring{R}_{\mu\alpha}$ of $\mathring{D}$ is necessarily symmetric, we can split Eq. (4.203) into:

$$\begin{aligned} R_{[\mu\alpha]} &= J_{[\mu\alpha]}, \\ R_{(\mu\alpha)} &= \mathring{R}_{\mu\alpha} + J_{(\mu\alpha)}. \end{aligned} \tag{4.205}$$

Now we specialize the above results for the case where the general connection $\nabla = D$ is the Levi-Civita connection of a bilinear form field $\boldsymbol{g} \in \sec T_2^0 M$, i.e., $\boldsymbol{\Theta} = 0$ and $\nabla \boldsymbol{g} = 0$. The results that we show next generalize and clear up those found in the formulations of the gravitational theory in a background metric space [13, 23, 35, 36].

First of all, note that the connection $\mathring{D}$ plays with respect to the tensor field $\mathring{\boldsymbol{g}}$ a role analogous to that played by the connection ∇ with respect to the metric tensor $\boldsymbol{g}$ and in consequence we shall have similar equations relating these two pairs of objects. In particular, the strain of $\mathring{D}$ with respect to $\boldsymbol{g}$ equals the negative of the strain of ∇ with respect to $\mathring{\boldsymbol{g}}$, since we have:

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = -(\mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha} - d^{\rho\cdot\cdot}_{\cdot\alpha\beta}) = S^{\rho\cdot\cdot}_{\cdot\beta\alpha},$$

where $b^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta} + \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\beta\alpha}$ and $d^{\rho\cdot\cdot}_{\cdot\alpha\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha}$ denote the Killing coefficients of the frame with respect to the tensors $\mathring{\boldsymbol{g}}$ and $\boldsymbol{g}$ respectively. Furthermore, in view of Eq. (4.197), we can write $K^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \frac{1}{2} S^{\rho\cdot\cdot}_{\cdot\alpha\beta}$ as:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= -\frac{1}{2} \mathring{g}^{\rho\sigma} (\nabla_\alpha \mathring{g}_{\beta\sigma} + \nabla_\beta \mathring{g}_{\alpha\sigma} - \nabla_\sigma \mathring{g}_{\alpha\beta}) \\ &= \frac{1}{2} g^{\rho\sigma} (\mathring{D}_\alpha g_{\beta\sigma} + \mathring{D}_\beta g_{\alpha\sigma} - \mathring{D}_\sigma g_{\alpha\beta}). \end{aligned} \tag{4.206}$$

Introducing the notation:

$$\varkappa = \sqrt{\frac{\det \boldsymbol{g}}{\det \mathring{\boldsymbol{g}}}}, \tag{4.207}$$

we have the following relations:

$$\begin{aligned} K^{\rho\cdot\cdot}_{\cdot\rho\sigma} &= -\frac{1}{2} \mathring{g}^{\alpha\beta} \nabla_\sigma \mathring{g}_{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \mathring{D}_\sigma g_{\alpha\beta} = \frac{1}{\varkappa} e_\sigma(\varkappa), \\ g^{\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= -\frac{1}{\varkappa} \mathring{D}_\sigma (\varkappa g^{\rho\sigma}), \\ \mathring{g}^{\alpha\beta} K^{\rho\cdot\cdot}_{\cdot\alpha\beta} &= \frac{1}{\varkappa^{-1}} \nabla_\sigma (\varkappa^{-1} \mathring{g}^{\rho\sigma}). \end{aligned} \tag{4.208}$$

Another important consequence of the assumption that ∇ is a Levi-Civita connection is that its Ricci tensor will then be symmetric. In view of Eqs. (4.205), this will be achieved, if and only if, the following equivalent conditions hold:

$$\begin{aligned} \mathring{D}_{\alpha} K^{\rho\cdot\cdot}_{\rho\beta} &= \mathring{D}_{\beta} K^{\rho\cdot\cdot}_{\cdot\rho\alpha}, \\ \nabla_{\alpha} K^{\rho}_{\cdot\rho\beta} &= \nabla_{\beta} K^{\rho\cdot}_{\cdot\rho\alpha}. \end{aligned} \tag{4.209}$$

4.8.9 Structure Equations II

With the results stated above, we can write down the structure equations of the RCWS structure defined by the connection ∇ in terms of the Riemannian structure defined by the metric $\mathring{\boldsymbol{g}}$. For this, let us write Eq. (4.192) in the form:

$$\omega^{\rho\cdot}_{\cdot\beta} = \hat{\omega}^{\rho\cdot}_{\cdot\beta} + w^{\rho\cdot}_{\cdot\beta} = \hat{\omega}^{\rho\cdot}_{\cdot\beta} + \tau^{\rho\cdot}_{\cdot\beta} + \sigma^{\rho\cdot}_{\cdot\beta}, \tag{4.210}$$

with $\omega^{\rho\cdot}_{\cdot\beta} = L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^{\alpha}$, $\hat{\omega}^{\rho\cdot}_{\cdot\beta} = \mathring{\Gamma}^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^{\alpha}$, $w^{\rho\cdot}_{\cdot\beta} = K^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^{\alpha}$, $\tau^{\rho\cdot}_{\cdot\beta} = \frac{1}{2}T^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^{\alpha}$ and $\sigma^{\rho\cdot}_{\cdot\beta} = \frac{1}{2}S^{\rho\cdot\cdot}_{\cdot\alpha\beta}\theta^{\alpha}$. Then, recalling Eq. (4.200) and the structure equations for both the RCWS and the Riemannian structures, we easily conclude that:

$$\begin{aligned} w^{\rho\cdot}_{\cdot\beta} \wedge \theta^{\beta} &= \Theta^{\rho}, \\ w^{\beta\cdot}_{\cdot\mu} \wedge \theta_{\beta} &= -\Phi_{\mu}, \\ \mathring{\mathbf{D}} w^{\rho\cdot}_{\cdot\mu} + w^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta\cdot}_{\cdot\mu} &= \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \end{aligned} \tag{4.211}$$

where $\mathring{\mathbf{D}}$ is the exterior covariant differential (of indexed form fields) associated to the Levi-Civita connection $\mathring{D}$ of $\mathring{\boldsymbol{g}}$. The third of these equations can also be written as:

$$\mathbf{D} w^{\rho\cdot}_{\cdot\mu} - w^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta\cdot}_{\cdot\mu} = \mathfrak{J}^{\rho\cdot}_{\cdot\mu}, \tag{4.212}$$

where $\mathbf{D}$ is the exterior covariant differential (of indexed form fields) associated to the connection ∇.

Now, the *Bianchi identities* for the RCWS structure are easily obtained by differentiating the above equations. We get:

$$\begin{aligned} &\text{(a)}\ \ \mathring{\mathbf{D}}\Theta^{\rho} = \mathfrak{J}^{\rho\cdot}_{\cdot\beta} \wedge \theta^{\beta} - w^{\rho\cdot}_{\cdot\beta} \wedge \Theta^{\beta}, \\ &\text{(b)}\ \ \mathring{\mathbf{D}}\Phi_{\mu} = \mathfrak{J}^{\beta\cdot}_{\cdot\mu} \wedge \theta_{\beta} + w^{\beta\cdot}_{\cdot\mu} \wedge \Phi_{\beta}, \\ &\text{(c)}\ \ \mathring{\mathbf{D}}\mathfrak{J}^{\rho\cdot}_{\cdot\mu} = \mathcal{R}^{\rho\cdot}_{\cdot\beta} \wedge w^{\beta}_{\cdot\mu} - w^{\rho\cdot}_{\cdot\beta} \wedge \mathcal{R}^{\beta\cdot}_{\cdot\mu}, \end{aligned} \tag{4.213}$$

or equivalently,

$$\begin{aligned} \mathbf{D}\Theta^{\rho} &= \mathfrak{J}_{\cdot\beta}^{\rho\cdot} \wedge \theta^{\beta}, \\ \mathbf{D}\Phi_{\mu} &= \mathfrak{J}_{\cdot\mu}^{\beta\cdot} \wedge \theta_{\beta}, \\ \mathbf{D}\mathfrak{J}_{\mu}^{\rho} &= \mathring{\mathcal{R}}_{\cdot\beta}^{\rho\cdot} \wedge w_{\cdot\mu}^{\beta\cdot} - w_{\cdot\beta}^{\rho\cdot} \wedge \mathring{\mathcal{R}}_{\cdot\mu}^{\beta\cdot} . \end{aligned} \tag{4.214}$$

4.8.10 D'Alembertian, Ricci and Einstein Operators

As we have seen in the Sect. 4.8.3 given the structure $(M, \mathring{D}, \mathring{\boldsymbol{g}})$ we can construct the Clifford algebra $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$ and the standard Dirac operator $\boldsymbol{\partial}$ given by (Eq. (4.152))

$$\boldsymbol{\partial} = d - \delta. \tag{4.215}$$

We investigate now the square of the standard Dirac operator. We shall see that this operator can be separated in some interesting parts that are related to the D'Alembertian, Ricci and Einstein operators of $(M, \mathring{D}, \mathring{\boldsymbol{g}})$.

Definition 4.130 The square of standard Dirac operator $\boldsymbol{\partial}$ is the operator, $\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial}$: $\sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}}) \rightarrow \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ given by:

$$\boldsymbol{\partial}^2 = (d - \delta)(d - \delta) = -(d\delta + \delta d). \tag{4.216}$$

We recognize that $\boldsymbol{\partial}^2 \equiv \Diamond$ is the *Hodge Laplacian* of the manifold introduced by (Eq. (4.92)). On the other hand, remembering also that Eq. (4.148)

$$\boldsymbol{\partial} = \theta^{\alpha} \mathring{D}_{\boldsymbol{e}_{\alpha}},$$

where $\{\theta^{\alpha}\}$ is an arbitrary reference frame on the manifold and $\mathring{D}$ is the Levi-Civita connection of the metric $\mathring{\boldsymbol{g}}$, we have:

$$\begin{aligned} \boldsymbol{\partial}^2 &= (\theta^{\alpha} \mathring{D}_{\boldsymbol{e}_{\alpha}})(\theta^{\beta} \mathring{D}_{\boldsymbol{e}_{\beta}}) = \theta^{\alpha}(\theta^{\beta} \mathring{D}_{\boldsymbol{e}_{\alpha}} \mathring{D}_{\boldsymbol{e}_{\beta}} + (\mathring{D}_{\boldsymbol{e}_{\alpha}} \theta^{\beta}) \mathring{D}_{\boldsymbol{e}_{\beta}}) \\ &= \mathring{g}^{\alpha\beta}(\mathring{D}_{\boldsymbol{e}_{\alpha}} \mathring{D}_{\boldsymbol{e}_{\beta}} - \mathring{\Gamma}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \mathring{D}_{\boldsymbol{e}_{\rho}}) + \theta^{\alpha} \wedge \theta^{\beta}(\mathring{D}_{\boldsymbol{e}_{\alpha}} \mathring{D}_{\boldsymbol{e}_{\beta}} - \mathring{\Gamma}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \mathring{D}_{\boldsymbol{e}_{\rho}}). \end{aligned}$$

Then defining the operators:

$$\begin{aligned} &\text{(a)}\ \ \boldsymbol{\partial} \cdot \boldsymbol{\partial} = \mathring{g}^{\alpha\beta}(\mathring{D}_{\boldsymbol{e}_{\alpha}} \mathring{D}_{\boldsymbol{e}_{\beta}} - \mathring{\Gamma}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \mathring{D}_{\boldsymbol{e}_{\rho}}), \\ &\text{(b)}\ \ \boldsymbol{\partial} \wedge \boldsymbol{\partial} = \theta^{\alpha} \wedge \theta^{\beta}(\mathring{D}_{\boldsymbol{e}_{\alpha}} \mathring{D}_{\boldsymbol{e}_{\beta}} - \mathring{\Gamma}_{\cdot\alpha\beta}^{\rho\cdot\cdot} \mathring{D}_{\boldsymbol{e}_{\rho}}), \end{aligned} \tag{4.217}$$

we can write:

$$\diamond = \boldsymbol{\partial}^2 = \boldsymbol{\partial} \cdot \boldsymbol{\partial} + \boldsymbol{\partial} \wedge \boldsymbol{\partial} \tag{4.218}$$

or,

$$\begin{aligned} \boldsymbol{\partial}^2 &= (\boldsymbol{\partial} \lrcorner + \boldsymbol{\partial} \wedge)(\boldsymbol{\partial} \lrcorner + \boldsymbol{\partial} \wedge) \\ &= \boldsymbol{\partial} \lrcorner \boldsymbol{\partial} \wedge + \boldsymbol{\partial} \wedge \boldsymbol{\partial} \lrcorner \,. \end{aligned} \tag{4.219}$$

Remark 4.131 It is important to observe that the operators $\boldsymbol{\partial} \cdot \boldsymbol{\partial}$ and $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ do not have anything analogous in the formulation of the differential geometry in the Cartan and Hodge bundles.

Remark 4.132 Moreover we write for $\omega \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, $\boldsymbol{\partial} \cdot \boldsymbol{\partial} \omega$ and $\boldsymbol{\partial} \wedge \boldsymbol{\partial} \omega$ to mean respectively $(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\omega$ and $(\boldsymbol{\partial} \wedge \boldsymbol{\partial})\omega$. The parenthesis will be included in a formula only if there is a risk of confusion.

The operator $\boldsymbol{\partial} \cdot \boldsymbol{\partial}$ can also be written as:

$$\boldsymbol{\partial} \cdot \boldsymbol{\partial} = \frac{1}{2} \mathring{g}^{\alpha\beta} \left[\mathring{D}_{\boldsymbol{e}_\alpha} \mathring{D}_{\boldsymbol{e}_\beta} + \mathring{D}_{\boldsymbol{e}_\beta} \mathring{D}_{\boldsymbol{e}_\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta} \mathring{D}_{\boldsymbol{e}_\rho} \right]. \tag{4.220}$$

Applying this operator to the 1-forms of the frame $\{\theta^\alpha\}$, we get:

$$\boldsymbol{\partial} \cdot \boldsymbol{\partial} \theta^\mu = -\frac{1}{2} \mathring{g}^{\alpha\beta} \mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta} \theta^\rho, \tag{4.221}$$

where:

$$\mathring{M}^{\mu\cdots}_{\cdot\rho\alpha\beta} = \boldsymbol{e}_\alpha(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) + \boldsymbol{e}_\beta(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma} \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\sigma} \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\rho} - b^{\sigma\cdot\cdot}_{\cdot\alpha\beta} \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho}. \tag{4.222}$$

The proof that an object with these components is a tensor is a consequence of the following proposition:

Proposition 4.133 *For every r-form field* $\omega \in \sec \bigwedge^r T^*M$, $\omega = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, *we have:*

$$\boldsymbol{\partial} \cdot \boldsymbol{\partial} \omega = \frac{1}{r!} \mathring{g}^{\alpha\beta} \mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}. \tag{4.223}$$

Proof We have $\mathring{D}_{\boldsymbol{e}_\beta} \omega = \frac{1}{r!} \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, with

$$\mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} = \boldsymbol{e}_\beta(\omega_{\alpha_1 \ldots \alpha_r}) - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\alpha_1} \omega_{\sigma\alpha_2 \ldots \alpha_r} - \cdots - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\alpha_r} \omega_{\alpha_1 \ldots \alpha_{r-1}\sigma}.$$

Observe moreover that we have

$$\mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} = \boldsymbol{e}_\alpha (\mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r}) - \mathring{\Gamma}^{\sigma \cdot\cdot}_{\cdot \beta \alpha_1} \mathring{D}_\sigma \omega_{\sigma \alpha_2 \ldots \alpha_r}$$
$$- \mathring{\Gamma}^{\sigma \cdot\cdot}_{\cdot \alpha \alpha_1} \mathring{D}_\beta \omega_{\sigma \alpha_2 \ldots \alpha_r} - \ldots \mathring{\Gamma}^{\sigma \cdot\cdot}_{\cdot \alpha \alpha_r} \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_{r-1} \sigma}$$

but

$$D_{\boldsymbol{e}_\alpha} D_{\boldsymbol{e}_\beta} \omega = D_{\boldsymbol{e}_\alpha} (\frac{1}{r!} \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r})$$
$$\frac{1}{r!} (\boldsymbol{e}_\alpha (\mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r}) - \mathring{\Gamma}^{\sigma}_{\alpha \alpha_1} \mathring{D}_\beta \omega_{\sigma \alpha_2 \ldots \alpha_r} - \cdots$$
$$- \mathring{\Gamma}^{\sigma \cdot\cdot}_{\cdot \alpha \alpha_r} \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_{r-1} \sigma}) \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Thus we conclude that:

$$(\mathring{D}_{\boldsymbol{e}_\alpha} \mathring{D}_{\boldsymbol{e}_\beta} - \mathring{\Gamma}^{\rho \cdot\cdot}_{\cdot \alpha \beta} \mathring{D}_{\boldsymbol{e}_\rho}) \omega = \frac{1}{r!} \mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_r}.$$

Finally, multiplying this equation by $\mathring{g}^{\alpha\beta}$ and using the Eq. (4.217a), we get the Eq. (4.223).■

In view of Eq. (4.223), we give the

Definition 4.134 The operator $\square = \partial \cdot \partial$ is called (covariant) *D'Alembertian*.

Note that the D'Alembertian of the 1-forms θ^μ can also be written as:

$$\partial \cdot \partial \theta^\mu = \mathring{g}^{\alpha\beta} \mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho \theta^\rho = \frac{1}{2} \mathring{g}^{\alpha\beta} (\mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho + \mathring{D}_\beta \mathring{D}_\alpha \delta^\mu_\rho) \theta^\rho$$

and therefore, taking into account the Eq. (4.221), we conclude that:

$$\mathring{M}^{\mu \cdots}_{\cdot \rho \alpha \beta} = -(\mathring{D}_\alpha \mathring{D}_\beta \delta^\mu_\rho + \mathring{D}_\beta \mathring{D}_\alpha \delta^\mu_\rho), \tag{4.224}$$

what proves our assertion that $\mathring{M}^{\mu \cdots}_{\cdot \rho \alpha \beta}$ are the components of a tensor.

By its turn, the operator $\partial \wedge \partial$ can also be written as:

$$\partial \wedge \partial = \frac{1}{2} \theta^\alpha \wedge \theta^\beta \left[\mathring{D}_{\boldsymbol{e}_\alpha} \mathring{D}_{\boldsymbol{e}_\beta} - \mathring{D}_{\boldsymbol{e}_\beta} \mathring{D}_{\boldsymbol{e}_\alpha} - c^{\rho \cdot\cdot}_{\cdot \alpha \beta} \mathring{D}_{\boldsymbol{e}_\rho} \right]. \tag{4.225}$$

Applying this operator to the 1-forms of the frame $\{\theta^\mu\}$, we get

$$\partial \wedge \partial \theta^\mu = -\frac{1}{2} \mathring{R}^{\mu \cdots}_{\cdot \rho \alpha \beta} (\theta^\alpha \wedge \theta^\beta) \theta^\rho = -\mathring{\mathcal{R}}^{\rho\mu} \theta_\rho, \tag{4.226}$$

where $\overset{\circ}{R}{}^{\mu\cdots}_{\cdot\rho\alpha\beta}$ are the components of the curvature tensor of the connection $\overset{\circ}{D}$. From Eq. (2.46), we get:

$$\overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho}\theta^{\rho} = \overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho}\llcorner\theta^{\rho} + \overset{\circ}{\mathcal{R}}{}^{\mu\cdot}_{\cdot\rho} \wedge \theta^{\rho}.$$

The second term in the r.h.s. of this equation is identically null because of the Bianchi identity given by Eq.(4.213a) for the particular case of a symmetric connection ($\Theta^{\mu} = 0$). Using Eqs. (2.35) and (2.37) we can write the first term in the r.h.s. as:

$$\begin{aligned}
\overset{\circ}{\mathcal{R}}{}^{\rho\mu}\llcorner\theta^{\rho} &= \frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}(\theta^{\alpha} \wedge \theta^{\beta})\llcorner\theta_{\rho} \\
&= -\frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta_{\rho}\lrcorner(\theta^{\alpha} \wedge \theta^{\beta}) \\
&= -\frac{1}{2}\overset{\circ}{R}{}^{\rho\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}(\delta^{\alpha}_{\rho}\theta^{\beta} - \delta^{\alpha}_{\beta}\theta^{\alpha}) \\
&= -\overset{\circ}{R}{}^{\alpha\mu\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta^{\beta} = \overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}\theta^{\beta},
\end{aligned} \tag{4.227}$$

where $\overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}$ are the components of the Ricci tensor of the Levi-Civita connection $\overset{\circ}{D}$ of $\overset{\circ}{\boldsymbol{g}}$. Thus we have:

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mu} = \overset{\circ}{\mathcal{R}}{}^{\mu}, \tag{4.228}$$

where $\overset{\circ}{\mathcal{R}}{}^{\mu} = \overset{\circ}{R}{}^{\mu\cdot}_{\cdot\beta}\theta^{\beta}$ are the Ricci 1-forms of the manifold. Because of this relation, we give the

Definition 4.135 The operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ is called the *Ricci operator* of the manifold associated to the Levi-Civita connection $\overset{\circ}{D}$ of $\overset{\circ}{\boldsymbol{g}}$.

The proposition below shows that the Ricci operator can be written in a purely algebraic way:

Proposition 4.136 *The Ricci operator* $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ *satisfies the relation:*

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} = \overset{\circ}{\mathcal{R}}{}^{\sigma} \wedge \mathbf{i}_{\sigma} + \overset{\circ}{\mathcal{R}}{}^{\rho\sigma} \wedge \mathbf{i}_{\rho}\mathbf{i}_{\sigma}, \tag{4.229}$$

where (keep in mind) $\overset{\circ}{\mathcal{R}}{}^{\rho\sigma} := \overset{\circ}{g}{}^{\sigma\mu}\overset{\circ}{\mathcal{R}}{}^{\rho\cdot}_{\cdot\mu} = \frac{1}{2}\overset{\circ}{g}{}^{\sigma\mu}\overset{\circ}{R}{}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha\beta}\theta^{\alpha} \wedge \theta^{\beta}$.

Proof The Hodge Laplacian of an arbitrary r-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$ is given by: (e.g., [3]—recall that our definition differs by a sign from that given there) $\Diamond\omega = \eth^2\omega = \frac{1}{r!}(\eth^2\omega)_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with:

$$\begin{aligned}(\Diamond\omega)_{\alpha_1\ldots\alpha_r} &= \mathring{g}^{\alpha\beta}\mathring{D}_\alpha\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r}\\ &\quad-\sum_p(-1)^p\mathring{R}^{\sigma\cdot}_{\cdot\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r}\\ &\quad-2\sum_{\substack{p,q\\ p<q}}(-1)^{p+q}\mathring{R}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha_q\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r},\end{aligned}\tag{4.230}$$

where the notation $\check{\alpha}$ means that the index α was exclude of the sequence.

The first term in the r.h.s. of this expression are the components of the D'Alembertian of the field ω.

Now, recalling that $\mathbf{i}_\sigma\omega = \theta_\sigma\lrcorner\omega$, we obtain:

$$\mathring{\mathcal{R}}^\sigma\wedge\mathbf{i}_\sigma\omega = -\frac{1}{r!}\left[\sum_p(-1)^p\mathring{R}^{\sigma\cdot}_{\cdot\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}$$

and also,

$$\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\omega = -\frac{2}{r!}\left[\sum_{\substack{p,q\\ p<q}}(-1)^{p+q}\mathring{R}^{\rho\sigma\cdot\cdot}_{\cdot\cdot\alpha_q\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}.$$

Hence, taking into account Eq. (4.218), we conclude that:

$$(\eth\wedge\eth)\omega = \mathring{\mathcal{R}}^\sigma\wedge\mathbf{i}_\sigma\omega + \mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\omega,$$

for every r-form field ω.■

Observe that applying the operator given by the second term in the r.h.s. of Eq. (4.229) to the dual of the 1-forms θ^μ, we get:

$$\begin{aligned}\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_\rho\mathbf{i}_\sigma\star\theta^\mu &= \mathring{\mathcal{R}}_{\rho\sigma}\star\theta^\rho\lrcorner(\theta^\sigma\lrcorner\theta^\mu))\\ &= -\mathring{\mathcal{R}}_{\rho\sigma}\wedge\star(\theta^\rho\wedge\theta^\sigma\theta^\mu)\\ &= \star(\mathring{\mathcal{R}}_{\rho\sigma}\lrcorner(\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu)),\end{aligned}\tag{4.231}$$

where we have used the Eq. (2.77). Then, recalling the definition of the curvature forms and using the Eq. (2.36), we conclude that:

$$\mathring{\mathcal{R}}^{\rho\sigma}\wedge\theta_\rho\lrcorner\theta_\sigma\lrcorner\star\theta^\mu = -2\star(\mathring{\mathcal{R}}^\mu - \frac{1}{2}\mathring{R}\theta^\mu) = -2\star\mathring{\mathcal{G}}^\mu,\tag{4.232}$$

where $\mathring{R}$ is the scalar curvature of the manifold and the $\mathring{\mathcal{G}}^{\mu}$ may be called the Einstein 1-form fields. That observation motivate us to give the

Definition 4.137 The *Einstein operator* of the Levi-Civita connection $\mathring{D}$ of $\mathring{\boldsymbol{g}}$ on the manifold M is the mapping $\blacksquare$: $\sec \mathcal{C}\ell(M, \mathring{g}) \to \sec \mathcal{C}\ell(M, \mathring{g})$ given by:

$$\blacksquare = -\frac{1}{2} \star^{-1} (\mathring{\mathcal{R}}^{\rho\sigma} \wedge \mathbf{i}_{\rho}\mathbf{i}_{\sigma}) \star . \tag{4.233}$$

Obviously, we have:

$$\blacksquare\theta^{\mu} = \mathring{\mathcal{G}}^{\mu} = \mathring{\mathcal{R}}^{\mu} - \frac{1}{2}\mathring{R}\theta^{\mu}. \tag{4.234}$$

In addition, it is easy to verify that $\star^{-1}(\boldsymbol{\partial}\wedge \boldsymbol{\partial})\star = -\boldsymbol{\partial}\wedge \boldsymbol{\partial}$ and $\star^{-1}(\mathring{\mathcal{R}}^{\sigma} \wedge \mathbf{i}_{\sigma})\star = \mathring{\mathcal{R}}^{\sigma} \lrcorner \mathbf{j}_{\sigma}$. Thus we can also write the Einstein operator as:

$$\blacksquare = \frac{1}{2}(\boldsymbol{\partial}\wedge \boldsymbol{\partial} - \mathring{\mathcal{R}}^{\sigma} \lrcorner \mathbf{j}_{\sigma}). \tag{4.235}$$

Another important result is given by the following proposition:

Proposition 4.138 *Let* $\mathring{\omega}^{\mu\cdot}_{\cdot\rho}$ *be the Levi-Civita connection 1-forms fields in an arbitrary moving frame* $\{\theta^{\mu}\}$ *on* $(M, \mathring{D}, \mathring{g})$*. Then:*

$$\begin{aligned} &(a)\ \ \boldsymbol{\partial}\cdot \boldsymbol{\partial}\theta^{\mu} = -(\boldsymbol{\partial}\cdot\mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma}_{\rho} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho} \\ &(b)\ \boldsymbol{\partial}\wedge \boldsymbol{\partial}\theta^{\mu} = -(\boldsymbol{\partial}\wedge\mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \wedge \mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho}, \end{aligned} \tag{4.236}$$

that is,

$$\boldsymbol{\partial}^{2}\theta^{\mu} = -(\boldsymbol{\partial}\mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho}\mathring{\omega}^{\mu\cdot}_{\cdot\sigma})\theta^{\rho}. \tag{4.237}$$

Proof We have

$$\begin{aligned} \boldsymbol{\partial}\cdot\mathring{\omega}^{\mu}_{\rho} &= \theta^{\alpha} \cdot \mathring{D}_{e_{\alpha}}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}\theta^{\beta}) \\ &= \theta^{\alpha} \cdot (\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho})\theta^{\beta} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\theta^{\beta}) \\ &= \mathring{g}^{\alpha\beta}(\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu}_{\beta\rho}) - \mathring{\Gamma}^{\mu}_{\sigma\rho}\mathring{\Gamma}^{\sigma}_{\alpha\beta}) \end{aligned}$$

and $\mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \cdot \mathring{\omega}^{\mu\cdot}_{\cdot\sigma} = (\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho}\theta^{\beta}) \cdot (\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\theta^{\alpha}) = \mathring{g}^{\beta\alpha}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho}$. Then,

$$\begin{aligned} &-(\boldsymbol{\partial}\cdot\mathring{\omega}^{\mu\cdot}_{\cdot\rho} - \mathring{\omega}^{\sigma\cdot}_{\cdot\rho} \cdot \mathring{\omega}^{\mu\cdot}_{\upsilon\sigma})\theta^{\rho} \\ &= \mathring{g}^{\alpha\beta}(\boldsymbol{e}_{\alpha}(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho})\theta^{\rho} \end{aligned}$$

$$= -\frac{1}{2}\mathring{g}^{\alpha\beta}(\boldsymbol{e}_\alpha(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\rho}) + \boldsymbol{e}_\beta(\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\rho}) - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\alpha\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\beta\rho} - \mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\beta\sigma}\mathring{\Gamma}^{\sigma\cdot\cdot}_{\cdot\alpha\rho} - b^{\sigma\cdot\cdot}_{\cdot\alpha\beta}\mathring{\Gamma}^{\mu\cdot\cdot}_{\cdot\sigma\rho})\theta^\rho$$
$$= \mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}}\theta^\mu.$$

Equation (4.236b) is proved analogously.■

Exercise 4.139 Show that $-(\theta_\rho \wedge \theta_\sigma)\lrcorner\mathring{\mathcal{R}}^{\rho\sigma} = \mathring{R}(\theta_\rho \wedge \theta_\sigma)\cdot\mathring{\mathcal{R}}^{\rho\sigma} = \mathring{R}$, where $\mathring{R}$ is the curvature scalar.

4.8.11 *The Square of a General Dirac Operator*

Consider the structure $(M, \nabla, \mathring{\boldsymbol{g}})$, where ∇ is an arbitrary Riemann-Cartan-Weyl connection and the Clifford algebra $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$. Let us now compute the square of the (general) Dirac operator $\boldsymbol{\partial} = \mathrm{tr}(u\nabla_{\boldsymbol{u}})$. As in the earlier section, we have, by one side,

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)(\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)\\ &= \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge\end{aligned}$$

and we write $\boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner \equiv \boldsymbol{\partial}^2\lrcorner$, $\boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge \equiv \boldsymbol{\partial}^2\wedge$ and

$$\mathcal{L}_+ = \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner, \tag{4.238}$$

so that:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2\lrcorner\boldsymbol{\partial} + \mathcal{L}_+\boldsymbol{\partial} + \boldsymbol{\partial}^2\wedge\ . \tag{4.239}$$

The operator $\mathcal{L}_+$ when applied to scalar functions corresponds, for the case of a Riemann-Cartan space, to the wave operator introduced in [30]. Obviously, for the case of the standard Dirac operator, $\mathcal{L}_+$ reduces to the usual Hodge Laplacian of the manifold, which preserve graduation of forms.

Now, a similar calculation for the product $\boldsymbol{\partial}\overset{\blacklozenge}{\boldsymbol{\partial}}$ of the Dirac derivative and the Dirac coderivative yields:

$$\boldsymbol{\partial}\overset{\blacklozenge}{\boldsymbol{\partial}} = \boldsymbol{\partial}\lrcorner\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner + \mathcal{L}_- + \boldsymbol{\partial}\wedge\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge, \tag{4.240}$$

with

$$\mathcal{L}_- = \boldsymbol{\partial}\lrcorner\overset{\blacklozenge}{\boldsymbol{\partial}}\wedge + \boldsymbol{\partial}\wedge\cdot\overset{\blacklozenge}{\boldsymbol{\partial}}\lrcorner\,. \tag{4.241}$$

On the other hand, we have also:

$$\begin{aligned}\blacklozenge &= (\theta^\alpha \nabla_{\boldsymbol{e}_\alpha})(\theta^\beta \nabla_{\boldsymbol{e}_\beta}) = \theta^\alpha(\theta^\beta \nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} + (\nabla_{\boldsymbol{e}_\alpha}\theta^\beta)\nabla_{\boldsymbol{e}_\beta}) \\ &= \mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) + \theta^\alpha \wedge \theta^\beta(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho})\end{aligned}$$

and we can then define:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \theta^\alpha \wedge \theta^\beta(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho})\end{aligned} \tag{4.242}$$

in order to have:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial} = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\ . \tag{4.243}$$

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \frac{1}{2}\theta^\alpha\cdot\theta^\beta(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) + \frac{1}{2}\theta^\beta\cdot\theta^\alpha(\nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - L^{\rho\cdot\cdot}_{\cdot\beta\alpha}\nabla_{\boldsymbol{e}_\rho}) \\ &= \frac{1}{2}\mathring{g}^{\alpha\beta}[\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} + \nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - (L^{\rho\cdot\cdot}_{\cdot\alpha\beta} + L^{\rho\cdot\cdot}_{\cdot\beta\alpha})\nabla_{\boldsymbol{e}_\rho}]\end{aligned}$$

or,

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}\mathring{g}^{\alpha\beta}(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} + \nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - b^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) - s^\rho\nabla_{\boldsymbol{e}_\rho}, \tag{4.244}$$

where s^ρ has been defined in Eq. (4.191).

By its turn, the operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \frac{1}{2}\theta^\alpha\wedge\theta^\beta(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - L^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) + \frac{1}{2}\theta^\beta\wedge\theta^\alpha(\nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - L^{\rho\cdot\cdot}_{\cdot\beta\alpha}\nabla_{\boldsymbol{e}_\rho}) \\ &= \frac{1}{2}\theta^\alpha\wedge\theta^\beta[\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - \nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - (L^{\rho\cdot\cdot}_{\cdot\alpha\beta} - L^{\rho\cdot\cdot}_{\cdot\beta\alpha})\nabla_{\boldsymbol{e}_\rho}]\end{aligned} \tag{4.245}$$

or,

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial} = \frac{1}{2}\theta^\alpha\wedge\theta^\beta(\nabla_{\boldsymbol{e}_\alpha}\nabla_{\boldsymbol{e}_\beta} - \nabla_{\boldsymbol{e}_\beta}\nabla_{\boldsymbol{e}_\alpha} - c^{\rho\cdot\cdot}_{\cdot\alpha\beta}\nabla_{\boldsymbol{e}_\rho}) - \Theta^\rho\nabla_{\boldsymbol{e}_\rho}. \tag{4.246}$$

Exercise 4.140 Prove that the Ricci and Einstein operators are (1, 1)-extensor fields on a Lorentzian spacetime, i.e., for any $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned}\boldsymbol{\partial}\wedge\boldsymbol{\partial}\, A &= \boldsymbol{\partial}\wedge\boldsymbol{\partial}\,(A_\mu\theta^\mu) = A_\mu\boldsymbol{\partial}\wedge\boldsymbol{\partial}\theta^\mu, \\ \blacksquare A &= \blacksquare(A_\mu\theta^\mu) = A_\mu\blacksquare\theta^\mu.\end{aligned} \tag{4.247}$$

Solution We prove the first formula, since after proving it the second one is obvious. We choose for simplicity an orthonormal cobasis $\{\theta^{\mathbf{a}}\}$ for T^*M dual to the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$ for TM, such that $[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}] = c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}$. Let ∇ be a connection on a Riemann-Cartan-Weyl spacetime, such that $\nabla_{\boldsymbol{e}_{\mathbf{a}}}\boldsymbol{e}_{\mathbf{b}} = L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}$. Recalling (Eq. (4.245)) we have

$$\begin{aligned}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A &= \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\{[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}](A_{\mathbf{k}}) - L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}(A_{\mathbf{k}}) - L_{\cdot\mathbf{ba}}^{\mathbf{d}\cdot\cdot}\boldsymbol{e}_{\mathbf{d}}(A_{\mathbf{k}})\}\theta^{\mathbf{k}}\} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}} \\ &= \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\{c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot} - L_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot} - L_{\cdot\mathbf{ba}}^{\mathbf{d}\cdot\cdot}\}\theta^{\mathbf{k}} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}} \\ &= \frac{1}{2}T_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} + A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}} = A_{\mathbf{k}}\boldsymbol{\partial} \wedge \boldsymbol{\partial}\theta^{\mathbf{k}},\end{aligned}$$

since for a Lorentzian spacetime the torsion tensor (with components $T_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}$) is null.

Exercise 4.141 Show that for any $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A = \mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}}\, A + \mathbf{J}^{\alpha} \cdot \theta_{\alpha}\check{A}, \tag{4.248}$$

where $\check{A} := \check{A}_{\sigma}\theta^{\sigma}$, $\check{A}_{\kappa} := \mathring{g}_{\beta\kappa}g^{\beta\sigma}A_{\sigma}$ and $\mathbf{J}^{\alpha} := \mathring{g}^{\alpha\beta}J_{\beta\sigma}\theta^{\sigma}$, where $J_{\beta\sigma}$ is given by

4.9 Some Applications

4.9.1 Maxwell Equations in the Hodge Bundle

The system of Maxwell equations has many faces.[27] Here we show how to express that system of equations in the Hodge bundle and then in the Clifford bundle. To start, let $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$ be an oriented Lorentzian manifold.

Maxwell equations on $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}})$ refers to an exterior system of differential equations given by a closed 2-form $F \in \sec \bigwedge^2 T^*M$ and a exact 3-form $\boldsymbol{J}_e \in \sec \bigwedge^3 T^*M$. Then there exists $G \in \sec \bigwedge^2 T^*M$ such that

$$dF = 0 \text{ and } dG = -\boldsymbol{J}_e. \tag{4.249}$$

It is postulated that in vacuum there is a relation between G and F (said constitutive relation) given by

$$G = \star F. \tag{4.250}$$

[27] Besides the ones presented in this chapter, others will be exhibited in Chap. 13.

In that case putting $\boldsymbol{J}_e = \star J_e$, $J_e \in \sec \bigwedge^1 T^*M$ and taking into account Eq. (4.91) we can write the system (4.249) as[28]

$$dF = 0 \text{ and } \delta F = -J_e. \tag{4.251}$$

F is called the Faraday field and J_e is called the electric current.

4.9.2 Charge Conservation

Of course, $\delta J_e = 0$, which means that charge is conserved. Indeed, let C_3 be a three dimensional volume contained in a space slice, i.e., in a spacelike surface. Then the electric flux contained in $C_2 = \partial C_3$ is

$$Q = \int_{C_3} \star J_e = -\int_{C_3} dG = -\int_{\partial C_3} \star F. \tag{4.252}$$

It is an empirical fact that all observable *free* charges are integer multiple of the electron charge. This phenomenon is called *charge quantization*. On the other hand consider a 4-volume C_4 with boundary given by $\partial C_4 = C_3^{(2)} - C_3^{(1)} + S$ where with the condition $J_e|_S = 0$ and where $C_3^{(2)}$ and $C_3^{(1)}$are three dimensional volumes contained in two different space slices. Then,

$$\int_{\partial C_4} \star J_e = \int_{\partial C_4} dG = \int_{C_4} d^2G = 0, \tag{4.253}$$

from where it follows that

$$\int_{C_3^{(1)}} \star J_e = \int_{C_3^{(2)}} \star J_e. \tag{4.254}$$

We postulate that F is closed but it may be (eventually) not exact. In that case it may have period integrals according to de Rham theorem, i.e.,

$$\int_{z_2^{(i)}} F = g_{(i)}, \tag{4.255}$$

where $z_2^{(i)} \in H_2(M)$ are cycles. It seems to be an empirical fact that all $g_{(i)} = 0$, at least for cycles in the region of the universe where men already did experiments. This means that F is exact, i.e., it is possible to define globally a differentiable

[28]Thirring [44] said that the two equations in Eq. (4.251) is the twentieth Century presentation of Maxwell equations.

potential $A \in \sec \bigwedge^1 T^*M$ such that $F = dA$. This also means that there are no magnetic monopoles in nature.[29] Indeed, if z_2 is a cycle (a closed surface) then we have

$$\int_{z_2} F = \int_{z_2} dA = \langle \partial z_2, A \rangle = \langle 0, A \rangle = 0. \tag{4.256}$$

4.9.3 Flux Conservation

Of course, A is only defined modulus a gauge, i.e., $A + A'$, with $A' \in \sec \bigwedge^1 T^*M$ a closed form. The period integrals of A' according to de Rham theorem are

$$\int_{z_i^{(i)}} A' = \Phi_{(i)}. \tag{4.257}$$

Now, it is an empirical fact that $\Phi_{(i)}$ is quantized in some (*but not all*) physical systems, like, e.g., in superconductors [16]. The phenomenon is then called flux quantization. In appropriate units

$$\int_{z_1} A' = nh/2e, \tag{4.258}$$

where n is an integer and h is Planck constant and e is the electron charge.

Note also that from $\boldsymbol{J}_e = -dG$ in Eq. (4.249) it follows that G is defined also only modulus a closed form G'. The period integrals of G' may eventually correspond to topological charges. Another possibility of having 'charge without charge' coming from statistical distributions of quantized flux loops has been investigated in [18, 19]. We shall not discuss these interesting issues in this book.

4.9.4 Quantization of Action

Finally we mention the following. As we shall see in Chap. 7 the Lagrangian *density* of the electromagnetic field in *free space* is given by

$$\mathcal{L}(A) = -\frac{1}{2} F \wedge \star F. \tag{4.259}$$

[29]See however the news in [31] where it is claimed that magnetic monopoles have been observed in a synthetic magnetic field.

Calling $\boldsymbol{K} = A \wedge \star F$, we can write

$$\mathcal{L}(A) = -\frac{1}{2} d\boldsymbol{K}. \tag{4.260}$$

Now, it seems an empirical fact that action is quantized, i.e., we have

$$\begin{aligned} a &= \int_{C_4} \mathcal{L}(A) \\ &= \int_{C_3 = \partial C_4} \boldsymbol{K} = nh. \end{aligned} \tag{4.261}$$

Remark 4.142 We observe that $\int_{C_3=\partial C_4} \boldsymbol{K}$ has been introduced by Kiehn (see [20]). However he called $A \wedge \star F$ the topological spin, which is not a good name (and identification of observable) in our opinion. The reason is that according to the Lagrangian formalism (see Chap. 8, Eq. (8.124))[30] the spin density is proportional to $A \wedge F$. This result and the other period integrals discussed above suggests that quantization may be linked to topology in a way not suspected by contemporary physicists. On this issue, see also [29].

4.9.5 A Comment on the Use of de Rham Pseudo-Forms and Electromagnetism

Besides the forms we have been working until now, in a famous book, de Rham [6] introduces also the concept of *impair* forms[31] in a n-dimensional manifold M, which is essential for the formulation of a theory of integration in a non orientable manifold.

Definition 4.143 An impair p-form in M is a pair of p-forms such that if its representative in a given $\mathfrak{A} \subset M$ in a cobasis $\{\theta^i\}$ for T^*U ($U \supset \mathfrak{A}$) is declared as being

$$\omega|_U = \frac{1}{p!} \omega_{i_1 \dots i_p} \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \in \sec \bigwedge^p T^*M$$

[30] See also [7].

[31] Also called by some authors pseudo forms.

then its representative $\omega|_V$ in $\mathfrak{A} \subset V \subset M$ in a cobasis $\{\bar{\theta}^i\}$, $\bar{\theta}^i = \Lambda^i_j \theta^j$, for T^*V ($V \cap U \supset \mathfrak{A}$) is

$$\omega|_V = \frac{1}{p!}\bar{\omega}_{j_1 ..j_p}\bar{\theta}^{i_1} \wedge \cdots \wedge \bar{\theta}^{i_p} \in \sec \bigwedge^p T^*M, \tag{4.262}$$

with

$$\bar{\omega}_{j_1 ..j_p} = \frac{\det\left[\Lambda^i_j\right]}{\left|\det\left[\Lambda^i_j\right]\right|}\omega_{i_1 ...i_p}\Lambda^{i_1}_{j_1} \cdots \Lambda^{i_1}_{j_1}. \tag{4.263}$$

The introduction of impair forms leads to the question of exterior (and interior) multiplication of forms of different parities (i.e., even and odd). The rule introduced by de Rham [6] is that the product of two forms of the same parity is a form, whereas the product of two forms of different parities is an impair form. Also de Rham introduces the rule that application of the differential operator d to a form preserves its parity.

We can verify that if we denote by $\bigwedge_{impair} T^*M = \sum_{p=0}^{n} \bigwedge^p_{impair} T^*M$ the real vector space of the pseudo forms we can give a structure of associative algebra to the (exterior) direct sum $\bigwedge T^*M \oplus \bigwedge_{impair} T^*M$ equipped with the exterior product satisfying the de Rham rules mentioned above.

Having introduced the concept of de Rham pseudo forms we call the reader's attention to the following remarks.

Remark 4.144 In our brief presentation above of Maxwell equations we introduced the electromagnetic current as $J_e = \star^{-1}\boldsymbol{J}_e$, $J_e \in \sec \bigwedge^1 T^*M$. Since until that point we have not introduced the concept of impair forms its is clear that we supposed that $\boldsymbol{J}_e$ is 3-form. This certainly means that the theory as presented presupposes that we use always bases with the same orientation in order to calculate the charge in a certain three dimensional volume contained in a given space slice (Eq. (4.253)). The use of bases with the same orientation presupposes that spacetime is an orientable manifold. As will be discussed in Chap. 7 orientability of a spacetime manifold is a necessary condition for the existence of spinor fields. Since these objects seems to be an essential tool for the understanding of the world we live in, we restrict all our considerations to orientable manifolds. Eventually, if is discovered some of these days that our universe cannot be represented by an orientable manifold, then it will be necessary to study deeply the theory of impair forms.

Remark 4.145 If the spacetime manifold is orientable we do not need to consider, as some authors claim (e.g., [20, 29]) that $\boldsymbol{J}_e$ and G must be considered as pseudo forms. A thoughtful discussion of this issue may be found in [5].

4.9.6 Maxwell Equation in the Clifford Bundle

Let now, $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ be a Lorentzian spacetime and let $\mathcal{C}\ell(M, \mathtt{g})$ be the Clifford bundle of differential forms. Since D is the Levi-Civita connection of $\boldsymbol{g}$ we know (Eq. (4.152)) that the action of the Dirac operator $\boldsymbol{\partial}$ on any $P \in \sec \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ is $\boldsymbol{\partial} P = (d - \delta)P$. So, let us suppose that the Faraday field and the electric current are sections of the Clifford bundle, i.e., $F \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, $J_e \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$. In that case, it is licit two sum the equations $dF = 0$ and $\delta F = -J_e$, which according to Eq. (4.251) represent the system of Maxwell equations in the Hodge bundle. We get, of course, the single equation

$$\boldsymbol{\partial} F = J_e, \tag{4.264}$$

which we be call *Maxwell equation*. Parodying Thirring [44] we may say that Eq. (4.264) the twenty-first century representation of Maxwell system of equations.

Exercise 4.146 Show that in Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ (Definition 4.108) Eq. (4.264) is equivalent to the standard vector form of Maxwell equations, that appears in elementary electrodynamics textbooks.

Solution We recall (see Table 3.1 in Chap. 3) that for any $x \in M$, $\mathcal{C}\ell(T_x^*M, \eta_x) \simeq \mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, is the so called spacetime algebra. The even elements of $\mathbb{R}_{1,3}$ close a subalgebra called the Pauli algebra. That subalgebra is denoted by $\mathbb{R}_{1,3}^0 \simeq \mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. Also, $\mathbb{H}(2)$ is the algebra of the 2×2 quaternionic matrices and $\mathbb{C}(2)$ is the algebra of the 2×2 complex matrices. As in Sect. 3.9.1 a convenient isomorphism $\mathbb{R}_{1,3}^0 \approx \mathbb{R}_{3,0}$ is easily exhibited. Choose a global orthonormal tetrad coframe $\{\gamma^\mu\}$, $\gamma^\mu = dx^\mu$, $\mu = 0, 1, 2, 3$, and let $\{\gamma_\mu\}$ be the reciprocal tetrad of $\{\gamma^\mu\}$, i.e., $\gamma_\nu \cdot \gamma^\mu = \delta_\nu^\mu$. Now, put

$$\sigma_i = \gamma_i \gamma_0, \; \mathbf{i} = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^5. \tag{4.265}$$

Observe that $\mathbf{i}$ commutes with bivectors and thus *acts* like the imaginary unity $\mathrm{i} = \sqrt{-1}$ in the subbundle $\mathcal{C}\ell^0(M, \eta) = \bigcup_{x \in M} \mathcal{C}\ell^0(T_x^*M, \eta_x) \hookrightarrow \mathcal{C}\ell(M, \eta)$, which we call *Pauli bundle*. Now, the electromagnetic field is represented in $\mathcal{C}\ell(M, \eta)$ by $F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta))$ with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{4.266}$$

where (E_1, E_2, E_3) and (B_1, B_2, B_3) are the *usual* Cartesian components of the electric and magnetic fields. Then, as it is easy to verify we can write

$$F = \vec{E} + \mathbf{i}\vec{B}, \tag{4.267}$$

with , $\vec{E} = \sum_{i=1}^3 E_i \sigma_i$, $\vec{B} = \sum_{i=1}^3 B_i \sigma_i$.

For the electric current density $J_e = \rho\gamma^0 + J^i\gamma_i$ we can write

$$\gamma_0 J_e = \rho - \vec{j} = \rho - J^i\sigma_i. \tag{4.268}$$

For the Dirac operator we have

$$\gamma_0 \boldsymbol{\partial} = \frac{\partial}{\partial x^0} + \sum_{i=1}^{3} \sigma_i \partial_i = \frac{\partial}{\partial t} + \nabla. \tag{4.269}$$

Multiplying both members of Eq. (4.264) on the left by γ_0 we obtain

$$\begin{aligned} \gamma_0 \boldsymbol{\partial} F &= \gamma_0 J_e, \\ (\frac{\partial}{\partial t} + \nabla)(\vec{E} + \mathbf{i}\vec{B}) &= \rho - \vec{j} \end{aligned} \tag{4.270}$$

From Eq. (4.270) we obtain

$$\partial_0 \vec{E} + \mathbf{i}\partial_0 \vec{B} + \nabla \cdot \vec{E} + \nabla \wedge \vec{E} + \mathbf{i}\nabla \cdot \vec{B} + \mathbf{i}\nabla \wedge \vec{B} = \rho - \vec{j}. \tag{4.271}$$

For any 'vector field' $\vec{A} \in \mathcal{C}\ell^0(M, \eta) \hookrightarrow \mathcal{C}\ell(M, \eta)$ we define the *rotational operator* $\nabla \times$ by

$$\nabla \times \vec{A} = -\mathbf{i}\nabla \wedge \vec{A}. \tag{4.272}$$

This relation follows once we realize that the usual *vector* product of two vectors $\vec{a} = \sum_{i=1}^{3} a_i\sigma_i$ and $\vec{b} = \sum_{i=1}^{3} b_i\sigma_i$ can be identified with the dual of the bivector $\vec{a} \wedge \vec{b}$ through the formula $\vec{a} \times \vec{b} = -\mathbf{i}\vec{a} \wedge \vec{b}$. Finally we obtain from Eq. (4.271) by equating terms with the same grades (in the Pauli subbundle)

$$\begin{array}{ll} \text{(a)} \quad \nabla \cdot \vec{E} = \rho, & \text{(b)} \; \nabla \times \vec{B} - \partial_0 \vec{E} = \vec{j}, \\ \text{(c)} \; \nabla \times \vec{E} + \partial_0 \vec{B} = 0, & \text{(d)} \quad \nabla \cdot \vec{B} = 0, \end{array} \tag{4.273}$$

which we recognize as the system of Maxwell equations in the usual vector notation.

We just exhibit three equivalent presentations of Maxwell systems of equations, namely Eqs. (4.251), (4.264), and (4.273). They are some of the many faces of Maxwell equations. Other faces exist as we shall see in Chap. 11.

4.9.7 Einstein Equations and the Field Equations for the $\theta^{\mathbf{a}}$

As, it is the case of Maxwell equations, also Einstein equations have many faces. Here we exhibit an interesting one which is possible once we have at our disposal the Clifford bundle formalism. So, let now $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ be a Lorentzian spacetime (Definition 4.107) modelling a gravitational field in the general theory of Relativity [38]. Let $\{\boldsymbol{e}_{\mathbf{a}}\}$ be an arbitrary orthonormal basis of TU (a tetrad[32]) and $\{\theta^{\mathbf{b}}\}$ of T^*M its dual basis (a cotetrad), with $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$. We recall that Einstein's equations relating the distribution of matter energy represented by the energy-momentum tensor $\boldsymbol{T} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \otimes \boldsymbol{e}_{\mathbf{a}} \in \sec T^1_1 U \subset \sec T^1_1 M$ can be written (in appropriated units)

$$R^{\mathbf{a}}_{\mathbf{b}} - \frac{1}{2}\delta^{\mathbf{a}}_{\mathbf{b}} R = -T^{\mathbf{a}}_{\mathbf{b}}, \tag{4.274}$$

where $R^{\mathbf{a}}_{\mathbf{b}}$ is the Ricci tensor and R is the scalar curvature. Multiplying both members of Eq. (4.274) by $\theta^{\mathbf{b}}$ and taking into account Eq. (4.228) defining the Ricci 1-forms in terms of the Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ (with $\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\boldsymbol{e}_{\mathbf{a}}}$) we can write after some trivial algebra

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ \theta^{\mathbf{a}} + \frac{T}{2}\theta^{\mathbf{a}} = -T^{\mathbf{a}}, \tag{4.275}$$

where[33] $T^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g})$ are the energy-momentum 1-form fields and $T = T^{\mathbf{a}}_{\mathbf{a}}$.

Now, taking into account Eqs. (4.218) and (4.219) we can write

$$-\boldsymbol{\partial} \cdot \boldsymbol{\partial}\ \theta^{\mathbf{a}} + \boldsymbol{\partial} \wedge (\boldsymbol{\partial} \cdot \theta^{\mathbf{a}}) + \boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \theta^{\mathbf{a}}) + \frac{1}{2} T\theta^{\mathbf{a}} = -T^{\mathbf{a}}. \tag{4.276}$$

Now, let $\{x^\mu\}$ be the coordinate functions of a local chart of the maximal atlas of M covering $U \subset M$. When $\theta^{\mathbf{a}}$ is an exact differential, and in that case we write $\theta^{\mathbf{a}} \mapsto \theta^\mu = dx^\mu$ and if the coordinate functions are *harmonic* [10], i.e., $\delta\theta^\mu = -\boldsymbol{\partial} \lrcorner \theta^\mu = g^{\alpha\beta}\Gamma^{\mu\cdot\cdot}_{\cdot\alpha\beta} = 0$, Eq. (4.276) becomes

$$\Box\theta^\mu - \frac{1}{2}R\theta^\mu = T^\mu, \tag{4.277}$$

where $\Box$ is the covariant D'Alembertian operator (Definition 4.134).

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[32]We shall see in Chap. 6 that any Lorentzian spacetime admitting spinor fields must have a global tetrad.

[33]Sometimes in the written of some formulas in the next chapters it is convenient to use the notation $\mathcal{T}^{\mathbf{a}} = -T^{\mathbf{a}}$.

4.9.8 Curvature of a Connection and Bending. The Nunes Connection of $\mathring{S}^2$

Consider the manifold $\mathring{S}^2 = \{S^2 \backslash \text{north pole + south pole}\} \subset \mathbb{R}^3$, it is an sphere of radius $\mathfrak{R} = 1$ excluding the north and south poles. Let $\boldsymbol{g} \in \sec T_2^0 \mathring{S}^2$ be a metric field for $\mathring{S}^2$, which is the pullback on it of the metric of the ambient space $\mathbb{R}^3$. Now, consider two different connections on $\mathring{S}^2$, D—the Levi-Civita connection— and $\boldsymbol{\nabla}^c$, a connection—here called the Nunes[34] (or navigator) connection[35]— defined by the following parallel transport rule: a vector is parallel transported along a curve, if at any $x \in S^2$ the angle between the vector and the vector tangent to the latitude line passing through that point is constant during the transport (see Fig. 4.5).

Exercise 4.147 (i) Show that the structure $(\mathring{S}^2, \boldsymbol{g}, D)$ is a Riemann geometry of constant curvature and;

(ii) that the structure $(\mathring{S}^2, \boldsymbol{g}, \boldsymbol{\nabla}^c)$ is a teleparallel geometry, with zero Riemann curvature tensor, but non zero tensor.

Solution The first part of the exercise is a standard one and can be found in many good textbooks on differential geometry. Here, we only show (ii). We clearly see from Fig. 4.5a that if we transport a vector along the infinitesimal quadrilateral *pqrs* composed of latitudes and longitudes, first starting from p along *pqr* and then starting from p along *psr* the parallel transported vectors that result in both cases will coincide. Using the definition of the Riemann curvature tensor, we see that it is null. So, we see that $\mathring{S}^2$ considered as part of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla^c)$ is flat!

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[34]Pedro Salacience Nunes (1502–1578) was one of the leading mathematicians and cosmographers of Portugal during the Age of Discoveries. He is well known for his studies in Cosmography, Spherical Geometry, Astronomic Navigation, and Algebra, and particularly known for his discovery of loxodromic curves and the nonius. Loxodromic curves, also called rhumb lines, are spirals that converge to the poles. They are lines that maintain a fixed angle with the meridians. In other words, loxodromic curves directly related to the construction of the Nunes connection. A ship following a fixed compass direction travels along a loxodromic, this being the reason why Nunes connection is also known as navigator connection. Nunes discovered the loxodromic lines and advocated the drawing of maps in which loxodromic spirals would appear as straight lines. This led to the celebrated Mercator projection, constructed along these recommendations. Nunes invented also the Nonius scales which allow a more precise reading of the height of stars on a quadrant. The device was used and perfected at the time by several people, including Tycho Brahe, Jacob Kurtz, Christopher Clavius and further by Pierre Vernier who in 1630 constructed a practical device for navigation. For some centuries, this device was called nonius. During the nineteenth century, many countries, most notably France, started to call it vernier. More details in http://www.mlahanas.de/Stamps/Data/Mathematician/N.htm.

[35]Some authors call the Columbus connection the Nunes connection. Such name is clearly unappropriated.

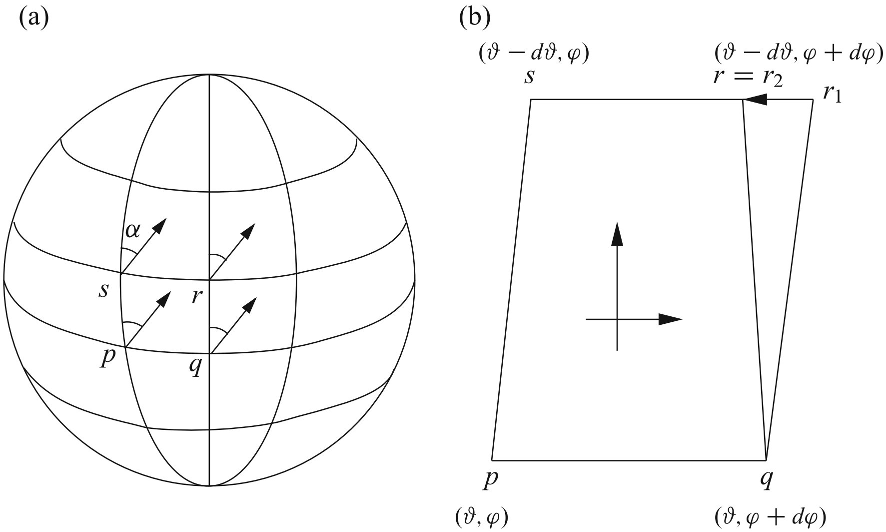

Fig. 4.5 Characterization of the Nunes connection

Let $(x^1, x^2) = (\vartheta, \varphi)$ $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, be the standard spherical coordinates of a $\mathring{S}^2$ or unitary radius, which covers all the open set U which is $\mathring{S}^2$ with the exclusion of a semi-circle uniting the north and south poles.

Introduce first the *coordinate bases*

$$\{\partial_\mu = \partial/\partial x^\mu\}, \{\theta^\mu = dx^\mu\} \tag{4.278}$$

for TU and T^*U.

Introduce next the *orthonormal bases* $\{\boldsymbol{e}_\mathbf{a}\}, \{\theta^\mathbf{a}\}$ for TU and T^*U with

$$\boldsymbol{e}_\mathbf{1} = \partial_1,\ \boldsymbol{e}_\mathbf{2} = \frac{1}{\sin x^1}\partial_2, \tag{4.279}$$

$$\theta^\mathbf{1} = dx^1, \theta^\mathbf{2} = \sin x^1 dx^2. \tag{4.280}$$

Then,

$$[\boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}] = c_{\cdot\mathbf{ij}}^{\mathbf{k}\cdot\cdot}\boldsymbol{e}_\mathbf{k}, \tag{4.281}$$
$$c_{\cdot\mathbf{12}}^{\mathbf{2}\cdot\cdot} = -c_{\cdot\mathbf{21}}^{\mathbf{2}\cdot\cdot} = -\cot x^\mathbf{1},$$

and

$$\begin{aligned}\boldsymbol{g} &= dx^1 \otimes dx^1 + \sin^2 x^1 dx^2 \otimes dx^2 \\ &= \theta^\mathbf{1} \otimes \theta^\mathbf{1} + \theta^\mathbf{2} \otimes \theta^\mathbf{2}.\end{aligned} \tag{4.282}$$

Now, it is obvious from what has been said above that our teleparallel connection is characterized by

$$\boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{e}_{\mathbf{i}} = 0. \tag{4.283}$$

Then taking into account the definition of the curvature operator (definition (4.104)), we have

$$\mathbf{R}(\theta^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{k}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{a}}\left(\left[\boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{i}}}\boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{j}}} - \boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{i}}} - \boldsymbol{\nabla}^{c}_{[\boldsymbol{e}_{\mathbf{i}},\boldsymbol{e}_{\mathbf{j}}]}\right]\boldsymbol{e}_{\mathbf{k}}\right) = 0. \tag{4.284}$$

Also, taking into account the definition of the torsion operation (definition (4.103)) we have

$$\begin{aligned}\boldsymbol{\tau}(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) &= \boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{j}}\boldsymbol{e}_{\mathbf{i}}} - \boldsymbol{\nabla}^{c}_{\boldsymbol{e}_{\mathbf{i}}\boldsymbol{e}_{\mathbf{j}}} - [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}] \\ &= [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}],\end{aligned} \tag{4.285}$$

and $T^{\mathbf{2}\cdot\cdot}_{\cdot\mathbf{21}} = -T^{\mathbf{2}\cdot\cdot}_{\cdot\mathbf{12}} = \cot\vartheta$.

If you still need more details, concerning this last result, consider Fig. 4.5b which shows the standard parametrization of the points p, q, r, s in terms of the spherical coordinates introduced above. According to the geometrical meaning of torsion, we determine its value at a given point by calculating the difference between the (infinitesimal)[36] segments (vectors) $pr_1$and pr_2 determined as follows. If we transport the vector pq along ps we get (recalling that $\mathfrak{R} = 1$) the vector $\vec{v} = sr_1$ such that $\left|\boldsymbol{g}(\vec{v}, \vec{v})\right|^{\frac{1}{2}} = \sin\vartheta\triangle\varphi$. On the other hand, if we transport the vector ps along pr we get the vector $qr_2 = qr$. Let $\vec{w} = sr$. Then,

$$\left|\boldsymbol{g}(\vec{w}, \vec{w})\right| = \sin(\vartheta - \triangle\vartheta)\triangle\varphi \simeq \sin\vartheta\triangle\varphi - \cos\vartheta\triangle\vartheta\triangle\varphi, \tag{4.286}$$

Also,

$$\vec{u} = r_1r_2 = -u(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}), u = \left|\boldsymbol{g}(\vec{u}, \vec{u})\right| = \cos\vartheta\triangle\vartheta\triangle\varphi \tag{4.287}$$

Then, the (Riemann-Cartan) connection $\boldsymbol{\nabla}^c$ of the structure $(\mathring{S}^2, \boldsymbol{g}, \boldsymbol{\nabla}^c, \tau_{\boldsymbol{g}})$ has a non null torsion tensor Θ. Indeed, the component of $\vec{u} = r_1r_2$ in the direction $\partial/\partial\varphi$ is precisely $T^{\varphi}_{\vartheta\varphi}\triangle\vartheta\triangle\varphi$. So, we get (recalling that $\boldsymbol{\nabla}^c{}_{\partial j}\partial_i = \Gamma^{k\cdot\cdot}_{\cdot ji}\partial_k$)

$$T^{\varphi\cdot\cdot}_{\cdot\vartheta\varphi} = \left(\Gamma^{\varphi\cdot\cdot}_{\cdot\vartheta\varphi} - \Gamma^{\varphi\cdot\cdot}_{\cdot\varphi\vartheta}\right) = -\cot\theta. \tag{4.288}$$

[36]This wording, of course, means that this vectors are identified as elements of the appropriate tangent spaces.

To complete the exercise we must show that $\boldsymbol{\nabla}^c \boldsymbol{g} = 0$. We have,

$$\begin{aligned} 0 &= \boldsymbol{\nabla}^c_{\boldsymbol{e}_\mathbf{c}} \boldsymbol{g}(\boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}) = (\boldsymbol{\nabla}^c_{\boldsymbol{e}_\mathbf{c}} \boldsymbol{g})(\boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}) + \boldsymbol{g}(\boldsymbol{\nabla}^c_{\boldsymbol{e}_\mathbf{c}} \boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}) + \boldsymbol{g}(\boldsymbol{e}_\mathbf{i}, \boldsymbol{\nabla}^c_{\boldsymbol{e}_\mathbf{c}} \boldsymbol{e}_\mathbf{j}) \\ &= (\boldsymbol{\nabla}^c_{\boldsymbol{e}_\mathbf{c}} \boldsymbol{g})(\boldsymbol{e}_\mathbf{i}, \boldsymbol{e}_\mathbf{j}). \end{aligned} \tag{4.289}$$

Remark 4.148 This exercise, shows clearly that we cannot *mislead* the Riemann curvature tensor of a connection with the fact that the manifold where that connection is defined may be bend as a surface in an Euclidean manifold where it is embedded. Bending is characterized by the shape operator[37] (a fundamental concept in differential geometry that will be presented in Chap. 5 using the Clifford bundle formalism). Neglecting this fact may generate a lot of wishful thinking. Taking it into account may suggest new formulations of the gravitational field theory as we will show in Chap. 11.

4.9.9 "Tetrad" Postulate? On the Necessity of Precise Notations

Given a differentiable manifold M, let $\boldsymbol{X}, \boldsymbol{Y} \in \sec TM$ be vector fields and $C \in \sec T^*M$ a covector field. Let $\mathcal{T}M = \bigoplus_{r,s=0}^{\infty} T^r_s M$ be the tensor bundle of M and $\mathbf{P} \in \sec \mathcal{T}M$ a general tensor field. We already introduced in M a rule for differentiation of tensor fields, namely the Lie derivative. Taking into account Appendix A.4 we introduce three covariant derivatives operators, $\boldsymbol{\nabla}^+$, $\boldsymbol{\nabla}^-$ and $\boldsymbol{\nabla}$, defined as follows:

$$\begin{aligned} \boldsymbol{\nabla}^+ &: \sec TM \times \sec TM \to \sec TM, \\ (\boldsymbol{X}, \boldsymbol{Y}) &\mapsto \boldsymbol{\nabla}^+_{\boldsymbol{X}} \boldsymbol{Y}, \end{aligned} \tag{4.290}$$

$$\begin{aligned} \boldsymbol{\nabla}^- &: \sec TM \times \sec T^*M \to \sec TM, \\ (\boldsymbol{X}, C) &\mapsto \boldsymbol{\nabla}^-_{\boldsymbol{X}} C, \end{aligned} \tag{4.291}$$

$$\begin{aligned} \boldsymbol{\nabla} &: \sec TM \times \sec \boldsymbol{\tau} M \to \sec TM, \\ (X, \mathbf{P}) &\mapsto \boldsymbol{\nabla}_X \mathbf{P}, \end{aligned} \tag{4.292}$$

Each one of the covariant derivative operators introduced above satisfy the following properties: Given, differentiable functions $f, g : M \to \mathbb{R}$, vector fields

[37] See, e.g., [17, 27, 34, 41] for details.

$X, Y \in \sec TM$ and $\mathbf{P}, \mathbf{Q} \in \sec \mathcal{T}M$ we have

$$\begin{aligned} \nabla_{fX+gY}\mathbf{P} &= f\nabla_X\mathbf{P} + g\nabla_Y\mathbf{P}, \\ \nabla_X(\mathbf{P}+\mathbf{Q}) &= \nabla_X\mathbf{P} + \nabla_X\mathbf{Q}, \\ \nabla_X(f\mathbf{P}) &= f\nabla_X(\mathbf{P}) + X(f)\mathbf{P}, \\ \nabla_X(\mathbf{P}\otimes\mathbf{Q}) &= \nabla_X\mathbf{P}\otimes\mathbf{Q} + \mathbf{P}\otimes\nabla_X\mathbf{Q}. \end{aligned} \tag{4.293}$$

The *absolute differential* of $\mathbf{P} \in \sec T_s^r M$ is given by the mapping

$$\begin{aligned} \nabla : \sec T_s^r M &\to \sec T_{s+1}^r M, \\ \nabla\mathbf{P}(\boldsymbol{X}, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s, \alpha_1, \ldots, \alpha_r) &= \nabla_X\mathbf{P}(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s, \alpha_1, \ldots, \alpha_r), \\ \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s &\in \sec TM, \alpha_1, \ldots \alpha_r \in \sec T^*M. \end{aligned} \tag{4.294}$$

To continue we must give the relationship between ∇^+, ∇^- and ∇. Let $U \subset M$ and consider a chart of the maximal atlas of M covering U coordinate functions $\{\mathbf{x}^\mu\}$. Let $\boldsymbol{g} \in \sec T_2^0 M$ be a metric field for TM and $g \in \sec T_0^2 M$ the corresponding metric for TM (as introduced previously). Let $\{\partial_\mu\}$ be a basis for TU, $U \subset M$ and let $\{\theta^\mu = dx^\mu\}$ be the dual basis of $\{\partial_\mu\}$. The reciprocal basis of $\{\theta^\mu\}$ is denoted $\{\theta_\mu\}$, and we have $\mathtt{g}(\theta^\mu, \theta_\nu) = \delta_\nu^\mu$. Introduce next a set of differentiable functions $h_\mu^{\mathbf{a}}, h_{\mathbf{b}}^\nu : U \to \mathbb{R}$ such that:

$$h_{\mathbf{a}}^\mu q_\mu^{\mathbf{b}} = \boldsymbol{\delta}_{\mathbf{a}}^{\mathbf{b}}, \qquad h_{\mathbf{a}}^\mu h_\nu^{\mathbf{a}} = \delta_\nu^\mu. \tag{4.295}$$

Define

$$\boldsymbol{e}_{\mathbf{b}} = h_{\mathbf{b}}^\nu \partial_\nu$$

where the set $\{\boldsymbol{e}_{\mathbf{a}}\}$ is an orthonormal basis[38] for TU, i.e., $\boldsymbol{g}(\boldsymbol{e}_{\mathbf{a},}\boldsymbol{e}_{\mathbf{b}}) = \eta^{\mathbf{ab}}$. The reciprocal basis of $\{\boldsymbol{e}_{\mathbf{a}}\}$ is $\{\boldsymbol{e}^a\}$ and $\boldsymbol{g}(\boldsymbol{e}^{\mathbf{a}}_{,}\boldsymbol{e}_{\mathbf{b}}) = \delta_{\mathbf{b}}^{\mathbf{a}}$ The dual basis of TU is $\{\boldsymbol{\theta}^{\mathbf{a}}\}$, with $\boldsymbol{\theta}^{\mathbf{a}} = h_\mu^{\mathbf{a}} dx^\mu$ and $\mathtt{g}(\boldsymbol{\theta}^{\mathbf{a}}, \boldsymbol{\theta}^{\mathbf{b}}) = \eta^{\mathbf{ab}}$. Also, $\{\boldsymbol{\theta}_{\mathbf{b}}\}$ is the reciprocal basis of $\{\boldsymbol{\theta}^{\mathbf{a}}\}$, i.e. $\mathtt{g}(\boldsymbol{\theta}^{\mathbf{a}}, \boldsymbol{\theta}_{\mathbf{b}}) = \delta_{\mathbf{b}}^{\mathbf{a}}$. It is trivial to verify the formulas

$$\begin{aligned} g_{\mu\nu} &= h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} \eta_{\mathbf{ab}}, & g^{\mu\nu} &= h_{\mathbf{a}}^\mu h_{\mathbf{b}}^\nu \eta^{\mathbf{ab}}, \\ \eta_{\mathbf{ab}} &= h_{\mathbf{a}}^\mu h_{\mathbf{b}}^\nu g_{\mu\nu}, & \eta^{\mathbf{ab}} &= h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} g^{\mu\nu}. \end{aligned} \tag{4.296}$$

[38] $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ is the orthonormal frame bundle (see Appendix A.1.2).

The connection coefficients associated to the respective covariant derivatives in the respective bases are denoted as:

$$\nabla^{+}_{\partial_\mu}\partial_\nu = \Gamma^{\rho\cdot\cdot}_{\cdot\mu\nu}\partial_\rho, \quad \nabla^{-}_{\partial_\sigma}\partial^\mu = -\Gamma^{\mu\cdot\cdot}_{\cdot\sigma\alpha}\partial^\alpha, \tag{4.297}$$

$$\nabla^{+}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{e}_\mathbf{b} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\boldsymbol{e}_\mathbf{c}, \qquad \nabla^{+}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{e}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\,\boldsymbol{e}^\mathbf{c}, \quad \nabla^{+}_{\partial_\mu}\,\boldsymbol{e}_\mathbf{b} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mu\mathbf{b}}\boldsymbol{e}_\mathbf{c}, \tag{4.298}$$

$$\nabla^{-}_{\partial_\mu}dx^\nu = -\Gamma^{\nu\cdot\cdot}_{\cdot\mu\alpha}dx^\alpha, \quad \nabla^{-}_{\partial_\mu}\theta_\nu = \Gamma^{\rho\cdot\cdot}_{\cdot\mu\nu}\theta_\rho, \tag{4.299}$$

$$\nabla^{-}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{\theta}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\boldsymbol{\theta}^\mathbf{c}, \quad \nabla^{-}_{\partial_\mu}\boldsymbol{\theta}^\mathbf{b} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mu\mathbf{a}}\boldsymbol{\theta}^\mathbf{a}, \tag{4.300}$$

$$\nabla^{-}_{\boldsymbol{e}_\mathbf{a}}\boldsymbol{\theta}^\mathbf{b} = -\omega_{\mathbf{cab}}\boldsymbol{\theta}^\mathbf{c}, \tag{4.301}$$

$$\omega_{\mathbf{abc}} = \eta_{\mathbf{ad}}\omega^{\mathbf{d}\cdot\cdot}_{\cdot\mathbf{bc}} = -\omega_{\mathbf{cba}},\ \omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot} = \eta^{\mathbf{bk}}\omega_{\mathbf{kal}}\eta^{\mathbf{cl}},\ \omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot} = -\omega^{\mathbf{c}\cdot\mathbf{b}}_{\cdot\mathbf{a}\cdot} \tag{4.302}$$

$$\text{etc.}\ldots \tag{4.303}$$

To understood how ∇ works, consider its action, e.g., on the sections of $T^1_1M = TM \otimes T^*M$. For that case, if $\boldsymbol{X} \in \sec TM$, $C \in \sec T^*M$, we have that

$$\nabla = \nabla^{+} \otimes \mathrm{Id}_{T^*M} + \mathrm{Id}_{TM} \otimes \nabla^{-}, \tag{4.304}$$

and

$$\nabla(\boldsymbol{X} \otimes C) = (\nabla^{+}\boldsymbol{X}) \otimes C + X \otimes \nabla^{-}C. \tag{4.305}$$

The general case, of ∇ acting on sections of $\mathcal{T}M$ is an obvious generalization of the previous one, and details are left to the reader.

For every vector field $\boldsymbol{V} \in \sec TU$ and a covector field $C \in \sec T^*U$ we have

$$\nabla^{+}_{\partial_\mu}\boldsymbol{V} = \nabla^{+}_{\partial_\mu}(V^\alpha\partial_\alpha), \qquad \nabla^{-}_{\partial_\mu}C = \nabla^{-}_{\partial_\mu}(C_\alpha\theta^\alpha) \tag{4.306}$$

and using the properties of a covariant derivative operator introduced above, $\nabla^{+}_{\partial_\mu}V$ can be written as:

$$\begin{aligned}\nabla^{+}_{\partial_\mu}\boldsymbol{V} &= \nabla^{+}_{\partial_\mu}(V^\alpha\partial_\alpha) = (\nabla^{+}_{\partial_\mu}V)^\alpha\partial_\alpha \\ &= (\partial_\mu V^\alpha)\partial_\alpha + V^\alpha\nabla^{+}_{\partial_\mu}\partial_\alpha \\ &= \left(\frac{\partial V^\alpha}{\partial x^\mu} + V^\rho\Gamma^{\alpha\cdot\cdot}_{\cdot\mu\rho}\right)\partial_\alpha := (\nabla^{+}_{\mu}V^\alpha)\partial_\alpha,\end{aligned} \tag{4.307}$$

where it is to be kept in mind that the symbol $\nabla^{+}_{\mu}V^\alpha$ is a short notation for

$$\nabla^{+}_{\mu}V^\alpha := (\nabla^{+}_{\partial_\mu}\boldsymbol{V})^\alpha. \tag{4.308}$$

Also, we have

$$\begin{aligned}\boldsymbol{\nabla}^-_{\partial_\mu} C = \boldsymbol{\nabla}^-_{\partial_\mu}(C_\alpha \theta^\alpha) &= (\boldsymbol{\nabla}^-_{\partial_\mu} C)_\alpha \theta^\alpha \\ &= \left(\frac{\partial C_\alpha}{\partial x^\mu} - C_\beta \Gamma^{\beta\cdot\cdot}_{\cdot\mu\alpha}\right)\theta^\alpha, \\ &:= (\boldsymbol{\nabla}^-_\mu C_\alpha)\theta^\alpha, \end{aligned} \tag{4.309}$$

where it is to be kept in mind that[39] that the symbol $\boldsymbol{\nabla}^-_\mu C_\alpha$ is a short notation for

$$\boldsymbol{\nabla}^-_\mu C_\alpha := (\boldsymbol{\nabla}^-_{\partial_\mu} C)_\alpha. \tag{4.310}$$

Remark 4.149 When there is no possibility of confusion, we shall use only the symbol $\boldsymbol{\nabla}$ to denote any one of the covariant derivatives introduced above. However, the standard practice of many Physics textbooks of representing, $\boldsymbol{\nabla}^+_\mu V^\alpha$ and $\boldsymbol{\nabla}^+_\mu V^\alpha$ by $\boldsymbol{\nabla}_\mu V^\alpha$ should be avoided whenever possible in order to not produce misunderstandings (see Exercise below).

Exercise 4.150 Calculate $\boldsymbol{\nabla}^-_\mu h^{\mathbf{a}}_\nu := (\boldsymbol{\nabla}^-_{\partial_\mu} \theta^{\mathbf{a}})_\nu = (\boldsymbol{\nabla}^-_{\partial_\mu} h^{\mathbf{a}}_\alpha \partial^\alpha)_\nu$ and $\boldsymbol{\nabla}^+_\mu h^{\mathbf{a}}_\nu := (\boldsymbol{\nabla}^+_{\partial_\mu} \partial_\nu)^{\mathbf{a}} = (\boldsymbol{\nabla}^+_{\partial_\mu} h^{\mathbf{b}}_\nu \boldsymbol{e}_{\mathbf{b}})^{\mathbf{a}}$. Show that in general $\boldsymbol{\nabla}^-_\mu h^{\mathbf{a}}_\nu \neq \boldsymbol{\nabla}^+_\mu h^{\mathbf{a}}_\nu \neq 0$ and that

$$\partial_\mu h^{\mathbf{a}}_\nu + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_\nu - \Gamma^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_\nu = 0. \tag{4.311}$$

Exercise 4.151 Define the object

$$\mathbf{e} = \boldsymbol{e}_{\mathbf{a}} \otimes \theta^{\mathbf{a}} = h^{\mathbf{a}}_\mu \partial_\mu \otimes dx^\mu \in \sec T^1_1 M, \tag{4.312}$$

which is clearly the identity endomorphism acting on sections of TU. Show that

$$\boldsymbol{\nabla}_\mu h^{\mathbf{a}}_\nu := (\boldsymbol{\nabla}_{\partial_\mu} \mathbf{e})^{\mathbf{a}}_\nu = \partial_\mu h^{\mathbf{a}}_\nu + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_\nu - \Gamma^{\mathbf{a}\cdot\cdot}_{\cdot\mu\mathbf{b}} h^{\mathbf{b}}_\nu = 0. \tag{4.313}$$

Remark 4.152 Equation (4.313) is presented in many textbooks (see., e.g., [2, 12, 37]) under the name 'tetrad postulate'. In that books, since authors do not distinguish clearly the derivative operators $\boldsymbol{\nabla}^+, \boldsymbol{\nabla}^-$ and $\boldsymbol{\nabla}$, Eq. (4.313) becomes sometimes misunderstood as meaning $\boldsymbol{\nabla}^-_\mu h^{\mathbf{a}}_\nu$ or $\boldsymbol{\nabla}^+_\mu h^{\mathbf{a}}_\nu$, thus generating a big confusion. For a discussion of this issue see [33].

[39] Recall that other authors prefer the notations $(\boldsymbol{\nabla}_{\partial_\mu} V)^\alpha := V^\alpha_{:\mu}$ and $(\boldsymbol{\nabla}_{\partial_\mu} C)_\alpha := C_{\alpha:\mu}$. What is important is always to have in mind the meaning of the symbols.

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Chapter 5
Clifford Bundle Approach to the Differential Geometry of Branes

Abstract Using the Clifford bundle formalism (CBF) of differential forms and the theory of extensors acting on $\mathcal{C}\ell(M, g)$ (introduced in Chap. 4) we first recall the formulation of the intrinsic geometry of a differential manifold M (a brane) equipped with a metric field $\boldsymbol{g}$ of signature (p, q) and an arbitrary metric compatible connection ∇ introducing the torsion $(2-1)$-extensor field τ, the curvature $(2-2)$ extensor field $\mathfrak{R}$ and (once fixing a gauge) the connection $(1-2)$-extensor ω and the Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ (where $\boldsymbol{\partial}$ is the Dirac operator acting on sections of $\mathcal{C}\ell(M, g)$) which plays an important role in this paper. Next, using the CBF we give a thoughtful presentation of the Riemann or the Lorentzian geometry of an orientable submanifold M ($\dim M = m$) living in a manifold $\mathring{M}$ (such that $\mathring{M} \simeq \mathbb{R}^n$ is equipped with a semi-Riemannian metric $\mathring{\boldsymbol{g}}$ with signature $(\mathring{p}, \mathring{q})$ and $\mathring{p} + \mathring{q} = n$ and its Levi-Civita connection $\mathring{D}$) and where there is defined a metric $\boldsymbol{g} = \boldsymbol{i}^* \mathring{g}$, where $\boldsymbol{i} : M \to \mathring{M}$ is the inclusion map. We prove several equivalent forms for the curvature operator $\mathfrak{R}$ of M. Moreover we show a very important result, namely that the Ricci operator of M is the (negative) square of the shape operator $\mathbf{S}$ of M (object obtained by applying the restriction on M of the Dirac operator $\mathring{\boldsymbol{\partial}}$ of $\mathcal{C}\ell(\mathring{M}, \mathring{g})$ to the projection operator $\mathbf{P}$). Also we disclose the relationship between the $(1-2)$-extensor ω and the shape biform $\mathcal{S}$ (an object related to $\mathbf{S}$). The results obtained are used in Chap. 11 to give a mathematical formulation to Clifford's theory of matter (Rodrigues and Wainer, Adv Appl Clifford Algebras 24:817–847, 2014).

5.1 Introduction

In this chapter we use the Clifford bundle formalism (CBF) developed previously in order to analyze the Riemann or the Lorentzian geometry of an orientable submanifold M ($\dim M = m$) living in a manifold $\mathring{M}$ such that $\mathring{M} \simeq \mathbb{R}^n$ is equipped with a semi-Riemannian metric $\mathring{\boldsymbol{g}}$ (with signature $(\mathring{p}, \mathring{q})$ and $\mathring{p} + \mathring{q} = n$) and its Levi-Civita connection $\mathring{D}$.

In order to achieve our objectives and exhibit some nice results that are not well known (and which, e.g., may possibly be of interest for the description and formulation of branes theories [15] and string theories [2]) we first recall in Sect. 5.2

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how to formulate using the CBF the intrinsic differential geometry of a structure $\langle M, \boldsymbol{g}, \nabla\rangle$ where ∇ is a general metric compatible Riemann-Cartan connection, i.e., $\nabla \boldsymbol{g} = \mathbf{0}$ and the Riemann and torsion tensors of ∇ are non null. We recall that we introduced in Chap. 4 (once we fix a gauge in the frame bundle) a $(1, 2)$-extensor field $\boldsymbol{\omega}$: sec $\bigwedge^1 T^*M \to \bigwedge^2 T^*M$ closed related with the connection 1-forms which permits to write a very nice formula for the covariant derivative [see Eq. (5.30)] of any section of the Clifford bundle of the structure $\langle M, \boldsymbol{g}, \nabla\rangle$. It will be shown that $\boldsymbol{\omega}$ is related to $\mathcal{S}$: sec $\bigwedge^1 T^*M \to \bigwedge^2 T^*M$ the shape operator biform of the manifold.

Then in Sect. 5.3, we suppose that M is a *proper* submanifold[1] of $\mathring{M}$ which $\boldsymbol{i}$: $M \mapsto \mathring{M}$ the inclusion map. Introducing natural *global* coordinates $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n)$ for $\mathring{M} \simeq \mathbb{R}^n$ we write $\mathring{\boldsymbol{g}} = \sum_{i,j=1}^n \eta_{ij} d\boldsymbol{x}^i \otimes d\boldsymbol{x}^j \equiv \eta_{ij} d\boldsymbol{x}^i \otimes d\boldsymbol{x}^j$ and equip $\mathring{M}$ with the pullback metric $\boldsymbol{g} := \boldsymbol{i}^* \mathring{\boldsymbol{g}}$. We then find the relation between the Levi-Civita connection D of $\boldsymbol{g}$ and $\mathring{D}$, the Levi-Civita connection of $\mathring{\boldsymbol{g}}$. We suppose that $\boldsymbol{g}$ is non degenerated of signature (p, q) with $p + q = m$.

In this chapter $\mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ and $\mathcal{C}\ell(M, \mathtt{g})$ denote respectively the Clifford bundles of differential forms of $\mathring{M}$ and M. Moreover, in what follows

$$\mathring{\mathtt{g}} = \sum_{i,j=1}^n \eta^{ij} \frac{\partial}{\partial \boldsymbol{x}^i} \otimes \frac{\partial}{\partial \boldsymbol{x}^j} \equiv \eta^{ij} \frac{\partial}{\partial \boldsymbol{x}^i} \otimes \frac{\partial}{\partial \boldsymbol{x}^j} \tag{5.1}$$

is the metric of the cotangent bundle. The Dirac operators[2] of $\mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ and $\mathcal{C}\ell(M, \mathtt{g})$ will be denoted by $\mathring{\boldsymbol{\partial}}$ and $\boldsymbol{\partial}$. Let $l = n - m$ and

$$\{\mathring{e}_1, \mathring{e}_2, \ldots, \mathring{e}_{\boldsymbol{m}}, \mathring{e}_{m+1}, \ldots, \mathring{e}_{m+l}\} \tag{5.2}$$

an orthonormal basis for $T\mathring{U}$ $(\mathring{U} \subset \mathring{M})$ such that

$$\{\boldsymbol{e}_{\mathbf{1}}, \boldsymbol{e}_{\mathbf{2}}, \ldots, \boldsymbol{e}_{\mathbf{m}}\} = \{\mathring{e}_1, \mathring{e}_2, \ldots, \mathring{e}_{\boldsymbol{m}}\} \tag{5.3}$$

is a basis for TU $(U \subset \mathring{U})$ and if

$$\{\mathring{\theta}^{\mathbf{1}}, \mathring{\theta}^{\mathbf{2}}, \ldots, \mathring{\theta}^{\mathbf{m}}, \mathring{\theta}^{m+1}, \ldots, \mathring{\theta}^{m+l}\} \tag{5.4}$$

is the dual basis of the $\{e_i\}$ we have that $\{\theta^{\mathbf{1}}, \theta^{\mathbf{2}}, \ldots, \theta^{\mathbf{m}}\} = \{\mathring{\theta}^1, \mathring{\theta}^2, \ldots, \mathring{\theta}^m\}$ is a basis for T^*U dual to the basis $\{\boldsymbol{e}_{\mathbf{1}}, \boldsymbol{e}_{\mathbf{2}}, \ldots, \boldsymbol{e}_{\mathbf{m}}\}$ of TU. We have, as well known

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[1]By a proper (or regular) submanifold M of $\mathring{M}$ we mean a subset $M \subset \mathring{M}$ such that for every $x \in M$ in the domain of a chart (U, σ) of $\mathring{M}$ such that $\sigma : \mathring{M} \cap U \to \mathbb{R}^n \times \{\mathbf{l}\}, \sigma(x) = (x^1, \ldots, x^n, l^1, \ldots, l^{m-n})$, where $\mathbf{l} = (l^1, \ldots, l^{m-n}) \in \mathbb{R}^{n-m}$.

[2]Take notice that the Dirac operators used in this chapter are acting on sections of the Clifford bundle. It is not to be confused with the Dirac operator which acts on sections of the spinor bundle (see details in [16]). According to [5], this last operator can also be used to probe the topology of the brane.

from Chap. 4

$$\mathring{\boldsymbol{\partial}} = \sum_{i=1}^{n} \mathring{\theta}^i \mathring{D}_{e_i} = \mathring{\theta}^i \mathring{D}_{e_i}, \quad \boldsymbol{\partial} = \sum_{i=1}^{m} \theta^i D_{e_i} = \theta^{\mathbf{i}} D_{e_{\mathbf{i}}}, \tag{5.5}$$

Remark 5.1 Take notice that the bold face sub and superscripts are used to denote bases $\{e_{\mathbf{i}}\}$ and $\{\theta^{\mathbf{i}}\}$ of the tangent and cotangent space of M. This notation is conveniently used in what follows.

The dual basis of the natural coordinate basis $\{\frac{\partial}{\partial \mathbf{x}^i}\}$ is denoted in what follows by $\{\gamma^i\}$ where, of course, $\gamma^i = d\mathbf{x}^i$. Moreover, $\{\mathring{e}^1, \mathring{e}^2, \ldots, \mathring{e}^m\}$ denotes the reciprocal frame of $\{\mathring{e}_i\}$, i.e., $\mathring{\boldsymbol{g}}(\mathring{e}^i, \mathring{e}_j) = \delta^i_j$ and by $\{\mathring{\theta}_i\}$ the reciprocal basis of $\{\mathring{\theta}^i\}$, i.e., $\mathring{g}(\mathring{\theta}^i, \mathring{\theta}_j) := \mathring{\theta}^i \cdot \mathring{\theta}_j = \delta^i_j$. Moreover, take into account that for $\mathbf{i}, \mathbf{j} = 1, \cdots, m$ it is $\mathtt{g}(\theta^{\mathbf{i}}, \theta_{\mathbf{j}}) = \mathring{\mathtt{g}}(\mathring{\theta}^i, \mathring{\theta}_j)$. So we will write also $\mathtt{g}(\theta^{\mathbf{i}}, \theta_{\mathbf{j}}) = \theta^{\mathbf{i}} \cdot \theta_{\mathbf{j}} = \delta^{\mathbf{i}}_{\mathbf{j}}$. The representation of the Dirac operator $\mathring{\boldsymbol{\partial}}$ in the natural coordinate basis of $\mathring{M}$ is of course, $\mathring{\boldsymbol{\partial}} = \sum_{\mathtt{i}=1}^{\mathtt{n}} {}^{i} \frac{\partial}{\partial \mathbf{x}^i}$. Note that we have $\mathring{\theta}^{m+1}\Big|_M = 0, \ldots, \mathring{\theta}^{m+l}\Big|_M = 0$, i.e., the $\{\mathring{\theta}^{m+1}, \ldots, \mathring{\theta}^{m+l}\}$ for any vector field $\boldsymbol{a} \in \sec TU$ and $d = 1, \ldots, m+l$ we have

$$\mathring{\theta}^{m+d}\Big|_M (\boldsymbol{a}) = 0.$$

We denote moreover

$$\mathring{\eth} = \mathring{\boldsymbol{\partial}}\Big|_M := \theta^{\mathbf{i}}\theta_{\mathbf{i}} \cdot \mathring{\boldsymbol{\partial}} = \sum_{i=1}^{m} \theta^i \mathring{D}_{e_{\mathbf{i}}} = \theta^{\mathbf{i}} \mathring{D}_{e_i} \tag{5.6}$$

the restriction of $\mathring{\boldsymbol{\partial}}$ on the submanifold M. The projection operator $\mathbf{P}$ (an extensor field) on M and the *shape operator* $\mathbf{S} = \mathring{\eth}\mathbf{P}$: $\sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}}) \to \sec \mathcal{C}\ell(M, \mathtt{g})$ and *shape biform* operator of the manifold M, $\mathcal{S} : \sec \bigwedge^1 T^*M \mapsto \bigwedge^2 T^*M$, $\mathcal{S}(a) := -(a \cdot \eth I_m) I_m^{-1}$ (where $\tau_{\mathtt{g}} = I_m = \theta^1\theta^2 \cdots \theta^m$ is the volume form[3] on $U \subset M$) are fundamental objects in our study. The definition of those objects are given in Sect. 5.3 and the main algebraic properties of $\mathbf{P}$, $\mathbf{S}$ and $\mathcal{S}$ besides all identities necessary for the present paper are given and proved at the appropriate places.

Section 5.4 is dedicated to find several equivalent expressions for the curvature biform $\mathfrak{R}(u, v)$ in terms of the shape operator. We recall that the square of the Dirac operator $\boldsymbol{\partial}$ acting on sections of the Clifford bundle has two different decompositions, namely

$$\boldsymbol{\partial}^2 = -(d\delta + \delta d) = \boldsymbol{\partial} \cdot \boldsymbol{\partial} + \boldsymbol{\partial} \wedge \boldsymbol{\partial}, \tag{5.7}$$

[3]The volume $\tau_{\mathring{g}}$ for on $\mathring{U} \subset \mathring{M}$ will be denoted by $I_n = \mathring{\boldsymbol{\theta}}^1 \mathring{\boldsymbol{\theta}}^2 \cdots \mathring{\boldsymbol{\theta}}^m$. The volume form $\tau_{\mathring{g}}$ on $\mathring{U} \subset M$ will be denoted $I_n = \mathring{\theta}^{\mathbf{1}}\mathring{\theta}^{\mathbf{2}} \cdots \mathring{\theta}^{\mathbf{m}}\mathring{\theta}^{m+1} \cdots \mathring{\theta}^{m+l} = I_m \mathring{\theta}^{\mathbf{m}}\mathring{\theta}^{m+1} \cdots \mathring{\theta}^{m+l}$.

where d and δ are respectively the exterior derivative and the Hodge coderivative and $\boldsymbol{\partial}\cdot\boldsymbol{\partial}+\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ are respectively the covariant Laplacian and the Ricci operator. The explicit forms of $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ and $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ are given in Chap. 4 where it is shown moreover that $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ is an extensorial operator and

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial}\;\theta^i = \mathcal{R}^i, \tag{5.8}$$

where the objects $\mathcal{R}^i = R^i_j\theta^j \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(\mathring{M},\mathring{\mathtt{g}})$ with R^i_j the components of the Ricci tensor associated with D are called the Ricci 1-form fields.

One of the objectives of the present chapter is to give (Sect. 5.5) a detailed proof of the remarkable equation

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial}\;(v) = -\mathbf{S}^2(v), \tag{5.9}$$

which says that the shape biform operator is the negative square root of the Ricci operator.[4] We moreover find the relation between $\mathcal{S}(v)$ and $\boldsymbol{\omega}(v)$ thus providing a very interesting geometrical meaning for the connection 1-forms $\omega^{i\cdot}_{\cdot j}$ of the Levi-Civita connection D, namely as the angular 'velocity' with which the pseudo scalar I_m when it slides on M.

In Sect. 5.5 we prove some identities involving the projection operator and its covariant derivative which are necessary, in particular, to prove Proposition 5.47.

In Sect. 5.6 we present our conclusions.

5.2 Torsion Extensor and Curvature Extensor of a Riemann-Cartan Connection

Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{t},\boldsymbol{z} \in \sec TM$ and $u,v,t,z \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ the physically equivalent 1-forms, i.e., $u = \boldsymbol{g}(\boldsymbol{u},\,)$, etc. Let moreover $\{\boldsymbol{e}_\mathbf{a}\}$ be an orthonormal basis for TM and $\{\theta^\mathbf{a}\}$, $\theta^\mathbf{a} \in \sec\bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$ the corresponding dual basis and consider the Riemann-Cartan structure $(M,\boldsymbol{g},\nabla)$.

Definition 5.2 The form derivative of M is the operator

$$\begin{aligned} \eth &: \sec\mathcal{C}\ell(M,\mathtt{g}) \to \sec\mathcal{C}\ell(M,\mathtt{g}),\\ \eth\mathcal{C} &:= \theta^\mathbf{a}\eth_{\boldsymbol{e}_\mathbf{a}}\mathcal{C} \end{aligned} \tag{5.10}$$

[4]This result appears [with a positive sign on the second member of Eq. (5.9)] in [11]. See also [18]. However, take into account that the methods used in those references use the Clifford algebra of multivectors and thus, comparison of the results there with the standard presentations of modern differential geometry using differential forms are not so obvious, this being probably one of the reasons why some important and beautiful results displayed in [11] are unfortunately ignored.

where $\eth_{e_{\mathbf{a}}}$ is the Pfaff derivative of form fields

$$\eth_{e_{\mathbf{a}}}\mathcal{C} :=\sum_{r=0}^{m}\eth_{e_{\mathbf{a}}}\langle\mathcal{C}\rangle_r \tag{5.11}$$

such that if $\langle\mathcal{C}\rangle_r$ is expanded in the basis generated by $\{\theta^{\mathbf{a}}\}$, i.e., $\langle\mathcal{C}\rangle_r = \mathcal{C}_r = \frac{1}{r!}\mathcal{C}_{\mathbf{i}_1\cdots\mathbf{i}_r}\theta^{\mathbf{i}_1\cdots\mathbf{i}_r} \in \sec\bigwedge^r T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ it is

$$\eth_{e_{\mathbf{a}}}\langle\mathcal{C}\rangle_r := \frac{1}{r!}e_{\mathbf{a}}(\mathcal{C}_{\mathbf{i}_1\cdots\mathbf{i}_r}\theta^{\mathbf{i}_1\cdots\mathbf{i}_r}) = \frac{1}{r!}e_{\mathbf{a}}(\mathcal{C}_{\mathbf{i}_1\cdots\mathbf{i}_r})\theta^{\mathbf{i}_1\cdots\mathbf{i}_r}. \tag{5.12}$$

Given two different pairs of basis $\{e_{\mathbf{a}},\theta^{\mathbf{a}}\}$ and $\{e'_{\mathbf{a}},\theta'^{\mathbf{a}}\}$ we have that

$$\theta^{\mathbf{a}}\eth_{e_{\mathbf{a}}}\mathcal{C} = \theta'^{\mathbf{a}}\eth'_{e_{\mathbf{a}}}\mathcal{C}, \tag{5.13}$$

since for all $\mathcal{C}_r$

$$\eth'\mathcal{C}_r = \theta'^{\mathbf{a}}\eth'_{e_{\mathbf{a}}}\mathcal{C}_r = \theta'^{\mathbf{a}}e'_{\mathbf{a}}(\frac{1}{r!}\mathcal{C}'_{\mathbf{i}_1\cdots\mathbf{i}_r}\theta'^{\mathbf{i}_1\cdots\mathbf{i}_r}) = \theta^{\mathbf{a}}e_{\mathbf{a}}(\frac{1}{r!}\mathcal{C}_{\mathbf{i}_1\cdots\mathbf{i}_r}\theta^{\mathbf{i}_1\cdots\mathbf{i}_r}). \tag{5.14}$$

Remark 5.3 We recall also that any biform $B \in \sec\bigwedge^2 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ and any $A_r \in \bigwedge^r T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ with $r \geq 2$ it holds that

$$BA_r = B\lrcorner A_r + B\times A_r + B\wedge A_r. \tag{5.15}$$

where for any $\mathcal{C},\mathcal{D} \in \sec\mathcal{C}\ell(M,\mathtt{g})$ it is $\mathcal{C}\times\mathcal{D} = \frac{1}{2}(\mathcal{C}\mathcal{D} - \mathcal{D}\mathcal{C})$.We observe that for $v \in \bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ it is

$$B\times v = B\llcorner v = -v\lrcorner B. \tag{5.16}$$

Call $\overset{\nabla}{\boldsymbol{\partial}} := \theta^{\mathbf{a}}\nabla_{e_{\mathbf{a}}}$ the Dirac operator associated with ∇, a general Riemann-Cartan connection. In Chap. 4 we introduced the Dirac commutator of two 1-form fields $u,v \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ associated with ∇ by

$$\begin{aligned}
[\![\ ,\ ]\!] &: \sec\bigwedge^1 T^*M \times \sec\bigwedge^1 T^*M \to \sec\bigwedge^1 T^*M \\
[\![u,v]\!] &= (u\cdot\overset{\nabla}{\boldsymbol{\partial}})v - (v\cdot\overset{\nabla}{\boldsymbol{\partial}})u - [\![u,v]\!]
\end{aligned}$$

where

$$[\![u,v]\!] = (u\cdot\boldsymbol{\partial})v - (v\cdot\boldsymbol{\partial})u, \tag{5.17}$$

is the Lie bracket of 1-form fields.[5]

[5]Recall that if $[e_{\mathbf{a}},e_{\mathbf{b}}] = c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}e_{\mathbf{d}}$, then $[\![\theta_{\mathbf{a}},\theta_{\mathbf{b}}]\!] = c_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}\theta_{\mathbf{d}}$.

Definition 5.4 For a metric compatible connection ∇, recalling the definition of the torsion operator we conveniently write

$$\boldsymbol{\tau}(u,v) = [\![u,v]\!], \tag{5.18}$$

which we call the (form) torsion operator.

Remark 5.5 We recall (see Chap. 2) the action of the operator ∂_u ($u \in \sec \bigwedge^1 T^*M \to \sec \mathcal{C}\ell(M,\mathtt{g})$) acting on an extensor field $F : \sec \bigwedge^1 T^*M \to \sec \bigwedge^r T^*M$, $u \mapsto F(u)$. If $u = u^{\mathbf{i}}\theta_{\mathbf{i}}$, $\partial_u := \theta^{\mathbf{k}} \frac{\partial}{\partial u^{\mathbf{k}}}$ acting on $F(u)$ is given by

$$\begin{aligned}\partial_u F(u) &:= \theta^{\mathbf{k}} \frac{\partial}{\partial u^{\mathbf{k}}} F(u^{\mathbf{i}}\theta_{\mathbf{i}}) := \theta^{\mathbf{k}} \frac{\partial}{\partial u^{\mathbf{k}}} u^{\mathbf{i}} F(\theta_{\mathbf{i}}) \\ &= \theta^{\mathbf{k}} F(\theta_{\mathbf{k}}) = \theta^{\mathbf{k}} \lrcorner F(\theta_{\mathbf{k}}) + \theta^{\mathbf{k}} \wedge F(\theta_{\mathbf{k}}).\end{aligned} \tag{5.19}$$

Also the action of the operator $\partial_u \wedge \partial_v$ ($u = u^{\mathbf{i}}\theta_{\mathbf{i}},, v = v^{\mathbf{i}}\theta_{\mathbf{i}}$) acting on an extensor field $G : \sec \bigwedge^1 T^*M \times \sec \bigwedge^1 T^*M \hookrightarrow \sec \bigwedge^r T^*M$, $(u,v) \mapsto G(u,v)$ is given by

$$\begin{aligned}\partial_u \wedge \partial_v G(u,v) &= \theta^{\mathbf{k}} \frac{\partial}{\partial u^{\mathbf{k}}} \wedge \theta^{\mathbf{l}} \frac{\partial}{\partial u^{\mathbf{l}}} u^{\mathbf{m}} u^{n} G(\theta^{\mathbf{m}}, \theta^{\mathbf{n}}) \\ &= \theta^{\mathbf{k}} \wedge \theta^{\mathbf{l}} G(\theta_{\mathbf{k}}, \theta_{\mathbf{l}}).\end{aligned} \tag{5.20}$$

Definition 5.6 The mapping

$$\begin{aligned}\mathfrak{t} &: \sec \bigwedge^2 T^*M \to \sec \bigwedge^1 T^*M, \\ \mathfrak{t}(B) &= \frac{1}{2} B \cdot (\partial_u \wedge \partial_v) \boldsymbol{\tau}(u,v).\end{aligned} \tag{5.21}$$

is called the (2-1)-extensorial torsion field and

$$\mathfrak{t}(u \wedge v) = \boldsymbol{\tau}(u,v). \tag{5.22}$$

Indeed, from Eq. (5.21) we have taking $B = a \wedge b$

$$\mathfrak{t}(a \wedge b) = \frac{1}{2}(a \wedge b) \cdot (\partial_u \wedge \partial_v) \boldsymbol{\tau}(u,v). \tag{5.23}$$

Now,

$$(\partial_u \wedge \partial_v)\, \tau(u,v) = (\theta^{\mathbf{k}} \wedge \theta^{\mathbf{l}}) \boldsymbol{\tau}(\theta_{\mathbf{k}}, \theta_{\mathbf{l}}). \tag{5.24}$$

Then,

$$\mathfrak{t}(a \wedge b) = \frac{1}{2}(a \wedge b) \cdot (\theta^{\mathbf{k}} \wedge \theta^{\mathbf{l}})\boldsymbol{\tau}(\theta_{\mathbf{k}}, \theta_{\mathbf{l}}) = \tau(a, b).$$

Definition 5.7 The extensor mapping

$$\begin{gathered}\boldsymbol{\Theta} : \sec \bigwedge^1 T^*M \to \sec \bigwedge^2 T^*M, \\ \boldsymbol{\Theta}(c) = \frac{1}{2}(\partial_u \wedge \partial_v)\tau(u, v) \cdot c,\end{gathered} \tag{5.25}$$

is called the Cartan torsion field.

We have that

$$\mathfrak{t}(u \wedge v) = \partial_c(u \wedge v) \cdot \boldsymbol{\Theta}(c)$$

and if $\nabla_{\boldsymbol{e}_{\mathbf{a}}}\theta^{\mathbf{b}} := -\omega_{\cdot\mathbf{ac}}^{\mathbf{b}\cdot\cdot}\theta^{\mathbf{c}}$ then

$$\begin{gathered}z \cdot \mathfrak{t}(u \wedge v) = z_{\mathbf{d}} u^{\mathbf{a}} v^{\mathbf{b}} T_{\cdot\mathbf{ab}}^{\mathbf{d}\cdot\cdot}, \\ T_{\cdot\mathbf{ab}}^{\mathbf{c}\cdot\cdot} = \omega_{\cdot\mathbf{ab}}^{\mathbf{c}\cdot\cdot} - \omega_{\cdot\mathbf{ba}}^{\mathbf{c}\cdot\cdot} - c_{\cdot\mathbf{ab}}^{\mathbf{c}\cdot\cdot} \,.\end{gathered} \tag{5.26}$$

Definition 5.8 The connection (1-2)-extensor field $\boldsymbol{\omega}$ in a given gauge is given by ($v = \boldsymbol{g}(\boldsymbol{v},)$)

$$\begin{gathered}\omega : \sec \bigwedge^1 T^*M \to \sec \bigwedge^2 T^*M, \\ v \mapsto \omega(v) = \frac{1}{2} v^{\mathbf{c}} \omega_{\cdot\mathbf{c}\cdot}^{\mathbf{a}\cdot\mathbf{b}} \theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}.\end{gathered} \tag{5.27}$$

We also introduce the operator

$$\begin{gathered}\boldsymbol{\omega} : \sec \bigwedge^1 TM \to \sec \bigwedge^2 T^*M, \\ \boldsymbol{v} \mapsto \boldsymbol{\omega}(\boldsymbol{v}) = \omega_{\boldsymbol{v}} := \frac{1}{2} v^{\mathbf{c}} \omega_{\cdot\mathbf{c}\cdot}^{\mathbf{a}\cdot\mathbf{b}} \theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\end{gathered} \tag{5.28}$$

and it is clear that

$$\omega(v) = \omega_{\boldsymbol{v}}. \tag{5.29}$$

We recall that it is proved in Chap. 7 (Sect. 7.5) using the theory of covariant derivatives in vector bundles that for any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned} \nabla_{\boldsymbol{v}}\mathcal{C} &= \eth_{\boldsymbol{v}}\mathcal{C} + \frac{1}{2}[\omega_{\boldsymbol{v}}, \mathcal{C}] \\ &= \eth_{\boldsymbol{v}}\mathcal{C} + \omega_{\boldsymbol{v}} \times \mathcal{C}, \end{aligned} \tag{5.30}$$

where $\omega_{\boldsymbol{v}} \times \mathcal{C} := \frac{1}{2}(\omega_{\boldsymbol{v}}\mathcal{C} - \mathcal{C}\omega_{\boldsymbol{v}})$ is the commutator of sections of the Clifford bundle.

Exercise 5.9 Choose a basis for $\mathcal{C}\ell(M, \mathtt{g})$ and verify by direct computation the validity of Eq. (5.30).

Remark 5.10 Note for future reference that if $v = \boldsymbol{g}(\boldsymbol{v}, \,)$ then

$$v \times \mathcal{C} = v \lrcorner \mathcal{C}. \tag{5.31}$$

Also take notice that

$$\nabla_{\boldsymbol{v}}\mathcal{C} = \boldsymbol{v} \cdot \overset{\nabla}{\boldsymbol{\partial}} \tag{5.32}$$

Definition 5.11 The form curvature operator is the mapping

$$\begin{aligned} \boldsymbol{\rho} &: \sec(\bigwedge\nolimits^1 T^*M \times \bigwedge\nolimits^1 T^*M) \to \operatorname{End} \bigwedge\nolimits^1 T^*M, \\ \boldsymbol{\rho}(u, v) &= [u \cdot \overset{\nabla}{\boldsymbol{\partial}}, v \cdot \overset{\nabla}{\boldsymbol{\partial}}] - [\![u, v]\!] \cdot \overset{\nabla}{\boldsymbol{\partial}} \\ &= [\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}} \end{aligned}$$

with $u = \boldsymbol{g}(\boldsymbol{u}, \,)$, $v = \boldsymbol{g}(\boldsymbol{v}, \,)$, $\boldsymbol{u}, \boldsymbol{v} \in \sec TU \subset \sec TM$

Definition 5.12 The form curvature extensor is the mapping

$$\begin{aligned} \rho &: \sec(\bigwedge\nolimits^1 T^*M \times \bigwedge\nolimits^1 T^*M \times \bigwedge\nolimits^1 T^*M) \to \sec \bigwedge\nolimits^1 T^*M, \\ \rho(u, v, w) &= [u \cdot \overset{\nabla}{\boldsymbol{\partial}}, v \cdot \overset{\nabla}{\boldsymbol{\partial}}]w - [\![u, v]\!] \cdot \overset{\nabla}{\boldsymbol{\partial}} w \\ &= [\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}]w - \nabla_{[\boldsymbol{u},\boldsymbol{v}]} w \end{aligned}$$

with $u = \boldsymbol{g}(\boldsymbol{u}, \,)$, $v = \boldsymbol{g}(\boldsymbol{v}, \,)$, $w = \boldsymbol{g}(\boldsymbol{w}, \,)$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \sec TU \subset TM$

It is obvious that for any Riemann-Cartan connection we have

$$\rho(u, v, w) = -\rho(v, u, w), \tag{5.33}$$

One can easily verify that for a Levi-Civita connection we have

$$\rho(u, v, w) + \rho(v, w, u) + \rho(w, u, v) = 0. \tag{5.34}$$

Note however that Eq. (5.34) is not true for a general connection.

Definition 5.13 The mapping

$$\begin{aligned} \mathbf{R} : \sec(\bigwedge^1 T^*M)^4 &\to \sec \bigwedge^0 T^*M, \\ \mathbf{R}(a, b, c, w) &= -\rho(a, b, c) \cdot w, \\ a = \boldsymbol{g}(\boldsymbol{a},), b &= \boldsymbol{g}(\boldsymbol{b},), c = \boldsymbol{g}(\boldsymbol{c},), w = \boldsymbol{g}(\boldsymbol{w},) \end{aligned} \tag{5.35}$$

with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{w} \in \sec TU \subset TM$ is called the curvature tensor.

One can verify that for the connection ∇

$$\mathbf{R}(a, b, c, w) = -\mathbf{R}(b, a, c, w), \tag{5.36}$$

$$\mathbf{R}(a, b, c, w) = -\mathbf{R}(a, b, w, c), \tag{5.37}$$

and that for a Levi-Civita connection

$$\mathbf{R}(a, b, c, w) = \mathbf{R}(c, w, a, b), \tag{5.38}$$

$$\mathbf{R}(a, b, c, w) + \mathbf{R}(b, c, a, w) + \mathbf{R}(c, a, b, w) = 0, \tag{5.39}$$

Eq. (5.39) is known as the first Bianchi identity.

Proposition 5.14 *There exists a smooth* (2-2)*-extensor field,*

$$\begin{aligned} \mathfrak{R} : \sec \bigwedge^2 T^*M &\to \bigwedge^2 T^*M, \\ B &\mapsto \mathfrak{R}(B) \end{aligned} \tag{5.40}$$

called the curvature biform such that for any $a, b, c, d \in \sec \bigwedge^1 T^*M$ *we have*

$$\mathbf{R}(a, b, c, d) = \mathfrak{R}(a \wedge b) \cdot (c \wedge d) = -(c \wedge d) \lrcorner \mathfrak{R}(a \wedge b) \tag{5.41}$$

Such $B \mapsto \mathfrak{R}(B)$ *is given by*

$$\mathfrak{R}(B) = -\frac{1}{4} B \cdot (\partial_a \wedge \partial_b) \partial_c \wedge \partial_d \rho(a, b, c) \cdot d, \tag{5.42}$$

and we also have

$$\mathfrak{R}(a \wedge b) = -\frac{1}{2}\partial_c \wedge \partial_d \rho(a, b, c) \cdot d. \tag{5.43}$$

Proof First, we verify that Eqs. (5.42) and (5.43) are indeed equivalent. Indeed, Eq. (5.42) implies Eq. (5.43) since we have

$$\begin{aligned}
\mathfrak{R}(a \wedge b) &= -\frac{1}{4}(a \wedge b) \cdot (\partial_p \wedge \partial_q)\partial_{\mathbf{c}} \wedge \partial_{\mathbf{d}} \rho(p, q, c) \cdot d \\
&= -\frac{1}{4} \det \begin{bmatrix} a \cdot \partial_p & a \cdot \partial_q \\ b \cdot \partial_p & b \cdot \partial_q \end{bmatrix} \partial_{\mathbf{c}} \wedge \partial_{\mathbf{d}} \rho(p, q, c) \cdot d \\
&= -\frac{1}{4} \left(a \cdot \partial_p b \cdot \partial_q - a \cdot \partial_q b \cdot \partial_p\right) \partial_{\mathbf{c}} \wedge \partial_{\mathbf{d}} \rho(p, q, c) \cdot d \\
&= -\frac{1}{2} \left(a \cdot \partial_p b \cdot \partial_q\right) \partial_{\mathbf{c}} \wedge \partial_{\mathbf{d}} \rho(p, q, c) \cdot d \\
&= -\frac{1}{2}\partial_c \wedge \partial_d \rho(a, b, c) \cdot d\ .
\end{aligned}$$

Also, Eq. (5.43) implies Eq. (5.42) since taking into account that

$$B = \frac{1}{2} B \cdot (\partial_a \wedge \partial_b) a \wedge b$$

we have

$$\begin{aligned}
\mathfrak{R}(B) &= \mathfrak{R}(\frac{1}{2} B \cdot (\partial_a \wedge \partial_b) a \wedge b) \\
&= \frac{1}{2} B \cdot (\partial_a \wedge \partial_b) \mathfrak{R}(a \wedge b) \\
&= -\frac{1}{4} B \cdot (\partial_a \wedge \partial_b) \partial_c \wedge \partial_d \rho(a, b, c) \cdot d\ .
\end{aligned}$$

Now, we show the validity of Eq. (5.41). We have taking into account Eq. (5.43)

$$\begin{aligned}
\mathfrak{R}(a \wedge b) \lrcorner (c \wedge d) &= -\frac{1}{2}(c \wedge d) \cdot (\partial_p \wedge \partial_q) \rho(a, b, p) \cdot q \\
&= -\frac{1}{2} \det \begin{bmatrix} c \cdot \partial_p & c \cdot \partial_q \\ d \cdot \partial_p & d \cdot \partial_q \end{bmatrix} \rho(a, b, p) \cdot q \\
&= -\frac{1}{2} \left(c \cdot \partial_p d \cdot \partial_q - c \cdot \partial_q d \cdot \partial_p\right) \rho(a, b, p) \cdot q
\end{aligned}$$

$$= -c \cdot \partial_p d \cdot \partial_q \rho(a, b, p) \cdot q$$
$$= -\rho(a, b, c) \cdot d = \mathbf{R}(a, b, c, d),$$

and the proposition is proved. ■

Proposition 5.15 *The curvature biform* $\mathfrak{R}(u \wedge v)$ *is given by*[6]

$$\mathfrak{R}(u \wedge v) = u \cdot \eth\omega\ (v) - v \cdot \eth\omega\ (u) + \omega(u) \times \omega(v). \tag{5.44}$$

Proof The proof is given in three steps (a)–(c)

(a) We first show that Eq. (5.44) can be written as

$$\mathfrak{R}(u \wedge v) = u \cdot \overset{\nabla}{\boldsymbol{\partial}} \boldsymbol{\omega_v} - v \cdot \overset{\nabla}{\boldsymbol{\partial}} \boldsymbol{\omega_u} - \frac{1}{2}[\boldsymbol{\omega_u}, \boldsymbol{\omega_v}] - \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]}$$
$$= \nabla_{\boldsymbol{u}} \boldsymbol{\omega_v} - \nabla_{\boldsymbol{v}} \boldsymbol{\omega_u} - \frac{1}{2}[\boldsymbol{\omega_u}, \boldsymbol{\omega_v}] - \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]}, \tag{5.45}$$

with $u = \boldsymbol{g}(\boldsymbol{u}, \,)$, $v = \boldsymbol{g}(\boldsymbol{v}, \,)$. Indeed,we have

$$u \cdot \overset{\nabla}{\boldsymbol{\partial}}(\omega(v)) = u \cdot \eth(\omega(v)) + \frac{1}{2}[\omega(u), \omega(v)]. \tag{5.46}$$

and recalling the definition of the derivative of an extensor field, it is:

$$(u \cdot \eth\omega)\ (v) \equiv u \cdot \eth\omega\ (v) := u \cdot \eth(\omega(v)) - \omega(u \cdot \eth v). \tag{5.47}$$

we have,

$$u \cdot \eth(\omega(v)) - v \cdot \eth(\omega(u)) = u \cdot \eth\ \omega(v) - v \cdot \eth\ \omega(u) + \omega(u \cdot \eth v) - \omega(v \cdot \eth u)$$
$$= u \cdot \eth\ \omega(v) - v \cdot \eth\ \omega(u) + \omega([\![u, v]\!])$$
$$= u \cdot \eth\ \boldsymbol{\omega_v} - v \cdot \eth\ \boldsymbol{\omega_u} + \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]}, \tag{5.48}$$

and using the above equations in Eq. (5.44) we arrive at Eq. (5.45).

(b) Next we show (by finite induction) that for any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$([\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]})\mathcal{C} = \frac{1}{2}[\mathfrak{R}(u \wedge v), \mathcal{C}], \tag{5.49}$$

with $\mathfrak{R}(u \wedge v)$ given by Eq. (5.45). Given that any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ is a sum of nonhomogeneous differential forms, i.e. $\mathcal{C} = \sum_{p=0}^{n} \mathcal{C}_p$ with

[6] Recall that in Eq. (5.45) $[\boldsymbol{u}, \boldsymbol{v}]$ is the standard Lie bracket of vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ and $[\![u, v]\!] := u \cdot \boldsymbol{\partial} v - v \cdot \boldsymbol{\partial} u$ is the commutator of the 1-form fields u and v.

$\mathcal{C}_p \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ and taking into account that $\mathcal{C}_p = \frac{1}{r!}\mathcal{C}_{\mathbf{i}_1 \cdots \mathbf{i}_p}\theta^{\mathbf{i}_1} \cdots \theta^{\mathbf{i}_p}$ it is enough to verify the formula for p-forms. We first verify the validity of the formula for a 1-form $\theta^i \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Using Eq. (5.45) and the Jacobi identity

$$[\![\boldsymbol{\omega}_v, [\![\boldsymbol{\omega}_u, \theta^{\mathbf{i}}]\!][\![\boldsymbol{\omega}_u, [\![\theta^{\mathbf{i}}, \boldsymbol{\omega}_v]\!][\![\theta^{\mathbf{i}}, [\![\boldsymbol{\omega}_v, \boldsymbol{\omega}_u]\!] \tag{5.50}$$

we have that

$$\begin{aligned}
&\frac{1}{2}[\mathfrak{R}(u \wedge v), \theta^{\mathbf{i}}] \qquad (5.51)\\
&= \frac{1}{2}\left\{(\nabla_u \omega_v \theta^{\mathbf{i}} - \theta^{\mathbf{i}} \nabla_u \omega_v + \frac{1}{2}[[\boldsymbol{\omega}_v, \boldsymbol{\omega}_u], \theta^{\mathbf{i}}] - (\nabla_v \boldsymbol{\omega}_u \theta^{\mathbf{i}} + \theta^{\mathbf{i}} \nabla_v \boldsymbol{\omega}_u - [\boldsymbol{\omega}_{[u,v]}, \theta^{\mathbf{i}}]\right\}\\
&= \frac{1}{2}\left\{\left[\nabla_u \boldsymbol{\omega}_v, \theta^{\mathbf{i}}\right] + \frac{1}{2}[\boldsymbol{\omega}_v, [\boldsymbol{\omega}_u, \theta^{\mathbf{i}}]] - \left[\nabla_v \boldsymbol{\omega}_v, \theta^{\mathbf{i}}\right] - \frac{1}{2}[\boldsymbol{\omega}_u, [\boldsymbol{\omega}_v, \theta^{\mathbf{i}}]] - [\boldsymbol{\omega}_{[u,v]}, \theta^{\mathbf{i}}]\right\}\\
&= \nabla_u \nabla_v \theta^{\mathbf{i}} - \nabla_v \nabla_u \theta^{\mathbf{i}} - \nabla_{[u,v]} \theta^{\mathbf{i}}. \qquad (5.52)
\end{aligned}$$

Now, suppose the formula is valid for p-forms. Let us calculate the first member of Eq. (5.49) for the $(r+1)$-form $\theta^{\mathbf{i}_1 \cdots \mathbf{i}_{r+1}} = \theta^{\mathbf{i}_1}\theta^{\mathbf{i}_2} \cdots \theta^{\mathbf{i}_{r+1}}$. We have

$$\begin{aligned}
&\nabla_u \nabla_v (\theta^{\mathbf{i}_1 \cdots \mathbf{i}_{r+1}}) - \nabla_v \nabla_u (\theta^{\mathbf{i}_1 \cdots \mathbf{i}_{r+1}}) - \nabla_{[u,v]} (\theta^{\mathbf{i}_1 \cdots \mathbf{i}_{r+1}})\\
&= \nabla_u ((\nabla_v \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} + \theta^{\mathbf{i}_1} \nabla_v \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}) - \nabla_v ((\nabla_u \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} + \theta^{\mathbf{i}_1} \nabla_u \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}})\\
&\quad - (\nabla_{[u,v]} \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}) - \theta^{\mathbf{i}_1} \nabla_{[u,v]} \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&= (\nabla_u \nabla_v \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} + \nabla_v \theta^{\mathbf{i}_1} \nabla_u \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} + \nabla_u \theta^{\mathbf{i}_1} \nabla_v \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} + \theta^{\mathbf{i}_1} \nabla_u \nabla_v \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&\quad - (\nabla_v \nabla_u \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} - \nabla_u \theta^{\mathbf{i}_1} \nabla_v \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} - \nabla_v \theta^{\mathbf{i}_1} \nabla_u \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} - \theta^{\mathbf{i}_1} \nabla_v \nabla_u \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&\quad - (\nabla_{[u,v]} \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}) - \theta^{\mathbf{i}_1} \nabla_{[u,v]} \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&= \theta^{\mathbf{i}_1} (\nabla_u \nabla_v \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} - \nabla_v \nabla_u \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}} - \nabla_{[u,v]} \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}})\\
&\qquad + (\nabla_u \nabla_v \theta^{\mathbf{i}_1} - \nabla_v \nabla_u \theta^{\mathbf{i}_1} - \nabla_{[u,v]} \theta^{\mathbf{i}_1})\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&= \theta^{\mathbf{i}_1} (\frac{1}{2}[\mathfrak{R}(u \wedge v), \theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}]) + (\frac{1}{2}[\mathfrak{R}(u \wedge v), \theta^{\mathbf{i}_1}])\theta^{\mathbf{i}_2 \cdots \mathbf{i}_{r+1}}\\
&= \frac{1}{2}[\mathfrak{R}(u \wedge v), \theta^{\mathbf{i}_1 \cdots \mathbf{i}_{r+1}}], \qquad (5.53)
\end{aligned}$$

where the last line of Eq. (5.53) is the second member of Eq. (5.49) evaluated for $\theta^{\mathbf{i}_1}\theta^{\mathbf{i}_2} \cdots \theta^{\mathbf{i}_{r+1}}$.

(c) Now, it remains to verify that

$$\mathbf{R}(u, v, t, z) = (t \wedge z) \cdot \mathfrak{R}(u \wedge v), \tag{5.54}$$

with $\mathfrak{R}(u \wedge v)$ given by Eq. (5.45). Indeed, from a well known identity, we have that for any $t, z \in \sec \bigwedge^1 T^*M$, and $\mathfrak{R}(u \wedge v) \in \sec \bigwedge^2 T^*M$ it is

$$
\begin{aligned}
(z \wedge t) \cdot \mathfrak{R}(u \wedge v) &= -z \lrcorner (t \lrcorner \mathfrak{R}(u \wedge v)) \\
&= z \cdot (\mathfrak{R}(u \wedge v) \llcorner t) \\
&= \frac{1}{2} z \cdot [\mathfrak{R}(u, v), t] \\
&\overset{\text{Eq. (5.49)}}{=} z \cdot (\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} t - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} t - \nabla_{[\boldsymbol{u}, \boldsymbol{v}]} t)
\end{aligned}
$$

and the proposition is proved. ■

In particular we have:

$$
\begin{aligned}
&\mathbf{R}(u, v, z, t) = z_{\mathbf{c}} t^{\mathbf{d}} u^{\mathbf{a}} v^{\mathbf{b}} R^{\mathbf{c} \cdots}_{\cdot \mathbf{dab}}, \\
&\quad R^{\mathbf{d} \cdots}_{\cdot \mathbf{cab}} = \boldsymbol{e}_{\mathbf{a}} (\omega^{\mathbf{d} \cdot \cdot}_{\cdot \mathbf{bc}}) - \boldsymbol{e}_{\mathbf{b}} (\omega^{\mathbf{d} \cdot \cdot}_{\cdot \mathbf{ac}}) + \omega^{\mathbf{d} \cdot \cdot}_{\cdot \mathbf{ak}} \omega^{\mathbf{k} \cdot \cdot}_{\cdot \mathbf{bc}} - \omega^{\mathbf{d} \cdot \cdot}_{\cdot \mathbf{bk}} \omega^{\mathbf{k} \cdot \cdot}_{\cdot \mathbf{ac}} - c_{\cdot \mathbf{ab}}{}^{k}{}_{\cdot \cdot} \omega^{d \cdot \cdot}_{\cdot \mathbf{kc}} \,.
\end{aligned} \tag{5.55}
$$

and

$$
\mathbf{R}(\theta^{\mathbf{a}}, \theta^{\mathbf{b}}, \theta_{\mathbf{a}}, \theta_{\mathbf{b}}) = (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \cdot \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) = R, \tag{5.56}
$$

where R is the curvature scalar.

Proposition 5.16 *For any* $v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

$$
[\nabla_{\boldsymbol{e}_{\mathbf{a}}}, \nabla_{\boldsymbol{e}_{\mathbf{b}}}] v = \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner v - (T^{\mathbf{c}}_{\mathbf{ab}} - \omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ab}} + \omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ba}}) \nabla_{\boldsymbol{e}_{\mathbf{c}}} v. \tag{5.57}
$$

Proof From Eq. (5.49) we can write

$$
\begin{aligned}
[\nabla_{\boldsymbol{e}_{\mathbf{a}}}, \nabla_{\boldsymbol{e}_{\mathbf{b}}}] v &= \frac{1}{2} [\mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}), v] + \nabla_{[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}]} v \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner v + \nabla_{([\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}] - \nabla_{\boldsymbol{e}_a} \boldsymbol{e}_{\mathbf{b}} + \nabla_{\boldsymbol{e}_{\mathbf{b}}} \boldsymbol{e}_{\mathbf{a}})} v + \nabla_{\nabla_{\boldsymbol{e}_{\mathbf{a}}} \boldsymbol{e}_{\mathbf{b}}} v - \nabla_{\nabla_{\boldsymbol{e}_{\mathbf{b}}} \boldsymbol{e}_{\mathbf{a}}} v \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner v + \nabla_{-T^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ab}} \boldsymbol{e}_{\mathbf{c}}} v + \nabla_{\omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ab}} \boldsymbol{e}_{\mathbf{c}}} v - \nabla_{\omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ba}} \boldsymbol{e}_{\mathbf{c}}} v \\
&= \mathfrak{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \llcorner v - (T^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ab}} - \omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ab}} + \omega^{\mathbf{c} \cdot \cdot}_{\cdot \mathbf{ba}}) \nabla_{\boldsymbol{e}_{\mathbf{c}}} v
\end{aligned}
$$

which proves the proposition. ■

Proposition 5.17

$$
\mathfrak{R}(\theta^{\mathbf{a}} \wedge \theta_{\mathbf{b}}) = \mathcal{R}^{\mathbf{a} \cdot}_{\cdot \mathbf{b}} = d\omega^{\mathbf{a} \cdot}_{\cdot \mathbf{b}} + \omega^{\mathbf{a} \cdot}_{\cdot \mathbf{c}} \wedge \omega^{\mathbf{c} \cdot}_{\cdot \mathbf{b}} \tag{5.58}
$$

Proof Recall that using

$$([\nabla_{e_\mathbf{k}}, \nabla_{e_\mathbf{l}}] - \nabla_{[e_\mathbf{k}, \mathbf{a_l}]})\theta^\mathbf{j} = \boldsymbol{\rho}(e_\mathbf{k}, e_\mathbf{l})\theta^\mathbf{j} = -R^{\mathbf{j}\cdots}_{\cdot\mathbf{ikl}}\theta^\mathbf{i},$$
$$([\nabla_{e_\mathbf{k}}, \nabla_{e_\mathbf{l}}] - \nabla_{[e_\mathbf{k}, \mathbf{a_l}]})\theta_\mathbf{j} = \boldsymbol{\rho}(e_\mathbf{k}, e_\mathbf{l})\theta_\mathbf{j} = R^{\mathbf{i}\cdots}_{\cdot\mathbf{jkl}}\theta_\mathbf{i}, \tag{5.59}$$

we have

$$\mathfrak{R}(\theta_\mathbf{a} \wedge \theta_\mathbf{b}) \llcorner v = v^\mathbf{m} \boldsymbol{\rho}(e_\mathbf{a}, e_\mathbf{b})\theta_\mathbf{m} = v^\mathbf{m} R^{\mathbf{i}\cdots}_{\cdot\mathbf{mab}}\theta_\mathbf{i}. \tag{5.60}$$

On the other hand, for a general connection, we must write

$$\mathcal{R}_\mathbf{ab} := \frac{1}{2} R_\mathbf{klab}\theta^\mathbf{k} \wedge \theta^\mathbf{l} \tag{5.61}$$

and then

$$\mathcal{R}_\mathbf{ab} \llcorner v = \frac{1}{2} v^\mathbf{m} R_\mathbf{klab}(\theta^\mathbf{k} \wedge \theta^\mathbf{l}) \llcorner \theta_\mathbf{m} = -v^\mathbf{m} R_\mathbf{mlab}\theta^\mathbf{l} = v^\mathbf{m} R_\mathbf{lmab}\theta^\mathbf{l} = v^\mathbf{m} R^{\mathbf{l}\cdots}_{\cdot\mathbf{mab}}\theta_\mathbf{l}$$

and the proposition is proved. ■

Proposition 5.18 *The Ricci 1-forms*[7] $\mathcal{R}^\mathbf{d} := R^\mathbf{d}_\mathbf{b}\theta^\mathbf{b}$ *and the curvature biform* $\mathfrak{R}(\theta_\mathbf{a} \wedge \theta_\mathbf{b})$ *for the Levi-Civita connection* D *of* $\boldsymbol{g}$ *are related by,*

$$\mathcal{R}^\mathbf{d} = \frac{1}{2}(\theta^\mathbf{a} \wedge \theta^\mathbf{b})(\mathfrak{R}(\theta_\mathbf{a} \wedge \theta_\mathbf{b}) \llcorner \theta^\mathbf{d}) \tag{5.62}$$

Proof Recalling that the Ricci operator is given by

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ \theta^\mathbf{d} = \frac{1}{2}(\theta^\mathbf{a} \wedge \theta^\mathbf{b}) \left([D_{e_\mathbf{a}}, D_{e_\mathbf{b}}]\theta^\mathbf{d} - c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} D_{e_\mathbf{c}}\theta^\mathbf{d}\right) \tag{5.63}$$

and moreover taking into account that by the first Bianchi identity it is $\mathcal{R}^{\mathbf{c}\cdot}_{\cdot d} \wedge \theta_\mathbf{c} = 0$, we have

$$\begin{aligned}
\frac{1}{2}(\theta^\mathbf{a} \wedge \theta^\mathbf{b}) \left([D_{e_\mathbf{a}}, D_{e_\mathbf{b}}]\theta^\mathbf{d} - c^{\mathbf{c}\cdot}_{\cdot\mathbf{ab}} D_{e_\mathbf{c}}\theta^\mathbf{d}\right) &= -\frac{1}{2}(\theta^\mathbf{a} \wedge \theta^\mathbf{b}) R^{\mathbf{d}\cdots}_{\cdot\mathbf{cab}}\theta^\mathbf{c} = \mathcal{R}^\mathbf{cd}\theta_\mathbf{c} \\
&= \mathcal{R}^\mathbf{cd} \lrcorner \theta_\mathbf{c} + \mathcal{R}^\mathbf{cd} \wedge \theta_\mathbf{c} = -\theta_\mathbf{c} \lrcorner \mathcal{R}^\mathbf{cd} \\
&= -\frac{1}{2}\theta_\mathbf{c} \lrcorner (\theta^\mathbf{a} \wedge \theta^\mathbf{b}) R^{\mathbf{cd}\cdot\cdot}_{\cdot\cdot\mathbf{ab}} \\
&= -R^{\mathbf{cd}\cdot\cdot}_{\cdot\cdot\mathbf{cb}}\theta^\mathbf{b} = \mathcal{R}^\mathbf{d}
\end{aligned} \tag{5.64}$$

which proves the proposition. ■

[7] The $R^\mathbf{a}_\mathbf{b} := \eta^\mathbf{ca} R^{\mathbf{k}\cdots}_{\cdot\mathbf{ckb}}$ are the components of the Ricci tensor.

Proposition 5.18 suggests the

Definition 5.19 The Ricci extensor is the mapping

$$\begin{aligned} \mathcal{R} : \sec \bigwedge^1 T^*M &\to \sec \bigwedge^1 T^*M, \\ \mathcal{R}(v) &= \partial_u \mathcal{R}(u \wedge v). \end{aligned} \tag{5.65}$$

Remark 5.20 Of course, we must have $\mathcal{R}(\theta^{\mathbf{d}}) = \mathcal{R}^{\mathbf{d}}$. Moreover, we have

$$\begin{aligned} \partial_u \mathfrak{R}(u \wedge v) &= \theta^{\mathbf{b}} \frac{\partial}{\partial u^{\mathbf{b}}} \mathfrak{R}(u_{\mathbf{k}} \theta^{\mathbf{k}} \wedge v) = \theta^{\mathbf{b}} \frac{\partial}{\partial u^{\mathbf{b}}} u^{\mathbf{k}} \mathfrak{R}(\theta_{\mathbf{k}} \wedge v) \\ &= \theta^{\mathbf{b}} \mathfrak{R}(\delta_{\mathbf{b}}^{\mathbf{k}} \theta_{\mathbf{k}} \wedge v) = \theta^{\mathbf{b}} \mathfrak{R}(\theta_{\mathbf{b}} \wedge v) \\ &= \theta^{\mathbf{b}} \lrcorner \mathfrak{R}(\theta_{\mathbf{b}} \wedge v) + \theta^{\mathbf{b}} \wedge \mathfrak{R}(\theta_{\mathbf{b}} \wedge v) \\ &= \theta^{\mathbf{b}} \lrcorner \mathfrak{R}(\theta_{\mathbf{b}} \wedge v). \end{aligned}$$

So,

$$\partial_u \mathfrak{R}(u \wedge v) = \partial_u \lrcorner \mathfrak{R}(u \wedge v) \text{ and } \partial_u \wedge \mathfrak{R}(u \wedge v) = 0. \tag{5.66}$$

5.3 The Riemannian or Semi-Riemannian Geometry of a Submanifold M of $\mathring{M}$

5.3.1 Motivation

Any manifold $M, \dim M = m$, according to Whitney's theorem (see, e.g., [1]), can be realized as a submanifold of $\mathbb{R}^n$, with $n = 2m$. However, if M carries additional structure the number n in general must be greater than $2m$. Indeed, it has been shown by Eddington [8] that if dim $M = 4$ and if M carries a Lorentzian metric $\boldsymbol{g}$ and which moreover satisfies Einstein's equations, then M can be *locally* embedded in a (pseudo)Euclidean space $\mathbb{R}^{1,9}$. Also, isometric embeddings of general Lorentzian spacetimes would require a lot of extra dimensions [4]. Indeed, a compact Lorentzian manifold can be embedded isometrically in $\mathbb{R}^{2,46}$ and a non-compact one can be embedded isometrically in $\mathbb{R}^{2,87}$! In particular this last result shows that the spacetime of M-theory [7, 14] may not be large enough to contain 4-dimensional branes (representing Lorentzian spacetimes) with arbitrary metric tensors. In what follows we show how to relate the intrinsic differential geometry of a structure $(M, \boldsymbol{g}, D)$ where $\boldsymbol{g}$ is a metric of signature (p,q), D is its Levi-Civita connection and M is an orientable *proper* submanifold of $\mathring{M}$, i.e., there is defined on M a global volume element $\tau_{\boldsymbol{g}} = I_m$ whose expression on $U \subset M$ is given by

$$I_m = \theta^1\theta^2 \cdots \theta^m. \tag{5.67}$$

We suppose moreover that $\mathring{M} \simeq \mathbb{R}^n$ and that it is equipped with a metric $\mathring{\boldsymbol{g}}$ of signature $(\mathring{p}, \mathring{q}) = n$. However, take notice that our presentation in the form of a local theory is easily adapted for a general manifold $\mathring{M}$.

Projection Operator P

Definition 5.21 Let $\mathcal{C} = \sum_{r=0}^{n} \mathcal{C}_r$, with $\mathcal{C}_r \in \sec \bigwedge^r T^*\mathring{M} \hookrightarrow \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$. The Projection operator on M is the extensor field

$$\begin{aligned} \mathbf{P} &: \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}}) \to \sec \mathcal{C}\ell(M, \mathtt{g}), \\ \mathbf{P}(\mathcal{C}) &= (\mathcal{C} \lrcorner I_m) I_m^{-1}. \end{aligned} \tag{5.68}$$

Remark 5.22 Note that for all $\mathcal{C}_k \in \sec \bigwedge^k T^*\mathring{M} \hookrightarrow \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$, if $k > m$ then $\mathbf{P}(\mathcal{C}_k) = 0$, but of course, it may happen that even if $A_r \in \sec \bigwedge^r T^*\mathring{M} \hookrightarrow \sec \mathcal{C}\ell(M, \mathring{\mathtt{g}})$ with $r \leq m$ we may have $\mathbf{P}(A_r) = 0$.

We define the complement of $\mathbf{P}$ by

$$\mathbf{P}_\perp(\mathcal{C}) = \mathcal{C} - \mathbf{P}(\mathcal{C}) \tag{5.69}$$

and it is clear that $\mathbf{P}_\perp(\mathcal{C})$ have only components lying outside $\mathcal{C}\ell(M, \mathtt{g})$. It is quite clear also that any $\mathcal{C}$ with components not all belonging to $\sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ will satisfy $\mathcal{C} \lrcorner I_m = 0$.

Having introduced in Sect. 5.1 the derivative operators $\mathring{\boldsymbol{\partial}}$ and its restriction $\mathring{\mathfrak{d}} = \mathring{\boldsymbol{\partial}}\big|_M$ (Eqs. (5.5) and (5.6)) we extend the action of $\mathbf{P}$ to act on the operator $\mathring{\mathfrak{d}}$, defining:

$$\mathbf{P}(\mathring{\mathfrak{d}}) = \textstyle\sum_{\mathbf{k}=1}^{m} \mathbf{P}(\theta^{\mathbf{k}} \mathring{D}_{\boldsymbol{e}_\mathbf{k}}) := \sum_{\mathbf{k}=1}^{m} \mathbf{P}(\theta^{\mathbf{k}}) \mathring{D}_{\boldsymbol{e}_\mathbf{k}} = \sum_{\mathbf{k}=1}^{m} \theta^{\mathbf{k}} \mathring{D}_{\boldsymbol{e}_\mathbf{k}} = \mathring{\mathfrak{d}}. \tag{5.70}$$

Shape Operator S

Definition 5.23 Given $\mathcal{C} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{g})$ we define the shape operator

$$\mathbf{S} : \sec \mathcal{C}\ell(\mathring{M}, \mathring{g}) \to \sec \mathcal{C}\ell(\mathring{M}, \mathring{g}),$$
$$\mathbf{S}(\mathcal{C}) = \mathring{\partial}\mathbf{P}\,(\mathcal{C}) = \mathring{\partial}(\mathbf{P}(\mathcal{C})) - \mathbf{P}(\mathring{\partial}\mathcal{C}). \tag{5.71}$$

5.3.2 *Induced Metric and Induced Connection*

For any $\mathcal{C} \in \sec \mathcal{C}\ell(M, g)$ and $\boldsymbol{v} \in \sec TM$ we write as usual [3, 13]

$$\mathring{D}_{\boldsymbol{v}}\mathcal{C} = (\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\parallel} + (\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\perp} \tag{5.72}$$

where $(\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\parallel} \in \sec \mathcal{C}\ell(M, g)$ and $(\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\perp} \in \sec[\mathcal{C}\ell(M, g)]_{\perp}$ where $[\mathcal{C}\ell(M, g)]_{\perp}$ is the orthogonal complement of $\mathcal{C}\ell(\mathring{M}, \mathring{g})$ in $\mathcal{C}\ell(M, g)$.

As it is very well known [3, 13] if $\boldsymbol{g} := \boldsymbol{i}^*\mathring{\boldsymbol{g}}$ and $\boldsymbol{v} \in \sec TM$ (and $v = \boldsymbol{g}(\boldsymbol{v},))$ and $\mathcal{C} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{g})$ the Levi-Civita connection D of $\boldsymbol{g} := \boldsymbol{i}^*\mathring{\boldsymbol{g}}$ is given by

$$D_{\boldsymbol{v}}\mathcal{C} := (\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\parallel} \tag{5.73}$$

and of course

$$D_{\boldsymbol{v}}\mathcal{C} := (v \cdot \mathring{\boldsymbol{\partial}}\mathcal{C})_{\parallel} \tag{5.74}$$

Moreover, note that we can write for any $\mathcal{C} \in \sec \mathcal{C}\ell(M, g)$

$$v \cdot \boldsymbol{\partial}\mathcal{C} = (v \cdot \mathring{\boldsymbol{\partial}}\mathcal{C})_{\parallel} = \mathbf{P}(v \cdot \mathring{\boldsymbol{\partial}}\mathcal{C}) \tag{5.75}$$

Also, writing

$$(\mathring{D}_{\boldsymbol{v}}\mathcal{C})_{\perp} := \mathbf{P}_{\perp}(v \cdot \mathring{\boldsymbol{\partial}}\mathcal{C}) \tag{5.76}$$

we have

$$v\cdot\boldsymbol{\partial} = \mathbf{P}(v\cdot\mathring{\boldsymbol{\partial}}) = (v\cdot\mathring{\boldsymbol{\partial}})_{\parallel} = (v\cdot\mathring{\partial})_{\parallel} = v\cdot\mathring{\boldsymbol{\partial}} - \mathbf{P}_{\perp}(v\cdot\mathring{\boldsymbol{\partial}}) = v\cdot\mathring{\partial} - \mathbf{P}_{\perp}(v\cdot\mathring{\partial}). \tag{5.77}$$

So, it is

$$v \cdot \mathring{\partial} I_m I_m^{-1} = \sum_{j=1}^{m} \theta^1 \cdots (D_{\boldsymbol{v}}\theta^j + \mathbf{P}_{\perp}(v \cdot \mathring{\partial}\theta^{\mathrm{j}})) \cdots \theta^m I_m^{-1}$$
$$= D_{\boldsymbol{v}} I_m I_m^{-1} + \mathbf{P}_{\perp}(v \cdot \mathring{\partial}\theta_j) \wedge \theta^j. \tag{5.78}$$

Now, $D_{\boldsymbol{v}} I_m \in \sec \bigwedge^m T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is a multiple of I_m and since $I_m^2 = \pm 1$ depending on the signature of the metric $\boldsymbol{g}$ we have that $D_{\boldsymbol{v}} I_m = 0$.

Indeed,

$$0 = D_{\boldsymbol{v}} I_m^2 = 2(D_{\boldsymbol{v}} I_m) I_m \tag{5.79}$$

and so

$$0 = (D_{\boldsymbol{v}} I_m) I_m I_m^{-1} = D_{\boldsymbol{v}} I_m.$$

In any Clifford algebra bundle, in particular $\mathcal{C}\ell(M, \mathtt{g})$ we can build multiples of the I_m not only multiplying it by an scalar function, but also multiplying it by a an appropriated biform. This result will be used below in the definition of the shape biform.

5.3.3 $\mathbf{S}(\mathring{\boldsymbol{v}}) = \mathbf{S}(\boldsymbol{v}) = \boldsymbol{\partial}_u \wedge \mathbf{P}_u\ (\mathring{\boldsymbol{v}})$ *and* $\mathbf{S}(\mathring{\boldsymbol{v}}_\perp) = \boldsymbol{\partial}_u \lrcorner \mathbf{P}_u\ (\mathring{\boldsymbol{v}})$

For any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ it is $\mathcal{C} = \mathbf{P}(\mathcal{C})$ we have (with $u \in \sec \bigwedge^1 T^* \mathring{M} \hookrightarrow \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$)

$$\begin{aligned}
\mathring{\boldsymbol{\partial}}(\mathbf{P}(\mathcal{C})) &= \mathring{\boldsymbol{\partial}}\mathbf{P}\ (\mathcal{C}) - \mathbf{P}(\mathring{\boldsymbol{\partial}}\mathcal{C}) \\
&= \partial_u \mathbf{P}_u\ (\mathcal{C}) - \mathbf{P}(\mathring{\boldsymbol{\partial}}\mathcal{C}) \\
&= \partial_u \wedge \mathbf{P}_u\ (\mathcal{C}) + \partial_u \lrcorner \mathbf{P}_u\ (\mathcal{C}) - \mathbf{P}(\mathring{\boldsymbol{\partial}}\mathcal{C})
\end{aligned} \tag{5.80}$$

where

$$\mathbf{P}_u\ (\mathcal{C}) := u \cdot \mathring{\boldsymbol{\partial}}\mathbf{P}\ (\mathcal{C}) = u \cdot \mathring{\boldsymbol{\partial}}(\mathbf{P}(\mathcal{C})) - (\mathbf{P}(u \cdot \mathring{\boldsymbol{\partial}}\mathcal{C})). \tag{5.81}$$

Recall that for $\mathring{v} \in \sec \bigwedge^1 T^* \mathring{M} \hookrightarrow \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ we can write

$$\mathbf{S}(\mathring{v}) = \mathring{\boldsymbol{\partial}}\mathbf{P}\ (\mathring{v}) = \partial_u \wedge \mathbf{P}_u\ (\mathring{v}) + \partial_u \lrcorner \mathbf{P}_u\ (\mathring{v}) \tag{5.82}$$

where we used that for any $\mathring{\mathcal{C}} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ it is

$$\partial_u \mathbf{P}_u\ (\mathring{\mathcal{C}}) := \sum_{i=1}^m \theta^{\mathbf{i}} \frac{\partial}{\partial u^{\mathbf{i}}} u \cdot \mathring{\boldsymbol{\partial}}\mathbf{P}\ (\mathring{\mathcal{C}}) = \sum_{i=1}^m \theta^{\mathbf{i}} \mathring{D}_{\mathring{\mathbf{e}}_{\mathbf{i}}} \mathbf{P}\ (\mathring{\mathcal{C}}) = \mathring{\boldsymbol{\partial}}\mathbf{P}\ (\mathring{\mathcal{C}}) \tag{5.83}$$

Putting $\mathring{v} = \mathring{v}_\parallel + \mathring{v}_\perp = v + \mathring{v}_\perp$ we have the

Proposition 5.24

$$\mathbf{S}(\mathring{v}) = \mathbf{S}(v) = \partial_u \wedge \mathbf{P}_u\ (\mathring{v}), \qquad \mathbf{S}(\mathring{v}_\perp) = \partial_u \lrcorner \mathbf{P}_u(\mathring{v}). \tag{5.84}$$

Proof Indeed,

$$\mathbf{S}(\mathring{v}) = \mathbf{S}(\mathring{v}_\parallel) + \mathbf{S}(\mathring{v}_\perp) = \partial_u \wedge \mathbf{P}_u\ (\mathring{v}) + \partial_u \lrcorner \mathbf{P}_u\ (\mathring{v}) \tag{5.85}$$

and so, it is enough to show that

$$\partial_u \lrcorner \mathbf{P}_u\ (\mathring{v}_\parallel) = 0 \quad \text{and} \quad \partial_u \wedge \mathbf{P}_u\ (\mathring{v}_\perp) = 0, \tag{5.86}$$

From $\mathbf{P}^2(\mathring{v}) = \mathbf{P}(\mathring{v})$ we get

$$\mathbf{P}_u \mathbf{P}(\mathring{v}_\parallel) + \mathbf{P}\mathbf{P}_u(\mathring{v}_\parallel) = \mathbf{P}_u(\mathring{v}_\parallel), \tag{5.87}$$

So, for $\mathring{v}_\parallel$ and $\mathring{v}_\perp$ it is

$$\mathbf{P}\mathbf{P}_u(\mathring{v}_\parallel) = 0 \quad \text{and} \quad \mathbf{P}\mathbf{P}_u(\mathring{v}_\perp) = \mathbf{P}_u(\mathring{v}_\perp). \tag{5.88}$$

Since $\mathbf{P}\mathbf{P}_u(\mathring{v}_\parallel) = 0$ we have that

$$\partial_u \lrcorner \mathbf{P}\mathbf{P}_u(\mathring{v}_\parallel) = \mathbf{P}(\partial_u) \lrcorner \mathbf{P}_u(\mathring{v}_\parallel) = \partial_u \lrcorner \mathbf{P}_u(\mathring{v}_\parallel) = 0. \tag{5.89}$$

From $\mathbf{P}\mathbf{P}_u(\mathring{v}_\perp) = \mathbf{P}_u(\mathring{v}_\perp)$ we can write

$$\partial_u \wedge \mathbf{P}_u(\mathring{v}_\perp) = \partial_u \wedge \mathbf{P}\mathbf{P}_u(\mathring{v}_\perp) = \mathbf{P}(\partial_u) \wedge \mathbf{P}\mathbf{P}_u(\mathring{v}_\perp) = \mathbf{P}(\partial_u \wedge \mathbf{P}_u(\mathring{v}_\perp)). \tag{5.90}$$

Now, take $t, y \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. We have

$$\begin{aligned}
(t \wedge y) \cdot (\partial_u \wedge \mathbf{P}\mathbf{P}_u(\mathring{v}_\perp)) &= (t \wedge y) \cdot (\partial_u \wedge \mathbf{P}_u(\mathring{v}_\perp)) \\
&= t \lrcorner ((y \cdot \boldsymbol{\partial}_u \wedge \mathbf{P}_u(\mathring{v}_\perp) - \boldsymbol{\partial}_u \wedge ((y \lrcorner \mathbf{P}_u(\mathring{v}_\perp)) \\
&= t \lrcorner (y \cdot \mathring{\boldsymbol{\eth}}(\mathbf{P}(\mathring{v}_\perp)) - \mathbf{P}(y \cdot \mathring{\boldsymbol{\eth}} \mathring{v}_\perp) - \boldsymbol{\theta}^{\mathbf{i}} \\
&\wedge (y \lrcorner (\boldsymbol{\theta}_{\mathbf{i}} \cdot \mathring{\boldsymbol{\eth}}(\mathbf{P}(\mathring{v}_\perp)) - \mathbf{P}(\boldsymbol{\theta}_{\mathbf{i}} \cdot \mathring{\boldsymbol{\eth}} \mathring{v}_\perp))) \\
&= t \lrcorner (\mathbf{P}(y \cdot \mathring{\boldsymbol{\eth}} \mathring{v}_\perp) - \boldsymbol{\theta}^{\mathbf{i}} \wedge (y \lrcorner \mathbf{P}(\boldsymbol{\theta}_{\mathbf{i}} \cdot \mathring{\boldsymbol{\eth}} \mathring{v}_\perp))) \\
&= t \lrcorner (D_y \mathring{v}_\perp) - \boldsymbol{\theta}^{\mathbf{i}} \wedge (y \lrcorner (D_{e_i} \mathring{v}_\perp))) = 0
\end{aligned} \tag{5.91}$$

from where it follows that

$$\partial_u \wedge \mathbf{P}_u(\mathring{v}_\perp) = 0 \tag{5.92}$$

and the proposition is proved. ■

Shape Biform $\mathcal{S}$

Definition 5.25 The shape biform (a $(1, 2)$-extensor field) is the mapping

$$\mathcal{S} : \sec \bigwedge^1 T^* M \to \bigwedge^2 T^* M, \qquad v \mapsto \mathcal{S}(v), \tag{5.93}$$

such that

$$v \cdot \mathring{\partial} I_m = -\mathcal{S}(v) I_m. \tag{5.94}$$

From Eq. (5.15) it follows that

$$\mathcal{S}(v) \lrcorner I_m = 0 \text{ and } \mathcal{S}(v) \wedge I_m = 0. \tag{5.95}$$

Since $\mathcal{S}(v) \lrcorner I_m = 0$ it follows from Remark 5.22 that

$$\mathbf{P}(\mathcal{S}(v)) = 0. \tag{5.96}$$

Now, using the fact that $D_{\boldsymbol{v}} I_m = 0$ and Eq. (5.94) it follows from Eq. (5.78) that

$$v \cdot \mathring{\partial} I_m = (\mathbf{P}_\perp (v \cdot \mathring{\partial} \theta_j) \wedge \theta^j) I_m = -\mathcal{S}(v) I_m,$$

i.e.,

$$\mathcal{S}(v) = -\mathbf{P}_\perp (v \cdot \mathring{\partial} \theta_j) \wedge \theta^j. \tag{5.97}$$

Proposition 5.26 *For any* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *we have*

$$D_{\boldsymbol{v}} \mathcal{C} = v \cdot \mathring{\partial} \mathcal{C} + \mathcal{S}(v) \times \mathcal{C} = \mathring{D}_{\boldsymbol{v}} \mathcal{C} + \mathcal{S}(v) \times \mathcal{C}. \tag{5.98}$$

Proof Taking into account Eq. (5.97) we have for any $v, w \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

$$\begin{aligned} v \lrcorner \mathcal{S}(w) &= -v \lrcorner (\mathbf{P}_\perp (w \cdot \mathring{\partial} \theta_j) \wedge \theta^j) \\ &= -(\boldsymbol{v} \lrcorner (\mathbf{P}_\perp (w \cdot \mathring{\partial} \theta_j)) \theta^j + v^j \mathbf{P}_\perp (w \cdot \mathring{\partial} \theta_j) \\ &= v^j \mathbf{P}_\perp (w \cdot \mathring{\partial} \theta_j) = \mathbf{P}_\perp (w \cdot \mathring{\partial} v) - \mathbf{P}_\perp [(w \cdot \mathring{\partial} v^j) \theta_j] \\ &= \mathbf{P}_\perp (w \cdot \mathring{\partial} v). \end{aligned} \tag{5.99}$$

So,

$$\begin{aligned} v \cdot \mathring{\boldsymbol{\partial}} w &= \mathbf{P}(v \cdot \mathring{\boldsymbol{\partial}} w) + \mathbf{P}_\perp(v \cdot \mathring{\boldsymbol{\partial}} w) \\ &= D_{\boldsymbol{v}} w + w \lrcorner \mathcal{S}(v) \\ &= D_{\boldsymbol{v}} w - \mathcal{S}(v) \llcorner w. \end{aligned} \tag{5.100}$$

Now, for $v, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned} D_{\boldsymbol{v}}(wu) = (D_{\boldsymbol{v}} w)u + w D_{\boldsymbol{v}} u &= (\mathring{D}_{\boldsymbol{v}} w)u + (\mathcal{S}(v) \times w)u + w\mathring{D}_{\boldsymbol{v}} u + w(\mathcal{S}(v) \times u) \\ &= (\mathring{D}_{\boldsymbol{v}} wu) + (\mathcal{S}(v) \times w)u - w(u \times \mathcal{S}(v)) \\ &= (\mathring{D}_{\boldsymbol{v}} wu) + (\mathcal{S}(v) \times wu), \end{aligned} \tag{5.101}$$

from where the proposition follows trivially by finite induction. ■

Of course, it is

$$D_{\boldsymbol{e}_i} \mathcal{C} = \mathring{D}_{\mathbf{e_i}} \mathcal{C} + \mathcal{S}(v) \times \mathcal{C} \tag{5.102}$$

Now, recalling Eq. (5.30) we have[8]

$$\mathring{D}_{\mathbf{e_i}} \mathcal{C} = \boldsymbol{\partial}_{\mathbf{e_i}} \mathcal{C} + \mathring{\omega}_{e_i} \times \mathcal{C} \tag{5.103}$$

where for $\mathbf{i}, \mathbf{j} = 1, \ldots, m$, $\mathring{D}_{\mathbf{e_i}} \theta^{\mathbf{j}} = \mathring{D}_{\boldsymbol{e}_i} \mathring{\theta}^j = -\sum_{\mathbf{k}=1}^n \mathring{\omega}_{\cdot\mathbf{ik}}^{\mathbf{j}\cdot\cdot} \mathring{\theta}^{\mathbf{k}}$, it is

$$\mathring{\omega}_{\boldsymbol{v}} = \frac{1}{2} v^{\mathbf{c}} \mathring{\omega}_{\cdot\mathbf{c}\cdot}^{\mathbf{a}\cdot\mathbf{b}} \theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}} \tag{5.104}$$

So, we get

$$\begin{aligned} D_{\boldsymbol{v}} \mathcal{C} &= v \cdot \mathring{\boldsymbol{\partial}} \mathcal{C} + \mathcal{S}(v) \times \mathcal{C} \\ &= \partial_{\boldsymbol{v}} \mathcal{C} + (\mathring{\omega}_{\boldsymbol{v}} + \mathcal{S}(v)) \times \mathcal{C} \end{aligned} \tag{5.105}$$

and in particular

$$D_{\boldsymbol{e}_i} \mathcal{C} = \boldsymbol{\partial}_{\mathbf{e_i}} \mathcal{C} + (\mathring{\omega}_{e_i} + \mathcal{S}(\boldsymbol{e}_i)) \times \mathcal{C}. \tag{5.106}$$

[8]Take notice that this formula being gauge dependent is not valid if $\boldsymbol{e}_i \mapsto \mathbf{x_i}$ where the $\mathbf{x_i}$ coordinate vector fields. See Corollary 5.27.

Comparison of Eq. (5.106) with Eq. (5.28) (valid for any metric compatible connection) implies the important result

$$\omega_{\boldsymbol{v}} = (\mathring{\omega}_{\boldsymbol{v}} + \mathcal{S}(v)) \tag{5.107}$$

We can easily find by direct calculation that in a gauge where $\mathring{\omega}_{\boldsymbol{v}} \neq 0$,

$$\omega_{\boldsymbol{v}} = \mathbf{P}(\mathring{\omega}_{\boldsymbol{v}}) \tag{5.108}$$

which is consistent with the fact that from Eq. (5.96) it is $\mathbf{P}(\mathcal{S}(v)) = 0$.

Let $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m, \ldots, \boldsymbol{x}^n)$ be the natural orthogonal coordinate functions of $\mathring{M} \simeq \mathbb{R}^n$.

Corollary 5.27 *For* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$

$$D_{\boldsymbol{v}}\mathcal{C} = v^{\mathbf{i}} \frac{\partial}{\partial \boldsymbol{x}^{\mathbf{i}}} \mathcal{C} + \mathcal{S}(v) \times \mathcal{C}. \tag{5.109}$$

Proof Taking into account that $D_{\frac{\partial}{\partial x^i}} d\boldsymbol{x}^j = 0$ follows that

$$\mathring{\omega}_{\frac{\partial}{\partial \boldsymbol{x}^{\mathbf{i}}}} = \frac{1}{2}(\mathring{\Gamma}_{\mathbf{kil}}) d\boldsymbol{x}^k \wedge d\boldsymbol{x}^{\mathbf{l}} = 0. \tag{5.110}$$

Using this result in Eq. (5.98) with $\boldsymbol{e}_{\mathbf{i}} \mapsto \frac{\partial}{\partial \boldsymbol{x}^{\mathbf{i}}}$ gives the desired result. ■

Remark 5.28 Comparison of Eqs. (5.105) and (5.109) shows that $\mathcal{S}(v)$ cannot always be identified with $\omega(v)$ which is a gauge dependent operator.

5.3.4 Integrability Conditions

Remark 5.29 Take into account that the commutator of Pfaff derivatives acting on any $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ is in general non null, i.e.,

$$[\eth_{\boldsymbol{e}_{\mathbf{i}}}, \eth_{\boldsymbol{e}_{\mathbf{j}}}]\mathcal{C} = \textstyle\sum_{r=0}^{m} c_{\cdot \mathbf{ij}}^{\mathbf{k}\cdot\cdot} \boldsymbol{e}_{\mathbf{k}}(\mathcal{C}_{\mathbf{i}_1 \cdots \mathbf{i}_m}) \theta^{\mathbf{i}_1 \cdots \mathbf{i}_m} \neq 0, \tag{5.111}$$

unless $\boldsymbol{e}_{\mathbf{i}}$ are coordinate vector fields, i.e., $\boldsymbol{e}_{\mathbf{i}} \mapsto \mathbf{x}_{\mathbf{i}}$.

Remark 5.30 Also, since the torsion of $\mathring{D}$ is null we have in general

$$[\theta_{\mathbf{i}} \cdot \mathring{\eth}, \theta_{\mathbf{j}} \cdot \mathring{\eth}]\mathcal{C} = [\theta_{\mathbf{i}}, \theta_{\mathbf{j}}] \cdot \mathring{\eth}\mathcal{C} = c_{\cdot \mathbf{ij}}^{\mathbf{k}\cdot\cdot} \theta_{\mathbf{k}} \cdot \mathring{\eth}\mathcal{C} = c_{\cdot \mathbf{ij}}^{\mathbf{k}\cdot\cdot} \mathring{D}_{\boldsymbol{e}_{\mathbf{k}}} \mathcal{C} \neq 0, \tag{5.112}$$

unless $\boldsymbol{e}_{\mathbf{i}}$ are coordinate vector fields. Moreover, for the case of orthonormal vector fields

$$[\theta_{\mathbf{i}} \cdot \mathring{\eth}, \theta_{\mathbf{j}} \cdot \mathring{\eth}]\mathcal{C} \neq [\eth_{\boldsymbol{e}_{\mathbf{i}}}, \eth_{\boldsymbol{e}_{\mathbf{j}}}]\mathcal{C}. \tag{5.113}$$

Remark 5.31 The integrability condition for the connection $\mathring{D}$ is expressed, given the previous results, by

$$\mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}} = 0 \tag{5.114}$$

which means that for any $\mathring{\mathcal{C}} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{g})$ it is

$$\mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}}\, \mathring{\mathcal{C}} = 0$$

For the manifold M recalling that $\mathbf{x_i} \equiv \eth_{\mathbf{x_i}}$ is a Pfaff derivative we have for any $\mathcal{C} \in \sec \mathcal{C}\ell(M, g)$

$$(\eth_{\mathbf{x_i}} \eth_{\mathbf{xj}} - \eth_{\mathbf{x_j}} \eth_{\mathbf{x_i}})\mathcal{C} = 0. \tag{5.115}$$

If we recall the definition of the form derivative [Eq. (5.10)], putting

$$\eth := \vartheta^{\mathbf{i}} \eth_{\mathbf{x_i}} \tag{5.116}$$

we can express the 'integrability' condition in M by

$$\eth \wedge \eth = 0. \tag{5.117}$$

Finally recalling Eqs. (5.63) and (5.64) for $v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, g)$ it is

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\; v = v_{\mathbf{i}} \mathcal{R}^{\mathbf{i}} \tag{5.118}$$

where $\mathcal{R}^{\mathbf{i}}$ are the Ricci 1-form fields.

5.3.5 $\mathbf{S}(v) = \mathcal{S}(v)$

Proposition 5.32 *Let* $\mathcal{C} = v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, g)$ *we have*

$$\mathbf{S}(v) = \mathcal{S}(v). \tag{5.119}$$

Proof We have

$$\begin{aligned} \mathbf{S}(v) &= \mathring{\eth}(\mathbf{P}(v)) - \mathbf{P}(\mathring{\eth} v) \\ &= \mathring{\eth} v - \mathbf{P}(\mathring{\eth} v). \end{aligned} \tag{5.120}$$

Now, $\mathring{\partial} v = \mathring{\partial} \wedge v + \mathring{\partial} \lrcorner v$ and since $\mathbf{P}(\mathring{\partial} \lrcorner v) = \mathring{\partial} \lrcorner v$ we have

$$\mathbf{S}(v) = \mathring{\partial} \wedge v - \mathbf{P}(\mathring{\partial} \wedge v) \tag{5.121}$$

It is only necessary due to the linearity $\mathbf{S}$ of to show Eq. (5.119) for $v = \theta_{\mathbf{d}}, \mathbf{d} = 1, \ldots, m$. We then evaluate

$$\begin{aligned} \mathring{\partial} \wedge \theta_{\mathbf{d}} &= \sum_{\mathbf{k}=1}^{m} \theta^{\mathbf{k}} \mathring{D}_{e_{\mathbf{k}}} \theta_{\mathbf{d}} = \sum_{\mathbf{k}=1}^{m} \sum_{t=1, t \neq k}^{n} \mathring{\omega}_{t\mathbf{kd}} \theta^{\mathbf{k}} \wedge \mathring{\theta}^{t} \\ &= \sum_{k=1}^{m} \sum_{t=1, t \neq k}^{m} \mathring{\omega}_{t\mathbf{kd}} \theta^{\mathbf{k}} \wedge \theta^{\mathbf{t}} + \sum_{\mathbf{k}=1}^{m} \sum_{t=m+1}^{m+l} \mathring{\omega}_{t\mathbf{kd}} \theta^{\mathbf{k}} \wedge \mathring{\theta}^{t}, \end{aligned} \tag{5.122}$$

from where it follows that

$$\mathbf{S}(\theta_{\mathbf{d}}) = \sum_{\mathbf{k}=1}^{m} \sum_{t=m+1}^{m+l} \mathring{\omega}_{t\mathbf{kd}} \theta^{\mathbf{k}} \wedge \mathring{\theta}^{t} = \frac{1}{2} \sum_{\mathbf{k}=1}^{m} \sum_{t=m+1}^{m+l} (\mathring{\omega}_{t\mathbf{kd}} - \mathring{\omega}_{\mathbf{k}t\mathbf{d}}) \theta^{\mathbf{k}} \wedge \mathring{\theta}^{t} \tag{5.123}$$

On the other hand

$$\begin{aligned} \theta_{\mathbf{d}} \cdot \mathring{\partial} I_m &= \eta^{11} \cdots \eta^{mm} \mathring{D}_{e_{\mathbf{d}}} (\theta_1 \wedge \cdots \wedge \theta_m) \\ &= \alpha \mathring{D}_{e_{\mathbf{d}}} (\theta_1 \cdots \theta_m) \\ &= \alpha \sum_{k=1}^{m} \sum_{t=1}^{n} \mathring{\omega}_{t\mathbf{dk}} \theta_1 \cdots \underbrace{\mathring{\theta}^{t}}_{k\text{-position}} \cdots \theta_m \\ &= \alpha \sum_{k=1}^{m} \sum_{t=1}^{m} \mathring{\omega}_{t\mathbf{dk}} \theta_1 \cdots \underbrace{\theta^{t}}_{k\text{-position}} \cdots \theta_m \\ &\quad + \alpha \sum_{k=1}^{m} \sum_{t=m+1}^{m+l} \mathring{\omega}_{t\mathbf{dk}} \theta_1 \cdots \underbrace{\mathring{\theta}^{t}}_{k\text{-position}} \cdots \theta_m \end{aligned} \tag{5.124}$$

and now we can easily see that

$$\mathbf{S}(\theta_{\mathbf{d}}) \times I_m = \theta_{\mathbf{d}} \cdot \mathring{\partial} I_m \tag{5.125}$$

and it follows that $\mathbf{S}(\theta_{\mathbf{d}}) = \mathcal{S}(\theta_{\mathbf{d}})$. ■

We also have the

Proposition 5.33 *Let* $v, w \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ *we have*

$$v \cdot \mathcal{S}(w) = w \cdot \mathcal{S}(v) \tag{5.126}$$

Proof Recalling Eq. (5.123) we can write

$$\begin{aligned} v \cdot \mathcal{S}(w) &= \sum_{\mathbf{i},\mathbf{d}=1}^{m} v^{\mathbf{i}} w^{\mathbf{d}} \theta_{\mathbf{i}} \lrcorner \sum_{\mathbf{k}=1}^{m} \sum_{t=m+1}^{m+l} \mathring{\omega}_{t\mathbf{kd}} \theta^{\mathbf{k}} \wedge \mathring{\theta}^{t} \\ &= \frac{1}{2} \sum_{\mathbf{i},\mathbf{d}=1}^{m} v^{\mathbf{i}} w^{\mathbf{d}} (\mathring{\omega}_{t\mathbf{id}} - \mathring{\omega}_{\mathbf{i}t\mathbf{d}}) \mathring{\theta}^{t} = w \cdot \mathcal{S}(v) \end{aligned}$$

and the proposition is proved. ■

5.3.6 $\mathring{\boldsymbol{\partial}} \wedge v = \boldsymbol{\partial} \wedge v + \mathcal{S}(v)$ *and* $\mathring{\boldsymbol{\partial}} \lrcorner v = \boldsymbol{\partial} \lrcorner v$

We first observe that since torsion is null for the Levi-Civita connection $\mathring{D}$ we have for any $u, v \in \sec T^* \mathring{M} \hookrightarrow \sec \mathcal{C}\ell(\mathring{M}, \mathring{g})$ we have

$$u \cdot \mathring{\boldsymbol{\partial}} v = v \cdot \mathring{\boldsymbol{\partial}} u + [\![u, v]\!] \tag{5.127}$$

from where it follows $\mathbf{P}_{\perp}([\![u, v]\!]) = 0$ when $u, v \in \sec T^* M \hookrightarrow \sec \mathcal{C}\ell(M, g)$ since calculating $[\![u, v]\!]$ with $\mathring{\boldsymbol{\partial}}$ expressed in the natural coordinates of $\mathring{M}$ we find that $[\![u, v]\!] \in \sec T^* M \hookrightarrow \sec \mathcal{C}\ell(M, g)$. From this it follows that

$$\mathbf{P}_{\perp}(u \cdot \mathring{\boldsymbol{\partial}} v) = \mathbf{P}_{\perp}(v \cdot \mathring{\boldsymbol{\partial}} u). \tag{5.128}$$

Then we can write

$$\begin{aligned} \mathring{\boldsymbol{\partial}} \wedge v &= \sum_{\mathbf{m},\mathbf{k}=1}^{m} \theta^{\mathbf{r}} \wedge \mathring{D}_{e_{\mathbf{r}}}(v^{\mathbf{k}} \theta_{\mathbf{k}}) \\ &= \sum_{\mathbf{r},\mathbf{k}=1}^{m} \theta^{\mathbf{r}} \wedge \{ \boldsymbol{e}_{\mathbf{r}}(v^{\mathbf{k}}) + v^{\mathbf{k}} \sum_{\mathbf{s}=1}^{\mathbf{m}} L^{\mathbf{s}}_{\mathbf{rk}} \theta_{\mathbf{s}}) \} + \sum_{\mathbf{r},\mathbf{k}=1}^{m} \theta^{\mathbf{r}} \wedge v^{\mathbf{k}} \sum_{\mathbf{s}=\mathbf{m}+1}^{\mathbf{m}+\mathbf{l}=\mathbf{n}} L^{\mathbf{s}}_{\mathbf{rk}} \mathring{\theta}_{\mathbf{s}} \\ &= \boldsymbol{\partial} \wedge v + \sum_{\mathbf{m},\mathbf{k}=1}^{m} \theta^{\mathbf{r}} \wedge v^{\mathbf{k}} \mathbf{P}_{\perp}(\mathring{D}_{\mathbf{r}} \theta_{\mathbf{k}}) \\ &= \boldsymbol{\partial} \wedge v + \sum_{\mathbf{m},\mathbf{k}=1}^{m} \theta^{\mathbf{r}} \wedge v^{\mathbf{k}} \mathbf{P}_{\perp}(\mathring{D}_{\mathbf{k}} \theta_{\mathbf{r}}) \\ &= \boldsymbol{\partial} \wedge v + \theta^{\mathbf{r}} \wedge \mathbf{P}_{\perp}(\mathring{D}_{\mathbf{v}} \theta_{\mathbf{r}}) \\ &= \boldsymbol{\partial} \wedge v + \theta^{\mathbf{r}} \wedge \mathbf{P}_{\perp}(v \cdot \mathring{\boldsymbol{\partial}} \theta_{\mathbf{r}}) \end{aligned}$$

and recalling that $\mathcal{S}(v) = -\mathbf{P}_{\perp}(v \cdot \mathring{\boldsymbol{\partial}} \theta_{\mathbf{r}}) \wedge \theta^{\mathbf{r}}$ we finally have

$$\mathring{\boldsymbol{\partial}} \wedge v = \boldsymbol{\partial} \wedge v + \mathcal{S}(v) \tag{5.129}$$

and the proposition is proved.

Also, from [Eq. (5.86)] we know that $\boldsymbol{\partial}_u \lrcorner \mathbf{P}_u\ (\mathring{v}_\parallel) = 0$. So,

$$\begin{aligned}\mathring{\boldsymbol{\partial}} v &= \mathring{\boldsymbol{\partial}}(\mathbf{P}(v)) = \mathring{\boldsymbol{\partial}}\mathbf{P}\ (v) + \mathbf{P}(\mathring{\boldsymbol{\partial}} v) \\ &= \partial_u \wedge \mathbf{P}_u\ (v) + \partial_u \lrcorner \mathbf{P}_u\ (v) + \boldsymbol{\partial} v \\ &= \partial_u \wedge \mathbf{P}_u\ (v) + \boldsymbol{\partial} \wedge v + \boldsymbol{\partial} \lrcorner v \\ &= \mathcal{S}(v) + \boldsymbol{\partial} \wedge v + \boldsymbol{\partial} \lrcorner v \end{aligned} \tag{5.130}$$

and thus we see that

$$\mathring{\boldsymbol{\partial}} \lrcorner v = \boldsymbol{\partial} \lrcorner v \tag{5.131}$$

We then can write

$$\mathring{\boldsymbol{\partial}} v = \boldsymbol{\partial} v + \mathcal{S}(v). \tag{5.132}$$

5.3.7 $\mathring{\boldsymbol{\partial}}\mathcal{C} = \boldsymbol{\partial}\mathcal{C} + \mathbf{S}(\mathcal{C})$

We can generalize Eq. (5.132), i.e., we have the

Proposition 5.34 *For any* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *we have*

$$\begin{gathered}\mathring{\boldsymbol{\partial}}\mathcal{C} = \boldsymbol{\partial}\mathcal{C} + \mathbf{S}(\mathcal{C}), \\ \mathring{\boldsymbol{\partial}} \wedge \mathcal{C} = \boldsymbol{\partial} \wedge \mathcal{C} + \mathbf{S}(\mathcal{C}), \quad \mathring{\boldsymbol{\partial}} \lrcorner \mathcal{C} = \boldsymbol{\partial} \lrcorner \mathcal{C}. \end{gathered} \tag{5.133}$$

Proof

(i) From the fact that for any $\mathcal{A}, \mathcal{B} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$ it is $\mathbf{P}(\mathcal{A} \wedge \mathcal{B}) = \mathbf{P}(\mathcal{A}) \wedge \mathbf{P}(\mathcal{B})$ we have differentiating with respect to $u \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

$$\mathbf{P}_u(\mathcal{A} \wedge \mathcal{B}) = \mathbf{P}_u(\mathcal{A}) \wedge \mathcal{B} + \mathcal{A} \wedge \mathbf{P}_u(\mathcal{B}) \tag{5.134}$$

and of course

$$\begin{gathered}\mathbf{P}_u(\mathcal{A}_\perp \wedge \mathcal{B}_\parallel) = \mathbf{P}_u(\mathcal{A}_\perp) \wedge \mathcal{B}_\parallel, \quad \mathbf{P}_u(\mathcal{A}_\perp \wedge \mathcal{B}_\perp) = 0, \\ \mathbf{P}_u(\mathcal{A}_\parallel \wedge \mathcal{B}_\parallel) = \mathbf{P}_u(\mathcal{A}_\parallel) \wedge \mathcal{B}_\parallel + \mathcal{A}_\parallel \wedge \mathbf{P}_u(\mathcal{B}_\parallel). \end{gathered} \tag{5.135}$$

(ii) For $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ it is $\mathcal{C} = \mathbf{P}(\mathcal{C})$ and we have using Eq. (5.71)

$$\begin{aligned}\mathring{\eth}\mathcal{C} &= \mathring{\eth}\mathbf{P}\,(\mathcal{C}) - \mathbf{P}(\mathring{\eth}\mathcal{C}) \\ &= \mathring{\eth}\wedge\mathbf{P}\,(\mathcal{C}) + \mathring{\eth}\lrcorner\mathbf{P}\,(\mathcal{C}) + \boldsymbol{\partial}\mathcal{C}\end{aligned} \tag{5.136}$$

(iii) Now, we can verify recalling that $\mathbf{S}(\mathcal{C}) = \mathbf{S}(\mathcal{C}_{\parallel} + \mathcal{C}_{\perp}) = \mathbf{S}(\mathcal{C}_{\parallel}) + \mathbf{S}(\mathcal{C}_{\perp})$ and following steps analogous to the ones used in the proof of Proposition 5.24 that

$$\mathbf{S}(\mathcal{C}_{\parallel}) = \mathbf{S}(\mathbf{P}(\mathcal{C}_{\parallel})) = \mathring{\eth}\wedge\mathbf{P}\,(\mathcal{C}_{\parallel}) \quad \mathbf{S}(\mathcal{C}_{\perp}) = \mathbf{P}(\mathbf{S}(\mathcal{C}_{\perp})) = \mathring{\eth}\lrcorner\mathbf{P}\,(\mathcal{C}_{\parallel}). \tag{5.137}$$

(iv) Using Eq. (5.137) in Eq. (5.135) we have

$$\mathring{\eth}\wedge\mathcal{C} + \mathring{\eth}\lrcorner\mathcal{C} = \mathring{\eth}\wedge\mathbf{P}\,(\mathcal{C}) + \mathring{\eth}\lrcorner\mathbf{P}\,(\mathcal{C}) + \boldsymbol{\partial}\wedge\mathcal{C} + \boldsymbol{\partial}\lrcorner\mathcal{C}$$

or

$$\begin{aligned}\mathring{\eth}\wedge\mathcal{C} + \mathring{\eth}\lrcorner\mathcal{C} &= \mathbf{S}(\mathcal{C}) + \mathbf{S}(\mathcal{C}_{\perp}) + \boldsymbol{\partial}\wedge\mathcal{C} + \boldsymbol{\partial}\lrcorner\mathcal{C} \\ &= \mathbf{S}(\mathcal{C}) + \boldsymbol{\partial}\wedge\mathcal{C} + \boldsymbol{\partial}\lrcorner\mathcal{C},\end{aligned} \tag{5.138}$$

which provides the proof of the proposition. ■

Proposition 5.35 *For any* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *we have:*

$$\boldsymbol{\partial}\mathcal{C} = \mathbf{P}(\mathring{\eth}\mathcal{C}) \tag{5.139}$$

Proof From Eq. (5.71) when $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ it is

$$\mathbf{P}(S(\mathcal{C})) = 0.$$

Then, applying $\mathbf{P}$ to both members of the first line of Eq. (5.133) we have

$$\mathbf{P}(\mathring{\eth}\mathcal{C}) = (\mathring{\eth}\mathcal{C})_{\parallel} = \mathbf{P}(\partial\mathcal{C}) + \mathbf{P}^2(\mathbf{S}(\mathcal{C})) = \mathbf{P}(\boldsymbol{\partial}\mathcal{C}) = \boldsymbol{\partial}\mathcal{C} \tag{5.140}$$

and the proposition is proved. ■

5.4 Curvature Biform $\mathfrak{R}(u \wedge v)$ Expressed in Terms of the Shape Operator

5.4.1 Equivalent Expressions for $\mathfrak{R}(u \wedge v)$

In this section we suppose that the structure $(M, \boldsymbol{g}, D)$ is such that is M a submanifold of $\mathring{M} \simeq \mathbb{R}^n$ and D the Levi-Civita connection of $\boldsymbol{g} = \boldsymbol{i}^*\mathring{\boldsymbol{g}}$. We obtained in Sect. 5.1 a formula [Eq. (5.44)] for the curvature biform $\mathfrak{R}(u \wedge v)$ of a general Riemann-Cartan connection. Of course, taking into account the fact that $\mathfrak{R}$ is an intrinsic object, the evaluation of $\mathfrak{R}(u \wedge v)$ does not depend on the coordinate chart and basis for vector and form fields used for its calculation. In what follows we take advantage of this fact choosing the basis $\{\frac{\partial}{\partial \boldsymbol{x}^{\mathbf{i}}}, d\boldsymbol{x}^{\mathbf{i}}\}$ as introduced above for which $\overset{\circ}{\omega}(u) = 0$. Thus, we have, recalling Eqs. (5.44) and (5.45) that

$$\begin{aligned}
\mathfrak{R}(u \wedge v) &= D_{\boldsymbol{u}}\, \omega(v) - D_{\boldsymbol{v}}\, \omega(u) + \omega(u) \times \omega(v) - \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]} \\
&= \overset{\circ}{D}_{\boldsymbol{u}}\, \omega(v) - \overset{\circ}{D}_{\boldsymbol{v}}\, \omega(u) + \omega(u) \times \omega(v) \\
&\quad - \omega(v) \times \omega(u) - \omega(u) \times \omega(v) - \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]} \\
&= \overset{\circ}{D}_{\boldsymbol{u}}\, \omega(v) - \overset{\circ}{D}_{\boldsymbol{v}}\, \omega(u) + \omega(u) \times \omega(v) - \boldsymbol{\omega}_{[\boldsymbol{u},\boldsymbol{v}]}.
\end{aligned} \tag{5.141}$$

On the other hand since in the gauge where $\overset{\circ}{\omega}(u) = 0$ we have that $\omega(u) = \mathcal{S}(u)$ and thus we can also write

$$\mathfrak{R}(u \wedge v) = \overset{\circ}{D}_{\boldsymbol{u}}\, \omega(v) - \overset{\circ}{D}_{\boldsymbol{v}}\, \omega(u) + \mathcal{S}(u) \times \mathcal{S}(v) - \mathcal{S}([u, v]). \tag{5.142}$$

Now, putting $\mathbf{x_i} = \partial/\partial \boldsymbol{x}^{\mathbf{i}}$ we have

$$\begin{aligned}
&\overset{\circ}{D}_{\boldsymbol{u}}\, \omega(v) - \overset{\circ}{D}_{\boldsymbol{v}}\, \omega(u) \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{\overset{\circ}{D}_{\mathbf{x_i}} \mathcal{S}(\vartheta_{\mathbf{j}}) - \overset{\circ}{D}_{\mathbf{x_J}} \mathcal{S}(\vartheta_{\mathbf{i}})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{\overset{\circ}{D}_{\mathbf{x_i}} (\overset{\circ}{D}_{\mathbf{x_j}} I_m\, I_m^{-1}) - \overset{\circ}{D}_{\mathbf{x_j}} (\overset{\circ}{D}_{\mathbf{x_i}} I_m\, I_m^{-1})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{\overset{\circ}{D}_{\mathbf{x_i}} \overset{\circ}{D}_{\mathbf{x_j}} I_m)\, I_m^{-1}) + (\overset{\circ}{D}_{\mathbf{x_j}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_i}} I_m^{-1}) \\
&\qquad - (\overset{\circ}{D}_{\mathbf{x_j}} \overset{\circ}{D}_{\mathbf{x_i}} I_m)\, I_m^{-1}) - (\overset{\circ}{D}_{\mathbf{x_i}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_j}} I_m^{-1})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{(\overset{\circ}{D}_{[\mathbf{x_i},\mathbf{x_j}]} I_m\, I_m^{-1}) - (\overset{\circ}{D}_{\mathbf{x_j}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_i}} I_m^{-1}) - (\overset{\circ}{D}_{\mathbf{x_i}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_j}} I_m^{-1})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{-(\overset{\circ}{D}_{\mathbf{x_j}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_i}} I_m^{-1}) - (\overset{\circ}{D}_{\mathbf{x_i}} I_m)\, (\overset{\circ}{D}_{\mathbf{x_j}} I_m^{-1})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{((\overset{\circ}{D}_{\mathbf{x_i}} I_m) I_m^{-1})\, ((\overset{\circ}{D}_{\mathbf{x}_j} I_m^{-1}) I_m) - ((\overset{\circ}{D}_{\mathbf{x_i}} I_m) I_m^{-1})\, ((\overset{\circ}{D}_{\mathbf{x}_j} I_m^{-1}) I_m)\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \{((\overset{\circ}{D}_{\mathbf{x_i}} I_m) I_m^{-1})\, ((\overset{\circ}{D}_{\mathbf{x}_j} I_m) I_m^{-1}) - ((\overset{\circ}{D}_{\mathbf{x_j}} I_m) I_m^{-1})\, ((\overset{\circ}{D}_{\mathbf{x_i}} I_m) I_m^{-1})\} \\
&\quad = -u^{\mathbf{i}} v^{\mathbf{j}} \mathcal{S}(\vartheta_{\mathbf{i}}) \mathcal{S}(\vartheta_{\mathbf{j}}) + u^{\mathbf{i}} v^{\mathbf{j}} \mathcal{S}(\vartheta_{\mathbf{j}}) \mathcal{S}(\vartheta_{\mathbf{i}}) \\
&\quad = -\mathcal{S}(u) \mathcal{S}(v) + \mathcal{S}(v) \mathcal{S}(u) = -2\mathcal{S}(u) \times \mathcal{S}(v).
\end{aligned} \tag{5.143}$$

Thus,we get

$$\mathfrak{R}(u \wedge v) = -\mathcal{S}(u) \times \mathcal{S}(v) - \mathcal{S}([u, v]). \tag{5.144}$$

Now take into account that since $\mathfrak{R}(u \wedge v) \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we must have, of course, $-\mathcal{S}(u) \times \mathcal{S}(v) - \mathcal{S}([u, v]) = \mathbf{P}(-\mathcal{S}(u) \times \mathcal{S}(v) - \mathcal{S}([u, v]))$ and since Eq. (5.96) tell us that $\mathbf{P}(\mathcal{S}([u, v]) = 0$ we have the nice formula[9]

$$\mathfrak{R}(u \wedge v) = -\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \tag{5.145}$$

which express the curvature biform in terms of the shape biform.

5.4.2 $\mathbf{S}^2(v) = -\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v)$

In this subsection we want to show the

Proposition 5.36 *Let* $v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. *Then,*

$$\mathbf{S}^2(v) = -\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v). \tag{5.146}$$

Equation (5.146) tell us that the square of the shape operator applied to a 1-form field v is equal to the Ricci operator applied to v. This result will play an important role in Chap. 11 where we give a formulation to Clifford intuition that matter is represented by little hills in a brane [17].

Now, to prove the Proposition 5.36 we need the following lemmas:

Lemma 5.37 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(\mathring{M}, \mathtt{g})$ *and* $v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. *Then*

$$\mathbf{P}_v(\mathcal{C}) = \mathbf{P}(\mathcal{C}) \times \mathcal{S}(v) - \mathbf{P}(\mathcal{C} \times \mathcal{S}(v)). \tag{5.147}$$

Proof Indeed,

$$\begin{aligned}
\mathbf{P}_v(\mathcal{C}) &= \mathring{D}_{\boldsymbol{v}}(\mathbf{P}(\mathcal{C})) - \mathbf{P}(\mathring{D}_{\boldsymbol{v}}\mathcal{C}) \\
&= D_{\boldsymbol{v}}(\mathbf{P}(\mathcal{C})) - \mathcal{S}(v) \times \mathbf{P}(\mathcal{C}) - \mathbf{P}(D_{\boldsymbol{v}}\mathcal{C} - \mathcal{S}(v) \times \mathcal{C}) \\
&= D_{\boldsymbol{v}}\mathcal{C} - \mathcal{S}(v) \times \mathbf{P}(\mathcal{C}) - D_{\boldsymbol{v}}\mathcal{C} + \mathbf{P}(\mathcal{S}(v) \times \mathcal{C}) \\
&= \mathbf{P}(\mathcal{C}) \times \mathcal{S}(v) - \mathbf{P}(\mathcal{C} \times \mathcal{S}(v))
\end{aligned} \tag{5.148}$$

which proves the lemma. ■

[9] Note that in [11, 18] the second member of Eq. (5.145) is the negative of what we found. Our result agrees with the one in [6].

Lemma 5.38 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *and* $v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$*.Then*

$$D_{\boldsymbol{v}}\mathcal{C} = \overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C} - \mathbf{P}_v(\mathcal{C}). \tag{5.149}$$

Proof Follows from the first line in Eq. (5.148). ∎

Lemma 5.39 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *and* $u, v \in \sec \bigwedge^1 T^*M. \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$*. Then*

$$D_{\boldsymbol{u}}D_{\boldsymbol{v}}\mathcal{C} = \mathbf{P}(\overset{\circ}{D}_{\boldsymbol{u}}\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) + \mathbf{P}_u\mathbf{P}_v(\mathcal{C}). \tag{5.150}$$

Proof Using Eq. (5.149) we have

$$\begin{aligned} D_{\boldsymbol{u}}(D_{\boldsymbol{v}}\mathcal{C}) &= D_{\boldsymbol{u}}(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C} - \mathbf{P}_v(\mathcal{C})) \\ &= \overset{\circ}{D}_{\boldsymbol{u}}\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C} - \mathbf{P}_u(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) - \overset{\circ}{D}_{\boldsymbol{u}}(\mathbf{P}_v(\mathcal{C})) + \mathbf{P}_u\mathbf{P}_v(\mathcal{C}). \end{aligned} \tag{5.151}$$

On the other hand we have

$$\begin{aligned} \mathbf{P}(\overset{\circ}{D}_{\boldsymbol{u}}\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) &= -\mathbf{P}_u(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) + \overset{\circ}{D}_{\boldsymbol{u}}(\mathbf{P}(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C})) \\ &= -\mathbf{P}_u(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) + \overset{\circ}{D}_{\boldsymbol{u}}(D_{\boldsymbol{v}}\mathcal{C}) \\ &= -\mathbf{P}_u(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) + \overset{\circ}{D}_{\boldsymbol{u}}(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C} - \mathbf{P}_v(\mathcal{C})) \\ &= \overset{\circ}{D}_{\boldsymbol{u}}\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C} - \mathbf{P}_u(\overset{\circ}{D}_{\boldsymbol{v}}\mathcal{C}) - \overset{\circ}{D}_{\boldsymbol{u}}(\mathbf{P}_v(\mathcal{C})). \end{aligned} \tag{5.152}$$

Putting Eq. (5.152) in Eq. (5.150) gives the desired result. ∎

Lemma 5.40 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *and* $u, v \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$*.Then*

$$\mathfrak{R}(u \wedge v) \times \mathcal{C} = [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C}. \tag{5.153}$$

Proof Using Eq. (5.150) we have

$$\begin{aligned} [D_{\boldsymbol{u}}, D_{\boldsymbol{v}}]\mathcal{C} &= \mathbf{P}([\overset{\circ}{D}_{\boldsymbol{u}}, \overset{\circ}{D}_{\boldsymbol{v}}]\mathcal{C}) + [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C} \\ &= \mathbf{P}(\overset{\circ}{D}_{[\boldsymbol{u},\boldsymbol{v}]}\mathcal{C}) + [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C} \\ &= D_{[\boldsymbol{u},\boldsymbol{v}]}\mathcal{C} + [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C} \end{aligned} \tag{5.154}$$

Thus we get that

$$([D_{\boldsymbol{u}}, D_{\boldsymbol{v}}] - D_{[\boldsymbol{u},\boldsymbol{v}]})\mathcal{C} = [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C} \tag{5.155}$$

Recalling now Eq. (5.49) we have

$$\mathfrak{R}(u \wedge v) \times \mathcal{C} = [\mathbf{P}_u, \mathbf{P}_v]\mathcal{C}$$

and the lemma is proved ■

Lemma 5.41 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *and* $u, v \in \sec \bigwedge^1 T^*M. \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. *Then,*

$$\mathfrak{R}(u \wedge v) \times \mathcal{C} = -\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times \mathcal{C} \tag{5.156}$$

Proof This follows directly from Eq. (5.145). ■

Remark 5.42 We shall now evaluate directly the first member of Eq. (5.157) to get Eq. (5.160) which when compared with Eq. (5.145) will furnish identities given by Eq. (5.161).

$$([D_{\boldsymbol{u}}, D_{\boldsymbol{v}}] - D_{[\boldsymbol{u},\boldsymbol{v}]})\mathcal{C} = \mathfrak{R}(u \wedge v) \times \mathcal{C}. \tag{5.157}$$

Given the linearity of $\mathfrak{R}(u \wedge v)$ we calculate the first member of Eq. (5.157) for the case $\boldsymbol{u} = \mathbf{x_i}$, $\boldsymbol{v} = \mathbf{x_j}$. Taking into account that $D_{\boldsymbol{u}}\mathcal{C} = \mathring{D}_{\boldsymbol{u}}\mathcal{C} + \mathcal{S}(u) \times C$ we get with calculations analogous to the ones in Eq. (5.143) that

$$[D_{\mathbf{x_i}}, D_{\mathbf{x_j}}]\mathcal{C} = -\mathcal{S}(\vartheta_\mathbf{i}) \times \mathcal{S}(\vartheta_\mathbf{j}) \times \mathcal{C} \tag{5.158}$$

and taking into account that it is $[\mathbf{x_i}, \mathbf{x_j}] = 0$ we can write the last equation as

$$([D_{\mathbf{x_i}}, D_{\mathbf{x_j}}] - D_{[\mathbf{x_i},\mathbf{x_j}]})\mathcal{C} = \mathfrak{R}(\vartheta_\mathbf{i} \wedge \vartheta_\mathbf{j}) \times \mathcal{C} = -\,\mathcal{S}(\vartheta_\mathbf{i}) \times \mathcal{S}(\vartheta_\mathbf{j}) \times \mathcal{C} \tag{5.159}$$

and so it follows that

$$\mathfrak{R}(u \wedge v) \times \mathcal{C} = -\,\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}. \tag{5.160}$$

Of course, we must have $\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}) \in \sec \mathcal{C}\ell(M, \mathtt{g})$. Since $\mathcal{S}(u) \times \mathcal{S}(v) = \mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) + \mathbf{P}_\perp(\mathcal{S}(u) \times \mathcal{S}(v))$ and we already know that $\mathfrak{R}(u \wedge v) = -\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v))$ it follows that $\mathbf{P}_\perp(\mathcal{S}(u) \times \mathcal{S}(v)) = 0$ and moreover we get that

$$\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}) = \mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times \mathcal{C} = \mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times \mathbf{P}(\mathcal{C}). \tag{5.161}$$

Lemma 5.43 *Let* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ *and* $u, v \in \sec \bigwedge^1 T^*M \hookrightarrow \in \sec \mathcal{C}\ell(M, \mathtt{g})$

$$\mathfrak{R}(u \wedge v) = \mathbf{P}_v(\mathcal{S}(u)) \tag{5.162}$$

Proof Taking $\mathcal{C} = \mathcal{S}(u)$ in Eq. (5.148) and recalling Eq. (5.96) $\mathbf{P}(\mathcal{S}(u)) = 0$.we get

$$\mathbf{P}_v(\mathcal{S}(u)) = -\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \tag{5.163}$$

which proves the lemma. ■

Remark 5.44 From Eq. (5.163) we immediately have

$$\mathbf{P}_u(\mathcal{S}(v)) = -\mathbf{P}(\mathcal{S}(v) \times \mathcal{S}(u)) = \mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) = -\mathbf{P}_v(\mathcal{S}(u)) = -\mathbf{P}_v(\mathbf{S}(u)) \tag{5.164}$$

where the last term follows from the fact that $\mathbf{S}(u) = \mathcal{S}(u)$.

Proof (of Proposition 5.36) We know that $\mathcal{R}(v) = \partial_u \mathfrak{R}(u \wedge v)$. Thus using Eq. (5.164) and recalling Eq. (5.71) we can write

$$\begin{aligned}\mathcal{R}(v) &= \partial_u \mathbf{P}_v(\mathcal{S}(u)) = -\partial_u \mathbf{P}_u(\mathcal{S}(v)) \\ &= -\mathring{\partial}\mathbf{P}\,(\mathbf{S}(v)) = -\mathbf{S}(\mathbf{S}(v)) = -\mathbf{S}^2(v).\end{aligned} \tag{5.165}$$

Since we already showed that $\mathcal{R}(v) = \boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v)$ we get

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v) = -\mathbf{S}^2(v)$$

and the proposition is proved. ■

Remark 5.45 Take notice that whereas $\mathbf{S}(v)$ is a section of $\mathcal{C}\ell(\mathring{M}, \mathring{\mathtt{g}})$, $\mathbf{S}^2(v) \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$.

Proposition 5.46 *Let* $u, v, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. *Then,*

$$\mathfrak{R}(u \wedge v) = \frac{1}{2}\partial_w \wedge [\mathbf{P}_v, \mathbf{P}_u](w). \tag{5.166}$$

Proof From Eq. (5.153) with $\mathcal{C} = w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\mathfrak{R}(u \wedge v) \times w = [\mathbf{P}_u, \mathbf{P}_v](w). \tag{5.167}$$

Now, the first member of Eq. (5.167) is

$$\mathfrak{R}(u \wedge v) \times w = -w \lrcorner \mathfrak{R}(u \wedge v)$$

Now, writing $\mathfrak{R}(u \wedge v) = \frac{1}{2} u^{\mathbf{i}} v^{\mathbf{j}} R_{\mathbf{ij}\cdot\cdot}^{\cdot\cdot\mathbf{kl}} \theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}}$ we have

$$\begin{aligned}\partial_w \wedge (\mathfrak{R}(u \wedge v) \times w) &= -\theta^{\mathbf{r}} \wedge (\theta_{\mathbf{r}} \lrcorner \mathfrak{R}(u \wedge v)) \\ -\frac{1}{2} u^{\mathbf{i}} v^{\mathbf{j}} \theta_{\mathbf{r}} \wedge (\theta^{\mathbf{r}} \lrcorner R_{\mathbf{ij}\cdot\cdot}^{\cdot\cdot\mathbf{kl}} \theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}}) &= -2\mathfrak{R}(u \wedge v).\end{aligned} \tag{5.168}$$

Taking into account Eqs. (5.167) and (5.168) the proof follows.

We can also prove the proposition as follows: directly from Eq. (5.155) we can write

$$[\mathbf{P}_u, \mathbf{P}_v](w) = u^{\mathbf{k}} v^{\mathbf{l}} ([D_{e_{\mathbf{k}}}, D_{e_{\mathbf{l}}}] - D_{[e_{\mathbf{k}}, e_{\mathbf{l}}]}) w = u^{\mathbf{k}} v^{\mathbf{l}} w^{\mathbf{j}} R^{\mathbf{i}\cdots}_{\cdot\mathbf{jkl}} \theta_{\mathbf{i}}. \tag{5.169}$$

Thus

$$\begin{aligned}
\frac{1}{2}\partial_w \wedge [\mathbf{P}_u, \mathbf{P}_v](w) &= \frac{1}{2}\theta^m \frac{\partial}{\partial w^m} \wedge (u^{\mathbf{k}} v^{\mathbf{l}} w^{\mathbf{j}} R^{\mathbf{i}\cdots}_{\cdot\mathbf{jkl}} \theta_{\mathbf{i}}) \\
&= \frac{1}{2} u^{\mathbf{k}} v^{\mathbf{l}} R_{\mathbf{imkl}} \theta^{\mathbf{m}} \wedge \theta^{\mathbf{i}} \\
&= -u^{\mathbf{k}} v^{\mathbf{l}} \mathcal{R}_{\mathbf{kl}} = -\mathfrak{R}(u \wedge v)
\end{aligned} \tag{5.170}$$

and the proof is complete. ■

Proposition 5.47 *Let* $u, v, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. *Then,*

$$\mathfrak{R}(u \wedge v) = \partial_w \wedge \mathbf{P}_v \mathbf{P}_u(w). \tag{5.171}$$

Proof Recall that we proved [Eq. (5.162)] that $\mathfrak{R}(u \wedge v) = \mathbf{P}_v(\mathcal{S}(u))$. Also Eq. (5.84) says that $\mathbf{S}(u) = \mathcal{S}(u) = \partial_w \wedge \mathbf{P}_w(u)$ for any $u, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Now from Eq. (5.179) we have

$$\mathbf{P}_w(u) = \mathbf{P}_u(w) = u \cdot \mathring{\boldsymbol{\partial}}\mathbf{P}\,(w) = \mathring{D}_{\mathbf{u}}\mathbf{P}\,(w) = \mathring{D}_{\mathbf{u}}(\mathbf{P}(w)) - \mathbf{P}(\mathring{D}_{\mathbf{u}} w) = (\mathring{D}_{\mathbf{u}} w)_{\perp} \tag{5.172}$$

which means that

$$\mathbf{P}_w(u) = (\mathbf{P}_w(u))_{\perp}. \tag{5.173}$$

Then, we have that

$$\begin{aligned}
\mathfrak{R}(u \wedge v) &= \mathbf{P}_v(\mathcal{S}(u)) = \mathbf{P}_v(\partial_w \wedge \mathbf{P}_w(u)) \\
&= \mathbf{P}_v(\partial_w \wedge \mathbf{P}_u(w)) = \mathbf{P}_v((\partial_w)_{\parallel} \wedge (\mathbf{P}_u(w))_{\perp}) \\
&\overset{\text{Eq.(5.182)}}{=} \partial_w \wedge \mathbf{P}_v \mathbf{P}_u(w))_{\perp}
\end{aligned}$$

and the proof is complete. ■

Remark 5.48 Since $\mathfrak{R}(u \wedge v) = -\mathfrak{R}(v \wedge u)$, Eq. (5.171) implies that $u, v, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

$$\partial_w \wedge \mathbf{P}_v \mathbf{P}_u(w) = -\partial_w \wedge \mathbf{P}_u \mathbf{P}_v(w), \tag{5.174}$$

thus exhibiting the consistency of Eq. (5.171) with Eq. (5.166).

5.5 Some Identities Involving P and $\mathbf{P}_u$

Here we derive some identities involving the projection operator that has been used in the above sections. The projection operator $\mathbf{P}$ has been defined by Eq. (5.68) and its covariant derivative $\mathbf{P}_u := u \cdot \mathring{\partial}\mathbf{P}$ has been defined by Eq. (5.81). Let $\mathcal{C}, \mathcal{D} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{\boldsymbol{g}})$. Since

$$\mathbf{P}(\mathcal{C} \wedge \mathcal{D}) = \mathbf{P}(\mathcal{C}) \wedge \mathbf{P}(\mathcal{D}), \tag{5.175}$$

we have for any $u \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g})$ that

$$\mathbf{P}_u(\mathcal{C} \wedge \mathcal{D}) = \mathbf{P}_u(\mathcal{C}) \wedge \mathbf{P}(\mathcal{D}) + \mathbf{P}(\mathcal{C}) \wedge \mathbf{P}_u(\mathcal{D}). \tag{5.176}$$

From $\mathbf{P}^2(\mathcal{C}) = \mathbf{P}(\mathcal{C})$ we have that

$$\mathbf{P}_u\mathbf{P}(\mathcal{C}) + \mathbf{P}\mathbf{P}_u(\mathcal{C}) = \mathbf{P}_u(\mathcal{C}). \tag{5.177}$$

Now, we easily verify that

$$\mathbf{P}_u(w) = \mathbf{P}_\perp(u \cdot \mathring{\partial} w), \quad \mathbf{P}_w(u) = \mathbf{P}_\perp(w \cdot \mathring{\partial} u). \tag{5.178}$$

Now, we already know from Eq. (5.128) that $\mathbf{P}_\perp(u \cdot \mathring{\partial} w)$ and $\mathbf{P}_\perp(w \cdot \mathring{\partial} u)$ are equal and thus

$$\mathbf{P}_u(w) = \mathbf{P}_w(u). \tag{5.179}$$

From this equation also follows immediately that

$$\mathbf{P}_u\mathbf{P}(w) = \mathbf{P}_w\mathbf{P}(u). \tag{5.180}$$

Now given that each $\mathcal{X} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{\boldsymbol{g}})$ can be written as $\mathcal{X} = \mathcal{X}_\parallel + \mathcal{X}_\perp$, with $\mathcal{X}_\parallel = \mathbf{P}(\mathcal{X})$ we get from Eq. (5.177) that

$$\mathbf{P}\mathbf{P}_u(\mathcal{X}_\parallel) = \mathbf{0}, \quad \mathbf{P}\mathbf{P}_u(\mathcal{X}_\perp) = \mathbf{P}_u(\mathcal{X}_\perp). \tag{5.181}$$

Also, from Eq. (5.176) we have immediately taking into account Eq. (5.181) for any $\mathcal{C}, \mathcal{D} \in \sec \mathcal{C}\ell(\mathring{M}, \mathring{\boldsymbol{g}})$ that

$$\mathbf{P}_u(\mathcal{C}_\parallel \wedge \mathcal{D}_\perp) = \mathcal{C}_\parallel \wedge \mathbf{P}_u(\mathcal{D}_\perp). \tag{5.182}$$

5.6 Conclusions

We presented a thoughtful presentation of the geometry of vector manifolds using the Clifford bundle formalism, hopping to provide a useful text for people (who know the Cartan theory of differential forms)[10] and are interested in the differential geometry of submanifolds M (of dimension m equipped with a metric[11] $\boldsymbol{g} = \boldsymbol{i}^*\mathring{\boldsymbol{g}}$ of signature (p, q) and its Levi-Civita connection D) of a manifold $\mathring{M} \simeq \mathbb{R}^n$ (of dimension n and equipped with a metric $\mathring{\boldsymbol{g}}$ of signature $(\mathring{p}, \mathring{q})$ and its Levi-Civita connection $\mathring{D}$). We proved in details several equivalent expressions for the curvature biforms $\mathfrak{R}(u \wedge v)$ and moreover proved that the Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ when applied to a 1-form field v is such that $\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v) = \mathcal{R}(v) = -\mathbf{S}^2(v)$ $(\mathcal{R}(v) = R^{\mathbf{a}}_{\mathbf{b}}\theta_{\mathbf{b}})$ is the negative of the square of the shape operator $\mathbf{S}$. It will be shown in Chap. 11 that when this result is applied to GRT it permits to give a mathematical realization to Clifford's theory of matter.

We observe that our methodology in this chapter differs considerably [11, 12, 18]. Indeed, we use in our approach the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathtt{g})$) and give detailed and (we hope) intelligible proofs of all formulas, clarifying some important issues, presenting, e.g., the precise relation between the shape biform S evaluate at v (a 1-form field) and the connection extensor ω evaluated at v [Eq. (5.107)]. In particular, our approach also generalizes for a general Riemann-Cartan connection the results in [12] which are valid only for the Levi-Civita connection D of a Lorentzian metric of signature $(1, 3)$. Moreover our approach makes rigorous the results in [12] which are valid only for 4-dimensional Lorentzian spacetimes admitting a spinor structure,[12] since in [12] it is postulated that the frame bundle of M has a global section (there called a fiducial frame).

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[10]This includes people, we think, interested in string and brane theories and GRT.

[11]$\boldsymbol{i} : M \to \mathring{M}$ is the inclusion map.

[12]This result follows from Geroch theorem (Chap. 7). See [9, 10] for details.

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Chapter 6
Some Issues in Relativistic Spacetime Theories

Abstract The chapter has as main objective to clarify some important concepts appearing in relativistic spacetime theories and which are necessary of a clear understanding of our view concerning the formulation and understanding of Maxwell, Dirac and Einstein theories. Using the definition of a Lorentzian spacetime structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ presented in Chap. 4 we introduce the concept of a *reference frame* in that structure which is an object represented by a given unit timelike vector field $\boldsymbol{Z} \in \sec TU$ $(U \subseteq M)$. We give two classification schemes for these objects, one according to the decomposition of $D\boldsymbol{Z}$ and other according to the concept of synchronizability of ideal clocks (at rest in $\boldsymbol{Z}$). The concept of a coordinate chart covering U and naturally adapted to the reference frame $\boldsymbol{Z}$ is also introduced. We emphasize that the concept of a *reference frame* is different (but related) from the concept of a *frame* which is a section of the frame bundle. The concept of Fermi derivative is introduced and the physical meaning of Fermi transport is elucidated, in particular we show the relation between the Darboux biform Ω of the theory of Frenet frames and its decomposition as an invariant sum of a Frenet biform Ω_F (describing Fermi transport) and a rotation biform $\Omega_{\mathbf{S}}$ such that the contraction of $\star\Omega_{\mathbf{S}}$ with the velocity field v of the spinning particle is directly associated with the so-called Pauli-Lubanski spin 1-form. We scrutinize the concept of diffeomorphism invariance of general spacetime theories and of General Relativity in particular, discuss what meaning can be given to the concept of *physically equivalent* reference frames and what one can understand by a *principle of relativity*. Examples are given and in particular, it is proved that in a general Lorentzian spacetime (modelling a gravitational field according to General Relativity) there is in general no reference frame with the properties (according to the scheme classifications) of the inertial referenced frames of special relativity theories. However there are in such a case reference frames called pseudo inertial reference frames (PIRFs) that have most of the properties of the inertial references frames of special relativity theories. We also discuss a formulation (that one can find in the literature) of a so-called *principle of local Lorentz invariance* and show that if it is interpreted as physical equivalence of PIRFs then it is not valid. The Chapter ends with a brief discussion of diffeormorphism invariance applied to Schwarzschild original solution and the Droste-Hilbert solution of Einstein equation which are shown to be *not* equivalent (the underlying manifolds have different topologies) and what these solutions have

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to do with the existence of blackholes in the "orthodox"interpretation of General Relativity.

6.1 Reference Frames on Relativistic Spacetimes

In this chapter $\mathfrak{M} = (M, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$ denotes a Lorentzian spacetime and $\mathcal{M} = (M, \boldsymbol{\eta}, D, \boldsymbol{\tau}_{\eta}, \uparrow)$ denotes Minkowski spacetime. We adopt, as before, *natural units* such that the value of the velocity of light is $c = 1$.

Proposition 6.1 *Let* $\mathbf{Q} \in \sec TU \subset \sec TM$ *be a time-like vector field such that* $\boldsymbol{g}(\mathbf{Q}, \mathbf{Q}) = 1$. *Then, there exist, in a coordinate neighborhood U, three space-like vector fields* $\boldsymbol{e}_{\mathbf{i}}$ *which together with* $\mathbf{Q}$ *form an orthogonal moving frame for* $x \in U$.

Proof Suppose that the metric of the manifold in a chart (U, ϕ) with coordinate functions $\{\mathbf{x}^\mu\}$ is $\boldsymbol{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$. Let $\mathbf{Q} = (Q^\mu \partial/\partial x^\mu) \in \sec TM$ be an arbitrary reference frame and $\alpha_{\mathbf{Q}} = \boldsymbol{g}(\mathbf{Q},) = Q_\mu dx^\mu$, $Q_\mu = g_{\mu\nu} Q^\nu$. Then, $g_{\mu\nu}(x) Q^\mu Q^\nu = 1$. Now, define

$$\begin{aligned} \theta^{\mathbf{0}} &= (\alpha_Q)_\mu dx^\mu = Q_\mu dx^\mu, \\ \gamma_{\mu\nu} &= Q_\mu Q_\nu - g_{\mu\nu}. \end{aligned} \tag{6.1}$$

Then the metric $\boldsymbol{g}$ can be written due to the hyperbolicity of the manifold as

$$\boldsymbol{g} = \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}},$$

$$\sum_{\mathbf{a}=1}^{3} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{a}} = \gamma_{\mu\nu}(x) dx^\mu \otimes dx^\nu. \tag{6.2}$$

Now, call $\boldsymbol{e}_{\mathbf{0}} = \mathbf{Q}$ and take $\boldsymbol{e}_{\mathbf{a}}$ such that $\theta^{\mathbf{a}}(\boldsymbol{e}_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$. It follows immediately that $\boldsymbol{g}(\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \eta_{\mathbf{ab}}$, $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$. ■

Before we proceed we need to know precisely how the metric $\boldsymbol{g}$ relates tangent space magnitudes to magnitudes on the manifold. Let $\sigma : \mathbb{R} \supset I \to M$, be a smooth curve, i.e., σ is C^0 and piecewise C^1. We denote the inclusion function $I \to \mathbb{R}$ by u, and the distinguished vector field on I by d/du. For each $u \in I$, $\sigma_* u$ denotes the tangent vectors at $\sigma(u) \in M$. Thus,

$$\sigma_{*u} = \left[\sigma_*(\frac{d}{du}) \right]_{\sigma(u)} \in T_{\sigma(u)} M, \tag{6.3}$$

where σ_* denotes the derivative mapping of the mapping σ.

Definition 6.2 A curve is said timelike (respectively lightlike, or respectively spacelike) if for all $u \in I$, $\boldsymbol{g}(\sigma_{*u}, \sigma_{*u}) > 0$ (respectively $\boldsymbol{g}(\sigma_{*u}, \sigma_{*u}) = 0$, or respectively $\boldsymbol{g}(\sigma_{*u}, \sigma_{*u}) < 0$).

Definition 6.3 The path length between events $\mathfrak{e}_1 = \sigma(a)$ and $\mathfrak{e}_2 = \sigma(b)$ along the curve σ, such that for all $u \in I$, $\boldsymbol{g}(\sigma_{*u}, \sigma_{*u})$ has the same signal at all points along $\sigma(u)$ is the quantity

$$\int_a^b du \, [|\boldsymbol{g}(\sigma_{*u}, \sigma_{*u})|]^{\frac{1}{2}} \, . \tag{6.4}$$

Observe that taking the point $\sigma(a)$ as a reference point we can use Eq. (6.4) to define a function with domain $\sigma(I)$ by

$$\mathtt{s} : \sigma(I) \to \mathbb{R}, \; \mathtt{s}(\sigma(u)) = s(u) = \int_r^u du' \, [|\boldsymbol{g}(\sigma_{*u'}, \sigma_{*u'})|]^{\frac{1}{2}} \, . \tag{6.5}$$

With Eq. (6.5) we can calculate the derivative of the function s (after introducing a local chart $\{\mathbf{x}^\mu\}$ covering the domain of interest)

$$\frac{ds}{du} = [|\boldsymbol{g}(\sigma_{*u'}, \sigma_{*u'})|]^{\frac{1}{2}} = \left[\left| g_{\mu\nu} \frac{d\mathbf{x}^\mu \circ \sigma}{du} \frac{d\mathbf{x}^\nu \circ \sigma}{du} \right| \right]^{\frac{1}{2}} \tag{6.6}$$

From Eq. (6.6) old books on differential geometry and almost all books on General Relativity writes the equation

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{6.7}$$

which is supposed to represent the square of the length of the 'infinitesimal' arc determined by the coordinate displacement

$$\mathbf{x}^\mu \circ \sigma(r) \mapsto \mathbf{x}^\mu \circ \sigma(r) + \frac{d\mathbf{x}^\mu \circ \sigma}{du}(r)\varepsilon, \tag{6.8}$$

where $\varepsilon << 1$ is an 'infinitesimal' number.

The above mathematically correct notation is somewhat cumbersome, and when no confusion arises $\mathbf{x}^\mu \circ \sigma(r)$ is denoted simply by $x^\mu(r)$.

We recall that a *moving frame* at $x \in M$ is a basis for the tangent space $T_x M$. An orthonormal frame at $x \in M$ is a basis of orthonormal vectors for $T_x M$.

Definition 6.4 An observer in a spacetime $\mathfrak{M}$ is a future pointing time-like curve $\sigma : \mathbb{R} \supset I \to M$ such that $\boldsymbol{g}(\sigma_*, \sigma_*) = 1$. The timelike curve σ is said to be the world line of the observer.

6.1.1 Bradyons and Luxons

One of the important ingredients of physical theory is the concept of particle. Roughly speaking a particle is defined by the attributes it carries and a classification of particles can only be given in the context of a particular theory. However, one of the attributes of a particle is its *inertial mass*, modelled classically by $m \in \mathbb{R}^+ + \{0\}$. We can give the

Definition 6.5 A massive scalar particle is a pair (m, σ), with $m > 0$ and $\sigma : I \to M$ a timelike curve pointing to the future such that $\boldsymbol{g}(\sigma_*, \sigma_*) = 1$.

A scalar particle as above defined is sometimes called a scalar *bradyon*. Relativistic quantum field theory admits the existence in nature zero mass particles, also called *luxons*, as, e.g., photons. Now, particles with $m > 0$ or $m = 0$ may also carry *intrinsic spin*, and the *classical* description of that property in the case $m > 0$ will be given in Sect. 6.1.6 and Chap. 6. Photons also carry spin, but in that case, a coherent description of their properties can only be given in the context of relativistic quantum field theory [137], which is a subject that will not be discussed in this book. However, a crucial property of photons is that their paths in spacetime are *null geodesics*. These null geodesics are also supposed in classical GRT to be the paths followed in spacetime by light rays. This assumption can be justified classically once we suppose light is a wave phenomenon described by Maxwell equations, and define *light rays* as normals to *wave fronts*. For details see, e.g., [30, 47].

Axiom 6.1 (Standard Clock Postulate) *Let σ be an observer. Then, there exists standard clocks that can be 'carried by σ' and such that they register (in σ) proper-time, i.e., the inclusion parameter u of the definition of observer.*

It seems that modern atomic clocks are standard clocks [4], and indeed they are used as such clocks in the GPS *system* [3, 10]. However we must call the readers attention that the earlier experiments by Hafelle-Keating always quoted in textbooks, like e.g., in [31] were not precise enough to verify any claim [43, 63]. In [31] it is also claimed that the lifetime of unstable elementary particles, like, e.g., the lifetimes of muons traveling at near the speed of light in a storage ring [12] are compatible with the standard clock postulate. However, as observed by Apsel [9], there are indeed some small discrepancies. On this issue see also [122, 123].

Definition 6.6 An instantaneous observer is an element of TM, i.e., a pair $(x, \mathcal{Q})$, where $z \in M$, and $\mathcal{Q} \in T_xM$ is a future pointing unit timelike vector. Span$\mathcal{Q} \subset T_xM$ is the local time axis of the observer and $\mathcal{Q}^\perp$ is the observer rest space.

Of course, $T_xM = \text{Span}\mathcal{Q} \oplus \mathcal{Q}^\perp$, and we denote in what follows $\text{Span}\mathcal{Q} = T$ and $\mathcal{Q}^\perp = H$, which is called the *rest space* of the instantaneous observer. If $\sigma : \mathbb{R} \supset I \to M$ is an observer, then $(\sigma u, \sigma_* u)$ is said to be the local observer at u and write

$$T_{\sigma u}M = T_u \oplus H_u \ , \ u \in I.$$

Definition 6.7 The orthogonal projections are the mappings

$$\mathbf{p}_u = T_{\sigma u}M \to H_u \ , \ \mathbf{q}_u : T_{\sigma u}M \to T_u. \tag{6.9}$$

Then if $\mathbf{Y}$ is a vector field over σ then $\mathbf{pY}$ and $\mathbf{qY}$ are vector fields over σ given by

$$(\mathbf{pY})_u = \mathbf{p}_u(\mathbf{Y}_u), \ (\mathbf{qY})_u = \mathbf{q}_u(\mathbf{Y}_u). \tag{6.10}$$

Definition 6.8 Let $(x, \mathcal{Q})$ be a instantaneous observer and $\mathbf{p}_x : T_xM \to H$ the orthogonal projection. The projection tensor is the symmetric bilinear mapping $\mathbf{h} : TM \times TM \to \mathbb{R}$ such that for any $\mathbf{U}, \mathbf{W} \in T_xM$ we have:

$$\mathbf{h}_x(\mathbf{U}, \mathbf{W}) = \boldsymbol{g}_x(\mathbf{pU}, \mathbf{pW}) \tag{6.11}$$

Let $\{x^\mu\}$ be coordinates of a chart covering $U \subset M$, with $x \in U$ and $\alpha_{\mathcal{Q}} = \boldsymbol{g}_x(\mathcal{Q},)$. We have the properties:

(a)	$\mathbf{h}_x = \boldsymbol{g}_x - \alpha_{\mathcal{Q}} \otimes \alpha_{\mathcal{Q}}$	
(b)	$\mathbf{h}\|_{\mathcal{Q}^\perp} = \boldsymbol{g}_x\|_{\mathcal{Q}^\perp}$	
(c)	$\mathbf{h}(\mathcal{Q},) = 0$	(6.12)
(d)	$\mathbf{h}(\mathbf{U},) = \boldsymbol{g}(\mathbf{U},) \Leftrightarrow \ \boldsymbol{g}(\mathbf{U}, \mathcal{Q}) = 0$	
(e)	$\mathbf{p} = h^\mu_\nu \left.\frac{\partial}{\partial x^\mu}\right\|_x \otimes dx^\nu\|_x$	
(f)	$\text{trace}(h^\mu_\nu \left.\frac{\partial}{\partial x^\mu}\right\|_x \otimes dx^\nu\|_x) = -3$	

The Proposition 6.1 together with the above definitions suggests:

Definition 6.9 A reference frame for $U \subseteq M$ in a spacetime $\mathfrak{M}$ is a time-like vector field which is a section of TU such that each one of its integral lines is an observer.

Definition 6.10 Let $\mathbf{Q} \in \sec TM$, be a reference frame. A chart in $U \subseteq M$ of an oriented atlas of M with coordinate functions $\{x^\mu\}$ such that $\partial/\partial x^0 \in \sec TU$ is a timelike vector field and the $\partial/\partial x^i \in \sec TU$ $(i = 1, 2, 3)$ are spacelike vector fields is said to be a possible naturally adapted coordinate chart to the frame $\mathbf{Q}$ (denoted $(nacs|\mathbf{Q})$ in what follows) if the space-like components of $\mathbf{Q}$ are null in the natural coordinate basis $\{\partial/\partial x^\mu\}$ of TU associated with the chart.

Note that such chart always exist [19].

6.1.2 *Classification of Reference Frames I*

An arbitrary reference frame $\mathbf{Q} \in \sec TU \subseteq \sec TM$ for a general *Lorentzian spacetime* may be classified according to: (1) a decomposition of $D\mathbf{Q}$ given by Eq. (6.42) below or, (2) according to its *synchronizability*. In order to present these concepts we shall need to introduce some additional mathematical tools, as the Fermi-Walker connection and infinitesimally nearby observers. However, even without that concepts we already may give the following definitions.

Definition 6.11 A reference frame $\mathbf{Q} \in \sec TU \subseteq \sec TM$ is in *free fall* (or geodetic) if and only if $\mathbf{a}_{\mathbf{Q}} = \boldsymbol{D}_{\mathbf{Q}}\mathbf{Q} = 0$.

Definition 6.12 A reference frame $\mathbf{Q} \in \sec TU \subseteq \sec TM$ such that $\boldsymbol{D}\mathbf{Q} = 0$, $\mathbf{Q}$ is said an inertial reference frame (IRF).

Remark 6.13 *IRF*s exist in Minkowski spacetime, as it is easy to verify. However, we can also easily show that in the *GRT* where each gravitational field is modelled by a Lorentzian spacetime there is in general no frame $\mathbf{Z} \in \sec TM$ satisfying $\boldsymbol{D}\mathbf{Z} = 0$. Indeed, existence of an *IRF* is possible [131] only in a spacetime where the Ricci tensor satisfies $Ricci(\mathbf{Z}, \mathbf{F}) = 0$, for any $\mathbf{F} \in \sec TM$. This excludes, e.g., Einstein-de Sitter spacetime. So, no *IRF* exist in many models of *GRT* considered to be of interest by one reason or another by 'professional relativists'. The best we can have are some reference systems maintaining some of the characteristics of an *IRF*. These reference frames are the pseudo inertial reference frames and the locally inertial reference frames associated to an observer in free fall. These important concepts will be discussed below (Sect. 6.9).

6.1.3 Rotation and Fermi Transport

Let $\sigma : \mathbb{R} \supset I \to M$, $\tau \mapsto \sigma(\tau)$ be an observer. Let $\mathbf{Y}$ be a vector field over σ. As well known [131] in order for the observer to decide when a unitary vector $\mathbf{Y} \in (\sigma_{*\tau})^{\perp}$ has the same spatial direction of the unitary vector $\mathbf{Y}' \in (\sigma_{*\tau'})^{\perp}$ $(\tau' \neq \tau)$, he has to introduce the concept of the Fermi-Walker connection.

Proposition 6.14 *There exists one and only one connection $\mathcal{F}$ over σ, such that*

$$\mathcal{F}_{\mathbf{X}}\mathbf{Y} = [\mathbf{p}(\sigma^*\boldsymbol{D})_{\mathbf{X}}\mathbf{p} + \mathbf{q}(\sigma^*\boldsymbol{D})_{\mathbf{X}}\mathbf{q}]\mathbf{Y} \tag{6.13}$$

for all vector fields $\mathbf{X}$ on I and for all vector fields $\mathbf{Y}$ over σ.

In Eq. (6.13) $\sigma^*\boldsymbol{D}$ is the *induced* connection over σ of the Levi-Civita connection $\boldsymbol{D}$ and $\mathcal{F}$ is called the Fermi-Walker connection over σ, and we shall use the notations $\mathcal{F}_{\sigma_*}$, $\mathcal{F}/d\tau$ or $\mathcal{F}_{\boldsymbol{\varepsilon}_0}$ (see below) when convenient. We also will write (by abuse of language) only $\boldsymbol{D}$, as usual, for $\sigma^*\boldsymbol{D}$ in what follows.

Proof The proof follows at once from the general properties concerning the behavior of connections under pullback mappings [19]. ■

Definition 6.15 A moving (orthonormal) frame $\{\epsilon_{\mathbf{a}}\}$ over σ is an orthonormal basis for $T_{\sigma(I)}M$ with $\epsilon_{\mathbf{0}} = \sigma_*$. The set $\{\varepsilon^{\mathbf{a}}\}$, $\varepsilon^{\mathbf{a}} \in \sec T^*_{\sigma(I)}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is the dual comoving frame on σ, i.e., $\varepsilon^{\mathbf{a}}(\epsilon_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$. The set $\{\varepsilon_{\mathbf{a}}\}$, $\varepsilon_{\mathbf{a}} \in \sec T^*_{\sigma(I)}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, with $\varepsilon^{\mathbf{a}} \cdot \varepsilon_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$ is the reciprocal frame of $\{\varepsilon^{\mathbf{a}}\}$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields over σ and $Y = \boldsymbol{g}(\mathbf{X},), Y = \boldsymbol{g}(\mathbf{Y},) \in \sec T^*_{\sigma(I)}M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ the physically equivalent 1-form fields.

Proposition 6.16 *Let be X, Y be form fields over σ, as defined above. The Fermi-Walker connection $\mathcal{F}$ satisfies the properties*

(a) $\mathcal{F}_{\epsilon_o} Y = \boldsymbol{D}_{\epsilon_0} Y - (\varepsilon_{\mathbf{0}} \cdot Y)a + (a \cdot Y)\varepsilon_0$,
  where $a = \boldsymbol{D}_{\epsilon_0}\varepsilon_0$, *is the* (1-*form*) *acceleration.*
(b) $\frac{d}{d\tau}(X \cdot Y) = \mathcal{F}_{\epsilon_0} X \cdot Y + X \cdot \mathcal{F}_{\epsilon_0} Y$
(c) $\mathcal{F}_{\epsilon_0}\varepsilon_{\mathbf{0}} = 0$
(d) *If* $X, \mathbf{Y}$ *are vector fields on* σ *such that* $X_u, \mathbf{Y}_u \in H_u$ $\forall u \in I$ *then*[1] $\mathtt{g}(\mathcal{F}_{\epsilon_0} X,)|_u$ *and* $g(\mathcal{F}_{\epsilon_0} Y,)|_u \in H_u$, $\forall u$ *and*

$$\mathcal{F}_{\epsilon_0} X \cdot Y = \boldsymbol{D}_{\epsilon_0} X \cdot Y \tag{6.14}$$

Proof We prove only (a). The other properties are trivial. First we prove that

$$\mathcal{F}_{\epsilon_0}\mathbf{Y} = \boldsymbol{D}_{\epsilon_0}\mathbf{Y} - (\epsilon_{\mathbf{0}} \cdot \mathbf{Y})\mathbf{a} + (\mathbf{a} \cdot \mathbf{Y})\epsilon_{\mathbf{0}} \tag{6.15}$$

with $\mathbf{a} = \boldsymbol{D}_{\epsilon_0}\epsilon_0$, and where we wrote $\epsilon_{\mathbf{0}} \cdot \mathbf{Y} = \boldsymbol{g}(\epsilon_{\mathbf{0}}, \mathbf{Y})$, $\mathbf{a} \cdot \mathbf{Y} = \boldsymbol{g}(\mathbf{a}, \mathbf{Y})$.

From Eq. (6.13) and the abuse of notation mentioned above and taking into account that $\mathbf{q}(\boldsymbol{D}_{\epsilon_0}\epsilon_{\mathbf{0}}) = 0$, we can write

$$\mathcal{F}_{\epsilon_o}\mathbf{Y} = \mathbf{p}[\boldsymbol{D}_{\epsilon_0}(\mathbf{pY})] + \mathbf{q}[\boldsymbol{D}_{\epsilon_0}(\mathbf{qY})] \tag{6.16}$$

Now,

$$\begin{aligned}
\mathbf{q}[\boldsymbol{D}_{\epsilon_0}(\mathbf{qY})] &= \mathbf{q}\{\boldsymbol{D}_{\epsilon_0}(\mathbf{Y}\cdot\epsilon_{\mathbf{0}})\epsilon_{\mathbf{0}}\} \\
&= \mathbf{q}\{(D_{\epsilon_0}\mathbf{Y})\cdot\epsilon_{\mathbf{0}})\boldsymbol{\varepsilon_0} + \mathbf{Y} \cdot (\boldsymbol{D}_{\epsilon_0}\epsilon_{\mathbf{0}})\epsilon_{\mathbf{0}} + (\mathbf{Y}\cdot\epsilon_{\mathbf{0}})\boldsymbol{D}_{\epsilon_0}\epsilon_{\mathbf{0}}\} \\
&= [(\boldsymbol{D}_{\epsilon_0}\mathbf{Y})\cdot\epsilon_{\mathbf{0}}]\epsilon_{\mathbf{0}} + (\mathbf{Y} \cdot \mathbf{a})\epsilon_{\mathbf{0}} \\
&= \mathbf{q}(\boldsymbol{D}_{\epsilon_0}\mathbf{Y}) + (\mathbf{Y} \cdot \mathbf{a})\epsilon_{\mathbf{0}}.
\end{aligned} \tag{6.17}$$

Also,

$$\begin{aligned}
\mathbf{p}[\boldsymbol{D}_{\epsilon_0}(\mathbf{pY})] &= \mathbf{p}[\boldsymbol{D}_{\epsilon_0}\mathbf{Y} - \boldsymbol{D}_{\epsilon_0}(\mathbf{qY})] \\
&= \mathbf{p}(\boldsymbol{D}_{\epsilon_0}\mathbf{Y}) - \mathbf{p}\{\boldsymbol{D}_{\epsilon_0}(\mathbf{Y} \cdot \epsilon_{\mathbf{0}})\epsilon_{\mathbf{0}}\} \\
&= \mathbf{p}(\boldsymbol{D}_{\epsilon_0}\mathbf{Y}) - \mathbf{p}\{[(\boldsymbol{D}_{\epsilon_0}\mathbf{Y}) \cdot \epsilon_{\mathbf{0}}]\epsilon_{\mathbf{0}} + (\mathbf{Y} \cdot \mathbf{a})\epsilon_{\mathbf{0}} + (\mathbf{Y} \cdot \epsilon_{\mathbf{0}})\mathbf{a}\} \\
&= \mathbf{p}(\boldsymbol{D}_{\epsilon_0}\mathbf{Y}) - (\mathbf{Y} \cdot \epsilon_{\mathbf{0}})\mathbf{a}.
\end{aligned} \tag{6.18}$$

[1] Recall the notations introduced in Chap. 4, where $\mathtt{g}$ denote the metric in the cotangent bundle.

Summing Eqs. (6.17) and (6.18) we get Eq. (6.15). Then (a) follows at once we take into account that for any $\mathbf{X}, \mathbf{Y} \in \sec TM$,

$$\boldsymbol{g}(\boldsymbol{D}_{\epsilon_\mathbf{0}}\mathbf{Y},) = \boldsymbol{D}_{\epsilon_\mathbf{0}}[\boldsymbol{g}(\mathbf{Y},\ )] = \boldsymbol{D}_{\epsilon_\mathbf{0}}Y, \tag{6.19}$$

which proves the result. ■

Now, let $Y_0 \in \sec T^*_{\sigma\tau_0}M$. Then, by a well known property of connections there exists one and only one 1-form field Y over σ such that $\mathcal{F}_{\epsilon_o}Y = 0$ and $Y(\tau_0) = Y_0$. So, if $\{\varepsilon_\mathbf{a}|_{\tau_0}\}$ is an orthonormal basis for $T^*_{\sigma\tau_0}M$ ($\varepsilon_\mathbf{a}|_{\tau_0} \in T^*_{\sigma\tau_0}M$, $\mathbf{a} = 0, 1, 2, 3$) we have that the $\varepsilon_\mathbf{a}$'s such that $\mathcal{F}_{\epsilon_\mathbf{0}}\varepsilon_\mathbf{a} = 0$ are orthonormal for any τ, as follows from (b) in Proposition 6.16. We then, give the

Definition 6.17 Let $Y_1 \in H^*_{\tau_1}$ and $Y_2 \in H^*_{\tau_2}$ are said to have the same spatial direction if and only if $Y_1 = a^i\ \varepsilon_i|_{\tau_1}$, $Y_2 = a^i\ \varepsilon_i|_{\tau_2}$.

This suggests the following definition:

Definition 6.18 We say that $Y \in \sec T^*_\sigma M$ is transported without rotation (*Fermi transported*) if and only if $\mathcal{F}_{\epsilon_\mathbf{0}}Y = 0$.

In that case we have

$$\begin{aligned}\boldsymbol{D}_{\epsilon_\mathbf{0}}Y = \varepsilon_\mathbf{0} \cdot \boldsymbol{\partial} Y \equiv \frac{D}{d\tau}Y &= (Y \cdot \varepsilon_\mathbf{0})a - (a \cdot Y)\varepsilon_\mathbf{0} \\ &= Y \lrcorner (\varepsilon_\mathbf{0} \wedge a) = (a \wedge \varepsilon_\mathbf{0}) \llcorner Y.\end{aligned} \tag{6.20}$$

6.1.4 *Frenet Frames over σ*

Proposition 6.19 *If* $\{\varepsilon_\mathbf{a}\}$ *is a comoving coframe over* σ (*Definition* 6.15), *then there exists a unique biform field* Ω_D *over* σ, *called the angular velocity* (*Darboux biform*) *such that the* $\varepsilon_\mathbf{a}$ *satisfy the following system of differential equations,*

$$\begin{aligned}\boldsymbol{D}_{\epsilon_\mathbf{0}}\varepsilon_\mathbf{a} &= \Omega_\mathrm{D} \llcorner \varepsilon_\mathbf{a}, \\ \Omega_\mathrm{D} &= \frac{1}{2}\omega_\mathbf{ab}\varepsilon^\mathbf{a} \wedge \varepsilon^\mathbf{b} = -\frac{1}{2}(\boldsymbol{D}_{\epsilon_\mathbf{0}}\varepsilon_\mathbf{b}) \wedge \varepsilon^\mathbf{b}.\end{aligned} \tag{6.21}$$

Proof It follows at once if we take into account that $\varepsilon_\mathbf{a} \cdot \varepsilon^\mathbf{b} = \delta^\mathbf{b}_\mathbf{a}$ and use Eqs. (2.36) and (2.37). ■

Corollary 6.20 *If the comoving coframe* $\{\varepsilon_\mathbf{a}\}$ *is Fermi transported then the angular velocity is* Ω_F,

$$\Omega_F = a \wedge \varepsilon_\mathbf{0}. \tag{6.22}$$

Proof The result follows from Eqs. (6.20) and (6.21). ■

6.1.5 *Physical Meaning of Fermi Transport*

Suppose that a comoving coframe $\{\varepsilon_{\mathbf{a}}\}$ is Fermi transported along a timelike curve σ, 'materialized' by some particle. The physical meaning associated to Fermi transport is that the spatial axis of the tetrad $\mathtt{g}(\varepsilon_i,) = \epsilon_i, i = 1, 2, 3$ are to be associated to the orthogonal spatial directions of three 'small' gyroscopes carried along σ.

Definition 6.21 A Frenet coframe $\{f_{\mathbf{a}}\}$ over σ is a moving coframe over σ such that $f_{\mathbf{0}} = \boldsymbol{g}(\sigma_*,) = \boldsymbol{g}(\epsilon_{\mathbf{0}},) = \varepsilon_{\mathbf{0}}$ and

$$\begin{aligned} \boldsymbol{D}_{\epsilon_{\mathbf{0}}} f_{\mathbf{a}} &= \Omega_{\mathrm{D}} \llcorner f_{\mathbf{a}}, \\ \Omega_{\mathrm{D}} &= \kappa_0 f^{\mathbf{1}} \wedge f^{\mathbf{0}} + \kappa_1 f^{\mathbf{2}} \wedge f^{\mathbf{1}} + \kappa_2 f^{\mathbf{3}} \wedge f^{\mathbf{2}}, \end{aligned} \tag{6.23}$$

where $\kappa_i, i = 0, 1, 2$ is the i-curvature, which is the projection of Ω_{D} in the $f^{\mathbf{i+1}} \wedge f^{I}$ plane.

Definition 6.22 We say that a 1-form field Y over σ is rotating if and only if it is rotating in relation to gyroscopes axis, i.e., if $\mathcal{F}_{\epsilon_{\mathbf{0}}} Y \neq 0$.

6.1.6 *Rotation 2-Form, Pauli-Lubanski Spin* **1**-*Form and Classical Spinning Particles*

From Eq. (6.23) taking into account that $a = \boldsymbol{D}_{\epsilon_{\mathbf{0}}} f_{\mathbf{0}} = \kappa_{\mathbf{0}} f^{\mathbf{1}}$ and Eq. (6.22) we can write

$$\Omega_{\mathrm{D}} = a \wedge f_{\mathbf{0}} + \Omega_{\mathbf{S}}. \tag{6.24}$$

We now show that the 2-form $\Omega_{\mathbf{S}}$ over σ is directly related (a dimensional factor apart) with the spin 2-form of a *classical spinning particle*. More, we show that the Hodge dual of $\Omega_{\mathbf{S}}$ is associated with the Pauli-Lubanski 1-form. To have a notation as closely as possible the usual ones of physical textbooks, let us put $f_{\mathbf{0}} = v$.

Call $\star\Omega_{\mathbf{S}}$ the Hodge dual of $\Omega_{\mathbf{S}}$. It is the 2-form over σ given by

$$\star\, \Omega_{\mathbf{S}} = -\Omega_{\mathbf{S}} f^{\mathbf{5}},\; f^{\mathbf{5}} = f^{\mathbf{0}} f^{\mathbf{1}} f^{\mathbf{2}} f^{\mathbf{3}}. \tag{6.25}$$

Define the *rotation* 1-form $\mathbf{S}$ over σ by

$$\mathbf{S} = - \star \Omega_{\mathbf{S}} \llcorner v. \tag{6.26}$$

Since $\Omega_{\mathbf{S}} \llcorner v = 0$, we have immediately that

$$\mathbf{S} \cdot v = - \star \Omega_{\mathbf{S}} \llcorner (v \wedge v) = 0. \tag{6.27}$$

Now, since $\boldsymbol{D}_{\boldsymbol{\epsilon_0}}(\mathbf{S} \cdot v) = 0$ we have that $(\boldsymbol{D}_{\boldsymbol{\epsilon_0}}\mathbf{S}) \cdot a = -\mathbf{S} \cdot a$ and then

$$\boldsymbol{D}_{\boldsymbol{\epsilon_0}}\mathbf{S} = -\mathbf{S} \lrcorner (a \wedge v), \tag{6.28}$$

and it follows that

$$\mathcal{F}_{\boldsymbol{\epsilon_0}}\mathbf{S} = 0. \tag{6.29}$$

It is intuitively clear that we must associate $\mathbf{S}$ with the spin of a classical spinning particle which follows the worldline σ. And, indeed, we define the *Pauli-Lubanski* spin 1-form by

$$W = k\hbar\mathbf{S},\ \hbar = 1, \tag{6.30}$$

where $k > 0$ is a real constant and $\hbar$ is Planck constant, which is equal to 1 in the natural system of units used here. We recall that as it is well-known $\boldsymbol{D}_{\boldsymbol{\epsilon_0}} W = -W \lrcorner (a \wedge v)$ is the equation of motion of the intrinsic spin of a classical spinning particle which is being accelerated by a force producing no torque [156].

6.2 Classification of Reference Frames II

6.2.1 Infinitesimally Nearby Observers, 3-Velocities and 3-Accelerations

Let $\mathbf{Q} \in \sec TU \subset \sec TM$ be a reference frame, μ_s its flux and σ an integral line of $\mathbf{Q}$ and suppose that we have a parametrization $\sigma : \mathbb{R} \supset I \to M$, $\boldsymbol{g}(\sigma_*, \sigma_*) = 1$. Then σ is an observer.

Definition 6.23 An infinitesimally nearby observer to σ is a vector field $\mathbf{W} : \mathbb{R} \supset I \to T_{\sigma(I)}M$ such that there exists a vector field $\mathbf{W}'$ over σ such that $\mathbf{W} = \mathbf{p}\mathbf{W}'$ and which is Lie parallel with respect to $\mathbf{Q}$, i.e., $\pounds_{\mathbf{Q}}\mathbf{W}' = 0$. $\mathbf{W}$ is sometimes called a Jacobi field.

Proposition 6.24 *Suppose that* $\mathbf{Q}$ *is a geodesic frame, i.e.,* $\boldsymbol{D}_{\mathbf{Q}}\mathbf{Q} = 0$. *Then,* $\pounds_{\mathbf{Q}}\mathbf{W} = 0$.

Proof Recall that in a Lorentzian manifold, where the torsion tensor is null the equation $\pounds_{\mathbf{Q}}\mathbf{W}' = 0$ implies that $\boldsymbol{D}_{\mathbf{W}'}\mathbf{Q} = \boldsymbol{D}_{\mathbf{Q}}\mathbf{W}'$. Now, $\mathbf{W} = \mathbf{W}' - \boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q}$ and then, $[\mathbf{W}, \mathbf{Q}] - [\boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q}, \mathbf{Q}] = 0$. We need then to prove that $[\boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q}, \mathbf{Q}] = 0$. We have,

$$\begin{aligned}[\boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q}, \mathbf{Q}] &= \boldsymbol{D}_{\boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q}}\mathbf{Q} - \boldsymbol{D}_{\mathbf{Q}}\boldsymbol{g}(\mathbf{W}', \mathbf{Q})\mathbf{Q} \\ &= [\boldsymbol{D}_{\mathbf{Q}}\boldsymbol{g}(\mathbf{W}', \mathbf{Q})]\mathbf{Q}.\end{aligned}$$

But, $\boldsymbol{D}_{\mathbf{Q}}\boldsymbol{g}(\mathbf{W}', \mathbf{Q}) = \boldsymbol{g}(\boldsymbol{D}_{\mathbf{Q}}\mathbf{W}', \mathbf{Q}) = \boldsymbol{g}(\boldsymbol{D}_{\mathbf{W}'}\mathbf{Q}, \mathbf{Q}) = \boldsymbol{D}_{\mathbf{W}'}\boldsymbol{g}(\mathbf{Q}, \mathbf{Q}) = 0$, and the proposition is proved. ■

Now, let us examine following [131] the geometrical meaning of an infinitesimally nearby observer. Recall that for $s \in I$ there is a neighborhood $\mathcal{E} = (s-\varepsilon, s+\varepsilon)$ of s and a neighborhood U of $\sigma(s)$ and a vector field $\mathbf{V} \in \sec TU$ such that $\mathbf{W} = \mathbf{V} \circ \sigma$ on $\mathcal{E}$ and $\pounds_{\mathbf{Q}}\mathbf{V} = 0$. Let (U, ϕ) be a map of the maximal atlas of M with coordinates[2] $\{\bar{x}^\mu\}$ covering $\phi(U)$ such that $\mathbf{Q}|_U = \frac{\partial}{\partial \bar{x}^0}\big|_U$. We write $\mathbf{W}|_{\mathcal{E}} = w^\mu(\frac{\partial}{\partial \bar{x}^\mu} \circ \sigma\big|_{\mathcal{E}})$. We may write for $p \in U$, $\phi(p) = \{(\bar{x}^0(p), \bar{x}^1(p), \bar{x}^2(p), \bar{x}^3(p)) \mid |\bar{x}^\mu(p)| < \varepsilon, \ \forall \mu\}$ and assume that for $p = \sigma(s)$, $\phi(\sigma(s)) = (s, 0, 0, 0)$. There is then a congruence of integral curves of $\mathbf{Q}$ determined by

$$(s, u) \mapsto (s + w^0 u, w^1 u, w^2 u, w^3 u). \tag{6.31}$$

Note that $u = 0$ gives $\sigma|_{\mathcal{E}}$ and u times a *constant* gives another curve of the congruence in U where the parametrization given by $\{\bar{x}^\mu\}$ holds. Now, when $\mathbf{W}|_{\mathcal{E}}$ and $\mathbf{Q} \circ \sigma$ are linearly independent, different curves have different images. This uniquely determines $\mathbf{W}|_{\mathcal{E}}$ as its *transversal* vector field, i.e., for any $f : U \to \mathbb{R}$ and $s \in \mathcal{E}$,

$$(\mathbf{W}f)(s) = \frac{\partial}{\partial u} f(s + w^0 u, w^1 u, w^2 u, w^3 u)\bigg|_{u=0}. \tag{6.32}$$

Conversely, once $\mathbf{W}|_{\mathcal{E}}$ is given, the family is determined up to first order in x^0 in the sense of a Taylor series.

In the coordinates $\{\bar{x}^\mu\}$ where $\mathbf{V} = \bar{V}^\mu \frac{\partial}{\partial \bar{x}^\mu}$, the equation $\pounds_{\mathbf{Q}}\mathbf{V} = 0$ implies that $\mathbf{Q}(\bar{V}^\mu) = 0$. Let $\{w^u\}$ be real constants such that $\mathbf{W}|_{\sigma\tau_0} = w^\mu(\frac{\partial}{\partial \bar{x}^\mu} \circ \sigma\big|_{\sigma\tau_0})$ for some $\tau_0 \in \varepsilon$. Then putting $\mathbf{V} = w^\mu \frac{\partial}{\partial \bar{x}^\mu}$ we have $\pounds_{\mathbf{Q}}\mathbf{V} = 0$. Then, the *interpretation* of $\mathbf{W}$ becomes clear. It represents the linearized version of a one parameter family of integral curves of $\mathbf{Q}$ near σ. Note however that in an arbitrary chart (U, ϕ') with

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[2] This chart always exist [19].

coordinates $\{x^\mu\}$ covering $\phi'(U)$, where $\mathbf{V} = V^\mu \frac{\partial}{\partial x^\mu}$, the equation $\pounds_\mathbf{Q}\mathbf{V} = 0$ implies that

$$\mathbf{Q}[V^\mu] = \frac{\partial Q^\mu}{\partial x^\nu} V^\nu, \tag{6.33}$$

which is nonzero in general.

Definition 6.25 Let σ be an observer on $\mathbf{Q}$ and $\mathbf{W}$ be an infinitesimally nearby observer to σ and $\mathcal{F}$ the Fermi-Walker connection over σ. Then, $\mathcal{F}_{\sigma*}\mathbf{W}$ and $\mathcal{F}^2_{\sigma*}\mathbf{W} = \mathcal{F}_{\sigma*}(\mathcal{F}_{\sigma*}\mathbf{W})$ are called respectively the 3-velocity and the 3-acceleration of the infinitesimally nearby observer $\mathbf{W}$ relative to σ.

We observe that we can show (see below) that for any $s \in I$, $(\mathcal{F}_{\sigma*}\mathbf{W})_u \in H_s$ and $(\mathcal{F}^2_{\sigma*}\mathbf{W}) \in H_s$, thus justifying the names 3-velocity and the 3-acceleration. The meaning of these concepts become clear with the aid of the following propositions.

Proposition 6.26 *Let* $\mathbf{Q} \in \sec TU \subset \sec TM$ *be a reference frame and* $\sigma : I \to M$ *an observer in* $\mathbf{Q}$. *Let also* $\mathbf{Y} \in H_s$. *Then, the mapping* $\mathbf{Y} \mapsto \boldsymbol{D}_\mathbf{Y}\mathbf{Q}$ *defines a linear transformation* $A_\mathbf{Q} : H_s \to H_s$, *such that if* $\mathbf{W}$ *is any infinitesimally nearby observer to* σ *we have*

$$\mathcal{F}_{\sigma*}\mathbf{W} = \mathcal{A}_\mathbf{Q}\mathbf{W} = \boldsymbol{D}_W\mathbf{Q}. \tag{6.34}$$

Proof First we need to show that $\mathcal{A}_\mathbf{Q}H_s \subset H_s$, $\forall s \in I$. But this is trivial since $\forall\mathbf{Y} \in H_s$ we have

$$\boldsymbol{g}(\mathcal{A}_\mathbf{Q}Y, \mathbf{Q}) = \boldsymbol{g}(\boldsymbol{D}_\mathbf{Y}\mathbf{Q}, \mathbf{Q}) = \frac{1}{2}\boldsymbol{D}_\mathbf{Y}\boldsymbol{g}(\mathbf{Q}, \mathbf{Q}) = 0. \tag{6.35}$$

Now, we need to show that $\forall s \in I$ $\mathcal{F}_{\sigma*_s}\mathbf{W} = \mathbf{D}_{\mathbf{W}_s}\mathbf{Q}$. To show that is, of course, equivalent to show that $\forall\mathbf{Y} \in H_s$,

$$\boldsymbol{g}(\mathcal{F}_{\sigma*S}\mathbf{W}, \mathbf{Y}) = \boldsymbol{g}(\boldsymbol{D}_{\mathbf{W}_s}\mathbf{Q}, \mathbf{Y}). \tag{6.36}$$

Now, recall (e) of Proposition 6.16 which says that

$$\boldsymbol{g}(\mathcal{F}_{\sigma*S}\mathbf{W}, \mathbf{Y}) = \boldsymbol{g}(\boldsymbol{D}_{\sigma*_s}\mathbf{Q}, \mathbf{Y}).$$

Then, we need only to prove that

$$\boldsymbol{g}(\boldsymbol{D}_{\sigma*S}\mathbf{W}, \mathbf{Y}) = \boldsymbol{g}(\boldsymbol{D}_{\sigma*_s}\mathbf{Q}, \mathbf{Y}), \forall\mathbf{Y} \in H_s. \tag{6.37}$$

Since $\mathbf{W}$ is an infinitesimally nearby observer to σ we know that there exists $\mathbf{W}'$, a vector field over σ, such that the projection $\mathbf{p}\mathbf{W}' = \mathbf{W}$ and such that $\pounds_\mathbf{Q}\mathbf{W}' = 0$. Of course, $\mathbf{W}' = \mathbf{W} - f\sigma_*$ for some smooth function f on σ. Now, let $\mathbf{V}'$ and F

be respectively a vector field and a smooth function defined on a sufficiently small open set U of $\sigma(s)$ such that

$$\mathbf{V}' \circ \sigma = \mathbf{W}', \quad [\mathbf{V}', \mathbf{Q}] = 0, \quad F \circ \sigma = f. \tag{6.38}$$

Moreover, define

$$\mathbf{V} = \mathbf{V}' - F\mathbf{Q} \tag{6.39}$$

such that $\mathbf{V} \circ \sigma = \mathbf{W}$. Now, since the torsion of a Lorentzian spacetime is zero, we have $D_{\mathbf{V}}\mathbf{Q} - D_{\mathbf{Q}}\mathbf{V} - [\mathbf{V}, \mathbf{Q}] = 0$, and we can write

$$\boldsymbol{D}_{\mathbf{V}}\mathbf{Q} = \boldsymbol{D}_{\mathbf{Q}}\mathbf{V} + \mathbf{Q}(F)\mathbf{Q}. \tag{6.40}$$

At s, we have $\boldsymbol{D}_{\mathbf{V}_s}\mathbf{Q} = D_{\sigma_{*s}}\mathbf{W} + \frac{df}{ds}(s)\sigma_{*s}$. This implies that $\boldsymbol{g}(\boldsymbol{D}_{\sigma_{*s}}\mathbf{W}, \mathbf{Y}) = \boldsymbol{g}(\boldsymbol{D}_{\sigma_{*s}}\mathbf{Q}, \mathbf{Y})$, $\forall \mathbf{Y} \in H_s$ and the proposition is proved. ■

Now, recall that since $\boldsymbol{g}(\mathbf{W}, \mathbf{Q}) = 0$ then $\mathbf{W}$ has only *spatial* components relative to an orthonormal frame $\{\boldsymbol{e}_{\mathbf{a}}\} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(U)$, with $\boldsymbol{e}_{\mathbf{0}} = \mathbf{Q}$ and $\mathcal{F}_{\mathbf{e}_0}\boldsymbol{e}_{\mathbf{a}} = 0$. Then from Eq. (6.34) we have $(i, j = 1, 2, 3)$

$$\frac{d}{ds}W_i(s) = (\boldsymbol{D}_{\mathbf{e}_j}\alpha_{\mathbf{Q}})_i W^j(s), \tag{6.41}$$

where $\alpha_{\mathbf{Q}} = \boldsymbol{g}(\mathbf{Q},\ )$. So, the first member of Eq. (6.41) clearly shows that the Fermi derivative $\mathcal{F}_{\sigma_*}\mathbf{W}$ is a measure of the rate of change of the coordinates of an infinitesimally nearby observer $\mathbf{W}$ to σ on $\mathbf{Q}$. To interpret the second member of Eq. (6.41) we need the

Proposition 6.27 *Given an arbitrary reference frame* $\mathbf{Q} \in \sec TU \subseteq \sec TM$ *for a Lorentzian spacetime* [131] *there exists a unique decomposition of* $\alpha_{\mathbf{Q}} = \boldsymbol{g}(\mathbf{Q},\ )$ *as*

$$\boldsymbol{D}\alpha_{\mathbf{Q}} = \mathbf{a}_{\mathbf{Q}} \otimes \alpha_{\mathbf{Q}} + \boldsymbol{\omega}_{\mathbf{Q}} + \boldsymbol{\sigma}_{\mathbf{Q}} + \frac{1}{3}\mathfrak{E}_{\mathbf{Q}}\mathbf{h}, \tag{6.42}$$

where $\mathbf{h} \in \sec T^0_2 M$ *is the projection tensor* (*Definition* 6.8), $\mathbf{a}_{\mathbf{Q}}$ *is the* (*form*) *acceleration of* $\mathbf{Q}$, $\boldsymbol{\omega}_{\mathbf{Q}}$ *is the rotation tensor* (*or vortex*) *of* $\mathbf{Q}$, $\boldsymbol{\sigma}_{\mathbf{Q}}$ *is the shear of* $\mathbf{Q}$ *and* $\mathfrak{E}_{\mathbf{Q}}$ *is the expansion ratio of* $\mathbf{Q}$. *In a coordinate chart* (U, ϕ) *with coordinates* x^μ *covering* $\phi(U)$, *writing* $\mathbf{Q} = Q^\mu \partial/\partial x^\mu$ *and* $\mathbf{h} = (g_{\mu\nu} - Q_\mu Q_\nu)dx^\mu \otimes dx^\nu$ *we have*

$$\begin{aligned}
\mathbf{a}_{\mathbf{Q}} &= \boldsymbol{g}(\boldsymbol{D}_{\mathbf{Q}}\mathbf{Q},\ ),\\
\boldsymbol{\omega}_{\mathbf{Q}\mu\nu} &= Q_{[\alpha;\beta]}h^\alpha_\mu h^\beta_\nu,\\
\boldsymbol{\sigma}_{\mathbf{Q}\alpha\beta} &= [Q_{(\mu;\nu)} - \frac{1}{3}\mathfrak{E}_{\mathbf{Q}}h_{\mu\nu}]h^\mu_\alpha h^\nu_\beta,\\
\mathfrak{E}_{\mathbf{Q}} &= Q^\mu{}_{;\mu}\,.
\end{aligned} \tag{6.43}$$

Proof It left as exercise. ■

Clearly, $\boldsymbol{\omega}_{\mathbf{Q}}$ measures the rotation that one of the infinitesimally nearby curves to σ had in an infinitesimal lapse of propertime with relation to an orthonormal basis Fermi transported by the observer at σ. Also, $\boldsymbol{\sigma}_{\mathbf{Q}}$ represents the ratio of change of the separation between σ and an infinitesimally nearby curves. Finally, as it is easy to verify, $\mathfrak{E}_{\mathbf{Q}} = \delta\alpha_{\mathbf{Q}}$ gives the fractional ratio expansion of the 3-dimensional volume element defined by σ and its nearby curves.

Remark 6.28 A preliminary classification of reference frames in Riemann-Cartan spacetimes is given in [52].

Exercise 6.29 Compute $\mathbf{a}_{\mathbf{Q}}, \boldsymbol{\omega}_{\mathbf{Q}}, \boldsymbol{\sigma}_{\mathbf{Q}}$ and $\mathfrak{E}_{\mathbf{Q}}$ in the orthonormal frame $\{\boldsymbol{e}_{\mathbf{i}}\} \in \mathbf{P}_{\mathrm{SO}^e_{1,3}}(U)$, with $\boldsymbol{e}_{\mathbf{0}} = \mathbf{Q}$ and $\mathcal{F}_{\boldsymbol{\epsilon}_{\mathbf{0}}}\boldsymbol{\epsilon}_{\mathbf{i}} = 0$, $\epsilon_{\mathbf{a}} = \boldsymbol{e}_{\mathbf{a}}|_{\sigma}$.

Exercise 6.30 Show that

$$\alpha_{\mathbf{Q}} \wedge d\alpha_{\mathbf{Q}} = 0 \Leftrightarrow \boldsymbol{\omega}_{\mathbf{Q}} = 0. \tag{6.44}$$

6.2.2 *Jacobi Equation*

Lemma 6.31 *Let* $\mathbf{Q} \in \sec TU \subset \sec TM$ *be a reference frame and* $\sigma : I \to M$ *an observer in* M*. Let also* $\mathbf{Y} \in \mathrm{H}_s$*. If* $\boldsymbol{\rho}$ *is the Riemann curvature operator see Eq. (4.102) then the mapping*

$$\mathbf{Y} \mapsto \boldsymbol{\rho}(\mathbf{Q},\mathbf{Y})\mathbf{Q} \tag{6.45}$$

defines a mapping $\mathrm{H}_s \to \mathrm{H}_s$.

Proof The result follows once we verify that $\boldsymbol{g}(\mathbf{Q}, \boldsymbol{\rho}(\mathbf{Q},\mathbf{Y})\mathbf{Q}) = 0$, which is a simple consequence of the symmetries of the Riemann tensor. ■

Proposition 6.32 *Let* $\mathbf{Q}$ *be a free fall frame,* σ *an observer on* $\mathbf{Q}$ *and* $\mathbf{W}$ *an infinitesimally nearby observer to* σ*. Then,*

$$\mathcal{F}^2_{\sigma_*}\mathbf{W} = \boldsymbol{\rho}(\boldsymbol{\sigma}_{*}, \boldsymbol{w})\sigma_*, \tag{6.46}$$

Proof Recall that Proposition 6.24 says that $\pounds_{\mathbf{Q}}\mathbf{W} = 0$ when $D_{\mathbf{Q}}\mathbf{Q} = \mathbf{0}$. Now, let $s \in I$ and let $\mathbf{V}$ be a vector field defined in some sufficiently small neighborhood of $\sigma(s)$ and such that $[\mathbf{V}, \mathbf{Q}] = 0$ and $\mathbf{W} = \mathbf{V} \circ \sigma$. Then, taking into account that in a Lorentzian manifold the torsion tensor is zero, which means that $\boldsymbol{D}_{\mathbf{Q}}\mathbf{V} - \boldsymbol{D}_{\mathbf{V}}\mathbf{Q} - [\mathbf{Q}, \mathbf{V}] = 0$, and that $\boldsymbol{D}_{\mathbf{Q}}\mathbf{Q} = 0$ by hypothesis, we can write

$$\begin{aligned}
\boldsymbol{D}^2_{\mathbf{Q}}\mathbf{V} &= \boldsymbol{D}_{\mathbf{Q}}(\boldsymbol{D}_{\mathbf{Q}}\mathbf{V}) = \boldsymbol{D}_{\mathbf{Q}}\{\boldsymbol{D}_{\mathbf{V}}\mathbf{Q} + [\mathbf{Q}, \mathbf{V}]\} = \boldsymbol{D}_{\mathbf{Q}}\boldsymbol{D}_{\mathbf{V}}\mathbf{Q} \\
&= \boldsymbol{D}_{\mathbf{Q}}\boldsymbol{D}_{\mathbf{V}}\mathbf{Q} - \boldsymbol{D}_{\mathbf{V}\boldsymbol{D}_{\mathbf{Q}}}\mathbf{Q} - \boldsymbol{D}_{[\mathbf{Q},\mathbf{V}]}\mathbf{Q} \\
&= \boldsymbol{\rho}(\mathbf{Q}, \mathbf{V})\mathbf{Q}.
\end{aligned} \tag{6.47}$$

Restricting Eq. (6.47) to σ and taking into account that $\mathcal{F}_{\sigma*} = \boldsymbol{D}_{\sigma*}$ since σ is geodesic, Eq. (6.46) follows. ■

Equation (6.46) is known as the geodesics separation equation or Jacobi equation.

Example 6.33 Consider a Friedmann-Robertson-Walker universe which is modelled by a Lorentzian spacetime with metric given by

$$\boldsymbol{g} = dt \otimes dt - \mathrm{R}(t)^2 \left(\frac{dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3}{(1 + Kr^2/4)^2} \right), \tag{6.48}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, K is a constant and $\mathrm{R}(t)$ a smooth function ($t > 0$). The reference frame $\mathbf{Q} = \frac{\partial}{\partial t}$ (the comoving reference frame) in a Friedmann-Robertson-Walker universe is supposed to be realized by a dust of 'particles' representing the matter of the universe. Choose an integral line σ of $\mathbf{Q}$. As an infinitesimally nearby observer to σ take, e.g., the vector field over σ given by

$$\mathbf{W} = \left. \frac{\partial}{\partial x^1} \right|_{\sigma(I)}. \tag{6.49}$$

Let us calculate $F^2_{\sigma_*}\mathbf{W}$. It is clear that we must take $\sigma_* = \mathbf{Q}|_{\sigma(I)}$ and $\mathbf{V} = \boldsymbol{\partial}/\boldsymbol{\partial} x^1$. We get:

$$\mathcal{F}^2_{\sigma_*}\mathbf{W} = \frac{\ddot{\mathrm{R}}}{\mathrm{R}}\mathbf{W}, \tag{6.50}$$

which means that nearby observers, even if they have constant coordinates in the chart $\{t, x^i\}$ naturally adapted to $\mathbf{Q}$ shows a 3-acceleration in that universe if $\ddot{\mathrm{R}} \neq 0$. Recall however that the acceleration of any observer σ on $\mathbf{Q}$ is $\boldsymbol{D}_{\sigma_*}\sigma_* = 0$.

6.3 Synchronizability

We can now give a classification of reference frames according to their synchronizability and discuss carefully the meaning of that concept.

Definition 6.34 Let $\mathbf{Q} \in \sec TU \subset \sec TM$ be an arbitrary reference frame and $\alpha_{\mathbf{Q}} = \boldsymbol{g}(\mathbf{Q},)$. We say that $\mathbf{Q}$ is locally synchronizable iff $\alpha_{\mathbf{Q}} \wedge d\alpha_{\mathbf{Q}} = 0$. The reference frame $\mathbf{Q}$ is said to be locally proper time synchronizable if and only if $d\alpha_{\mathbf{Q}} = 0$. $\mathbf{Q}$ is said to be synchronizable if and only if there are C^∞ functions $h, t : M \to \mathbb{R}$ such that $\alpha_{\mathbf{Q}} = hdt$ and $h > 0$. $\mathbf{Q}$ is proper time synchronizable if and only if $\alpha_{\mathbf{Q}} = dt$.

At first sight, the classification of reference frames according to their synchronizability does not look intuitive, so it is a good idea to see from where it came from.

6.3.1 *Einstein Synchronization Procedure*

To start, let $\mathbf{Q} \in \sec TU \subset \sec TM$ be an arbitrary reference frame, and let (U, φ) be a chart covering U with coordinate functions $\{\mathbf{x}^{\mu}\}$ which are used by the observers in $\mathbf{Q}$ to label events.

Now, consider two clocks A and B at rest on $\mathbf{Q} \in \sec TU \subset \sec TM$, and let σ and σ' be their world lines (which are integral lines of $\mathbf{Q}$) and which moreover we suppose to be worldlines of two *nearby* observers in the sense of Definition 6.23. According to the standard clock postulate (Axiom 6.1) the events on σ and σ' can be ordered. Let $\mathfrak{e}_1, \mathfrak{e}, \mathfrak{e}_2$ be three events on σ ordered as

$$\mathfrak{e}_1 < \mathfrak{e} < \mathfrak{e}_2, \tag{6.51}$$

and let $\mathfrak{e}'$ be an event on σ' (see Fig. 6.1). At event $\mathfrak{e}_1$ a light signal is sent from clock A to clock B where it arrives at the event $\mathfrak{e}'$ and is instantaneously reflected back to clock A where it arrives at event $\mathfrak{e}_2$.

The question arises: which event $\mathfrak{e}$ on σ is simultaneous to the event $\mathfrak{e}'$? The answer to that question depends on a *definition* as realized by Einstein [39] and long before him by Poincaré [105]. Let[3]

$$\begin{aligned} \varphi(\mathfrak{e}_1) &= (x^0_{e_1}, x^1, x^2, x^3), \quad \varphi(\mathfrak{e}_2) = (x^0_{e_2}, x^1, x^2, x^3), \\ \varphi(\mathfrak{e}) &= (x^0_{\mathfrak{e}}, x^1, x^2, x^3), \quad \varphi(\mathfrak{e}') = (x^0_{\mathfrak{e}'}, x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3), \end{aligned} \tag{6.52}$$

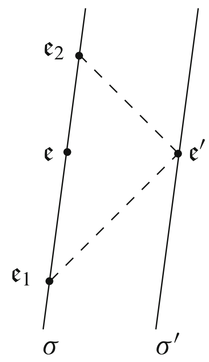

Fig. 6.1 Einstein synchronization of standard clocks

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[3] We are using an obvious sloppy notation.

be the coordinates of the pertinent events in chart (U, φ). The following definition is known as Einstein's synchronization procedure (or Einstein's convention).

Definition 6.35 The event $\mathfrak{e}$ in σ which is simultaneous to the event $\mathfrak{e}'$ in $\acute{\sigma}$ is the one such that its timelike coordinate is given by

$$x^0_{\mathfrak{e}} = x^0_{\mathfrak{e}_1} + \frac{1}{2}\left[\left(x^0_{\mathfrak{e}'} - x^0_{\mathfrak{e}_1}\right) + (x^0_{\mathfrak{e}_2} - x^0_{\mathfrak{e}'})\right]. \tag{6.53}$$

Remark 6.36 The factor $1/2$ in Eq. (6.53) is purely conventional. Non standard synchronizations can be used as explained in details, e.g., by Reichenbach [113]. In a non standard synchronization we change the factor $1/2$ by any function κ of the coordinates such that $0 < \kappa < 1$.

Our next task is to determine the relation between the timelike coordinates of events $\mathfrak{e}$ and $\mathfrak{e}'$ as functions of $\triangle x^i$ and the metric coefficients. Recall then that, as already observed in Sect. 6.2.1, in Relativity Theory the motion of a ray of light is supposed to take place along a null geodesic. So, let $\ell : \mathbb{R} \supset I \to M$, $u \mapsto \ell(u)$ be a null geodesic passing through events $\mathfrak{e}_1$ and $\mathfrak{e}'$ and $\mathfrak{e}_2$ (see Fig. 6.1). Then,

$$\boldsymbol{g}(\ell_*, \ell_*) = 0. \tag{6.54}$$

Calling $\triangle x^0_1 = x^0_{\mathfrak{e}'} - x^0_{\mathfrak{e}_1}$, $\triangle x^0_2 = x^0_{\mathfrak{e}_2} - x^0_{\mathfrak{e}'}$, and taking into account that the 'infinitesimal' arcs $\mathfrak{e}_1\mathfrak{e}'$ and $\mathfrak{e}'\mathfrak{e}_2$ in ℓ can be represented by the vectors

$$\mathbf{W}_1 = (-\triangle x^0_1, \triangle x_1, \triangle x_2, \triangle x_3),\ \mathbf{W}_2 = (\triangle x^0_2, \triangle x_1, \triangle x_2, \triangle x_3), \tag{6.55}$$

their arc length in γ are given by $\boldsymbol{g}(W_1, \mathbf{W}_1) = \boldsymbol{g}(W_2, W_2) = 0$. Solving that equations for $\triangle x^0_1$ and for $\triangle x^0_2$, we get

$$x^0_{\mathfrak{e}} = x^0_{\mathfrak{e}'} + \frac{g_{i0}}{g_{00}} \triangle x^i. \tag{6.56}$$

6.3.2 Locally Synchronizable Reference Frame

Let us now explain the genesis of the definition of a locally synchronizable frame $\mathbf{Q} \in \sec TM$, for which $\alpha_{\mathbf{Q}} \wedge d\alpha_{\mathbf{Q}} = 0$. We need to recall first some purely mathematical results related to Frobenius theorem [11].

(i) A 3-direction vector field H_x in a 4-dimensional manifold M for $x \in M$ is a 3-dimensional vector subspace H_x of T_xM which satisfies a *differentiability condition*. This condition is usually expressed as one of the two equivalent propositions that follows:

Proposition 6.37 *For each point $x_0 \in M$ there is a neighborhood $U \subset M$ of x_0 such that there exist three differential vector fields, $\mathbf{Y}_i \in \sec TM$ $(i = 1, 2, 3)$ such that $\mathbf{Y}_1(x), \mathbf{Y}_2(x), \mathbf{Y}_3(x)$, is a basis of H_x, $\forall x \in U$.*

Proof see [11]. ■

Proposition 6.38 *For each point $x_0 \in M$ there is a neighborhood $U \subset M$ of x_0 such that there exist a one form $\alpha \in \sec T^*M$ such that for all $\mathbf{Y} \in \mathrm{H}_x \Leftrightarrow \alpha(\mathbf{Y}) = 0$.*

Proof see [11]. ■

(ii) We now recall Frobenius theorem:

Proposition 6.39 (Frobenius) *Let be $x_0 \in M$ and let be U a neighborhood of x_0. In order that, for each $x_0 \in M$ there exist a 3-dimensional manifold $\Pi_{x_0} \ni x_0$ (called the integral manifold through x_0) of the neighborhood U, tangent to H_x for all $x \in \Pi_{x_0}$ it is necessary and sufficient that $[\mathbf{Y}_i, \mathbf{Y}_j]_x \in H_x$, (for all $i, j = 1, 2, 3$ and $\forall x \in \Pi_{x_0}$) if we consider the condition in Proposition 6.37. If we consider the condition on Proposition 6.38 then a necessary and sufficient condition for the existence of Π_{x_0} is that*

$$\alpha \wedge d\alpha = 0. \tag{6.57}$$

(iii) Now, let us apply Frobenius theorem for the case of a Lorentzian manifold. Let $\mathbf{Q} \in \sec TM$ be a reference frame for which $\alpha_{\mathbf{Q}} \wedge d\alpha_{\mathbf{Q}} = 0$.

Then, from the condition for the existence of a integral manifold through $x_0 \in M$ we can write,

$$\alpha_{\mathbf{Q}}(\mathbf{Y}_1) = \alpha_{\mathbf{Q}}(\mathbf{Y}_2) = \alpha_{\mathbf{Q}}(\mathbf{Y}_3) = 0. \tag{6.58}$$

Now, since $\mathbf{Q}$ is a time like vector field, Eq. (6.58) implies that the $\mathbf{Y}_i$ $(i = 1, 2, 3)$ are spacelike vector fields. It follows that the vector field $\mathbf{Q}$ is orthogonal to the integral manifold Π_{x_0} which is in this case a spacelike surface.

Now the meaning of a synchronizable reference frame (which is given by Eq. (6.57)) becomes clear. Observers in such a frame can locally separate any neighborhood U of x_0 (where they "are") in time×space.

6.3.3 Synchronizable Reference Frame

Now, suppose that $\mathbf{Q} = Q^\mu \partial/\partial x^\mu \in TU$ is synchronizable, i.e., there exists a function $t : U \to \mathbb{R}$ such that $\alpha_{\mathbf{Q}} = h dt$. In this case we can choose a (nacs|$\mathbf{Q}$), with $x^0 = t$ as the time like coordinate in U. We have

$$(\alpha_{\mathbf{Q}}) = g_{\mu\nu} Q^\nu dx^\mu = h dx^0$$

whose solution is $\mathbf{Q} = \frac{1}{\sqrt{g_{00}(x)}}\partial/\partial x^0$ with $g_{00}(x) = h^2$ and that $g_{i0}(x) = 0$. This can be seen as follows.

When $\mathbf{Q} = \frac{1}{\sqrt{g_{00}(x)}}\partial/\partial x^0$, we have from Proposition 6.1 that $\theta^{\mathbf{0}} = \boldsymbol{g}(\mathbf{Q},)$ can be written as

$$\theta^{\mathbf{0}} = \sqrt{g_{00}(x)}dx^0 + \frac{g_{i0}(x)}{\sqrt{g_{00}(x)}}dx^i, \tag{6.59}$$

and also

$$\begin{aligned}\sum_{\mathbf{i}=1}^{\mathbf{3}} \theta^{\mathbf{i}} \otimes \theta^{\mathbf{i}} &= \gamma_{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu} \\ &= \left(\frac{g_{i0}(x)g_{j0}(x)}{g_{00}(x)} - g_{ij}(x)\right) dx^i \otimes dx^j,\ i,j = 1,2,3. \end{aligned} \tag{6.60}$$

Now, since $\alpha_{\mathbf{Q}} = \theta^{\mathbf{0}}$ the metric $\boldsymbol{g}$ in the coordinates $\{x^{\mu}\}$ must be diagonal with $g_{00}(x) = h^2$ and that $g_{i0}(x) = 0$. In that case Eq. (6.56) implies in that case

$$x^0_{\mathfrak{e}} = x^0_{\mathfrak{e}'}, \tag{6.61}$$

which justifies the definition of a synchronizable frame.

It is an appropriate time to work out some examples of the above formalism, which, we admit, may look very abstract at first sight.

6.4 Sagnac Effect

As an important application of the concepts introduced above, we give in this section a kinematical analysis of the Sagnac effect [132]. Our description illustrates typical problems that arises when we deal with reference frames that are not locally synchronizable (i.e., frames $\mathbf{Q}$ for which $\alpha_{\mathbf{Q}} \wedge d\alpha_{\mathbf{Q}} \neq 0$). To go directly to the essentials, in this section $(M, \boldsymbol{\eta}, D, \boldsymbol{\tau}_{\boldsymbol{\eta}}, \uparrow)$ is Minkowski spacetime.

We recall that according to Definition 6.12, a reference frame $\mathbf{I} \in \sec TM$ on Minkowski spacetime is inertial if and only if $D\mathbf{I} = 0$.

Now, let (t, r, ϕ, z) be cylindrical coordinates of a chart (U, ϕ) for M such that $\mathbf{I} = \partial/\partial t$. Then

$$\boldsymbol{\eta} = dt \otimes dt - dr \otimes dr - r^2 d\phi \otimes d\phi - dz \otimes dz. \tag{6.62}$$

Let $\mathbf{P} \in TM$ be another reference frame on M given by

$$\mathbf{P} = (1-\omega^2 r^2)^{-1/2}\frac{\partial}{\partial t} + \omega(1-\omega^2 r^2)^{-1/2}\frac{\partial}{\partial \phi} \tag{6.63}$$

P is well defined in an open set $U \subset M$ defined through the coordinates by

$$\phi(U) = \{-\infty < t < \infty; 0 < r < 1/\omega; 0 \leq \phi < 2\pi; -\infty < z < \infty\}. \tag{6.64}$$

Then,

$$\alpha_{\mathbf{P}} = \boldsymbol{\eta}(\mathbf{P},) = (1 - \omega^2 r^2)^{-1/2} dt - \omega r^2 (1 - \omega^2 r^2)^{-1/2} d\phi \tag{6.65}$$

and

$$\alpha_{\mathbf{P}} \wedge d\alpha_{\mathbf{P}} = -\frac{2\omega r}{(1 - \omega^2 r^2)} dt \wedge dr \wedge d\phi \tag{6.66}$$

The rotation (or vortex) *vector* can be calculated from the formula $\mathbf{w_Q} = \eta(\omega_{\mathbf{Q}},) = \eta(*(\alpha_{\mathbf{P}} \wedge d\alpha_{\mathbf{P}}),)]$, η being here the cotangent bundle Minkowski metric. We have

$$\mathbf{w_Q} = \omega (1 - \omega^2 r^2)^{-1} \frac{\partial}{\partial z}. \tag{6.67}$$

Since $\mathbf{w_Q} \neq 0$, for $0 < r < 1/\omega$. This means that **P** is rotating with (classical) angular velocity ω (as measured) according to **I** in the z direction.

Now **P** can be realized on $U \subset M$ by a rotating platform, but is obvious that at the same 'time' on U, **I** cannot be realized by any physical system.

P is a typical example of a reference frame for which it does *not* exist a $(nacs|\mathbf{P})$ such that the time like coordinate of the frame has the meaning of proper time registered by standard clocks at rest on **P**.

According to the classification of reference frames given above it is indeed trivial to see that **P** is *not* proper time synchronizable or even locally synchronizable.

This fact is *not* very well known as it should be and leads some people from time to time to claim that optical experiments done on a rotating platform *disproves* the Special Relativity Theory (SRT). Recent claims of this kind has been done by Selleri and collaborators on a series of papers [36, 53, 139, 140] and by Vigier [154]. This last author wrongly stated that in order to explain the effects observed it is necessary to attribute a non-zero mass to the photon and that this fact, by its turn, implies in the existence of a fundamental reference frame.

There are also some other misleading papers that claim that in order to describe the Sagnac effect it is necessary to build a "non-abelian electrodynamics" [8, 13–17]. This is only a small sample of papers containing *very* wrong statements concerning the Sagnac experiment.[4] Now, the Sagnac effect is a well established fact (used in the technology of the gyro-ring) that the transit time employed for a light ray to go around a closed path enclosing a non-null area in a non inertial reference frame depends on the sense of the curve followed by the light ray.

[4] Of course, there are good papers on the Sagnac effect, and [107] is one of them.

Selleri arguments that such a fact implies that the velocity of light as determined by observers on the rotating platform cannot be constant, and must depend on the direction, implying that nature realizes a synchronization of the clocks which are at rest on the platform different from the one given by the Einstein method.

Selleri arguments also that each small segment of the periphery of the rotating platform of radius R can be thought as at rest on an inertial reference frame moving with speed ωR relative to the laboratory (here modeled by $\mathbf{I} = \boldsymbol{\partial}/\boldsymbol{\partial} t$). In that way, he argues that if the synchronization is done à l'Einstein between two clocks at neighboring points of a small segment, the resulting measured value of the one way velocity of light must result constant in both directions ($c = 1$), thus contradicting the empirical fact demonstrated by the Sagnac effect.

Now, we have already said that **P** is not proper time synchronizable, nor is **P** *locally* synchronizable, as can be verified.

However, for two neighboring clocks at rest on the periphery of a uniformly rotating platform an Einstein's synchronization can be done. Let us see what we get.

First, let $\{\widehat{\mathbf{x}}^{\mu}\}$ be coordinate functions for U such that[5]

$$\hat{t} = t, \quad \hat{r} = r, \quad \hat{\phi} = \phi - \omega t, \quad \hat{z} = z \tag{6.68}$$

In these coordinates $\boldsymbol{\eta}$ is written as

$$\begin{aligned}\boldsymbol{\eta} &= (1 - \omega^2 \hat{r}^2) d\hat{t} \otimes d\hat{t} - 2\omega \hat{r}^2 d\hat{\phi} \otimes dt - d\hat{r} \otimes d\hat{r} - r^2 d\hat{\phi} \otimes d\hat{\phi} - d\hat{z} \otimes d\hat{z} \\ &= g_{\mu\nu} d\hat{x}^{\mu} \otimes d\hat{x}^{\nu}.\end{aligned} \tag{6.69}$$

Now take two standard clocks A and B, at rest on **P**. Suppose they follow the world lines ρ and ρ' which are infinitesimally close.

As we know (Axiom 6.1) the events on ρ or ρ' can be ordered. Let $\mathfrak{e}_1, \mathfrak{e}, \mathfrak{e}_2 \in \rho$ with $\mathfrak{e}_1 < \mathfrak{e} < \mathfrak{e}_2$, where

$$\phi(\mathfrak{e}_1) = (\hat{x}^0_{\mathfrak{e}_1}, \hat{x}^1, \hat{x}^2, \hat{x}^3), \quad \phi(\mathfrak{e}_2) = (\hat{x}^0_{\mathfrak{e}_2}, \hat{x}^1, \hat{x}^2, \hat{x}^3),$$

$$\phi(\mathfrak{e}) = (\hat{x}^0_{\mathfrak{e}}, \hat{x}^1, \hat{x}^2, \hat{x}^3), \quad \phi(\mathfrak{e}') = (\hat{x}^0_{\mathfrak{e}'}, \hat{x}^1 + \Delta\hat{x}^1, \hat{x}^2 + \Delta\hat{x}^2, \hat{x}^3 + \Delta\hat{x}^3), \tag{6.70}$$

with $\mathfrak{e}_1$ is the event on ρ when a light signal is sent from clock A to clock B, $\mathfrak{e}'$ is the event when the light signal arrives at clock B on ρ' and is (instantaneously) reflected back to clock A where it arrives at event $\mathfrak{e}_2$ and finally $\mathfrak{e}$ is the event simultaneously on ρ to the event $\mathfrak{e}'$ according to Einstein's convention introduced above. Then according to Eq. (6.56)

$$\hat{x}^0_{\mathfrak{e}} = \hat{x}^0_{\mathfrak{e}'} + \frac{g_{0i}}{g_{00}} \Delta\hat{x}^i \neq \hat{x}^0_{\mathfrak{e}'} \tag{6.71}$$

[5]Note that $\{\hat{x}^{\mu}\}$ *is* a $(nacs|\mathbf{P})$ since in these coordinates $\mathbf{P} = (1 - \omega^2 r^2)^{-\frac{1}{2}} \partial/\partial\hat{t}$.

We emphasize that Eq. (6.71) does not mean that we achieved a process permitting the synchronization of two *standard* clocks following the world lines ρ and ρ', *because* standard clocks in general do not register the "flow" of the time-like coordinate x^0. However, in some particular cases such as when $g_{\mu\nu}$ is independent of x^0 and for the specific case where the clocks are very near (see below) and at rest on the periphery of a uniformly rotating platform this can be done.

This is so because two standard clocks at rest at the periphery of a uniformly rotating platform "tick tack" at the same ratio relative to **I**. Once synchronized they will *remain* synchronized. It follows that the velocity of light measured by these two clocks will be independent of the direction followed by the light signal and will result to be $c = 1$ every time that the measurement will be done. This statement can be *trivially* verified [124] and is in complete disagreement with a proposal of Chiu et al. [28]. We now analyze with more details what will happen if we try the *impossible* task (since **P** is not proper time synchronizable, as already said above) of synchronizing standard clocks at rest at the ring of a rotating platform which is the material support of the reference frame **P**.

Suppose that we synchronize (two by two) a series of standard clocks (such that any two are *very close*[6]) at rest and living on a closed curve along the periphery of a rotating platform. Let us number the clocks as $0, 1, 2, \ldots, n$. Clocks 0 and 1 are supposed "be" at the same point p_1 and are the beginning of our experiment synchronized. After that we synchronize, clock 1 with 2, 2 with 3, ... and finally n with 0. From Eq. (6.71) we get immediately that at the end of the experiment clocks 0 and 1 will not be synchronized and the coordinate time difference between them will be

$$\Delta\hat{t} = -\oint \frac{g_{0i}}{g_{00}} d\hat{x}^i = \oint \frac{\omega R^2}{1-\omega^2 R^2} d\hat{\phi}. \tag{6.72}$$

For $\omega R << 1$ we have $\Delta\hat{t} = \pm 2\omega S$ where S is the area of the rotating platform and the signals $\pm$ refer to the two possible directions in each we can follow around the rotating platform.

The correct relativistic explanation of the Sagnac experiment is as follows. Suppose (accepting the validity of the geometrical optics approximation) that the world line of a light signal that follows the periphery of the rotating platform of radius R is the curve $\sigma : \mathbb{R} \supset I \to M$ such that σ_* is a null vector. Using the coordinate $\hat{t}$ as a curve parameter we have

$$\boldsymbol{\eta}(\sigma_*, \sigma_*) = (1-\omega^2 R^2)(d\hat{t}\circ\sigma)^2 - 2\omega R^2 (d\hat{\phi}\circ\sigma)(d\hat{t}\circ\sigma) - R^2 (d\hat{\phi}\circ\sigma)^2 = 0. \tag{6.73}$$

[6]Very close means that $l/R \ll 1$, where l is the distance between the clocks and R is the radius of the platform, both distances being determined in the frame **P**.

Then

$$\frac{d\hat{\phi} \circ \sigma}{d\hat{t}} = -\omega \pm 1/R. \quad (6.74)$$

It follows that the coordinate times for a complete round are

$$\widehat{T}_{\pm} = 2\pi R/1 \mp \omega R, \quad (6.75)$$

where the signals $\pm$ refer to the two possible paths around the periphery, with $-$ when the signal goes in the direction of rotation and + in the other case. It is quite obvious that $\widehat{T}_{\pm}$ can be measured by a single clock.

6.4.1 Length of the Periphery of a Rotating Disk

How observers on board of a rotating platform will define the length of the periphery of their disc? We think that a reasonable answer is that they will define such a length as an appropriate sum of the infinitesimal distances determined by them at a given coordinate time t (instantaneous observers). This means the following. Each observer is equipped with a standard ruler, device realized by a clock and a light emitter and receiver at rest on the platform, which permits the determination of the time of flight of a light signal that are emitted from the position of the standard clock, go to the infinitesimal point in the disc whose distance is to be measured and is reflected back to the clock. At coordinate time t all observers make measurements of the infinitesimal distances that they are supposed to measure and then one of them collect the results and effectuates the appropriate sum. From the above calculations it is clear that the metric of the rest space corresponds to the projection tensor **h** (6.11) and we get[7]

$$L = 2\pi R(1 - \omega^2 R^2)^{-1/2}. \quad (6.76)$$

Being $\tau_{\pm}$ the proper times measured by standard clock at rest at the periphery of the rotating platform, corresponding to $\widehat{T}_{\pm}$, we have

$$\tau_{\pm} = (1 - \omega^2 R^2)^{1/2} \widehat{T}_{\pm} = L(1 \pm \omega R). \quad (6.77)$$

This equation explains trivially the Sagnac effect according to Special Relativity.

[7] This result follows at once from the definition Note that the claim by Klauber [65] that the space geometry of the disc is flat is in contradiction with the way measurements are realized in Relativity Theory.

Selleri [36, 53, 139, 140] calls the quantities $c_{\pm}$

$$\frac{1}{c_{\pm}} = \frac{L}{\tau_{\pm}} \tag{6.78}$$

the *global* velocities of light in the rotating platform for motions of light in the directions of rotation $(-)$ and in the contrary sense $(+)$. He then argues that these values must also be the local values of the *one-way* velocities of light, i.e., the values that an observer would necessarily measure for light going from a point p_1 in the periphery of a rotating platform to a neighboring point p_2. He even believes to have presented an ontological argument that implies that Special Relativity is not true. Well, what is wrong with Selleri's argument is the following. Although it is true that the global velocities $c_{\pm}$ can be measured with a single clock, the measurement of the local one way velocity of light transiting between p_1 and p_2 requires two standard clocks synchronized at that points. Local Einstein synchronization is possible and as described above gives a local velocity equal to $c = 1$, and this fact leads to no contradiction.

Before concluding this section it is very much important to recall that a reference frame field as introduced above is a mathematical *instrument*. It did not necessarily need to have a material substratum (i.e., to be realized as a material physical system) in the points of the spacetime manifold where it is defined. More properly, we state that the integral lines of the vector field representing a given reference frame do not need to correspond to world lines of real particles. If this crucial aspect is not taken into account we may incur in serious misunderstandings.

Exercise 6.40 Do solid rulers on board of a rotating platform suffers the Lorentz contraction? Suppose that the platform and the small solid rulers on board are of the same material. Suppose that when the platform is at rest the maximum integral number of the small rulers around the periphery of the disc is p_0 and the maximum integral number of small rulers around the diameter is d_0. Let p and q be the respective number of rulers when the platform is rotating. How did you model the platform in both cases? Is $p/q = p_0/d_0$?

6.5 Characterization of a Spacetime Theory

In this section we define what we mean by a general relativistic spacetime theory [124]. In our approach a physical theory F is characterized by:

(i) a theory of a certain *species of structure* in the sense of Boubarki [21];
(ii) its physical interpretation;
(iii) its present meaning and present applications.

We recall that in the mathematical exposition of a given physical theory F, the postulates or basic axioms are presented as definitions. Such definitions mean that the physical phenomena described by F behave in a certain way. Then, the definitions require more motivation than the pure mathematical definitions. We call coordinative definitions the physical definitions, a term introduced by Reichenbach [113]. Also, according to Sachs and Wu [131] it is necessary to make clear that completely *convincing* and *genuine* motivations for the coordinative definitions cannot be given, since they refer to nature as a whole and to the physical theory as a whole.

The theoretical approach to Physics behind (i)–(iii) above is then to admit the mathematical concepts of the *species of structure* defining F as primitives, and define coordinately the observational entities from them. Reichenbach assumes that "*physical knowledge* is characterized by the fact that concepts are not only defined by other concepts, but are also coordinated to real objects". However, in our approach, each physical theory, when characterized as a species of structure, contains some *implicit* geometric objects, like some of the reference frame fields defined above, that cannot in general be coordinated to real objects. Indeed, it would be an absurd to suppose that the infinity number of IRFs that (mathematically) exist in a Minkowski spacetime are simultaneously realized as physical systems.

Definition 6.41 A general relativistic *spacetime* theory is a theory of a species of structure such that:

(i) If Mod F is the class of models of F, then each $\Upsilon \in$ Mod F contains as substructure a Lorentzian spacetime $\mathfrak{M} = (M, \boldsymbol{D}, \boldsymbol{g}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$. We recall here that $\boldsymbol{g}$ is a Lorentz metric and $\boldsymbol{D}$ is the Levi-Civita connection of $\boldsymbol{g}$ on M. More precisely, we have

$$\Upsilon = ((M, \boldsymbol{D}, \boldsymbol{g}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow), \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_m) \ , \tag{6.79}$$

(ii) The objects $S_i, i = 1, 2$, ($S_1 = \boldsymbol{g}, S_2 = \boldsymbol{D}$) of the substructure $\mathfrak{M}$ characterize the geometry of a spacetime. The $\boldsymbol{\phi}_i \in \sec \mathcal{T}M$ (the tensor bundle),[8] $i = 1, \ldots, m$ are (explicit) geometrical objects defined in $U \subseteq M$ characterizing the physical fields and particle trajectories that *cannot* be geometrized in the theory. Here, to be geometrizable means to be a metric field or a connection on M or objects derived from these concepts as, e.g., the Riemann tensor (or the torsion tensor in more general spacetime theories).

(iii) The mathematical objects $o_i = (S_1, S_2, \boldsymbol{\phi}_k, k = 1, \ldots, m)$ describing any particular model $\Upsilon \in$ Mod F are supposed to satisfy the proper axioms of the theory F, also called the equations of motion (or dynamical laws) of F. The equations of motion for all spacetime theories analyzed in this work are intrinsic equations, i.e., they do not need the introduction of a particular chart of the manifold to be presented.[9] Here we write the equations of motion

[8] Some of the $\boldsymbol{\phi}_i$ may be sections of spinor bundles. See Chap. 6.

[9] The intrinsic equations, for any coordinate chart of the maximal atlas of M with coordinate functions $\{\boldsymbol{x}^{\mu}\}$ translate as a set of partial differential equations, which may be nonlinear.

symbolically as $\mathfrak{D}o_i = f(o_1, ..o_j)$ where $\mathfrak{D}$ is some (differential) operator and f a 'function' of the o_i.

(iv) Each spacetime theory has some *implicit* geometrical objects that do not appear explicitly in Eq. (6.79). These objects are the reference frame fields (which we already defined above).

Remark 6.42 We shall investigate some crucial issues associated with the fact that the equations of motion of the principal physical theories possesses the mathematical property of being diffeomorphically invariant, a concept to be introduced below.

6.5.1 Diffeomorphism Invariance

Definition 6.43 Let $\Upsilon, \Upsilon' \in \text{Mod}\, F$, $\Upsilon = ((M, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow), \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_m)$ and $\Upsilon' = ((M, \boldsymbol{g}', \boldsymbol{D}', \boldsymbol{\tau}_{\boldsymbol{g}'}, \uparrow), \boldsymbol{\phi}'_1, \ldots, \boldsymbol{\phi}'_m)$ with the $\boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau}_{\boldsymbol{g}}$ and the $\boldsymbol{\phi}_i$, $i = 1, \ldots, m$ defined in $U \subseteq M$ and $\boldsymbol{g}', \boldsymbol{D}', \boldsymbol{\tau}_{\boldsymbol{g}'}$ and the $\boldsymbol{\phi}'_i$, $i = 1, \ldots, m$ defined in $V \subseteq M$. We say that Υ and Υ' are mathematically equivalent[10] (and denotes $\Upsilon \sim \Upsilon'$) if

(i) there exists $\mathtt{h} \in \mathfrak{G}_M$ such that $\Upsilon' = \mathtt{h}^*\Upsilon$, i.e., $V \subseteq \mathtt{h}(U)$ and

$$\boldsymbol{D}' = \mathtt{h}^*\boldsymbol{D}, \;\; \boldsymbol{g}' = \mathtt{h}^*\boldsymbol{g}, \;\; \boldsymbol{\phi}'_1 = \mathtt{h}^*\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}'_m = \mathtt{h}^*\boldsymbol{\phi}_m, \tag{6.80}$$

The models Υ and Υ' are:

(ii) free boundary solutions of the equations of motion of F, or (more important)

(iii) solutions of well posed initial and boundary valued problems to the equations of motion of F. This means that if the mathematical objects defining Υ satisfy the equations of motion of the theory with initial and boundary conditions $B(U)$ in $U \subset M$, then the mathematical objects defining Υ' satisfy diffeomorphically equivalent equations of motion (to the ones satisfied by Υ) and verify the initial and boundary conditions $\mathtt{h}_*B(U)$ in $\mathtt{h}(U) \subset V$.

In Physics literature any spacetime theory satisfying Definition 6.43 is said to be *diffeomorphism invariant* or *generally covariant.*

However, in the literature of GRT it is also stated that $\Upsilon, \Upsilon' \in \text{Mod}\, F$ do represent the *same* physical model. Moreover, due to the acceptance of that statement *sometimes* it is stated that general covariance is to be identified with a Principle of *General* Relativity. What are the meaning of these statements? Let us analyze them in detail.

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[10]In fact we can be a little bit more general and define as equivalent $\Upsilon = ((M, \boldsymbol{g}, D, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow), \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_m)$ and $\Upsilon' = ((M', \boldsymbol{g}', D', \boldsymbol{\tau}_{\boldsymbol{g}'}, \uparrow), \boldsymbol{\phi}'_1, \ldots, \boldsymbol{\phi}'_m)$, where $\mathtt{h}\colon M \to M'$ is any diffeomorphism between two manifolds M and M'.

6.5.2 Diffeomorphism Invariance and Maxwell Equations

To understand the meaning to be attached to diffeomorphism invariance let us start with a simple example, i.e., F_{LME}, the Lorentz-Maxwell *electrodynamics* (LME) in *Minkowski* spacetime. We are here interested in knowing under which conditions and in which sense any two $\Upsilon, \Upsilon' \in \text{Mod } F_{LME}$ related by a diffeomorphism do represent the *same* physical model. Now, F_{LME} has as model

$$\Upsilon_{LME} = (M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow, F, J, \{\sigma_i, m_i, q_i\}), \tag{6.81}$$

where $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$ is Minkowski spacetime, $\{\sigma_i, m_i, q_i\}$, $i = 1, 2, \ldots, N$ is a set of all charged particles (generating J), m_i and q_i being their masses and charges with $\sigma_i : \mathbb{R} \supset I \to M$ being their world lines. Also, $F \in \sec \bigwedge^2 T^*M$ is the electromagnetic field and $J \in \sec \bigwedge^1 T^*M$ is the electromagnetic current. Denoting $v_{(i)} = \boldsymbol{\eta}(\sigma_{*i}, \ )$, the proper axioms of the theory are

$$dF = 0, \ \underset{\boldsymbol{\eta}}{\delta} F = -J, \tag{6.82}$$

$$m_i D_{\sigma_{i*}} v_{(i)} = q_i v_{(i)} \underset{\boldsymbol{\eta}}{\lrcorner} F, \tag{6.83}$$

Consider the substructure $(M, \boldsymbol{\eta})$ and a general diffeomorphism $\mathtt{h} : M \to M$. Under this diffeomorphism $\boldsymbol{\eta} \mapsto \mathtt{h}^* \boldsymbol{\eta} = \boldsymbol{g}$.

The metric field $\boldsymbol{g}$ in M, defines a Lorentzian manifold $(M, \boldsymbol{g})$. Now, since $\boldsymbol{g} = \mathtt{h}^* \boldsymbol{\eta}$, we know that in this case for any $L \in \sec \bigwedge^r T^*M$, $K \in \sec \bigwedge^p T^*M$, $r \leq p$, we have

$$\begin{aligned} d\mathtt{h}^* K &= \mathtt{h}^* dK, \\ \underset{\boldsymbol{g}}{\star} d \underset{\boldsymbol{g}}{\star} \mathtt{h}^* K &= \mathtt{h}^* \underset{\boldsymbol{\eta}}{\star} d \underset{\boldsymbol{\eta}}{\star} K, \\ \mathtt{h}^* L \underset{\boldsymbol{g}}{\lrcorner} \mathtt{h}^* K &= \mathtt{h}^* (L \underset{\boldsymbol{\eta}}{\lrcorner} K), \end{aligned} \tag{6.84}$$

where the first two lines in the above equation follows directly from Eq. (4.146) and the last line from a convenient use of the second identity in Eq. (2.130) and from Eq. (4.145).

Thus, from a mathematical point of view it is a trivial result that F_{LME} has the following important property.

Proposition 6.44 *If Eqs.* (6.82) *and* (6.83) *have a solution* $(F, J, (\sigma_i, m_i, q_i))$ *in* $\mathtt{h}(U) \subseteq M$ *in the 'universe'* $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$*, then* $(\mathtt{h}^* F, \mathtt{h}^* J, (h_* \sigma_i, m_i, q_i))$ *is also a solution of modified equations of motion*

$$d\mathtt{h}^* F = 0, \ \underset{\boldsymbol{g}}{\delta} \mathtt{h}^* F = -\mathtt{h}^* J, \tag{6.85}$$

$$m_i \mathtt{h}^* D_{\mathtt{h}_*^{-1}\sigma_{i*}} \mathtt{h}^* v_{(i)} = q_i \mathtt{h}^* v_{(i)} \underset{\boldsymbol{g}}{\lrcorner} \mathtt{h}^* F, \tag{6.86}$$

in U in the 'universe' $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \mathtt{h}^* D, \uparrow)$.[11]

Proof It follows trivially from Eq. (6.84). ■

To fix the ideas imagine an electromagnetic current $J \in \sec \bigwedge^1 T^*M$ with has support only in a region $U \subset M$. Let $C \subset U$ be a Cauchy surface and ∂C its boundary, where Cauchy data and boundary conditions are given. Consider a series of diffeormorphisms $\{\mathtt{h}\}$ such that each $\mathtt{h} : M \to M$ is equal to the identity in the region $U \subset M$ and nonzero in the region $M \backslash U$. Then $\mathtt{h}^* J = J$. Let $F \in \sec \bigwedge^2 T^*M$ be a solution of the proper axioms of the theory,[12] i.e., $dF = 0$, $\delta F = -J$, for a well posed initial and boundary valued problem obtained in a global coordinate chart (M, ϕ) of an atlas of M with coordinate functions $\{\boldsymbol{x}^\mu\}$ in Einstein-Lorentz-Poincaré gauge. As well known, since Maxwell equations are a hyperbolic system of differential equations a well posed problem consists in the Cauchy problem (initial data on a Cauchy surface) with appropriate boundary conditions given at the boundary of the Cauchy surface [58]. It is obvious that any $\mathtt{h}^* F \in \sec \bigwedge^2 T^*M$ will satisfy the *transformed* Maxwell equations (6.85) in the coordinate chart (M, ϕ) with coordinate functions $\{\boldsymbol{x}^\mu\}$ in Einstein-Lorentz-Poincaré gauge. Moreover $\mathtt{h}^* F$ will satisfy the *same* initial and boundary conditions in C and ∂C, since $\mathtt{h}^* F|_C = F|_C$ and $\mathtt{h}^* F|_{\partial C} = F|_{\partial C}$ This is no contradiction because the solution $\mathtt{h}^* F$ refers to a field which moves in another structure, i.e., the structure $(M, \boldsymbol{g})$, not the original Minkowski structure $(M, \boldsymbol{\eta})$.

So, we claim that for an arbitrary diffeomorphism $\mathtt{h} : M \to M$, the structures $(F, J, (\sigma_i, m_i, q_i))$ and $(\mathtt{h}^* F, \mathtt{h}^* J, (h_* \sigma_i, m_i, q_i))$ do represent the same model if we *identify* the 'universes' $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ and $(M, \boldsymbol{g}, \mathtt{h}^* D, \tau_{\boldsymbol{g}}, \uparrow)$, and indeed, more generally, we identify all universes that are related by diffeomorphisms of the type described above. This is a perfect procedure from the mathematical point of view. However we need to investigate if what is mathematically perfect is also correct from the *physical* point of view. So we need to have an answer for the following question:

Question 1 Is our *identification* of 'universes' related by diffeomorphisms in the case of electromagnetic phenomena also compatible with other known physical phenomena taking place in these 'universes'? This will be investigated in the next section. However, first we introduce yet another important question, which is presented after the following remark.

Remark 6.45 Return to $\mathtt{h}^* F$. It is obvious that in general $\mathtt{h}^* F$ is not a solution of the *original* Maxwell equations (6.82) in the universe $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$, for indeed

[11] We observe also that although we restrict our example to Minkowski spacetime it is easy to prove that Maxwell equations are diffeomorphism invariant in any Lorentzian spacetime.

[12] We write for simplicity $\underset{\boldsymbol{\eta}}{\delta} = \delta$, $\underset{\boldsymbol{\eta}}{\lrcorner} = \lrcorner$.

if $\mathtt{h}^*F$ would be a solution we would have two different solutions $\mathtt{h}^*F$ and F satisfying the originally given initial and boundary conditions in that universe. This is impossible as a consequence of the uniqueness theorem for the solutions of hyperbolic differential equations satisfying Cauchy conditions and appropriated boundary conditions, once we exclude, on physical grounds *advanced* solutions [92].

Question 2 When do F and $\mathtt{h}^*F$ under the above conditions do represent possible physical phenomena in the same 'universe' $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$?

To answer that question we introduce the concept of Poincaré diffeomorphisms.

Poincaré Diffeomorphisms

Since the diffeomorphism invariance of Maxwell equations is true for any $\mathtt{h} \in \mathfrak{G}_M$, it is true for $\ell \in \mathcal{P}_+^{\uparrow} \subset \mathfrak{G}_M$, *i.e.*, for any Poincaré diffeomorphism. We have

$$\ell^* \boldsymbol{\eta} = \boldsymbol{\eta}, \;\; \ell^* D = D. \tag{6.87}$$

In this case, $(\ell^* F, \ell^* J, (\ell_* \varphi_i, m_i, e_i))$ can be considered a solution of the equations of motion (6.82) in the universe $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$ but the coordinate representations of the original equations of motion in the Einstein-Lorentz-Poincaré coordinates $\{x^\mu\}$ must satisfy appropriate *transformed* initial and boundary conditions (*of course*).

Remark 6.46 Take into account that $(\ell^* F, \ell^* J, (\ell_* \varphi_i, m_i, e_i))$ will represent for observers at rest in an inertial reference frame $\mathbf{I}$ a phenomenon distinct from the one described by $(F, J, (\varphi_i, m_i, e_i))$. A typical example is the field of an electric charge at rest in $\mathbf{I}$ and the one of a charge moving at constant velocity (relative to $\mathbf{I}$). See Example 6.58.

Exercise 6.47 Let $(M, \boldsymbol{g}_1)$ and $(N, \boldsymbol{g}_2)$ be two 4-dimensional Lorentzian manifolds. A diffeomorphism $\mathtt{h} : M \to N$ is called a conformal map if

$$\mathtt{h}^* \boldsymbol{g}_2 = \Omega^2 \boldsymbol{g}_1, \tag{6.88}$$

where $\Omega^2 : M \to \mathbb{R}^+$. If $\mathtt{h}$ is a orientation-preserving conformal map show that:

(a) If $F \in \sec \bigwedge^2 T^* M$, then

$$\mathtt{h}^* (\underset{\boldsymbol{g}_2}{\star} F) = \underset{\boldsymbol{g}_1}{\star} (\mathtt{h}^* F) \tag{6.89}$$

(b) Put $M = N \simeq \mathbb{R}^4$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2 = \boldsymbol{\eta}$, a Minkowski metric. Show that the free Maxwell equations are invariant under conformal transformations, i.e., $dF = 0$, $\delta F = 0 \Rightarrow d\mathtt{h}^* F = 0, \delta \mathtt{h}^* F = 0$.
(c) Discuss the initial value problem for the situation described in *(b)*.

Enter the Electromagnetic Potential A

The homogeneous Maxwell equation $dF = 0$ in Minkowski spacetime implies that there exists $A \in \sec \bigwedge^1 T^*M$ such that

$$F = dA \tag{6.90}$$

As already discussed A is defined modulus a closed form ψ ($d\psi = 0$). In terms of A, Eq. (6.82) read

$$\Diamond A + d\delta A = J, \tag{6.91}$$

where $\Diamond = -(\delta d + d\delta)$ is the Hodge Laplacian. Equation (6.91) is diffeomorphism invariant (in the sense explained above), since the corresponding equation for $\mathtt{h}^*A$ [see Eq. (6.85)] is

$$-(\underset{g}{\delta}d + d\underset{g}{\delta})\mathtt{h}^*A + d\underset{g}{\delta}\mathtt{h}^*A = \mathtt{h}^*J. \tag{6.92}$$

Note that even the Lorenz gauge $\delta A = 0$ is diffeomorphism invariant, since

$$\underset{g}{\star} d \underset{g}{\star} \mathtt{h}^*A = \mathtt{h}^* \star d \star A = 0. \tag{6.93}$$

Taking into account Eq. (4.93) we have from Eq. (6.91), taking into account that $\delta J = 0$, that

$$\Diamond \delta A + \delta d\delta A = 0. \tag{6.94}$$

This equation shows that we indeed have only three independent equations for the components of A in any given coordinate chart. The missing degree of freedom, corresponds to gauge invariance. This means that Eq. (6.91) is not enough to determine A for a Cauchy problem, unless we fix the gauge, a procedure that eliminates one degree of freedom. In particular, if we work in the *Lorenz* gauge $\delta A = 0$, Eq. (6.91) is the wave equation, which has a unique solution for a Cauchy problem.

Remark 6.48 Sometimes we find in the literature the statement that if A is solution of Eq. (6.91) so is $(A + \psi)$. Well, this is true only for *boundary free* solutions and if ψ is harmonic, i.e., $\Diamond\psi = 0$. The same is not true for a Cauchy problem, for indeed, if $(A + \psi)$ is to satisfy the same initial conditions as A on a Cauchy surface, it is necessary that ψ satisfies homogeneous boundary conditions on that Cauchy surface, and since ψ solves the wave equation we must have that $\psi = 0$.

6.5.3 Diffeomorphism Invariance and GRT

Let us now analyze a problem in GRT which is analogous to the one in electrodynamics just discussed in the previous section. This will permit us to give an answer to the **question** just introduced above. Here, one of the S_i is $\boldsymbol{g}$, which is a *dynamic* variable determined by the distribution of the energy momentum tensor $\mathcal{T}$ of all fields and particles in M through Einstein's equations,

$$Ricci - \frac{1}{2}gR = -\boldsymbol{T}. \tag{6.95}$$

This equation is manifestly diffeomorphism invariant (Definition 6.43). Consider then a series of diffeormorphisms $\mathtt{h}_\alpha$ (where α belongs to a given index set) such that each $\mathtt{h}_\alpha$ is equal to the identity in the region $U \subset M$ where the energy momentum tensor of matter is different of zero and nonzero in the region $M \backslash U$. If the above conditions are satisfied we have that $\mathtt{h}_{\alpha*}\boldsymbol{T} = \boldsymbol{T}$. If we accept that $\Upsilon = ((M, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow), \boldsymbol{T})$ and $\Upsilon' = ((M, \mathtt{h}_\alpha^*\boldsymbol{g}, \mathtt{h}_\alpha^*\boldsymbol{D}, \boldsymbol{\tau}_{\mathtt{h}_\alpha^*\, \boldsymbol{g}}, \acute{\uparrow}), \mathtt{h}_\alpha^*\boldsymbol{T})$ do represent the same physical model (as suggested by the diffeomorphism invariance of the theory) we see that it is necessary that the diffeomorphism *invariance* of Einstein's equations implies that if the metric field $\boldsymbol{g}$ solves a well posed initial value and boundary problem for Einstein's equations[13] for a given $\boldsymbol{T}$, then $\mathtt{h}_{\alpha*}\boldsymbol{g}$ must also solve the diffeomorphic initial and boundary valued problem for Einstein's equations for the same $\boldsymbol{T} = \mathtt{h}_{\alpha*}\boldsymbol{T}$.[14] This may only be true, of course, if the mathematical nature of the Einstein's equations do not allow the complete determination of the ten functions $g_{\mu\nu}(x)$ once that equations are written in a coordinate chart $\{\boldsymbol{x}^\mu\}$ of the maximal atlas of M (covering a region big enough for our considerations to make sense). And indeed, that is the case,[15] since we have ten equations for the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}$, but they are not independent since we have four constraints, coming from $\boldsymbol{D}_\mu G^\mu_\nu = 0$.

This implies according to the majority view that in GRT any particular gravitational field must be described by an element of the quotient space LorM/G_M where Lor$M$ is the space of all spacetimes associated to Lorentzian manifolds $(M, \boldsymbol{g})$ for all possible Lorentzian metrics $\boldsymbol{g}$ on M and G_M is the group of diffeomorphisms of M. More on this issue may be found in [109].

[13]The initial value problem (Cauchy problem) in GRT is very subtle one and difficult one, and the reader interested in details must consult, e.g., [29, 55].

[14]This situation is known as Einstein's hole argument. A very detailed discussion of the argument is given in [119], where many important references can be found.

[15]As known since a long time ago. See, e.g., [74, 91, 156] for details.

6.5.4 A Comment on Logunov's Objection

The above arguments are so simple and appealing that is hard to imagine any objection. However, according to Logunov [74, 75] in practice, when we introduce an arbitrary chart for $U \subset M$ in order to find a gravitational field for a given matter distribution, we have a problem. Indeed, Logunov argues: which $\mathtt{h}_{\alpha *}\boldsymbol{g}$ do we choose as solution for Einstein's equation? He argues that GRT has no *internal* answer for that question and this is true. In GRT [55, 91, 156] a unique $\mathtt{h}_{\alpha *}\boldsymbol{g}$ is specified only by arbitrarily selecting four unknown components of the metric tensor for a given problem. Experts in GRT say that procedure corresponds in fixing a *coordinate* gauge. Other authors are more subtle and only say that a 'gauge condition' must be given. What is the meaning of such statements? Hawking and Ellis, e.g., [55] select the 'gauge' by *fixing* a background metric $\mathring{\boldsymbol{g}}$ and imposing a condition on the covariant derivative of $\mathtt{h}_{\alpha *}\boldsymbol{g}$ relative to the Levi-Civita connection determined[16] by $\mathring{\boldsymbol{g}}$. In particular they choose the *harmonic gauge*. Such a gauge has been used by Fock [46] for the particular case where $\mathring{\boldsymbol{g}}$ is a constant Minkowski metric on a manifold M diffeomorphic[17] to $\mathbb{R}^4$. So, the above procedure furnishes a solution $\boldsymbol{g}$ which satisfies Einstein's equations, the given initial and boundary valued conditions (supposed physically realizable in nature) and the gauge condition. To see the importance of the above remarks, let us analyze a simple problem, namely the solutions of Einstein's equations for a static and spherically symmetric distribution of matter with its energy momentum tensor having support in an open set $U \subset M$. Introducing a chart (U, φ) with coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$ for $\varphi(U)$, and being the image of U under the coordinate mapping given by $\{-\infty < t < \infty, (0 < r < r_0) \cup (r_0 < r < \infty)\}$ the form of the metric for $r > r_0$ must be[18]

$$\boldsymbol{g} = g_{00} dt \otimes dt + 2g_{01} dt \otimes dr + g_{11} dr \otimes dr + g_{22} d\theta \otimes d\theta + g_{33} d\varphi \otimes d\varphi, \tag{6.96}$$

where all the metric coefficients can be functions only of the radial coordinate. An analysis of the possible solutions of Einstein's equations for the above problem has been given by several authors, the presentation of [74, 75] being particularly well done and careful. The conclusion is that there are infinite possible metrics that have

[16]This of course, implies in four conditions to be satisfied by the components of the metric $h_{\alpha *}\boldsymbol{g}$.

[17]You may argue that in so doing we have fixed the topology of the spacetime. Well, this is true, but do not forget that any solution of Einstein's equations is obtained in a local chart of an abstract manifold, and in general there are many different topologies consistent with the metric obtained in the particular chart. This means that in truth, the topology of a solution to Einstein's equations is fixed by hand. This will become clear in Sect. 6.9.

[18]See, e.g., [71].

the same asymptotic behavior when $r \to \infty$. Two of them are:

$$\boldsymbol{g}_s = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi), \tag{6.97}$$

and

$$\boldsymbol{g}_i = \left(\frac{r-m}{r+m}\right) dt \otimes dt - \left(\frac{r+m}{r-m}\right) dr \otimes dr - (r+m)^2 (d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi), \tag{6.98}$$

where m is a parameter with the meaning of the mass of the system in the natural systems of units.

Equation (6.97) is (wrongly) known (see Sect. 6.9) as Schwarzschild solution and Eq. (6.98) is another valid solution, for which

$$D_\mu (\sqrt{\det \boldsymbol{g}_i} \boldsymbol{g}_i^{\mu\nu}) = 0, \tag{6.99}$$

where $D = \mathring{D}$ is the Levi-Civita covariant derivative of a Minkowski metric $\boldsymbol{\eta} = \mathring{\boldsymbol{g}}$, which in the coordinates (t, r, θ, φ) is written as

$$\mathring{\boldsymbol{g}} = dt \otimes dt - dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi). \tag{6.100}$$

Equation (6.99)[19] is not satisfied by $\boldsymbol{g}_s$. So, which one are we going to use in order to compare empirical data with predictions of the theory? Any expert on GRT will, of course, answer that question saying that both metrics are permitted, because, the *meaning* of the coordinates in each one are *different* (even if they are presented by the same letter). The reasoning behind that answer is that we can know what the spacetime labels mean, only after we fix a metric on it. With this answer, diffeomorphism invariance holds, if we do not include a *particular gauge* fixing equations as part of the theory.

Indeed, we can perform a coordinate transformation in Eq. (6.97) which makes it in the new variables to have the appearance of Eq. (6.98).

However, according to Logunov, to *not* know a priori the meaning of the coordinates leads to ambiguities in the predictions of experiments, e.g., in the time delay of radio signals in the solar system.[20] The reason is that the time delay of a radio signal that goes, e.g., from Earth to Mercury and comes back flying in the background gravitational field generated by the Sun, is defined as the amount of extra time (as measured with a clock on Earth) to do the same path in the absence of the Sun's gravitational field. Now, both metrics ($\boldsymbol{g}_s$ and $\boldsymbol{g}_i$) reduce to Minkowski

[19]Equation (6.99) is similar but not identical to *harmonic gauge condition*.

[20]Recently Scharf [133] also puts in doubt the validity of diffeomorphism invariance. We will not comment on his paper here.

metric in the absence of the Sun's gravitational field and so, we have two *different* predictions using the labels (t, r, θ, φ), if we suppose that in $\boldsymbol{g}_s$ and $\boldsymbol{g}_i$ they have the same meaning.

We think that only meaningful question that can be done at this point is the following: what is the meaning attributed by *astronomers* to the coordinate functions (t, r, θ, φ)?

This is an important question since once we know that answer only one of the predictions, using $\boldsymbol{g}_s$ or $\boldsymbol{g}_i$ will agree with empirical data once we use the astronomers' interpretation of (t, r, θ, φ). Logunov [74, 75] claims that under these conditions only the metric $\boldsymbol{g}_i$ is compatible with experimental data. More, according to him, his theory (with a zero graviton mass) predicts that data because in it there is an analogous to Einstein's equation plus Eq. (6.99), which is an *integral* part of his theory.

Logunov's conclusion would be correct only if one can claim that the astronomers meaning of the coordinates (t, r, θ, φ) for the time delay experiment is the one that those coordinates have in a flat Minkowski spacetime. But this can only be confirmed if: someone can do the time delay radio signal experiment putting off the gravitational field, and this no astronomer can do, of course.

According to our view, what can be inferred is the following: if the data favors the use of $\boldsymbol{g}_i$, this only says that the procedure astronomers use to put labels to spacetime events make the coordinates (t, r, θ, φ) to have the meaning that they have as encoded in $\boldsymbol{g}_i$.

Remark 6.49 Logunov emphasizes that there are compelling reasons to claim that the Minkowski structure of spacetime manifests itself in some identifiable way. Indeed, in his theory it is claimed that manifestation of the Minkowski spacetime structure arises in the empirical validity of the energy-momentum and angular-momentum conservation laws for all physical phenomena. In Chap. 9 we show in details that in *GRT* there are no genuine energy-momentum and angular-momentum conservation laws.

Remark 6.50 It is also said that diffeomorphism invariance is a crucial requirement that *GRT* must satisfy in order to avoid indeterminism [38, 109, 119]. We prefer not to go on the that discussion here. The reason is that as it will become clear in Chap. 9, where we study the shameful problem of the 'energy-momentum conservation' in *GRT*, we do not think that this theory, with its orthodox interpretation, is one worthy to be preserved anymore. One way to have trustful energy-momentum conservation is to suppose that the arena of physical phenomena is Minkowski spacetime and that the gravitational field is a field in the Faraday sense, which no special distinction in relation to the other fields. This, of course do not implies that we necessarily need to have Logunov's theory as the unique possibility. However, whatever theory we decide to use it must give the results predicted by *GRT* in the case that we know these predictions are good ones. In that case, we may say that the geometrical description "à la General Relativity" is a coincidence, which may be valid as a first approximation (see Chap. 11). We mention yet, that diffeomorphism invariance is also said to play a crucial role in some recent tentatives of formulation of a quantum theory of gravity, like loop gravity. Here is not the place to venture on that subject. The interested reader may consult [129] and the careful analysis presented in [119] of many of Rovelli's claims.

6.5.5 Spacetime Symmetry Groups

Let $\mathtt{h} \in \mathfrak{G}_M$. We recall (Definition 4.35) that if for a geometrical object $\mathbf{T}$ we have $\mathtt{h}^*\mathbf{T} = \mathbf{T}$ or equivalently

$$\mathtt{h}_*\mathbf{T} = \mathbf{T} \tag{6.101}$$

then $\mathtt{h}$ is said to be a symmetry of $\mathbf{T}$.

Definition 6.51 The set of all $\{\mathtt{h} \in \mathfrak{G}_M\}$ such that Eq. (6.101) holds is said to be the symmetry group of $\mathbf{T}$.

Definition 6.52 Let $\Upsilon, \bar{\Upsilon} \in \operatorname{Mod} F$, $\Upsilon = \langle (M, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow), \mathbf{T}_1, \ldots, \mathbf{T}_m \rangle$, $\bar{\Upsilon} = \langle (M, \mathtt{h}_*\boldsymbol{D}, \mathtt{h}_*\boldsymbol{g}, \mathtt{h}_*\boldsymbol{\tau}_{\boldsymbol{g}}, \mathtt{h}_*\uparrow), \mathtt{h}_*\mathbf{T}_1, \ldots, \mathtt{h}_*\mathbf{T}_m \rangle$ with the $\mathbf{T}_i$, $i = 1, \ldots, m$ defined in $U \subseteq M$ and $\mathbf{T}'_i$, $i = 1, \ldots, m$ defined in $V \subseteq \mathtt{h}(U) \subseteq M$ and such that

$$\boldsymbol{D} = \mathtt{h}_*\boldsymbol{D}, \quad \boldsymbol{g} = \mathtt{h}_*\boldsymbol{g}. \tag{6.102}$$

Then $\bar{\Upsilon}$ is said to be the $\mathtt{h}$-deformed version of Υ.

Remark 6.53 Take notice that—as will be clear in a while—$(\mathbf{T}_1, \ldots, \mathbf{T}_m)$ and $(\mathtt{h}_*\mathbf{T}_1, \ldots, \mathtt{h}_*\mathbf{T}_m)$ in general will correspond to different phenomena as registered by observers at rest in a given arbitrary reference frame $\mathbf{Q} \in \sec TM$.

Definition 6.54 Let $\mathbf{Q} \in \sec TU \subseteq \sec TM$, $\bar{\mathbf{Q}} \in \sec TV \subseteq \sec TM$, $U \cap V \neq \emptyset$ and let $\{\mathbf{x}^\mu\}$, $\{\overline{\mathbf{x}}^\mu\}$ (the coordinate functions associated respectively to the charts (U, φ) and $(V, \bar{\varphi})$) be respectively a $(nacs|\mathbf{Q})$ and a $(nacs|\bar{\mathbf{Q}})$ and suppose that $\overline{\mathbf{x}} = \mathbf{x}^\mu \circ \bar{\mathtt{h}}^{-1} : \bar{\mathtt{h}}(U) \to \mathbb{R}$. Thus, $\bar{\mathbf{Q}} = \bar{\mathtt{h}}_*\mathbf{Q}$ and $\bar{\mathbf{Q}}$ is said to be a $\bar{\mathtt{h}}$-*deformed version* of $\mathbf{Q}$.

Let $\Upsilon, \bar{\Upsilon} \in \operatorname{Mod} F$ be as in Definition 6.52. Call $o = (\boldsymbol{D}, \boldsymbol{g}, \ldots, \mathbf{T}_1, \ldots, \mathbf{T}_m)$ and $\bar{o} = (\boldsymbol{D}, \boldsymbol{g}, , \ldots, \bar{\mathtt{h}}_*\mathbf{T}_1, \ldots, \bar{\mathtt{h}}_*\mathbf{T}_m)$. Now, o is such that it solves a set of differential equations in $\varphi(U) \subset \mathbb{R}^4$ with a given set of boundary conditions denoted $b^{o\{x^\mu\}}$, which we write as

$$\mathfrak{D}^{\alpha}_{\{x^\mu\}}(o_{\{x^\mu\}})_{\mathfrak{e}} = 0\,;\ b^{o\{x^\mu\}}\,; \quad \mathfrak{e} \in U, \tag{6.103}$$

and $\bar{o}$ defined in $\bar{\mathtt{h}}(U) \subseteq V$ solves

$$\mathfrak{D}^{\alpha}_{\{\bar{x}^\mu\}}(\bar{\mathtt{h}}_* o_{\{\bar{x}^\mu\}})_{|\bar{\mathtt{h}}\ \mathfrak{e}} = 0\,;\ b^{\bar{\mathtt{h}}_* o\{\bar{x}^\mu\}}\,; \quad \bar{\mathtt{h}}\,\mathfrak{e} \in \bar{\mathtt{h}}(U) \subseteq V. \tag{6.104}$$

In Eqs. (6.103) and (6.104) $\mathfrak{D}^{\alpha}_{\{x^\mu\}}$ and $\mathfrak{D}^{\alpha}_{\{\bar{x}^\mu\}}$, $\alpha = 1, 2, \ldots, m$, mean sets of differential equations in $\mathbb{R}^4$.

How can observers living on the universe $(M, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\tau_g}, \uparrow)$ discover that $\Upsilon, \bar{\Upsilon} \in$ Mod F are deformed versions of each other? In order to answer this question we need some additional definitions.

6.5.6 Physically Equivalent Reference Frames

Definition 6.55 Let $\mathbf{Q}, \bar{\mathbf{Q}}$ be as in Definition 6.54. We say that $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ are physically equivalent or indistinguishable according to theory F (and we denote $\bar{\mathbf{Q}} \sim \mathbf{Q}$) if and only if there exist a (nacs$|\mathbf{Q}$) and a (nacs$|\bar{\mathbf{Q}}$) such that: (i) the functions $D_\mu \mathbf{Q}_\nu$ and $D_{\bar{\mu}} \bar{\mathbf{Q}}_{\bar{\nu}}$ have the same functional form and (ii) the system of differential equations (6.103) have the same functional form as the system of differential equations (6.104) and $b^{\bar{\mathtt{h}}_* o\{\bar{x}^\mu\}}$ must be relative to $\{\bar{x}^\mu\}$ the same as $b^{o\{x^\mu\}}$ is relative to $\{x^\mu\}$ and if $b^{o\{x^\mu\}}$ is physically realizable then $b^{\bar{\mathtt{h}}_* o\{\bar{x}^\mu\}}$ must also be physically realizable.

Definition 6.56 Given a reference frame $\mathbf{Q} \in \sec TU \subseteq \sec TM$ the set of all diffeomorphisms $\{\mathtt{h} \in \mathfrak{G}_M\}$ such that $\mathtt{h}_* \mathbf{Q} \sim \mathbf{Q}$ forms a subgroup of $\mathfrak{G}_M$ called the equivalence group of the class of reference frames of kind $\mathbf{Q}$ according to the theory F.

Remark 6.57 In the next section we establish what is the meaning of a Principle of Relativity for a spacetime theory based on Minkowski spacetime. One of the meanings of this principle is that all inertial reference frames are physically equivalent. It is very important also to realize that *general* covariance is not to be identified with a Principle of *General* Relativity. Indeed, if such a principle is to have the meaning that all arbitrary reference frames are physically equivalent then it does not hold in *GRT* as will become clear in Sect. 6.8.2.

6.6 Principle of Relativity

In this section the arena where physical phenomena occur is supposed to be Minkowski spacetime $\mathcal{M} = (M, \boldsymbol{\eta}, D, \boldsymbol{\tau_\eta}, \uparrow)$. Let then $\mathbf{I}, \mathbf{I}' \in \sec \mathbf{T}M$ be two distinct IRF in $\mathcal{M}$, which we recall are frames such that $D\mathbf{I} = D\mathbf{I}' = 0$. According to Definition 6.34 any IRF is *proper time synchronizable*. A global (*nacs*$|\mathbf{I}$) $\{\mathbf{x}^\mu\}$ is said to be in the Einstein-Lorentz-Poincaré gauge if $\mathbf{I} = \frac{\partial}{\partial x^0}$ and the set $\{\frac{\partial}{\partial x^\mu}\}$ is an orthonormal frame, i.e., $\boldsymbol{\eta}(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) = \eta_{\mu\nu}$. It is crucial to have in mind that, given the *natural* system of units used in this book where the light velocity $c = 1$, the coordinate $x^0 = \mathbf{x}(\mathfrak{e})$ has the meaning of *proper time* as measured by clocks at rest in $\mathbf{I}$ and synchronized à l'Einstein. Also, the spatial coordinates $x^i = \mathbf{x}^i(\mathfrak{e})$,

$i = 1, 2, 3$ have the meaning of *proper distances* as measured along the spatial axis from a fixed origin. Let $\{x'^{\mu}\}$ be a (nacs$|\mathbf{I}'$) also in the Einstein-Lorentz-Poincaré gauge. Then the coordinates $x^{\mu} = \mathbf{x}(\mathfrak{e})$ and $x'^{\mu} = \mathbf{x}'^{\mu}(\mathfrak{e})$ of any event $\mathfrak{e} \in M$ are related by

$$x'^{\mu} = \mathbf{L}^{\mu}_{\nu} x^{\nu} + a^{\mu}, \tag{6.105}$$

where $\mathbf{L}^{\mu}_{\nu}$ are the matrix components of a Lorentz transformation $\mathbf{L} \in \mathcal{L}^{\uparrow}_{+}$ and a^{μ} are real constants. When $a^{\mu} = 0$, Eq. (6.105) is a special orthochronous Lorentz mapping. According to Eq. (4.29) the set of all coordinate transformations of the form given by Eq. (6.105) can be associated to a subset of the diffeomorphism group, namely $\mathcal{P}^{\uparrow}_{+} = \{\ell\} \subset \mathfrak{G}_M$. Any $\ell \in \mathcal{P}^{\uparrow}_{+}$ is a *Poincaré diffeomorphism*. If $\mathbf{T} \in \sec \mathcal{T}M$ (or is a connection) we call $\ell_* \mathbf{T}$ a *Poincaré deformed version* of $\mathbf{T}$. When, ℓ induces only a Lorentz transformation, then $\ell_* \mathbf{T}$ is called a Lorentz deformed version of $\mathbf{T}$.

We can verify that

$$\ell_* \boldsymbol{\eta} = \boldsymbol{\eta}, \quad \ell_* D = D, \quad \forall \ell \in \mathcal{P}^{\uparrow}_{+}, \tag{6.106}$$

i.e., according to Definition 6.51 the Poincaré group (and in particular its subgroup $\mathcal{L}^{\uparrow}_{+} = SO^{e}_{1,3}$) is a symmetry group of $\boldsymbol{\eta}$ and D.

Now, the following statement denoted PR_1 is usually presented as the Principle of Relativity in *active* form [127].

PR_1: *Let* $\ell \in \mathcal{P}^{\uparrow}_{+} \subset \mathfrak{G}_M$. *If for any possible physical theory* F, if $\Upsilon \in F$, $\Upsilon = ((M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow), \mathbf{T}_1, \ldots, \mathbf{T}_m)$ *is a possible phenomenon, then* $\Upsilon' = \ell_* \Upsilon$ *is also a possible physical phenomenon.*

The following statement denoted PR_2 is known as the Principle of Relativity in *passive* form [127].

PR_2: *All inertial reference frames are physically equivalent or indistinguishable for any possible physical theory* F.

PR_1 and PR_2 are equivalent statements of the Principle of Relativity and we believe that they capture the ideas of Poincaré [106] and Einstein [39] (see also on this respect [73, 86, 157]). The existence of a Principle of Relativity for a physical theory permit us to find nontrivial solutions for the equations of motion of the theory once very simple solutions are known. We illustrate this case in the following

Example 6.58 As in Sect. 6.5 let F_{LME} be *classical electrodynamics* taking place Minkowski spacetime. More precisely, consider Lorentz-Maxwell electrodynamics (*LME*) F_{LME} as a theory of a species of structure. We already know that *LME* has as model

$$\Upsilon_{LME} = (M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow, F, J, \{\sigma_i, m_i, e_i\}). \tag{6.107}$$

We already know from Sect. 6.5.1 the precise sense in which Υ_{LME} may be considered diffeomorphism invariant for $\mathtt{h} \in \mathfrak{G}_M$.

Now, let us study the case where $\mathtt{h} = \ell \in \mathcal{P}_+^{\uparrow} \subset \mathfrak{G}_M$, *i.e.*, for a Poincaré diffeomorphism for which

$$\ell^*\boldsymbol{\eta} = \boldsymbol{\eta}, \ell^* D = D. \tag{6.108}$$

In this case, as we learned in Sect. 6.5 $(\ell^* F, \ell^* J, (\ell_* \varphi_i, m_i, e_i))$ can be considered a solution of the equations of motion (6.82) in the universe $(M, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, D, \uparrow)$, but with appropriate transformed initial and boundary conditions (of course).

The explicit form of Poincaré diffeomorphisms is introduced as follows. Let $\{\mathbf{x}^\mu\}$ and $\{\mathbf{x}'^\mu\}$ be two coordinate charts covering M naturally adapted to the global IRFs $\mathbf{I} = \partial/\partial x^0$ and $\mathbf{I}' = \partial/\partial x'^0$, such that, e.g.,

$$\mathbf{I}' = \frac{1}{\sqrt{1-\mathrm{v}^2}} \frac{\partial}{\partial x^0} + \frac{v^i}{\sqrt{1-\mathrm{v}^2}} \frac{\partial}{\partial x^i}, \tag{6.109}$$

where $\mathrm{v} = (v^1, v^2, v^3)$ and $\mathrm{v}^2 = \sum_{i=1}^{3} (v^i)^2$. Then, a Poincaré mapping $\ell : \mathfrak{e} \mapsto \ell\mathfrak{e}$ is defined by the following coordinate transformation

$$\mathbf{x}'^\mu(\mathfrak{e}) = \mathbf{x}^\mu(\ell\mathfrak{e}) \equiv \Lambda^\mu_\nu \mathbf{x}^\nu(\mathfrak{e}) + a^\mu, \tag{6.110}$$

Suppose we have a charge at rest at the origin of the $\mathbf{I}'$ frame and let be $F = \frac{1}{2} F_{\mu\nu}(x'^\alpha) dx'^\mu \wedge dx'^\nu$ be the electromagnetic field generated by this charge. As it is well known,

$$F_{0i}(x'^\alpha) = \frac{e x'^i}{\sqrt{\sum_{i=1}^{3} (x^i)^2}}, \; F_{ij}(x'^\alpha) = 0. \tag{6.111}$$

Then, we have

$$\bar{F} := \ell^* F = \frac{1}{2} \left(\bar{F}(x^\rho)\right)_{\alpha\beta} dx^\alpha \wedge dx^\beta,$$
$$\left(\bar{F}(x^\rho)\right)_{\alpha\beta} = F_{\mu\nu}(x'^\beta(x^\rho)) \Lambda^\mu_\alpha \Lambda^\nu_\beta. \tag{6.112}$$

To simplify the calculations, let $\mathrm{v} = (v, 0, 0)$. Writing, as usual, in Cartesian notation, $F = (\mathbf{E}, \mathbf{0})$ and $\bar{F} = (\bar{\mathbf{E}}, \bar{\mathbf{B}})$, we have

$$\bar{\mathbf{E}}(x^\mu) = \frac{e}{\mathsf{R}^3 \sqrt{1-v^2}} \left(x^1 - v x^0, x^2, x^3\right), \; \bar{\mathbf{B}} = \mathrm{v} \times \bar{\mathbf{E}}(x^\mu),$$
$$\mathsf{R} = \sqrt{\frac{(x^1 - v x^0)}{1-v^2} + (x^2)^2 + (x^3)^2} \tag{6.113}$$

The field $\bar{F} = \ell^* F$ describes the field of a charge moving with constant velocity along the x-axis. We have obtained this result in a practically trivial way using PR_1. The reader will certainly appreciate the power of PR_1 once he tries to solve directly Maxwell equation for that problem (see, e.g., [76]).

6.6.1 Internal and External Synchronization Processes

We recall once more that a IRF on Minkowski spacetime is mathematically described by a unit time like vector field $\mathbf{I} \in \sec TM$ such that $D\mathbf{I} = 0$. We also defined a set $\{\mathbf{x}^\mu\}$ of coordinate functions covering M which is moreover a $(nacs|\mathbf{I})$ in the Einstein-Lorentz-Poincaré gauge, i.e., in these coordinates $\mathbf{I} = \boldsymbol{\partial}/\boldsymbol{\partial}x^0$and $\boldsymbol{\eta}(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) = \eta_{\mu\nu}$. We recall that $x^0 = \mathbf{x}(\mathfrak{e})$ has the meaning of proper time at $\mathfrak{e} \in M$ as measured by a clocks at rest in the frame and which have been synchronized by Einstein's method described above. Such a procedure is an *internal* operation in the $\mathbf{I}$ frame, and as such it is more properly called an *internal synchronization* process.[21] Of course, Einstein's method using light signals, is not the only possible internal synchronization procedure, and indeed, several other methods are known, like, e.g., the ones described in [61, 62, 79, 80, 159].

As discussed in [126] the validity of the Principle of Relativity implies that any possible internal synchronization procedure of clocks must be equivalent to Einstein's method and no method can realize absolute synchronization relative to the time of a chosen 'preferred' reference frame. This statement means the following. Let $\mathbf{I}_0 = \partial/\partial\bar{x}^0$ be a 'preferred' inertial reference frame and $\{\bar{\mathbf{x}}^\mu\}$ be a $(nacs|\mathbf{I}_0)$ in the Einstein-Lorentz-Poincaré gauge and where it is experimentally verified that all internal synchronization procedures agree with the one obtained through Einstein's synchronization procedure. Let, e.g.,

$$\mathbf{I} = \frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial\bar{x}^0} - \frac{v}{\sqrt{1-v^2}}\frac{\partial}{\partial\bar{x}^1} \tag{6.114}$$

be another inertial reference frame. Choose a $(nacs|\mathbf{I})$ $\{\mathbf{x}^\mu\}$ in the Einstein-Lorentz-Poincaré gauge such that

$$x^0 = \frac{\bar{x}^0 - v\bar{x}^1}{\sqrt{1-v^2}},\ x^1 = \frac{\bar{x}^1 - v\bar{x}^0}{\sqrt{1-v^2}},\ x^2 = \bar{x}^2, x^3 = \bar{x}^3. \tag{6.115}$$

Let A and B clocks at rest in $\mathbf{I}$ following world lines σ_A and σ_B parametrized by $\bar{x}^0$. When the standard clocks at $\mathbf{I}_0$ reads $\bar{x}^0$ the spatial coordinates of the clocks are $\bar{x}^i{}_A = \bar{\mathbf{x}}^i \circ \sigma_A(\bar{x}_0)$ and $\bar{x}^i{}_B = \bar{\mathbf{x}}^i \circ \sigma_A(\bar{x}_0)$. Then from Eq. (6.115) it follows that the

[21] It is crucial to have in mind that the synchronabilty to which Definition 6.34 refers is internal synchronization.

readings of the standard clocks A and B are (according to the observers in $\mathbf{I}_0$) out of phase by

$$x_A^0 - x_B^0 = -\frac{v}{\sqrt{1-v^2}}(\bar{x}^1{}_A - \bar{x}^1{}_B) = -\frac{v}{\sqrt{1-v^2}}\triangle\bar{x}^1 \tag{6.116}$$

in relation to the standard clocks synchronized "à l'Einstein" in $\mathbf{I}_0$. The phase difference depends explicitly on the relative velocity of the frames.

Suppose now that it would be possible with an *internal* synchronization procedure, *different* from Einstein's method, to synchronize clocks in $\mathbf{I}$ such that the time registered by standard clocks (say at A and B) at rest in that frame are given by coordinates x_A^0 and x_B^0 and such that when the standard clocks at $\mathbf{I}_0$ reads $\bar{x}^0$ we have

$$\mathrm{x}_A^0 - \mathrm{x}_B^0 = 0. \tag{6.117}$$

The coordinate x^0 may be named *absolute* synchronization as defined by $\mathbf{I}_0$.

Then, by comparing set of clocks synchronized with the different synchronization procedures it would be possible to determine the velocity of the $\mathbf{I}$ reference frame relative to the $\mathbf{I}_0$ reference frame. If for all possible inertial frames it would be possible to find coordinate functions that realize absolute synchronization in the sense of Eq. (6.117), this will select $\mathbf{I}_0$ as a preferred one, and we would have a breakdown of the Principle of Relativity. This is because, the phenomenon involved in the alternative synchronization procedure in $\mathbf{I}$ will not be a Lorentz deformed version of the same phenomenon in $\mathbf{I}_0$.

6.6.2 External Synchronization

Suppose that we eventually identify in the universe we live a given IRF, say $\mathbf{I}_0$ as having some cosmic significance. Let $\{\bar{\mathbf{x}}^\mu\}$ be a $(nacs|\mathbf{I}_0)$ in Einstein-Lorentz-Poincaré coordinate gauge. Let $\mathbf{I}$ [given by Eq. (6.114)] be another IRF whose observers have determined the velocity of their frame $\mathbf{I}$ relative to $\mathbf{I}_0$ by realizing experiments involving some phenomena generated *external* to that frame. Of course, nothing prevents those observers to use $\mathrm{x}^0 = \bar{x}^0\sqrt{1-v^2}$ as time coordinate function representing the reading of standard clocks at rest in $\mathbf{I}$ which are synchronized according to the readings of the clocks at rest in $\mathbf{I}_0$. A natural set of global coordinates which could be used by observers in $\mathbf{I}$ are then

$$\mathrm{x}^0 = \bar{x}^0\sqrt{1-v^2}, \mathrm{x}^1 = \frac{\bar{x}^1 - v\bar{x}^0}{\sqrt{1-v^2}},\ \mathrm{x}^2 = \bar{x}^2, \mathrm{x}^3 = \bar{x}^3, \tag{6.118}$$

which we call the *absolute gauge coordinates*, and which have been used by many authors in the past, as, e.g., in [26, 81–83, 115, 117, 118, 126, 150]. Now, the coordinate functions $\{\mathbf{x}^\mu\}$ such that $\mathbf{x}^\mu(\mathfrak{e}) = \mathrm{x}^\mu$ *is* a $(nacs|\mathbf{I})$, since in that coordinates we have

$$\mathbf{I} = \frac{\partial}{\partial \mathrm{x}^0}. \tag{6.119}$$

Also, in these coordinates the Minkowski metric tensor reads,

$$\boldsymbol{\eta} = d\mathrm{x}^0 \otimes d\mathrm{x}^0 - 2v d\mathrm{x}^0 \otimes d\mathrm{x}^1 - (1 - v^2) d\mathrm{x}^1 \otimes d\mathrm{x}^1 - d\mathrm{x}^2 \otimes d\mathrm{x}^2 - d\mathrm{x}^2 \otimes d\mathrm{x}^2, \tag{6.120}$$

being non diagonal.

Recall that the set of naturally adapted coordinates $\{x^\mu\}$ (Einstein-Lorentz-Poincaré gauge) and $\{\mathrm{x}^\mu\}$ (absolute gauge) to $\mathbf{I}$ are related by

$$x^i = \mathrm{x}^i,\ x^0 = \mathrm{x}^0 - v\mathrm{x}^1. \tag{6.121}$$

Some authors, as e.g., [20, 32, 81, 82, 114–116, 118] claims that $\mathbf{I}_0$ may be identified as the reference frame where the cosmic background radiation is isotropic. However, if the reference frame where the cosmic background radiation is isotropic is to be understood as a reference frame on a Lorentzian spacetime modeling a cosmological model according to GRT, then this identification is *not* possible since as already said in Remark 6.13 there are in general no inertial frames in a general Lorentzian spacetime.

The use of the coordinates $\{\mathrm{x}^\mu\}$ will be a useful one indeed only in the case that we can identify a preferred IRF by internal experiments breaking Lorentz invariance in the frame $\mathbf{I}$. More on this issue will be discussed below and in our book yet in preparation [100], where we shall need the results of the next section.

6.6.3 A Non Standard Realization of the Lorentz Group

Let

$$\mathbf{I} = \frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial \bar{x}^0} + \frac{v^i}{\sqrt{1-v}}\frac{\partial}{\partial \bar{x}^i},$$
$$\mathbf{I}' = \frac{1}{\sqrt{1-(v')^2}}\frac{\partial}{\partial \bar{x}^0} + \frac{v'^i}{\sqrt{1-(v')^2}}\frac{\partial}{\partial \bar{x}^i}, \tag{6.122}$$

be two IRFs. Suppose that observers at the IRF $\mathbf{I}$ and $\mathbf{I}'$ can measure their velocities relative to a given preferred frame $\mathbf{I}_0$. Observers in the frames $\mathbf{I}$ and $\mathbf{I}'$ may use coordinates in the Einstein-Lorentz-Poincaré gauge, denoted respectively by $\{x^\mu\}$ and $\{x'^\mu\}$ or they can use coordinates in the absolute gauge, respectively $\{\mathrm{x}^\mu\}$ and $\{\mathrm{x}'^\mu\}$. If they use absolute gauge coordinates, the velocity of the preferred frame as measured by those observers can be written as

$$\mathbf{I}_0 = u^0 \frac{\partial}{\partial \mathrm{x}^0} + u^i \frac{\partial}{\partial \mathrm{x}^i}$$
$$\mathbf{I}_0 = u'^0 \frac{\partial}{\partial \mathrm{x}'^0} + u'^i \frac{\partial}{\partial \mathrm{x}'^i}. \tag{6.123}$$

Also, the velocity of frame $\mathbf{I}'$ as determined by the observers in the $\mathbf{I}$ frame will be written as

$$\mathbf{I}' = w^0 \frac{\partial}{\partial \mathrm{x}^0} + w^i \frac{\partial}{\partial \mathrm{x}^i} \tag{6.124}$$

We write $x^t = (x^0, x^1, x^2, x^3)$, $\bar{x}^t = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$, $\mathrm{x}^t = (\mathrm{x}^0, \mathrm{x}^1, \mathrm{x}^2, \mathrm{x}^3)$, $\mathrm{x}'^t = (\mathrm{x}'^0, \mathrm{x}'^1, \mathrm{x}'^2, \mathrm{x}'^3)$, $u^t = (u^0, u^i)$ and $u'^t = (u'^0, u'^i)$, $w^t = (w^0, w^i)$ Then the set of coordinates in the absolute gauge are related by

$$\mathrm{x} = \mathbf{\Lambda}(\bar{\mathbf{L}}, u)\bar{x},\ \mathrm{x}' = \mathbf{\Lambda}(\bar{\mathbf{L}}', u')\bar{x},$$
$$\mathrm{x}' = \mathbf{\Lambda}(\mathbf{L}, u)x, \tag{6.125}$$

and

$$u' = \mathbf{\Lambda}(\mathbf{L}, u)u. \tag{6.126}$$

In Eq. (6.124), $\bar{\mathbf{L}}, \bar{\mathbf{L}}', \mathbf{L}$ are elements of the Lorentz group, u and u' are, of course, the components (i.e., the velocity) of the $\mathbf{I}_0$ frame as determined by $\mathbf{I}$ and $\mathbf{I}'$ [see Eq. (6.124)] in absolute gauge coordinates. Finally, $\mathbf{\Lambda}(\mathbf{L}, u)$ is a 4×4 matrix, whose explicit form (has been found in [83] and with more details in [114–116, 118]) is[22]:

(a) For rotations we have

$$\mathbf{\Lambda}(\mathbf{R}, u) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \tag{6.127}$$

where $\mathbf{R} \in \mathrm{SO}_3$ is a standard rotation matrix.

(b) For boosts we have

$$\mathbf{\Lambda}(\mathbf{L}_w, u) = \begin{pmatrix} (w^0)^{-1} & 0 \\ -\mathbf{w} & \mathbf{1}_3 + \frac{\mathbf{w} \otimes \mathbf{w}^t}{1 + \sqrt{1 + |\mathbf{w}^2}} - u^0 \mathbf{w} \otimes \mathbf{w}^t \end{pmatrix} \tag{6.128}$$

[22] We use for future reference the notations of [114–116, 118].

For boosts (with "parallel" space axis) the coordinates introduced above are related as follows:

$$\mathrm{x}^0 = \frac{1}{u^0}x^0,\ \mathbf{x} = \bar{\mathbf{x}} - \mathbf{u}\left(\bar{x}^0 - \frac{\mathbf{u}\cdot\bar{\mathbf{x}}}{1+\sqrt{1+\mathbf{u}^2}}\right),$$
$$u^0 = \frac{1}{\sqrt{1-\boldsymbol{v}^2}},\ \mathbf{u} = u^0\boldsymbol{v} \tag{6.129}$$

$$\mathrm{x}'^0 = \frac{1}{u'^0}x^0,\ \mathbf{x}' = \bar{\mathbf{x}} - \mathbf{u}\left(\bar{x}^0 - \frac{u'\cdot\bar{\mathbf{x}}}{1+\sqrt{1+\mathbf{u}^2}}\right),$$
$$u'^0 = \frac{1}{\sqrt{1-\boldsymbol{v}'^2}},\ \mathbf{u}' = u'^0\boldsymbol{v}' \tag{6.130}$$

and

$$\mathrm{x}'^0 = \frac{1}{w^0}x^0,\ \mathbf{x}' = \mathbf{x} - \mathbf{w}\left(x^0 - \frac{\mathbf{w}\cdot\ \mathbf{x}}{1+\sqrt{1+\mathbf{w}^2}}\right),$$
$$w^0 = \frac{u^0}{u'^0},\ \mathbf{w} = \frac{(u^0+u'^0)(\mathbf{u}-\mathbf{u}')}{1+u^0u'^0(1+\delta_{ij}u^iu^j)} \tag{6.131}$$

The metric tensor reads, e.g., in coordinates $\{\mathrm{x}^\mu\}$ as $\boldsymbol{\eta} = g_{\mu\nu}d\mathrm{x}^\mu \otimes d\mathrm{x}^\nu$, with

$$g_{\mu\nu} = \left(\begin{array}{c|c} 1 & u^0\mathbf{u}^t \\ \hline u^0\mathbf{u} & -\mathbf{1}_3 + (u^0)^2\mathbf{u}\otimes\mathbf{u}^t \end{array}\right). \tag{6.132}$$

Also, as a generalization of Eq. (6.121) we also have

$$x^0 = \mathrm{x}^0 - \delta_{ij}v^i\mathrm{x}^j,\ x^i = \mathrm{x}^i. \tag{6.133}$$

which gives the relation between coordinates in the Lorentz and absolute coordinate gauges.

Finally, the *nonstandard* realization of the Lorentz group is given by the following rules:

$$\begin{aligned}\boldsymbol{\Lambda}(\mathbf{L}_2, \boldsymbol{\Lambda}(\mathbf{L}_1, u)u)\boldsymbol{\Lambda}(\mathbf{L}_1, u) &= \boldsymbol{\Lambda}(\mathbf{L}_2\mathbf{L}_1, u),\\ \boldsymbol{\Lambda}^{-1}(\mathbf{L}, u) &= \boldsymbol{\Lambda}(\mathbf{L}^{-1}, \boldsymbol{\Lambda}(\mathbf{L}, u)u),\\ \boldsymbol{\Lambda}(\mathbf{L}, u) &= \mathbf{1}_4.\end{aligned} \tag{6.134}$$

6.6.4 Status of the Principle of Relativity

The Principle of Relativity according to textbooks and in particular in Roberts review [120] titled: '*What is the Experimental Basis of Special Relativity?'* is supposed to be one of the most well tested principles of Physics (see also [160]). High Energy physicists proudly claimed during all the twentieth century that PR_1 is routinely verified in any high energy physical laboratory[23] and that PR_2 is routinely used when data of different laboratories are compared. The validity of the Principle of Relativity is known also as Poincaré invariance of physical laws. According to the definitions given above the reason for that is obvious.

Despite all enthusiasm, the question arises? Is the Principle of Relativity a true law on nature valid under all conditions? Well, *if* the GRT is a correct description of the nature of the gravitational field the answer is *no*. The reason is that in this theory there are no inertial reference frames (in general) as we already remarked. Even a so called *Principle of Local Lorentz Invariance* (PLLI), stated in many books and articles (see references below) is not true, as it will be proved in Sect. 6.8 below. Finally, even if we could formulate (as indeed we can, see, e.g., Chap. 11, and also [45, 73, 127, 138, 156]) a theory of the gravitational field in Minkowski spacetime (satisfying the Principle of Relativity), no one can warrant that this Principle is a true law of nature valid under all conditions, for we do not know all laws of nature. Having said that the reader must be informed that:

(a) From time to time there are claims in the literature that certain very low energy experiments involving the propagation of light and the roto-translational motion of solid bodies violate Lorentz invariance. We quote in that class of experiments[24] [81, 82] and [66, 152, 153]. The data described in [152] can be explained with a very simple model, where it is postulated a breakdown of PR_1 for solids in roto-translational motion [126]. However it seems that the data in [153] is more compatible with a null result. Breakdown of Lorentz invariance in the roto-translational motion of solid bodies as described in [126] may also explain the data in [81, 82]. However, these experiments (to the best of the authors' knowledge) have not been duplicated and we have serious doubts about those results. In this respect see also the thoughtful analysis in [77, 151].
(b) Also, recently Cahill [24, 25] and also Consoli [33] and Consoli and Constanzo [34, 35] claim that the *small*, but *not* null results in Michelson-Morley like experiments done in the past can be accounted by explicitly postulating that the propagation of light in a medium breaks Lorentz invariance. For the case of the anomalous experimental results in the Brillet and Hall experiment (which is

[23] Which they wrongly suppose are inertial reference frames. Indeed, all known high energy laboratories are located on the Earth, which is not an inertial reference frame.

[24] We must say that we do not know if these experiments have been duplicated. In [126] it is analyzed a possible breakdown of PR_1 for solids in rototranslational motion which accounts for the results of the Kolen-Torr experiment [66, 152].

a Michelson-Morley like experiment done in vacuum) Cahill postulated that the Schwarzschild gravitational field of the Earth accounts for an effective medium. With that hypothesis his calculations at first sight seems to explain those ‘anomalous’ results. On the other hand, Klauber [65] claims to explain the anomalous Brillet and Hall [22] experimental results by postulating that the devices used in the experiment does not suffer Lorentz contraction when turned. Also it is worth to quote here that a reanalysis of the data of Miller’s classical experiment [85] by Allais[25] [2], show the existence of correlations that are hard to believe to have the simple explanations given by Shankland [141]. All these claims need, of course, to be more carefully analyzed, but in particular it is necessary to take into account that Earth is *not* an inertial reference frame according to both the SRT and the GRT. Recall that inertial reference frames did not exist (in general) in GRT as already remarked several times. This fact, it seems to us, has not been properly taken into account by those authors in their analysis of the classical experiments. In particular, in any real Michelson-Morley like experiment the light paths enclose a finite area and then, those experiments are indeed analogous to a Sagnac experiment, and a non null (but very small) phase shift may be predict for them. This observation has already been remarked by Post [107, 108]. However, if the authors quoted above are correct, since along time ago a breakdown of PR_1 has already been found. This would be *disturbing* to say the less.[26]

(c) There are also many conjectures that a possible *breakdown of Lorentz invariance* will happen in phenomena involving low and very high energies (see, e.g., [1, 5–7, 67–69, 78, 155] and references therein[27]) and/or cosmological scales [98]

(d) Also, there are claims of a possible breakdown of Lorentz invariance in the phenomenon of the wave packet reduction of an entangled quantum state of two identical particles (see, e.g., [23, 54, 103, 104, 114–118, 142, 146, 161] and references therein).[28]

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[25]Maurice Allais is a French physicist that won Noble Prize in Economics. It is worth to visit his home page at: http://allais.maurice.free.fr/Science.htm.

[26]We advise the reader keep an open eye on this issue following the articles on the subject appearing at the arXiv.

[27]In fact, a google search for “breakdown of Lorentz invariance” will show several hundred of serious articles on the subject. Of course, we do have the opportunity to analyze all such possibilities in our book.

[28]Cases (c) and (d) will discussed in our planned book [100], whose proposal is among others to show that none of the experiments claiming superluminal propagation (as e.g.: **(A)** Superluminal group velocities of voltage and currents configurations propagating in wires [64, 87, 88, 102, 121]; **(B$_1$)** Superluminal group velocities of electromagnetic field configurations propagating in dispersive media with absorption or gain [27, 143]; **(B$_2$)** Superluminal group velocities of tunneling microwaves [40–42, 56, 95–97]; **(B$_3$)**Superluminal group velocities of a single tunneling photon [144]; **(B$_4$)** Superluminal group velocities of tunneling electrons [72]; **(C$_1$)** Superluminal group velocities of microwaves launched and received by non *axially aligned* horn antennas [51, 59, 60, 110–112]; **(C$_2$)** Direct measurements of superluminal velocities of peaks of finite

For the rest of this book we *assume* the validity of Principle of Relativity (PR_1 or PR_2) in our discussions.

6.7 Principle of Local Lorentz Invariance in GRT?

From time to time we find in the literature the statement that the existence of the cosmic background radiation defines a kind of *preferred inertial reference frame*. However, as we already know, in GRT in a general Lorentzian spacetime modelling a gravitational field generated by a given energy momentum tensor, in general, inertial reference frames do not exist.

In view of that fact, the following question arises naturally: which characteristics a reference frame on a GRT spacetime model must have in order to reflect as much as possible the properties of an IRF that exists in Minkowski spacetime?

The answer to the question is that there are two kind of frames in GRT, namely PIRFs (Definition 6.59) and LLRFs (Definition 6.61), such that each frame in one of these classes share some important aspects of the IRFs of SRT. Both concepts are useful and it is worth to distinguish between them in order to avoid misunderstandings. A thoughtful discussion of these concepts was presented in [124] and we follow that exposition.

Definition 6.59 A reference frame $\mathfrak{I} \in \sec TU, U \subset M$ is said to be a pseudo inertial reference frame *(PIRF)* if $\boldsymbol{D}_{\mathfrak{I}}\mathfrak{I} = 0$ and $\alpha_{\mathfrak{I}} \wedge d\alpha_{\mathfrak{I}} = 0$, with $\alpha_{\mathfrak{I}} = \boldsymbol{g}(\mathfrak{I},)$.

This definition means that a PIRF is in *free fall* and is non rotating. It means also that it is at least *locally synchronizable*.

Definition 6.60 A chart (U, φ) of an oriented atlas of M with coordinates $\xi^\mu$is said to be a local Lorentzian coordinate chart (*LLCC*) and $\{\xi^\mu\}$ are said to be local Lorentz coordinates (*LLC*) in $p_0 \in U$ if and only if

$$\boldsymbol{g}(\partial/\partial\xi^\mu, \partial/\partial\xi^\nu) \mid_{p_0} = \eta_{\mu\nu}, \tag{6.135}$$

$$\Gamma^{\alpha\cdot\cdot}_{\cdot\beta\mu}(\xi^\mu) \mid_{p_0} = 0, \quad \Gamma^{\alpha\cdot\cdot}_{\cdot\beta\gamma,\mu}(\xi^\mu) \mid_p = -\frac{1}{3}(R^{\alpha\cdots}_{\cdot\beta\gamma\mu}(\xi^\mu) + R^{\alpha\cdots}_{\cdot\gamma\beta\mu}(\xi^\mu)) \mid_p, \; p \neq p_0. \tag{6.136}$$

Let (V, χ) $(V \cap U \neq \emptyset)$ be an arbitrary chart with coordinates $\{x^\mu\}$. Then, supposing that p_0 is at the origin of both coordinate systems the following relations

aperture approximations to electromagnetic Bessel beams (X-waves), both in the optical and in the microwave region [93, 130]) do not imply in *any* violation of the Principle of Relativity.

holds (approximately)

$$\xi^{\mu} = x^{\mu} + \frac{1}{2}\Gamma^{\mu\cdot\cdot}_{\cdot\alpha\beta}(p_0)x^{\alpha}x^{\beta},$$
$$x^{\mu} = \xi^{\mu} - \frac{1}{2}\Gamma^{\mu\cdot\cdot}_{\cdot\alpha\beta}(p_0)\xi^{\alpha}\xi^{\beta}, \tag{6.137}$$

where in Eq. (6.137) $\Gamma^{\mu\cdot\cdot}_{\cdot\alpha\beta}(p_0)$ are the values of the connection coefficients at p_0 expressed in the coordinates $\{x^{\mu}\}$.

The coordinates $\{\xi^{\mu}\}$ are also known as *Riemann normal coordinates* and the explicit methods for obtaining them are described in many texts of Riemannian geometry as, e.g., in [94] and of GRT, as e.g., in [156].

Let $\gamma \in U \subset M$ be the world line of an observer in *geodetic motion* in spacetime, i.e., $\boldsymbol{D}_{\gamma_*}\gamma_* = 0$. Then as it is well known we can introduce in U LLC $\{\xi^{\mu}\}$ such that for every $p \in \gamma$ we have

$$\left.\frac{\partial}{\partial\xi^0}\right|_{p\in\gamma} = \gamma_{*|p}; \qquad \boldsymbol{g}(\partial/\partial\xi^{\mu}, \partial/\partial\xi^{\nu})|_{p\in\gamma} = \eta_{\mu\nu},$$
$$\left.\Gamma^{\mu\cdot\cdot}_{\cdot\nu\rho}(\xi^{\mu})\right|_{p\in\gamma} = g^{\mu\alpha}\boldsymbol{g}(\partial/\partial\xi^{\alpha}, \boldsymbol{D}_{\partial/\partial\xi^{\nu}}\partial/\partial\xi^{\rho})\big|_{p\in\gamma} = 0. \tag{6.138}$$

Take into account for future reference that if the $\{\xi^{\mu}\}$ are LLC then it is clear from Definition 6.60 that in general $\Gamma^{\nu\cdot\cdot}_{\cdot\mu\rho}(\xi^{\mu})\ |_p \neq 0$ for all $p \notin \gamma$.

Definition 6.61 Given a geodetic line $\gamma \subset U \subset M$ and *LLCC* (U, ξ^{μ}) we say that a reference frame $\mathbf{L} = \partial/\partial\xi^0 \in \sec TU$ is a Local Lorentz Reference Frame Associated to γ (*LLRF*γ) if and only if

$$\mathbf{L}|_{p\in\gamma} = \left.\frac{\partial}{\partial\xi^0}\right|_{p\in\gamma} = \gamma_*|_p\,,$$
$$\alpha_{\mathbf{L}} \wedge d\alpha_{\mathbf{L}}|_{p\in\gamma} = 0. \tag{6.139}$$

Moreover, we say also that the Riemann normal coordinate functions or Lorentz coordinate functions (LLC) $\{\xi^{\mu}\}$ are *associated* with the LLRFγ.

Remark 6.62 It is very important to have in mind that for a *LLRF*γ $\mathbf{L}$ in general $\boldsymbol{D}_{\mathbf{L}}\mathbf{L}|_{p\notin\gamma} \neq 0$ (i.e., only the integral line γ of $\mathbf{L}$ in free fall in general), and also eventually $\alpha_{\mathbf{L}} \wedge d\alpha_{\mathbf{L}}|_{p\notin\gamma} \neq 0$, which may be a surprising result for many readers. In contrast, a *PIRF* $\mathfrak{I}$ such that $\mathfrak{I}|_{\gamma} = \mathbf{L}|_{\gamma}$ has all its integral lines in free fall and the rotation of the frame is always null in all points where the frame is defined. Finally its is worth to recall that both $\mathfrak{I}$ and $\mathbf{L}$ may eventually have shear and expansion even at the points of the geodesic line γ that they have in common [124].

Definition 6.63 Let γ be a geodetic line as in Definition 6.61. A section s of the orthogonal frame bundle $\mathbf{P}_{SO^e_{1,3}}U, U \subset M$ is called an inertial moving tetrad along γ (IMTγ) when the set

$$s_\gamma = \{(\boldsymbol{\epsilon_0}(p), \boldsymbol{\epsilon_1}(p), \boldsymbol{\epsilon_2}(p), \boldsymbol{\epsilon_3}(p)), p \in \gamma \cap U\} \subset s, \tag{6.140}$$

it such that $\forall p \in \gamma$

$$\boldsymbol{\epsilon_0}(p) = \gamma_*|_p, \quad \boldsymbol{g}(\boldsymbol{\epsilon}_\mu, \boldsymbol{\epsilon}_\nu)\big|_{p\in\gamma} = \eta_{\mu\nu} \tag{6.141}$$

with

$$\Gamma^{\mu\cdot\cdot}_{\cdot\nu\rho}(p) = g^{\mu\alpha}, \quad \boldsymbol{g}(\boldsymbol{\epsilon}_\alpha(p), \boldsymbol{D}_{\boldsymbol{\varepsilon}_\nu(p)}\boldsymbol{\epsilon}_\rho(p)) = 0. \tag{6.142}$$

The existence of $s \in \sec \mathbf{P}_{SO^e_{1,3}}U$ satisfying the above conditions can be easily proved. Introduce coordinates $\{\xi^\mu\}$ for U such that at $p_0 \in \gamma, \boldsymbol{\epsilon}_0(p_0) = \left.\frac{\partial}{\partial \xi^0}\right|_{p_o} = \gamma_{*|p_0}$, and $\boldsymbol{\epsilon}_i(p_0) = \left.\frac{\partial}{\partial \xi^i}\right|_{p_o}, i = 1, 2, 3$ (three orthonormal vectors) satisfying Eq. (6.138) and parallel transport the set $\boldsymbol{\epsilon}_\mu(p_0)$ along γ. The set $\boldsymbol{\epsilon}_\mu(p_0)$ will then also be *Fermi* transported since γ is a geodesic and as such they define the standard of *no* rotation along γ.

Remark 6.64 Let $\mathfrak{I} \in \sec TV$ be a *PIRF* and $\gamma \subset U \subset V$ one of its integral lines and let $\{\xi^\mu\}$, $U \subset M$ be a *LLC* through all the points of the world line γ such that $\gamma_* = \mathfrak{I}|_\gamma$. Then, in general $\{\xi^\mu\}$ is not a (*nacs*$|\mathfrak{I}$) in U, i.e., $\mathfrak{I}|_{p\notin\gamma} \neq \partial/\partial\xi^0\big|_{p\notin\gamma}$ even if $\mathfrak{I}|_{p\in\gamma} = \partial/\partial\xi^0\big|_{p\in\gamma}$.

6.7.1 LLRFγs and the Equivalence Principle

There are many presentations of the EP and even very strong criticisms against it, the most famous being probably the one offered by Synge [148]. We are not going to bet on this particular issue. Our intention here is to prove that there are models of GRT where the so called Principle of Local Lorentz Invariance (PLLI) which according to several authors (see below) follows from the Equivalence Principle is not valid in general. Our strategy to prove this strong statement is to give a precise mathematical wording to the PLLI (which formalizes the PLLI verbally introduced by several authors) in terms of a physical equivalence of LLRFγs (see below) and then prove that PLLI is a false statement according to GRT. We start by recalling formulations and comments concerning the EP and the PLLI.

According to Friedman [48] the

> Standard formulation of the *EP* characteristically obscure [the] crucial distinction between first order laws and second order laws by blurring the distinction between infinitesimal laws, holding at a single point, and local laws, holding on a neighborhood of a point. . . .

According to our point of view, in order to give a mathematically precise formulation of Einstein's EP besides the distinctions mentioned above between infinitesimal and local laws, it is also necessary to distinguish between some very different (but related) concepts, namely,[29]

1. The concept of an observer (Definition 6.4);
2. The general concept of a reference frame in *a Lorentzian spacetime* (Definition 6.9);
3. The concept of a natural adapted coordinate chart to a reference frame (Definition 6.10);
4. The concept of PIRF*s* (Definition 6.59) and LLRFγs (Definition 6.61) on $U \subset M$;
5. The concept of an inertial moving *observer* carrying a tetrad along γ (a geodetic curve), a concept we abbreviate by calling it an IMTγ (Definition 6.63).

Einstein's EP is formulated by Misner, Thorne and Wheeler (MTW)[89] as follows:

> in any and every Local Lorentz Frame (*LLF*), anywhere and anytime in the universe, all the (non-gravitational) laws of physics must take on their familiar special relativistic forms. Equivalently, there is no way, by experiments confined to small regions of spacetime to distinguish one *LLF* in one region of spacetime from any other *LLF* in the same or any other region.

We comment here that those authors[30] did not give a formal definition of a LLF. They try to make intelligible the EP by formulating its wording in terms of a LLCC (see Definition 6.60) and indeed these authors as many others do not distinguish the concept of a reference frame $\mathbf{Z} \in \sec TM$ from that of a ($nacs|\mathbf{Z}$). This may generate misunderstandings. The mathematical formalization of a LLF used by MTW (and many other authors) corresponds to the concept of LLRF introduced in Definition 6.61.

In [31] Ciufolini and Wheeler call the above statement of MTW the medium strong form of the EP. They introduced also what they called the strong EP as follows:

> in a sufficiently small neighborhood of any spacetime event, in a locally falling frame, no gravitational effects are observable.

[29] These concepts are in general used without distinction by different authors leading to misunderstandings and misconceptions.

[30] For the best of our knowledge no author gave until now the formal definition of a LLRF as in Definition 6.61.

Again, no mathematical formalization of a locally falling frame is given, the formulation uses only LLCC. Worse, if local means in a neighborhood of a given spacetime event this principle must be false. For example, it is well known fact that the Riemann tensor couples locally with spinning particles. Moreover, the neighborhood must be at least large enough to contain an experimental physicist and the devices of his laboratory and must allow for enough time for the experiments. With a gradiometer built by Hughes corporation which has an area of approximately 400 cm^2 any researcher can easily discover if he is leaving in a region of spacetime with a gravitational field or if he is living in an accelerated frame in a region of spacetime free with a zero gravitational field.

Following [31, 89] recently several authors as, e.g., Will [158], Bertotti and Grishchuk [18] and Gabriel and Haugan [49] (see also Weinberg [156]) claim that Einstein EP requires a sort of local Lorentz invariance. This concept is introduced in, e.g., [18] with the following arguments.

To start we are told that to state the Einstein EP we need to consider a laboratory that falls freely through an external gravitational field. Moreover, such a laboratory must be shielded, from external non-gravitational fields and must be small enough such that effects due to the non homogeneity of the field are negligible through its volume. Then, they say, that the local non-gravitational test experiments are experiments performed within such a laboratory and in which self-gravitational interactions play no significant part. They define:

> The Einstein *EP* states that the outcomes of such experiments are independent of the velocity of the apparatus with which they are performed and when in the universe they are performed.

This statement is then called the *Principle of Local Lorentz Invariance* (PLLI) and 'convincing' proofs of its validity are offered, and there is no need to repeat that 'convincing' proofs here. Prugovecki [109] endorses the PLLI and also said that it can be experimentally verified. In his formulation he translates the statements of [18, 31, 48, 49, 89, 156, 158] in terms of Lorentz and Poincaré covariance of measurements done in two different IMFγ (see Definition 6.63). Based on these past tentatives of formalization[31] we give the following one.

Einstein EP: Let γ be a timelike geodetic line on the world manifold M. For any LLRFγ (see Definition 6.61) all non gravitational laws of physics, expressed through the coordinates $\{\xi^{\mu}\}$ which are LLC associated with the LLRFγ (Definition 6.61) should at each point along γ be *equal* (up to terms in first order in those coordinates) to their special relativistic counterparts when the mathematical objects appearing in these special relativistic laws are expressed through a coordinates in Einstein-Lorentz-Poincaré gauge naturally adapted to an arbitrary inertial frame $\mathbf{I} \in \sec TM'$, where $M' = \mathbb{R}^4$ is the manifold of a Minkowski spacetime structure $\mathcal{M} = (M', \boldsymbol{\eta}, D, \boldsymbol{\tau}_{\boldsymbol{\eta}}, \uparrow)$.

[31] See [99] for a history of the subject.

Also, if the PLLI would be a true law of nature it could be formulated as follows:

Principle of Local Lorentz Invariance: Any two LLRFγ and LLRFγ' associated with the timelike geodetic lines γ and γ' of two observers such that $\gamma \cap \gamma' = p$ are *physically equivalent* (according to Definition 6.55) at event p.

Of course, if PLLI is correct, it must follow that from experiments done by observers inside some LLRFγ'—say $\mathbf{L}'$ that is moving relative to another *LLRF* $\mathbf{L}$—there is no means for that observers to determine that $\mathbf{L}'$ is in motion relative to $\mathbf{L}$.

Unfortunately the PLLI is *not* true. To show that it is only necessary to find a model of GRT where the statement of the PLLI is false. Before proving this result we shall need to prove that there are models for GRT were PIRFs are not physically equivalent also.

6.8 PIRFs on a Friedmann Universe

Recall that GRT is a theory of the gravitational field [131] where a typical model $F \in ModF_E$ is of the form

$$F = ((M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow), \mathbf{T}, (m, \sigma)), \tag{6.143}$$

where $\mathfrak{M} = (M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ is a relativistic spacetime and $\mathbf{T} \in sec T_2^0 M$ is called the energy-momentum tensor. The tensor $\mathbf{T}$ represents the material and energetic content of spacetime, including contributions from all physical fields (with exception of the gravitational field and test particles). For what follows we do not need to know the explicit form of $\mathbf{T}$. Also, (m, σ) represents a test particle, whose world line is σ. The proper axioms of F_E are:

$$\boldsymbol{D}\boldsymbol{g} = 0, \boldsymbol{\Theta}(\boldsymbol{D}) = 0, \mathbf{G} = Ricci - \frac{1}{2}R\boldsymbol{g} = -\mathbf{T}, \tag{6.144}$$

where $\boldsymbol{\Theta}$ is the torsion tensor, $\mathbf{G}$ is the Einstein tensor, *Ricci* is the Ricci tensor and r is the Ricci scalar. The equation of motion of the test particle (m, σ) that moves only under the influence of gravitation is:

$$D_{\sigma *}\sigma_* = 0. \tag{6.145}$$

$\mathfrak{M}$ is in general not flat, which implies that in general (see Remark 6.13) there is no IRF $\mathbf{I}$, i.e., a reference frame such that $D\mathbf{I} = 0$.

Now, the physical universe we live in is reasonably described by metric tensors of the Robertson-Walker-Friedmann type [131]. In particular, a very simple spacetime structure $\mathfrak{M}$ that represents the main properties observed (after the big-bang) is formulated as follows: Let $M = \mathbb{R}^3 \times I$, $I \subset \mathbb{R}$ and $\mathrm{R} : I \to (0, \infty), t \to \mathrm{R}(t)$

and define $\boldsymbol{g}$ in M (considering M as subset of $\mathbb{R}^4$) by:

$$\boldsymbol{g} = dt \otimes dt - \mathrm{R}(t)^2 \sum dx^i \otimes dx^i, i = 1, 2, 3. \tag{6.146}$$

Then $\boldsymbol{g}$ is a Lorentzian metric in M and $\mathbf{V} = \partial/\partial t$ is a time-like vector field in $(M, \boldsymbol{g})$. Let $\mathfrak{M}$ be oriented in time by $\partial/\partial t$ and spacetime oriented by $dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. Then $\mathfrak{M}$ is a relativistic spacetime for $I = (0, \infty)$.

Now, $\mathbf{V} = \partial/\partial t$ is a reference frame. Taking into account that the connection coefficients in a $(nacs|\mathbf{V})$ given by the coordinate system in Eq. (6.146) are

$$\Gamma^{i\cdot\cdot}_{\cdot kl} = 0,\ \Gamma^{0\cdot\cdot}_{\cdot kl} = \mathrm{R}\dot{\mathrm{R}}\delta_{kl}, \Gamma^{k\cdot\cdot}_{\cdot 0l} = \frac{\dot{\mathrm{R}}}{\mathrm{R}}\delta^k_l$$
$$\Gamma^{i\cdot\cdot}_{\cdot 00} = \Gamma^{0\cdot\cdot}_{\cdot 0l} = \Gamma^{0\cdot\cdot}_{\cdot 00} = 0, \tag{6.147}$$

we can easily verify that $\mathbf{V}$ is a PIRF (according to Definition 2.6) since $D_{\mathbf{V}}\mathbf{V} = 0$ and $d\alpha_{\mathbf{V}} \wedge \alpha_{\mathbf{V}} = 0$, $\alpha_{\mathbf{V}} = \boldsymbol{g}(\mathbf{V},)$. Also, since $\alpha_{\mathbf{V}} = dt$, $\mathbf{V}$ is proper time synchronizable.

Proposition 6.65 *In a spacetime defined by Eq.* (6.146) *which is a model of* F_E *there exists a* PIRF $\mathbf{Z} \in \sec TU$ *which is not physically equivalent to* $\mathbf{V} = \partial/\partial t$.

Proof Let $\mathbf{Z} \in \sec TU$ be given by

$$\mathbf{Z} = \frac{(\mathrm{R}^2 + u^2)^{1/2}}{\mathrm{R}}\partial/\partial t + \frac{u}{\mathrm{R}^2}\partial/\partial x^1 \tag{6.148}$$

where in Eq. (6.148) $u \neq 0$ is a real constant.

Since $D_{\mathbf{Z}}\mathbf{Z} = 0$ and $d\alpha_{\mathbf{Z}} \wedge \alpha_{\mathbf{Z}} = 0$, $\alpha_{\mathbf{Z}} = \boldsymbol{g}(\mathbf{Z},)$, it follows that $\mathbf{Z}$ is a PIRF.[32] All that is necessary in order to prove our proposition is to show that $\mathbf{Z}$ and $\mathbf{V}$ are not equivalent. It is enough to prove that the expansion ratios $\mathfrak{E}_{\mathbf{Z}} \neq \mathfrak{E}_{\mathbf{V}}$. Indeed, Eq. (6.43) gives

$$\mathfrak{E}_{\mathbf{V}} = 3\dot{\mathrm{R}}/\mathrm{R},$$
$$\mathfrak{E}_{\mathbf{Z}} = \frac{\left[\mathrm{R}\dot{\mathrm{R}} + 2\dot{\mathrm{R}}(\mathrm{R}^2 + u^2)^{1/2}\right]}{\mathrm{R}^2\,(\mathrm{R}^2 + u^2)^{1/2}}, \tag{6.149}$$

where

$$v = \mathrm{R}(\frac{d}{dt}x^1 \circ \gamma)\bigg|_{t=0} = u(1 + u^2)^{-1/2} \tag{6.150}$$

[32]Introducing the (nacs|$\mathbf{Z}$) given by Eq. (6.153) we can show that $\alpha_{\mathbf{Z}} = dt'$ and it follows that is also proper time synchronizable.

is the initial metric velocity of **Z** relative to **V**, since we choose in what follows the coordinate function t such that $\mathrm{R}(0) = 1$, $t = 0$ being taken as the present epoch where the experiments are done. Then, $\mathfrak{E}_{\mathbf{V}}(p_0) = 3a$, and for $v << 1$, $\mathfrak{E}_{\mathbf{Z}}(p_0) = 3a - av^2$.■

6.8.1 Mechanical Experiments Distinguish PIRFs

The question arises: can mechanical experiments (distinct from the one designed to measure the expansion ratio) distinguish between the PIRF*s* **V** and **Z?** The answer is yes. To prove our statement we proceed as follows.

(1) We start by finding a $(nacs|\mathbf{Z})$. To do that we note if γ is an integral curve of **Z**, we can write

$$\mathbf{Z}_{|\gamma} = [\frac{d}{ds}(\mathbf{x}^{\mu} \circ \gamma)\frac{\partial}{\partial x^{\mu}}]_{|\gamma} \tag{6.151}$$

where s is the proper time parameter along γ. Then, we can write [taking into account Eqs. (6.147)] its parametric equations as

$$\frac{d}{dt}\mathbf{x}^1 \circ \gamma = \frac{(\frac{d}{ds}\mathbf{x}^1 \circ \gamma)}{(\frac{d}{ds}\boldsymbol{t} \ \circ \gamma)} = \frac{u}{\mathrm{R}(\mathrm{R}^2 + u^2)^{1/2}}; \quad \mathbf{x}^2 \circ \gamma = 0; \quad \mathbf{x}^3 \circ \gamma = 0 \tag{6.152}$$

(The direction $x^1 \circ \gamma = 0$ is obviously arbitrary.) We then choose as a $(nacs|\mathbf{Z})$ the coordinate functions $(\boldsymbol{t}', \mathbf{x}'^1, \mathbf{x}'^2, \mathbf{x}'^3)$ such that:

$$\begin{aligned} x'^1 &= x^1 - u\int_0^t dr\frac{1}{\mathrm{R}(r)[\mathrm{R}^2(r) + u^2]^{1/2}}; \quad x'^2 = x^2; \\ x'^3 &= x^3; \quad t' = \int_0^t dr\frac{[\mathrm{R}^2(r) + u^2]^{1/2}}{\mathrm{R}(r)} - ux^1 \end{aligned} \tag{6.153}$$

We then get:

$$\boldsymbol{g} = dt' \otimes dt' - \overline{\mathrm{R}}(t')^2 \left\{ \begin{array}{l} \left[\frac{1-v^2(1-\overline{R}(t')^{-2})}{1-v^2}\right] dx'^1 \otimes dx'^1 \\ +dx'^2 \otimes dx'^2 + dx'^3 \otimes dx'^3 \end{array} \right\}, \tag{6.154}$$

and the connection coefficients in the $(nacs|\mathbf{Z})$ are,

$$\begin{aligned} &\bar{\Gamma}^{0\cdot\cdot}_{\cdot kl} = \frac{\dot{\bar{\mathrm{R}}}\bar{\mathrm{R}}^2}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{1}{2}}}\,\delta_{kl}, \bar{\Gamma}^{1\cdot\cdot}_{\cdot 01} = \frac{\dot{\bar{\mathrm{R}}}\bar{\mathrm{R}}^2}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{3}{2}}}, \bar{\Gamma}^{2\cdot\cdot}_{\cdot 02} = \bar{\Gamma}^{3\cdot\cdot}_{\cdot 03} = \frac{\dot{\bar{\mathrm{R}}}}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{1}{2}}}, \\ &\bar{\Gamma}^{i\cdot\cdot}_{\cdot kl} = 0, \bar{\Gamma}^{i\cdot\cdot}_{\cdot 00} = \bar{\Gamma}^{0\cdot\cdot}_{\cdot 0l} = \bar{\Gamma}^{0\cdot\cdot}_{\cdot 00} = 0. \end{aligned} \tag{6.155}$$

where $\overline{\mathrm{R}}(t') = \mathrm{R}(t(t'))$ and v given by Eq. (6.150) is the initial metric velocity of **Z** relative to **V**, since we choose in what follows the coordinate function t such that $\mathrm{R}(0) = 1$, $t = 0$ being taken as the present epoch where the experiments are done. $\mathbf{Z} = \partial/\partial t'$ is a proper time synchronizable reference frame and we can verify that t' is the time shown by standard clocks at rest in the **Z** reference frame and which are synchronized à l'Einstein. Notice that an observer at rest in **Z** does *not* know a priori the value of v. He can discover this value as follows:

(2) The solution of the equation of motion for a free particle (m, σ) in **V** with the initial conditions at $p_0 = (0,\ \mathbf{x}^i \circ \sigma(0) = 0),\ i = 1, 2, 3$ and $\frac{d}{dt}\mathbf{x}^i \circ \sigma(0) = \bar{u}^i$ for a fixed i and $\frac{d}{dt}\mathbf{x}^i \circ \sigma(0) = 0, j \neq i$, is given by an equation analogous to Eq. (6.152). The accelerations are such that

$$\left.\frac{d^2}{ds^2}\mathbf{x}^j \circ \sigma(t))\right|_{p_0} = 0,\ j \neq i. \tag{6.156}$$

(3) The equation of motion for a free particle (m, σ') in **Z** , can be written as (we write for simplicity in what follows $\frac{d^2}{ds^2}\mathbf{x}'^1 \circ \sigma'(t') \equiv \frac{d^2}{ds^2}x'^1(t') \equiv \frac{d^2}{ds^2}x'^1$, etc...)

$$\begin{aligned}
\frac{d^2x'^1}{ds^2} &= -2\frac{\dot{\bar{\mathrm{R}}}\bar{\mathrm{R}}^2}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{3}{2}}}\frac{dx'^1}{dt'}\left(\frac{dt'}{ds}\right)^2, \\
\frac{d^2x'^i}{ds^2} &= -2\frac{\dot{\bar{\mathrm{R}}}}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{1}{2}}}\frac{dx'^i}{dt'}\left(\frac{dt'}{ds^2}\right),\ i = 2, 3, \\
\frac{d^2t'}{ds^2} &= -2\frac{\dot{\bar{\mathrm{R}}}\bar{\mathrm{R}}^2}{(\bar{\mathrm{R}}^2 + u^2)^{\frac{1}{2}}}\left[\left(\frac{dx'^1}{dt'}\right)^2 + \left(\frac{dx'^2}{dt'}\right)^2 + \left(\frac{dx'^3}{dt'}\right)^2\right] \\
\frac{dt'}{ds} &= \left[1 + \widehat{\bar{\mathrm{R}}}^2\left(\frac{dx'^1}{dt'}\right)^2 + \bar{\mathrm{R}}^2\left(\frac{dx'^2}{dt'}\right)^2 + \bar{\mathrm{R}}^2\left(\frac{dx'^3}{dt'}\right)^2\right]^{-\frac{1}{2}}
\end{aligned} \tag{6.157}$$

where the dot over $\bar{\mathrm{R}}$ in Eq. (6.157) means derivative with respect to t' and $\widehat{\bar{\mathrm{R}}}$ denotes the square root of the coefficient of $dx'^1 \otimes dx'^1$ term in Eq. (6.154).

From these equations it is easy to verify that the two situations:

(a) motion in the (x'^1, x'^2) plane with initial conditions at p_0 with coordinates $(t' = 0, x'^1 = x'^2 = 0 = x'^3)$ given by

$$\left.\frac{dx'^1(t')}{dt'}\right|_{p_0} = v'_1,\ \left.\frac{dx'^2(t')}{dt}\right|_{p_0} = 0, \tag{6.158}$$

and

(b) motion in the (x'^1, x'^2) plane with initial conditions at p_0 with coordinates $(t' = 0, x'^1 = x'^2 = 0 = x'^3)$ given by

$$\left.\frac{dx'^1(t')}{dt'}\right|_{p_0} = 0, \qquad \left.\frac{dx'^2(t')}{dt'}\right|_{p_0} = v_{2'}, \tag{6.159}$$

produce asymmetrical outputs for the measured accelerations along x'^1 and x'^2. The explicit values depends of course of the function $\mathrm{R}(t)$. If we take $\mathrm{R}(t) = 1 + at$, the asymmetrical accelerations will be given in terms of $a << 1$ and v. This would permit in principle for the *eventual* observers living in the PIRF $\mathbf{Z}$ to infer the value of u (or v).

6.8.2 LLRFγ and LLRFγ′ Are Not Physically Equivalent on a Friedmann Universe

Proposition 6.66 [33]*There are models of* GRT *for which two* Local Lorentz Reference Frames *are not physically equivalent.*

Proof Take as model of GRT the one just described above where $\boldsymbol{g}$ is given by Eq. (6.146) and take as before, $\mathrm{R}(t) = 1 + at$. Consider two integral lines γ and γ' of $\mathbf{V}$ and $\mathbf{Z}$ such that $\gamma \cap \gamma' = p$.

We can associate with these two integral lines the LLRFγ $\mathbf{L}$ and the LLFRγ' $\mathbf{L}'$ as in Definition 6.60. Observe that $\mathbf{V}|_\gamma = \mathbf{L}|_\gamma$ and $\mathbf{Z}|_{\gamma'} = \mathbf{L}'|_{\gamma'}$.

Definition 6.56 says that if $\mathbf{L}$ and $\mathbf{L}'$ are physically equivalent then we must have $D\mathbf{L} = D\mathbf{L}'$. However, a simple calculation shows that in general $\boldsymbol{D}\mathbf{L} \neq \boldsymbol{D}\mathbf{L}'$ even at p! Indeed, we have

$$\mathfrak{E}_{\mathbf{L}} = -3t\left(\frac{\dot{\mathrm{R}}}{\mathrm{R}}\right)^2, \tag{6.160}$$

$$\mathfrak{E}_{\mathbf{L}'} = 2\dot{\bar{\mathrm{R}}}(\bar{\mathrm{R}}^2 + u^2)^{1/2} + \dot{\mathrm{R}} - \frac{\bar{\mathrm{R}}\dot{\bar{\mathrm{R}}}^2}{(\bar{\mathrm{R}}^2 + u^2)^{3/2}} - \frac{2\dot{\bar{\mathrm{R}}}}{(\bar{\mathrm{R}}^2 + u^2)^{1/2}}$$

[33]The suggestion of the validity of a proposition like the one formalized by Proposition 6.66 has been first proposed by Rosen [128]. However, he has not been able to identify the true nature of the $\mathbf{V}$ and $\mathbf{Z}$ which he thought as representing 'inertial' frames. He tried to show the validity of the proposition by analyzing the output of mechanical and optical experiments done inside the frames $\mathbf{V}$ and $\mathbf{Z}$. We present below a simplified version of his suggested mechanical experiment. It is important to emphasize here that from the validity of the Proposition 6.66 Rosen suggested that it implies in a breakdown of the PLLI. Of course, the PLLI refers to the physical equivalence of LLRFγs. Also the proof of Proposition 6.66 given above appeared originally in [125].

$$-\frac{2\dot{\bar{\mathrm{R}}}^2\bar{\mathrm{R}}^4}{(\bar{\mathrm{R}}^2+u^2)^3}tx'^1-\frac{2\,\dot{\bar{\mathrm{R}}}^2}{(\bar{\mathrm{R}}^2+u^2)}tx'^2-\frac{2\,\dot{\bar{\mathrm{R}}}^2}{(\bar{\mathrm{R}}^2+u^2)}tx'^3. \tag{6.161}$$

From Eqs. (6.160) and (6.161) we see that the expansions ratios $\mathfrak{E}_{\mathbf{L}}$ and $\mathfrak{E}_{\mathbf{L}'}$ are different in our model and then our claim is proved. At p, we have $\mathfrak{E}_{\mathbf{L}}(p) = 0$ and $\mathfrak{E}_{\mathbf{L}'}(p) = 2av^2$. ■

Remark 6.67 Proposition 6.66 establishes that in a Friedmann universe there is a *LLRF*γ (say $\mathbf{L}$) whose expansion ratio at p is zero. Any other *LLRF*γ' (say $\mathbf{L}'$) at p will have an expansion ratio at p given by $2av^2$, where $a \ll 1$ and v is the metric velocity of $\mathbf{L}'$ relative to $\mathbf{L}$ at p. This expansion ratio can in principle be measured and this is the reason for the non validity of the *PLLI* as formulated by many contemporary physicists and formalized above. Note that all experimental verifications of the *PLLI* mentioned by the authors that endorse the *PLLI* have been obtained for *LLRF*γ*s* moving with $v << 1$, and have no accuracy in order to contradict the result we found. We do not know of any experiment that has been done on a *LLRF*γ which enough precision to verify the effect. Anyway the non physical equivalence between $\mathbf{L}$ and $\mathbf{L}'$ is a *prediction* of *GRT* and must be accepted if this theory is right. In conclusion, *PLLI* is only approximately valid.

We recall that Friedman [48] formulates the PLLI by saying that if (U, ξ^μ), $(U', \bar{\xi}^\mu)$ $(U \cap U \neq \emptyset)$ are LLCC adapted to the $\mathbf{L}$ and $\mathbf{L}'$ respectively, then the PLLI implies that two experiments whose initial conditions read alike in terms of $\{\xi^\mu\}$ and $\{\bar{\xi}^\mu\}$ will also have the same *outcome* in terms of these coordinates .

Friedman's statement is not correct, of course, in view of Proposition 6.66 above, for measurement of the *expansion ratio* of a real reference frame which has material support is something objective and, of course, can be in principle, be determined with an appropriate physical experiment. However, for experiments different from this one (measurement of the expansion ratio) we can accept Friedman's formulation of the PLLI as an approximately true statement.

Recall the expansion ratios calculated for $\mathbf{V}, \mathbf{Z}, \mathbf{L}, \mathbf{L}'$. Now, $a << 1$. Then, if $v << 1$ the LLRFγ $\mathbf{L}$ and the LLRFγ' $\mathbf{L}'$ will be almost 'rigid' whereas the $\mathbf{V}$ and $\mathbf{Z}$ are expanding. In other words, the $\mathbf{L}$ and $\mathbf{L}'$ frames can be thought as being physically materialized in their domain by real solid bodies and thus correspond to small real laboratories, the one used by physicists. On the other hand it is well known that the $\mathbf{V}$ frame is an idealization, since only the center of mass of the galactic clusters are supposed to be comoving with the $\mathbf{V}$ frame, i.e., each center of mass of a galactic cluster follows some particular integral line of $\mathbf{V}$. Concerning the $\mathbf{Z}$ frame, in order for it to be realized as a physical system it must be build with a special matter that suffers in all points of its domain an expansion a little bit greater than the cosmic expansion. Of course, such a frame would be a very artificial one, and we suspect that such a special matter cannot be prepared in our universe.

6.8.3 No Generalization of the Principle of Relativity for GRT

In the previous sections we presented a careful analysis of the concept of a reference frames in GRT. These objects have been modelled as certain unit timelike vector fields. We gave physically motivated and mathematical rigorous definition of physically equivalent reference frames in a relativistic spacetime theory. We investigate which are the reference frames in GRT which share some of the properties of the inertial reference frames of the Special Theory of Relativity. We found that in GRT there are two classes of frames that have some of the properties of the inertial frames of Special Theory of Relativity. These are the class of the pseudo inertial reference frames (PIRFs) and the class of the local Lorentz reference frames (LLRFγs). We showed that LLRFγs are *not* physically equivalent in general and this implies that the so called Principle of Local Lorentz invariance (PLLI) which several authors state as meaning that LLRFγs are equivalent is *false*. It can only be used as an approximation in experiments that do not have enough accuracy to measure the effect we found. We prove moreover that there are models of GRT where PIRFs are not physically equivalent also. Our results show without any doubt that there is no generalization of the Principle of Relativity, understood as a generalization of PR_2, i.e., that there exists physical equivalence of all reference frames in GRT. Indeed, in the structure $(M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ given two arbitrary reference frames $\mathbf{Z}, \mathbf{Z}'$ it is not the even the case in general that $\boldsymbol{D}\mathbf{Z} = \boldsymbol{D}\mathbf{Z}'$ and so they are not physically equivalent.

6.9 Schwarzschild Original Solution and the Existence of Black Holes in GRT

1. Schwarzschild [136] looked for a solution of Einstein equations supposing a priori that the spacetime manifold where a point mass and the gravitational field it generated live is $M = \mathbb{R} \times \mathbb{R}^3$ where time takes values in $\mathbb{R}$ and $\mathbb{R}^3$ denotes the usual three-dimensional Euclidean space. Indeed, he equipped $\mathbb{R} \times \mathbb{R}^3$ with coordinates (t, x, y, z) explicitly saying that (x, y, z) are *rectangular Cartesian coordinates*. After that, as a second step he introduced usual polar coordinate functions in the game. In so doing, he correctly left out from $\mathbb{R}^3$ the origin $\mathbf{O} = (0, 0, 0)$ where the point mass particle is supposed to be located at any instant of coordinate time t and start solving Einstein equations in the manifold $\mathbb{R} \times (\mathbb{R}^3 - \mathbf{O}) \approx \mathbb{R} \times (0, \infty) \times \boldsymbol{S}^2$, introducing standard spherical coordinates $(\boldsymbol{r}, \vartheta, \phi)$. However, since those coordinate functions do not satisfy the Einstein coordinate gauge[34] that Schwarzschild chose to use, he introduced *spherical coordinates with determinant* 1. After some mathematical tricks, he found as

[34]Einstein coordinate gauge fixes $\sqrt{|\det \boldsymbol{g}|} = 1$.

solution of his problem a metric field $\boldsymbol{g}_{os}$, which has a unique singularity at $\mathbf{O}$ and as such his solution *does not imply in any black hole* (defined in **4** below).[35]

Schwarzschild original solution reads

$$\boldsymbol{g}_{os} = h(\mathbf{r})dt \otimes dt - h(\mathbf{r})^{-1} f'(\mathbf{r}) d\mathtt{r} \otimes d\mathbf{r} - F(\mathbf{r})(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi), \tag{6.162}$$

where

$$h(\mathbf{r}) = \left(1 - \frac{2m}{f(\mathbf{r})}\right), \quad f(\mathbf{r}) = (\mathbf{r}^3 + 8m^3)^{1/3} \tag{6.163}$$

2. However, in fact Schwarzschild, in order to determine one of the integration constants of the differential equations he was solving and that leads him to Eq. (6.162), needed for his calculations to use the *manifold with boundary*[36] $\mathbb{R} \times [0, \infty) \times \boldsymbol{S}^2$ and thus his original mass point supposed to be located for any instant of time at the point $\mathbf{O} \in \mathbb{R}^3$ ended to be represented by the manifold[37] $\{0\} \times \boldsymbol{S}^2$ (something obviously odd that Hilbert elegantly, without criticizing Schwarzschild, observed in a footnote of his paper on the Schwarzschild solution).
3. Before proceeding, and in order to avoid any confusion note that despite the fact that the original manifold postulated as model of space-time by Schwarzschild is $\mathbb{R} \times \mathbb{R}^3$ this does not imply that this manifold or the manifold $\mathbb{R} \times [0, \infty) \times \boldsymbol{S}^2$ equipped with the Levi-Civita connection $\boldsymbol{D}$ of $\boldsymbol{g}_S$ (that solves Einstein equation) is *flat*. In fact, the connection $\boldsymbol{D}$ for Schwarzschild problem is curved, this statement meaning, of course, that its Riemann curvature tensor is non null. Please, take always the following statement into account [44]:

> Manifolds do not have curvature, it is the connection imposed on a manifold that may or may not have non null curvature (and/or non null torsion, non null nonmetricity). Some manifolds may be bended surfaces in a Euclidean (or pseudo-Euclidean) space of appropriate dimension. But to be **bended** (a property described by the shape operator introduced in Chap. 5) has in general nothing to do with the fact that a connection defined in the manifold is curved.

4. Droste [37] and Hilbert [57] found independently another solution of Einstein equations based on different assumptions than the ones used by

[35] One can easily verify that the coordinate time for a radial light ray to go from any $\boldsymbol{r} > 0$ to a point such that $2m < \boldsymbol{r} < \boldsymbol{\infty}$ is finite.

[36] For the use of the concept of manifold with boundary to present singularities in General Relativity, see, [50, 55, 134, 135]. For some skilful commentaries on the peril of using boundary manifolds in General Relativity without taking due care see [145].

[37] $\{0\}$ denotes the set whose unique element is $0 \in [0, \infty)$, i.e., the *boundary* of the semi-line $[0, \infty)$.

Schwarzschild.[38] Modern relativists[39] (following Droste and Hilbert) find as solution[40] with *rotational symmetry* of Einstein equation in vacuum a metric field $\boldsymbol{g}_{DH}$ (at least C^2) defined in the manifold $\mathbb{R} \times (0, 2m) \cup (2m, \infty) \times \boldsymbol{S}^2$. Relativists say that the "part" $\mathbb{R} \times (0, 2m) \times \boldsymbol{S}^2$ where the solution is valid defines a black hole [131].

5. It is crucial to have in mind that the *quasi spherical coordinates functions* (r, ϑ, ϕ) used by modern relativists are such that the coordinate function r is not the Schwarzschild spherical coordinate function $\boldsymbol{r}$, i.e.,

$$r \neq \boldsymbol{r}.$$

6. Schwarzschild wrote his final formula for $\boldsymbol{g}_{os}$ using a function $f(\boldsymbol{r})$ which is formally identical to the Droste-Hilbert formula for $\boldsymbol{g}_{DH}$ if $f(\boldsymbol{r})$ is read as the coordinate function r. However, Schwarzschild solution is valid only for $f(\boldsymbol{r}) > 2m$ whereas the Droste-Hilbert solution is valid for any $r \in (0, 2m) \cup (2m, \infty)$.
7. Of course, there is no sense in supposing that space-time has a *disconnected* topology. Thus, under the present *ideology* of finding maximal extension of manifolds equipped with Lorentzian metrics as the true representatives of gravitational fields, relativists maximally extend the solution $\boldsymbol{g}_{DH}$ to a solution $\boldsymbol{g}$ valid in a connected manifold called the *Kruskal* (sometimes, Kruskal-Szekeres) spacetime [70, 149]. The total Kruskal manifold which has an exotic topology is usually associated to a hypothetical object called the *wormhole*. The final solution $\boldsymbol{g}$ is presented as a function of coordinate functions $(u, v, \vartheta, \varphi)$ and r which (*keep this in mind*) becomes an implicit function of the coordinate functions (u, v).
8. It is assumed by relativists that a connected "part" of the Kruskal manifold describes a black-hole where $\boldsymbol{g}$ has a real singularity only at the place defined by the function $r(u, v) = 0$.
9. In conclusion, Schwarzschild original solution and the Kruskal extension of the Droste-Hilbert solution define space-times with very *different* topologies, so they are not the same solution of Einstein equation. In the former the topology of the manifold has been fixed a priori, in the latter the topology of the manifold has been fixed a posteriori by the process of maximal extension.
10. There are some published papers that do not properly distinguish these two different solutions,[41] moreover, there are some authors stating (explicitly, or in a disguised way) that it is possible to extend the Schwarzschild original

[38] In the words of Synge [147]: Schwarzschild imposed spherical symmetry, whereas Droste and Hilbert imposed rotational symmetry, a subtle but crucial detail.

[39] See, e.g., [55, 101].

[40] Eventually, it would better to say, as did O'Neill in his book [101] that we start looking for *one* solution of a problem and ended with *two* solutions.

[41] Take notice that there are also some non sequitur mathematical statements in some of those papers. See http://www.physicsforums.com/showthread.php?t=124277.

solution that was written in terms of the function f for the domain $0 \leq f(\mathbf{r}) < \infty$, but this idea is, of course, a logical non sequitur, since for Schwarzschild the manifold is fixed a priori. Any appropriate discussion of the mathematical aspect of the back-hole solution of Einstein equations clearly requires a reasonable understanding of differential geometry, and of course, of topology.[42] And it is also important, to advise that everyone that wants to discuss the black-hole issue and did not read the original Schwarzschild paper (or its English version, available at the **arXiv**) must do that in a hurry.

11. Failing to properly understand the different topologies of the two solutions mentioned above (Schwarzschild and the maximal extension of the Droste-Hilbert) is thus making some people (including some that say to be relativist physicists) not to discuss contemporary GRT, but some other things, believing to be the same thing.
12. *The question if black holes exist or not is, of course, not a mathematical one,*[43] *it is a physical question* and presently at least one of the present authors believe that they do not exist, leaving this clear in [44], where it is argued that it is necessary to construct a theory of the gravitational field where that field is to be regarded as a field in the sense of Faraday (like the electromagnetic field and the weak and strong force fields) "living" in Minkowski space-time (see also Chap. 11). *Thus, that "part" of the maximal extension of the Droste-Hilbert solution of Einstein equations* (*describing a black hole*) *probably does not describe anything real in the physical world.*
13. It is indeed out of question, the fact that Einstein equations (according to the contemporary interpretation of GRT) have solutions describing black holes. Of course, this does not leave everyone happy and many physicists have proposed and are proposing alternative solutions of Einstein equations capable of describing the final stage of super dense stars and which according to them looks more "realistic". See, e.g., [84].

Exercise 6.68

(i) Show that under the transformation

$$t' = t,\ r = f(\mathbf{r}) = (\mathbf{r}^3 + 8m^3)^{1/3},\ \theta' = \theta,\ \varphi' = \varphi \tag{6.164}$$

the expression of $\boldsymbol{g}_{os}$ is the one given by $\boldsymbol{g}_s$ in Eq. (6.97).

(ii) Are $\boldsymbol{g}_{os}$ and $\boldsymbol{g}_s$ diffeomorphically equivalent?

[42] No more than what may be found in [55] or [101]. A thoughtful mathematical discussion (and historical review) of the Schwarzschild,the Droste-Hilbert and Kruskal-Szekeres solutions may be found in [90].

[43] We mean that black holes exist as legitimate solutions of the Einstein equation, e.g., the ones described by a "part"[101] of the Kruskal-Szekeres solution.

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Chapter 7
Clifford and Dirac-Hestenes Spinor Fields

Abstract This chapter presents an original approach to the theory of Dirac-Hestenes spinor fields. After recalling details the structure of the Clifford bundle of a Lorentzian manifold structure $(M, \boldsymbol{g})$ we introduce the concept of spin structures on $(M, \boldsymbol{g})$, define spinor bundles and spinor fields. Left and right spin-Cillford bundles are presented and the concept of Dirac-Hestenes spinor fields (as sections of a special spin-Clifford bundle denoted by $\mathcal{C}\ell^r_{Spin^e_{1,3}}(M, \mathtt{g})$) is investigated in details disclosing their real nature and showing how these objects can be represented as some equivalence classes of even sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$. We introduce and obtain the connection (covariant derivative) acting on spin-Clifford bundles $\mathcal{C}\ell^r_{Spin^e_{1,3}}(M, \mathtt{g})$ from a Levi-Civita connection acting on the tensor bundle of $(M, \boldsymbol{g})$. Associated with that connection we next introduce the standard spin-Dirac operator acting on sections of $\mathcal{C}\ell^r_{Spin^e_{1,3}}(M, \mathtt{g})$ (not to be confused with the Dirac operator acting on sections of the Clifford bundle defined in Chap. 4). We discuss how to write Dirac equation using the spin-Clifford bundle formalism and clarifies its properties and many misunderstandings relating to such a notion that are spread in the literature. We also discuss the notion of what is known as *amorphous* spinor fields showing that these objects cannot represent fermion fields. Finally, the Chapter also presents a simple proof of the famous Lichnerowicz formula which relates the square of the standard spin-Dirac operator to the curvature of the (spin) connection and also we obtain a generalization of that formula for the square of a general spin-Dirac operator associated to a general (metric compatible) Riemann-Cartan connection defined in $(M, \boldsymbol{g})$.

7.1 The Clifford Bundle of Spacetime

In this Chapter, M refers (unless otherwise stated) to a four dimensional, real, connected, paracompact and non-compact manifold. We recall that in Sect. 4.7.1 of Chap. 4 we defined a Lorentzian manifold as a pair $(M, \boldsymbol{g})$, where $\boldsymbol{g} \in \sec T^0_2 M$ is a Lorentzian metric of signature $(1, 3)$, i.e., for all $x \in M$, $T_x M \simeq T^*_x M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space. Moreover, we also defined in Sect. 4.7.1 a *spacetime* $\mathfrak{M}$ as pentuple $(M, \boldsymbol{g}, \boldsymbol{\nabla}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$, where $(M, \boldsymbol{g}, \boldsymbol{\tau}_{\boldsymbol{g}}, \uparrow)$ is an oriented

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Lorentzian manifold (oriented by $\boldsymbol{\tau}_{\boldsymbol{g}}$) and time oriented by $\uparrow$, and $\boldsymbol{\nabla}$ is a linear connection on M such that $\boldsymbol{\nabla g} = 0$. Also, $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and curvature tensors of $\boldsymbol{\nabla}$. If $\mathbf{T}(\boldsymbol{\nabla}) = 0$, then $\mathfrak{M}$ has been called in Chap. 4 a Lorentzian spacetime. The particular Lorentzian spacetime where $M \simeq \mathbb{R}^4$ and such that $\mathbf{R}(\boldsymbol{\nabla}) = 0$ has been called Minkowski spacetime and will be denoted by $\mathcal{M}$. We recall also that when $\boldsymbol{\nabla g} = 0$, $\mathbf{R}(\boldsymbol{\nabla}) \neq 0$ and $\mathbf{T}(\boldsymbol{\nabla}) \neq 0$, $\mathfrak{M}$ has been called a RCST. The particular RCST such that $\mathbf{R}(\boldsymbol{\nabla}) = 0$ has been called a teleparallel spacetime.

In what follows $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ ($P_{\mathrm{SO}^e_{1,3}}(M)$) denotes the principal bundle of oriented *Lorentz tetrads* (*cotetrads*). We assume that the reader is acquainted with the structure of $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ ($P_{\mathrm{SO}^e_{1,3}}(M)$) (as recalled, e.g., in Sect. A.1.1) whose local sections in $U \subset M$ are the time oriented and oriented orthonormal frames (coframes) in U, or simply *frame* (*coframe*) when no confusion arises [6, 9, 19, 20, 23]. Also, since we work with the Clifford bundle of nonhomogeneous differential forms, given a choice of a frame $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ we immediately choose for almost all considerations in this chapter the corresponding dual frame $\Sigma \in \sec P_{\mathrm{SO}^e_{1,3}}(M)$. We recall also that $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M) \simeq P_{\mathrm{SO}^e_{1,3}}(M)$. We already defined the Clifford bundle of differential forms on a structure $(M, \boldsymbol{g})$ on Chap. 4. We recall that the Clifford bundle of differential forms of a Lorentzian manifold $(M, \boldsymbol{g})$ is the bundle of algebras

$$\mathcal{C}\ell(M, \mathtt{g}) = \bigcup_{x \in M} \mathcal{C}\ell(T_x^*M, \mathtt{g}_x), \tag{7.1}$$

where $\mathcal{C}\ell(T_x^*M, \mathtt{g}_x) \simeq \mathbb{R}_{1,3}$ the spacetime algebra introduce in Chap. 3 and $\mathtt{g}$ is the metric of the cotangent bundle such that $\mathtt{g}_x$ is related to $\boldsymbol{g}_x$ as introduced in Chap. 4.

7.1.1 Details of the Bundle Structure of $\mathcal{C}\ell(M, \mathtt{g})$

We mention in Chap. 4 that a Clifford bundle is a vector bundle associated to the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ of orthonormal frames. More specifically we have that

$$\mathcal{C}\ell(M, \mathtt{g}) = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} \mathbb{R}_{1,3} \simeq \boldsymbol{P}_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} \mathbb{R}_{1,3}$$

We now give the details of the bundle structure.

(i) Let $\boldsymbol{\pi}_c : \mathcal{C}\ell(M, \mathtt{g}) \to M$ be the canonical projection of $\mathcal{C}\ell(M, \mathtt{g})$ and let $\{U_\alpha\}$ be an open covering of M. There are trivialization mappings $\boldsymbol{\psi}_i : \boldsymbol{\pi}_c^{-1}(U_i) \to U_i \times \mathbb{R}_{1,3}$ of the form $\boldsymbol{\psi}_i(p) = (\boldsymbol{\pi}_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p))$. If $x \in U_i \cap U_j$ and $p \in \boldsymbol{\pi}_c^{-1}(x)$, then

$$\psi_{i,x}(p) = h_{ij}(x)\psi_{j,x}(p) \tag{7.2}$$

for $h_{ij}(x) \in \mathrm{Aut}(\mathbb{R}_{1,3})$, where $h_{ij} : U_i \cap U_j \to \mathrm{Aut}(\mathbb{R}_{1,3})$ are the transition mappings of $\mathcal{C}\ell(M, \mathtt{g})$. We know from Chap. 3 that every automorphism of $\mathbb{R}_{1,3}$ is *inner*. Then,

$$h_{ij}(x)\psi_{j,x}(p) = g_{ij}(x)\psi_{i,x}(p)g_{ij}(x)^{-1} \tag{7.3}$$

for some $g_{ij}(x) \in \mathbb{R}^{\star}_{1,3}$, the group of invertible elements of $\mathbb{R}_{1,3}$.

(ii) Now, as we learned in Chap. 3, the group $\mathrm{SO}^e_{1,3}$ has a natural extension in the Clifford algebra $\mathbb{R}_{1,3}$. Indeed we know that $\mathbb{R}^{\star}_{1,3}$ acts naturally on $\mathbb{R}_{1,3}$ as an algebra automorphism through its adjoint representation. A set of *lifts* of the transition functions of $\mathcal{C}\ell(M, \mathtt{g})$ is a set of elements $\{g_{ij}\} \subset \mathbb{R}^{\star}_{1,3}$ such that if[1]

$$\begin{aligned} \mathrm{Ad} &: g \mapsto \mathrm{Ad}_g, \\ \mathrm{Ad}_g(a) &= gag^{-1}, \forall a \in \mathbb{R}_{1,3}, \end{aligned} \tag{7.4}$$

then $\mathrm{Ad}_{g_{ij}} = h_{ij}$ in all intersections.

(iii) Also $\sigma = \mathrm{Ad}|_{\mathrm{Spin}^e_{1,3}}$ defines a group homeomorphism $\sigma : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}$ which is onto with kernel $\mathbb{Z}_2$. We have that Ad_{-1} = identity, and so $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3})$ descends to a representation of $\mathrm{SO}^e_{1,3}$. Let us call Ad' this representation, i.e., $\mathrm{Ad}' : \mathrm{SO}^e_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3})$. Then we can write $\mathrm{Ad}'_{\sigma(g)}a = \mathrm{Ad}_g a = gag^{-1}$.

(iv) It is clear then, that the structure group of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ is reducible from $\mathrm{Aut}(\mathbb{R}_{1,3})$ to $\mathrm{SO}^e_{1,3}$. Thus the transition maps of the principal bundle of oriented Lorentz cotetrads $P_{\mathrm{SO}^e_{1,3}}(M)$ can be (through Ad') taken as transition maps for the Clifford bundle. We then have [15]

$$\mathcal{C}\ell(M, \mathtt{g}) = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} \mathbb{R}_{1,3}, \tag{7.5}$$

i.e., the Clifford bundle is an associated vector bundle to the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ of orthonormal Lorentz coframes.

7.1.2 Clifford Fields

Recall that $\mathcal{C}\ell(T^*_x M, \mathtt{g}_x)$ is also a vector space over $\mathbb{R}$ which is isomorphic to the exterior algebra $\bigwedge T^*_x M$ of the cotangent space and $\bigwedge T^*_x M = \bigoplus_{k=0}^4 \bigwedge^k T^*_x M$, where $\bigwedge^k T^*_x M$ is the $\binom{4}{k}$-dimensional space of k-forms. Then, as we already saw in Sect. 4.6.1, there is a natural embedding $\bigwedge T^* M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ [15] and sections of

[1] Recall that $\mathrm{Spin}^e_{1,3} = \{a \in \mathbb{R}^+_{1,3} : a\tilde{a} = 1\} \simeq \mathrm{Sl}(2, \mathbb{C})$ is the universal covering group of the restricted Lorentz group $\mathrm{SO}^e_{1,3}$.

$\mathcal{C}\ell(M, \mathtt{g})$—Clifford fields (Definition 4.111)—can be represented as a sum of non homogeneous differential forms. Let $\{\boldsymbol{e}_{\mathbf{a}}\} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ (the orthonormal frame bundle) be a tetrad basis for $TU \subset TM$, i.e., $\boldsymbol{g}(\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \eta_{\mathbf{ab}} = \mathrm{diag}(1, -1, -1, -1)$ and $(\mathbf{a}, \mathbf{b} = 0, 1, 2, 3)$. Moreover, let $\{\theta^{\mathbf{a}}\} \in \sec P_{\mathrm{SO}^e_{1,3}}(M)$. Then, for each $\mathbf{a} = 0, 1, 2, 3$, $\theta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, i.e., $\{\theta^{\mathbf{b}}\}$ is the dual basis of $\{\boldsymbol{e}_{\mathbf{a}}\}$. Finally, let $\{\theta_{\mathbf{a}}\}$, $\theta_{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ be the *reciprocal basis* of $\{\theta^{\mathbf{b}}\}$, i.e., $\theta_{\mathbf{a}} \cdot \theta^{\mathbf{b}} = \delta^{\mathbf{b}}_{\mathbf{a}}$.

Recall also that the fundamental *Clifford product* is generated by

$$\theta^{\mathbf{ab}} + \theta^{\mathbf{ba}} = 2\eta^{\mathbf{ab}}. \tag{7.6}$$

If $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ is a Clifford field, we have:

$$\mathcal{C} = s + v_{\mathbf{i}}\theta^{\mathbf{i}} + \frac{1}{2!}b_{\mathbf{ij}}\theta^{\mathbf{i}}\theta^{\mathbf{j}} + \frac{1}{3!}t_{\mathbf{ijk}}\theta^{\mathbf{i}}\theta^{\mathbf{j}}\theta^{\mathbf{k}} + p\theta^{\mathbf{5}}, \tag{7.7}$$

where $\theta^{\mathbf{5}} = \theta^{\mathbf{0}}\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{\mathbf{3}}$ is the volume element and

$$s, v_{\mathbf{i}}, b_{\mathbf{ij}}, t_{\mathbf{ijk}}, p \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}). \tag{7.8}$$

7.2 Spin Structure

Definition 7.1 A spin structure on M consists of a principal fibre bundle $\boldsymbol{\pi}_s : \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \to M$ (called the Spin Frame Bundle) with group $\mathrm{Spin}^e_{1,3}$ and a map

$$s : \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \to \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M) \tag{7.9}$$

satisfying the following conditions:

(i) $\boldsymbol{\pi}(s(p)) = \boldsymbol{\pi}_s(p)\ \forall p \in \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M), \pi$ is the projection map of the bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$.

(ii) $s(pu) = s(p)Ad_u$, $\forall p \in \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ and $Ad : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3})$, $Ad_u : \mathbb{R}_{1,3} \ni x \mapsto uxu^{-1} \in \mathbb{R}_{1,3}$.

Definition 7.2 Any section of $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ is called a spin frame field (or simply a spin frame).

Remark 7.3 We define also a Spin *Coframe* Bundle $P_{\mathrm{Spin}^e_{1,3}}(M)$ by substituting in Eq. (7.9) $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ by $P_{\mathrm{SO}^e_{1,3}}(M)$. Of course, $P_{\mathrm{Spin}^e_{1,3}}(M) \simeq \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$. Any section of $P_{\mathrm{Spin}^e_{1,3}}(M)$ is called a spin coframe field (or simply a spin coframe). We shall use the symbol $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ to denoted a spin coframe dual to a spin frame $\boldsymbol{\Xi} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$.

Recall that we learned in Chap. 3 that the minimal left (right) ideals of $\mathbb{R}_{p,q}$ are left (right) modules for $\mathbb{R}_{p,q}$. There, covariant, algebraic and Dirac-Hestenes spinors [when $(p, q) = (1, 3)$] were defined as certain equivalence classes in appropriate sets. We are now interested in defining algebraic Dirac spinor fields and also Dirac-Hestenes spinor fields, on a general RCST, as sections of appropriate vector bundles (spinor bundles) associated to $P_{\mathrm{Spin}^e_{1,3}}(M)$. The compatibility between $P_{\mathrm{Spin}^e_{1,3}}(M)$ and $P_{\mathrm{SO}^e_{1,3}}(M)$, as captured in Definition 7.1 (and taking into account Remark 7.3), is essential for that matter.

It is therefore natural to ask: When does a *spin structure* exist on an oriented manifold M? The answer, which is a classical result [2, 3, 7, 9, 11, 12, 15, 18–21, 23, 24], is that a necessary and sufficient condition for the existence of a spin structure on M (given the already assumed properties of M) is that the second Stiefel-Whitney class $w_2(M)$ of M is trivial. Moreover, when a spin structure exists, one can show that it is unique (modulo isomorphisms), if and only if, $H^1(M, \mathbb{Z}_2)$, the de-Rham cohomology group with values in $\mathbb{Z}_2$ is trivial. We now, introduce without proof a theorem that is crucial for our theory.

Theorem 7.4 *For a Lorentz manifold* $(M, \boldsymbol{g})$*, a spin structure exists if and only if* $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ *is a trivial bundle.*

Proof See Geroch [11]. ■

Remark 7.5 Recall that global sections $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ are Lorentz frames and global sections $\boldsymbol{\Xi} \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ are *spin frames*. We recall from Sect. A.1.3 that each $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ is a basis for TM, which is completely specified once we give an element of the Lorentz group for each $x \in M$ and fix a fiducial frame. Each $\boldsymbol{\Xi} \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ is also a basis for TM and is completely identified once give an element of the $\mathrm{Spin}^e_{1,3}$ for each $x \in M$ and fix a fiducial frame. Note that two ordered basis for TM when considered as spin frames are to be considered *different* even if consisting of the same vector fields, which are related by multiples of a 2π rotation. Also, two ordered basis for TM are considered *equal* when considered as spin frames, if they consist of the same vector fields related by multiples of a 4π rotation. Although we recognize that this mathematical construction seems at first sight impossible of experimental detection, Aharonov and Susskind [1] warrants that with clever experiments the spinor structure can be detected.

Remark 7.6 Recall that a principal bundle is trivial, if and only if, it admits a global section. Therefore, Geroch's result says that a (non-compact) spacetime admits a spin structure, if and only if, it admits a (globally defined) Lorentz frame. In fact, it is possible to replace $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ by $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ in Remark 7.4 (see footnote 25 in [11]). In this way, when a (non-compact) spacetime admits a spin structure, the bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ is trivial and, therefore, every bundle associated to it is also trivial. This result is indeed a very important one, because it says to us that the real spacetime of our universe (that, of course, is inhabited by several different types of spinor fields) must have a topology that admits a global tetrad field, which is defined only modulus a local Lorentz transformation. A dual cotetrad has been associated to the gravitational field in Chap. 4 (see more details in Chaps. 9 and 11), where we wrote wave equations for them. In a certain sense that cotetrad field is a representation of the substance of physical spacetime.

Definition 7.7 A oriented manifold endowed with a spin structure will be called a spin manifold.

Exercise 7.8 Show that Minkowski and de Sitter spacetimes are spin manifolds.

7.3 Spinor Bundles and Spinor Fields

We now present the most usual definitions of spinor bundles appearing in the literature[2] and next we find appropriate vector bundles such that particular sections are LIASF (Definition 7.16) or DHSF (Definition 7.27)

Definition 7.9 A real (left) spinor bundle for M is a vector bundle

$$S(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbf{M} \tag{7.10}$$

where $\mathbf{M}$ is a left module for $\mathbb{R}_{1,3}$ and μ_l is a representation of $\mathrm{Spin}^e_{1,3}$ on $End(\mathbf{M})$ given by left multiplication by elements of $\mathrm{Spin}^e_{1,3}$.

Definition 7.10 The dual bundle $S^\star(M, \mathtt{g})$ is a real (right) spinor bundle

$$S^\star(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_r} \mathbf{M}^\star \tag{7.11}$$

where $\mathbf{M}^\star$ is a right module for $\mathbb{R}_{1,3}$ and μ_r is a representation $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M})$ given by right multiplication by (inverse) elements of $\mathrm{Spin}^e_{1,3}$. By right multiplication we mean that given $a \in \boldsymbol{M}^\star$, $\mu_r(u)a = au^{-1}$, then

$$\mu_r(u\acute{u})a = \psi(u\acute{u})^{-1} = \psi\acute{u}^{-1}u^{-1} = \mu_r(u)\mu_r(\acute{u})a. \tag{7.12}$$

Definition 7.11 A complex spinor bundle for M is a vector bundle

$$S_c(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}_c \tag{7.13}$$

where $\mathbf{M}_c$ is a complex left module for $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M}_c)$ given by left multiplication by elements of $\mathrm{Spin}^e_{1,3}$.

[2] We recall that there are some other (equivalent) definitions of spinor bundles that we are not going to introduce in this book as, e.g., the one given in [5] in terms of mappings from $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}$ to some appropriate vector space.

Definition 7.12 The dual complex spinor bundle for M is a vector bundle

$$S_c^{\star}(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}_c^{\star} \tag{7.14}$$

where $\mathbf{M}_c^{\star}$ is a complex right module for $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, and where μ_c is a representation of $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathbf{M}_c)$ given by right multiplication by (inverse) elements of $\mathrm{Spin}^e_{1,3}$.

Taking, e.g., $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Spin}^e_{1,3} \cong \mathrm{Sl}(2, \mathbb{C})$ in $\mathrm{End}(\mathbb{C}^4)$, we immediately recognize the usual definition of the (Dirac) covariant spinor bundle of M, as given, e.g., in [6, 9, 20].

7.3.1 Left and Right Spin-Clifford Bundles

We saw in Chap. 3 that besides the ideal $I = \mathbb{R}_{1,3}\frac{1}{2}(1 + E_0)$, other ideals exist in $\mathbb{R}_{1,3}$ that are only *algebraically* equivalent to this one. In order to capture all possibilities we recall that $\mathbb{R}_{1,3}$ can be considered as a module over itself by left (or right) multiplication. We are thus led to the

Definition 7.13 The left real spin-Clifford bundle of M is the vector bundle

$$\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_l \mathbb{R}_{1,3} \tag{7.15}$$

where l is the representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$. Sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ are called left spin-Clifford fields.

Remark 7.14 $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ is a 'principal $\mathbb{R}_{1,3}$- bundle', i.e., it admits a free action of $\mathbb{R}_{1,3}$ on the right [15], which is denoted by R_g, $g \in \mathbb{R}_{1,3}$. This will be discussed in detail below.

Remark 7.15 There is a *natural* embedding $P_{\mathrm{Spin}^e_{1,3}}(M) \hookrightarrow \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ which comes from the embedding $\mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}^0_{1,3}$. Hence (as we shall see in more details below), every real left spinor bundle (see Definition 7.13) for M can be captured from $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, which is a vector bundle very different from $\mathcal{C}\ell(M, \mathtt{g})$. Their relation is presented below, but before that we give the

Definition 7.16 Let $I(M, \mathtt{g})$ be a subbundle of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ such that there exists a primitive idempotent e of $\mathbb{R}_{1,3}$ with

$$R_{\mathrm{e}}\Psi = \Psi \tag{7.16}$$

for all $\Psi \in \sec I(M,\mathsf{g}) \hookrightarrow \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$. Then, $I(M,\mathsf{g})$ is called a subbundle of ideal left algebraic spinor fields. *Any* $\Psi \in \sec I(M,\mathsf{g}) \hookrightarrow \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$ is called a left ideal algebraic spinor field (*LIASF*).

$I(M,\mathsf{g})$ can be thought of as a real spinor bundle for M such that $\mathbf{M}$ in Eq. (7.13) is a minimal left ideal of $\mathbb{R}_{1,3}$.

Definition 7.17 Two subbundles $I(M,\mathsf{g})$ and $I'(M,\mathsf{g})$ of *LIASF are said to be geometrically equivalent if the idempotents* $\mathrm{e}, \mathrm{e}' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \mathrm{Spin}^{e}_{1,3}$, i.e., $\mathrm{e}' = u\mathrm{e}u^{-1}$.

Definition 7.18 The *right* real spin-Clifford bundle of M is the vector bundle

$$\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g}) = P_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{r} \mathbb{R}_{1,3}. \tag{7.17}$$

Sections of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$ are called right spin-Clifford fields.

In Eq. (7.17) r refers to a representation of $\mathrm{Spin}^{e}_{1,3}$ on $\mathbb{R}_{1,3}$, given by $r(a)x = xa^{-1}$. As in the case for the left real spin-Clifford bundle, there is a *natural* embedding $P_{\mathrm{Spin}^{e}_{1,3}}(M) \hookrightarrow \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$ which comes from the embedding $\mathrm{Spin}^{e}_{1,3} \hookrightarrow \mathbb{R}^{0}_{1,3}$. There exists also a natural left L_a action of $a \in \mathbb{R}_{1,3}$ on $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$. This will be proved in Sect. 7.4.

Definition 7.19 Let $I^{\star}(M,\mathsf{g})$ be a subbundle of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$ such that there exists a primitive idempotent element e of $\mathbb{R}_{1,3}$ with

$$L_{\mathrm{e}}\Psi = \Psi \tag{7.18}$$

for any $\Psi \in \sec I^{\star}(M,\mathsf{g}) \hookrightarrow \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})\}$. Then, $I^{\star}(M,\mathsf{g})$ is called a subbundle of right ideal algebraic spinor fields. Any $\Psi \in \sec I^{\star}(M,\mathsf{g}) \hookrightarrow \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M,\mathsf{g})$ is called a *RIASF*. $I^{\star}(M,\mathsf{g})$ can be thought of as a real spinor bundle for M such that $\mathbf{M}^{\star}$ in Eq. (7.14) is a minimal right ideal of $\mathbb{R}_{1,3}$.

Definition 7.20 Two subbundles $I^{*}(M,\mathsf{g})$ and $I^{*\prime}(M,\mathsf{g})$ of *RIASF are said to be geometrically equivalent if the idempotents* $\mathrm{e},\mathrm{e}' \in \mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u \in \mathrm{Spin}^{e}_{1,3}$, i.e., $\mathrm{e}' = u\mathrm{e}u^{-1}$.

Proposition 7.21 *In a spin manifold, we have*

$$\mathcal{C}\ell(M,\mathsf{g}) = P_{\mathrm{Spin}^{e}_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}.$$

Proof Remember once again that the representation

$$\mathrm{Ad} : \mathrm{Spin}^{e}_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3}) \quad \mathrm{Ad}_{u}a = uau^{-1} \qquad u \in \mathrm{Spin}^{e}_{1,3}$$

is such that Ad_{-1} = identity and so Ad descends to a representation Ad′ of $\mathrm{SO}^e_{1,3}$ which we considered above. It follows that when $P_{\mathrm{Spin}^e_{1,3}}(M)$ exists $\mathcal{C}\ell(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$. ■

7.3.2 Bundle of Modules over a Bundle of Algebras

Proposition 7.22 *$S(M, \mathtt{g})$ (or $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) is a bundle of (left)* modules *over the bundle of algebras $\mathcal{C}\ell(M, g)$. In particular, the sections of the spinor bundle ($S(M, \mathtt{g})$ or $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) are a module over the sections of the Clifford bundle.*

Proof For the proof, see [15]. ■

Corollary 7.23 *Let $\Phi, \Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and $\Psi \neq 0$. Then there exists $\psi \in \sec \mathcal{C}\ell(M, \mathtt{g})$ such that*

$$\Psi = \psi \Phi. \tag{7.19}$$

Proof It is an immediate consequence of Proposition 7.22. ■

So, the corollary allows us to identify a *correspondence* between some sections of $\mathcal{C}\ell(M, \mathtt{g})$ and some sections of $I(M, \mathtt{g})$ or $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ once we fix a section on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$. This and other correspondences are essential for our theory of Dirac-Hestenes spinor fields. Once we clarified what is the meaning of a bundle of modules $S(M, \mathtt{g})$ over a bundle of algebras $\mathcal{C}\ell(M, \mathtt{g})$, we can give the following

Definition 7.24 Two real left spinor bundles are equivalent, if and only if, they are equivalent as bundles of $\mathcal{C}\ell(M, \mathtt{g})$ modules.

Remark 7.25 Of course, geometrically equivalently real left spinor bundles are equivalent.

Remark 7.26 In what follows we denote the complexified left spin Clifford bundle and the complexified right spin Clifford bundle by

$$\begin{aligned}
\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M) &= P_{\mathrm{Spin}^e_{1,3}}(M) \times_l \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\mathrm{Spin}^e_{1,3}}(M) \times_r \mathbb{R}_{4,1}, \\
\mathbb{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M) &= P_{\mathrm{Spin}^e_{1,3}}(M) \times_r \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\mathrm{Spin}^e_{1,3}}(M) \times_r \mathbb{R}_{4,1}.
\end{aligned} \tag{7.20}$$

7.3.3 Dirac-Hestenes Spinor Fields (DHSF)

Let $\mathbf{e^a}$, $\mathbf{a} = 0, 1, 2, 3$ be the canonical basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ which generates the algebra $\mathbb{R}_{1,3}$. They satisfy the basic relation $\mathbf{e^a e^b} + \mathbf{e^b e^a} = 2\eta^{\mathbf{ab}}$. We recall that we showed in Chap. 3 that

$$\mathrm{e} = \frac{1}{2}(1 + \mathbf{e}^0) \in \mathbb{R}_{1,3} \tag{7.21}$$

is a primitive idempotent of $\mathbb{R}_{1,3}$ and

$$\mathrm{f} = \frac{1}{2}(1 + \mathbf{e}^0)\frac{1}{2}(1 + \mathrm{i}\,\mathbf{e}^2\mathbf{e}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3} \tag{7.22}$$

is a primitive idempotent of $\mathbb{C} \otimes \mathbb{R}_{1,3}$. Now, let $\mathbf{I} = \mathbb{R}_{1,3}\mathrm{e}$ and $\mathbf{I}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}_{1,3}\mathrm{f}$ be respectively the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C} \otimes \mathbb{R}_{1,3}$ generated by e and f. Let $\Phi = \Phi\mathrm{e} \in \mathbf{I}$ and $\boldsymbol{\Psi} = \boldsymbol{\Psi}\mathrm{f} \in \mathbf{I}_{\mathbb{C}}$. Then, any $\phi \in \mathbf{I}$ can be written as

$$\Phi = \boldsymbol{\Phi}\mathrm{e} \tag{7.23}$$

with $\boldsymbol{\Phi} \in \mathbb{R}^0_{1,3}$. Analogously, any $\Phi \in \mathbf{I}_{\mathbb{C}}$ can be written as

$$\Phi = \boldsymbol{\Phi}\mathrm{e}\frac{1}{2}(1 + \mathrm{i}\mathbf{e}^2\mathbf{e}^1), \tag{7.24}$$

with $\boldsymbol{\Phi} \in \mathbb{R}^0_{1,3}$.

Recall moreover that $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, where $\mathbb{C}(4)$ is the algebra of the 4×4 complexes matrices. We can verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.25}$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of f. In that way as proved in Chap. 3 there is a bijection between column spinors, i.e., elements of $\mathbb{C}^4$ (the complex 4-dimensional vector space) and the elements of $\mathbf{I}_{\mathbb{C}}$. All that, plus the definitions of the left real and complex spin bundles and the subbundle $I(M, \mathtt{g})$ suggests the

Definition 7.27 Let $\Phi \in \sec I(M, \mathtt{g}) \hookrightarrow \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ be as in Definition 7.16, i.e.,

$$R_{\mathrm{e}}\Phi = \Phi\mathrm{e} = \Phi, \mathrm{e}^2 = \mathrm{e} = \frac{1}{2}(1+\mathbf{e}^{\mathbf{0}}) \in \mathbb{R}_{1,3}. \tag{7.26}$$

A *DHSF* associated with Φ is an *even* section $\boldsymbol{\Phi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ such that

$$\Phi = \boldsymbol{\Phi}\mathrm{e}. \tag{7.27}$$

Remark 7.28 An equivalent definition of a *DHSF* is the following. Let $\Psi \in \sec \mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ be such that

$$R_{\mathrm{f}}\Psi = \Psi\mathrm{f} = \Psi,\ \mathrm{f}^2 = \mathrm{f} = \frac{1}{2}(\mathbf{1} + \mathbf{e^0})\frac{1}{2}(\mathbf{1} + \mathbf{i}\mathbf{e}^2\mathbf{e}^1) \in \mathbb{C}\otimes\mathbb{R}_{1,3}. \tag{7.28}$$

Then, a *DHSF* associated to Ψ is an even section $\mathbf{\Phi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ such that

$$\Psi = \mathbf{\Phi}\mathrm{f}. \tag{7.29}$$

Remark 7.29 In what follows, when we refer to a *DHSF* $\mathbf{\Phi}$ we omit for simplicity the wording associated with Φ (or Ψ). It is very important to observe that a *DHSF* is not a sum of even multivector fields although, under a local trivialization, $\mathbf{\Phi}$ which is a section of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ is for each $x \in M$ mapped on an even element[3] of $\mathbb{R}_{1,3}$. We emphasize that a *DHSF* is a particular section of a spinor bundle, not of the Clifford bundle. However, we show below a very important fact, namely that *DHSFs* have representatives in the Clifford bundle.

7.4 Natural Actions on Some Vector Bundles Associated with $P_{\mathrm{Spin}^e_{1,3}}(M)$

Recall that, when M is a spin manifold:

(i) The elements of $\mathcal{C}\ell(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{Ad} \mathbb{R}_{1,3}$ are equivalence classes $[(p, a)]$ of pairs (p, a), where $p \in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1},\ a' = uau^{-1}$, for some $u \in \mathrm{Spin}^e_{1,3}$;

(ii) The elements of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ are equivalence classes of pairs (p, a), where $p \in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1},\ a' = ua$, for some $u \in \mathrm{Spin}^e_{1,3}$;

(iii) The elements of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ are equivalence classes of pairs (p, a), where $p \in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a \in \mathbb{R}_{1,3}$ and $(p, a) \sim (p', a') \Leftrightarrow p' = pu^{-1},\ a' = au^{-1}$, for some $u \in \mathrm{Spin}^e_{1,3}$.

In this way, it is possible to define the following natural actions on these associated bundles.

Proposition 7.30 *There is a natural right action of* $\mathbb{R}_{1,3}$ *on* $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ *and a natural left action of* $\mathbb{R}_{1,3}$ *on* $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$.

[3]Note that it is meaningful to speak about even (or odd) elements in $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ since $\mathrm{Spin}^e_{1,3} \subseteq \mathbb{R}^0_{1,3}$.

Proof Given $b \in \mathbb{R}_{1,3}$ and $\alpha \in \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$, select a representative (p, a) for α and define $\alpha b := [(p, ab)]$. If another representative (pu^{-1}, ua) is chosen for α, we have $(pu^{-1}, uab) \sim (p, ab)$ and thus αb is a well defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$. ■

Denote the space of $\mathbb{R}_{1,3}$-valued smooth functions on M by $\mathcal{F}(M, \mathbb{R}_{1,3})$. Then, the above proposition immediately yields the following

Corollary 7.31 *There is a natural right action of* $\mathcal{F}(M, \mathbb{R}_{1,3})$ *on the sections of* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *and a natural left action of* $\mathcal{F}(M, \mathbb{R}_{1,3})$ *on the sections of* $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$.

Proposition 7.32 *There is a natural left action of* $\sec \mathcal{C}\ell(M, \mathtt{g})$ *on the sections of* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *and a natural right action of* $\sec \mathcal{C}\ell(M, \mathtt{g})$ *on sections of* $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$.

Proof Given $\alpha \in \sec \mathcal{C}\ell(M, \mathtt{g})$ and $\beta \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p, ab)] \in \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$. If alternative representatives (pu^{-1}, uau^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have

$$(pu^{-1}, uau^{-1}ub) = (pu^{-1}, uab) \sim (p, ab)$$

and thus $(\alpha\beta)(x)$ is a well defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$. ■

Proposition 7.33 *There is a natural pairing*

$$\sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \times \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \to \sec \mathcal{C}\ell(M, \mathtt{g}).$$

Proof Given $\alpha \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ and $\beta \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := [(p, ab)] \in \mathcal{C}\ell(M, \mathtt{g})$. If alternative representatives (pu^{-1}, ua) and (pu^{-1}, bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$, we have $(pu^{-1}, uabu^{-1}) \sim (p, ab)$ and thus $(\alpha\beta)(x)$ is a well defined element of $\mathcal{C}\ell(M, \mathtt{g})$. ■

Proposition 7.34 *There is a natural pairing*

$$\sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \times \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \to \mathcal{F}(M, \mathbb{R}_{1,3}).$$

Proof Given $\alpha \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ and $\beta \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$, select representatives (p, a) for $\alpha(x)$ and (p, b) for $\beta(x)$ (with $p \in \pi^{-1}(x)$) and define $(\alpha\beta)(x) := ab \in \mathbb{R}_{1,3}$. If alternative representatives (pu^{-1}, au^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have $au^{-1}ub = ab$ and thus $(\alpha\beta)(x)$ is a well defined element of $\mathbb{R}_{1,3}$. ■

7.4.1 Fiducial Sections Associated with a Spin Frame

We start by exploring the possibility of defining "unit sections" on the various vector bundles associated with the principal bundle $P_{\mathrm{Spin}^e_{1,3}}(M)$.

Let

$$\Phi_i : \boldsymbol{\pi}^{-1}(U_i) \to U_i \times \mathrm{Spin}^e_{1,3}, \quad \Phi_j : \boldsymbol{\pi}^{-1}(U_j) \to U_j \times \mathrm{Spin}^e_{1,3}$$

be two local trivializations for $P_{\mathrm{Spin}^e_{1,3}}(M)$, with

$$\Phi_i(u) = (\pi(u) = x, \phi_{i,x}(u)), \quad \Phi_j(u) = (\pi(u) = x, \phi_{j,x}(u)).$$

Recall that the transition function on $g_{ij} : U_i \cap U_j \to \mathrm{Spin}^e_{1,3}$ is then given by

$$g_{ij}(x) = \phi_{i,x} \circ \phi_{j,x}{}^{-1},$$

which does not depend on u.

Proposition 7.35 *$\mathcal{C}\ell(M, \mathtt{g})$ has a naturally defined global unit section.*

Proof For the associated bundle $\mathcal{C}\ell(M, \mathtt{g})$, the transition functions corresponding to local trivializations

$$\Psi_i : \boldsymbol{\pi}_c^{-1}(U_i) \to U_i \times \mathbb{R}_{1,3}, \quad \Psi_j : \boldsymbol{\pi}_c^{-1}(U_j) \to U_j \times \mathbb{R}_{1,3}, \tag{7.30}$$

are given by $h_{ij}(x) = \mathrm{Ad}_{g_{ij}(x)}$. Define the local sections

$$\mathbf{1}_i(x) = \Psi_i^{-1}(x, 1), \quad \mathbf{1}_j(x) = \Psi_j^{-1}(x, 1), \tag{7.31}$$

where 1 is the unit element of $\mathbb{R}_{1,3}$. Since $h_{ij}(x) \cdot 1 = \mathrm{Ad}_{g_{ij}(x)}(1) = g_{ij}(x) 1 g_{ij}(x)^{-1} = 1$, we see that the expressions above uniquely define a global section $\mathbf{1}$ for $\mathcal{C}\ell(M, \mathtt{g})$ with $\mathbf{1}|_{U_i} = \mathbf{1}_i$. ■

Remark 7.36 It is clear that such a result can be immediately generalized for the Clifford bundle $\mathcal{C}\ell_{p,q}(M, \mathtt{g})$, of any n-dimensional manifold endowed with a metric of arbitrary signature (p, q) (where $n = p + q$). Now, we observe also that the left (and also the right) spin-Clifford bundle can be generalized in an obvious way for any spin manifold of arbitrary finite dimension $n = p + q$, with a metric of arbitrary signature (p, q). However, another important difference between $\mathcal{C}\ell(M, \mathtt{g})$ and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$ or $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) is that these latter bundles only admit a global unit section if they are *trivial*.

Proposition 7.37 *There exists an unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$ (and also on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$), if and only if, $\mathbf{P}_{\mathrm{Spin}^e_{p,q}}(M)$ is trivial.*

Proof We show the necessity for the case of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$,[4] the sufficiency is trivial. For $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$, the transition functions corresponding to local trivializations

$$\zeta_i : \boldsymbol{\pi}_{sc}^{-1}(U_i) \to U_i \times \mathbb{R}_{p,q}, \quad \zeta_j : \boldsymbol{\pi}_{sc}^{-1}(U_j) \to U_j \times \mathbb{R}_{p,q}, \tag{7.32}$$

are given by $k_{ij}(x) = R_{g_{ij}(x)}$, with $R_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, x \mapsto xa^{-1}$. Let 1 be the unit element of $\mathbb{R}_{1,3}$. An unit section in $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$—if it exists—is written in terms of these two local trivializations as

$$\mathbf{1}^r_i(x) = \zeta_i^{-1}(x,1), \quad \mathbf{1}^r_j(x) = \zeta_j^{-1}(x,1), \tag{7.33}$$

and we must have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x)\ \forall x \in U_i \cap U_j$. As $\zeta_i(\mathbf{1}^r_i(x)) = (x,1) = \zeta_j(\mathbf{1}^r_j(x))$, we have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x) \Leftrightarrow 1 = k_{ij}(x)\cdot 1 \Leftrightarrow 1 = 1g_{ij}(x)^{-1} \Leftrightarrow g_{ij}(x) = 1$. This proves the proposition. ■

Remark 7.38 For general spin manifolds, the bundle $P_{\mathrm{Spin}^e_{p,q}}(M)$ is not necessarily trivial for arbitrary (p,q), but Geroch's theorem (Remark 7.4) warrants that, for the special case $(p,q) = (1,3)$ with M non-compact, $P_{\mathrm{Spin}^e_{1,3}}(M)$ is trivial. The above proposition implies that $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ and also $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$ have global "unit sections". It is most important to note, however, that each different choice of a (global) trivialization ζ_i on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ (respectively $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M,\mathtt{g})$) induces a different global unit section $\mathbf{1}^r_i$ (respectively $\mathbf{1}^l_i$). Therefore, even in this case there is no canonical unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ (respectively on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$).

By Remark 7.6, when the (non-compact) spacetime M is a spin manifold, the bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ admits global sections. With this in mind, let us fix a *spin frame* $\boldsymbol{\Xi}$ and its dual spin *coframe* Ξ for M. This induces a global trivialization for $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ and of course of $P_{\mathrm{Spin}^e_{1,3}}(M)$. We denote the trivialization of $P_{\mathrm{Spin}^e_{1,3}}(M)$ by

$$\Phi_\Xi : P_{\mathrm{Spin}^e_{1,3}}(M) \to M \times \mathrm{Spin}^e_{1,3},$$

with $\Phi_\Xi^{-1}(x,1) = \Xi(x)$. We recall that a spin coframe $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ (Remark A.17) can also be used to induce certain fiducial global sections on the various vector bundles associated to $P_{\mathrm{Spin}^e_{1,3}}(M)$:

(i) $\mathcal{C}\ell(M,\mathtt{g})$

Let $\{\mathbf{e_a}\}$ be a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$ (which can be thought of as the *canonical* basis of $\mathbb{R}^{1,3}$). We define basis sections in $\mathcal{C}\ell(M,\mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{Ad} \mathbb{R}_{1,3}$

[4]The proof for the case of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M)$ is analogous.

by $\theta_{\mathbf{a}}(x) = [(\Xi(x), \mathbf{e_a})]$. Of course, this induces a *multiform* basis $\{\theta_I(x)\}$ for each $x \in M$. Note that a more precise notation for $\theta_{\mathbf{a}}$ would be, for instance, $\theta_{\mathbf{a}}^{(\Xi)}$.

(ii) $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$

Let $\mathbf{1}^l_\Xi$ be a section of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ defined by $\mathbf{1}^l_\Xi(x) = [(\Xi(x), 1)]$. Then the natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ leads to $\mathbf{1}^l_\Xi(x)a = [(\Xi(x), a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from Corollary 7.31 that an arbitrary section α of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ can be written as $\alpha = \mathbf{1}^l_\Xi f$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

(iii) $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$

Let $\mathbf{1}^r_\Xi$ be a section of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ defined by $\mathbf{1}^r_\Xi(x) = [(\Xi(x), 1)]$. Then the natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ leads to $a\mathbf{1}^r_\Xi(x) = [(\Xi(x), a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from Corollary 7.31 that an arbitrary section α of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ can be written as $\alpha = f\mathbf{1}^r_\Xi$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

Now recall (Definition 7.1) that a spin structure on M is a 2–1 bundle map $s : P_{\mathrm{Spin}^e_{1,3}}(M) \to \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ such that $s(pu) = s(p)Ad_u$, $\forall p \in \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$, $u \in \mathrm{Spin}^e_{1,3}$, where $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}$, $\mathrm{Ad}_u : x \mapsto uxu^{-1}$. We see that the specification of the global section in the case (i) above is compatible with the Lorentz coframe $\{\theta_a\} = s(\Xi)$ assigned by s. More precisely, for each $x \in M$, the element $s(\Xi(x)) \in P_{\mathrm{SO}^e_{1,3}}(M)$ is to be regarded as proper isometry $s(\Xi(x)) : \mathbb{R}^{1,3} \to T_xM$, so that $\theta_{\mathbf{a}}(x) := s(p) \cdot \mathbf{e_a}$ yields a Lorentz coframe $\{\theta_{\mathbf{a}}\}$ on M, which we denoted by $s(\Xi)$. On the other hand, $\mathcal{C}\ell(M, \mathtt{g})$ is isomorphic to $P_{\mathrm{Spin}^e_{1,3}}(M) \times_{Ad} \mathbb{R}_{1,3}$, and we can always arrange things so that $\theta_{\mathbf{a}}(x)$ is represented in this bundle as $\theta_{\mathbf{a}}(x) = [(\Xi(x), \mathbf{e_a})]$. In fact, all we have to do is to verify that this identification is covariant[5] under a change of coframes. To see that, let Ξ' be another section of $P_{\mathrm{Spin}^e_{1,3}}(M)$, i.e., another spin coframe on M. From the principal bundle structure of $P_{\mathrm{Spin}^e_{1,3}}(M)$, we know that, for each $x \in M$, there exists (an unique) $u(x) \in \mathrm{Spin}^e_{1,3}$ such that $\Xi'(x) = \Xi(x)u^{-1}(x)$. If we define, as above, $\theta'_{\mathbf{a}}(x) = s(\Xi'(x)) \cdot \mathbf{e_a}$, then $\theta'_{\mathbf{a}}(x) = s(\Xi(x)u^{-1}(x)) \cdot \mathbf{e_a} = s(\Xi(x))Ad_{u(x)} \cdot \mathbf{e_a} = \left[(\Xi(x), Ad_{u(x)} \cdot \mathbf{e_a})\right] = \left[(\Xi(x)u^{-1}(x),\ \mathbf{e_a})\right] = [(\Xi'(x), \mathbf{e_a})]$, which proves our claim.

[5] Being covariant means here that the identification does not depend on the choice of the coframe used in the computations.

Proposition 7.39

$$\begin{array}{ll}\text{(i)} & \mathbf{e_a} = \mathbf{1}^r_\Xi(x)\theta_\mathbf{a}(x)\mathbf{1}^l_\Xi(x), \forall x \in M,\\ \text{(ii)} & \mathbf{1}^l_\Xi \mathbf{1}^r_\Xi = 1, \textit{global unity section of } \mathcal{C}\ell(M,g),\\ \text{(iii)} & \mathbf{1}^r_\Xi \mathbf{1}^l_\Xi = 1 \in \mathbb{R}_{1,3}.\end{array}$$

Proof This follows from the form of the various actions defined in Propositions 7.30–7.34. For example, for each $x \in M$, we have $\mathbf{1}^r_\Xi(x)\theta_\mathbf{a}(x) = [(\Xi(x), 1\mathbf{e_a})] = [(\Xi(x), \mathbf{e_a})]$, a section of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathfrak{g})$ (from Proposition 7.32). Then, it follows from Proposition 7.34 that $\mathbf{1}^r_\Xi(x)\theta_\mathbf{a}(x)\mathbf{1}^l_\Xi(x) = \mathbf{e_a}1 = \mathbf{e_a}$, $\forall x \in M$. ■

Let us now consider how the various global sections defined above transform when the spin coframe Ξ is changed. Let $\Xi' \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ be another spin coframe with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \mathrm{Spin}^e_{1,3}$. Let $\theta_\mathbf{a}$, $\mathbf{1}^r_\Xi$, $\mathbf{1}^l_\Xi$ and $\theta'_\mathbf{a}$, $\mathbf{1}^r_{\Xi'}$, $\mathbf{1}^l_{\Xi'}$ be the global sections respectively defined by Ξ and Ξ' (as above). We then have

Proposition 7.40 *Let Ξ and Ξ' be two spin coframes (sections of $P_{\mathrm{Spin}^e_{1,3}}(M)$) related by $\Xi' = \Xi u^{-1}$, where $u : M \to \mathrm{Spin}^e_{1,3}$. Then*

$$\begin{aligned}\text{(i) } \theta'_\mathbf{a} &= U\theta_\mathbf{a}U^{-1}\\ \text{(ii) } \mathbf{1}^l_{\Xi'} &= \mathbf{1}^l_\Xi u = U\mathbf{1}^l_\Xi,\\ \text{(iii) } \mathbf{1}^r_{\Xi'} &= u^{-1}\mathbf{1}^r_\Xi = \mathbf{1}^r_\Xi U^{-1},\end{aligned} \tag{7.34}$$

where $U \in \sec \mathcal{C}\ell(M, \mathfrak{g})$ is the Clifford field associated to u by $U(x) = [(\Xi(x), u(x))]$. Also, in (ii) *and* (iii)*, u and u^{-1} respectively act on $\mathbf{1}^l_\Xi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathfrak{g})$ and $\mathbf{1}^r_\Xi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathfrak{g})$ according to Proposition 7.31.*

Proof (i) We have

$$\begin{aligned}\theta'_\mathbf{a}(x) &= [(\Xi'(x), \mathbf{e_a}] = [(\Xi(x)u(x), \mathbf{e_a})]\\ &= [(\Xi(x), u(x)\mathbf{e_a}u^{-1}(x))]\\ &= [(\Xi(x), u(x))][(\Xi(x), \mathbf{e_a})][(\Xi(x), u^{-1}(x))]\\ &= U(x)\theta_\mathbf{a}(x)U^{-1}(x).\end{aligned} \tag{7.35}$$

(iii) It follows from Proposition 7.32 that

$$\begin{aligned}\mathbf{1}^r_{\Xi'}(x) &= \big[(\Xi'(x), 1)\big] = [(\Xi(x)u(x), 1)]\\ &= \big[(\Xi(x), 1u(x)^{-1})\big] = \big[(\Xi(x), u^{-1}(x))\big] = u^{-1}(x)\mathbf{1}^r_\Xi(x),\end{aligned} \tag{7.36}$$

where in the last step we used Proposition 7.31 and the fact that $\mathbf{1}^r_\Xi(x) = [(\Xi(x), 1)]$.

To prove that $\mathbf{1}^r_{\Xi'} = \mathbf{1}^r_{\Xi} U^{-1}$ we observe that

$$\begin{aligned} u^{-1}(x)\mathbf{1}^r_{\Xi}(x) &= \left[(\Xi(x), u(x)^{-1})\right] \\ &= \left[(\Xi(x), 1u(x)^{-1})\right] = [(\Xi(x), 1)]\left[(\Xi(x), u(x)^{-1})\right] \\ &= \mathbf{1}^r_{\Xi}(x)U^{-1}(x), \end{aligned} \tag{7.37}$$

for all $x \in M$. It is important to note that in the last step we have a product between an element of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ (i.e. $[(\Xi(x), 1)]$) and an element of $\mathcal{C}\ell(M, \mathtt{g})$ (i.e. $\left[(\Xi(x), u(x)^{-1})\right]$). ■

We emphasize that the right unit sections associated with spin coframes are *not* constant in any covariant way. To understand the meaning of this statement we now investigate the theory of the covariant derivatives of Clifford and spinor fields.

7.5 Representatives of DHSF in the Clifford Bundle

Let $\mathbf{\Psi}^l$, a section of $\mathcal{C}\ell^{0l}_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ be a Dirac-Hestenes spinor field associated to Ψ^l a (ideal) section of $I(M, \mathtt{g})$ (Definition 7.27). Recalling that $\mathbf{\Psi}^l = [(\Xi, \mathbf{\Psi}^l_{\Xi})]$ and Corollary 7.23 we give the

Definition 7.41 For a given spin frame Ξ, a representative of $\mathbf{\Psi}^l$ in the Clifford bundle is the even section ψ_{Ξ} of $\mathcal{C}\ell(M, \mathtt{g})$ such that

$$\mathbf{\Psi}^l := \psi_{\Xi}\mathbf{1}^l_{\Xi} \tag{7.38}$$

Recalling Proposition 7.39 which says that $\mathbf{1}^l_{\Xi}\mathbf{1}^r_{\Xi} = 1$, the global unity section of $\mathcal{C}\ell(M, \mathtt{g})$ we have that

$$\psi_{\Xi} = \mathbf{\Psi}^l\mathbf{1}^r_{\Xi}. \tag{7.39}$$

Given another spin frame $\Xi' = \Xi u$ the representative in $\mathcal{C}\ell(M, \mathtt{g})$ of $\mathbf{\Psi}^l$ is the even section ψ_{Ξ} of $\mathcal{C}\ell(M, \mathtt{g})$ such that

$$\psi_{\Xi'} = \mathbf{\Psi}^l\mathbf{1}^r_{\Xi'}. \tag{7.40}$$

To find the relation between ψ_{Ξ} and $\psi_{\Xi'}$ we recall Eq. (7.34) and write

$$\begin{aligned} \psi_{\Xi'}(x) &= \mathbf{\Psi}^l(x)\mathbf{1}^r_{\Xi'}(x) = [(\Xi(x), \mathbf{\Psi}^r_{\Xi}(x))][(\Xi'(x), 1)] \\ &= [(\Xi(x), \mathbf{\Psi}^r_{\Xi}(x))][(\Xi(x)u(x), 1)] \\ &= [(\Xi(x), \mathbf{\Psi}^r_{\Xi}(x))][(\Xi(x), u^{-1}(x))] \end{aligned}$$

$$
\begin{aligned}
&= [(\Xi(x), \boldsymbol{\Psi}^r_\Xi(x))][(\Xi(x), 1)][(\Xi(x), u^{-1}(x))] \\
&= [(\Xi(x), \boldsymbol{\Psi}^l_\Xi(x))]\mathbf{1}^r_\Xi U^{-1}(x) \\
&= \psi_\Xi((x)U^{-1}(x). \qquad (7.41)
\end{aligned}
$$

Remark 7.42 A Right DHSF $\boldsymbol{\Phi}^r$, a section of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ associated to Φ^* a (ideal) section of $I^*(M, \mathtt{g})$ also has representatives in the Clifford bundle and we have in obvious notation

$$\phi_\Xi := \mathbf{1}^l_\Xi \boldsymbol{\Phi}^r, \quad \phi_{\Xi'} := \mathbf{1}^l_{\Xi'} \boldsymbol{\Phi}^r \qquad (7.42)$$

and two different representatives ϕ'_Ξ and ϕ_Ξ are related by

$$\phi'_\Xi = U\phi_\Xi. \qquad (7.43)$$

7.6 Covariant Derivatives of Clifford Fields

Since the Clifford bundle of differential forms is $\mathcal{C}\ell(M, \mathtt{g}) = \mathcal{T}M/J_\mathtt{g}$, it is clear that any linear connection $\boldsymbol{\nabla}$ on $\mathcal{T}M$ which is metric compatible ($\boldsymbol{\nabla}\mathtt{g} = 0$) passes to the quotient $\mathcal{T}M/J_\mathtt{g}$, and thus define an algebra bundle connection [7]. In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M, \mathtt{g})$ is completely determined.

We now prove a very important formula for the covariant derivative of Clifford fields and of *DHSF* using the general theory of connections in principal bundles and covariant derivatives in associate vector bundles, which we recalled in Appendix A.

Proposition 7.43 *The covariant derivative* (*in a given gauge*) *of a Clifford field* $A \in \sec \mathcal{C}\ell(M, \mathtt{g})$, *in the direction of the vector field* $V \in \sec TM$ *is given by*

$$\boldsymbol{\nabla}_V A = \eth_V(A) + \frac{1}{2}[\omega_V, A], \qquad (7.44)$$

where ω_V *is the usual* ($\bigwedge^2 T^*M$*-valued*) *connection 1-form written in the basis* $\{\theta_\mathbf{a}\}$ *and, if* $A = A^\mathbf{a}\theta_\mathbf{a}$, *then* $\eth_V$ *is the* (*Pfaff*) *derivative operator such that* $\eth_V(A) := V(A^I)\theta_I$ *introduced in Chap.* 5.[6]

Proof Writing $A(t) = A(\sigma(t))$ in terms of the multiform basis $\{\theta_I\}$ of sections associated to a given spin coframe, as in Sect. 7.4.1, we have $A(t) = A^I(t)\theta_I(t) = A^I(t)[(\Xi(t), \mathbf{e}_I)] = [(\Xi(t), A_I(t)\mathbf{e}_I)] = [(\Xi(t), a(t))]$, with $a(t) := A^I(t)\mathbf{e}_I \in \mathbb{R}_{1,3}$. If

[6] I denotes collective indexes of a basis of $\mathcal{C}\ell(M, \mathtt{g})$ [see Eq. (7.7)].

follows from item (ii) Definition A.54 in Sect. A.5 that

$$A^0_{\|t} = [(\Xi(0), g(t)a(t)g(t)^{-1})] \tag{7.45}$$

for some $g(t) \in \mathrm{Spin}^e_{1,3}$, with $g(0) = 1$. Thus

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\left[g(t)a(t)g(t)^{-1} - a(0)\right] &= \left[\frac{dg}{dt}ag^{-1} + g\frac{da}{dt}g^{-1} + ga\frac{dg_t^{-1}}{dt}\right]_{t=0}\\ &= \dot{a}(0) + \dot{g}(0)a(0) - a(0)\dot{g}(0)\\ &= V(A^I)E_I + [\dot{g}(0), a(0)],\end{aligned}$$

where $\dot{g}(0) \in \mathrm{spin}^e_{1,3}(\simeq \bigwedge^2 \mathbb{R}^{1,3})$ is the Lie algebra of $\mathrm{Spin}^e_{1,3}$. Therefore with

$$\begin{aligned}\omega_V = [(p, \dot{g}(0))] &= \frac{1}{2}\omega^{\mathbf{a}\cdot\cdot}_{\cdot V\mathbf{b}}\theta_{\mathbf{a}} \wedge \theta^{\mathbf{b}} = \frac{1}{2}\omega^{\mathbf{a}\cdot\ \mathbf{b}}_{\cdot V\cdot}\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}} \\ &= \frac{1}{2}V^{\mathbf{c}}\omega_{\cdot\mathbf{c}\cdot}^{\ \mathbf{a}\cdot\mathbf{b}}\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}} = \frac{1}{2}V^{\mathbf{c}}\omega_{\mathbf{acb}}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}.\end{aligned} \tag{7.46}$$

we have

$$\nabla_V A = V(A^I)\theta_I + \frac{1}{2}[\omega_V, A],$$

which proves the proposition. ■

Remark 7.44 In particular, calculating the covariant derivative of the basis 1-covector fields θ_a yields $\frac{1}{2}[\omega_{e_{\mathbf{c}}}, \theta_{\mathbf{a}}] = \omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ca}}\theta_{\mathbf{b}}$. Note that

$$\omega_{\mathbf{acb}} = \eta_{\mathbf{ad}}\omega^{\mathbf{d}\cdot\cdot}_{\cdot\mathbf{cb}} = -\omega_{\mathbf{bca}} \tag{7.47}$$

and

$$\omega^{\mathbf{a}\cdot\mathbf{b}}_{\cdot\mathbf{c}\cdot} = \eta^{\mathbf{ka}}\omega_{\mathbf{kcl}}\eta^{\mathbf{lb}} = -\omega^{\mathbf{b}\cdot\mathbf{a}}_{\cdot\mathbf{c}\cdot} \tag{7.48}$$

In this way, $\omega : V \mapsto \omega_V$ is the usual ($\bigwedge^2 T^*M$-valued) connection 1-form on M written in a given gauge (i.e., relative to a spin coframe).

Remark 7.45 Equation (7.44) shows that the covariant derivative preserves the degree of an homogeneous Clifford field, as can be easily verified.

The general formula Eq. (7.44) and the associative law in the Clifford algebra immediately yields the

Corollary 7.46 *The covariant derivative* $\boldsymbol{\nabla}_V$ *on* $\mathcal{C}\ell(M,\mathtt{g})$ *acts as a derivation on the algebra of sections, i.e., for* $A, B \in \sec \mathcal{C}\ell(M,\mathtt{g})$ *and* $V \in \sec TM$*, it holds*

$$\boldsymbol{\nabla}_V(AB) = (\boldsymbol{\nabla}_V A)B + A(\boldsymbol{\nabla}_V B). \tag{7.49}$$

Proof It is a trivial calculation using Eq. (7.44). ■

Corollary 7.47 *Under a change of gauge* (*local Lorentz transformation*) ω_V*, transforms as*

$$\frac{1}{2}\omega_V \mapsto U\frac{1}{2}\omega_V U^{-1} + (\boldsymbol{\nabla}_V U)U^{-1}, \tag{7.50}$$

Proof Using Eq. (7.44) we can calculate $\boldsymbol{\nabla}_V A$ in two different gauges as

$$\boldsymbol{\nabla}_V A = \eth_V(A) + \frac{1}{2}[\omega_V, A], \tag{7.51}$$

or

$$\boldsymbol{\nabla}_V A = \eth'_V(A) + \frac{1}{2}[\omega'_V, A], \tag{7.52}$$

where by definition $\eth_V(A) = V(A_I)\theta^I$ and $\eth'_V(A) = V(A'_I)\theta'^I$. Now, we observe that since $\theta'^I = U\theta^I U^{-1}$, we can write

$$\begin{aligned} U\eth_V(U^{-1}A) &= U\eth_V(U^{-1}A'_I\varepsilon'^I) = V(A'_I)\varepsilon'^I + A'_I\varepsilon'^I U\eth_V(U^{-1}) \\ &= \eth'_V(A) + AU\eth_V(U^{-1}). \end{aligned}$$

Now, $U\eth_V(U^{-1}A) = \eth_V A + U(\eth_V U^{-1})A$ and it follows that

$$\eth'_V(A) = \eth_V(A) - [(\eth_V U)\, U^{-1}, A].$$

Then, we see comparing the second members of Eqs. (7.51) and (7.52) that

$$\begin{aligned} \frac{1}{2}\omega_V &= \frac{1}{2}\omega'_V + U\left(\eth_V U^{-1}\right), \\ \frac{1}{2}\omega'_V &= \frac{1}{2}\omega_V + (\eth_V U)\, U^{-1}. \end{aligned} \tag{7.53}$$

Finally, we have

$$\frac{1}{2}\omega'_V = \frac{1}{2}\omega_V + \left[\nabla_V U - \frac{1}{2}\omega_V U + \frac{1}{2}U\omega_V\right]U^{-1} \quad (7.54)$$
$$= \frac{1}{2}U\omega_V U^{-1} + (\nabla_V U)U^{-1},$$

and the proposition is proved. ■

7.7 Covariant Derivative of Spinor Fields

The spinor bundles introduced in Sect. 5.5, like $I(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_\ell I$, $I = \mathbb{R}_{1,3}\frac{1}{2}(1 + E_0)$, and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ (and subbundles) are vector bundles. Thus, as in the case of Clifford fields we can use the general theory of covariant derivative operators on associate vector bundles to obtain formulas for the covariant derivatives of sections of these bundles. Given $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, we denote the corresponding covariant derivatives by[7] $\nabla^s_V \Psi$ and $\nabla^s_V \Phi$.

Proposition 7.48 *Given* $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ *and* $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ *we have,*

$$\nabla^s_V \Psi = \eth_V(\Psi) + \frac{1}{2}\omega_V \Psi, \quad (7.55)$$

$$\nabla^s_V \Phi = \eth_V(\Phi) - \frac{1}{2}\Phi\omega_V. \quad (7.56)$$

Proof It is analogous to that of Proposition 7.43, with the difference that Eq. (7.45) should be substituted by $\Psi^0_{||t} = [(\Xi(0), g(t)a(t))]$ and $\Phi^0_{||t} = [(\Xi(0), a(t)g(t)^{-1})]$. ■

Proposition 7.49 *Let* ∇ *be the connection on* $\mathcal{C}\ell(M, \mathtt{g})$ *to which* ∇^s *is related. Then, for any* $V \in \sec TM$, $A \in \sec \mathcal{C}\ell(M, \mathtt{g})$, $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ *and* $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$,

$$\nabla^s_V(A\Psi) = A(\nabla^s_V \Psi) + (\nabla_V A)\Psi, \quad (7.57)$$

$$\nabla^s_V(\Phi A) = \Phi(\nabla_V A) + (\nabla^s_V \Phi)A. \quad (7.58)$$

[7] Recall that $I^l(M, \mathtt{g}) \hookrightarrow \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and $I^r(M, \mathtt{g}) \hookrightarrow \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$.

Proof Recalling that $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ ($\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$) is a module over $\mathcal{C}\ell(M,\mathtt{g})$, the result follows from a simple computation. ■

Finally, let $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ be such that $\Psi\mathrm{e}{=}\Psi$ where $\mathrm{e}^2 = \mathrm{e} \in \mathbb{R}_{1,3}$ is a primitive idempotent. Then, since $\Psi\mathrm{e} = \Psi$,

$$\begin{aligned}\boldsymbol{\nabla}^s_V\Psi &= \boldsymbol{\nabla}^s_V(\Psi\mathrm{e}) = \eth_V(\Psi\mathrm{e}) + \frac{1}{2}\omega_V\Psi\mathrm{e}\\ &= [\eth_V(\Psi) + \frac{1}{2}\omega_V\Psi]\mathrm{e} = (\boldsymbol{\nabla}^s_V\Psi)\mathrm{e},\end{aligned} \tag{7.59}$$

from where we verify that the covariant derivative of a LIASF is indeed a LIASF.

Finally we can prove a statement done at the end of the previous section: that the right unit sections associated with spin coframes are *not* constant in any covariant way. In fact, we have the

Proposition 7.50 *Let* $\mathbf{1}^r_{\Xi} \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ *be the right unit section associated to the spin coframe* Ξ. *Then*

$$\boldsymbol{\nabla}^s_{\boldsymbol{e}_\mathbf{a}}\mathbf{1}^r_{\Xi} = -\frac{1}{2}\mathbf{1}^r_{\Xi}\omega_{\boldsymbol{e}_\mathbf{a}}. \tag{7.60}$$

Proof It follows directly from Eq. (7.56). ■

Exercise 7.51 Calculate the covariant derivative of $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ in the direction of the vector field $V \in \sec TM$ and confirm the validity of Eq. (7.50).

Solution Let $u : M \to \mathrm{Spin}^e_{1,3} \subset \mathbb{R}_{1,3}$ be such that in the spin coframe $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ we have for $U \in \sec \mathcal{C}\ell^{(0)}(M,\mathtt{g})$, $UU^{-1} = 1$, $U(x) = [(\Xi(x), u(x))]$.We can write the covariant derivative of $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ in two different gauges $\Xi, \Xi' \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ as

$$\boldsymbol{\nabla}^s_V\Psi = \eth_V(\Psi) + \frac{1}{2}\omega_V\Psi, \tag{7.61}$$

$$\boldsymbol{\nabla}^s_V\Psi = \eth'_V(\Psi) + \frac{1}{2}\omega'_V\Psi. \tag{7.62}$$

Now,

$$\Psi = \Psi_I s^I = \Psi'_I s'^I,$$

where s^I, s'^I are the following equivalent classes in $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$:

$$s^I = \left[\left(\Xi(x), S^I\right)\right]_\ell,\ s'^I = \left[\left(\Xi'(x), S^I\right)\right]_\ell,$$

with S^I a spinor basis in an minimal left ideal in $\mathbb{R}_{1,3}$. Now, if $\Xi' = \Xi u$ we can write using the fact that $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ is a module over $\mathcal{C}\ell(M, \mathtt{g})$

$$\begin{aligned} s'^I &= \left[\left(\Xi'(x), S^I\right)\right]_\ell = \left[\left(\Xi(x)u(x), S^I\right)\right]_\ell = \left[\left(\Xi(x), uS^I\right)\right]_\ell \\ &= \left[(\Xi(x), u(x))\right]_{\mathcal{C}\ell} \left[\left(\Xi(x), S^I\right)\right]_\ell . \end{aligned}$$

Recalling that $U = [(\Xi(x), u(x))]_{\mathcal{C}\ell} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ we can write

$$s'^I = Us^I. \tag{7.63}$$

Moreover, since

$$\begin{aligned} \eth_V(\Psi) &= V(\Psi_I)s^I, \\ \eth'_V(\Psi) &= V(\Psi'_I)s'^I, \end{aligned} \tag{7.64}$$

we get

$$\eth'_V(\Psi) = \eth_V(\Psi) + U(\eth_V U^{-1})\Psi. \tag{7.65}$$

Indeed, since $\Psi = [(\Xi, \Psi_\Xi)]_\ell = \left[\left(\Xi', \Psi'_\Xi\right)\right]_\ell$ we can write

$$\left[\left(\Xi', \Psi'_\Xi\right)\right]_\ell = \left[\left(\Xi u, \Psi'_\Xi\right)\right]_\ell = \left[\left(\Xi, u\Psi'_\Xi\right)\right]_\ell ,$$

i.e.,

$$\Psi'_\Xi = u^{-1}\Psi_\Xi. \tag{7.66}$$

Then,

$$\eth_V(\Psi) = V(\Psi_I)\left[\left(\Xi, S^I\right)\right]_\ell = \left[\left(\Xi, V(\Psi_I)S^I\right)\right]_\ell = [(\Xi, \eth_V(\Psi_\Xi)] \tag{7.67}$$

$$\begin{aligned} \eth'_V(\Psi) &= V(\Psi'_I)\left[\left(\Xi', S^I\right)\right]_\ell = \left[\left(\Xi', V(\Psi'_I)S^I\right)\right]_\ell = \left[\left(\Xi', \eth_V(\Psi'_\Xi)\right)\right] \\ &= \left[\left(\Xi, u\eth_V(\Psi'_\Xi)\right)\right] = \left[\left(\Xi, u\eth_V(u^{-1}\Psi_\Xi)\right)\right] \end{aligned} \tag{7.68}$$

from where Eq. (7.65) follows. Also,

$$\nabla^s_V\Psi = \eth_V(\Psi) + \frac{1}{2}\omega_V\Psi = \left[\left(\Xi, \eth_V(\Psi_\Xi) + \frac{1}{2}w_V\Psi_\Xi\right)\right]_\ell , \tag{7.69}$$

$$\nabla^s_V\Psi = \eth'_V(\Psi) + \frac{1}{2}\omega'_V\Psi = \left[\left(\Xi', \eth_V(\Psi'_\Xi) + \frac{1}{2}w'_V\Psi'_\Xi\right)\right]_\ell , \tag{7.70}$$

with

$$\omega_V = [(\Xi, w_V)]_{\mathcal{C}\ell}\ , \tag{7.71}$$

$$\omega'_V = \left[\left(\Xi', w'_V\right)\right]_{\mathcal{C}\ell}. \tag{7.72}$$

Then we can write using Eq. (7.70),

$$\begin{aligned}\nabla^s_V \Psi &= \left[\left(\Xi u, \eth_V(u^{-1}\Psi_\Xi) + \frac{1}{2}w'_V u^{-1}\Psi_\Xi\right)\right]_\ell \\ &= \left[\left(\Xi, u\eth_V(u^{-1}\Psi_\Xi) + \frac{1}{2}uw'_V u^{-1}\Psi_\Xi\right)\right]_\ell \\ &= \left[\left(\Xi, \eth_V\Psi_\Xi + u\eth_V(u^{-1})\Psi_\Xi + \frac{1}{2}uw'_V u^{-1}\Psi_\Xi\right)\right]_\ell \end{aligned} \tag{7.73}$$

Comparing Eqs. (7.69) and (7.70) we get

$$\frac{1}{2}w'_V = \frac{1}{2}u^{-1}w_V u - \eth_V(u^{-1})u. \tag{7.74}$$

We now must verify that Eqs. (7.74) and (7.50) are compatible. To do this, we use Eq. (7.72) to write

$$\begin{aligned}\frac{1}{2}\omega'_V &= \frac{1}{2}\left[\left(\Xi', w'_V\right)\right]_{\mathcal{C}\ell} = U\frac{1}{2}\omega_V U^{-1} + (\boldsymbol{\nabla}_V U)U^{-1} \\ &= \frac{1}{2}\omega_V + (\eth_V U)U^{-1} \\ &= \left[\left(\Xi, \frac{1}{2}w_V + \eth_V(u)u^{-1}\right)\right]_{\mathcal{C}\ell} \\ &= \left[\left(\Xi' u^{-1}, \frac{1}{2}w_V + \eth_V(u)u^{-1}\right)\right]_{\mathcal{C}\ell} \\ &= \left[\left(\Xi', \frac{1}{2}u^{-1}w_V u + u^{-1}\eth_V(u)\right)\right]_{\mathcal{C}\ell}. \end{aligned} \tag{7.75}$$

Comparing Eqs. (7.75) and (7.74), we see that Eqs. (7.74) and (7.50) are indeed compatible.

7.8 The Many Faces of the Dirac Equation

As it is the case of Maxwell and Einstein equations also the Dirac equation has many faces. In this section we exhibit two of them.

7.8.1 Dirac Equation for Covariant Dirac Fields

As well known [6], a *covariant* Dirac spinor field is a section $\boldsymbol{\Psi} \in \sec S_c(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbb{C}^4$. Let $(U = M, \Phi)$, $\Phi(\boldsymbol{\Psi}) = (x, |\Psi(x)\rangle)$ be a *global* trivialization corresponding to a spin coframe Ξ (Remark 7.5), such that for $\{\theta_{\mathbf{a}}\} \in P_{\mathrm{SO}^e_{1,3}}(M)$ and $\{\boldsymbol{e}_{\mathbf{a}}\} \in \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ it is

$$s(\Xi) = \{\theta^{\mathbf{a}}\}, \theta^{\mathbf{a}} \in \sec \mathcal{C}\ell(M, \mathtt{g}), \theta^{\mathbf{a}}(\boldsymbol{e}_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$$
$$\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \theta^{\mathbf{b}}\theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}}, \mathbf{a}, \mathbf{b} = 0, 1, 2, 3. \tag{7.76}$$

The usual Dirac equation in a Lorentzian spacetime for the spinor field $\boldsymbol{\Psi}$ in interaction with an electromagnetic field $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is then

$$i\underline{\boldsymbol{\gamma}}^{\mathbf{a}}(\boldsymbol{\nabla}^s_{\boldsymbol{e}_{\mathbf{a}}} + iqA_{\mathbf{a}})|\Psi(x)\rangle - m|\Psi(x)\rangle = 0, \tag{7.77}$$

where $\underline{\boldsymbol{\gamma}}^{\mathbf{a}} \in \mathbb{C}(4)$, $\mathbf{a} = 0, 1, 2, 3$ is a set of *constant* Dirac *matrices* satisfying

$$\underline{\boldsymbol{\gamma}}^{\mathbf{a}}\underline{\boldsymbol{\gamma}}^{\mathbf{b}} + \underline{\boldsymbol{\gamma}}^{\mathbf{b}}\underline{\boldsymbol{\gamma}}^{\mathbf{a}} = 2\eta^{\mathbf{ab}} \tag{7.78}$$

and $|\Psi(x)\rangle \in \mathbb{C}^4$ for $x \in M$.

7.8.2 Dirac Equation in $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$

Due to the one-to-one correspondence between *ideal* sections of the vector bundles $\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and of $S_c(M, \mathtt{g})$ as explained above, we can *translate* the Dirac Eq. (7.77) for a covariant spinor field into an equation for a spinor field, which is a section of $\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, and finally write an equivalent equation for a DHSF $\psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$. In order to do that we introduce the spin-Dirac operator.

7.8.3 Spin Dirac Operator

Definition 7.52 The spin Dirac operator acting on sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ is the first order differential operator [15]

$$\boldsymbol{\partial}^s = \theta^{\mathbf{a}}\boldsymbol{\nabla}^s_{\boldsymbol{e}_{\mathbf{a}}}. \tag{7.79}$$

where $\{\theta^{\mathbf{a}}\}$ is a basis as defined in Eq. (7.76).

Remark 7.53 It is crucial to keep in mind the distinction between the Dirac operator $\boldsymbol{\partial}$ introduce through Definition 4.125 and the spin Dirac operator $\boldsymbol{\partial}^s$ just defined.

We now write Dirac equation in $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$, denoted DE$\mathcal{C}\ell^l$. It is

$$\boldsymbol{\partial}^s \boldsymbol{\psi} \boldsymbol{e}^{\mathbf{21}} - m\boldsymbol{\psi}\boldsymbol{e}^{\mathbf{0}} - qA\boldsymbol{\psi} = 0 \tag{7.80}$$

where $\boldsymbol{\psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$ is a DHSF and the $\mathbf{e^a} \in \mathbb{R}_{1,3}$ are such that $\mathbf{e^a e^b} + \mathbf{e^b e^a} = 2\eta^{\mathbf{ab}}$. Multiplying Eq. (7.80) on the right by the idempotent $\mathrm{f} = \frac{1}{2}(1 + \mathbf{e^0})\frac{1}{2}(1 + \mathrm{i}\mathbf{e^1 e^2}) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$ we get after some simple algebraic manipulations the following equation for the (complex) ideal left spin-Clifford field $\Psi\mathrm{f} = \Psi \in \sec \mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$,

$$\mathrm{i}\boldsymbol{\partial}^s \Psi - m\Psi - qA\Psi = 0. \tag{7.81}$$

Now we can easily show, using the methods of Chap. 3, that given any global trivializations $(U = M, \Delta)$ and $(U = M, \Gamma)$, of $\mathcal{C}\ell(M,\mathtt{g})$ and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$, there exists matrix representations of the $\{\theta^a\}$ that are equal to the Dirac matrices $\underline{\boldsymbol{\gamma}}^{\mathbf{a}}$ (appearing in Eq. (7.77)). In that way the correspondence between Eqs. (7.77), (7.80) and (7.81) is proved.

Remark 7.54 We must emphasize at this point that we call Eq. (7.80) the *DE*$\mathcal{C}\ell^l$. It looks similar to the Dirac-Hestenes equation (on Minkowski spacetime) discussed, e.g., in [26], but it is indeed very different regarding its mathematical nature. It is an intrinsic equation satisfied by a legitimate spinor field, namely a *DHSF* $\boldsymbol{\psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathtt{g})$. The question naturally arises: May we write an equation with the same mathematical information of Eq. (7.80) but satisfied by objects living on the Clifford bundle of an arbitrary Lorentzian spacetime, admitting a spin structure? Below we show that the answer to that question is yes. But before we prove that result let us recall how to formulate the electromagnetic gauge invariance for the *DE*$\mathcal{C}\ell^l$.

7.8.4 Electromagnetic Gauge Invariance of the DE$\mathcal{C}\ell^l$

Proposition 7.55 *The* DE$\mathcal{C}\ell^l$ *is invariant under electromagnetic gauge transformations*

$$\boldsymbol{\psi} \mapsto \boldsymbol{\psi}' = \boldsymbol{\psi} e^{q\mathbf{e^{21}}\chi}, \tag{7.82}$$

$$A \mapsto A + \boldsymbol{\partial}\chi, \tag{7.83}$$

$$\omega_{\boldsymbol{e}_\mathbf{a}} \mapsto \omega_{\boldsymbol{e}_\mathbf{a}} \tag{7.84}$$

$$\boldsymbol{\psi}, \boldsymbol{\psi}' \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M) \tag{7.85}$$

$$A \in \sec \bigwedge^{1} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}) \tag{7.86}$$

with $\boldsymbol{\psi}, \boldsymbol{\psi}'$ *distinct* DHSF, $\boldsymbol{\partial}$ *is the Dirac operator and* $\chi : M \to \mathbb{R} \hookrightarrow \mathbb{R}_{1,3}$ *is a gauge function.*

Proof The proof is obtained by direct verification. ■

Remark 7.56 It is important to note that although local rotations and electromagnetic gauge transformations look similar, they are indeed very different mathematical transformations, without any obvious geometrical link between them, differently of what seems to be the case for the Dirac-Hestenes equation which we introduce in the next section.

7.9 The Dirac-Hestenes Equation (DHE)

We obtained above a Dirac equation, which we called DE$\mathcal{C}\ell^{l}$, describing the motion of spinor fields represented by sections $\boldsymbol{\Psi}$ of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ in interaction with an electromagnetic field $A \in \sec \mathcal{C}\ell(M, \mathtt{g})$,

$$\boldsymbol{\partial}^{s}\boldsymbol{\Psi}\boldsymbol{e}^{\mathbf{21}} - qA\boldsymbol{\Psi} = m\boldsymbol{\Psi}\boldsymbol{e}^{\mathbf{0}}, \tag{7.87}$$

where $\boldsymbol{\partial}^{s} = \theta^{\mathbf{a}}\boldsymbol{\nabla}^{s}_{\boldsymbol{e}_{\mathbf{a}}}$, $\{\theta^{\mathbf{a}}\}$ is given by Eq. (7.76), $\boldsymbol{\nabla}^{s}_{\boldsymbol{e}_{\mathbf{a}}}$ is the natural spinor covariant derivative acting on $\sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ and $\{\mathbf{e}^{\mathbf{a}}\} \in \mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$ is such that $\mathbf{e}^{\mathbf{a}}\mathbf{e}^{\mathbf{b}} + \mathbf{e}^{\mathbf{b}}\mathbf{e}^{\mathbf{a}} = 2\eta^{\mathbf{ab}}$. As we already mentioned, although Eq. (7.87) is written in a kind of Clifford bundle (i.e., $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$), it does not suffer from the inconsistency of representing spinors as pure differential forms and, in fact, the object $\boldsymbol{\Psi}$ behaves as it should under Lorentz transformations.

As a matter of fact, Eq. (7.87) can be thought of as a mere *rewriting* of the usual Dirac equation, where the role of the constant gamma matrices is undertaken by the constant elements $\{\mathbf{e}^{\mathbf{a}}\}$ in $\mathbb{R}_{1,3}$ and by the set $\{\theta^{\mathbf{a}}\}$. In this way, Eq. (7.87) is *not* a kind of Dirac-Hestenes equation as discussed, e.g., in [26]. It suffices to say that (i) the state of the electron, represented by $\boldsymbol{\Psi}$, is not a *Clifford field* and (ii) the $\mathbf{e}^{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$ are just *constant* elements of $\mathbb{R}_{1,3}$ and not sections of 1-form fields in $\mathcal{C}\ell(M, \mathtt{g})$. Nevertheless, as we show in the following, Eq. (7.87) does lead to a multiform Dirac equation once we carefully employ the theory of right and left actions on the various Clifford bundles introduced earlier. It is that multiform equation to be derived below that we call the DHE.

7.9.1 Representative of the DHE in the Clifford Bundle

Let $\{\mathbf{e^a}\}$ be, as before, a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$. Remember that these objects are fundamental to the Dirac equation (7.87) in terms of sections $\boldsymbol{\Psi}$ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$:

$$\boldsymbol{\partial}^s \boldsymbol{\Psi} \boldsymbol{e}^{\mathbf{21}} - qA\boldsymbol{\Psi} = m\boldsymbol{\Psi}\boldsymbol{e}^{\mathbf{0}}.$$

Let $\boldsymbol{\Xi} \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$ be a spin frame and $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ its dual spin coframe on M and define the sections $\mathbf{1}^l_\Xi$, $\mathbf{1}^r_\Xi$ and $\varepsilon_{\mathbf{a}}$, respectively on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and $\mathcal{C}\ell(M, \mathtt{g})$, as above. Now we can use Proposition 7.39 to write the above equation in terms of sections of $\mathcal{C}\ell(M, \mathtt{g})$:

$$(\boldsymbol{\partial}^s \boldsymbol{\Psi})\mathbf{1}^r_\Xi \theta^{\mathbf{21}} \mathbf{1}^l_\Xi - qA\boldsymbol{\Psi} = m\boldsymbol{\Psi}\mathbf{1}^r_\Xi\, \varepsilon^{\mathbf{0}} \mathbf{1}^l_\Xi. \tag{7.88}$$

Right-multiplying by $\mathbf{1}^r_\Xi$ yields, using Proposition 7.39,

$$\theta^{\mathbf{a}}(\boldsymbol{\nabla}^s_{\boldsymbol{e}_\mathbf{a}} \boldsymbol{\Psi})\mathbf{1}^r_\Xi \theta^{\mathbf{21}} - qA\boldsymbol{\Psi}\mathbf{1}^r_\Xi = m\boldsymbol{\Psi}\mathbf{1}^r_\Xi \theta^{\mathbf{0}}. \tag{7.89}$$

It follows from Propositions 7.33 and 7.49 that

$$\begin{aligned}(\boldsymbol{\nabla}^s_{\boldsymbol{e}_\mathbf{a}} \boldsymbol{\Psi})\mathbf{1}^r_\Xi &= \boldsymbol{\nabla}_{\boldsymbol{e}_\mathbf{a}}(\boldsymbol{\Psi}\mathbf{1}^r_\Xi) - \boldsymbol{\Psi}\boldsymbol{\nabla}^s_{\boldsymbol{e}_\mathbf{a}}(\mathbf{1}^r_\Xi) \\ &= \boldsymbol{\nabla}_{\boldsymbol{e}_\mathbf{a}}(\boldsymbol{\Psi}\mathbf{1}^r_\Xi) + \frac{1}{2}\boldsymbol{\Psi}\mathbf{1}^r_\Xi \omega_{\boldsymbol{e}_\mathbf{a}},\end{aligned} \tag{7.90}$$

where Proposition 7.50 was employed in the last step. Therefore

$$\theta^{\mathbf{a}}\left[\boldsymbol{\nabla}_{\boldsymbol{e}_\mathbf{a}}(\boldsymbol{\Psi}\mathbf{1}^r_\Xi) + \frac{1}{2}\boldsymbol{\Psi}\mathbf{1}^r_\Xi \omega_{\boldsymbol{e}_\mathbf{a}}\right]\theta^{\mathbf{21}} - qA(\boldsymbol{\Psi}\mathbf{1}^r_\Xi) = m(\boldsymbol{\Psi}\mathbf{1}^r_\Xi)\theta^{\mathbf{0}}. \tag{7.91}$$

To proceed we recall Definition 7.41 which says that

$$\psi_\Xi := \boldsymbol{\Psi}\mathbf{1}^r_\Xi \tag{7.92}$$

is a representative in $\mathcal{C}\ell(M, \mathtt{g})$ of $\boldsymbol{\Psi}$ associated to the spin coframe Ξ. We then have

$$\theta^{\mathbf{a}}\left[\boldsymbol{\nabla}_{e_a}\psi_\Xi + \frac{1}{2}\psi_\Xi \omega_{\boldsymbol{e}_\mathbf{a}}\right]\theta^{\mathbf{21}} - qA\psi_\Xi = m\psi_\Xi\theta^{\mathbf{0}}. \tag{7.93}$$

7.9.2 A Comment About the Nature of Spinor Fields

A comment about the nature of spinors is in order. As we repeatedly said in the previous sections, spinor fields should not be ultimately regarded as fields of multiforms, for their behavior under Lorentz transformations is not tensorial (they are able to distinguished between 2π and 4π rotations). So, how can the identification above be correct? The answer is that the definition in Eq. (7.92) is intrinsically spin coframe dependent. Clearly, this is the price one ought to pay if one wants to make sense of the procedure of representing spinors by differential forms.

Note also that the covariant derivative acting on ψ_Ξ in Eq. (7.93) is the tensorial covariant derivative $\boldsymbol{\nabla}_V$ on $\mathcal{C}\ell(M, \mathtt{g})$, as it should be. However, we see from the expression above that $\boldsymbol{\nabla}_V$ acts on ψ_Ξ together with the term $\frac{1}{2}\psi_\Xi\omega_V$. Therefore, it is natural to define an "effective covariant derivative" $\boldsymbol{\nabla}^{(s)}_V$ acting on ψ_Ξ by

$$\boldsymbol{\nabla}^{(s)}_{\boldsymbol{e}_\mathbf{a}}\psi_\Xi := \boldsymbol{\nabla}_{\boldsymbol{e}_\mathbf{a}}\psi_\Xi + \frac{1}{2}\psi_\Xi\omega_{\boldsymbol{e}_\mathbf{a}}. \tag{7.94}$$

Then, Proposition 7.43 yields

$$\boldsymbol{\nabla}^{(s)}_{\boldsymbol{e}_\mathbf{a}}\psi_\Xi = \mathfrak{d}_{\boldsymbol{e}_\mathbf{a}}(\psi_\Xi) + \frac{1}{2}\omega_{\boldsymbol{e}_\mathbf{a}}\psi_\Xi, \tag{7.95}$$

which emulates the spinorial covariant derivative,[8] as it should. We observe moreover that if $\mathcal{C} \in \sec\mathcal{C}\ell(M, \mathtt{g})$ and if $\psi_\Xi \in \sec\mathcal{C}\ell(M, \mathtt{g})$ is a representative of a Dirac-Hestenes spinor field then

$$\boldsymbol{\nabla}^{(s)}_{\boldsymbol{e}_\mathbf{a}}(\mathcal{C}\psi_\Xi) = (\boldsymbol{\nabla}_{\boldsymbol{e}_\mathbf{a}}\mathcal{C})\,\psi_\Xi + \mathcal{C}\boldsymbol{\nabla}^{(s)}_{\boldsymbol{e}_\mathbf{a}}\psi_\Xi. \tag{7.96}$$

With this notation, we finally have the DHE for the Clifford field *representative* $\psi_\Xi \in \sec\mathcal{C}\ell(M, \mathtt{g})$ relative to the spin coframe Ξ on a Lorentzian spacetime[9] of a DHSF $\boldsymbol{\Psi} \in \sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$:

$$\boldsymbol{\partial}^{(s)}\psi_\Xi\theta^{\mathbf{21}} - qA\psi_\Xi = m\psi_\Xi\theta^{\mathbf{0}}, \tag{7.97}$$

where

$$\boldsymbol{\partial}^{(s)} = \theta^{\mathbf{a}}\boldsymbol{\nabla}^{(s)}_{\boldsymbol{e}_\mathbf{a}} \tag{7.98}$$

will be called the representative of the spin-Dirac operator in the Clifford bundle.

[8]This is the derivative used in [26], there introduced in an ad hoc way.

[9]The DHE on a Riemann-Cartan spacetime is discussed in Chap. 10.

Let us finally show that this formulation recovers the usual transformation properties characteristic of the Hestenes's formalism as described, e.g., in [26]. For that matter, consider two spin coframes $\Xi, \Xi' \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$, with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \mathrm{Spin}^e_{1,3}$. We already know that $\psi_{\Xi'} = \psi_{\Xi} U^{-1}$. Therefore, the various spin coframe dependent Clifford fields from Eq. (7.93) transform as

$$\theta'_{\mathbf{a}} = U\theta_{\mathbf{a}}U^{-1}, \tag{7.99}$$
$$\psi_{\Xi'} = \psi_{\Xi}U^{-1}.$$

These are exactly the transformation rules one expects from fields satisfying the Dirac-Hestenes equation.

7.9.3 *Passive Gauge Invariance of the DHE*

Exercise 7.57 Show that if

$$\theta'^{\mathbf{a}}\nabla^{(s)}_{e'_{\mathbf{a}}}\psi_{\Xi'}\theta'^{\mathbf{21}} - qA\psi_{\Xi'} = m\psi_{\Xi'}\theta'^{\mathbf{0}}$$

then if the connection ω_V transforms as in Eq. (7.50) then

$$\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi_{\Xi}\theta^{\mathbf{21}} - qA\psi_{\Xi} = m\psi_{\Xi}\theta^{\mathbf{0}}.$$

This property will be referred as *passive gauge invariance* of the DHE. It shows that the fact that even if writing of the Dirac-Hestenes is, of course, frame dependent, this fact does not implies in the selection of any preferred *reference frame*.[10]

The concept of active gauge invariance under local rotations of the DHE will be studied in Chap. 10.

7.10 Amorphous Spinor Fields

Crumeyrolle [7] gives the name of *amorphous* spinor fields to ideal sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$. Thus an amorphous spinor field $\boldsymbol{\phi}$ is a section of $\mathcal{C}\ell(M, \mathtt{g})$ such that $\boldsymbol{\phi}\mathrm{P} = \boldsymbol{\phi}$, where $\mathrm{P} = \mathrm{P}^2$ is an idempotent section of $\mathcal{C}\ell(M, \mathtt{g})$. However, these fields and also the so-called Dirac-Kähler [13, 14] fields, which are also sections of $\mathcal{C}\ell(M, \mathtt{g})$, cannot be used in a physical theory of fermion fields since they do not have the correct transformation law under a Lorentz rotation

[10]The fundamental concept of reference frame in a Lorentzian spacetime has been discussed in details in Chap. 6.

of the local *spin coframe*. However, amorphous spinor fields appears in many 'Dirac like' representations of Maxwell equations and are often confounded with authentic spinor fields (i.e., sections of spinor bundles). Some of these 'Dirac like' representations of Maxwell equations will be discussed in Chap. 13.

7.11 Bilinear Invariants

7.11.1 Bilinear Invariants Associated to a DHSF

We are now in position to give a precise definition of the bilinear invariants fields of Dirac theory associated to a given DHSF. Recalling that $\bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, $p = 0, 1, 2, 3, 4$, and recalling Propositions 7.33 and 7.34, we have the

Definition 7.58 Let $\boldsymbol{\Psi} \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ be a DHSF. The bilinear invariants associated to $\boldsymbol{\Psi}$ are the following sections of $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$:

$$\boldsymbol{\Psi}\tilde{\boldsymbol{\Psi}} = \sigma + \theta_{\mathbf{5}}\omega \in \sec \left(\bigwedge^0 T^*M + \bigwedge^4 T^*M\right), \tag{7.100}$$

$$J = \boldsymbol{\Psi} \boldsymbol{e_0} \tilde{\boldsymbol{\Psi}} \in \sec \bigwedge^1 T^*M,\; K = \boldsymbol{\Psi} \boldsymbol{e_3} \tilde{\boldsymbol{\Psi}} \in \sec \bigwedge^1 T^*M$$

$$S = \boldsymbol{\Psi} \boldsymbol{e_{12}} \tilde{\boldsymbol{\Psi}} \in \sec \bigwedge^2 T^*M,$$

where $\boldsymbol{\Psi} = \boldsymbol{\Psi}\frac{1}{2}(1 + \mathbf{e_0})$, $\tilde{\boldsymbol{\Psi}} \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$ and $\theta_{\mathbf{5}} = \theta_{\mathbf{0123}} \in \sec \bigwedge^4 T^*M$.

Remark 7.59 Of course, since all bilinear invariants in Eq. (7.100) are sections of $\mathcal{C}\ell(M, \mathtt{g})$ they have the right transformations properties under arbitrary local Lorentz transformations, as required. As recalled in Chap. 3 these bilinear invariants and their Hodge duals satisfy a set of identities, called the Fierz identities. They are crucial for the physical interpretation of the Dirac equation (in first and second quantizations).

7.11.2 Bilinear Invariants Associated to a Representative of a DHSF

We note that the bilinear invariants, when written in terms of $\psi_{\Xi} := \boldsymbol{\Psi}\mathbf{1}^{r}_{\Xi}$, read (from Proposition 7.39):

$$S = \psi_{\Xi}\tilde{\psi}_{\Xi} = \sigma + \theta_{\mathbf{5}}\omega \in \sec(\bigwedge^0 T^*M + \bigwedge^4 T^*M),$$

$$J = \psi_\Xi \theta_{\mathbf{0}} \tilde{\psi}_\Xi \in \sec \bigwedge^1 T^*M, \; K = \psi_\Xi \theta_{\mathbf{3}} \tilde{\psi}_\Xi \in \sec \bigwedge^1 T^*M$$
$$S = \psi_\Xi \theta_{\mathbf{1}} \theta_{\mathbf{2}} \tilde{\psi}_\Xi \in \sec \bigwedge^2 T^*M,$$

where $\theta_{\mathbf{5}} = \theta_{\mathbf{0123}} = \theta_{\mathbf{0}}\theta_1\theta_{\mathbf{2}}\theta_{\mathbf{3}}$. These are all intrinsic quantities, as they should be. For a discussion on the Fierz identities satisfied by these bilinear covariants, written with the representatives of Dirac-Hestenes spinor fields, see Sect. 3.7.3 and [17, 26].

7.11.3 Electromagnetic Gauge Invariance of the DHE

Proposition 7.60 *The* DHE *is invariant under electromagnetic gauge transformations*

$$\psi_\Xi \mapsto \psi'_\Xi = \psi_\Xi e^{q_1 \theta_{\mathbf{21}} \chi}, \tag{7.101}$$
$$A \mapsto A + \partial\chi, \tag{7.102}$$
$$\omega_{\boldsymbol{e}_{\mathbf{a}}} \mapsto \omega_{\boldsymbol{e}_{\mathbf{a}}} \tag{7.103}$$

where $\psi_\Xi, \psi'_\Xi \in \sec \mathcal{C}\ell^+(M, \mathtt{g})$, $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ *and where* $\chi \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ *is a gauge function.*

Proof It is a direct calculation. ■

But, what are the meaning of these transformations? Equation (7.101) looks similar to Eq. (7.99) defining the change of a representative of a DHSF once we change spin coframe, but here we have an *active* transformation, since we did *not* change the spin coframe. On the other hand Eq. (7.102) does not correspond either to a passive (no transformation at all) or active local Lorentz transformation for A. Nevertheless, writing $\chi = \vartheta/2$ yields

$$e_{\mathbf{1}}^{-q\theta^{\mathbf{21}}\vartheta/2} \theta^{\mathbf{0}} e^{q\theta^{\mathbf{21}}\vartheta/2} = \theta'^{\mathbf{0}} = \theta^{\mathbf{0}}$$
$$e^{-q\theta^{\mathbf{21}}\vartheta/2} \theta^1 e^{q\theta^{\mathbf{21}}\vartheta/2} = \theta'^{\mathbf{1}} = \theta^{\mathbf{1}} \cos q\vartheta + \theta^{\mathbf{2}} \sin q\theta,$$
$$e^{-q\theta^{\mathbf{21}}\vartheta/2} \theta^3 e^{q\theta^{\mathbf{21}}\vartheta/2} = \theta'^{\mathbf{2}} = -\theta^{\mathbf{1}} \sin q\theta + \theta^{\mathbf{2}} \cos q\theta^{\mathbf{2}},$$
$$e^{-q\theta^{\mathbf{21}}\vartheta/2} \theta^3 e^{q\theta^{\mathbf{21}}\vartheta/2} = \theta'^{\mathbf{3}} = \theta^{\mathbf{3}}. \tag{7.104}$$

We see that Eqs. (7.104) define a spin coframe Ξ' to which corresponds, as we already know, a basis $\{\theta'^{\mathbf{0}}, \theta'^{\mathbf{1}}, \theta'^{\mathbf{2}}, \theta'^{\mathbf{3}}\}$ for $\bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$. We can then think of the electromagnetic gauge transformation as a rotation in the spin plane $\theta^{\mathbf{21}}$ by identifying ψ'_Ξ in Eq. (7.101) with $\psi_{\Xi'}$, the representative of the *DHSF* in the spin coframe Ξ' and by supposing that instead of transforming the spin connection

$\omega_{\boldsymbol{e}_\mathbf{a}}$ as in Eq. (7.50) it is taken as fixed and instead of maintaining the electromagnetic potential A fixed it is transformed as in Eq. (7.102).

We observe that, since in GRT and also in Riemann-Cartan theory (see Chap. 9) the field $\omega_{\boldsymbol{e}_\mathbf{a}}$ is associated with some aspects of that field, our interpretation for the electromagnetic gauge transformation suggests a possible non trivial coupling between electromagnetism and gravitation, *if* the Dirac-Hestenes equation is taken as a more fundamental representation of fermionic matter than the usual Dirac equation. We will not have space and time to explore that possibility in this book.

7.12 Commutator of Covariant Derivatives of Spinor Fields and Lichnerowicz Formula

In this section we complement the results of Sect. 5.2 and calculate the commutator of the covariant derivatives of spinor fields and the square of the spin-Dirac operator of a Riemann-Cartan connection leading to the generalized Lichnerowicz formula. Let $\psi \in \sec \mathcal{C}\ell^{(0)}(M, \mathtt{g})$ be a representative of a DHSF in a given spin frame Ξ defining the orthonormal basis $\{\boldsymbol{e}_\mathbf{a}\}$ for TM and let $\{\theta^\mathbf{a}\}$, $\theta^\mathbf{a} \in \sec \bigwedge^1 T^*M \hookrightarrow\in \sec \mathcal{C}\ell(M, \mathtt{g})$ be the corresponding dual basis. Let moreover $\{\theta_\mathbf{a}\}$ be the reciprocal basis of $\{\theta^\mathbf{a}\}$. We show that[11]

$$[\nabla^{(s)}_{\boldsymbol{e}_\mathbf{a}}, \nabla^{(s)}_{\boldsymbol{e}_\mathbf{b}}]\psi = \frac{1}{2}\mathfrak{R}(\theta_\mathbf{a} \wedge \theta_\mathbf{b})\psi - (T^{\mathbf{c}\cdot}_{\cdot\mathbf{ab}} - \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{ab}} + \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{ba}})\nabla^{(s)}_{\boldsymbol{e}_\mathbf{c}}\psi \tag{7.105}$$

Let $\boldsymbol{u} = \mathtt{g}(u,)$, $\boldsymbol{v} = \mathtt{g}(v,) \in \sec TM$. We calculate $[\nabla^{(s)}_{\boldsymbol{u}}, \nabla^{(s)}_{\boldsymbol{v}}]\psi$. Taking into account that, e.g., $\nabla^{(s)}_{\boldsymbol{u}}\psi = \nabla_{\boldsymbol{u}}\psi + \frac{1}{2}\psi\omega_{\boldsymbol{u}}$, we have

$$\nabla^{(s)}_{\boldsymbol{u}}\nabla^{(s)}_{\boldsymbol{v}}\psi = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\psi + \frac{1}{2}(\nabla_{\boldsymbol{v}}\psi)\omega_{\boldsymbol{u}} + \frac{1}{2}(\nabla_{\boldsymbol{u}}\psi)\omega_{\boldsymbol{v}} + \frac{1}{4}\omega_{\boldsymbol{v}}\omega_{\boldsymbol{u}} + \frac{1}{2}\psi\nabla_{\boldsymbol{u}}\omega_{\boldsymbol{v}}.$$

Then, using some of the results of Chap. 5, we have

$$\begin{aligned}
[\nabla^{(s)}_{\boldsymbol{u}}, \nabla^{(s)}_{\boldsymbol{v}}]\psi &= [\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}]\psi + \frac{1}{2}\psi(\nabla_{\boldsymbol{u}}\omega_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\omega_{\boldsymbol{u}} - \frac{1}{2}[\omega_{\boldsymbol{u}}, \omega_{\boldsymbol{v}}]) \\
&= \frac{1}{2}[\mathfrak{R}(u \wedge v), \psi] + \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\psi \\
&\quad + \frac{1}{2}\psi(\nabla_{\boldsymbol{u}}\omega_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\omega_{\boldsymbol{u}} - \frac{1}{2}[\omega_{\boldsymbol{u}}, \omega_{\boldsymbol{v}}])
\end{aligned}$$

---

[11] Compare Eq. (7.105) with Eq. (5.57). Also, compare Eq. (7.105) with Eq. (6.4.54) of Rammond's book [25].

$$
\begin{aligned}
&= \frac{1}{2}[\mathfrak{R}(u \wedge v), \psi] + \nabla^{(s)}_{[u,v]}\psi - \frac{1}{2}\psi\omega_{[u,v]} \\
&\quad + \frac{1}{2}\psi(\nabla_u\omega_v - \nabla_v\omega_u - \frac{1}{2}[\omega_u, \omega_v]) \\
&= \frac{1}{2}[\mathfrak{R}(u \wedge v), \psi] + \nabla^{(s)}_{[u,v]}\psi \\
&\quad + \frac{1}{2}\psi(\nabla_u\omega_v - \nabla_v\omega_u - \frac{1}{2}[\omega_u, \omega_v] - \omega_{[u,v]}) \\
&= \frac{1}{2}[\mathfrak{R}(u \wedge v), \psi] + \nabla^{(s)}_{[u,v]}\psi + \frac{1}{2}\psi\mathfrak{R}(u \wedge v) \\
&= \frac{1}{2}\mathfrak{R}(u \wedge v)\psi + \nabla^{(s)}_{[u,v]}\psi.
\end{aligned}
\tag{7.106}
$$

From Eq. (7.106), Eq. (7.105) follows trivially.

7.12.1 The Generalized Lichnerowicz Formula

In this section we calculate the square of the spin-Dirac operator on a Riemann-Cartan spacetime acting on a representative ψ of the DHSF. We have the

Proposition 7.61

$$
\left(\boldsymbol{\partial}^{(s)}\right)^2\psi = \left(\eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}} - \eta^{\mathbf{ac}}\omega^{\mathbf{b}\cdot}_{\cdot\mathbf{ac}}\right)\nabla^{(s)}_{e_\mathbf{b}}\psi + \frac{1}{4}R\psi + \mathbf{J}\psi - \Theta^{\mathbf{c}}\nabla^{(s)}_{e_\mathbf{c}}\psi
$$

Proof Taking notice that since $\varepsilon^{\mathbf{b}} \in \sec \mathcal{C}\ell(M, g)$, then $\nabla^{(s)}_{e_\mathbf{a}}\theta^{\mathbf{b}} = \nabla_{e_\mathbf{a}}\theta^{\mathbf{b}}$, we have

$$
\begin{aligned}
\left(\boldsymbol{\partial}^{(s)}\right)^2 &= \left(\theta^{\mathbf{a}}\nabla^{(s)}_{e_\mathbf{a}}\right)\left(\theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{b}}\right) \\
&= \theta^{\mathbf{a}}\left[\left(\nabla_{e_\mathbf{a}}\theta^{\mathbf{b}}\right)\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\right] \\
&= \theta^{\mathbf{a}}\lrcorner\left[\left(\nabla_{e_\mathbf{a}}\theta^{\mathbf{b}}\right)\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\right] \\
&\quad + \theta^{\mathbf{a}} \wedge \left[\left(\nabla_{e_\mathbf{a}}\theta^{\mathbf{b}}\right)\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\right]
\end{aligned}
\tag{7.107}
$$

and since $\nabla_{e_\mathbf{a}}\theta^{\mathbf{b}} = -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\theta^{\mathbf{c}}$ we get

$$
\begin{aligned}
\left(\boldsymbol{\partial}^{(s)}\right)^2 &= \theta^{\mathbf{a}}\lrcorner\left[\left(-\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\theta^{\mathbf{c}}\right)\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\right] \\
&\quad + \theta^{\mathbf{a}} \wedge \left[\left(-\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\theta^{\mathbf{c}}\right)\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\right],
\end{aligned}
$$

or

$$\begin{aligned}\left(\partial^{(s)}\right)^2 &= -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\eta^{\mathbf{ac}}\nabla^{(s)}_{e_\mathbf{b}} + \eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} \\ &\quad -\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\theta^{\mathbf{a}}\wedge\theta^{\mathbf{c}}\nabla^{(s)}_{e_\mathbf{b}} + \theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} \\ &= -\omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}} + \eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} \\ &\quad -\omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{c}} + \theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}}\end{aligned},$$

or

$$\left(\boldsymbol{\partial}^{(s)}\right)^2 = \eta^{\mathbf{ab}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right] + \theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right].$$

Now we can define the operator

$$\boldsymbol{\partial}^{(s)}\cdot\boldsymbol{\partial}^{(s)} = \eta^{\mathbf{ab}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right], \tag{7.108}$$

called the *generalized spin D'Alembertian* and

$$\boldsymbol{\partial}^{(s)}\wedge\boldsymbol{\partial}^{(s)} = \theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right], \tag{7.109}$$

which will be called *twisted curvature operator*. Then, we can write

$$\left(\boldsymbol{\partial}^{(s)}\right)^2 = \boldsymbol{\partial}^{(s)}\cdot\boldsymbol{\partial}^{(s)} + \boldsymbol{\partial}^{(s)}\wedge\boldsymbol{\partial}^{(s)}. \tag{7.110}$$

On the other hand, we can write

$$\begin{aligned}\boldsymbol{\partial}^{(s)}\cdot\boldsymbol{\partial}^{(s)} &= \eta^{\mathbf{ab}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right] \\ &= \eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \eta^{\mathbf{ab}}\omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}} \\ &= \eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \eta^{\mathbf{ac}}\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\nabla^{(s)}_{e_\mathbf{b}} \\ &= \left[\eta^{\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{a}} - \eta^{\mathbf{ac}}\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\right]\nabla^{(s)}_{e_\mathbf{b}}\end{aligned} \tag{7.111}$$

and

$$\begin{aligned}\boldsymbol{\partial}^{(s)}\wedge\boldsymbol{\partial}^{(s)} &= \frac{1}{2}\boldsymbol{\partial}^{(s)}\wedge\boldsymbol{\partial}^{(s)} + \frac{1}{2}\boldsymbol{\partial}^{(s)}\wedge\boldsymbol{\partial}^{(s)} \\ &= \frac{1}{2}\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}\nabla^{(s)}_{e_\mathbf{c}}\right] + \frac{1}{2}\theta^{\mathbf{b}}\wedge\theta^{\mathbf{a}}\left[\nabla^{(s)}_{e_\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ba}}\nabla^{(s)}_{e_\mathbf{c}}\right] \\ &= \frac{1}{2}\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\left[\nabla^{(s)}_{e_\mathbf{a}}\nabla^{(s)}_{e_\mathbf{b}} - \nabla^{(s)}_{e_\mathbf{b}}\nabla^{(s)}_{e_\mathbf{a}} - \left(\omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ba}}\right)\nabla^{(s)}_{e_\mathbf{c}}\right]\end{aligned}$$

$$= \frac{1}{2}\boldsymbol{\theta}^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \left[\nabla^{(s)}_{e_{\mathbf{a}}} \nabla^{(s)}_{e_{\mathbf{b}}} - \nabla^{(s)}_{e_{\mathbf{b}}} \nabla^{(s)}_{e_{\mathbf{a}}} - (c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} + T^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}) \nabla^{(s)}_{e_{\mathbf{c}}} \right]. \tag{7.112}$$

Taking into account that $T^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ba}} - c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}$, we have from Eqs. (7.111) and (7.112) that

$$\begin{aligned} \left(\boldsymbol{\partial}^{(s)}\right)^2 &= \left[\eta^{\mathbf{ab}} \nabla^{(s)}_{e_{\mathbf{a}}} - \eta^{\mathbf{ac}} \omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}} \right] \nabla^{(s)}_{e_{\mathbf{b}}} \\ &\quad + \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \left[\nabla^{(s)}_{e_{\mathbf{a}}} \nabla^{(s)}_{e_{\mathbf{b}}} - \nabla^{(s)}_{e_{\mathbf{b}}} \nabla^{(s)}_{e_{\mathbf{a}}} - (c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} + T^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}) \nabla^{(s)}_{e_{\mathbf{c}}} \right]. \end{aligned} \tag{7.113}$$

On the other hand, from Eq. (7.105), we have

$$\left[\nabla^{(s)}_{e_{\mathbf{a}}}, \nabla^{(s)}_{e_{\mathbf{b}}} \right] \psi = \frac{1}{2}\mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \psi + c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} \nabla^{(s)}_{e_{\mathbf{c}}} \psi$$

and then Eq. (7.113) becomes

$$\begin{aligned} \left(\boldsymbol{\partial}^{(s)}\right)^2 \psi &= \left[\eta^{\mathbf{ab}} \nabla^{(s)}_{e_{\mathbf{a}}} - \eta^{\mathbf{ac}} \omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}} \right] \nabla^{(s)}_{e_{\mathbf{b}}} \psi + \frac{1}{4}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \psi \\ &\quad - \frac{1}{2}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} T^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} \nabla^{(s)}_{e_{\mathbf{c}}} \psi \\ &= \left[\eta^{\mathbf{ab}} \nabla^{(s)}_{e_{\mathbf{a}}} - \eta^{\mathbf{ac}} \omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}} \right] \nabla^{(s)}_{e_{\mathbf{b}}} \psi + \frac{1}{4}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \psi - \Theta^{\mathbf{c}} \nabla^{(s)}_{e_{\mathbf{c}}} \psi. \end{aligned}$$

We need to compute $(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right)$. From (Eq. (2.70)) we have

$$\begin{aligned} (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) &= \langle (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \rangle_0 + \langle (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \rangle_2 \\ &\quad + \langle (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \rangle_4 \end{aligned}$$

Now, recalling Eq. (5.56), we can have

$$\begin{aligned} \langle (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \rangle_0 &:= (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \lrcorner \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \\ &= -(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \cdot \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) = R. \end{aligned}$$

Next, we recall the identity (Eq. (2.72))

$$\langle (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right) \rangle_2 = \theta^{\mathbf{a}} \wedge (\theta^{\mathbf{b}} \lrcorner \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right)) + \theta^{\mathbf{a}} \lrcorner (\theta^{\mathbf{b}} \wedge \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right))$$

and also the identity Eq. (2.60)

$$\theta^{\mathbf{a}} \lrcorner (\theta^{\mathbf{b}} \wedge \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right)) - \theta^{\mathbf{a}} \wedge (\theta^{\mathbf{b}} \lrcorner \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right)) = (\theta^{\mathbf{a}} \cdot \theta^{\mathbf{b}}) \mathcal{R}\left(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}\right).$$

Then, it follows that

$$\langle(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})\mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}})\rangle_2 = (\theta^{\mathbf{a}} \cdot \theta^{\mathbf{b}})\mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) = \eta^{\mathbf{ab}}\mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) = 0$$

It remains to calculate $\langle\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}})\rangle_4$. We have

$$\begin{aligned}\langle\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}\mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}})\rangle_4 = \theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \mathcal{R}(\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) &= \frac{1}{2}R_{\mathbf{abcd}}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}} \\ &= \frac{1}{6}(R_{\mathbf{abcd}}\theta^{\mathbf{abcd}} + R_{\mathbf{acdb}}\theta^{\mathbf{acdb}} + R_{\mathbf{adbc}}\theta^{\mathbf{adbc}}) \\ &= \frac{1}{6}(R_{\mathbf{abcd}} + R_{\mathbf{acdb}} + R_{\mathbf{adbc}})\theta^{\mathbf{abcd}}.\end{aligned}$$

Now, using Eq. (4.198) we can write

$$R_{\mathbf{abcd}} = \mathring{R}_{\mathbf{abcd}} + J_{\mathbf{ab[cd]}}$$

where $\mathring{R}_{\mathbf{abcd}}$ are the components of the Riemann tensor of the Levi-Civita connection of $\boldsymbol{g}$ and from Eq. (4.199)

$$\begin{aligned}J^{\mathbf{b}\cdots}_{\cdot\mathbf{acd}} &= \nabla_{\mathbf{c}}K^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{da}} - K^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ck}}K^{\mathbf{k}\cdot\cdot}_{\cdot\mathbf{da}} + K^{\mathbf{k}\cdot\cdot}_{\cdot\mathbf{cd}}K^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ka}}, \\ J^{\mathbf{b}\cdots}_{\cdot\mathbf{a[cd]}} &= J^{\mathbf{b}\cdots}_{\cdot\mathbf{acd}} - J^{\mathbf{b}\cdots}_{\cdot\mathbf{adc}},\end{aligned}$$

with $K^{\mathbf{k}\cdot\cdot}_{\cdot\mathbf{cd}}$ given by Eq. (4.197), i.e.,

$$K^{\mathbf{k}\cdot\cdot}_{\cdot\mathbf{cd}} = -\frac{1}{2}\eta^{\mathbf{km}}(\eta_{\mathbf{nc}}T^{\mathbf{n}\cdot\cdot}_{\cdot\mathbf{md}} + \eta_{\mathbf{nd}}T^{\mathbf{n}\cdot\cdot}_{\cdot\mathbf{mc}} - \eta_{\mathbf{nm}}T^{\mathbf{n}\cdot\cdot}_{\cdot\mathbf{cd}}).$$

Taking into account the well known first Bianchi identity $\mathring{R}_{\mathbf{abcd}} + \mathring{R}_{\mathbf{acdb}} + \mathring{R}_{\mathbf{adbc}} = 0$, we have

$$\left(\boldsymbol{\partial}^{(s)}\right)^2\psi = \left[\eta^{\mathbf{ab}}\nabla^{(s)}_{e_{\mathbf{a}}} - \eta^{\mathbf{ac}}\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\right]\nabla^{(s)}_{e_{\mathbf{b}}}\psi + \frac{1}{4}R\psi + \mathbf{J}\psi - \Theta^{\mathbf{c}}\nabla^{(s)}_{e_{\mathbf{c}}}\psi, \tag{7.114}$$

where

$$\begin{aligned}\mathbf{J} &= \frac{1}{6}(J_{\mathbf{ab[cd]}} + J_{\mathbf{ac[db]}} + J_{\mathbf{ad[bc]}})\theta^{\mathbf{abcd}} \\ &= \frac{1}{6}(J_{\mathbf{ab[cd]}} + J_{\mathbf{ac[db]}} + J_{\mathbf{ad[bc]}})\epsilon^{\mathbf{abcd}}_{\mathbf{0123}}\tau_{g},\end{aligned} \tag{7.115}$$

and the proposition is proved. ■

Remark 7.62 Equation (7.114) may be called the generalized Lichnerowicz formula and (equivalent expressions) appears in the case of a totally skew-symmetric torsion in many different contexts, like, e.g., in the geometry of moduli spaces of a class of black holes, the geometry of NS-5 brane solutions of type II supergravity theories and BPS solitons in some string theories [8] and many important topics of modern mathematics (see [4, 10]). For a Levi-Civita connection we have that $\mathbf{J} = 0$ and $\Theta^{\mathbf{c}} = 0$ and we obtain the famous Lichnerowicz formula [16].

Exercise 7.63 Describe the bundles of dotted and undotted algebraic spinor fields and their tensor product.[12]

Solution We start by considering the bundle

$$\mathcal{C}\ell^{(0)}(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{Ad} \mathbb{R}^{(0)}_{1,3}, \tag{7.116}$$

which may be called the bundle of Pauli fields. Next we define the spinor bundles

$$\mathcal{S}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_l \mathbf{I}, \quad \dot{\mathcal{S}}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_r \dot{\mathbf{I}} \tag{7.117}$$

where $\mathbf{I} = \mathbb{R}^{(0)}_{1,3}\mathrm{e}$ is the ideal in $\mathbb{R}^{(0)}_{1,3}$ defined in Eq. (3.129), $\dot{\mathbf{I}}' = \mathbb{R}^{(0)}_{1,3}\mathrm{e}$. Also, l stands for a left modular representation of $\mathrm{Spin}^e_{1,3}$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(\frac{1}{2},0)}$ representation of $\mathrm{Sl}(2, \mathbb{C})$ and r stands for a right modular representation of $\mathrm{Spin}^e_{1,3}$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(0,\frac{1}{2})}$ representation of $\mathrm{Sl}(2, \mathbb{C})$.

We then have the obvious isomorphism

$$\begin{aligned}\mathcal{C}\ell^{(0)}(M, \mathtt{g}) &= P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}^{(0)}_{1,3} \\ &= P_{\mathrm{Spin}^e_{1,3}}(M) \times_{l \otimes r} \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}} \\ &= \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M).\end{aligned} \tag{7.118}$$

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[12] For a detailed discussion of the theory of dotted and undotted *algebraic* spinor fields on arbitrary Lorentzian space times the reader may consult [22].

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Chapter 8
A Clifford Algebra Lagrangian Formalism in Minkowski Spacetime

Abstract Using tools introduced in previous chapters, particularly the concept of Clifford and spin-Clifford bundles (and the representations of sections of the spin-Clifford bundles as equivalence classes of sections the Clifford bundle), extensor fields and the Dirac operator we give an unified and original approach to the Lagrangian field theory in Minkowski spacetime with special emphasis on the Maxwell and Dirac-Hestenes fields. We derive for these fields their canonical energy-momentum extensor fields and also their angular momentum and spin momentum extensor fields. In particular we show that the antisymmetric part of the canonical energy-momentum tensor is the "source" of spin of the field. Several nontrivial exercises are solved with details in order to help the reader to familiarize with the formalism and to make contact with standard formulations of field theory.

8.1 Some Preliminaries

Recall that in Definition (4.108) Minkowski spacetime has been introduced as the structure $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ where $M \simeq \mathbb{R}^4$ and $\mathbf{R}(D) = 0$, $\mathbf{T}(D) = 0$. As we know from Chap. 6, Minkowski spacetime possess an infinity of physically equivalent inertial reference frames. These are frames $\mathbf{I} \in \sec TM$ such that $D\mathbf{I} = 0$. Given an inertial frame $\mathbf{I}$, a global coordinate chart with coordinate functions in the Einstein-Lorentz-Poincaré coordinate gauge $\{\boldsymbol{x}^\mu\}$ for M is a $(nacs|\mathbf{I})$ if $\mathbf{I} = \frac{\partial}{\partial x^0}$. We choose a global basis of the *orthonormal* frame bundle $\boldsymbol{\Sigma} = \{\boldsymbol{e}_\mu\}$, with $\boldsymbol{e}_0 = \mathbf{I}$ and $\boldsymbol{e}_i = \frac{\partial}{\partial x^i}$. The frame $\boldsymbol{\Sigma}$ can be used in order to define an equivalence relation between tensors located at different points $\mathfrak{e} \in M$. This can be done as follows.

The dual basis of $\{\boldsymbol{e}_\mu\}$ will be denoted by $\{\gamma^\mu\}$, i.e., $\gamma^\mu = dx^\mu \in \sec \bigwedge^1 T^*M$ and $\gamma^\mu(\boldsymbol{e}_\nu) = \delta^\mu_\nu$. As previously introduced, the set $\{\gamma_\mu\}$, with $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$ is called the *reciprocal* basis of the basis $\{\gamma^\mu\}$. Recall that $\{\boldsymbol{e}_\mu\big|_\mathfrak{e}\}$ and $\{dx^\mu|_\mathfrak{e}\}$ are the natural basis for the tangent vector space $T_\mathfrak{e}M$ and the tangent covector space $T^*_\mathfrak{e}M$. In what follows we write,

$$\boldsymbol{x}^\mu(\mathfrak{e}) = x^\mu, \boldsymbol{x}^\mu(\mathfrak{e}') = x'^\mu, \tag{8.1}$$

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and recall that

$$D_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = 0, \tag{8.2}$$

$$\boldsymbol{\eta} = \eta_{\mu\nu} dx^\mu \otimes dx^\nu, \quad \eta_{\mu\nu} = \boldsymbol{\eta}(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}), \tag{8.3}$$

where the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix diag$(1, -1, -1, -1)$.

Definition 8.1 Let $\mathbf{T}, \mathbf{T}' \in \sec T^r_s M$. We say that $\mathbf{T}_e \in T^r_{s\mathfrak{e}} M$ is equivalent to $\mathbf{T}_{\mathfrak{e}'} \in T^r_{s\mathfrak{e}'} M$ (written $\mathbf{T}_\mathfrak{e} = \mathbf{T}'_{\mathfrak{e}'}$) if and only if

$$\mathbf{T}^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(\boldsymbol{x}^\mu(\mathfrak{e})) = \mathbf{T}'^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(\boldsymbol{x}^\mu(\mathfrak{e}')) \tag{8.4}$$

As usual, the set of vectors equivalent to $\mathbf{v}_\mathfrak{e} \in T_\mathfrak{e} M$ will be denoted by $[\mathbf{v}_\mathfrak{e}]$. The set of equivalent classes of tangent vectors over the tangent bundle is a vector space over the reals, that we denote by

$$\mathbf{M} = \{[\mathbf{v}_\mathfrak{e}] \mid \text{for all } \mathbf{v}_\mathfrak{e} \in T_\mathfrak{e} M\}. \tag{8.5}$$

Note that $\{[\boldsymbol{e}_\mu\big|_\mathfrak{e}]\}$ is a natural basis for $\mathbf{M}$. With the notations $\vec{v} = [\mathbf{v}_\mathfrak{e}]$ and $\vec{e}_\mu = [\boldsymbol{e}_\mu\big|_\mathfrak{e}]$ we can write $\vec{v} = v^\mu \vec{e}_\mu$. The dual space of $\mathbf{M}$ will be denoted by $\boldsymbol{M} = \mathbf{M}^*$. The dual basis of $\{\vec{e}_\mu\}$ will be denoted by $\{\boldsymbol{\gamma}^\mu\}$. A scalar product in $\boldsymbol{M}$, $\cdot : \boldsymbol{M} \times \boldsymbol{M} \to \mathbb{R}$ can be defined in a canonical way using η (the metric of the cotangent bundle) and the equivalence relation for tensors at different spacetime points. Recall moreover that the Clifford algebra generated by the structure $(\boldsymbol{M}, \cdot)$ is $\mathcal{C}\ell(\boldsymbol{M}, \cdot) \simeq \mathbb{R}_{1,3}$.

The pair $(M, \boldsymbol{M})$ has the structure of an *affine* space, and $(M, \boldsymbol{M}, \cdot)$ is a *metric affine space.*

Definition 8.2 Fix a point $\mathfrak{e}_o \in M$. Then for any $\mathfrak{e} \in M$, we call $x \in \boldsymbol{M}$ such that

$$x := (\mathfrak{e} - \mathfrak{e}_o) \in \boldsymbol{M}$$

the position covector of $\mathfrak{e}$ relative to $\mathfrak{e}_o$.

Given two different Einstein-Lorentz-Poincaré coordinate charts for M, with coordinate functions $\{\boldsymbol{x}^\mu\}$ and $\{\boldsymbol{y}^\mu\}$, $\mathfrak{e} \in M$ is represented by the coordinates

$$\{\boldsymbol{x}^\mu(\mathfrak{e})\} \text{ and } \{\boldsymbol{y}^\mu(\mathfrak{e})\}, \tag{8.6}$$

which are related in general by Poincaré transformation $(\mathbf{L}, a) \equiv (L^\mu_\nu, a^\rho)$. Of course, when we introduce coordinates using the Einstein-Lorentz-Poincaré coordinate functions $\boldsymbol{x}^\mu$, a given event, say $\mathfrak{e}_o$ has coordinates $\boldsymbol{x}^\mu(\mathfrak{e}_o) = (0, 0, 0, 0)$ and since $\boldsymbol{y}^\mu(\mathfrak{e}_o) = L^\mu_\nu x^\mu(\mathfrak{e}_o) + a^\mu$ it follows that $y^\mu(\mathfrak{e}_o) = a^\mu$. So, we can write

$$x = \mathfrak{e} - \mathfrak{e}_0 = \boldsymbol{x}^\mu(\mathfrak{e})\boldsymbol{\gamma}_\mu = x^\mu \boldsymbol{\gamma}_\mu = \boldsymbol{y}^\mu(\mathfrak{e})\boldsymbol{\gamma}'_\mu = y^\mu \boldsymbol{\gamma}'_\mu, \tag{8.7}$$

where[1]

$$\boldsymbol{\gamma}'^{\mu} = L^{-1\mu}_{\ \ \nu}\boldsymbol{\gamma}^{\nu}. \tag{8.8}$$

Remark 8.3 It is clear that once we fix a coordinate chart for M with coordinate functions in Einstein-Lorentz-Poincaré gauge $\{\boldsymbol{x}^{\mu}\}$, with $\boldsymbol{x}^{\mu}(\mathfrak{e}_0) = 0$, there is a correspondence between the position covector $x = (\mathfrak{e} - \mathfrak{e}_0)$ and the object $\boldsymbol{x}^{\mu}(\mathfrak{e})\gamma_{\mu} \in \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M,\eta)$. However take notice that $\boldsymbol{x}^{\mu}(\mathfrak{e})\gamma_{\mu}$ is not a legitimate 1-form field because the $\boldsymbol{x}^{\mu}(\mathfrak{e})$, $\mu = 0,1,2,3$ are not the components of a vector field. However, if we restrict ourselves to charts such that the coordinates are related by Poincaré transformations then we can call $\boldsymbol{x}^{\mu}(\mathfrak{e})\gamma_{\mu}$ the *position* 1*-form* of the event $\mathfrak{e} \in M$ relative to event $\mathfrak{e}_o \in M$. When there is no chance of confusion we denote $\boldsymbol{x}^{\mu}(\mathfrak{e})\gamma_{\mu}$ by the same symbol x used to denote the position (co)vector $(\mathfrak{e} - \mathfrak{e}_0) \in \boldsymbol{M}$.

The introduction of the position covector and the position "1-form field", permit us to associate to each Clifford field $C \in \sec \mathcal{C}\ell(M,\eta)$ a multiform (valued) function $\boldsymbol{C}$ of the position covector x. Indeed, the representation of C in a Einstein-Lorentz-Poincaré chart with coordinate functions $\{\boldsymbol{x}^{\mu}\}$ is

$$C = \frac{1}{\nu(J)} C_J(x^{\alpha})\gamma^J, \tag{8.9}$$

where $C_J : \mathbb{R}^4 \to \mathbb{R}$ are (smooth) functions. The multiform (valued) function associated to C is

$$\boldsymbol{C} : \boldsymbol{M} \to \mathcal{C}\ell(\boldsymbol{M},\cdot),\ x \mapsto \boldsymbol{C}(x), \tag{8.10}$$

such that in the Einstein-Lorentz-Poincaré chart with coordinate functions $\{\boldsymbol{x}^{\mu}\}$ we have

$$\boldsymbol{C} = \frac{1}{\nu(J)} \boldsymbol{C}_J(x)\boldsymbol{\gamma}^J, \tag{8.11}$$

where the $\boldsymbol{C}_J : \boldsymbol{M} \to \mathbb{R}$ are (smooth) functions such that (recalling that $x = \boldsymbol{x}^{\alpha}(\mathfrak{e})\boldsymbol{\gamma}_{\alpha} = x^{\alpha}\boldsymbol{\gamma}_{\alpha}$) we have

$$C_J(x^{\alpha}) = \boldsymbol{C}_J(x). \tag{8.12}$$

Remark 8.4 Since in this chapter we shall use only coordinate charts in Einstein-Lorentz-Poincaré gauge, taking into account the identification provided by Eq. (8.12), $(\mathfrak{e} - \mathfrak{e}_0) \leftrightarrow x$, $C \leftrightarrow \boldsymbol{C}$ every time we introduce a Clifford field $C \in \sec \mathcal{C}\ell(M,\eta)$ we immediately identify it with the corresponding multiform

[1] $L^{-1\mu}_{\ \ \nu}$ refers, of course, to the elements of the Lorentz transformation represented by the matrix $\mathbf{L}^{-1}$.

(valued) function $\boldsymbol{C} : \boldsymbol{M} \to \mathcal{C}\ell(\boldsymbol{M}, \cdot)$ and when no confusion arises we use the same symbol for both. The importance of the identification $(\mathfrak{e} - \mathfrak{e}_0) \leftrightarrow x$, $C \leftrightarrow \boldsymbol{C}$ is that it permits the use all identities of the multiform functions of multiform variables, developed in Chap. 2 in an obvious way.

We recall now that a (p, q)-extensor (Definition 2.45) is a linear mapping

$$t : \bigwedge^p \boldsymbol{M} \to \bigwedge^q \boldsymbol{M} \tag{8.13}$$

and that the set of all (p, q)-extensors is denoted $ext(\bigwedge^p \boldsymbol{M}, \bigwedge^q \boldsymbol{M})$.

Definition 8.5 A smooth (p, q)-extensor *field* t on Minkowski spacetime is a differentiable (p, q)-extensor valued function on M,

$$M \ni \mathfrak{e} \mapsto t_{\mathfrak{e}} \in ext(\bigwedge^p T^*_{\mathfrak{e}} M, \bigwedge^q T^*_{\mathfrak{e}} M). \tag{8.14}$$

Remark 8.6 As it was the case for sections of $\mathcal{C}\ell(M, \eta)$, it is sometimes convenient to associate (in an obvious way) to a smooth (p, q)-extensor field t a (p, q)-extensor valued function $\boldsymbol{t}$,

$$\boldsymbol{M} \ni x \mapsto \boldsymbol{t}_x \in ext(\bigwedge^p \boldsymbol{M}, \bigwedge^q \boldsymbol{M}). \tag{8.15}$$

This again permit us to use the identities presented in the extensor calculus developed in Chap. 2 in the computations that follows. Once again we remark that since we are to use only charts in the Einstein-Lorentz-Poincaré gauge we shall use (when no confusion arises) the same symbol for both t and $\boldsymbol{t}$.

8.2 Lagrangians and Lagrangian Densities for Multiform Fields

Let Y be a Clifford field, i.e., $Y = X \in \sec \mathcal{C}\ell(M, \eta)$, or a *representative* of a Dirac-Hestenes spinor field, i.e., $Y = \psi \in \sec \mathcal{C}\ell^{(0)}(M, \eta)$.[2] Since we *restrict* ourselves to Einstein-Lorentz-Poincaré coordinate charts and choose a gauge where the connection coefficients are null, the Dirac operator and the representative of the spin-Dirac operator acting on sections of $\mathcal{C}\ell(M, \eta)$ has a very simple representation indeed, and in order to leave no chance for confusion with the general case, we

[2]To shorten the notation, when Y is a homogeneous section of the Clifford bundle, we write $Y \in \sec \bigwedge^p T^*M$, instead $Y \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$.

write, $\boldsymbol{\partial} \mapsto \boldsymbol{\partial}_x$, $\boldsymbol{\partial}^{(s)} \mapsto \boldsymbol{\partial}_x^{(s)}$. Then, with the above assumption

$$\boldsymbol{\partial}_x = \boldsymbol{\partial}_x^{(s)} = \gamma^\mu \frac{\partial}{\partial x^\mu}. \tag{8.16}$$

The operator $\boldsymbol{\partial}_x$ acts on the multiform (valued) function $\boldsymbol{Y} : \boldsymbol{M} \to \mathcal{C}\ell(\boldsymbol{M}, \cdot)$ associated to $Y \in \sec \mathcal{C}\ell(M, \eta)$ as the *vector derivative* relative to the position covector x also denoted $\boldsymbol{\partial}_x$ introduced in Sect. 2.11.2. We write

$$(\mathfrak{e} - \mathfrak{e}_0) \leftrightarrow x, \qquad \boldsymbol{\partial}_x Y \leftrightarrow \boldsymbol{\partial}_x \boldsymbol{Y}(x),$$

and if

$$Y = \sum_J \frac{1}{\nu(J)} \gamma^J Y_J(x^\mu) \tag{8.17}$$

we have that

$$\boldsymbol{\partial}_x Y = \gamma^\mu \sum_J \frac{1}{\nu(J)} \gamma^J \frac{\partial}{\partial x^\mu} Y_J(x^\mu). \tag{8.18}$$

Remark 8.7 In what follows we shall need to consider also the derivatives $\boldsymbol{\partial}_x \circledast Y$, where $\circledast$ is any one of the product of multiforms (Clifford, $\lrcorner$, $\llcorner$ or $\wedge$) introduced in Sect. 2.12 and also derivatives of functionals of multiform (valued) functions.

Definition 8.8 The Clifford jet bundle[3] $\mathbf{J}_\circledast(\mathcal{C}\ell(M, \eta))$ of M is the (trivial) vector bundle given by

$$\mathbf{J}_\circledast(\mathcal{C}\ell(M, \eta)) = \bigcup\nolimits_{x \in M} J_x^1(\mathcal{C}\ell(M, \eta)), \tag{8.19}$$

such that for $X, Y \in \sec \mathcal{C}\ell(M, \eta)$ such that $X \underset{x}{\overset{1}{\sim}} Y$ we have

$$\boldsymbol{\partial}_x \circledast Y(x) = \boldsymbol{\partial}_x \circledast X(x). \tag{8.20}$$

Definition 8.9 We call a functional of the Clifford field $Y \in \sec \mathcal{C}\ell(M, \eta)$ any mapping $\mathcal{F} : \mathbf{J}_\circledast(\mathcal{C}\ell(M, \eta)) \to \mathcal{C}\ell(M, \eta)$.

Remark 8.10 Given any field $Y \in \sec \mathcal{C}\ell(M, \eta)$ we abbreviate by $\mathcal{F}_\circledast(Y)$ (or $\mathcal{F}(x, Y, \boldsymbol{\partial}_x \circledast Y)$ or *yet*[4] $\mathcal{F}(Y(x), \boldsymbol{\partial}_x \circledast Y(x))$) the mapping

$$\mathcal{F} \circ \mathbf{J}_\circledast(Y). \tag{8.21}$$

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[3]For the definition of jet bundles and notation employed see Sect. A.1.3.

[4]In this case we are using a common sloppy notation where the section Y of the Clifford bundle is written as $Y(x)$. Note that $\boldsymbol{Y}(x)$ is also a sloppy notation for the mutiform (valued) function $\boldsymbol{Y}$.

Remark 8.11 To any functional of the Clifford field Y corresponds in an obvious way a functional of the corresponding multiform (valued) function $\boldsymbol{Y}$ with values in $\mathcal{C}\ell(\boldsymbol{M},\cdot)$ and both objects are identified (when no confusion arises) in order to be possible to use in a standard way the theory of multiform functions of multiform variables described in Chap. 2.

Remark 8.12 Clearly, given two functionals $\mathcal{F}$, $\mathcal{F}'$: $\mathbf{J}_{\circledast}(\mathcal{C}\ell(M,\eta)) \to \mathcal{C}\ell(M,\eta)$ any one of the products $\mathcal{F} \circledast \mathcal{F}'$ is well defined. We write

$$(\mathcal{F} \circledast \mathcal{F}')(Y) = \mathcal{F}(Y) \circledast \mathcal{F}'(Y). \tag{8.22}$$

Definition 8.13 In the case $\mathcal{L} : \mathbf{J}_{\circledast}(\mathcal{C}\ell(M,\eta)) \to \bigwedge^4 T^*M$, $\mathcal{L}$ is said a Lagrangian density for the field Y. In the case, $\mathfrak{L} : \mathbf{J}_{\circledast}(\mathcal{C}\ell(M,\eta)) \to \bigwedge^0 T^*M$, $\mathfrak{L}$ is said a Lagrangian for the field Y.

For any Lagrangian density $\mathcal{L}(Y, \boldsymbol{\partial}_x \circledast Y)$ there corresponds a well defined Lagrangian $\mathfrak{L}(Y, \boldsymbol{\partial}_x \circledast Y)$ and they are related by

$$\mathcal{L}(Y, \boldsymbol{\partial}_x \circledast Y) = \mathfrak{L}(Y, \boldsymbol{\partial}_x \circledast Y)\tau_{\boldsymbol{\eta}}, \tag{8.23}$$

where τ_η is the volume element in M.

Definition 8.14 To any Lagrangian density $\mathcal{L}_{\circledast}(Y)$ or Lagrangian $\mathfrak{L}_{\circledast}(Y)$ the action for the multiform field $Y \in T^*M$ on $U \subseteq M$ is the real number

$$\mathcal{A}(Y) = \int_U \mathcal{L}_{\circledast}(Y) = \int_U \mathfrak{L}_{\circledast}(Y)\tau_{\boldsymbol{\eta}}. \tag{8.24}$$

8.2.1 Variations

Vertical Variation

In this book, we restrict our investigation only to theories where the fields Y are Clifford fields ($Y = X$) or Dirac-Hestenes spinor fields ($Y = \psi$). As we learned in Chap. 7 these fields carry different representations of $\mathrm{Spin}^e_{1,3}$,which is the universal covering group of the homogeneous restrict and orthochronous Lorentz group $\mathcal{L}_0^{\uparrow}$. Let be $u \in \mathrm{Spin}^e_{1,3}(M) \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$, i.e., for any $\mathrm{x} \in M$, $u(\mathrm{x}) \in \mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Then, if $Y = X \in \sec \mathcal{C}\ell(M,\eta)$ is a Clifford field, an active local Lorentz transformation sends $X \mapsto X' \in \sec \mathcal{C}\ell(M,\eta)$, with

$$X' = uX\tilde{u}. \tag{8.25}$$

If $Y = \psi \in \sec \mathcal{C}\ell^{(0)}(M, \eta)$ is a representative of a Dirac-Hestenes spinor field, then an active local transformation sends $\psi \mapsto \psi'$, with

$$\psi' = u\psi. \tag{8.26}$$

From Sect. 3.3.4 we know that each $u \in \mathrm{Spin}^e_{1,3}(M)$ can be written as $\pm$ the exponential of a biform field $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$. For infinitesimal transformations we must choose the $+$ sign and write $F = \alpha f$, $\alpha \ll 1$, $F^2 \neq 0$.

Definition 8.15 Let $Y = X$ be a Clifford field or $Y = \psi$ a representative of a Dirac-Hestenes spinor field. The vertical variation of Y is the field $\boldsymbol{\delta}_v Y$ (of the same nature of Y) such that

$$\boldsymbol{\delta}_v Y = Y' - Y. \tag{8.27}$$

Remark 8.16 The case where F is independent of $x \in M$ is said to be a gauge transformation of the first kind, and the general case is said to be a gauge transformation of the second kind.

Horizontal Variation

Let σ_t be a one-parameter group of diffeomorphisms of M and let $\xi \in \sec TM$ be the vector field that generates σ_t, i.e., in local coordinates in Einstein-Lorentz-Poincaré gauge, we have

$$\xi^\mu(x^\alpha) = \left.\frac{d\sigma_t^\mu(x^\alpha)}{dt}\right|_{t=0}. \tag{8.28}$$

Definition 8.17 We call the horizontal variation of ϕ induced by a one-parameter group of diffeomorphisms of M the quantity

$$\boldsymbol{\delta}_h Y = \lim_{t\to 0} \frac{Y - \sigma_t^* Y}{t} = -\pounds_\xi Y. \tag{8.29}$$

Definition 8.18 We call total variation of a multiform field ϕ the quantity

$$\boldsymbol{\delta} Y = \boldsymbol{\delta}_v Y + \boldsymbol{\delta}_h Y = \boldsymbol{\delta}_v Y - \pounds_\xi Y. \tag{8.30}$$

Remark 8.19 It is *crucial* to distinguish between the variations defined above, something that sometimes is not done appropriately in textbooks. We denote in what follows by $\boldsymbol{\delta}$ a generic variation (horizontal or vertical). In particular such a distinction is essential for the developments in Chap. 9.

Now, when we have a field theory formulated on *Minkowski* spacetime, diffeomorphisms associated with *global* Lorentz transformations have a very important physical meaning, as we learned in Chap. 6. How to describe such a diffeomorphism using the Clifford bundle formalism? Recall that a diffeomorphism related to a global Lorentz transformation is given by

$$\ell : \mathfrak{e} \mapsto \mathfrak{e}',\ x = \mathfrak{e} - \mathfrak{e}_o = \boldsymbol{x}^{\mu}(\mathfrak{e})\gamma_{\mu} \mapsto x' = lx = \boldsymbol{x}^{\mu}(\mathfrak{e}')\gamma_{\mu}, \tag{8.31}$$

with

$$\boldsymbol{x}^{\mu}(\mathfrak{e}') = L^{\mu}_{\nu}\boldsymbol{x}^{\nu}(\mathfrak{e}), \tag{8.32}$$

with $L^{\mu}_{\nu} \in \mathrm{S0}^{e}_{1,3}$ independent of $\mathfrak{e} \in M$. We can then write

$$x' = L^{\mu}_{\nu}\boldsymbol{x}^{\nu}(\mathfrak{e})\gamma_{\mu} = \boldsymbol{x}^{\nu}(\mathfrak{e})u^{-1}\gamma_{\nu}u = \boldsymbol{x}^{\nu}(\mathfrak{e})\gamma'_{\nu} = u^{-1}xu \tag{8.33}$$

for some constant section $u \in \mathrm{Spin}^{e}_{1,3}(M) \hookrightarrow \sec\ \mathcal{C}\ell(M,\eta)$, i.e., independent of $\mathfrak{e} \in M$ such that the $2-1$ homomorphism $h : \mathrm{Spin}^{e}_{1,3} \to \mathrm{S0}^{e}_{1,3}$ (Chap. 3) gives $h(u^{-1}) = l$. Recalling now our discussion of Sect. 4.5 concerning the pullback mapping, we have: putting $\ell^{*}X = X'$, $\ell^{*}\psi = \psi'$ that

$$\begin{aligned} X'(x) &= X\left(x'\right) = uX(x)u^{-1}, \\ \psi'(x) &= \psi\left(x'\right) = u\psi(x). \end{aligned} \tag{8.34}$$

Remark 8.20 Equation (8.34) shows that the pullback mapping corresponding to a global Lorentz transformation in the present formalism has the same form as a local constant rotation, but care must be taken in using such formulas in order to avoid misconceptions.

8.3 Stationary Action Principle and Euler-Lagrange Equations

Let U be an open set of M, with boundary ∂U and consider an arbitrary multiform field $A \in \sec \mathcal{C}\ell(M,\eta)$ such that $A|_{\ \partial U} = 0$. Given a Lagrangian $\mathfrak{L}_{\circledast}(Y) \equiv \mathcal{L}(Y, \boldsymbol{\partial}_x \circledast Y)$ we introduce [2] the mapping[5] $\mathfrak{L}_A : M \times I \to \mathbb{R}$

$$\mathfrak{L}_A(x,\lambda) := \mathfrak{L}(Y(x) + \lambda\langle A(x)\rangle_Y, \boldsymbol{\partial}_x \circledast Y(x) + \lambda\langle \boldsymbol{\partial}_x \circledast A(x)\rangle_{\boldsymbol{\partial}_x \circledast Y}), \tag{8.35}$$

[5]Recall that in Eq. (8.35) $\langle A\rangle_X$ means the projection of A in the grades of X. See Definition 2.4, Chap. 2.

where $\lambda \in I$ and I is a subset of $\mathbb{R}$ containing zero. Note that $\mathfrak{L}_A(x,0) = \mathfrak{L}(Y(x), \boldsymbol{\partial}_x \circledast Y(x))$. In view of this fact we call $\mathfrak{L}_A(x,\lambda)$ the *varied* Lagrangian.

Definition 8.21 Given a Lagrangian $\mathfrak{L}_{\circledast}(Y)$ we define its variation by

$$\boldsymbol{\delta}\mathfrak{L}_{\circledast}(Y) := \frac{d}{d\lambda}\mathfrak{L}(Y(x) + \lambda\langle A(x)\rangle_Y, \boldsymbol{\partial}_x \circledast Y(x) + \lambda\langle \boldsymbol{\partial}_x \circledast A(x)\rangle_{\boldsymbol{\partial}_x \circledast Y})\Big|_{\lambda=0}. \tag{8.36}$$

Definition 8.22 The variation of the action is

$$\boldsymbol{\delta}\mathcal{A}(Y) = \boldsymbol{\delta}\int_U \mathfrak{L}_{\circledast}(Y) = \int_U \boldsymbol{\delta}\mathfrak{L}_{\circledast}(Y). \tag{8.37}$$

Axiom 8.23 *In Lagrangian field theory the dynamics of a field Y is supposed to be derived from the Stationary Action Principle, hereafter denoted* SAP*, i.e., we suppose that the laws of motion are to be deduced from*

$$\boldsymbol{\delta}\mathcal{A}(Y) = 0 \textit{ for all } A \textit{ such that } A|_{\partial U} = 0\,. \tag{8.38}$$

The SAP implies the so called Euler-Lagrange equations (ELE) for the field Y.

Proposition 8.24 *Given a field Y on $U \subseteq M$ and postulated a Lagrangian $\mathfrak{L}(Y, \boldsymbol{\partial}_x \circledast Y)$ the* SAP *implies for the cases:* (a) $\boldsymbol{\partial}_x \circledast Y = \boldsymbol{\partial}_x \lrcorner Y$; (b) $\boldsymbol{\partial}_x \circledast Y = \boldsymbol{\partial}_x \wedge Y$; (c) $\boldsymbol{\partial}_x \circledast Y = \boldsymbol{\partial}_x Y$ *the following* ELE*s.*

$$\begin{array}{ll}\text{(a)} & \partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x \lrcorner Y) - \boldsymbol{\partial}_x \wedge \partial_{\boldsymbol{\partial}_x \lrcorner Y}\mathfrak{L}(Y, \boldsymbol{\partial}_x \lrcorner Y) = 0,\\ \text{(b)} & \partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x \wedge Y) - \boldsymbol{\partial}_x \lrcorner \partial_{\boldsymbol{\partial}_x \wedge Y}\mathfrak{L}(Y, \boldsymbol{\partial}_x \wedge Y) = 0,\\ \text{(c)} & \partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) - \boldsymbol{\partial}_x(\partial_{\boldsymbol{\partial}_x Y}\mathfrak{L}(Y, \boldsymbol{\partial}_x Y)) = 0.\end{array} \tag{8.39}$$

Proof Here, we prove only case (c), leaving the proof of the other cases as exercises for the reader. The Y-variation of $\mathfrak{L}(Y, \boldsymbol{\partial}_x Y)$ gives immediately, using the definition of derivatives of multiform functions of multiform variables (see Eq. (2.165)),

$$\boldsymbol{\delta}\mathfrak{L}(Y, \boldsymbol{\partial}_x Y) = A \cdot \partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) + \boldsymbol{\partial}_x A \cdot \partial_{\boldsymbol{\partial}_x Y}\mathfrak{L}(Y, \boldsymbol{\partial}_x Y) \tag{8.40}$$

Now, recall the identity (c) in Eq. (2.111) which says that for any multiform fields Y and 1-form $a = a^\mu \gamma_\mu$, with a^μ constant functions, we have

$$(\boldsymbol{\partial}_x Y) \cdot X + Y \cdot (\boldsymbol{\partial}_x X) = \boldsymbol{\partial}_x \cdot [\partial_a (aY) \cdot X] \tag{8.41}$$

where $\partial_a = \gamma^\mu \frac{\partial}{\partial a^\mu}$. Using this result we can write Eq. (8.40) as

$$\begin{aligned}\boldsymbol{\delta}\mathfrak{L}(Y, \boldsymbol{\partial}_x Y) = A \cdot [\partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) - \boldsymbol{\partial}_x \cdot \partial_Y \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y)]\\ + \boldsymbol{\partial}_x \cdot [\partial_a (aA) \cdot \partial_{\partial Y}\mathfrak{L}(Y, \boldsymbol{\partial}_x Y)].\end{aligned} \tag{8.42}$$

Then the SAP gives

$$\int_U \{A \cdot [\partial_Y \mathfrak{L} - \boldsymbol{\partial}_x \cdot \partial_Y \mathfrak{L}] + \boldsymbol{\partial}_x \cdot [\partial_a (aA) \cdot \partial_{\partial Y} \mathfrak{L}]\} \tau_{\boldsymbol{\eta}} = 0, \tag{8.43}$$

for all A such that $A|_{\partial U} = 0$. Now, given any form field v we write Stokes theorem for $d(\star v)$ in the form

$$\int_U (\boldsymbol{\partial}_x \cdot v) \tau_{\boldsymbol{\eta}} = \int_{\partial U} v \lrcorner \tau_{\boldsymbol{\eta}}. \tag{8.44}$$

Then, the second term in Eq. (8.43) taking into account the boundary condition can be written (in obvious notation)

$$\begin{aligned} \int_U \boldsymbol{\partial}_x \cdot [\partial_a (aA) \cdot \partial_{\partial Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x Y)] \tau_{\eta} &= \int_{\partial U} \gamma^{\mu} \cdot [\partial_a (aA) \cdot \partial_{\partial Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x Y)](\gamma_{\mu} \lrcorner \tau_{\boldsymbol{\eta}}) \\ &= \int_{\partial U} A \cdot (\gamma^{\mu} \partial_{\partial_Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) d^3 S_{\mu} = 0. \end{aligned} \tag{8.45}$$

Using Eq. (8.45) in Eq. (8.43) we get

$$\int_U A \cdot [\partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) - \boldsymbol{\partial}_x \partial_Y \mathfrak{L}(Y, \boldsymbol{\partial}_x Y)] d^4 x = 0, \tag{8.46}$$

for all A. By the arbitrariness of A we get

$$\partial_Y \mathcal{L}(Y, \boldsymbol{\partial}_x Y) - \boldsymbol{\partial}_x (\partial_{\partial Y} \mathcal{L}(Y, \boldsymbol{\partial}_x Y)) = 0,$$

and the proposition is proved. ■

8.4 Some Important Lagrangians

8.4.1 Maxwell Lagrangian

The Lagrangian associated with the Maxwell field $A \in \sec \bigwedge^1 T^*M$ (i.e., the electromagnetic *potential*) generated by a current $J_e \in \sec \bigwedge^1 T^*M$ is

$$\mathfrak{L}(A, \boldsymbol{\partial}_x \wedge A) = -\frac{1}{2} (\boldsymbol{\partial}_x \wedge A) \cdot (\boldsymbol{\partial}_x \wedge A) - A \cdot J_e. \tag{8.47}$$

Using Eq. (8.39b) we get immediately

$$\boldsymbol{\partial}_x \lrcorner (\boldsymbol{\partial}_x \wedge A) = J_e. \tag{8.48}$$

The electromagnetic field is (see Sect. 3.6) $F = \boldsymbol{\partial}_x \wedge A \in \sec \bigwedge^2 T^*M$. Then taking into account that $\boldsymbol{\partial}_x \wedge (\boldsymbol{\partial}_x \wedge A) = 0$, we can write Eq. (8.48) as

$$\boldsymbol{\partial}_x F = J_e, \tag{8.49}$$

which is, of course, equivalent Eq. (4.264) of Chap. 4.

8.4.2 Dirac-Hestenes Lagrangian

We recall that once a global spin coframe $\Xi \in P_{\mathrm{Spin}^e_{1,3}}(M)$ is fixed, a Dirac-Hestenes spinor field can be represented by a even section of $\sec \mathcal{C}\ell(M, \eta)$. Let then $\psi \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M)$ be a *representative* of a Dirac-Hestenes spinor field $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \eta)$ relative to a global spin coframe Ξ. A possible Lagrangian for such a field in interaction with an electromagnetic field $A \in \sec \bigwedge^1 T^*M$ is

$$\mathfrak{L}(\psi, \boldsymbol{\partial}_x \psi) = (\boldsymbol{\partial}_x \psi \mathbf{i}\gamma_3) \cdot \psi - \mathrm{q}(A\psi\gamma_0) \cdot \psi - m\psi \cdot \psi, \tag{8.50}$$

where the real parameters $m \in \mathbb{R}^+$ and q$\in \mathbb{R}$ are called the mass and electric charge of the Dirac-Hestenes field and $\mathbf{i} := \gamma_5$. Lagrangian (8.50) is of type (c) in Eq. (8.39). Then, the respective ELE is

$$\partial_\psi \mathfrak{L}(\psi, \boldsymbol{\partial}_x \psi) - \boldsymbol{\partial}_x (\partial_{\partial \psi} \mathfrak{L}(x, Y, \boldsymbol{\partial}_x \psi)) = 0. \tag{8.51}$$

We have

$$\begin{aligned} \partial_\psi \mathfrak{L}(\psi, \boldsymbol{\partial}_x \psi) &= \langle \boldsymbol{\partial}_x \psi \mathbf{i}\gamma_3 \rangle_\psi - \mathrm{q}\langle A\psi\gamma_0 + \tilde{A}\psi\tilde{\gamma}_0 \rangle_\psi - 2m\psi \\ &= \boldsymbol{\partial}_x \psi \mathbf{i}\gamma_3 - 2A\mathrm{q}\psi\gamma_0 - 2m\psi. \end{aligned} \tag{8.52}$$

Also,

$$\partial_{\boldsymbol{\partial}_x \psi} \mathfrak{L}(Y, \boldsymbol{\partial}_x \psi) = -\partial_{\boldsymbol{\partial}_x \psi} (\boldsymbol{\partial}_x \psi \cdot \psi \mathbf{i}\gamma_3) = -\psi \mathbf{i}\gamma_3. \tag{8.53}$$

Then, we have with $\sigma_3 = \gamma_3\gamma_0$

$$\boldsymbol{\partial}_x \psi \mathbf{i}\sigma_3 - m\psi\gamma_0 - \mathrm{q}A\psi = 0, \tag{8.54}$$

or, equivalently σ_3

$$\boldsymbol{\partial}_x \psi \gamma_2\gamma_1 - m\psi\gamma_0 - \mathrm{q}A\psi = 0, \tag{8.55}$$

which we recognize as equivalent to the Dirac-Hestenes equation (Eq. (7.93)) introduced in Chap. 7, since the term containing the *connection biform* in Eq. (7.93) is null in our case, because we are using only Einstein-Lorentz-Poincaré coordinate charts and using only exact global cotetrad fields (as introduced above), for which the connection coefficients are all null.

Exercise 8.25 Show that $(A\psi\gamma_0)\cdot\psi = A\cdot\psi\gamma_0\tilde{\psi}$ see Eq. (2.69).

Exercise 8.26 Show that for the above Lagrangian

$$\boldsymbol{\partial}_x(\partial_{\boldsymbol{\partial}_x\psi}\mathfrak{L}(Y,\boldsymbol{\partial}_x\psi)) = \partial_\mu(\partial_{\partial_\mu\psi}\mathfrak{L}(Y,\boldsymbol{\partial}_x\psi)). \tag{8.56}$$

Solution: We have from Eq. (8.53) that

$$\boldsymbol{\partial}_x(\partial_{\boldsymbol{\partial}_x\psi}\mathfrak{L}(Y,\boldsymbol{\partial}_x\psi)) = -\gamma^\mu\partial_\mu\psi\mathbf{i}\gamma_3.$$

On the other hand we can write the term $(\boldsymbol{\partial}_x\psi\mathbf{i}\gamma_3)\cdot\psi$ in the Lagrangian as

$$(\gamma^\mu\partial_\mu\psi\mathbf{i}\gamma_3)\cdot\psi. \tag{8.57}$$

Then, using Eqs. (2.68) and (2.69) we can write

$$(\gamma^\mu\partial_\mu\psi\mathbf{i}\gamma_3)\cdot\psi = (\partial_\mu\psi\mathbf{i}\gamma_3)\cdot\gamma^\mu\psi = -\partial_\mu\psi\cdot(\gamma^\mu\psi\mathbf{i}\gamma_3) = -(\gamma^\mu\psi\mathbf{i}\gamma_3)\cdot\partial_\mu\psi. \tag{8.58}$$

Using now Eq. (2.192) we get

$$\partial_{\partial_\mu\psi}\left(-(\gamma^\mu\psi\mathbf{i}\gamma_3)\cdot\partial_\mu\psi\right) = -\gamma^\mu\psi\mathbf{i}\gamma_3,$$

and then

$$\partial_\mu\left(\partial_{\partial_\mu\psi}\left(-(\gamma^\mu\psi\mathbf{i}\gamma_3)\cdot\partial_\mu\psi\right)\right) = -\gamma^\mu\partial_\mu\psi\mathbf{i}\gamma_3.$$

8.5 Canonical Energy-Momentum Extensor Field

We now find the canonical energy-momentum extensor for each one of the types of Lagrangians in Proposition 8.24.

We are interested here in diffeomorphisms that are simply translations in the Minkowski manifold. These are generated by very simple one-parameter group of diffeomorphisms. Indeed, under a translation in the constant 'direction' γ_μ the position 1-form x (of a spacetime point $\mathfrak{e}$, with coordinates $\boldsymbol{x}^\mu(\mathfrak{e}) = x^\mu$) goes in the position 1-form x' (of a spacetime point $\mathfrak{e}' = \sigma_t\mathfrak{e}$, with coordinates $\boldsymbol{x}'^\mu(\mathfrak{e}) = x'^\mu$)

and we write

$$x' = x + \varepsilon\gamma_\mu. \tag{8.59}$$

Given the rule for the pullbacks $Y' = \sigma_t^* Y$ for any $Y \in \sec \bigwedge^k T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$, the equivalence relation (Definition 8.1), and the convention given by Eq. (8.1), we can write $\sigma_t^* Y(x) = Y(x')$. Then, we have

$$Y'(x) = Y(x'). \tag{8.60}$$

Now, a Lagrangian $\mathfrak{L}_\circledast(Y)$ is invariant under the action of a one-parameter group of diffeomorphisms generated by a vector field ξ if the Lie derivative $\pounds_\xi \mathfrak{L}_\circledast(Y) = 0$. This means that we must have

$$\mathcal{L}(Y'(x), \boldsymbol{\partial}_x \circledast Y'(x)) = \mathcal{L}(Y(x'), \boldsymbol{\partial}_{x'} \circledast Y(x')). \tag{8.61}$$

To proceed we detail *only* the case where $\circledast$ refers to the Clifford product. We derive both members of Eq. (8.61) in relation to the parameter ε obtaining,

$$\begin{aligned} &\partial_\varepsilon Y'(x) \cdot \partial_{Y'} \mathfrak{L}(Y'(x), \boldsymbol{\partial}_x Y'(x)) + \boldsymbol{\partial}_x \partial_\varepsilon Y'(x) \cdot \partial_{\boldsymbol{\partial}_x Y'} \mathfrak{L}(Y'(x), \boldsymbol{\partial}_x Y'(x)) \\ &\quad = \partial_\varepsilon x' \cdot \boldsymbol{\partial}_{x'} \mathfrak{L}(Y(x'), \boldsymbol{\partial}_{x'} Y(x')). \end{aligned} \tag{8.62}$$

In writing Eq. (8.62) we used the fact that both members can be considered (composite) functions of ε, some chain rules for composition of multiform functions (Sect. 2.11) and that $\boldsymbol{\partial}_x \partial_\varepsilon = \partial_\varepsilon \boldsymbol{\partial}_x$. Now, we calculate both members of Eq. (8.62) for $\varepsilon = 0$. Noting that

$$Y(x')\big|_{\varepsilon=0} = Y(x), \boldsymbol{\partial}_{x'}\ Y(x')\big|_{\varepsilon=0} = \boldsymbol{\partial}_x Y(x), \tag{8.63}$$

and at $\varepsilon = 0$,

$$\partial_\varepsilon x' = \gamma_\mu, \partial_\varepsilon Y'(x) = \partial_\mu Y(x) = \gamma_\mu \cdot \boldsymbol{\partial}_x Y(x), \tag{8.64}$$

we can write Eq. (8.62) as

$$\begin{aligned} &\boldsymbol{\partial}_\mu Y \cdot \partial_Y \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) + \boldsymbol{\partial}_x \partial_\mu Y \cdot \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \partial_{\mathbf{x}} Y(x)) \\ &\quad = \gamma_\mu \cdot \boldsymbol{\partial}_x \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)). \end{aligned} \tag{8.65}$$

Now, the first member of Eq. (8.65) can be written as

$$\begin{aligned} &\partial_\mu Y \cdot \partial_Y \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) + \boldsymbol{\partial}_x \partial_\mu Y \cdot \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) \\ &\quad = \boldsymbol{\partial}_\mu Y \cdot \partial_Y \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) + \boldsymbol{\partial}_x \partial_\mu Y \cdot \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial} Y(x)) \end{aligned}$$

$$+ \partial_\mu \mathbf{Y} \cdot \boldsymbol{\partial}_x \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) - \partial_\mu Y \cdot \partial_{\boldsymbol{\partial}_x Y} \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \partial_{\mathbf{x}} Y(x))$$
$$= \boldsymbol{\partial}_\mu Y \cdot [\partial_{Y'} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) - \boldsymbol{\partial}_x \boldsymbol{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)]$$
$$+ \boldsymbol{\partial}_x \langle \partial_\mu Y (\partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(\widetilde{Y(x)}, \boldsymbol{\partial} Y(x)) \rangle_1. \tag{8.66}$$

In writing Eq. (8.66) we use the identity $\boldsymbol{\partial}_x \langle A\tilde{B} \rangle = \boldsymbol{\partial}_x A \cdot B + A \cdot \boldsymbol{\partial}_x B$, valid for arbitrary $A, B \in \sec \mathcal{C}\ell(M, \eta)$. Moreover, to simplify the notation we write in what follows

$$(\boldsymbol{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(\widetilde{Y(x)}, \boldsymbol{\partial}_x Y(x)) = \tilde{\boldsymbol{\partial}}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)). \tag{8.67}$$

Observe also that the right side member of Eq. (8.65) can be written as

$$\begin{aligned} \gamma_\mu \cdot \boldsymbol{\partial}_x \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) &= \boldsymbol{\partial}_\mu \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) \\ &= \gamma^\alpha \cdot \gamma_\mu \partial_\alpha \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) \\ &= \gamma^\alpha [\partial_\alpha \gamma_\mu \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x))] \\ &= \boldsymbol{\partial}_x \cdot [\gamma_\mu \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x))]. \end{aligned} \tag{8.68}$$

Now, if we suppose that the field satisfy the Euler-Lagrange equations, using Eqs. (8.66)–(8.68), we can write Eq. (8.65) as

$$\boldsymbol{\partial}_x [\langle (\gamma_\mu \cdot \boldsymbol{\partial}_x Y) \tilde{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x)) \rangle_1 - \gamma_\mu \mathfrak{L}(Y(x), \boldsymbol{\partial}_x Y(x))] = 0. \tag{8.69}$$

So, we conclude that associated to the Lagrangian $\mathfrak{L}(Y, \boldsymbol{\partial}_x Y)$ there exists a $(1,1)$-extensor field such that for any $n \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$, with *constant* coefficients we have a $(1,1)$- extensor field

$$T : \sec \mathcal{C}\ell(M, \eta) \hookleftarrow \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$$
$$T(n) = [\langle (n \cdot \boldsymbol{\partial}_x Y) \tilde{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) \rangle_1 - n \mathfrak{L}(Y, \boldsymbol{\partial}_x Y)], \tag{8.70}$$

such that

$$\boldsymbol{\partial}_x \cdot T(\gamma_\mu) = 0. \tag{8.71}$$

Putting $T^\mu = T(\gamma^\mu)$, we have

$$\boldsymbol{\partial}_x \cdot T^\mu = -\delta T^\mu = 0 \tag{8.72}$$

or

$$d \star T^\mu = 0. \tag{8.73}$$

Definition 8.27 The extensor field

$$T : \sec \mathcal{C}\ell(M,\eta) \hookleftarrow \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta) \tag{8.74}$$

is called the (canonical)energy-momentum extensor field of the field $Y \in \sec \mathcal{C}\ell(M,\eta)$.

Taking into account the definition of $T^\dagger$(the adjoint of T) satisfies

$$[\partial_n \cdot \boldsymbol{\partial}_x T^\dagger(n)] \cdot \gamma_\mu = \boldsymbol{\partial}_x \cdot T(\gamma_\mu), \tag{8.75}$$

we can write equivalently.

$$\partial_n \cdot \boldsymbol{\partial} T^\dagger(n) = 0. \tag{8.76}$$

Remark 8.28 The $T^\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ are called (canonical) energy-momentum 1-forms and $\star T^\mu \in \sec \bigwedge^3 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ the (canonical) energy-momentum 3-forms. Note that some authors (e.g., [3]) call $T^{\dagger\mu}$, instead of T^μ the (canonical) energy-momentum 1-forms.

Exercise 8.29 Show that the following conserved energy-momentum extensors may be derived from the Lagrangians $\mathfrak{L}(Y, \boldsymbol{\partial}_x \lrcorner Y)$ and $\mathfrak{L}(Y, \boldsymbol{\partial}_x \wedge Y)$.

$$\begin{aligned} &(a)\ T(n) = [\langle \bar{\tilde{\partial}}_{\boldsymbol{\partial}_x \lrcorner Y} \mathfrak{L}(x, Y, \boldsymbol{\partial}_x Y) \lrcorner (n \cdot \boldsymbol{\partial}_x Y)\rangle_1 - n\mathfrak{L}(Y, \boldsymbol{\partial}_x \lrcorner Y)],\\ &(b)\ T(n) = [(n \cdot \boldsymbol{\partial}_x \widetilde{Y}) \lrcorner \langle \partial_{\boldsymbol{\partial}_x Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x \wedge Y)\rangle_1 - n\mathfrak{L}(Y, \boldsymbol{\partial}_x \wedge Y)], \end{aligned} \tag{8.77}$$

where $\bar{\tilde{\partial}}_{\boldsymbol{\partial}_x \lrcorner Y} \mathfrak{L}(Y, \boldsymbol{\partial}_x Y) \equiv \left(\partial_{\boldsymbol{\partial}_x \lrcorner Y} \widetilde{\mathfrak{L}(Y, \boldsymbol{\partial}_x Y}\right)^{-}$.

8.5.1 *Canonical Energy-Momentum Extensor of the Free Electromagnetic Field*

The Lagrangian of the Free Electromagnetic field is

$$\mathfrak{L}(A, \boldsymbol{\partial}_x \wedge A) = -\frac{1}{2}(\boldsymbol{\partial}_x \wedge A) \cdot (\boldsymbol{\partial}_x \wedge A). \tag{8.78}$$

Using Eq. (8.77b) we get

$$T_A(n) = (n \cdot \boldsymbol{\partial}_x A) \lrcorner F + \frac{1}{2} nF \cdot F. \tag{8.79}$$

The adjoint of the energy-momentum extensor of the electromagnetic field is given by

$$T_A^\dagger(n) = \partial_b n \cdot T_A(b) \tag{8.80}$$
$$= \partial_b n \cdot (b \cdot \boldsymbol{\partial}_x A) \lrcorner F + \frac{1}{2} \partial_b n F \cdot F.$$

Exercise 8.30 Show that

$$\frac{1}{2} F n \tilde{F} = (n \lrcorner F) \lrcorner F + \frac{1}{2} n (F \cdot F) \tag{8.81}$$

Solution:

$$(n \lrcorner F) \lrcorner F + \frac{1}{2} n (F \cdot F) = \frac{1}{2} [(n \lrcorner F) F - F (n \lrcorner F)] + \frac{1}{2} n (F \cdot F)$$
$$= \frac{1}{4} [nFF - FnF - FnF + FFn] + \frac{1}{2} n (F \cdot F)$$
$$= -\frac{1}{2} FnF + \frac{1}{4} [-2n(F \cdot F) + n(F \wedge F) + (F \wedge F) n]$$
$$+ \frac{1}{2} n (F \cdot F)$$
$$= -\frac{1}{2} FnF - \frac{1}{2} n (F \cdot F) + \frac{1}{2} n \wedge (F \wedge F) + \frac{1}{2} n (F \cdot F)$$
$$= -\frac{1}{2} FnF = \frac{1}{2} F n \tilde{F}.$$

Exercise 8.31 Show that

$$T_A(n) = \frac{1}{2} F n \tilde{F} + [d(n \cdot A)] \lrcorner F. \tag{8.82}$$

Solution: From Eq. (8.79) we can write

$$T_A(n) = (n \cdot \boldsymbol{\partial}_x A) \lrcorner F + \frac{1}{2} n F \cdot F \cdot (n \cdot \boldsymbol{\partial}_x A) \lrcorner F - (n \lrcorner F) \lrcorner F$$
$$+ (n \lrcorner F) \lrcorner F + \frac{1}{2} n F \cdot F. \tag{8.83}$$

Now, recalling the identity given by Eq. (4.154) we have taking into account that $\boldsymbol{\partial}_x n = 0$ we have

$$\boldsymbol{\partial}_x (n \cdot A) = (n \cdot \boldsymbol{\partial}_x) A - n \lrcorner (\boldsymbol{\partial}_x \wedge A)$$
$$= (n \cdot \boldsymbol{\partial}_x) A - n \lrcorner F. \tag{8.84}$$

Then,

$$T_A(n) = [\boldsymbol{\partial}_x(n \cdot A)]\lrcorner F + (n\lrcorner F)\lrcorner F + \frac{1}{2}nF \cdot F$$
$$= [d(n \cdot A)]\lrcorner F + (n\lrcorner F)\lrcorner F + \frac{1}{2}nF \cdot F, \tag{8.85}$$

and taking into account Eq. (8.81), we can also write

$$T_A(n) = \frac{1}{2}Fn\tilde{F} + [d(n \cdot A)]\lrcorner F,$$

which is what we wanted to show.

Exercise 8.32 Show that

$$T_A^{\dagger}(n) = \frac{1}{2}Fn\tilde{F} - (n\lrcorner F) \cdot \boldsymbol{\partial}_x A. \tag{8.86}$$

Exercise 8.33 Show that

$$\star\, T_A(n) = -\frac{1}{2}(n\lrcorner F) \wedge \star F + \frac{1}{2}F \wedge (n\lrcorner \star F) - d(n \cdot A) \wedge \star F. \tag{8.87}$$

Solution: First, observe that (for any $n \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$),

$$d(n \cdot A) \wedge \star F = d(n \cdot A) \wedge \tilde{F}\tau_\eta = -d(n \cdot A) \wedge F\tau_\eta$$
$$= -\frac{1}{2}[d(n \cdot A)F\tau_\eta + F\tau_\eta d(n \cdot A)]$$
$$= -\frac{1}{2}[d(n \cdot A)F - Fd(n \cdot A)]\tau_\eta$$
$$= -\{[d(n \cdot A)]\lrcorner F\}\tau_\eta. \tag{8.88}$$

Also

$$(n\lrcorner F) \wedge \star F = n\lrcorner(F \wedge \star F) - F \wedge (n\lrcorner \star F)$$
$$= n[(F \cdot F)\tau_\eta] - F \wedge (n\lrcorner \tilde{F}\tau_\eta) \tag{8.89}$$

and

$$(n\lrcorner F) \wedge \star F = -(n\lrcorner F) \wedge F\tau_\eta \tag{8.90}$$
$$= -\frac{1}{2}[(n\lrcorner F)F\tau_\eta + F\tau_\eta(n\lrcorner F)]$$

$$
\begin{aligned}
&= -\frac{1}{2}[(n\lrcorner F)F - F(n\lrcorner F)]\tau_\eta \\
&= -[(n\lrcorner F)\lrcorner F]\tau_\eta .
\end{aligned}
$$

Then, using Eq. (8.85) and the identities just derived above, we can write

$$
\begin{aligned}
\star T_A(n) &= \star\left[(n\lrcorner F)\lrcorner F + \frac{1}{2}nF\cdot F + d(n\cdot A)]\lrcorner F\right] \\
&= -(n\lrcorner F)\wedge\star F + \frac{1}{2}n(F\cdot F)\tau_\eta + \{[d(n\cdot A)]\lrcorner F\}\tau_\eta \\
&= -(n\lrcorner F)\wedge\star F + \frac{1}{2}(n\lrcorner F)\wedge\star F + \frac{1}{2}F\wedge(n\lrcorner\star F) + \{[d(n\cdot A)]\lrcorner F\}\tau_\eta \\
&= -\frac{1}{2}(n\lrcorner F)\wedge\star F + \frac{1}{2}F\wedge(n\lrcorner\star F) + \{[d(n\cdot A)]\lrcorner F\}\tau_\eta \\
&= -\frac{1}{2}(n\lrcorner F)\wedge\star F + \frac{1}{2}F\wedge(n\lrcorner\star F) - d(n\cdot A)\wedge\star F. \qquad (8.91)
\end{aligned}
$$

Remark 8.34 We easily verify that $T_A^{\mu\nu} = T_A(\gamma^\mu)\cdot\gamma^\nu$ is not symmetric, but

$$
T_{AB}(n) = \frac{1}{2}(F\cdot F)n + (n\lrcorner F)\lrcorner F = \frac{1}{2}Fn\tilde{F}, \qquad (8.92)
$$

called the *Belinfante energy-momentum extensor* is symmetric. The non symmetric part is also gauge dependent, but as can be easily verified the term $d(n\lrcorner A)\wedge\star F$ in $\star T_A(n)$ can be written as an exact differential, since taking into account that $d\star F = 0$ we have

$$
\begin{aligned}
d(n\lrcorner A)\wedge\star F &= d(n\cdot A)\wedge\star F \\
&= d(n\cdot A\wedge\star F) + n\cdot A\wedge d\star F \\
&= d(n\cdot A\wedge\star F). \qquad (8.93)
\end{aligned}
$$

So, the gauge dependent term does not change the computation of the energy-momentum of a free electromagnetic field, but as we are going to see, it has a crucial role in the computation of the spin of the electromagnetic field and consequently in the law of conservation of angular momentum. This important issue is not discussed appropriately in textbooks.

8.5.2 Canonical Energy-Momentum Extensor of the Free Dirac-Hestenes Field

The Lagrangian for the (representative) of a free Dirac-Hestenes spinor field $\psi \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M)$ is

$$\mathfrak{L}_0(\psi, \boldsymbol{\partial}_x\psi) = (\boldsymbol{\partial}_x\psi\mathbf{i}\gamma_3)\cdot\psi - m\psi\cdot\psi, \tag{8.94}$$

In this case, we get from Eq. (8.70) the energy-momentum extensor of the Dirac-Hestenes field which, once we take into account that for any solution of the ELE associated to $\mathfrak{L}_0(\psi, \boldsymbol{\partial}_x\psi)$. We have,

$$\begin{aligned} T_\psi(n) &= \langle (n\cdot\boldsymbol{\partial}_x\psi)\tilde{\partial}_{\boldsymbol{\partial}_x\psi}\mathfrak{L}_0(\psi(,\boldsymbol{\partial}_x\psi))\rangle_1 - n\mathfrak{L}_0(\psi, \boldsymbol{\partial}_x\psi) \\ &= -\langle (n\cdot\boldsymbol{\partial}_x\psi)\mathbf{i}\gamma_3\tilde{\psi}\rangle_1. \end{aligned} \tag{8.95}$$

This extensor is called the *free Tetrode* energy-momentum extensor of the Dirac-Hestenes field.

8.6 Canonical Orbital Angular Momentum and Spin Extensors

In this section we show that the *global* rotational invariance of the Lagrangian of a Clifford field $Y = X \in \sec\mathcal{C}\ell(M,\eta)$ or of a representative of a Dirac-Hestenes spinor field, $Y = \psi \in \sec\mathcal{C}\ell(M,\eta)$, implies the existence of a conserved $(2,1)$-extensor field. From the possible Lagrangians $\mathfrak{L}_0(\psi, \boldsymbol{\partial}_x \circledast \psi)$, we detail only the calculations for the case where $\circledast$ refers to the Clifford product and Y is a Clifford field. For the other cases, we give only the final results.

By *rotational invariance* here we mean that the Lagrangian is invariant by diffeomorphisms in Minkowski spacetime generated by one-parameter group of diffeomorphisms of M generated by the six vector fields $\xi_{(\alpha\beta)} \in \sec TM$ such that

$$\xi_{(\alpha\beta)} = \eta_{\alpha\zeta}x^\zeta\frac{\partial}{\partial x^\beta} - \eta_{\beta\zeta}x^\zeta\frac{\partial}{\partial x^\alpha},\ \alpha,\beta = 0,1,2,3, \tag{8.96}$$

which close the Lie algebra of the homogeneous Lorentz group. As we already learned in Sect. 8.3, the representation of a diffeomorphism in M (generated a rotation by ϑ in the fixed direction $\xi_{(\mu\nu)}$) can be written as

$$x \mapsto x' = e^{-\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2}}xe^{\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2}}, \tag{8.97}$$

with $x = x^\mu(\mathfrak{e})\gamma_\mu$ and $x' = x^\mu(\mathfrak{e}')\gamma_\mu \equiv x'^\mu\gamma_\mu$.

Recall that for any Clifford field $X \in \sec \mathcal{C}\ell(M,\eta)$ or representative of a Dirac-Hestenes field $\psi \in \sec \mathcal{C}\ell^{(0)}(M,\eta)$ their pullbacks $X' \in \sec \mathcal{C}\ell(M,\eta)$, $\psi' \in \sec \mathcal{C}\ell^{(0)}(M,\eta)$ under a diffeomorphism generated by $\xi_{(\mu\nu)}$ satisfy

$$X'(x) = X(x'), \psi'(x) = \psi(x'). \tag{8.98}$$

and according to Eq. (8.34) we have,

$$\begin{aligned}&\text{(a) } X'(x) = e^{(\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2})}X(x)e^{(-\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2})},\\ &\text{(b) } \qquad \psi'(x) = e^{(\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2})}\psi(x').\end{aligned} \tag{8.99}$$

Then, we must have

$$\mathfrak{L}(Y'(x), \boldsymbol{\partial}_x Y'(x)) = \mathfrak{L}(Y(x'), \boldsymbol{\partial}_{x'} Y(x')), \tag{8.100}$$

for $Y = X$ or $Y = \psi$. Since $x = e^{(\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2})}x'e^{(-\gamma_\mu\wedge\gamma_\nu\frac{\vartheta}{2})}$, we can write

$$\begin{aligned}&\partial_\vartheta Y'(x)\cdot\partial_{Y'}\mathfrak{L}(Y'(x),\boldsymbol{\partial}_x Y'(x)) + \boldsymbol{\partial}_x\partial_\vartheta Y'(x)\cdot\partial_{\boldsymbol{\partial}_x Y'}\mathfrak{L}(Y'(x),\boldsymbol{\partial}_x Y'(x))\\ &\quad = \partial_\vartheta x'\cdot\boldsymbol{\partial}_{x'}\mathfrak{L}(Y(x'),\boldsymbol{\partial}_{x'}Y(x')).\end{aligned} \tag{8.101}$$

Next, we evaluate in details both members of Eq. (8.101) at $\vartheta = 0$ for the case $Y = X$ (*Clifford fields*). The result for a Dirac-Hestenes spinor field will be given in the next subsection. We need the results

$$\begin{aligned}\partial_\vartheta x'\big|_{\vartheta=0} &= x\lrcorner\left(\gamma_\mu\wedge\gamma_\nu\right),\\ \partial_\vartheta X'(x)\big|_{\vartheta=0} &= [\frac{1}{2}\gamma_\mu\wedge\gamma_\nu X(x) + x\lrcorner(\gamma_\mu\wedge\gamma_\nu)\cdot\boldsymbol{\partial}_x X(x) - \frac{1}{2}\gamma_\mu\wedge\gamma_\nu X(x)]\Big|_{\vartheta=0}\\ &= (\gamma_\mu\wedge\gamma_\nu)\times X(x) + x\lrcorner(\gamma_\mu\wedge\gamma_\nu)\cdot\boldsymbol{\partial}_x X(x),\end{aligned} \tag{8.102}$$

where we recall that $(\gamma_\mu\wedge\gamma_\nu)\times X = [(\gamma_\mu\wedge\gamma_\nu), X]$.

Now, the first member of Eq. (8.101) ($Y = X$) at $\vartheta = 0$ can be written as a divergence, i.e.,

$$\begin{aligned}&\partial_\vartheta X(x)\cdot\partial_{X'}\mathfrak{L}(X'(x),\boldsymbol{\partial}_x X'(x)) + \boldsymbol{\partial}_x\partial_\vartheta X'(x)\cdot\partial_{\boldsymbol{\partial}_x X'}\mathfrak{L}(X'(x),\boldsymbol{\partial}_x X'(x))\big|_{\vartheta=0}\\ &\quad = \boldsymbol{\partial}_x\cdot\langle\,\partial_\vartheta X'(x)\big|_{\vartheta=0}\tilde{\boldsymbol{\partial}}_{\boldsymbol{\partial}_x X'}\mathfrak{L}(X(x),\boldsymbol{\partial}_x X(x)\rangle_1\\ &\quad = \boldsymbol{\partial}_x\cdot\langle[(\gamma_\mu\wedge\gamma_\nu)\times X(x) + x\lrcorner(\gamma_\mu\wedge\gamma_\nu)\cdot\boldsymbol{\partial}_x X(x)]\tilde{\partial}_{\boldsymbol{\partial}X'}\mathfrak{L}(X(x),\boldsymbol{\partial}X(x)\rangle_1,\end{aligned} \tag{8.103}$$

and for the second member we have

$$\partial_\vartheta x' \cdot \boldsymbol{\partial}_{x'} \mathfrak{L}(X(x'), \boldsymbol{\partial}_{x'} X(x'))\big|_{\vartheta=0} = x\lrcorner\left(\gamma_\mu \wedge \gamma_\nu\right) \cdot \partial_X \mathfrak{L}(X(x), \boldsymbol{\partial}_x X(x))$$
$$= \boldsymbol{\partial}_x \cdot [x\lrcorner\left(\gamma_\mu \wedge \gamma_\nu\right) \mathfrak{L}(X(x), \boldsymbol{\partial}_x X(x))]. \quad (8.104)$$

Then, we get

$$\boldsymbol{\partial}_x \cdot [x\lrcorner\left(\gamma_\mu \wedge \gamma_\nu\right) \mathfrak{L}(X, \boldsymbol{\partial}_x X)]$$
$$- \boldsymbol{\partial}_x \cdot [\langle[(\gamma_\mu \wedge \gamma_\nu) \times X + x\lrcorner(\gamma_\mu \wedge \gamma_\nu) \cdot \boldsymbol{\partial}_x X \tilde{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(X, \boldsymbol{\partial}_x X)]\rangle_1] = 0. \quad (8.105)$$

Equation (8.105) implies that there exists a conserved $(2,1)$-extensor field,

$$\mathbf{J}_X : \sec \mathcal{C}\ell(M,\eta) \hookleftarrow \sec \bigwedge^2 TM \to \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta), \quad (8.106)$$

$$\mathbf{J}_X(B) = x\lrcorner\,(B)\,\mathfrak{L} - \langle[B \times X + x\lrcorner B \cdot \boldsymbol{\partial}_x X \tilde{\partial}_{\boldsymbol{\partial}_x X} \mathfrak{L}]\rangle_1, \quad (8.107)$$

such that with $B \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ a constant biform

$$\boldsymbol{\partial}_x \cdot \mathbf{J}_X(B) = 0. \quad (8.108)$$

Definition 8.35 The $(2,1)$ extensor field $\mathbf{J}_X$ given by Eq. (8.107) is called the canonical angular momentum extensor of the field $X \in \sec \mathcal{C}\ell(M,\eta)$.

Exercise 8.36 Show that adjoint extensor of $\mathbf{J}_X(B)$, i.e., the $(1,2)$-extensor $\mathbf{J}_X^\dagger(n)$, $n \in \sec \bigwedge^1 T^*M$ with constant coefficients is given by

$$\mathbf{J}_X^\dagger(n) = \partial_B n \cdot \mathbf{J}_X(B)$$
$$= T_X^\dagger(n) \wedge x + \langle X \times \tilde{\partial}_{\boldsymbol{\partial}_x X} \mathfrak{L}(X, \boldsymbol{\partial}_x X) n\rangle_2. \quad (8.109)$$

Exercise 8.37 Show that

$$\boldsymbol{\partial}_{\mathbf{n}} \cdot \boldsymbol{\partial}_x \mathbf{J}_X^\dagger(n) = 0. \quad (8.110)$$

Remark 8.38 Sometimes, the $(1,2)$-extensor

$$\mathbf{J}_X^\dagger : \sec \mathcal{C}\ell(M,\eta) \hookleftarrow \sec \bigwedge^2 TM \to \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(M,\eta),$$

$$\mathbf{J}_X^\dagger(n) = T_X^\dagger(n) \wedge x + \langle X(x) \times \tilde{\partial}_{\boldsymbol{\partial}_x X} \mathfrak{L}(x, X, \boldsymbol{\partial}_x X) n\rangle_2. \quad (8.111)$$

instead of $\mathbf{J}_X$ is called the canonical angular momentum extensor of a multiform field $X \in \sec \mathcal{C}\ell(M,\eta)$. We use that wording indifferently since this will generate no confusion.

Equations (8.109) and (8.110) suggests the

Definition 8.39 The orbital angular momentum extensor and the spin extensor of the field $X \in \sec \mathcal{C}\ell(M, \eta)$ are respectively

$$\mathbf{L}_X^\dagger(n) = T_{\mathbf{X}}^\dagger(n) \wedge x, \tag{8.112}$$

with $x = x^\mu \gamma_\mu$ and

$$\mathbf{S}_X^\dagger(n) = \langle X(x) \times \tilde{\partial}_{\boldsymbol{\partial}_x X} \mathfrak{L}(x, X, \boldsymbol{\partial}_x X) n \rangle_2. \tag{8.113}$$

Exercise 8.40 Show that the adjoint of $\mathbf{S}_X^\dagger(n)$, i.e., the $(2, 1)$-extensor field $\mathbf{S}_{\mathbf{X}}(B)$ such that $\mathbf{S}_X^\dagger(n) = \boldsymbol{\partial}_{\mathbf{B}} n \cdot \mathbf{S}_{\mathbf{X}}(B)$ is given by

$$\mathbf{S}_{\mathbf{X}}(B) = -\langle (B \times X) \tilde{\partial}_{\boldsymbol{\partial}_{\mathbf{x}} \mathbf{X}} \mathfrak{L}(x, X, \boldsymbol{\partial}_x X) \rangle_1. \tag{8.114}$$

Now, take $B = \gamma^\mu \wedge \gamma_\nu$.

Definition 8.41 The six $\mathbf{S}_X(\gamma^\mu \wedge \gamma_\nu) = \mathbf{S}_{\nu X}^\mu$ are called spin 1-form fields of the field X and the six $\star \mathbf{S}_{\nu X}^\mu$ are called spin 3-form fields (densities) of the field X.

As promised, we give next the angular momentum and spin extensors for Lagrangians of the types (a) and (b) in Proposition 8.24, i.e., $\mathfrak{L}(x, Y, \boldsymbol{\partial}_x \lrcorner Y)$ and $\mathfrak{L}(x, Y, \boldsymbol{\partial}_x \wedge Y)$, and also for the case of the Dirac-Hestenes spinor field $\mathfrak{L}(x, \psi, \boldsymbol{\partial}_x \psi)$.

8.6.1 Canonical Orbital and Spin Density Extensors for the Dirac-Hestenes Field

For the case of a Dirac-Hestenes field $\psi \in \sec \mathcal{C}\ell^{(0)}(M, \eta)$, repeating the calculations done in the previous section, we find the following conserved angular momentum extensor

$$\mathbf{J}_\psi^\dagger(n) = T_\psi^\dagger(n) \wedge x + \langle \frac{1}{2} \psi \tilde{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(x, \psi, \boldsymbol{\partial}_x \psi) n \rangle_2. \tag{8.115}$$

The density of orbital angular momentum and spin extensors for the Dirac-Hestenes field are respectively,

$$\mathbf{L}_\psi^\dagger(n) = T_\psi^\dagger(n) \wedge x \tag{8.116}$$

and

$$\mathbf{S}_{\psi}^{\dagger}(n) = \langle \frac{1}{2}\psi \tilde{\partial}_{\boldsymbol{\partial}_x Y} \mathfrak{L}(x, \psi, \boldsymbol{\partial}_x \psi) n \rangle_2$$
$$= \frac{1}{2}\langle \psi \mathbf{i}\gamma_3 \tilde{\psi} n \rangle_2 = \frac{1}{2}\mathbf{i}(s \wedge n). \tag{8.117}$$

Definition 8.42 The bilinear invariant

$$s = \psi \gamma_3 \tilde{\psi} \tag{8.118}$$

is called the spin covector of the Dirac-Hestenes field.

Exercise 8.43 Show that the adjoint of the canonical $(1, 2)$-extensor $\mathbf{S}_{\psi}^{\dagger}(n)$ of the Dirac-Hestenes field is the $(2, 1)$-extensor field $\mathbf{S}_{\psi}$ given by

$$S_{\psi}(B) = \langle \frac{1}{2} B\psi + (x \lrcorner B)\tilde{\partial}_{\boldsymbol{\partial}_x \psi} \mathfrak{L}(\psi, \boldsymbol{\partial}_x \psi) n \rangle_1. \tag{8.119}$$

8.6.2 *Case* $\mathfrak{L}(X, \boldsymbol{\partial}_x \lrcorner X)$

In this case, calculations analogous to the ones of the previous sections give

$$\mathbf{S}_X(B) = -\langle \widetilde{\bar{\partial}}_{\boldsymbol{\partial}_x \lrcorner X} \mathfrak{L}(X, \boldsymbol{\partial}_x \lrcorner X) \lrcorner (B \times X) \rangle_1 \tag{8.120}$$

and

$$\mathbf{S}_X^{\dagger}(n) = \frac{1}{2}\langle X \times \widetilde{\partial}_{\boldsymbol{\partial}_x \lrcorner X} \mathfrak{L}(x, X, \boldsymbol{\partial}_x \lrcorner X) n - X \times n \widetilde{\partial}_{\boldsymbol{\partial}_x \lrcorner X} \mathfrak{L}(X, \boldsymbol{\partial}_x \lrcorner X) \rangle_2. \tag{8.121}$$

8.6.3 *Case* $\mathfrak{L}(X, \boldsymbol{\partial}_x \wedge X)$

In this case we obtain,

$$\mathbf{S}_X(B) = -\langle \widetilde{(B \times X)} \tilde{\partial}_{\boldsymbol{\partial}_x \wedge X} \mathcal{L}(x, X, \boldsymbol{\partial} \wedge X) \rangle_1, \tag{8.122}$$

$$\mathbf{S}_X^{\dagger}(n) = \frac{1}{2}\langle X \times \tilde{\boldsymbol{\partial}}_{\boldsymbol{\partial}_x \wedge X} \mathcal{L}(X, \boldsymbol{\partial}_x \wedge X) n + \bar{X} \times n \tilde{\boldsymbol{\partial}}_{\boldsymbol{\partial}_x \wedge X} \mathcal{L}(X, \boldsymbol{\partial}_x \wedge X \rangle_2. \tag{8.123}$$

8.6.4 Spin Extensor of the Free Electromagnetic Field

In this case, using the Lagrangian given by Eqs. (8.47) and (8.123) we get immediately,

$$
\begin{aligned}
\mathbf{S}_A^\dagger(n) &= -\frac{1}{2}\langle A \times \tilde{F}n + \bar{A} \times n\tilde{F}\rangle_2 \\
&= \langle A \times F \llcorner n\rangle_2 = A \wedge F \llcorner n \\
&= (n \lrcorner F) \wedge A \\
&= n \lrcorner (A \wedge F) - (n \cdot A)F
\end{aligned}
\tag{8.124}
$$

This formula shows that the spin density of the electromagnetic field is indeed *gauge dependent* and we think that it is just here that the reality of A becomes apparent.[6]

Exercise 8.44 Show that

$$
\star\, \mathbf{S}_A(\gamma^\mu \wedge \gamma^\nu) \equiv \star \mathbf{S}^{\mu\nu} = \star[(A \lrcorner (\gamma^\mu \wedge \gamma^\nu)) \lrcorner F]. \tag{8.125}
$$

8.7 The Source of Spin

We showed that for any Clifford field $X \in \sec \mathcal{C}\ell(M,\eta)$ or representative $\psi \in \sec \mathcal{C}\ell^{(0)}(M,\eta)$ of a Dirac-Hestenes spinor field there exists a $(1,2)$-extensor field $\mathbf{J}^\dagger(n) = T^\dagger(n) \wedge x + \mathbf{S}^\dagger(n)$ such that $\partial_n \cdot \boldsymbol{\partial} \mathbf{J}^\dagger(n) = 0$. Recalling Definition 2.54 that says that $bif(T) = T(\gamma^\mu) \wedge \gamma_\mu)$ we can write

$$
\begin{aligned}
\partial_n \cdot \boldsymbol{\partial}_x \mathbf{J}^\dagger(n) &= \partial_n \cdot \boldsymbol{\partial}_x [T^\dagger(n) \wedge x + \mathbf{S}^\dagger(n)] \\
&= \gamma^\mu \cdot \partial_n[\partial_\mu T^\dagger(n) + T^\dagger(n) \wedge \partial_\mu x] + \partial_n \cdot \boldsymbol{\partial}_x \mathbf{S}^\dagger(n) \\
&= \partial_n \cdot \boldsymbol{\partial}_x T^\dagger(n) + bif(T^\dagger) + \partial_n \cdot \boldsymbol{\partial} \mathbf{S}^\dagger(n) = 0.
\end{aligned}
\tag{8.126}
$$

Taking into account that $\partial_n \cdot \boldsymbol{\partial}_x T^\dagger(n) = 0$ we get the fundamental result

$$
\partial_n \cdot \boldsymbol{\partial}_x \mathbf{S}^\dagger(n) = -bif(T^\dagger), \tag{8.127}
$$

which says that $\mathbf{S}^\dagger(n)$ alone is not conserved when the energy-momentum extensor is not symmetric, its source being $-bif(T^\dagger)$.

[6] We observe that this density of spin is analogous to the corresponding one used in Relativistic Quantum Field Theory, which is also gauge dependent. See, e.g., the discussion in [1].

8.8 Coupled Maxwell and Dirac-Hestenes Fields

In this case, the Lagrangian for the coupled system of fields is:

$$\mathfrak{L}(\psi, \boldsymbol{\partial}_x\psi, A, \boldsymbol{\partial}_x A) = (\boldsymbol{\partial}_x\psi\mathbf{i}\gamma_3)\cdot\psi - A\cdot(q\psi\gamma_0\tilde{\psi}) - m\psi\cdot\psi - \frac{1}{2}(\boldsymbol{\partial}_x\wedge A)\cdot(\boldsymbol{\partial}_x\wedge A). \tag{8.128}$$

Our Lagrangian depends then on two dynamic fields, A and ψ, which is a combination of Lagrangians of types (a) and (c). A straightforward computation gives as equations of motion for the dynamic fields

$$\begin{aligned} &\text{(a)}\ \boldsymbol{\partial}_x\psi\gamma_2\gamma_1 - m\psi\gamma_0 - eA\psi = 0,\\ &\text{(b)}\qquad\qquad \boldsymbol{\partial}_x F = J_e. \end{aligned} \tag{8.129}$$

This is a nonlinear system of partial differential equations, since

$$J_e = e\psi\gamma_0\tilde{\psi}. \tag{8.130}$$

Also, a straightforward computation gives for the energy-momentum extensor of the coupled fields

$$\begin{aligned} T(n) &= \langle(n\cdot\boldsymbol{\partial}_x\psi)\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 + (n\cdot\boldsymbol{\partial}_x A)\lrcorner F + \frac{1}{2}nF\cdot F\\ &= T_\psi(n) + T_A(n), \end{aligned} \tag{8.131}$$

which is the sum of the energy-momentum extensors of the free Dirac-Hestenes and free electromagnetic fields. We show below that there exists an extensor field, called *Tetrode* extensor field $T^\dagger_{Tetr}(n)$, with

$$T_{Tetr}(n) = T_\psi(n) - (n\cdot J_e)A, \tag{8.132}$$

such that

$$\partial_n\cdot\boldsymbol{\partial}_x T^\dagger_{Tetr}(n) = F\llcorner J_e. \tag{8.133}$$

Exercise 8.45 Show that

$$\begin{aligned} &(i)\quad bif(T^\dagger_{Tetr}) = -\tfrac{1}{2}\boldsymbol{\partial}_x\lrcorner\mathbf{i}s = \tfrac{1}{2}\mathbf{i}\boldsymbol{\partial}_x\wedge s,\\ &(ii)\ \ T_{Tetr}(n) - T^\dagger_{Tetr}(n) = n\lrcorner bif(T^\dagger_{Tetr}),\\ &(iii)\ \ \partial_n\cdot\boldsymbol{\partial}_x T^\dagger_{Tetr}(n) = \partial_n\cdot\boldsymbol{\partial}_x T_{Tetr}(n). \end{aligned} \tag{8.134}$$

Solution: We show (i). From the Dirac-Hestenes equation (Eq. (8.54)) we have

$$[\boldsymbol{\partial}_x\psi\mathbf{i}\gamma_3]\tilde{\psi} - m\psi\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} = 0. \tag{8.135}$$

Now,

$$\partial_\mu(\psi\mathbf{i}\gamma_3\tilde{\psi}) = \partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi} + \psi\mathbf{i}\gamma_3\partial_\mu\tilde{\psi} \tag{8.136}$$

and taking into account the notable identity valid in the spacetime algebra which says that for any odd $c \in \mathbb{R}_{1,3}$, $\langle c\rangle_1 = \frac{1}{2}(c + \tilde{c})$ we have

$$\langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 = \frac{1}{2}(\partial_\mu\psi\gamma_3\tilde{\psi} - \psi\mathbf{i}\gamma_3\partial_\mu\tilde{\psi}) \tag{8.137}$$

and then

$$\frac{1}{2}\boldsymbol{\partial}_x(\psi\mathbf{i}\gamma_3\tilde{\psi}) + \gamma^\mu\langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 = [\boldsymbol{\partial}_x\psi\mathbf{i}\gamma_3]\tilde{\psi}. \tag{8.138}$$

Using Eq. (8.136) in the second member of Eq. (8.138) yields

$$\frac{1}{2}\boldsymbol{\partial}_x(\mathbf{i}s) + \gamma^\mu\langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 = AJ_e + m\psi\tilde{\psi}, \tag{8.139}$$

where $J_e = e\psi\gamma_0\tilde{\psi}$ is the electromagnetic current and $s = \psi\gamma_3\tilde{\psi}$ is the spin covector field introduced in Eq. (8.118). Note also that the $\langle\rangle_2$ part of Eq. (8.139) yields

$$\frac{1}{2}\boldsymbol{\partial}_x\lrcorner(\mathbf{i}s) + \gamma^\mu \wedge \langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 = A \wedge J_e. \tag{8.140}$$

Now,

$$\begin{aligned} bif(T^\dagger_{Tetr}) &= -bif(T_{Tetr}) = \gamma^\mu \wedge T_{Tetr}(\gamma^\mu) \\ &= \gamma^\mu \wedge \big(\langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 - \gamma_\mu \cdot AJ_e\big) \\ &= \gamma^\mu \wedge \langle\partial_\mu\psi\mathbf{i}\gamma_3\tilde{\psi}\rangle_1 - A \wedge J_e \\ &= \frac{1}{2}\boldsymbol{\partial}_x\lrcorner(\mathbf{i}s) = \frac{1}{2}\mathbf{i}\boldsymbol{\partial}_x\wedge s, \end{aligned} \tag{8.141}$$

and (i) in Eq. (8.134) is proved.

To prove Eq. (8.133) we start from the conservation of $T(n)$,

$$\partial_n \cdot \boldsymbol{\partial}_x T_\psi(n) + \partial_n \cdot \boldsymbol{\partial}_x T_A(n) = 0. \tag{8.142}$$

We have,

$$\partial_n \cdot \boldsymbol{\partial}_x T_A(n) = \partial_n \cdot \boldsymbol{\partial}_x (\frac{1}{2} F n \tilde{F} - (n \lrcorner F) \cdot \boldsymbol{\partial}_x A). \tag{8.143}$$

Moreover,

$$\begin{aligned} \partial_n \cdot \boldsymbol{\partial}_x \frac{1}{2} F n \tilde{F} &= \frac{1}{2} \gamma^\mu \cdot \partial_n [(\partial_\mu \tilde{F}) n F + \tilde{F}(\partial_\mu n) F + \tilde{F} n \partial_\mu F] \\ &= \frac{1}{2} (\widetilde{\boldsymbol{\partial}_x F} F + \tilde{F} \boldsymbol{\partial}_x F) \\ &= \frac{1}{2} (J_e F - F J_e) = J_e \lrcorner F. \end{aligned} \tag{8.144}$$

Also,

$$\begin{aligned} \partial_n \cdot \boldsymbol{\partial}_x (n \lrcorner F) \cdot \boldsymbol{\partial}_x A) &= \gamma^\mu \cdot \partial_n [(\partial_\mu n \lrcorner F) \cdot \boldsymbol{\partial}_x A + (n \lrcorner \partial_\mu F) \cdot \boldsymbol{\partial}_x A + (n \lrcorner F) \cdot \partial_\mu \boldsymbol{\partial}_x A] \\ &= (\gamma^\mu \lrcorner \partial_\mu F) \cdot \boldsymbol{\partial}_x A + (\gamma^\mu \lrcorner F) \cdot \partial_\mu \boldsymbol{\partial}_x A \\ &= J_e \cdot \boldsymbol{\partial}_x A \\ &= \boldsymbol{\partial}_x \cdot J_e A + J_e \cdot \boldsymbol{\partial}_x \frac{1}{2} A \\ &= \gamma^\mu \cdot [(\partial_\mu J_e) A + J_e (\partial_\mu A)] \cdot \gamma^\mu \cdot \partial_n [(\partial_\mu n) J_e A + n \cdot (\partial_\mu J_e) A \\ &\quad + n \cdot J_e (\partial_\mu A) \\ &= \partial_n \cdot \boldsymbol{\partial}_x (n \cdot J_e) A. \end{aligned} \tag{8.145}$$

Using Eqs. (8.144) and (8.145) in Eq. (8.142) gives

$$\partial_n \cdot \boldsymbol{\partial}_x [T_\psi(n) - (n \cdot J_e) A] = F \llcorner J_e. \tag{8.146}$$

Finally taking into account (iii) in Eq. (8.134) we get

$$\partial_n \cdot \boldsymbol{\partial}_x [T_\psi(n) - (n \cdot J_e) A] = F \llcorner J_e, \tag{8.147}$$

which we recognize as Eq. (8.133).

Remark 8.46 This equation says that even if the Dirac-Hestenes field is moving in a region where $F = 0$, but $A \neq 0$ the consideration of the coupling term $-(n \cdot J_e) A$ is necessary in order for energy-momentum conservation to take place. Of course, this is related to the well known Bohm-Aharonov effect.

Exercise 8.47 Show that $\mathbf{J}^{\dagger}_{\psi A}(n) = T^{\dagger}_{\psi}(n) \wedge x + S^{\dagger}_{\psi}(n)$

$$\partial_n \cdot \boldsymbol{\partial}_x \mathbf{J}^{\dagger}_{\psi A}(n) = (F \llcorner \underset{e}{J}) \wedge x. \tag{8.148}$$

Exercise 8.48 Show that $\mathbf{J}_{\psi A}(\gamma^{\mu} \wedge \gamma_{\nu}) = \mathbf{J}^{\mu}_{\nu \psi A}$ satisfy

$$d \star \mathbf{J}^{\mu}_{\nu \psi A} = - \star \{[(F \llcorner \underset{e}{J}) \wedge x] \cdot (\gamma^{\mu} \wedge \gamma_{\nu})\}. \tag{8.149}$$

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Chapter 9
Conservation Laws on Riemann-Cartan and Lorentzian Spacetimes

Abstract In this chapter we examine in details the conditions for existence of conservation laws of energy-momentum and angular momentum for the matter fields of on a general Riemann-Cartan spacetime $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ and also in the particular case of Lorentzian spacetimes $\mathfrak{M} = (M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ which as we already know model gravitational fields in the GRT [3]. Riemann-Cartan spacetimes are supposed to model generalized gravitational fields in so called Riemann-Cartan theories. In what follows, we suppose that in $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ (or $\mathfrak{M}$) a set of dynamic fields live and interact. Of course, we want that the Riemann-Cartan spacetime admits spinor fields, and from what we learned in Chap. 7, this implies that the orthonormal frame bundle must be *trivial*. This permits a great simplification in our calculations. Moreover, we will suppose, for simplicity that the dynamic fields ϕ^A, $A = 1, 2, \ldots, n$, are in general distinct r-forms,[1] i.e., each $\phi^A \in \sec \bigwedge^r T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, for some $r = 0, 1, \ldots, 4$. Before we start our enterprise we think it is useful to recall some results which serve also the purpose to fix the notation for this chapter.

9.1 Preliminaries

9.1.1 Functional Derivatives on Jet Bundles

Let $J^1(\bigwedge T^*M)$ be the 1-jet bundle[2] over $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, i.e., the vector bundle defined by

$$J^1(\bigwedge T^*M) = \bigcup\nolimits_{x \in M} J^1_x(\bigwedge T^*M). \tag{9.1}$$

[1] This is not a serious restriction in the formalism since we already learned how (given a spinorial frame) to represent spinor fields by sums of even multiform fields.

[2] For the definition of jet bundles and notations see Sect. A.3.1.

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Then, with each local section $\phi \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})\}$ we may associate a local section $j_1(\phi) \in \sec J^1(\bigwedge T^*M)$. Let $\{\theta^{\mathbf{a}}\}$, $\theta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, $\mathbf{a} = 0, 1, 2, 3$, be an orthonormal basis of T^*M dual to the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$ of TM and let $\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ be the connection 1-forms of the connection ∇ such that $\nabla_{\boldsymbol{e}_{\mathbf{a}}} \boldsymbol{e}_{\mathbf{b}} = \omega_{\cdot\mathbf{ab}}^{\mathbf{c}\cdot\cdot} \boldsymbol{e}_{\mathbf{c}}$. We introduce also the 1-jet bundle $J^1[(\bigwedge T^*M)^{n+2}]$, i.e.,

$$J^1[(\bigwedge T^*M)^{n+2}] := \bigcup_{x \in M} J^1_x(\bigwedge T^*M \times \bigwedge T^*M \times \ldots \times \bigwedge T^*M) \tag{9.2}$$

over the configuration space $(\bigwedge T^*M)^{n+2} \hookrightarrow (\mathcal{C}\ell(M, \mathtt{g}))^{n+2}$ of a field theory describing n different fields ϕ^A belonging to sections of $\bigwedge T^pM \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ on a RCST, where for each different value of A we have in general a different value of p. We denote sections of $J^1[(\bigwedge T^*M)^{n+2}]$ by $j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi)$ or when no confusion arises simply by $j_1(\phi)$.

A functional for a field $\phi \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ in $J^1(\bigwedge T^*M)$ is a mapping $\mathcal{F} : \sec J^1(\bigwedge T^*M) \to \sec \bigwedge T^*M, j_1(\phi) \mapsto \mathcal{F}(j_1(\phi))$.

A Lagrangian density for a field theory described by fields $\phi^A \in \sec \bigwedge T^*M$, $A = 1, 2, \ldots, n$ over a Riemann-Cartan spacetime is a mapping

$$\mathcal{L}_m : \sec J^1[(\bigwedge T^*M)^{n+2}] \to \sec \bigwedge^4 T^*M, \tag{9.3}$$

$$j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi) \mapsto \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi)). \tag{9.4}$$

Remark 9.1 When convenient and context is clear, we eventually use instead of $\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi))$ the sloppy notations $\mathcal{L}_m(x, \theta^{\mathbf{a}}, d\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, d\omega^{\mathbf{a}}_{\mathbf{b}}, \phi, d\phi)$ or $\mathcal{L}_m(\theta^{\mathbf{a}}, d\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, d\omega^{\mathbf{a}}_{\mathbf{b}}, \phi, d\phi)$ when the Lagrangian density does not depend explicitly on x, or simply $\mathcal{L}_m[\phi]$ and even only $\mathcal{L}_m$. Moreover, in the calculations done below $\mathcal{L}_m(\theta^{\mathbf{a}}, d\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, d\omega^{\mathbf{a}}_{\mathbf{b}}, \phi, d\phi)$ will be considered as a multiform functional of the field variables $(\theta^{\mathbf{a}}, d\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, d\omega^{\mathbf{a}}_{\mathbf{b}}, \phi, d\phi)$.

9.1.2 Algebraic Derivatives

Definition 9.2 Let $X \in \sec \bigwedge^p T^*U$. *A multiform functional* F of X (not depending *explicitly* on $x \in M$) is a mapping

$$F : \sec \bigwedge^p T^*U \to \sec \bigwedge^r T^*U$$

Let $w := \boldsymbol{\delta} X \in \sec \bigwedge^p T^*U$. Write the *variation* of F in the direction of $\boldsymbol{\delta} X$ as the functional $\boldsymbol{\delta} F : \bigwedge^p U \to \bigwedge^r U$ given by

$$\boldsymbol{\delta} F = \lim_{\lambda \to 0} \frac{F(X + \lambda \boldsymbol{\delta} X) - F(X)}{\lambda}. \tag{9.5}$$

Remark 9.3 The algebraic derivative of F relative to X is given by

$$\boldsymbol{\delta} F := \boldsymbol{\delta} X \wedge \frac{dF}{dX}. \tag{9.6}$$

Remark 9.4 We recall (see Remark 2.99) that $\frac{dF}{dX}$ is related $\partial_X F$ defined in Chap. 2, but take into account that these objects in general differ. See Remark 2.99.

Remark 9.5 Sometimes, even if F depends only on X we write $\frac{\partial F}{\partial X}$ instead of $\frac{dF}{dX}$, i.e., we write Eq. (9.6) as

$$\boldsymbol{\delta} F := \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X}. \tag{9.7}$$

When a functional depends on two independent variables, e.g., $K(X, Y)$ we define the (partial) derivative $\frac{\partial K}{\partial X}$ by

$$\boldsymbol{\delta}_X K = \boldsymbol{\delta} X \wedge \frac{\partial K}{\partial X}. \tag{9.8}$$

Moreover, for a composed functional $F \circ G(X) = F(G(X))$ we gave the chain rule::

$$\frac{\partial}{\partial X} F(G(X)) = \frac{\partial G}{\partial X} \wedge \frac{\partial F}{\partial G} \tag{9.9}$$

and for a functional $L^p(X^r, G^k(X^r))$ where the superindices indicates here the grade of the object, we have immediately

$$\frac{dL^p}{dX^r} = \frac{\partial L^p}{\partial X^r} + \frac{\partial G^k}{\partial X^r} \wedge \frac{\partial L^p}{\partial G^k}. \tag{9.10}$$

Moreover, let $X \in \sec \bigwedge^p T^*U$. Given the functionals $F : \sec \bigwedge^p T^*U \to \sec \bigwedge^r T^*U \to$ and $G : \sec \bigwedge^p T^*U \to \sec \bigwedge^s T^*U$ the variation $\boldsymbol{\delta}$ satisfies

$$\boldsymbol{\delta}(F \wedge G) = \boldsymbol{\delta} F \wedge G + F \wedge \boldsymbol{\delta} G, \tag{9.11}$$

and the algebraic derivative (as is trivial to verify) satisfies

$$\frac{\partial}{\partial X}(F \wedge G) = \frac{\partial F}{\partial X} \wedge G + (-1)^{rp} F \wedge \frac{\partial G}{\partial X}. \tag{9.12}$$

Another important property of $\boldsymbol{\delta}$ is that it commutes with the exterior derivative operator d, i.e., for any given functional F

$$d\boldsymbol{\delta} F = \boldsymbol{\delta} dF. \tag{9.13}$$

In general, we may have functionals depending on several different multiform fields, say, $F : \sec(\bigwedge^p T^*U \times \bigwedge^q T^*U) \to \sec \bigwedge^r T^*U$, with $(X, Y) \mapsto F(X, Y) \in \sec \bigwedge^r T^*U$. In this case, we have:

$$\boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} Y \wedge \frac{\partial F}{\partial Y}. \tag{9.14}$$

We are particularly interested in the important case where the functional F is such that $F(X, dX) : \sec(\bigwedge^p T^*U \times \bigwedge^{p+1} T^*U) \to \sec \bigwedge^4 T^*U$, $(X, dX) \mapsto F(X, dX)$. Supposing that the variation $\boldsymbol{\delta} X$ is chosen to be null on the boundary $\partial U'$, $U' \subset U$ (or that $\frac{\partial F}{\partial dX}\big|_{\partial U'} = 0$) and taking into account Stokes theorem, we can write defining $\mathcal{F} : j^1(X) \to \mathbb{R}$ by

$$\mathcal{F}(X) := \int_{U'} F(X) \tag{9.15}$$

that

$$\begin{aligned}
\boldsymbol{\delta}\mathcal{F}(X) &= \int_{U'} \boldsymbol{\delta} F(X) = \int_{U'} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} dX \wedge \frac{\partial F}{\partial dX} \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + d\left(\boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX}\right) \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + \int_{\partial U'} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX} \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \frac{\boldsymbol{\delta}\mathcal{F}(X)}{\boldsymbol{\delta} X},
\end{aligned} \tag{9.16}$$

where

$$\frac{\boldsymbol{\delta}\mathcal{F}(X)}{\boldsymbol{\delta} X} = \frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right). \tag{9.17}$$

with $\frac{\boldsymbol{\delta}\mathcal{F}}{\boldsymbol{\delta} X} : \sec(\bigwedge^p T^*U \times \bigwedge^{p+1} T^*U) \to \bigwedge^{4-p} T^*U$ called the *functional derivative*[3] of $\mathcal{F}$.

[3]We observe that some authors (e.g.,[1, 7, 11]) denote $\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)$ by $\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta} X}$, something we also did in the first edition of our book.

9.2 Euler-Lagrange Equations from Lagrangian Densities

The principle of stationary action, is here formulated as the statement that the variation of the action integral (see Eq. (8.24)) written in terms of a Lagrangian density $\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))$ is null for arbitrary variations of ϕ which vanish in the boundary ∂U of the open set $U \subset M$ (i.e., $\boldsymbol{\delta}\phi|_{\partial U} = 0$)

$$\boldsymbol{\delta}\mathcal{A}(\phi) = \boldsymbol{\delta}\int_U \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) = \int_U \boldsymbol{\delta}\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) = 0. \tag{9.18}$$

Using Eq. (9.17) gives immediately

$$\boldsymbol{\delta}\mathcal{A}(\phi) = \int_U \boldsymbol{\delta}\phi \wedge \frac{\boldsymbol{\delta}\mathcal{A}(\phi)}{\boldsymbol{\delta}\phi} \tag{9.19}$$

where

$$\star\,\boldsymbol{\Sigma}(\phi) = \frac{\boldsymbol{\delta}\mathcal{A}(\phi)}{\boldsymbol{\delta}\phi} := \frac{\partial\mathcal{L}_m(\phi)}{\partial\phi} - (-1)^r d\left(\frac{\partial\mathcal{L}_m(\phi)}{\partial d\phi}\right). \tag{9.20}$$

is known as the *Euler-Lagrange functional* for the field ϕ Since $\boldsymbol{\delta}\phi$ is arbitrary in Eq. (9.19), the stationary action principle implies that

$$\star\boldsymbol{\Sigma}(\phi) = 0,$$

is the corresponding ELE for the field ϕ.

We recall also that if $\mathcal{G}(j_1(\phi)) \in \sec \bigwedge^p T^*M$ is an arbitrary functional and $\sigma : M \to M$ a diffeomorphism, then $\mathcal{G}(j_1(\phi))$ is said to be invariant under σ if and only if $\sigma^*\mathcal{G}(j_1(\phi)) = \mathcal{G}(j_1(\phi))$. Also, it is a well known result that $\mathcal{G}(j_1(\phi))$ is invariant under the action of a one parameter group of diffeomorphisms σ_t if and only if

$$\pounds_\xi\mathcal{G}(j_1(\phi)) = 0, \tag{9.21}$$

where $\xi \in \sec TM$ is the infinitesimal generator of the group σ_t.

As an example, the Lagrangian density for the electromagnetic field generated by a current $J_e \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ in a Riemann-Cartan spacetime where $F = dA$, $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is

$$\mathcal{L}_{\mathrm{em}}(A) = -\frac{1}{2}F \wedge \star F - \star J_e \wedge A. \tag{9.22}$$

The ELE (see Exercise 9.15) gives $\delta F = -J_e$ and since $dF = 0$ we have Maxwell equations

$$dF = 0,\ \delta F = -J_e. \tag{9.23}$$

9.3 Invariance of the Action Integral Under the Action of a Diffeomorphism

Proposition 9.6 *The action $\mathcal{A}(\phi)$ for any field theory formulated in terms of fields that are differential forms is invariant under the action of one parameters groups of diffeomorphisms if $\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))\big|_{\partial U} = 0$ on the boundary of ∂U of a domain $U \subset M$.*

Proof Let $\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))$ be the Lagrangian density of the theory. The variation of the action which we are interested is the horizontal variation, i.e.:

$$\boldsymbol{\delta}_h \mathcal{A}(\phi) = -\int_U £_\xi \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)). \tag{9.24}$$

Let

$$\xi^* = \boldsymbol{g}(\xi,) \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}). \tag{9.25}$$

Then we have (from a well known property of the Lie derivative) that

$$£_\xi \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) = d[\xi^* \lrcorner \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))] + \xi^* \lrcorner [d\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))]. \tag{9.26}$$

But, since $\mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) \in \sec \bigwedge^4 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have $d\mathcal{L}_m = 0$ and then $£_\xi \mathcal{L}_m = d[\xi^* \lrcorner \mathcal{L}_m]$. It follows, using Stokes theorem that

$$\begin{aligned}\int_U £_\xi \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) &= \int_U d[\xi^* \lrcorner \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi))] \\ &= \int_{\partial U} \xi^* \lrcorner \mathcal{L}_m(j_1(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)) = 0, \end{aligned} \tag{9.27}$$

since $\mathcal{L}_m(\phi)|_{\partial U} = 0$. ■

Remark 9.7 It is important to emphasize that the action integral is always invariant under the action of a one parameter group of diffeomorphisms even if the corresponding Lagrangian density is not invariant (in the sense of Eq. (9.21)) under the action of that one parameter group of diffeomorphisms.

9.4 Covariant 'Conservation' Laws

Let $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ denotes a general Riemann-Cartan *spacetime*. As stated above we suppose that the dynamic fields ϕ^A, $A = 1, 2, \ldots, n$, are r-forms, i.e., each $\phi^A \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, for some $r = 0, 1, \ldots, 4$.

Let $\{e_{\mathbf{a}}\}$ be an arbitrary global orthonormal basis for TM, and let $\{\theta^{\mathbf{a}}\}$ be the dual basis.[4] We suppose that $\theta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Let moreover $\{\theta_{\mathbf{a}}\}$ be the reciprocal basis to $\{\theta^{\mathbf{a}}\}$. In Chap. 4 we learned how to represent the gravitational field using $\{\theta^{\mathbf{a}}\}$ and how to write Einstein equations for such objects.[5]

Here, we make the hypothesis that a Riemann-Cartan spacetime models a generalized gravitational field which must be described by $\{\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\}$, where $\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ are the connection 1-forms (in a given gauge). Thus, we suppose that a dynamic theory for the matter fields $\phi^A \in \sec \bigwedge^r T^*M$ is obtained through the introduction a Lagrangian density, which is a functional on $J^1[(\bigwedge T^*M)^{2+n}]$ as previously discussed.

Active *local* Lorentz transformations are represented by *even* sections of the Clifford bundle $U \in \sec \mathrm{Spin}^e_{1,3}(M) \hookrightarrow \sec \mathcal{C}\ell^{(0)}(M, \mathtt{g})$, such that $U\tilde{U} = \tilde{U}U = 1$, i.e., $U(x) \in \mathrm{Spin}^e_{1,3} \simeq \mathrm{Sl}(2, \mathbb{C})$. Under a local Lorentz transformation the fields transform as

$$\begin{aligned} \theta^{\mathbf{a}} &\mapsto \theta'^{\mathbf{a}} = U\theta^{\mathbf{a}}U^{-1} = \Lambda^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}, \\ \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} &\mapsto \omega'^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \Lambda^{\mathbf{a}}_{\mathbf{c}}\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{d}}(\Lambda^{-1})^{\mathbf{d}}_{\mathbf{b}} + \Lambda^{\mathbf{a}}_{\mathbf{c}}(d\Lambda^{-1})^{\mathbf{c}}_{\mathbf{b}}, \\ \phi^A &\mapsto \phi'^A = U\phi^A U^{-1}, \end{aligned} \tag{9.28}$$

where $\Lambda^{\mathbf{a}}_{\mathbf{b}}(x) \in \mathrm{SO}^e_{1,3}$. In our formalism it is easy to see that $\mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)$ is invariant under local Lorentz transformations. Indeed, since $\tau_{\boldsymbol{g}} := \theta^{\mathbf{5}} = \theta^{\mathbf{0}}\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{\mathbf{3}} \in \sec \bigwedge^4 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ commutes with even multiform fields, we have that a local Lorentz transformation produces no changes in $\mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)$, i.e.,

$$\mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi) \mapsto U\mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi)U^{-1} = \mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi). \tag{9.29}$$

However, this does not implies necessarily that the variation of the Lagrangian density $\mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi)$ obtained by variation of the fields $(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi)$ is null, since $\boldsymbol{\delta}_v\mathcal{L}_m = \mathcal{L}_m(\theta^{\mathbf{a}} + \boldsymbol{\delta}_v\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}} + \boldsymbol{\delta}_v\omega^{\mathbf{a}}_{\mathbf{b}}, \phi + \boldsymbol{\delta}_v\phi) - \mathcal{L}_m(\theta^{\mathbf{a}}, \omega^{\mathbf{a}}_{\mathbf{b}}, \phi) \neq 0$, unless it happens that for an infinitesimal Lorentz transformation,

$$\begin{aligned} \mathcal{L}_m(\theta^{\mathbf{a}} + \boldsymbol{\delta}_v\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \boldsymbol{\delta}_v\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi + \boldsymbol{\delta}_v\phi) &= \mathcal{L}_m(U\theta^{\mathbf{a}}U^{-1}, U\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}U^{-1}, U\phi U^{-1}) \\ &= U\mathcal{L}_m U^{-1} = \mathcal{L}_m. \end{aligned} \tag{9.30}$$

As we just showed above the action of any Lagrangian density is invariant under diffeomorphisms. Let us calculate the total variation of the Lagrangian density $\mathcal{L}_m$, arising from a one-parameter group of diffeomorphisms generated by a vector field

[4] In this Chapter boldface latin indices, say **a**, take the values 0, 1, 2, 3.

[5] A Lagrangian density for the $\{\theta^{\mathbf{a}}\}$ for the case of GRT will be introduced in Sect. 8.5 and explored in details in the next Chapter.

$\xi \in \sec TM$ and by a local Lorentz transformation, when we vary $\theta^{\mathbf{a}}, \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}, \phi^A, d\phi^A$ independently. We have (recalling Eq. (8.30))

$$\boldsymbol{\delta}\mathcal{L}_m = \boldsymbol{\delta}_v\mathcal{L}_m - \pounds_\xi\mathcal{L}_m. \tag{9.31}$$

In what follows we suppose that the Lagrangian of the matter field is invariant under local Lorentz transformations,[6] i.e., $\boldsymbol{\delta}_v\mathcal{L}_m = 0$. Now, note that $\mathcal{L}_m$ depends on the $\theta^{\mathbf{a}}$ due to the dependence of the fields $\phi^{\mathbf{a}}$ on these variables and on the $\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ because eventual covariant derivatives of the fields $\phi^{\mathbf{a}}$ must appears in it. We suppose moreover that $\mathcal{L}_m$ does not depend explicitly on $d\theta^{\mathbf{a}}$ and $d\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$. Then

$$\boldsymbol{\delta}\mathcal{L}_m = -\pounds_\xi\mathcal{L}_m = \boldsymbol{\delta}\theta^{\mathbf{a}} \wedge \frac{\partial\mathcal{L}_m}{\partial\theta^{\mathbf{a}}} + \boldsymbol{\delta}\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \frac{\partial\mathcal{L}_m}{\partial\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}} + \boldsymbol{\delta}\phi^A \wedge \frac{\boldsymbol{\delta}\mathcal{A}(\phi)}{\boldsymbol{\delta}\phi^A} \tag{9.32}$$

$$= -\pounds_\xi\theta^{\mathbf{a}} \wedge \star\mathcal{T}_{\mathbf{a}} - \pounds_\xi\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \star\boldsymbol{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} - \pounds_\xi\phi^A \wedge \star\boldsymbol{\Sigma}_A, \tag{9.33}$$

where $\star\boldsymbol{\Sigma}_A = \frac{\boldsymbol{\delta}\mathcal{A}(\phi)}{\boldsymbol{\delta}\phi^A}$ are the Euler-Lagrange functionals of the fields ϕ^A and we have:

Definition 9.8 The negative of the coefficients of $\boldsymbol{\delta}\theta^{\mathbf{a}} = -\pounds_\xi\theta^{\mathbf{a}}$, i.e.

$$\star\, T_{\mathbf{a}} = -\star\,\mathcal{T}_{\mathbf{a}} := -\frac{\partial\mathcal{L}_m}{\partial\theta^{\mathbf{a}}} \in \sec \bigwedge\nolimits^3 T^*M \tag{9.34}$$

are called the energy-momentum densities of the matter fields, and the $T_{\mathbf{a}} = -\mathcal{T}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M$ are called the energy momentum 1-form fields of the matter fields. The negative of coefficients of $\boldsymbol{\delta}\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = -\pounds_\xi\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$, i.e.,

$$\star\,\boldsymbol{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} = -\star\,\mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} = -\frac{\partial\mathcal{L}_m}{\partial\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}} \in \sec \bigwedge\nolimits^3 T^*M, \tag{9.35}$$

are called the angular momentum densities of the matter fields.

Taking into account that each one of the fields ϕ^A obey a Euler-Lagrange equation, $\star\boldsymbol{\Sigma}_A = 0$, we can write

$$\int \boldsymbol{\delta}\mathcal{L}_m = -\int \pounds_\xi\mathcal{L}_m = \int \star\mathcal{T}_{\mathbf{a}} \wedge \pounds_\xi\theta^{\mathbf{a}} + \star\mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} \wedge \pounds_\xi\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\,. \tag{9.36}$$

Now, since all geometrical objects in the above formulas are sections of the Clifford bundle, recalling Eq. (4.61), we can write

$$\pounds_\xi\theta^{\mathbf{a}} = \xi^* \lrcorner d\theta^{\mathbf{a}} + d(\xi^* \lrcorner \theta^{\mathbf{a}}). \tag{9.37}$$

[6] We discuss further the issue of local Lorentz invariance and its hidden consequence in [5, 28].

Moreover, recalling also the first Cartan's structure equation,

$$d\theta^{\mathbf{a}} + \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} = \Theta^{\mathbf{a}}, \tag{9.38}$$

we get

$$\begin{aligned}
\pounds_{\xi}\theta^{\mathbf{a}} &= \xi^* \lrcorner \Theta^{\mathbf{a}} - \xi^* \lrcorner \left(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}}\right) + d(\xi^* \lrcorner \theta^{\mathbf{a}}) \\
&= \xi^* \lrcorner \Theta^{\mathbf{a}} - (\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}})\theta^{\mathbf{b}} + \left(\xi^* \cdot \theta^{\mathbf{b}}\right)\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + d(\xi^* \lrcorner \theta^{\mathbf{a}}) \\
&= \mathbf{D}(\xi^* \lrcorner \theta^{\mathbf{a}}) + \xi^* \lrcorner \Theta^{\mathbf{a}} - (\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}})\theta^{\mathbf{b}},
\end{aligned} \tag{9.39}$$

where $\mathbf{D}$ is the covariant exterior derivative of indexed p-form fields introduced by (Eq. (4.118)). To continue we need the following

Proposition 9.9 *Let* $\boldsymbol{\omega}$ *be the* 4×4 *matrix whose entries are the connection* 1*-forms. For any* $x \in M$, $\xi^* \lrcorner \omega^{\mathbf{a}}_{\mathbf{b}} \in \mathrm{spin}^e_{1,3} \simeq \mathrm{sl}(2,\mathbb{C}) = \mathrm{so}^e_{1,3}$, *the Lie algebra of* $\mathrm{Spin}^e_{1,3}$ *(or of* $\mathrm{SO}^e_{1,3}$*).*

Proof Recall that any infinitesimal local Lorentz transformation (at $x \in M$) $\Lambda^{\mathbf{a}}_{\mathbf{b}} \in \mathrm{SO}^e_{1,3}$ can be written as

$$\begin{aligned}
&\Lambda^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \delta^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \chi^{\mathbf{a}\cdot}_{\cdot b}, \ \left|\chi^{\mathbf{a}}_{\mathbf{b}}\right| \ll 1, \\
&\chi_{\mathbf{ab}} = -\chi_{\mathbf{ba}}.
\end{aligned} \tag{9.40}$$

Now,

$$\begin{aligned}
\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} &= \xi^* \cdot (L^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{cb}}\theta^{\mathbf{c}}) = (\xi_{\mathbf{d}}\theta^{\mathbf{d}}) \cdot (L^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{cb}}\theta^{\mathbf{c}}) \\
&= \xi^{\mathbf{c}} L^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{cb}}
\end{aligned} \tag{9.41}$$

and we see that $\xi^* \cdot \omega_{\mathbf{ab}}$ satisfy

$$\xi^* \cdot \omega_{\mathbf{ab}} + \xi^* \cdot \omega_{\mathbf{ba}} = \xi^{\mathbf{c}}(L_{\mathbf{acb}} + L_{\mathbf{bca}}) = 0, \tag{9.42}$$

since in an orthonormal basis the connection coefficients satisfy $L_{\mathbf{acb}} = -L_{\mathbf{bca}}$. We see then that we can identify if $|\xi^{\mathbf{c}}| << 1$

$$\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} := -\xi^* \cdot \omega^{\mathbf{a}}_{\cdot\mathbf{b}}. \tag{9.43}$$

as the generator of an infinitesimal Lorentz transformation, and the proposition is proved. ■

Now, the term $-\left(\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right)\theta^{\mathbf{b}}$ has the form of a local *vertical* variation of the $\theta^{\mathbf{a}}$ and thus we write

$$\boldsymbol{\delta}_v \theta^{\mathbf{a}} := -\left(\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right)\theta^{\mathbf{b}} = \chi^{a\cdot}_{\cdot b}\theta^{\mathbf{b}} \tag{9.44}$$

Using Eq. (9.44) we can rewrite Eq. (9.39) as

$$\pounds_\xi \theta^{\mathbf{a}} = \mathbf{D}(\xi^* \cdot \theta^{\mathbf{a}}) + \xi^* \lrcorner \Theta^{\mathbf{a}} + \boldsymbol{\delta}_v \theta^{\mathbf{a}}. \tag{9.45}$$

We see that $\pounds_\xi \theta^{\mathbf{a}} = \boldsymbol{\delta}_v \theta^{\mathbf{a}}$ only if we have the following constraint

$$\mathbf{D}(\xi^* \cdot \theta^{\mathbf{a}}) + \xi^* \lrcorner \Theta^{\mathbf{a}} = 0. \tag{9.46}$$

A necessary and sufficient condition for the validity of Eq. (9.46) is given by Lemma 9.11 below.

Now, let us calculate $\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$. By definition,

$$\begin{aligned} \pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} &= \xi^* \lrcorner (d\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) + d(\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) \\ &= \xi^* \lrcorner (\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) - (\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}})\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} + (\xi^* \cdot \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}})\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} + d(\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}), \end{aligned} \tag{9.47}$$

where in writing the second line in Eq. (9.47) we used Cartan's second structure equation,

$$d\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} = \mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}. \tag{9.48}$$

Under an infinitesimal Lorentz transformation $\Lambda = 1 + \chi$, recalling Eq. (9.28), we can write (in obvious matrix notation)

$$\boldsymbol{\delta}_v \omega = -d\chi + \chi\omega - \omega\chi, \tag{9.49}$$

which using Eq. (9.43) gives for Eq. (9.47)

$$\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \xi^* \lrcorner (\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) + \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}. \tag{9.50}$$

Now, for a vertical variation we have:

$$\int_U \boldsymbol{\delta}_v \mathcal{L}_m := \int_U \boldsymbol{\delta}_v \theta^{\mathbf{a}} \wedge \frac{\partial \mathcal{L}_m}{\partial \theta^{\mathbf{a}}} + \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \frac{\partial \mathcal{L}_m}{\partial \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}} + \boldsymbol{\delta}_v \phi^A \wedge \frac{\boldsymbol{\delta} \mathcal{L}_m}{\boldsymbol{\delta} \phi^A}. \tag{9.51}$$

Then, if we recall that we assumed that $\int \boldsymbol{\delta}_v \mathcal{L}_m = 0$ and if we suppose that the field equations are satisfied, i.e., $\star\boldsymbol{\Sigma}_A = \frac{\boldsymbol{\delta}\mathcal{L}_m}{\boldsymbol{\delta}\phi^A} = 0$, Eq. (9.36) becomes,

$$\int \boldsymbol{\delta} \mathcal{L}_m = -\int \pounds_\xi \mathcal{L}_m = \int \star\mathcal{T}_{\mathbf{a}} \wedge \pounds_\xi \theta^{\mathbf{a}} + \star\mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} \wedge \pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\,. \tag{9.52}$$

Then

$$\begin{aligned}
-\int \pounds_\xi \mathcal{L}_m &= -\int \left[\mathbf{D}(\xi^* \cdot \theta^{\mathbf{a}}) + (\boldsymbol{\xi}^* \lrcorner \Theta^{\mathbf{a}}) + \boldsymbol{\delta}_v \theta^{\mathbf{a}}\right] \wedge \star \mathcal{T}_{\mathbf{a}} \\
&\quad - \int \left[\xi^* \lrcorner (\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) + \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right] \wedge \star \mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} \\
&= \int \star \mathcal{T}_{\mathbf{a}} \wedge (\boldsymbol{\xi}^* \lrcorner \Theta^{\mathbf{a}}) + \star \mathcal{J}^{\mathbf{b}\cdot}_{\cdot\mathbf{a}} \wedge (\xi^* \lrcorner (\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) \\
&\quad + \mathbf{D}[\star \mathcal{T}_{\mathbf{a}} (\xi^* \cdot \theta^{\mathbf{a}})] - (\mathbf{D} \star \mathcal{T}_{\mathbf{a}})(\xi^* \cdot \theta^{\mathbf{a}})) \\
&= \int \star \mathcal{T}_{\mathbf{a}} \wedge (\boldsymbol{\xi}^* \lrcorner \Theta^{\mathbf{a}}) + \star \mathcal{J}^{\mathbf{b}\cdot}_{\cdot\mathbf{a}} \wedge (\xi^* \lrcorner (\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}) - (\mathbf{D} \star \mathcal{T}_{\mathbf{a}})(\xi^* \cdot \theta^{\mathbf{a}}))
\end{aligned} \tag{9.53}$$

where we used that $\mathbf{D}[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] = d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}]$, that $\star \mathcal{T}_{\mathbf{a}}|_{\partial U} = 0$ and

$$\int_U d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] = \int_{\partial U} (\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}} = \mathbf{0}. \tag{9.54}$$

Now, writing $\xi^* = \xi^{\mathbf{a}} \theta_{\mathbf{a}} = \xi_{\mathbf{a}} \theta^{\mathbf{a}}$, and recalling that the action is invariant under diffeomorphisms we have (if as usual we suppose that $\mathcal{L}_m|_{\partial U} = 0$):

$$\int \boldsymbol{\delta} \mathcal{L}_m = -\int \pounds_\xi \mathcal{L}_m = \left[\star \mathcal{T}_{\mathbf{a}} \wedge (\theta_{\mathbf{c}} \lrcorner \Theta^{\mathbf{a}}) + \star \mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} \wedge (\theta_{\mathbf{c}} \lrcorner \mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) - \mathbf{D} \star \mathcal{T}_{\mathbf{c}}\right] \xi^{\mathbf{c}} = 0, \tag{9.55}$$

and since the $\xi^{\mathbf{c}}$ are arbitrary, we end with

$$\mathbf{D} \star \mathcal{T}_{\mathbf{c}} + \star \mathcal{T}_{\mathbf{a}} \wedge (\theta_{\mathbf{c}} \lrcorner \boldsymbol{\Theta}^{\mathbf{a}}) + \star \mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} \wedge (\theta_{\mathbf{c}} \lrcorner \mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}) = 0. \tag{9.56}$$

Also, using the explicit expressions for $\boldsymbol{\delta}_v \theta^{\mathbf{a}}$ and $\boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ given by Eqs. (9.44) and (9.50) in Eq. (9.51) for $\int \boldsymbol{\delta}_v \mathcal{L}_m$ we get,

$$\begin{aligned}
&\int \star \mathcal{T}_{\mathbf{a}} \wedge \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \theta^{\mathbf{b}} + \star \mathcal{J}^{\mathbf{b}\cdot}_{\cdot\mathbf{a}} \wedge \left(\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} - \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \chi^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} - d\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right) \\
&= \int \left[\frac{1}{2}\left(\star \mathcal{T}_{\mathbf{a}} \wedge \theta^{\mathbf{b}} - \star \mathcal{T}^{\mathbf{b}} \wedge \theta_{\mathbf{a}}\right) - d \star \mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{a}\cdot} - \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \star \mathcal{J}^{\cdot\mathbf{b}}_{\mathbf{c}\cdot} - \star \mathcal{J}^{\mathbf{c}\cdot}_{\cdot\mathbf{a}} \wedge \omega^{\mathbf{b}\cdot}_{\cdot\mathbf{c}}\right] \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \\
&= 0,
\end{aligned} \tag{9.57}$$

and since the coefficients $\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ are arbitrary we end with

$$\mathbf{D} \star \mathcal{J}^{\mathbf{b}\cdot}_{\cdot\mathbf{a}} + \frac{1}{2}\left(\star \mathcal{T}^{\mathbf{b}} \wedge \theta_{\mathbf{a}} - \star \mathcal{T}_{\mathbf{a}} \wedge \theta^{\mathbf{b}}\right) = 0. \tag{9.58}$$

Equations (9.56) and (9.58) are known as *covariant conservation laws* [3]. They are simply identities that follows from the hypothesis utilized, namely that the theory is invariant under diffeomorphisms and also invariant under the local action of the group $\mathrm{Spin}^e_{1,3}$. Equations (9.56) and (9.58) do *not* encode genuine conservation laws and a memorable number of nonsense have been generated along the years, by authors that use in a naive way those equations. Some examples of the nonsense is discussed in the specific case of Einstein's theory in Sect. 8.7.

9.5 When Genuine Conservation Laws Do Exist?

We show now that when the Riemann-Cartan *spacetime* $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ admits symmetries, then Eqs. (9.56) and (9.58) can be used for the construction of closed 3-forms, which then provides genuine conservation laws for the matter fields. We present that result in the form of the following [3]

Proposition 9.10 *For each Killing vector field* $\xi \in \sec TM$*, such that* $\pounds_\xi \boldsymbol{g} = 0$ *and* $\pounds_\xi \Theta = 0$*, where* $\Theta = \boldsymbol{e}_{\mathbf{a}} \otimes \Theta^{\mathbf{a}}$ *is the* torsion *tensor of* ∇*, and* $\Theta^{\mathbf{a}}$ *the torsion 2-forms, we have*

$$d\left[\left(\xi^* \cdot \theta^{\mathbf{a}}\right) \star \mathcal{T}_{\mathbf{a}} + \left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \star \mathcal{J}^{\mathbf{b}\cdot}_{\cdot\mathbf{a}}\right] = 0, \tag{9.59}$$

where $\mathbf{L}_\xi = \xi^* \lrcorner \mathbf{D} + \mathbf{D}\mathbf{i}_\xi$ *is the so called Lie covariant derivative.*

In order to prove the Proposition 9.10, we need some preliminary results, which we recall in the form of lemmas.

Lemma 9.11 $\pounds_\xi \theta^{\mathbf{a}} = \boldsymbol{\delta}_v \theta^{\mathbf{a}}$ *and* $\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ *if and only if* $\pounds_\xi \boldsymbol{g} = 0$ *and* $\pounds_\xi \Theta = 0$.

Proof Let us show first that if $\pounds_\xi \theta^{\mathbf{a}} = \boldsymbol{\delta}_v \theta^{\mathbf{a}}$ then $\pounds_\xi \boldsymbol{g} = 0$. We have

$$\pounds_\xi \boldsymbol{g} = \eta_{\mathbf{ab}} \left(\pounds_\xi \theta^{\mathbf{a}}\right) \otimes \theta^{\mathbf{b}} + \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes (\pounds_\xi \theta^{\mathbf{b}}). \tag{9.60}$$

On the other since $\boldsymbol{g}$ is invariant under local Lorentz transformations, we have

$$\boldsymbol{\delta}_v \boldsymbol{g} = \eta_{\mathbf{ab}} \left(\boldsymbol{\delta}_v \theta^{\mathbf{a}}\right) \otimes \theta^{\mathbf{b}} + \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes (\boldsymbol{\delta}_v \theta^{\mathbf{b}}) = 0. \tag{9.61}$$

Then, it follows from Eqs. (9.60) and (9.61) that if $\pounds_\xi \theta^{\mathbf{a}} = \boldsymbol{\delta}_v \theta^{\mathbf{a}}$ then $\pounds_\xi \boldsymbol{g} = 0$.

Taking into account the definition of Lie derivative we can write

$$\begin{aligned} \pounds_\xi \boldsymbol{e}_{\mathbf{a}} &= -\varkappa^{\mathbf{b}\cdot}_{\cdot\mathbf{a}} \boldsymbol{e}_{\mathbf{b}}, \quad \pounds_\xi \theta^{\mathbf{a}} = \varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \theta^{\mathbf{b}}, \\ \varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} &= [\boldsymbol{e}_{\mathbf{a}}(\xi^{\mathbf{b}}) - \xi^{\mathbf{m}} c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{am}}]. \end{aligned} \tag{9.62}$$

Now, if $\pounds_\xi \boldsymbol{g} = 0$ we have from Eq. (9.60) that $\left(\eta_{\mathbf{cb}}\varkappa^{\mathbf{c}\cdot}_{\cdot\mathbf{a}} + \eta_{\mathbf{ac}}\varkappa^{\mathbf{c}\cdot}_{\cdot\mathbf{b}}\right)\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} = 0$, i.e., in this case we need to have

$$\varkappa_{\mathbf{ab}} + \varkappa_{\mathbf{ba}} = 0, \tag{9.63}$$

and then it follows that for any $x \in M$, $\varkappa_{\mathbf{ab}} \in \mathrm{spin}^e_{1,3}$. Using Proposition 9.9 and identifying $\varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = -\xi^* \cdot \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ the vertical variation can be written as $\boldsymbol{\delta}_v \theta^{\mathbf{a}} = \pounds_\xi \theta^{\mathbf{a}}$.

The proof that if $\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\mathbf{b}\cdot\mathbf{v}}$ then $\pounds_\xi \Theta = 0$ is trivial. In the following we prove the reciprocal, i.e., if $\pounds_\xi \Theta = 0$ then $\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$. We have,

$$\pounds_\xi \Theta = \pounds_\xi \boldsymbol{e}_{\mathbf{a}} \otimes \Theta^{\mathbf{a}} + \boldsymbol{e}_{\mathbf{a}} \otimes \pounds_\xi \Theta^{\mathbf{a}}. \tag{9.64}$$

Then, if $\pounds_\xi \Theta = 0$ we conclude that

$$\pounds_\xi \Theta^{\mathbf{a}} = \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \Theta^{\mathbf{b}}, \tag{9.65}$$

which is an infinitesimal Lorentz transformation of the torsion 2-forms. On the other hand, taking into account Cartan's first structure equation, Eq. (9.62) and the fact that $\pounds_\xi d\theta^{\mathbf{a}} = d(\pounds_\xi \theta^{\mathbf{a}})$, we can write

$$\begin{aligned} \pounds_\xi \Theta^{\mathbf{a}} &= \pounds_\xi d\theta^{\mathbf{a}} + \pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} + \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \pounds_\xi \theta^{\mathbf{b}} \\ &= d\left(\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \theta^{\mathbf{b}}\right) + \pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} + \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \chi^{\mathbf{b}\cdot}_{\cdot\mathbf{c}} \theta^{\mathbf{c}} \\ &= d\left(\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right) \wedge \theta^{\mathbf{b}} + \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} d\theta^{\mathbf{b}} + \pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} + \chi^{\mathbf{b}\cdot}_{\cdot\mathbf{c}} \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}}. \end{aligned} \tag{9.66}$$

Also, using Eq. (9.65) we have

$$\pounds_\xi \Theta^{\mathbf{a}} = \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} d\theta^{\mathbf{b}} + \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \omega^{\mathbf{b}\cdot}_{\cdot\mathbf{c}} \wedge \theta^{\mathbf{c}}. \tag{9.67}$$

From Eqs. (9.66) and (9.67) it follows that

$$\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} = \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} - \chi^{\mathbf{b}\cdot}_{\cdot\mathbf{c}} \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \theta^{\mathbf{b}} - d\left(\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\right) \wedge \theta^{\mathbf{b}}, \tag{9.68}$$

or

$$\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \chi^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \omega^{\mathbf{c}}_{\cdot\mathbf{b}} - \chi^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} - d\chi^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\,. \tag{9.69}$$

Thus, recalling Eq. (9.49) we finally have that $\pounds_\xi \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \boldsymbol{\delta}_v \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$. ■

Corollary 9.12 *For any $x \in M$, $\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}$ is an element of* $\mathrm{spin}^e_{1,3}$, *if and only if,* $\pounds_\xi \boldsymbol{g} = 0$.

Proof The Lie covariant derivative of $\theta^{\mathbf{a}}$ is given by

$$\begin{aligned}
\mathbf{L}_{\xi}\theta^{\mathbf{a}} &= \xi^{*}\lrcorner\mathbf{D}\theta^{\mathbf{a}} + \mathbf{D}\left(\boldsymbol{\xi}^{*}\cdot\theta^{\mathbf{a}}\right)\\
&= \xi^{*}\lrcorner\left(d\theta^{\mathbf{a}} + \omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}\wedge\theta^{\mathbf{b}}\right) + d\left(\boldsymbol{\xi}^{*}\cdot\theta^{\mathbf{a}}\right) + \omega_{\mathbf{b}}^{\mathbf{a}}\left(\boldsymbol{\xi}^{*}\cdot\theta^{\mathbf{b}}\right)\\
&= \pounds_{\xi}\theta^{\mathbf{a}} + (\xi^{*}\cdot\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot})\theta^{\mathbf{b}} - (\xi^{*}\cdot\theta^{\mathbf{b}})\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} + \omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}\left(\xi^{*}\cdot\theta^{\mathbf{b}}\right)\\
&= \pounds_{\xi}\theta^{\mathbf{a}} + (\xi^{*}\cdot\omega_{\cdot\mathbf{k}}^{\mathbf{a}\cdot})\theta^{\mathbf{k}} = \left(\chi_{\cdot\mathbf{k}}^{\mathbf{a}\cdot} + \xi^{*}\cdot\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}\right)\theta^{\mathbf{k}},
\end{aligned}$$

where we put $\pounds_{\xi}\theta^{\mathbf{a}} = \chi_{\cdot\mathbf{k}}^{\mathbf{a}\cdot}\theta^{\mathbf{k}}$. Then,

$$\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}} = \chi_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} + \xi^{*}\cdot\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}. \tag{9.70}$$

Now, we already showed above that for any $x \in M$, the matrix of the $\xi^{*}\cdot\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}$ is an element of $\mathrm{spin}_{1,3}^{e}$ and then, $\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}$ will be an element of $\mathrm{spin}_{1,3}^{e}$ if and only if the matrix of the $\varkappa_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}$ is an element of $\mathrm{spin}_{1,3}^{e}$. The corollary is proved. ■

Lemma 9.13 *If* $\pounds_{\xi}\boldsymbol{g} = 0$ *and* $\pounds_{\xi}\Theta = 0$ *then we have the identity*

$$\mathbf{D}\left(\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}\right) + \xi^{*}\lrcorner\mathcal{R}_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} = 0. \tag{9.71}$$

Proof Using the definitions of the exterior covariant derivative and the Lie covariant derivative we have

$$\begin{aligned}
\mathbf{D}\left(\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}\right) &= d\left(\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}\right) + \omega_{\mathbf{b}}^{\mathbf{c}}(\theta_{\mathbf{c}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}) - \omega_{\mathbf{c}}^{\mathbf{a}}(\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{c}})\\
&= d\left\{\theta_{\mathbf{b}}\cdot\left[\pounds_{\xi}\theta^{\mathbf{a}} + (\xi^{*}\cdot\omega_{\cdot\mathbf{c}}^{\mathbf{a}\cdot})\theta^{\mathbf{c}}\right]\right\} + \left\{\theta_{\mathbf{d}}\cdot\left[\pounds_{\xi}\theta^{\mathbf{a}} + (\xi^{*}\cdot\omega_{\cdot\mathbf{c}}^{\mathbf{a}\cdot})\theta^{\mathbf{c}}\right]\right\}\omega_{\cdot\mathbf{b}}^{\mathbf{d}\cdot}\\
&\quad - \left\{\theta_{\mathbf{b}}\cdot\left[\pounds_{\xi}\theta^{\mathbf{d}} + (\xi^{*}\cdot\omega_{\cdot\mathbf{c}}^{\mathbf{d}\cdot})\theta^{\mathbf{c}}\right]\right\}\omega_{\cdot\mathbf{d}}^{\mathbf{a}\cdot},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\mathbf{D}\left(\theta_{\mathbf{b}}\cdot\mathbf{L}_{\xi}\theta^{\mathbf{a}}\right) &= \pounds_{\xi}\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} - \xi^{*}\lrcorner\left(d\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} + \omega_{\cdot\mathbf{c}}^{\mathbf{a}\cdot}\wedge\omega_{\cdot\mathbf{b}}^{\mathbf{c}\cdot}\right)\\
&\quad + d(\theta_{\mathbf{b}}\cdot\pounds_{\xi}\theta^{\mathbf{a}}) + \omega_{\cdot\mathbf{b}}^{\mathbf{c}\cdot}(\theta_{\mathbf{c}}\cdot\pounds_{\xi}\theta^{\mathbf{a}}) - (\theta_{\mathbf{b}}\cdot\pounds_{\xi}\theta^{\mathbf{c}})\omega_{\cdot\mathbf{c}}^{\mathbf{a}\cdot}.
\end{aligned} \tag{9.72}$$

If, $\pounds_{\xi}\boldsymbol{g} = 0$, then for any $x \in M$, $\theta_{\mathbf{b}}\cdot\pounds_{\xi}\theta^{\mathbf{a}} \in \mathrm{spin}_{1,3}^{e}$ and the second line of Eq. (9.72) is an infinitesimal Lorentz transformation of the $\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}$. If besides that, also $\pounds_{\xi}\Theta = 0$ then $\pounds_{\xi}\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot} = \boldsymbol{\delta}_{v}\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}$ and then the first term on the second member of Eq. (9.72) cancels the term in the second line. Then, taking into account Cartan's second structure equation the proposition is proved. ■

Proof (Proposition 9.10) We are now in conditions of proving the Proposition 9.10. In order to do that we combine the results of Lemmas 9.11 and 9.13 with the identities given by Eqs. (9.56) and (9.58). We get,

$$\begin{aligned} d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] &= \mathbf{D}[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] \\ &= \mathbf{D}(\xi^* \cdot \theta^{\mathbf{a}}) \wedge \star \mathcal{T}_{\mathbf{a}} + (\xi^* \cdot \theta^{\mathbf{a}}) \mathbf{D} \star \mathcal{T}_{\mathbf{a}} \\ &= \mathbf{L}_\xi \theta^{\mathbf{a}} \wedge \star \mathcal{T}_{\mathbf{a}} - \left(\xi^* \lrcorner \Theta^{\mathbf{a}}\right) \wedge \star \mathcal{T}_{\mathbf{a}} + (\xi^* \cdot \theta^{\mathbf{a}}) \mathbf{D} \star \mathcal{T}_{\mathbf{a}}, \end{aligned}$$

i.e., using Eq. (9.56),

$$d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] = \mathbf{L}_\xi \theta^{\mathbf{a}} \wedge \star \mathcal{T}_{\mathbf{a}} - \star \mathcal{J}_{\mathbf{b}\cdot}^{\cdot\mathbf{a}} \wedge \left(\xi^* \lrcorner \mathcal{R}_{\cdot\mathbf{a}}^{\mathbf{b}\cdot}\right). \tag{9.73}$$

Observe now that if $A \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ then, $\theta^{\mathbf{a}} \wedge (\theta_{\mathbf{a}} \cdot A) = A$. This permit us to write Eq. (9.73) as

$$d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] = -\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \star \mathcal{T}_{\mathbf{a}} \wedge \theta^{\mathbf{b}} - \star \mathcal{J}_{\mathbf{b}\cdot}^{\cdot\mathbf{a}} \wedge \left(\xi^* \lrcorner \mathcal{R}_{\cdot\mathbf{a}}^{\mathbf{b}\cdot}\right). \tag{9.74}$$

If $\pounds_\xi \boldsymbol{g} = 0$, we have by the Corollary of Lemma 9.11 that for any $x \in M$, $\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}} \in \mathrm{spin}_{1,3}^e$. In that case, we can write Eq. (9.74) as

$$\begin{aligned} d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] &= -\frac{1}{2}\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge [\star \mathcal{T}_{\mathbf{a}} \wedge \theta^{\mathbf{b}} - \star \mathcal{T}^{\mathbf{b}} \wedge \theta_{\mathbf{a}}] - \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}} \wedge \left(\xi^* \lrcorner \mathcal{R}_{\cdot\mathbf{a}}^{\mathbf{b}\cdot}\right) \\ &= -\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \mathbf{D} \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}} - \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}} \wedge \left(\xi^* \lrcorner \mathcal{R}_{\cdot\mathbf{a}}^{\mathbf{b}\cdot}\right). \end{aligned} \tag{9.75}$$

On the other hand, if $\pounds_\xi \Theta = 0$, in view of Proposition 9.13 we can write

$$\begin{aligned} d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}}] &= -\mathbf{D}\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}} - \left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \mathbf{D} \star \mathcal{J}_{\cdot\mathbf{a}}^{\mathbf{b}\cdot} \\ &= -\mathbf{D}[\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}}] = -d[\left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}}]. \end{aligned} \tag{9.76}$$

Finally, if $\pounds_\xi \boldsymbol{g} = 0$ and $\pounds_\xi \Theta = 0$ we have

$$d[(\xi^* \cdot \theta^{\mathbf{a}}) \star \mathcal{T}_{\mathbf{a}} + \left(\theta_{\mathbf{b}} \cdot \mathbf{L}_\xi \theta^{\mathbf{a}}\right) \wedge \star \mathcal{J}_{\mathbf{a}\cdot}^{\cdot\mathbf{b}}] = 0, \tag{7.40bis}$$

which is the result we wanted to prove. ■

The fact that the existence of symmetries implies in the existence of closed 3-forms has been originally demonstrated by Trautman [34–37]. See also [3]

9.6 Pseudopotentials in GRT

As we already said in Chap. 4, in Einstein's gravitational theory, i.e., GRT each gravitational field is modelled by a Lorentzian spacetime $\mathfrak{M} = (M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$. The 'gravitational field' $\boldsymbol{g}$ is determined through Einstein's equation by the energy-momentum of the matter fields ϕ^A, $A = 1, 2, \ldots, m$, living in $\mathfrak{M}$. As we already know from Chap. 4, Einstein's equation (Eq. (4.276)) can be written in terms of the fields $\theta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, where $\{\theta^{\mathbf{a}}\}$ is an *orthonormal* basis of T^*M as

$$-(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\theta^{\mathbf{a}} + \boldsymbol{\partial} \wedge (\boldsymbol{\partial} \cdot \theta^{\mathbf{a}}) + \boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \theta^{\mathbf{a}}) + \frac{1}{2}\mathcal{T}\theta^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}, \tag{9.77}$$

where $\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{e_{\mathbf{a}}}$, $T^a = -\mathcal{T}^{\mathbf{a}} = -T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$ are the energy-momentum 1-form fields and $\mathcal{T} = -T^{\mathbf{a}}_{\mathbf{a}}$. An explicit Lagrangian density giving that equation, which differs from the original Einstein-Hilbert Lagrangian $\mathcal{L}_{EH}$ by an exact differential is (See Exercise 9.17).

$$\mathcal{L}_g = -\frac{1}{2}d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + \frac{1}{2}\delta\theta^{\mathbf{a}} \wedge \star\delta\theta_{\mathbf{a}} + \frac{1}{4}\left(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}\right) \wedge \star\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right), \tag{9.78}$$

with

$$\mathcal{L}_{EH} : -\frac{1}{2} \star R = -d(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) + \mathcal{L}_g \tag{9.79}$$

The total Lagrangian density of the gravitational field and the matter fields can then be written as

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m, \tag{9.80}$$

where $\mathcal{L}_m(\phi^A, d\phi^A) = L_m(\phi^A, d\phi^A)\tau_g = L_m(\phi^A, d\phi^A)\tau_{\mathbf{g}}$ is the matter Lagrangian. So, it depends on the $\theta^{\mathbf{a}}$ but does not depends on the $d\theta^{\mathbf{a}}$.

Now, variation of $\mathcal{L}$ with respect to the fields $\theta^{\mathbf{a}}$ yields

$$\begin{aligned}
\boldsymbol{\delta}\textstyle\int\mathcal{L} &= \boldsymbol{\delta}\mathcal{A}_g + \boldsymbol{\delta}\mathcal{A}_m = \textstyle\int\boldsymbol{\delta}\mathcal{L}_g + \int\boldsymbol{\delta}\mathcal{L}_m \\
&= \textstyle\int\boldsymbol{\delta}\theta_{\mathbf{a}} \wedge \dfrac{\boldsymbol{\delta}\mathcal{A}_g}{\boldsymbol{\delta}\theta_{\mathbf{a}}} + \int\boldsymbol{\delta}\theta_a \wedge \dfrac{\boldsymbol{\delta}\mathcal{A}_m}{\boldsymbol{\delta}\theta_{\mathbf{a}}} \\
&= \textstyle\int\boldsymbol{\delta}\theta_{\mathbf{a}} \wedge \left(\dfrac{\partial\mathcal{L}_g}{\partial\theta_a} + d\left(\dfrac{\partial\mathcal{L}_g}{\partial d\theta_a}\right)\right) + \int\boldsymbol{\delta}\theta_{\mathbf{a}} \wedge \dfrac{\partial\mathcal{L}_m}{\partial\theta_{\mathbf{a}}}.
\end{aligned} \tag{9.81}$$

We define

$$\star\, t^{\mathbf{a}} := \frac{\partial\mathcal{L}_g}{\partial\theta_{\mathbf{a}}}, \quad \star\,\mathcal{S}^{\mathbf{c}} := \frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{a}}}, \quad \star\, T^{\mathbf{a}} := -\frac{\partial\mathcal{L}_m}{\partial\theta_{\mathbf{a}}} \tag{9.82}$$

where the $T^{\mathbf{a}} = -\star^{-1}\frac{\partial\mathcal{L}_m}{\partial\theta_{\mathbf{a}}}$ are the matter energy momentum 1-form fields. We will show in a while that

$$\textstyle\int\boldsymbol{\delta}\mathcal{L}_g = -\int\boldsymbol{\delta}\theta_{\mathbf{a}} \wedge \star\mathcal{G}^{\mathbf{a}}, \tag{9.83}$$

where $\mathcal{G}^{\mathbf{a}} = (\mathcal{R}^{\mathbf{a}} - \frac{1}{2}R\theta^{\mathbf{a}}) \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$ are the Einstein 1-form fields, with the $\mathcal{R}^{\mathbf{a}} = R^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$ the Ricci 1-form fields and R the scalar curvature.

Moreover, a calculation (see Exercise 9.18) gives for $\star t^{\mathbf{c}} \in \sec \bigwedge^3 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$ and $\star\mathcal{S}^{\mathbf{c}} \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$:

$$\begin{aligned} \star t^{\mathbf{c}} &:= \frac{\partial \mathcal{L}_g}{\partial \theta_c} = -\frac{1}{2}\omega_{\mathbf{ab}} \wedge [\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{d}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) - \omega^{\mathbf{b}\cdot}_{\cdot\mathbf{d}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \wedge \theta^{\mathbf{c}})], \\ \star\mathcal{S}^{\mathbf{c}} &:= \frac{\partial \mathcal{L}_g}{\partial d\theta_c} = \frac{1}{2}\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}). \end{aligned} \tag{9.84}$$

So, we have with

$$\star\, \mathcal{G}^{\mathbf{a}} := -\left(\frac{\partial \mathcal{L}_g}{\partial \theta_a} + d\left(\frac{\partial \mathcal{L}_g}{\partial d\theta_a}\right)\right) = -\star t^{\mathbf{a}} - d \star \mathcal{S}^{\mathbf{a}} = -\star T^{\mathbf{a}} = \star\mathcal{T}^{\mathbf{a}} \tag{9.85}$$

We give now a proof that the second and third members of Eq. (9.85) are equal starting from the Einstein-Hilbert Lagrangian density $\mathcal{L}_{EH}$ instead of using $\mathcal{L}_g$.[7] We have

$$\mathcal{L}_{EH} := -\frac{1}{2}R\tau_{\mathbf{g}} = -\frac{1}{2} \star R. \tag{9.86}$$

Now,

$$-\star R = \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) = -\star [\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})]. \tag{9.87}$$

Indeed,

$$\begin{aligned} -\star [\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] &= -\star \frac{1}{2} R^{\cdot\cdot\mathbf{cd}}_{\mathbf{ab}\cdot\cdot}[(\theta_{\mathbf{c}} \wedge \theta_{\mathbf{d}}) \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] \\ &= -\star \frac{1}{2} R^{\cdot\cdot\mathbf{cd}}_{\mathbf{ab}\cdot\cdot}[(\theta_{\mathbf{c}} \lrcorner (\theta_{\mathbf{d}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] \\ &= -\star \frac{1}{2} R^{\cdot\cdot\mathbf{cd}}_{\mathbf{ab}\cdot\cdot}(\delta^{\mathbf{b}}_{\mathbf{c}}\delta^{\mathbf{a}}_{\mathbf{d}} - \delta^{\mathbf{a}}_{\mathbf{c}}\delta^{\mathbf{b}}_{\mathbf{d}}) = -\star R^{\cdot\cdot\mathbf{cd}}_{\mathbf{dc}\cdot\cdot} = -\star R. \end{aligned} \tag{9.88}$$

Moreover

$$\delta\mathcal{L}_{EH} = \frac{1}{2}\delta\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \delta[\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] \tag{9.89}$$

[7] For a derivation using $\mathcal{L}_g$ see Exercise 9.18

and taking into account Cartan's second structure equation $\mathcal{R}_{\mathbf{ab}} = d\omega_{\mathbf{ab}} + \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{d}}$, we have

$$\begin{aligned}\boldsymbol{\delta}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) &= d\boldsymbol{\delta}\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + [\boldsymbol{\delta}\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot d}] \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \\ &\quad + [\omega_{\mathbf{ac}} \wedge \boldsymbol{\delta}\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{d}}] \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}). \end{aligned} \tag{9.90}$$

Now, taking into account Eq. (4.113) we have

$$\begin{aligned} d\boldsymbol{\delta}\omega_{\mathbf{ab}} \wedge d \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] &= d\boldsymbol{\delta}[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] + \boldsymbol{\delta}\omega_{\mathbf{ac}} \wedge d[\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] \\ &= d\boldsymbol{\delta}[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] - \boldsymbol{\delta}\omega_{\mathbf{ac}} \wedge \{[\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}})] \\ &\quad + \omega^{\mathbf{b}\cdot}_{\cdot\mathbf{c}} \wedge (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{c}})\} \end{aligned}$$

and thus

$$\boldsymbol{\delta}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) = d\boldsymbol{\delta}[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})].$$

Also, taking into account that

$$\boldsymbol{\delta}[\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] = \boldsymbol{\delta}\theta_{\mathbf{c}} \wedge [\theta^{\mathbf{c}} \lrcorner \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})]$$

it is

$$\mathcal{R}_{\mathbf{ab}} \wedge \boldsymbol{\delta}[\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] = \boldsymbol{\delta}\theta_{\mathbf{c}} \wedge \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \tag{9.91}$$

and so

$$\boldsymbol{\delta}\mathcal{L}_{EH} = \frac{1}{2}d[\boldsymbol{\delta}\omega_{\mathbf{ab}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] + \frac{1}{2}\boldsymbol{\delta}\theta_{\mathbf{c}} \wedge \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}). \tag{9.92}$$

Then under the usual assumption that variations vanish at the boundary we get

$$\boldsymbol{\delta}\int \mathcal{L}_{EH} = \frac{1}{2}\int \boldsymbol{\delta}\theta_{\mathbf{c}} \wedge \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}). \tag{9.93}$$

Now,

$$\begin{aligned} \frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) &= -\frac{1}{2} \star [\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= -\frac{1}{4} R_{\mathbf{abck}} \star [(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{k}}) \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= - \star (\mathcal{R}^{\mathbf{d}} - \frac{1}{2}R\theta^{\mathbf{d}}), \end{aligned} \tag{9.94}$$

i.e.,

$$\star\,\mathcal{G}^{\mathbf{d}} = -\frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}). \tag{9.95}$$

and we get

$$\delta \int \mathcal{L}_{EH} = -\int \delta\theta_{\mathbf{a}} \wedge \star\mathcal{G}^{\mathbf{a}} = -\int \delta\theta^{\mathbf{a}} \wedge \star\mathcal{G}_{\mathbf{a}}. \tag{9.96}$$

Next we use in Eq. (9.95) Cartan's second structure equation. We get

$$\begin{aligned}
2\star\mathcal{G}^{\mathbf{d}} &= -d\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) - \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= -d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] + \omega_{\mathbf{ab}} \wedge d\star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&\quad - \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= -d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] + \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{p}} \wedge \star(\theta^{\mathbf{p}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&\quad + \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{b}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}}) + \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{d}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}})] \\
&\quad - \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\
&= \omega_{\mathbf{ab}} \wedge [\omega^{\mathbf{d}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}}) - \omega^{\mathbf{b}}_{\mathbf{p}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}})] \\
&\quad - d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})].
\end{aligned} \tag{9.97}$$

Then, taking into account Eq. (9.84) and that from Eq. (9.79) it is $\delta\int\mathcal{L}_{EH} = \delta\int\mathcal{L}_g$, the dual of Einstein 1-form fields can be written as:

$$\star\,\mathcal{G}^{\mathbf{d}} = -\star t^{\mathbf{d}} - d\star S^{\mathbf{d}}. \tag{9.98}$$

9.6.1 Pseudopotentials Are Not Uniquely Defined

Now, we can write Einstein's equation in a very interesting, but eventually *dangerous* form, i.e.:

$$-\,d\star\mathcal{S}^{\mathbf{a}} = \star\mathcal{T}^{\mathbf{a}} + \star t^{\mathbf{a}}. \tag{9.99}$$

In writing Einstein's equations in that way, we have associated to the gravitational field a set of 2-form fields $\star\mathcal{S}^{\mathbf{a}}$ called *pseudopotentials* that have as sources the currents $(\star\mathcal{T}^{\mathbf{a}} + \star t^{\mathbf{a}})$. However, pseudopotentials are not uniquely defined since, e.g., pseudopotentials $(\star\mathcal{S}^{\mathbf{a}} + \star\alpha^{\mathbf{a}})$, with $\star\alpha^{\mathbf{a}}$ closed, i.e., $d\star\alpha^{\mathbf{a}} = 0$ give the same second member for Eq. (9.99).

9.7 Is There Any Energy-Momentum Conservation Law in GRT?

Why did we say that Eq. (9.99) is a dangerous one?

The reason is that we may be led to think that we have discovered a conservation law for the energy momentum of matter plus gravitational field independent of the existence of appropriated Killing vector fields,[8] since from Eq. (9.99) it follows that

$$d(\star\mathcal{T}^{\mathbf{a}} + \star t^{\mathbf{a}}) = 0. \tag{9.100}$$

This thought however is only an example of wishful thinking, because in Einstein theory the $\star t^{\mathbf{a}}$ depends on the connection (see Eq. (9.84)) and thus *gauge dependent*. They do not have the same tensor transformation law as the $\star\mathcal{T}^{\mathbf{a}}(= - \star T^{\mathbf{a}})$, i.e., there is no tensor field associated to the $t^{\mathbf{a}}$. So, Stokes theorem cannot be used to derive from Eq. (9.100) conserved quantities that are independent of the gauge, which is clear. However, and this is less known, for this specific problem, Stokes theorem, also cannot be used to derive conclusions that are independent of the local coordinate *chart* used to perform calculations [4]. In fact, the currents $\star t^{\mathbf{a}}$ are nothing more than an old pseudo energy momentum tensor in a new dress. Non recognition of this fact can lead to many misunderstandings. We present some of them in what follows, in order to call our readers' attention of potential errors of inference that can be done when we use sophisticated mathematical formalisms without a perfect domain of their contents.[9]

(i) First, it is easy to see that from Eq. (9.85) it follows that [19]

$$\mathbf{D}\star\mathfrak{G} = \mathbf{D}\star\mathfrak{T} = 0, \tag{9.101}$$

where $\star\mathfrak{G} = \boldsymbol{e}_{\mathbf{a}} \otimes \star\mathcal{G}^{\mathbf{a}} \in \sec TM \otimes \sec \bigwedge^3 T^*M$ and $\star\mathfrak{T} = \boldsymbol{e}_{\mathbf{a}} \otimes \star\mathcal{T}^{\mathbf{a}} \in \sec TM \otimes \sec \bigwedge^3 T^*M$ and where

$$\mathbf{D}\star\mathfrak{G} := \boldsymbol{e}_{\mathbf{a}} \otimes \mathbf{D}\star\mathcal{G}^{\mathbf{a}}, \mathbf{D}\star\mathfrak{T} = -\boldsymbol{e}_{\mathbf{a}} \otimes \mathbf{D}\star\mathcal{T}^{\mathbf{a}} \tag{9.102}$$

and $\mathbf{D}$ is the exterior covariant derivative of index valued forms (Definition 4.89). Now, in [19] it is written (without proof) a 'Stokes like theorem'

$$\boxed{\int_{\text{4-cube}} \mathbf{D}\star\mathfrak{T} = \int_{\substack{\text{3 boundary}\\ \text{of this 4-cube}}} \star\mathfrak{T}.} \tag{9.103}$$

[8]Recall that from the previous section we learned that energy-momentum conservation law for the matter fields alone exist only when appropriated Killing vector fields exist.

[9]More details on this issue may be found in [22].

We searched in the literature for a proof of Eq. (9.103) which appears also in many other texts and scientific papers as, e.g., in [6, 38] but could, of course, find none, which may be considered as valid.[10] The reason is simply. If expressed in details, e.g., the first member of Eq. (9.103) reads

$$\int_{\text{4-cube}} \boldsymbol{e}_{\mathbf{a}} \otimes (d \star \mathcal{T}^{\mathbf{a}} + \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \star \mathcal{T}^{\mathbf{b}}), \tag{9.104}$$

and it is necessary to explain what is the meaning (if any) of the integral. Since the integrand is a sum of tensor fields, this integral says that we are *adding* tensors belonging to the tensor spaces of different spacetime points. As, well known, this cannot be done in general, unless there is a way for identification of the tensor spaces at different spacetime points. This requires, of course, the introduction of additional structure on the spacetime representing a given gravitational field, and such extra structure is lacking in Einstein theory. We unfortunately, must conclude that Eq. (9.103) do not express any conservation law, for it lacks a precise mathematical meaning.

In Einstein theory possible pseudopotentials are, of course, the $\star \mathcal{S}^{\mathbf{a}}$ that we identified above (Eq. (9.84)), with

$$\star\, \mathcal{S}_{\mathbf{c}} = [\frac{1}{2}\omega_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta_{\mathbf{c}})]\theta^{\mathbf{5}}. \tag{9.105}$$

Then, if we integrate Eq. (9.99) over a 'certain finite 3-dimensional volume', say a ball B, and use Stokes theorem we have[11]

$$P^{\mathbf{a}} := -\frac{1}{8\pi}\int_B \star\,(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = \frac{1}{8\pi}\int_{\partial B} \star\, \mathcal{S}^{\mathbf{a}}. \tag{9.106}$$

In particular the energy or (*inertial mass*) of the gravitational field plus matter generating the field is defined by[12]

$$P^{\mathbf{0}} = E = m_{\mathbf{I}} = \frac{1}{8\pi} \lim_{R\to\infty} \int_{\partial B} \star\, \mathcal{S}^{\mathbf{0}}. \tag{9.107}$$

(ii) Now, a frequent misunderstanding is the following. Suppose that in a *given* gravitational theory there exists an energy-momentum conservation law for

[10] In particular, on this issue the reader should read page 108 of Parrot's book [23].

[11] The reason for the factor 8π in Eq. (9.106) is that we choose units where the numerical value gravitational constant $8\pi G/c^4$ is 1, where G is Newton gravitational constant.

[12] See the details of the calculation, e.g., in [22]

matter plus the gravitational field expressed in the form of Eq. (9.100), where $\mathcal{T}^{\mathbf{a}}$ are the energy-momentum 1-forms of matter and $t^{\mathbf{a}}$ are *true*[13] energy-momentum 1-forms of the gravitational field. This means that the 3-forms $(\star T^{\mathbf{a}} + \star t^{\mathbf{a}})$ are closed, i.e., they satisfy Eq. (9.100). Is this enough to warrant that the energy of a closed universe is zero? Well, that would be the case if starting from Eq. (9.100) we could jump to an equation like Eq. (9.99) and then to Eq. (9.107) (as done, e.g., in [33]). But that sequence of inferences in general cannot be done, for indeed, as it is well known, it is not the case that closed three forms are always exact. Take a closed universe with topology, say $\mathbb{R} \times S^3$. In this case $B = S^3$ and we have $\partial B = \partial S^3 = \varnothing$. Now, as it is well known (see, e.g., [21]), the third de Rham cohomology group of $\mathbb{R} \times S^3$ is $H^3\left(\mathbb{R} \times S^3\right) = H^3\left(S^3\right) = \mathbb{R}$. Since this group is non trivial it follows that in such manifold closed forms are not exact. Then from Eq. (9.100) it did not follow the validity of an equation analogous to Eq. (9.99). So, in that case an equation like Eq. (9.106) cannot even be written.

Despite that commentary, keep in mind that in Einstein's theory the 'energy' of a closed universe[14] supposed to be given by Eq. (9.107) is indeed zero, since in that theory the 3-forms $(\star\mathcal{T}^{\mathbf{a}} + \star t^{\mathbf{a}})$ are indeed exact (see Eq. (9.99)). This means that accepting $t^{\mathbf{a}}$ as the energy-momentum 1-form fields of the gravitational field, it follows that gravitational energy must be *negative* in a closed universe.

(iii) But, is the above formalism a consistent one? Given a coordinate chart $\{x^\mu\}$ of the maximal atlas of M, with some algebra (left as exercise to the reader) one can show that for a gravitational model represented by a diagonal asymptotic flat metric,[15] the inertial mass $E = m_{\mathbf{I}}$ is given by

$$m_{\mathbf{I}} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\partial B} \frac{x_i}{r} \frac{\partial}{\partial x^j}(g_{11}g_{22}g_{33}g^{ij}) r^2 d\Omega, \tag{9.108}$$

where $\partial B = S^2(r)$ is a 2-sphere of radius r, $g_{ij}x^j = x_i$ and $d\Omega$ is the element of solid angle. If we apply Eq. (9.108) to calculate, e.g., the energy of the Schwarzschild space time[16] generate by a gravitational mass m, we expect to have one unique and unambiguous result, namely $m_{\mathbf{I}} = m$.

[13] This means that the $t^{\mathbf{a}}$ are no in this case pseudo 1-forms, as in Einstein's theory.

[14] Note that if we suppose that the universe contains spinor fields, then it must be a spin manifold, i.e., it is parallelizable according to Geroch's theorem [12, 13], as we already know from Chap. 5.

[15] A metric is said to be asymptotically flat in given coordinates, if $g_{\mu\nu} = n_{\mu\nu}(1 + \mathrm{O}\left(r^{-k}\right))$, with $k = 2$ or $k = 1$ depending on the author. See, e.g., [30, 31, 39].

[16] For a Schwarzschild spacetime we have $g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)$.

However, as showed in details, e.g., in [4] the calculation of E depends on the spatial coordinate system naturally adapted to the reference frame $\boldsymbol{Z} = \frac{1}{\sqrt{\left(1-\frac{2m}{r}\right)}}\frac{\partial}{\partial t}$, even if these coordinates produce asymptotically flat metrics. Then, even if in one given chart we may obtain $m_{\mathbf{I}} = m$ there are others where $m_{\mathbf{I}} \neq m$![17]

Moreover, note also that, as showed above, for a closed universe, Einstein's theory implies on general grounds (once we accept that the $t^{\mathbf{a}}$ describes the energy-momentum distribution of the gravitational field) that $m_{\mathbf{I}} = 0$. This result, it is important to quote, does not contradict the so called "positive mass theorems" of, e.g., references [30, 31, 42], because that theorems refers to the total energy of an isolated system. A system of that kind is supposed to be modelled by a Lorentzian spacetime having a spacelike, asymptotically Euclidean hypersurface.[18] However, we want to emphasize here, that although the energy results positive, its value is not unique, since depends on the asymptotically flat coordinates chosen to perform the calculations, as it is clear from the elementary example of the Schwarzschild field commented above and detailed in [4].

In a book written in 1970, Davis [7] said:

> Today, some 50 years after the development of Einstein's generally covariant field theory it appears that no general agreement regarding the proper formulation of the conservation laws has been reached.

Well, we hope that the reader has been convinced that the fact is: there are *in general* no conservation laws of energy-momentum in GRT. Moreover, all discourses (based on Einstein's equivalence principle)[19] concerning the use of pseudo-energy momentum tensors as *reasonable* descriptions of energy and momentum of gravitational fields in Einstein's theory are not convincing.

And, at this point it is better to quote page 98 of Sachs and Wu [29]:

> As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage.

In GRT, we already said, every gravitational field is modelled (module diffeomorphisms, according to present wisdom) by a Lorentzian spacetime. In that particular case, when this spacetime structure admits a *timelike* Killing vector field, we may formulate a law of energy conservation for the matter fields. Also, if the Lorentzian spacetime admits three linearly independent *spacelike* Killing vectors, we have a law of conservation of momentum for the matter fields.

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[17]This observation is true even if we use the so called ADM formalism [2] to be presented in Chap. 11. To be more precise, let us recall that we have a well defined ADM energy only if the fall off rate of the metric is in the interval $1/2 < k < 1$. For details, see [20].

[18]The proof also uses as hypothesis the so called energy dominance condition [14].

[19]Like, e.g., in [1, 19, 24] and many other textbooks. It is worth to quote here that, at least, Anderson [1] explicitly said: "In an interaction that involves the gravitational field a system can loose energy without this energy being transmitted to the gravitational field."

This follows at once from the theory developed in the previous section. Indeed, in the *particular* case of GRT, the Lagrangian density is not supposed to be explicitly dependent on the $\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$. Then, the term $\frac{\partial \mathcal{L}_m}{\partial \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}} = 0$ in Eq. (9.59) is null. Then writing $\mathcal{T}(\xi) = \xi^\mu \mathcal{T}_\mu$ Eq. (9.59) becomes $d \star \mathcal{T}(\xi) = 0$ or and Eq. (9.59) becomes writing $\mathcal{T}(\xi) = \xi^\mu \mathcal{T}_\mu$,

$$\delta \mathcal{T}(\xi) = 0. \tag{9.109}$$

The crucial fact to have in mind here is that a general Lorentzian spacetime, does *not* admits such Killing vector fields in general as it is the case, e.g., of the popular Friedmann-Robertson-Walker expanding universes models.

At present, the authors know only one possibility of resurrecting a *trustworthy* conservation law for the energy-momentum of matter plus the gravitational field valid in all circumstances in a theory of the gravitational field that *resembles* GRT (in the sense of keeping Einstein's equation).[20] It consists in reinterpreting that theory as a field theory in flat Minkowski spacetime. Theories using Minkowski spacetime have been proposed in the past by, e.g., Feynman [10], Schwinger [32], Thirring [33] and Weinberg [40, 41] among others and have been extensively studied by Logunov and collaborators in a series of papers summarized in the monographs [16, 17]. In the Chap. 11 we discuss the nature of the gravitational field and give Clifford bundle approach to the theory of the gravitational field in Minkowski spacetime,[21] following [26]. We also qualify a statement in [8] that in the theory called teleparallel equivalent of GRT [18] there is an energy-momentum conservation law.

Remark 9.14 As a final remark, we make the important observation that even if a given Lorentziam spacetime modelling a gravitational field has one timelike and three spacelike Killing vector fields and so we can defined four conserved quantities $P^{\mathbf{a}}$ (see Eq. (9.106)) we cannot in general define an energy-momentum covector $\boldsymbol{P}$ (not a covector field) *for the system as in special relativistic field theories. For a thoughtful discussion of this issue see [27].*

[20] On this issue, see also [25].

[21] Another presentation the theory of the gravitational field in Minkowski spacetime employing Clifford algebra techniques has been given in [15]. However, that work, which contains many interesting ideas, unfortunately contains also some equivocated statements that make (in our opinion) the theory, as originally presented by those authors invalid. This has been discussed with details in [9].

9.8 Is There Any Angular Momentum Conservation Law in the GRT

If the $\{\theta^{\mathbf{a}}\}$ and the $\{\omega^{\mathbf{a}}_{\mathbf{b}}\}$ are varied independently in the sum of the Einstein-Hilbert Lagrangian plus the matter Lagrangian then, as it is easy to verify we get the additional field equation

$$\mathbf{D}\star\theta^{\mathbf{ab}} = \boldsymbol{J}^{\mathbf{ab}} = -\star 2\mathcal{J}^{\mathbf{ab}} \tag{9.110}$$

From this equation we get immediately

$$d\star\theta^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \boldsymbol{J}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} - \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \star\theta^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} + \star\theta^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \tag{9.111}$$

and one is tempted to define $\mathbf{S}^{\mathbf{ab}} = (-\omega^{\mathbf{cb}} \wedge \star\theta^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} + \star\theta^{\mathbf{cb}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}})$ as 2-form densities of spin angular momentum of the gravitational field and to the define the orbital angular momentum of the system as

$$L^{\mathbf{ab}} := \int_{S^2} \star\,\theta^{\mathbf{ab}}. \tag{9.112}$$

This definition, of course, has the same problems as the definition of energy in the GRT because the 2-form fields $\mathbf{S}^{\mathbf{ab}}$ are gauge dependent. Moreover, the scalars $L^{\mathbf{ab}}$ cannot be considered as components of any tensor field in the spacetime manifold.

9.9 Some Non Trivial Exercises

Exercise 9.15

(a) Show that the energy-momentum densities $\star T_{\mathbf{a}}$ of the Maxwell field are given by

$$\star\, T_{\mathbf{a}} = \star\frac{1}{2}F\theta_{\mathbf{a}}\tilde{F}. \tag{9.113}$$

(b) Show also that $T_{\mathbf{a}} \cdot \theta_{\mathbf{b}} = T_{\mathbf{b}} \cdot \theta_{\mathbf{a}}$.

Solution:

(a) The Maxwell Lagrangian, here considered as the matter field coupled to the background gravitational field must be taken (due to our convention in the writing of Einstein equations and the definition of $\star T_{\mathbf{a}}$) as

$$\mathcal{L}_m = -\frac{1}{2}F \wedge \star F, \tag{9.114}$$

where $F = \frac{1}{2}F_{\mathbf{ab}}\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} = \frac{1}{2}F_{\mathbf{ab}}\theta^{\mathbf{ab}} \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is the electromagnetic field. Now, recall that

$$\boldsymbol{\delta} \star \theta^{\mathbf{ab}} = \boldsymbol{\delta}\theta^{\mathbf{c}} \wedge [\theta_{\mathbf{c}} \lrcorner \star \theta^{\mathbf{ab}}]$$

and that in general $\boldsymbol{\delta}$ and $\star$ do not commute. Indeed, for any $A_p \in \sec \bigwedge^p TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned}[\boldsymbol{\delta}, \star]A_p &= \boldsymbol{\delta} \star A_p - \star\boldsymbol{\delta}A_p \\ &= \boldsymbol{\delta}\theta^{\mathbf{a}} \wedge \left(\theta_{\mathbf{a}} \lrcorner \star A_p\right) - \star\left[\boldsymbol{\delta}\theta^{\mathbf{a}} \wedge \left(\theta_{\mathbf{a}} \lrcorner A_p\right)\right].\end{aligned} \tag{9.115}$$

Multiplying both members of Eq. (9.115) with $A_p = F$ on the right by $F\wedge$ we get

$$F \wedge \boldsymbol{\delta} \star F = F \wedge \star\boldsymbol{\delta}F + F \wedge \{\boldsymbol{\delta}\theta^{\mathbf{a}} \wedge (\theta_{\mathbf{a}} \lrcorner \star F) - \star[\boldsymbol{\delta}\theta^{\mathbf{a}} \wedge (\theta_{\mathbf{a}} \lrcorner F)]\}.$$

Next we sum $\boldsymbol{\delta}F \wedge \star F$ to both members of the above equation obtaining

$$\boldsymbol{\delta}\,(F \wedge \star F) = 2\boldsymbol{\delta}F \wedge \star F + \boldsymbol{\delta}\theta^{\mathbf{a}} \wedge [F \wedge (\theta_{\mathbf{a}} \lrcorner \star F) - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F].$$

or,

$$\boldsymbol{\delta}\left(-\frac{1}{2}F \wedge \star F\right) = -\boldsymbol{\delta}F \wedge \star F - \frac{1}{2}\boldsymbol{\delta}\theta^{\mathbf{a}} \wedge [F \wedge (\theta_{\mathbf{a}} \lrcorner \star F) - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F].$$

It then follows from Eq. (9.33) that if $\boldsymbol{\delta}\theta^{\mathbf{a}} = -\pounds_{\xi}\theta^{\mathbf{a}}$ for some diffeomorphism generated by the vector field ξ that

$$\star T_{\mathbf{a}} = -\frac{\partial \mathcal{L}_m}{\partial \theta^{\mathbf{a}}} = \frac{1}{2}\left[F \wedge (\theta_{\mathbf{a}} \lrcorner \star F) - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F\right].$$

Now,

$$(\theta_{\mathbf{a}} \lrcorner F) \wedge \star F = -\star[(\theta_{\mathbf{a}} \lrcorner F) \lrcorner F] = -[(\theta_{\mathbf{a}} \lrcorner F) \lrcorner F]\tau_g$$

and also using Eq. (2.60) we can write

$$(\theta_{\mathbf{a}} \lrcorner F) \wedge \star F = \theta_{\mathbf{a}}(F \cdot F)\tau_g - F \wedge (\theta_{\mathbf{a}} \lrcorner \star F).$$

Using these results, we have

$$\begin{aligned}&\frac{1}{2}\left[F \wedge (\theta_{\mathbf{a}} \lrcorner \star F) - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F\right] \\ &\quad = \frac{1}{2}\left\{\theta_{\mathbf{a}}(F \cdot F)\tau_g - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F - (\theta_{\mathbf{a}} \lrcorner F) \wedge \star F\right\}\end{aligned}$$

$$= \frac{1}{2}\left\{\theta_{\mathbf{a}}(F\cdot F)\tau_g - 2(\theta_{\mathbf{a}}\lrcorner F)\wedge \star F\right\}$$
$$= \frac{1}{2}\left\{\theta_{\mathbf{a}}(F\cdot F)\tau_g + 2[(\theta_{\mathbf{a}}\lrcorner F)\lrcorner F]\tau_g\right\}$$
$$= \star\left(\frac{1}{2}\theta_{\mathbf{a}}(F\cdot F) + (\theta_{\mathbf{a}}\lrcorner F)\lrcorner F\right) = \frac{1}{2}\star(F\theta_{\mathbf{a}}\tilde{F}),$$

where in writing the last line we used the identity given by Eq. (8.81).

(b) To prove that $T_{\mathbf{a}}\cdot\theta_{\mathbf{b}} = T_{\mathbf{b}}\cdot\theta_{\mathbf{a}}$ we write:

$$T_{\mathbf{a}}\cdot\theta_{\mathbf{b}} = -\frac{1}{2}\langle F\theta_{\mathbf{a}}F\theta_{\mathbf{b}}\rangle_0 = -\langle(F\llcorner\theta_{\mathbf{a}})F\theta_{\mathbf{b}}\rangle_0 + \frac{1}{2}\langle(\theta_{\mathbf{a}}\lrcorner F + \theta_{\mathbf{a}}\wedge F)\,F\theta_{\mathbf{b}}\rangle_0$$
$$= -\langle(F\llcorner\theta_{\mathbf{a}})F\theta_{\mathbf{b}}\rangle_0 - \frac{1}{2}\langle(\theta_{\mathbf{a}}FF\theta_{\mathbf{b}}\rangle_0 = -\langle(F\llcorner\theta_{\mathbf{a}})(F\llcorner\theta_{\mathbf{b}}) + (F\llcorner\theta_{\mathbf{a}})(F\wedge\theta_{\mathbf{b}})\rangle_0$$
$$+ \frac{1}{2}\langle\theta_{\mathbf{a}}(F\cdot F)\theta^{\mathbf{b}}\rangle_0 - \frac{1}{2}\langle\,\theta_{\mathbf{a}}(F\wedge F)\,\theta_{\mathbf{b}}\rangle_0$$
$$= -\langle(F\llcorner\theta_{\mathbf{a}})(F\llcorner\theta_{\mathbf{b}})\rangle_0 + \frac{1}{2}\langle(F\cdot F)(\theta_{\mathbf{a}}\cdot\theta_{\mathbf{b}})\rangle_0$$
$$= -(F\llcorner\theta_{\mathbf{b}})\cdot(F\llcorner\theta_{\mathbf{a}}) + \frac{1}{2}(F\cdot F)(\theta_{\mathbf{b}}\cdot\theta_{\mathbf{a}}) = T_{\mathbf{b}}\cdot\theta_{\mathbf{a}}.$$

Note moreover that

$$T_{\mathbf{ab}} = T_{\mathbf{a}}\cdot\theta_{\mathbf{b}} = -\eta^{\mathbf{cl}}F_{\mathbf{ac}}F_{\mathbf{bl}} + \frac{1}{4}F_{\mathbf{cd}}F^{\mathbf{cd}}\eta_{\mathbf{ab}}, \tag{9.116}$$

a well known result.

Exercise 9.16 Show that the connection 1-forms can be written as

$$\omega_{\mathbf{ab}} = -\frac{1}{2}[\theta_{\mathbf{a}}\lrcorner d\theta_{\mathbf{b}} - \theta_{\mathbf{b}}\lrcorner d\theta_{\mathbf{a}} - (\theta_{\mathbf{a}}\lrcorner(\theta_{\mathbf{b}}\lrcorner d\theta_{\mathbf{c}})\theta^{\mathbf{c}}] \tag{9.117}$$

or

$$\omega_{\mathbf{ab}} = \theta_{\mathbf{b}}\lrcorner d\theta_{\mathbf{a}} - \theta_{\mathbf{a}}\lrcorner d\theta_{\mathbf{b}} + \frac{1}{2}(\theta_{\mathbf{a}}\lrcorner(\theta_{\mathbf{b}}\lrcorner(\theta^{\mathbf{c}}\wedge d\theta_{\mathbf{c}})] \tag{9.118}$$

Solution: We prove Eq. (9.117). From Cartan's first structure equation and Eq. (2.60) we have

$$\theta_{\mathbf{a}}\lrcorner d\theta_{\mathbf{b}} = \theta_{\mathbf{a}}\lrcorner(\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}}\wedge\theta_{\mathbf{c}}) = (\theta_{\mathbf{a}}\lrcorner\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}})\wedge\theta_{\mathbf{c}} - \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}}\wedge(\theta_{\mathbf{a}}\lrcorner\theta_{\mathbf{c}})$$
$$= (\theta_{\mathbf{a}}\lrcorner\omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}})\theta_{\mathbf{c}} - \omega_{\mathbf{ab}}.$$

Moreover recalling Eq. (2.64) we have

$$
\begin{aligned}
\theta_{\mathbf{a}} \lrcorner (\theta_{\mathbf{b}} \lrcorner d\theta_{\mathbf{c}}) = (\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \lrcorner d\theta_{\mathbf{c}} &= (\theta_{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \cdot (\omega^{\mathbf{d}\cdot}_{\cdot\mathbf{c}} \wedge \theta_{\mathbf{d}}) \\
&= \det \begin{pmatrix} \theta_{\mathbf{a}} \lrcorner \omega^{\mathbf{d}\cdot}_{\cdot\mathbf{c}} & \theta_{\mathbf{a}} \lrcorner \theta_{\mathbf{d}} \\ \theta_{\mathbf{b}} \lrcorner \omega^{\mathbf{d}\cdot}_{\cdot\mathbf{c}} & \theta_{\mathbf{b}} \lrcorner \theta_{\mathbf{d}} \end{pmatrix} \\
&= (\theta_{\mathbf{a}} \lrcorner \omega^{\mathbf{d}\cdot}_{\cdot\mathbf{c}})(\theta_{\mathbf{b}} \lrcorner \theta_{\mathbf{d}}) - (\theta_{\mathbf{a}} \lrcorner \theta_{\mathbf{d}})(\theta_{\mathbf{b}} \lrcorner \omega^{\mathbf{d}\cdot}_{\cdot\mathbf{c}}) \\
&= \theta_{\mathbf{a}} \lrcorner \omega_{\mathbf{bc}} - \theta_{\mathbf{b}} \lrcorner \omega_{\mathbf{ac}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\theta_{\mathbf{a}} \lrcorner d\theta_{\mathbf{b}} - \theta_{\mathbf{b}} \lrcorner d\theta_{\mathbf{a}} - (\theta_{\mathbf{a}} \lrcorner (\theta_{\mathbf{b}} \lrcorner d\theta_{\mathbf{c}}) \theta^{\mathbf{c}} &= (\theta_{\mathbf{a}} \lrcorner \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}}) \theta_{\mathbf{c}} - \omega_{\mathbf{ab}} - (\theta_{\mathbf{b}} \lrcorner \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{a}}) \theta_{\mathbf{c}} + \omega_{\mathbf{ba}} \\
&\quad - (\theta_{\mathbf{a}} \lrcorner \omega_{\mathbf{bc}}) \theta^{\mathbf{c}} + (\theta_{\mathbf{b}} \lrcorner \omega_{\mathbf{ac}}) \theta^{\mathbf{c}} = -2\omega_{\mathbf{ab}}.
\end{aligned}
$$

and using the above results Eq. (9.117) is proved.

Exercise 9.17 Show that the Einstein Hilbert Lagrangian $\mathcal{L}_{EH} := -\frac{1}{2} R \tau_g$ can be written as

$$
\mathcal{L}_{EH} = -d(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) - \frac{1}{2} d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + \frac{1}{2} \delta\theta^{\mathbf{a}} \wedge \star \delta\theta_{\mathbf{a}} + \frac{1}{4} (d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \star \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}} \right) \tag{9.119}
$$

Solution: We already know from Eq. (9.79) that

$$
-\frac{1}{2} R \tau_g = \frac{1}{2} \mathcal{R}_{\mathbf{ab}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}).
$$

We use now Cartan's second structure equation to write

$$
\begin{aligned}
\mathcal{L}_{EH} &= \frac{1}{2} d\omega_{\mathbf{ab}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} (\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}}) \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \\
&= \frac{1}{2} d[\omega_{\mathbf{ab}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] + \frac{1}{2} \omega_{\mathbf{ab}} \wedge d \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \\
&= \frac{1}{2} d[\omega_{\mathbf{ab}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] - \frac{1}{2} \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}).
\end{aligned}
$$

Now, we have

$$
\omega_{\mathbf{ab}} \wedge \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) = - \star [\omega_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] = 2 \star [(\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{b}}) \theta^{\mathbf{a}}]
$$

Moreover, from Cartan's first structure equation and Eq. (2.60) we have immediately that

$$
\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} = - \star [(\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{b}})] \theta^{\mathbf{a}},
$$

from where it follows that

$$\frac{1}{2}d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] = -d[\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}]. \tag{9.120}$$

Also,

$$\begin{aligned}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) &= -\star[(\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}}) \lrcorner (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}})] = -\star(\omega_{\mathbf{ab}} \lrcorner [\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \lrcorner (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}})] \\ &= -\star[(\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{c}})(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \lrcorner \theta^{\mathbf{b}}) - (\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{b}})(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \lrcorner \theta^{\mathbf{c}})].\end{aligned}$$

But,

$$\begin{aligned}(\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{c}})(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \lrcorner \theta^{\mathbf{b}}) &= \omega_{\mathbf{ab}} \lrcorner [(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \lrcorner \theta^{\mathbf{c}})\theta^{\mathbf{b}}] \\ &= \omega_{\mathbf{ab}} \lrcorner [(\theta^{\mathbf{c}} \lrcorner (\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \theta^{\mathbf{b}}) + \omega^{\mathbf{ab}}) \\ &= (\omega_{\mathbf{ab}} \wedge \theta^{\mathbf{c}}) \lrcorner (\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \theta^{\mathbf{b}}) + \omega_{\mathbf{ab}} \lrcorner \omega^{\mathbf{ab}}\end{aligned}$$

and then

$$\begin{aligned}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) &= -(\theta^{\mathbf{b}} \cdot \omega_{\mathbf{ab}})(\theta^{\mathbf{c}} \wedge \star\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}}) \\ &\quad + (\omega_{\mathbf{ab}} \wedge \theta^{\mathbf{b}}) \wedge \star(\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \theta^{\mathbf{c}}) + \omega_{\mathbf{ab}} \wedge \star\omega^{\mathbf{ab}}.\end{aligned}$$

Hence recalling that $d\theta^{\mathbf{a}} = -\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \theta^{\mathbf{c}}$ and $d \star \theta^{\mathbf{a}} = -\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star\theta^{\mathbf{c}}$ and that $\delta\theta_{\mathbf{a}} = -\star^{-1} d \star \theta_{\mathbf{a}} = -\omega_{\mathbf{ab}} \lrcorner \theta^{\mathbf{b}}$, we have

$$\begin{aligned}(\theta^{\mathbf{b}} \cdot \omega_{\mathbf{ab}})(\theta^{\mathbf{c}} \wedge \star\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}}) &= -\delta\theta_{\mathbf{a}} \wedge d \star \theta^{\mathbf{a}} \\ &= \delta\theta_{\mathbf{a}} \wedge \star\star^{-1} d \star \theta^{\mathbf{a}} = -\delta\theta_{\mathbf{a}} \wedge \star\delta\theta^{\mathbf{a}}\end{aligned}$$

and

$$\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) = -\delta\theta_{\mathbf{a}} \wedge \star\delta\theta^{\mathbf{a}} - d\theta_{\mathbf{a}} \wedge \star d\theta^{\mathbf{a}} + \omega_{\mathbf{ab}} \wedge \star\omega^{\mathbf{ab}}$$

Now, using Eq. (9.118) we can write

$$\begin{aligned}\omega_{\mathbf{ab}} \wedge \star\omega^{\mathbf{ab}} &= \omega_{\mathbf{ab}} \wedge \theta^{\mathbf{a}} \wedge \star d\theta^{\mathbf{b}} - \omega_{\mathbf{ab}} \wedge \theta^{\mathbf{b}} \wedge \star d\theta^{\mathbf{a}} \\ &\quad + \frac{1}{2}\omega_{\mathbf{ab}} \wedge \star[\theta^{\mathbf{a}} \lrcorner (\theta^{\mathbf{b}} \lrcorner (\theta_{\mathbf{c}} \wedge d\theta^{\mathbf{c}})] \\ &= d\theta_{\mathbf{b}} \wedge \star d\theta^{\mathbf{b}} + d\theta_{\mathbf{a}} \wedge \star d\theta^{\mathbf{a}} - \frac{1}{2}d\theta_{\mathbf{a}} \wedge \theta^{\mathbf{a}} \wedge \star(d\theta^{\mathbf{c}} \wedge \theta_{\mathbf{c}})\end{aligned}$$

and get

$$\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) = -\delta\theta_{\mathbf{a}} \wedge \star\delta\theta^{\mathbf{a}} - d\theta_{\mathbf{a}} \wedge \star d\theta^{\mathbf{a}}$$
$$+ 2d\theta_{\mathbf{a}} \wedge \star d\theta^{\mathbf{a}} - \frac{1}{2}d\theta_{\mathbf{a}} \wedge \theta^{\mathbf{a}} \star (d\theta^{\mathbf{c}} \wedge \theta_{\mathbf{c}}). \quad (9.121)$$

Using Eqs. (9.120) and (9.121) we finally have

$$\mathcal{L}_{EH} = -d(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) - \frac{1}{2}d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + \frac{1}{2}\delta\theta^{\mathbf{a}} \wedge \star\delta\theta_{\mathbf{a}} + \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \star\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right).$$

and Eq. (9.119) is proved.

Exercise 9.18 Find the algebraic derivatives $\frac{\partial \mathcal{L}_g}{\partial \theta^{\mathbf{d}}}$ and $\frac{\partial \mathcal{L}_g}{\partial d\theta^{\mathbf{d}}}$ of Einstein-Hilbert Lagrangian density $\mathcal{L}_{EH} := -\frac{1}{2}R\tau_{\mathbf{g}}$ which can be written as

$$\mathcal{L}_{EH} = -d(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) - \frac{1}{2}d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + \frac{1}{2}\delta\theta^{\mathbf{a}} \wedge \star\delta\theta_{\mathbf{a}}$$
$$+ \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \star\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right) = -d(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) + \mathcal{L}_g.$$

necessary to obtain Eq. (9.84).

Solution: We first show that $\mathcal{L}_g$ can be written as

$$\mathcal{L}_g = -\frac{1}{2}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \wedge \star\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{a}}\right) + \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \star\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right) \quad (9.122)$$

Indeed, using the identities in Eq. (2.77) we can write:

$$\begin{aligned}
(d\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \wedge \star(d\theta_{\mathbf{b}} \wedge \theta_{\mathbf{a}}) &= d\theta^{\mathbf{a}} \wedge [\theta^{\mathbf{b}} \wedge \star(d\theta_{\mathbf{b}} \wedge \theta_{\mathbf{a}})] \\
&= d\theta^{\mathbf{a}} \wedge \star[(\theta^{\mathbf{b}} \lrcorner d\theta_{\mathbf{b}}) \wedge \theta_{\mathbf{a}} + d\theta_{\mathbf{a}}] \\
&= d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + d\theta^{\mathbf{a}} \wedge \star[\theta^{\mathbf{b}} \lrcorner d\theta_{\mathbf{b}}) \wedge \theta_{\mathbf{a}})] \\
&= d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + [(\theta^{\mathbf{b}} \lrcorner d\theta_{\mathbf{b}}) \wedge \theta_{\mathbf{a}}] \wedge \star d\theta^{\mathbf{a}} \\
&= d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} + (\theta^{\mathbf{b}} \lrcorner d\theta_{\mathbf{b}}) \wedge (\theta_{\mathbf{a}} \wedge \star d\theta^{\mathbf{a}}) \\
&= d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} - (\theta^{\mathbf{b}} \lrcorner d\theta_{\mathbf{b}}) \wedge \star(\theta_{\mathbf{a}} \lrcorner d\theta^{\mathbf{a}}) \\
&= d\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}} - \underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} \wedge \star\underset{\mathbf{g}}{\delta}\theta_{\mathbf{a}}. \quad (9.123)
\end{aligned}$$

from where Eq. (9.122) follows.

Now, we write

$$\mathcal{L}_g = -\frac{1}{2} d\theta_{\mathbf{a}} \wedge [\theta^{\mathbf{b}} \wedge \underset{g}{\star}(d\theta_{\mathbf{b}} \wedge \theta^{\mathbf{a}}) - \frac{1}{2}\theta^{\mathbf{a}} \wedge \star \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right)]$$
$$:= \frac{1}{2} d\theta_{\mathbf{a}} \wedge \star \mathcal{S}^{\mathbf{a}}. \tag{9.124}$$

Then we have from Eq. (2.60)

$$\theta_{\mathbf{k}} \lrcorner (\mathcal{L}_g) = -\frac{1}{2}\theta_{\mathbf{k}} \lrcorner (d\theta_{\mathbf{a}} \wedge \star \mathcal{S}^{\mathbf{a}})$$
$$= -\frac{1}{2}(\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}} - \frac{1}{2} d\theta_{\mathbf{a}} \wedge (\theta_{\mathbf{k}} \lrcorner \star \mathcal{S}^{\mathbf{a}}) \tag{9.125}$$

and subtracting $(\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}}$ on both sides of the last equation we get

$$\theta_{\mathbf{k}} \lrcorner (\mathcal{L}_g) - (\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}} = \frac{1}{2}[(\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}} - \frac{1}{2} d\theta_{\mathbf{a}} \wedge (\theta_{\mathbf{k}} \lrcorner \star \mathcal{S}^{\mathbf{a}})]. \tag{9.126}$$

Now, we recall from the properties of the algebraic derivatives and of contraction operators (see Exercise 9.19) and taking also into account Eq. (9.10) that for $\mathcal{L}_g$ it holds

$$\theta_{\mathbf{k}} \lrcorner (\mathcal{L}_g) = \frac{d\mathcal{L}_g}{d\theta_{\mathbf{k}}} = \frac{\partial \mathcal{L}_g}{\partial \theta_{\mathbf{k}}} + \frac{\partial d\theta_{\mathbf{a}}}{\partial \theta_{\mathbf{k}}} \wedge \frac{\partial \mathcal{L}_g}{\partial d\theta_{\mathbf{a}}} \tag{9.127}$$

and we have immediately that

$$\frac{\partial \mathcal{L}_g}{\partial \theta_{\mathbf{k}}} = \star t_{\mathbf{k}} :, \quad \frac{\partial d\theta_{\mathbf{a}}}{\partial \theta_{\mathbf{k}}} \wedge \frac{\partial \mathcal{L}_g}{\partial d\theta_{\mathbf{a}}} = (\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \frac{\partial \mathcal{L}_g}{\partial d\theta_{\mathbf{a}}} = (\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}}. \tag{9.128}$$

Since

$$\boldsymbol{\delta} \mathcal{L}_g = \boldsymbol{\delta} \theta_{\mathbf{k}} \wedge \frac{\boldsymbol{\delta} \mathcal{L}_g}{\boldsymbol{\delta} \theta_{\mathbf{k}}} + \boldsymbol{\delta} d\theta_{\mathbf{k}} \wedge \frac{\partial \mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}$$

we finally get

$$\star\, t_{\mathbf{k}} = \theta_{\mathbf{k}} \lrcorner (\mathcal{L}_g) - (\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \frac{\partial \mathcal{L}_g}{\partial d\theta_{\mathbf{a}}} = \theta_{\mathbf{k}} \lrcorner \mathcal{L}_g - (\theta_{\mathbf{k}} \lrcorner d\theta_{\mathbf{a}}) \wedge \star \mathcal{S}^{\mathbf{a}}.$$

and we recall that according to Eq. (9.124)

$$\star\, \mathcal{S}_{\mathbf{k}} = -\theta^{\mathbf{a}} \wedge \underset{g}{\star}(d\theta_{\mathbf{a}} \wedge \theta_{\mathbf{k}}) - \frac{1}{2}\theta_{\mathbf{k}} \wedge \star\, (d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}})\,, \tag{9.129}$$

which using the identities in Exercise 2.39 may be written in a suggestive form (to be used in Chap. 11) as

$$\star\,\mathcal{S}_{\mathbf{k}} = -\star d\theta_{\mathbf{k}} - (\theta_{\mathbf{k}} \lrcorner \star \theta^{\mathbf{a}}) \wedge \star d \star \theta_{\mathbf{a}} + \frac{1}{2}\theta_{\mathbf{k}} \wedge \star\,(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}})\,. \tag{9.130}$$

Finally we recall that

$$\begin{aligned}\boldsymbol{\delta}\mathcal{L}_g &= \boldsymbol{\delta}\theta_{\mathbf{k}} \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\theta_{\mathbf{k}}} + d\left(\boldsymbol{\delta}\theta_{\mathbf{k}} \wedge \frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}\right) \\ &= \boldsymbol{\delta}\theta_{\mathbf{k}} \wedge \left[\frac{d\mathcal{L}_g}{d\theta_{\mathbf{k}}} + d\left(\frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}\right)\right] + d(\boldsymbol{\delta}\theta_{\mathbf{k}} \wedge \frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}).\end{aligned} \tag{9.131}$$

and for variations that are null on the boundary

$$\boldsymbol{\delta}\mathcal{L}_g = \int \boldsymbol{\delta}\theta_{\mathbf{k}} \wedge \left[\frac{d\mathcal{L}_g}{d\theta_{\mathbf{k}}} + d\left(\frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}\right)\right] \tag{9.132}$$

and taking into account Eq. (9.96) we get immediately

$$\frac{d\mathcal{L}_g}{d\theta_{\mathbf{k}}} + d\left(\frac{\partial\mathcal{L}_g}{\partial d\theta_{\mathbf{k}}}\right) = -\mathcal{G}_{\mathbf{k}}.$$

Exercise 9.19 Verify that

$$\frac{d}{d\theta^{\mathbf{k}}}(\theta_{\mathbf{a}} \wedge d\theta^{\mathbf{a}}) = \theta_{\mathbf{k}} \lrcorner (\theta_{\mathbf{a}} \wedge d\theta^{\mathbf{a}}) \tag{9.133}$$

Since $d\theta^{\mathbf{a}} = \frac{1}{2}c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{mn}}\theta^{\mathbf{m}} \wedge \theta^{\mathbf{n}}$ we have $\frac{\partial\theta_{\mathbf{a}}}{\partial\theta^{\mathbf{k}}} = \eta_{\mathbf{ak}}$ and $\frac{\partial d\theta^{\mathbf{a}}}{\partial\theta^{\mathbf{k}}} = c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{kn}} \wedge \theta^{\mathbf{n}} = c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{kn}}\theta^{\mathbf{n}}$ and thus

$$\begin{aligned}\frac{d}{d\theta_{\mathbf{k}}}(\theta_{\mathbf{a}} \wedge d\theta^{\mathbf{a}}) &= \frac{\partial(\theta_{\mathbf{a}} \wedge d\theta^{\mathbf{a}})}{\partial\theta^{\mathbf{k}}} + \frac{\partial d\theta^{\mathbf{a}}}{\partial\theta_{\mathbf{k}}} \wedge \frac{\partial(\theta_{\mathbf{b}} \wedge d\theta^{\mathbf{b}})}{\partial d\theta^{\mathbf{a}}} \\ &= d\theta_{\mathbf{k}} + c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{kn}}\theta^{\mathbf{n}} \wedge \theta_{\mathbf{a}}.\end{aligned}$$

On the other hand we have from[22] Eq. (2.59)

$$\begin{aligned}\theta_{\mathbf{k}} \lrcorner (\theta_{\mathbf{a}} \wedge d\theta^{\mathbf{a}}) &= (\theta_{\mathbf{k}} \lrcorner \theta_{\mathbf{a}}) \wedge d\theta^{\mathbf{a}} - \theta_{\mathbf{a}} \wedge (\theta_{\mathbf{k}} \lrcorner d\theta^{\mathbf{a}}) \\ &= d\theta_{\mathbf{k}} - \theta_{\mathbf{a}} \wedge \theta_{\mathbf{k}} \lrcorner \left(\frac{1}{2}c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{mn}}\theta^{\mathbf{m}} \wedge \theta^{\mathbf{n}}\right) = d\theta_{\mathbf{k}} + c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{kn}}\theta^{\mathbf{n}} \wedge \theta_{\mathbf{a}}.\end{aligned}$$

and Eq. (9.133) is proved.

[22] Recall that $\lrcorner$ is an antiderivation.

Exercise 9.20 Prove that Eqs. (9.129) and (9.130) are equivalent.

Solution: It is only necessary to verify that

$$\theta^{\mathbf{a}} \wedge \star(d\theta_{\mathbf{a}} \wedge \theta_{\mathbf{k}}) = \star d\theta_{\mathbf{k}} + (\theta_{\mathbf{k}} \lrcorner \star \theta^{\mathbf{a}}) \wedge \star d \star \theta_{\mathbf{a}}$$

Since $\star d \star \theta_{\mathbf{a}} = \delta\theta_{\mathbf{a}} = -\Gamma^{\mathbf{l}\cdot\cdot}_{\cdot\mathbf{la}}$ (where $D_{e_{\mathbf{a}}}\theta^{\mathbf{b}} := -\Gamma^{\mathbf{b}}_{\mathbf{ac}}\theta^{\mathbf{c}}$)we have that

$$\theta^{\mathbf{a}} \wedge \star(d\theta_{\mathbf{a}} \wedge \theta_{\mathbf{k}}) = \star d\theta_{\mathbf{k}} + \delta\theta_{\mathbf{a}}\,(\theta_{\mathbf{k}} \lrcorner \star \theta^{\mathbf{a}})$$

and

$$\delta\theta_{\mathbf{a}} \star (\theta_{\mathbf{k}} \wedge \theta^{\mathbf{a}}) = -\Gamma^{\mathbf{l}\cdot\cdot}_{\cdot\mathbf{la}} \star (\theta_{\mathbf{k}} \wedge \theta^{\mathbf{a}})$$

On the other hand

$$\begin{aligned}\theta^{\mathbf{a}} \wedge \star(d\theta_{\mathbf{a}} \wedge \theta_{\mathbf{k}}) &= \theta^{\mathbf{a}} \wedge \star(\theta_{\mathbf{k}} \wedge d\theta_{\mathbf{a}}) = \star[\theta^{\mathbf{a}} \lrcorner(\theta_{\mathbf{k}} \wedge d\theta_{\mathbf{a}})\\ &= \star[d\theta_{\mathbf{k}} - \theta_{\mathbf{k}} \wedge (\theta^{\mathbf{a}} \lrcorner d\theta_{\mathbf{a}})] = \star d\theta_{\mathbf{k}} - \star\theta_{\mathbf{k}} \wedge (\theta^{\mathbf{a}} \lrcorner d\theta_{\mathbf{a}})\\ &= \star d\theta_{\mathbf{k}} - \theta_{\mathbf{k}} \lrcorner \star (\theta^{\mathbf{a}} \lrcorner d\theta_{\mathbf{a}})\end{aligned}$$

So, it remains to prove that

$$\theta_{\mathbf{k}} \lrcorner \star (\theta^{\mathbf{a}} \lrcorner d\theta_{\mathbf{a}}) = \delta\theta_{\mathbf{a}}\,(\theta_{\mathbf{k}} \lrcorner \star \theta^{\mathbf{a}}) = \delta\theta_{\mathbf{a}} \star (\theta_{\mathbf{k}} \wedge \theta^{\mathbf{a}}).$$

But as easily verified $\theta^{\mathbf{a}} \lrcorner d\theta_{\mathbf{a}} = -\Gamma^{\mathbf{k}\cdot\cdot}_{\cdot\mathbf{kc}}\theta^{\mathbf{c}}$ and thus

$$-\Gamma^{\mathbf{l}\cdot\cdot}_{\cdot\mathbf{lc}}\theta_{\mathbf{k}} \lrcorner \star \theta^{\mathbf{c}} = -\Gamma^{\mathbf{l}\cdot\cdot}_{\cdot\mathbf{lc}} \star (\theta_{\mathbf{k}} \wedge \theta^{\mathbf{c}})$$

and the equivalence of Eqs. (9.129) and (9.130) is proven.

Exercise 9.21 (a) Let (σ_a, m_a) and (σ_a, m_a) be two particles living in Minkowski spacetime. Let $p_a = m_a\boldsymbol{g}(\sigma_{a*},) \in \sec T_{\sigma_a}M$ and $p_b = m_b\boldsymbol{g}(\sigma_{b*},) \in \sec T_{\sigma_b}M$ the momentum covectors of the particles. How you would define the total momentum of the two particles. If this object exists, where is the space where it lives? (b) May you find a way to define the total momentum covector of the two particles if they live in a general Lorentzian manifold? (c) Has this question something to do which the absence of conservation laws of energy-momentum in *GRT*?

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Chapter 10
The DHE on a RCST and the Meaning of Active Local Lorentz Invariance

Abstract In this chapter we give a formulation of the Dirac-Hestenes equation on a Riemann-Cartan manifold $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ using the Clifford and spin-Clifford bundles formalism. We show that the obtained equation which follows for a properly chosen Lagrangian density (heuristically based on the principle of minimum coupling) agrees with the one proposed by some authors using the standard concept of covariant spinor fields. However, we do more: we show that postulating invariance under active rotational gauge transformations of the Dirac-Hestenes Lagrangian implies in the equivalence (in a precise sense) of torsion free and non torsion free connections. Such a result suggests that the choice of a particular connection in order to formulate spacetime field theories (which includes the gravitational field) is somewhat arbitrary. This issue is deeply investigated in Chap. 11.

10.1 Formulation of the DHE on a RCST

Let $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ be a general Riemann-Cartan spacetime. In this section we investigate how to obtain a generalization in $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ of the Dirac-Hestenes equation (see Eq. (7.93)) for a representative $\psi_{\Xi} \in \sec \mathcal{C}\ell(M, \mathtt{g})$ of a Dirac-Hestenes spinor field $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$. In order to do that we first introduce a chart (φ, U) from the maximal atlas of M, with coordinates $\{x^{\mu}\}$. The associate coordinate basis of TU is denoted by $\{e_{\mu} = \frac{\partial}{\partial x^{\mu}} = \partial_{\mu}\}$ and we denoted by $\{\gamma^{\mu} = dx^{\mu}\}$ its dual basis. Moreover, we suppose that the $\gamma^{\mu} \in \sec \bigwedge^{1} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Also, let $\{\boldsymbol{e}_{\mathbf{a}}\} \in \sec \mathbf{P}_{\mathrm{SO}^{e}_{1,3}}(M)$ an orthonormal frame and $\{\theta^{\mathbf{a}}\} \in \sec P_{\mathrm{SO}^{e}_{1,3}}(M)$ the dual coframe. Note that, for each $\mathbf{a} = 0, 1, 2, 3$, $\theta^{\mathbf{a}} \in \sec \bigwedge^{1} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. In what follows we are going to work in a *fixed* spin coframe $\Xi \in \sec P_{\mathrm{Spin}^{e}_{1,3}}(M)$ such that $s(\pm\Xi) = \{\theta^{\mathbf{a}}\}$ and so, in order to simplify the notation we write simply ψ instead of ψ_{Ξ}.

Recall now that the Lagrangian density for the free Dirac-Hestenes spinor in Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ (see Eq. (8.50)) can be written with

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coordinates $\{\mathring{x}^{\mu}\}$ in Einstein-Lorentz-Poincaré gauge as

$$\begin{aligned}\mathring{\mathcal{L}}(\psi, \mathring{\partial}\psi) &= \mathring{\mathfrak{L}}(, \psi, \mathring{\partial}\psi) d\mathring{x}^0 \wedge d\mathring{x}^1 \wedge d\mathring{x}^2 \wedge d\mathring{x}^3 \\ &= \left[(\mathring{\partial}\psi \mathbf{i}\gamma_3) \cdot \psi - m\psi \cdot \psi\right] d\mathring{x}^0 \wedge d\mathring{x}^1 \wedge d\mathring{x}^2 \wedge d\mathring{x}^3, \end{aligned} \tag{10.1}$$

where $\mathring{\partial} = d\mathring{x}^{\mu} D_{\frac{\partial}{\partial \mathring{x}^{\mu}}}$. The usual prescription of *minimal coupling* between a given spinor field and the (generalized) gravitational field,

$$d\mathring{x}^{\mu} \mapsto \theta^{\mathbf{a}}, \quad D_{\frac{\partial}{\partial \mathring{x}^{\mu}}} \mapsto \nabla_{\mathbf{e_a}},$$

suggests that we take the following Lagrangian for a representative ψ of a Dirac-Hestenes spinor field in a Riemann-Cartan spacetime,

$$\begin{aligned}\mathcal{L}(\psi, \boldsymbol{\partial}^{(s)}\psi) &= \mathfrak{L}(\psi, \boldsymbol{\partial}^{(s)})\psi) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= \left[(\boldsymbol{\partial}^{(s)}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}}) \cdot \psi - m\psi \cdot \psi\right] \sqrt{|\det \boldsymbol{g}|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{10.2}$$

where (with the notations of Chap. 7)

$$\boldsymbol{\partial}^{(s)}\psi = \theta^{\mathbf{a}}\nabla^{(s)}_{\boldsymbol{e}_{\mathbf{a}}}\psi = \theta^{\mathbf{a}}\left(\boldsymbol{\eth}_{\boldsymbol{e}_{\mathbf{a}}}\psi + \frac{1}{2}\omega_{\boldsymbol{e}_a}\psi\right). \tag{10.3}$$

$$= \theta^{\mathbf{a}}\left(h_a^{\mu}\partial_{\mu}\psi + \frac{1}{2}\omega_{\boldsymbol{e}_a}\psi\right) \tag{10.4}$$

Then $\boldsymbol{\delta}\int \mathcal{L}(\psi, \boldsymbol{\partial}^{(s)}\psi) = 0$ gives the Euler-Lagrange equation

$$\partial_{\psi}\mathfrak{L} - \partial_{\mu}\left(\partial_{\partial_{\mu}\psi}\mathfrak{L}\right) = 0. \tag{10.5}$$

Using the identities given by Eqs. (2.68) and (2.69) we can write the term $\theta^{\mathbf{a}}h_a^{\mu}\partial_{\mu}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} \cdot \psi$ as

$$\begin{aligned}\boldsymbol{\eth}_{\boldsymbol{e}_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} \cdot \psi &= \boldsymbol{\eth}_{\boldsymbol{e}_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} \cdot \theta^{\mathbf{a}}\psi \\ &= -\boldsymbol{\eth}_{\boldsymbol{e}_{\mathbf{a}}}\psi \cdot \theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} \end{aligned} \tag{10.6}$$

and

$$\partial_{\boldsymbol{\eth}_{\boldsymbol{e}_{\mathbf{a}}}\psi}\mathfrak{L} = -\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} \tag{10.7}$$

We now must express $\partial_\mu \left(\partial_{\partial_\mu \psi} \mathfrak{L}\right)$ in terms of the Pfaff derivatives $\boldsymbol{\eth}_{\boldsymbol{e}_\mathbf{a}} \psi$. To do that we first write

$$\theta^\mathbf{a} = h^\mathbf{a}_\mu dx^\mu. \tag{10.8}$$

Then, putting $\det \boldsymbol{g} := \det[\boldsymbol{g}(\partial_\mu, \partial_\nu)]$ since

$$\sqrt{|\det \boldsymbol{g}|} = \frac{1}{\det[h^\mathbf{a}_\mu]} \equiv \mathfrak{h}^{-1} \tag{10.9}$$

we have

$$\mathfrak{L}(\psi, \boldsymbol{\partial}^{(s)}\psi) = \left[(\boldsymbol{\partial}^{(s)}\psi\theta^\mathbf{0}\theta^\mathbf{2}\theta^\mathbf{1}) \cdot \psi - m\psi \cdot \psi\right] \mathfrak{h}^{-1}. \tag{10.10}$$

Then,

$$\partial_{\partial_\mu \psi} \mathfrak{L} = (\partial_{\partial_\mu \psi} \boldsymbol{\eth}_{\boldsymbol{e}_\mathbf{a}} \psi)(\partial_{\eth_{\boldsymbol{e}_\mathbf{a}} \psi} \mathfrak{L}) = h^\mu_\mathbf{a} \partial_{\boldsymbol{\eth}_{\boldsymbol{e}_\mathbf{a}} \psi} \mathfrak{L} \tag{10.11}$$

and Eq. (10.5) becomes

$$\partial_\psi \mathfrak{L} - (\partial_\mu h^\mu_\mathbf{a}) \partial_{\boldsymbol{\eth}_{\boldsymbol{e}_\mathbf{a}} \psi} \mathfrak{L} - \boldsymbol{\eth}_{e_\mathbf{a}} \left(\partial_{\boldsymbol{\eth}_{e_\mathbf{a}} \psi} \mathfrak{L}\right) = 0. \tag{10.12}$$

Now, we can verify that

$$\mathfrak{h}^{-1} \partial_\mu \mathfrak{h} = h^\mathbf{a}_\sigma \partial_\mu h^\sigma_\mathbf{a}, \tag{10.13}$$

and taking into account that

$$[\boldsymbol{e}_\mathbf{a}, \boldsymbol{e}_\mathbf{b}] = (h^\nu_\mathbf{a} \boldsymbol{e}_\mathbf{b}(h^\mathbf{c}_\nu) - h^\nu_\mathbf{b} \boldsymbol{e}_\mathbf{a}(h^\mathbf{c}_\nu)) \boldsymbol{e}_\mathbf{c} = c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} \boldsymbol{e}_\mathbf{c} \tag{10.14}$$

we can write

$$\partial_\mu h^\mu_\mathbf{a} = -c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}} + \mathfrak{h}^{-1} \boldsymbol{e}_\mathbf{a}(\mathfrak{h}), \tag{10.15}$$

and Eq. (10.12) becomes

$$\partial_\psi \mathfrak{L} - [\partial_{\boldsymbol{e}_\mathbf{a}} - c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}} + \mathfrak{h}^{-1} \boldsymbol{e}_\mathbf{a}(\mathfrak{h})] \left(\partial_{\boldsymbol{\eth}_{\boldsymbol{e}_\mathbf{a}} \psi} \mathfrak{L}\right) = 0. \tag{10.16}$$

Now, we have taking into account Eq. (2.194)

$$\partial_\psi \left(\theta^\mathbf{a} \omega_{\boldsymbol{e}_\mathbf{a}} \psi \theta^\mathbf{0}\theta^\mathbf{2}\theta^\mathbf{1} \cdot \psi\right) = \langle \theta^\mathbf{a} \omega_{\boldsymbol{e}_\mathbf{a}} \psi \theta^\mathbf{0}\theta^\mathbf{2}\theta^\mathbf{1} + \omega_{\boldsymbol{e}_\mathbf{a}} \theta^\mathbf{a} \psi \theta^\mathbf{0}\theta^\mathbf{2}\theta^\mathbf{1} \rangle_\psi,$$

and then

$$\begin{aligned}\mathfrak{h}\partial_{\psi}\mathfrak{L} &= \left(\theta^{\mathbf{a}}\eth_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}\theta^{\mathbf{a}}\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}\omega_{e_{\mathbf{a}}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - 2m\psi\right),\\ \partial_{\eth_{e_{\mathbf{a}}}\psi}\mathfrak{L} &= -\mathfrak{h}^{-1}\left(\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}}\right),\\ \eth_{e_{\mathbf{a}}}\left(\partial_{\eth_{e_{\mathbf{a}}}\psi}\mathfrak{L}\right) &= -\mathfrak{h}^{-1}\boldsymbol{e}_{\mathbf{a}}(\mathfrak{h})(\partial_{\eth_{e_{\mathbf{a}}}\psi}\mathfrak{L}) - \mathfrak{h}^{-1}\theta^{\mathbf{a}}\eth_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}},\end{aligned} \tag{10.17}$$

and Eq. (10.16) becomes

$$\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{4}\theta^{\mathbf{a}}\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{4}\theta^{\mathbf{a}}\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}\theta^{\mathbf{a}}c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\psi - m\psi = 0 \tag{10.18}$$

or, recalling that the components of the torsion tensor in an orthonormal basis is given by

$$T^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}} - \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ba}} - c^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}, \tag{10.19}$$

and that, in particular[1] $\omega^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}} = \eta_{\mathbf{bb}}\omega^{\mathbf{b}\cdot\mathbf{b}}_{\cdot\mathbf{a}\cdot} = 0$, we have

$$\begin{aligned}&\theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{4}\omega_{e_{\mathbf{a}}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi\\ &\quad= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{4}\theta^{\mathbf{a}}\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}}\\ &\qquad\frac{1}{4}\omega_{e_{\mathbf{a}}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi\\ &\quad= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}\theta^{\mathbf{a}}\lrcorner\omega_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - \frac{1}{2}c^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi\\ &\quad= \theta^{\mathbf{a}}\nabla^{(s)}_{e_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}T^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\theta^{\mathbf{a}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi = 0.\end{aligned} \tag{10.20}$$

Finally we write the Dirac-Hestenes equation in a general Riemann-Cartan spacetime as [3]

$$\boldsymbol{\partial}^{(s)}\psi\theta^{\mathbf{2}}\theta^{\mathbf{1}} + \frac{1}{2}\mathrm{T}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}} - m\psi\theta^{\mathbf{0}} = 0, \tag{10.21}$$

where

$$\mathrm{T} = T^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ab}}\theta^{\mathbf{a}}. \tag{10.22}$$

[1]No sum in **b**.

is (sometimes) called the *torsion covector*. Note that in a Lorentzian manifold $\mathrm{T} = 0$ and we come back to the Dirac-Hestenes equation given by Eq. (7.93). We observe moreover that the matrix representation of Eq. (10.21) coincides with an equation first proposed by Hehl and Datta [2].

We observe yet that if we tried to get the equation of motion of a Dirac-Hestenes spinor field on a Riemann-Cartan spacetime directly from that equation on Minkowski spacetime by using the principle of minimal coupling, we would miss the term $\frac{1}{2}\mathrm{T}\psi\theta^{\mathbf{2}}\theta^{\mathbf{1}}$ appearing in Eq. (10.21). This would be very bad indeed, because in a complete theory where the $\{\theta^{\mathbf{a}}\}$ and the $\{\omega_{\boldsymbol{e}_{\mathbf{a}}}\}$ are dynamical fields we can easily show that spinor fields generate torsion (details in [2]).

10.2 Meaning of Active Lorentz Invariance of the Dirac-Hestenes Lagrangian

In the proposed gauge theories of the gravitational field, it is said that the Lagrangians and the corresponding equations of motion of physical fields must be invariant under arbitrary *active* local Lorentz rotations. In this section we briefly investigate how to mathematically implement such an hypothesis and what is its meaning for the case of a Dirac-Hestenes spinor field on a Riemann-Cartan spacetime. The Lagrangian we shall investigate is the one given by Eq. (10.10), i.e.,

$$\mathfrak{L}(\psi, \boldsymbol{\partial}^{(s)}\psi) = \left[(\theta^{\mathbf{a}}\nabla^{(s)}_{\boldsymbol{e}_{\mathbf{a}}}\psi\theta^{\mathbf{0}}\theta^{\mathbf{2}}\theta^{\mathbf{1}})\cdot\psi - m\psi\cdot\psi\right]\mathfrak{h}^{-1}. \qquad \text{(Dirac-Hestenes)}$$

Observe that the Dirac-Hestenes Lagrangian has been written in a fixed gauge individualized by a spin coframe $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ and we already know after Exercise 7.57, that it is invariant under passive gauge transformations $\psi \mapsto \psi U^{-1}$ ($UU^{-1} = 1$, $U \in \sec \mathrm{Spin}^e_{1,3}(M) \subset \sec \mathcal{C}\ell(M, \mathtt{g})$), once the 'connection' 2-form ω_V transforms as given in Eq. (7.50), i.e.,

$$\frac{1}{2}\omega_V \mapsto U\frac{1}{2}\omega_V U^{-1} + (\boldsymbol{\nabla}_V U)U^{-1}. \tag{10.23}$$

Under an active rotation (gauge) transformation the fields transform in new fields given by

$$\begin{aligned}
\psi &\mapsto \psi' = U\psi,\\
\theta^{\mathbf{m}} &\mapsto \theta'^{\mathbf{m}} = U\theta^{\mathbf{m}}U^{-1} = \Lambda^{\mathbf{m}}_{\mathbf{n}}\theta^{\mathbf{n}},\\
\boldsymbol{e}_{\mathbf{m}} &\mapsto \boldsymbol{e}'_{\mathbf{m}} = (\Lambda^{-1})^{\mathbf{n}}_{\mathbf{m}}\boldsymbol{e}_{\mathbf{n}}.
\end{aligned} \tag{10.24}$$

Now, according to the mathematical ideas behind gauge theories briefly outlined in Appendix A.4, we must search for a new connection ∇'^{s} such that the Lagrangian results invariant. This will be the case if connections ∇^s and ∇'^s are generalized G-connections[2] (Definition A.66), i.e.,

$$\nabla'^{(s)}_{e'_{\mathbf{m}}}(U\psi) = U\nabla^{(s)}_{e_{\mathbf{m}}}\psi,$$

or

$$\nabla'^{(s)}_{e_{\mathbf{n}}}(U\psi) = \Lambda^{\mathbf{m}}_{\mathbf{n}} U\nabla^{(s)}_{e_{\mathbf{m}}}\psi. \tag{10.25}$$

Also, taking into account the structure of a representative of a spinor covariant derivative in the Clifford bundle (see Eq. (7.55)) we must have for the Pfaff derivative

$$\eth_{e_{\mathbf{n}}} \mapsto \eth_{e'_{\mathbf{a}}} = \Lambda^{\mathbf{m}}_{\mathbf{n}} \eth_{e_{\mathbf{n}}}, \tag{10.26}$$

and for the connection

$$\omega'_{e_{\mathbf{n}}} = \Lambda^{\mathbf{m}}_{\mathbf{n}} \left(U\omega_{e_{\mathbf{m}}} U^{-1} - 2\eth_{e_{\mathbf{m}}}(U)U^{-1} \right),$$

or

$$\omega'_{e'_{\mathbf{m}}} = U\omega_{e_{\mathbf{m}}} U^{-1} - 2\eth_{e_{\mathbf{m}}}(U)U^{-1}. \tag{10.27}$$

Write

$$\begin{aligned}
\omega'_{e_{\mathbf{n}}} &= \frac{1}{2}\omega'^{\mathbf{k\cdot l}}_{\cdot\mathbf{m}\cdot}\theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}} = \frac{1}{2}\omega'^{\mathbf{k\cdot l}}_{\cdot\mathbf{m}\cdot}\theta_{\mathbf{kl}} \in \sec \mathcal{C}\ell M, \mathtt{g}),\\
\omega_{e_{\mathbf{n}}} &= \frac{1}{2}\omega^{\mathbf{k\cdot l}}_{\cdot\mathbf{m}\cdot}\theta_{\mathbf{k}} \wedge \theta_{\mathbf{l}} = \frac{1}{2}\omega^{\mathbf{k\cdot l}}_{\cdot\mathbf{m}\cdot}\theta_{\mathbf{kl}} \in \sec \mathcal{C}\ell M, \mathtt{g}),\\
U &= e^F, F = \frac{1}{2}F^{\mathbf{rs}}\theta_{\mathbf{rs}} \in \sec \mathcal{C}\ell M, \mathtt{g}).
\end{aligned} \tag{10.28}$$

Recall that

$$\begin{aligned}
\omega^{\mathbf{r\cdot s}}_{\cdot\mathbf{n}\cdot} &= \eta^{\mathbf{ra}}\omega_{\mathbf{anb}}\eta^{\mathbf{sb}} = \omega^{\mathbf{r\cdot\cdot}}_{\cdot\mathbf{nb}}\eta^{\mathbf{sb}},\\
\omega^{\mathbf{r\cdot\cdot}}_{\cdot\mathbf{nk}} &= \omega^{\mathbf{r\cdot s}}_{\cdot\mathbf{n}\cdot}\eta_{\mathbf{sk}}.
\end{aligned} \tag{10.29}$$

Then, from Eqs. (10.27)–(10.29) we get

$$\omega'^{\mathbf{r\cdot\cdot}}_{\cdot\mathbf{nk}} = \Lambda^{\mathbf{b}}_{\mathbf{q}}\omega^{\mathbf{p\cdot\cdot}}_{\cdot\mathbf{mb}}\Lambda^{\mathbf{r}}_{\mathbf{p}}\Lambda^{\mathbf{m}}_{\mathbf{k}} - \eta_{\mathbf{sk}}\Lambda^{\mathbf{m}}_{\mathbf{k}}e_{\mathbf{m}}\left(F^{\mathbf{rs}}\right). \tag{10.30}$$

[2]Note that ∇^s and ∇'^s are connection in the spin-Clifford bundle of Dirac-Hestenes spinor fields whereas $\nabla^{(s)}$ and $\nabla'^{(s)}$ the effective covariant derivative operators acting on the representatives of Dirac-Hestenes spinor fields in the Clifford bundle.

Now, we recall that the components of the torsion tensors, $\mathbf{\Theta}$ and $\mathbf{\Theta}'$of the connections ∇ and ∇' in the orthonormal basis $\{\boldsymbol{e}_\mathbf{r} \otimes \theta^\mathbf{n} \wedge \theta^\mathbf{k}\}$ are given by

$$\begin{aligned} T_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot} &= \omega_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot} - \omega_{\cdot\mathbf{kn}}^{\mathbf{r}\cdot\cdot} - c_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot}, \\ T_{\cdot\mathbf{nk}}^{\prime\mathbf{r}\cdot\cdot} &= \omega_{\cdot\mathbf{nk}}^{\prime\mathbf{r}\cdot\cdot} - \omega_{\cdot\mathbf{kn}}^{\prime\mathbf{r}\cdot\cdot} - c_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot}, \end{aligned} \tag{10.31}$$

where $[\boldsymbol{e}_\mathbf{n}, \boldsymbol{e}_\mathbf{k}] = c_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot}\boldsymbol{e}_\mathbf{r}$.

Let us suppose that we start with a torsion free connection ∇. This means that $c_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot} = \omega_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot} - \omega_{\cdot\mathbf{kn}}^{\mathbf{r}\cdot\cdot}$. Then,

$$T_{\cdot\mathbf{nk}}^{\prime\mathbf{r}\cdot\cdot} = \Lambda_\mathbf{n}^\mathbf{b}\Lambda_\mathbf{k}^\mathbf{m}\Lambda_\mathbf{p}^\mathbf{r}c_{\cdot\mathbf{mb}}^{\mathbf{p}\cdot\cdot} - c_{\cdot\mathbf{nk}}^{\mathbf{r}\cdot\cdot} - \boldsymbol{e}_\mathbf{m}\left(F^{\mathbf{rs}}\right)\left[\eta_{\mathbf{sk}}\Lambda_\mathbf{n}^\mathbf{m} - \eta_{\mathbf{sn}}\Lambda_\mathbf{k}^\mathbf{m}\right], \tag{10.32}$$

and we see that $\mathbf{\Theta}' = 0$ only for very particular gauge transformations.

We then arrive at the conclusion that to suppose the Dirac-Hestenes Lagrangian is invariant under active rotational gauge transformations imply in an equivalence between torsion free and non torsion free connections. It is always emphasized that in a theory where besides ψ, also the tetrad fields $\theta^\mathbf{a}$ and the connection 1-forms $\omega_{\cdot\mathbf{b}}^{\mathbf{a}\cdot}$ are dynamical variables, the torsion is not zero, because its source is the spin of the ψ field. Well, this is true in particular gauges, because as showed above it seems that it is always possible to find gauges where the torsion is null. The reader is invited to reflect on this result, taking also into account a result proved in Chap. 6 that says that distinct LLRFγ and LLRFγ' that meet at $p \in M$ are not physically equivalent.

Exercise 10.1 Show that whereas Maxwell Lagrangian density is invariant under local Lorentz rotations, Maxwell equations (in general) are not [1].

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Chapter 11
On the Nature of the Gravitational Field

Abstract In this chapter we investigate the nature of the gravitational field. We first give a formulation for the theory of that field as a field in Faraday's sense (i.e., as of the same nature as the electromagnetic field) on a 4-dimensional parallelizable manifold M. The gravitational field is represented through the 1-form fields $\{\mathfrak{g}^{\mathbf{a}}\}$ dual to the parallelizable vector fields $\{\boldsymbol{e}_{\mathbf{a}}\}$. The $\mathfrak{g}^{\mathbf{a}}$'s ($\mathbf{a} = 0, 1, 2, 3$) are called gravitational potentials, and it is imposed that at least for one of them, $d\mathfrak{g}^{\mathbf{a}} \neq 0$. A metric *like* field $\boldsymbol{g} = \eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ is introduced in M with the purpose of permitting the construction of the Hodge dual operator and the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathtt{g})$, where $\mathtt{g} = \eta^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{b}}$. Next a Lagrangian density for the gravitational potentials is introduced with consists of a *Yang-Mills* term plus a *gauge fixing* term and an *auto-interacting* term. Maxwell like equations for $F^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}}$ are obtained from the variational principle and a *legitimate* energy-momentum tensor for the gravitational field is identified which is given by a formula that at first look seems very much complicated. Our theory does not uses any connection in M and we clearly demonstrate that representations of the gravitational field as Lorentzian, teleparallel and even general Riemann-Cartan-Weyl geometries depend only on the arbitrary particular connection (which may be or not to be metrical compatible) that we may define on M. When the Levi-Civita connection of $\boldsymbol{g}$ in M is introduced we prove that the postulated Lagrangian density for the gravitational potentials differs from the Einstein-Hilbert Lagrangian density of General Relativity only by a term that is an exact differential. The theory proceeds choosing the most simple topological structure for M, namely that it is $\mathbb{R}^4$, a choice that is compatible with present experimental data. With the introduction of a Levi-Civita connection for the structure $(M = \mathbb{R}^4, \boldsymbol{g})$ as a mathematical aid we can exhibit a nice short formula for the genuine energy-momentum of the gravitational field. Next, we introduce the Hamiltonian formalism and discuss possible generalizations of the gravitational field theory (as a field in Faraday's sense) when the graviton mass is not null. Also we show using the powerful Clifford calculus developed in previous chapters that if the structure $(M = \mathbb{R}^4, \boldsymbol{g})$ possess at least one Killing vector field, then the gravitational field equations can be written as a single Maxwell like equation, with a well defined current like term (of course, associated to the energy-momentum tensor of matter and the gravitational field). This result is further generalized for arbitrary vector fields generating one-parameter groups of diffeomorphisms of M in Chap. 14. Chapter 11 ends with another possible interpretation of the gravitational

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field, namely that it is represented by a particular geometry of a brane embedded in a high dimensional pseudo-Euclidean space. Using the theory developed in Chap. 5 we are able to write Einstein equation using the Ricci operator in such a way that its second member (of "wood" nature, according to Einstein) is transformed (also according to Einstein) in the "marble"nature of its first member. Such a form of Einstein equation shows that the energy momentum quantities $-T^{\mathbf{a}} + \frac{1}{2}T\mathfrak{g}^{\mathbf{a}}$ (where $T^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}}\mathfrak{g}^{\mathbf{b}}$ are the energy momentum 1-form fields of matter and $T = T^{\mathbf{a}}_{a}$) which characterize matter is represented by the negative square of the shape operator $(\mathbf{S}^2(\mathfrak{g}^{\mathbf{a}}))$ of the brane. Such a formulation thus give a mathematical expression for the famous Clifford "little hills" as representing matter.

11.1 Introduction

As well known, in GRT, each gravitational field generated by a given energy-momentum tensor $\boldsymbol{T}$ is represented by an equivalence class[1] of *Lorentzian spacetimes* $[(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)]$, where we recall once again that a Lorentzian spacetime is a structure $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ where M is a non compact (locally compact) 4-dimensional Hausdorff manifold, $\boldsymbol{g}$ is a Lorentzian metric on M and $\boldsymbol{D}$ is its Levi-Civita connection. Moreover M is supposed to be oriented by the volume form $\tau_{\boldsymbol{g}}$ and the symbol $\uparrow$ means that the spacetime is time orientable. From the geometrical objects in the structure $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ we can calculate the *Riemann curvature tensor* $\mathbf{R}$ of $\boldsymbol{D}$ and a nontrivial GRT model is one in which $\mathbf{R} \neq 0$. In that way textbooks often say that in GRT *spacetime is curved*. Unfortunately many people mislead the curvature of a connection $\boldsymbol{D}$ on M with the fact that M can eventually be a bent surface in a (pseudo)Euclidean space with a sufficient number of dimensions.[2] This confusion according to our view leads to all sort of wishful thinking because many forget that GRT does not fix the topology of M that often must be put "by hand" when solving a problem, and thus think that they can *bend* spacetime or even change its topology if they have an appropriate kind of some exotic matter. Worse, the insistence in supposing that the gravitational field is *geometry* lead the majority of physicists to relegate the search for the real *physical nature* of the gravitational field as not important at all (see a nice discussion of this issue in [19]). As discussed with details in Chap. 9 what most textbooks with a few exceptions (see, e.g., the excellent book by Sachs and Wu [34]) forget to say and give a proof to their readers is that in the standard formulation of GRT there are *no* genuine conservation laws of energy-momentum and angular momentum *unless* spacetime

[1] $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ is said to be equivalent to $(M', \boldsymbol{D}', \boldsymbol{g}', \tau'_{\boldsymbol{g}}, \uparrow')$ if there exists a diffeomorphism $\mathtt{h} : M \to M'$ such that $M =' \mathtt{h}M$ $\boldsymbol{D}' = \mathtt{h}^*\boldsymbol{D}, \boldsymbol{g}' = \mathtt{h}^*\boldsymbol{g}, \tau'_{\boldsymbol{g}} = \mathtt{h}^*\tau_{\boldsymbol{g}}, \uparrow' = \mathtt{h}^* \uparrow$ and $T' = \mathtt{h}^*T$.

[2] Recall that bending is characterized by the shape operator introduced in Chap. 5. Recall moreover that, e.g., the shape operator for a punctured sphere viewed as a submanifold embedded in 3-dimensional Euclidean space is non null, but its Nunes connection has zero curvature (Sect. 4.9.8).

has some *additional* structure which is not present in a general Lorentzian spacetime [26]. Some textbooks e.g., [24] even claim that energy-momentum conservation for matter plus the gravitational fields is forbidden due the *equivalence principle*[3] because the energy-momentum of the gravitational field must be non localizable. Only a few people tried to develop consistent theories where the gravitational field (at least from a classical point of view) is simple another field, which like the electromagnetic field lives in Minkowski spacetime (see a list of references in [11]). A field of that nature will be called, in what follows, a field in Faraday's sense.

Here we want to recall that: (1) the representation of gravitational fields by Lorentzian spacetimes is not a necessary one, for indeed, there are some geometrical structures different from $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ that can equivalently represent such a field; (2) That any realistic gravitational field can also be nicely represented as a field living in a fixed background spacetime.

The preferred one which seems to describe all realistic situations is, of course, Minkowski spacetime[4] $(M \simeq \mathbb{R}^4, D, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$.

Concerning the possible alternative geometrical models, the particular case where the connection is *teleparallel* (i.e., it is metric compatible, has *null* Riemann curvature tensor and *non null* torsion tensor) will be briefly addressed below (for other possibilities see [27]). What we will show, is that starting with a thoughtful representation of the gravitational field in terms of *gravitational potentials* $\mathfrak{g}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, $\mathbf{a} = 0, 1, 2, 3$ and postulating a convenient Lagrangian density for the gravitational potentials which *does not use any connection* there is a posteriori different ways of geometrically representing the gravitational field, such that the field equations in each representation result equivalent in a precise mathematical sense to Einstein's field equations. Explicitly we mean by this statement the following: any *realistic* model of a gravitational field in GRT where that field is represented by a *Lorentzian spacetime* (*with non null Riemann curvature tensor and null torsion tensor*) which is also parallelizable, i.e., admits four *global* linearly independent vector fields) is *equivalent* to a teleparallel spacetime (i.e., a spacetime structure equipped with a metrical compatible teleparallel connection, which has *null* Riemann curvature tensor and *non null* torsion tensor).[5] The teleparallel possibility follows almost directly from the results in Sect. 11.1 and a recent claim that it can give a mathematical representation to "Einstein most happy though" is discussed in Sect. 11.2. Comments about possible conservation laws in the teleparallel equivalent of GRT is discussed in Sect. 11.6.

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[3]We recall here that most presentations of the equivalence principle are according to our view devoid from mathematical and physical sense. See, e.g., [32, 38] and our discussion in Sect. 6.7.

[4]Of course, the true background spacetime may be eventually a more complicated one, since that manifold must represent the global topological structure of the universe, something that is not known at the time of this writing [43]. We do not study this possibility here, but the results we are going to present can be easily generalized for more general spacetime backgrounds.

[5]There are hundreds of papers (as e.g., [10]) on the subject, but none (to the best of our knowledge) develop the theory from the point of view presented here and originally in [31].

Equipped with the powerful Clifford bundle formalism developed in previous chapters we give in Sect. 11.3 a field theory for the gravitational field in Minkowski spacetime (with field equations equivalent to Einstein equation in a precise mathematical sense). In our theory we are able to identify in Sect. 11.4.1 a *legitimate* energy momentum tensor for the gravitational field expressible in a very short and elegant formula.

Besides this important result we think that another important feature of this chapter is that our representation of the gravitational field by the global 1-form fields potentials $\{\mathfrak{g}^{\mathbf{a}}\}$ living on a manifold M and coupled among themselves and with the matter fields in a specific way (see below) shows that we can *dispense* with the concept of a connection and a corresponding geometrical description for that field. The simplest case is when M is part of Minkowski spacetime structure, in which case the gravitational field is (like the electromagnetic field) a field in Faraday's sense.[6] In Sect. 11.4.1 we present the Hamiltonian formalism for our theory and discuss the relation of one possible energy concept[7] naturally appearing in it and its relation to the concept of ADM energy. In Sect. 11.5 we discuss the role of a possible graviton mass in the formulation of the field theory of gravitation. Despite our present opinion that gravitation is a plastic distortion of the Lorentz vacuum [11] and thus is to be described by a field in the sense of Faraday living in Minkowski spacetime in Sect. 11.7 (titled *On Clifford Little Hills*) using a nice result proved in Chap. 5 (namely, that the Ricci operator is the negative square of the shape operator) we show a way to represent as "marble" the "wood" part of Einstein equation, i.e., we show how the phenomenological energy-momentum tensor can be explicitly represented by a geometrical property of a 4-dimensional Lorentzian brane embedded as a submanifold in a pseudo-Euclidean space of large dimension. In Sect. 11.8 we present our conclusions.

11.2 Representation of the Gravitational Field

Suppose that a 4-dimensional M manifold is parallelizable, thus admitting a set of four global linearly independent vector $\boldsymbol{e}_{\mathbf{a}} \in \sec TM$, $\mathbf{a} = 0,1,2,3$ fields such $\{\boldsymbol{e}_{\mathbf{a}}\}$ is a basis for TM and let $\{\mathfrak{g}^{\mathbf{a}}\}$, $\mathfrak{g}^{\mathbf{a}} \in \sec T^{*}M$ be the corresponding dual basis ($\mathfrak{g}^{\mathbf{a}}(\boldsymbol{e}_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$). Suppose also that not all the $\mathfrak{g}^{\mathbf{a}}$ are closed, i.e., $d\mathfrak{g}^{\mathbf{a}} \neq 0$, for a

[6] In Chap. 15 (see also [33]) we even show that when a Lorentzian spacetime structure $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ representing a gravitational field in GRT possess a Killing vector field $\boldsymbol{K}$, then there are Maxwell like equations with well determined source term satisfied for $F = dA$ with $A = \boldsymbol{g}(\boldsymbol{K},\,)$ encoding Einstein equation and more, there is a Navier-Stokes equation encoding the Maxwell (like) and Einstein equations.

[7] This other possibility does not define in general a legitimate energy-momentum tensor for the gravitational field in GRT, but it defines a legitimate energy-momentum tensor in our theory in which the gravitational field is interpreted as a field in the sense of Faraday living in Minkowski spacetime.

least some $\mathbf{a} = 0, 1, 2, 3$. This will be necessary for the possible interpretations we have in mind for our theory. The 4-form field $\mathfrak{g}^{\mathbf{0}} \wedge \mathfrak{g}^{\mathbf{1}} \wedge \mathfrak{g}^{\mathbf{2}} \wedge \mathfrak{g}^{\mathbf{3}}$ defines a (positive) orientation for M.

Now, the $\{\mathfrak{g}^{\mathbf{a}}\}$ can be used to define a Lorentzian "metric" field in M by defining $\boldsymbol{g} \in \sec T_2^0 M$ by $\boldsymbol{g} := \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$, with the matrix with entries $\eta_{\mathbf{ab}}$ being the diagonal matrix $(1, -1, -1, -1)$. Then, according to $\boldsymbol{g}$ the $\{\mathbf{e}_{\mathbf{a}}\}$ are orthonormal, i.e., $\boldsymbol{e}_{\mathbf{a}} \underset{\boldsymbol{g}}{\cdot} \boldsymbol{e}_{\mathbf{b}} := \boldsymbol{g}(\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}) = \eta_{\mathbf{ab}}$.

Since the $\boldsymbol{e}_{\mathbf{0}}$ is a *global* time like vector field it follows that it defines a *time orientation* in M which we denote by $\uparrow$. Then, the 4-tuple $(M, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ is part of a structure defining a *Lorentzian spacetime* and can eventually serve as a *substructure* to model a gravitational field in GRT.

For future use we also introduce $\mathtt{g} \in \sec T_0^2 M$ by $\mathtt{g} := \eta^{\mathbf{ab}} \boldsymbol{e}_{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{b}}$, and we write $\mathfrak{g}^{\mathbf{a}} \underset{g}{\cdot} \mathfrak{g}^{\mathbf{b}} := \mathtt{g}(\mathfrak{g}^{\mathbf{a}}, \mathfrak{g}^{\mathbf{b}}) = \eta^{\mathbf{ab}}$.

Due to the hypothesis that $d\mathfrak{g}^{\mathbf{a}} \neq 0$ the commutator of vector fields $\mathbf{e}_{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$ will in general satisfy $[\boldsymbol{e}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}] = c_{\cdot \mathbf{ab}}^{\mathbf{k} \cdot\cdot} \boldsymbol{e}_{\mathbf{k}}$,where the $c_{\cdot \mathbf{ab}}^{\mathbf{k} \cdot\cdot}$, the structure coefficients of the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$, and we. easily show that $d\mathfrak{g}^{\mathbf{a}} = -\frac{1}{2} c_{\cdot \mathbf{kl}}^{\mathbf{a} \cdot\cdot} \mathfrak{g}^{\mathbf{k}} \wedge \mathfrak{g}^{\mathbf{l}}$.

Next, we introduce two different metric compatible connections on M, namely $\boldsymbol{D}$ (the Levi-Civita connection of $\boldsymbol{g}$) and a *teleparallel* connection $\boldsymbol{\nabla}$. Metric compatibility means that for both connections it is $\boldsymbol{D}\boldsymbol{g} = 0$, $\boldsymbol{\nabla}\boldsymbol{g} = 0$. Now, we put

$$
\begin{aligned}
&\boldsymbol{D}_{\boldsymbol{e}_{\mathbf{a}}} \boldsymbol{e}_{\mathbf{b}} = \boldsymbol{\omega}_{\cdot \mathbf{ab}}^{\mathbf{c} \cdot\cdot} \boldsymbol{e}_{\mathbf{c}}, \ \boldsymbol{D}_{\boldsymbol{e}_{\mathbf{a}}} \mathfrak{g}^{\mathbf{b}} = -\boldsymbol{\omega}_{\cdot \mathbf{ac}}^{\mathbf{b} \cdot\cdot} \mathfrak{g}^{\mathbf{c}}, \\
&\boldsymbol{\nabla}_{\boldsymbol{e}_{\mathbf{a}}} \boldsymbol{e}_{\mathbf{b}} = 0, \qquad \boldsymbol{\nabla}_{\boldsymbol{e}_{\mathbf{a}}} \mathfrak{g}^{\mathbf{b}} = 0.
\end{aligned} \tag{11.1}
$$

As we know, the objects $\boldsymbol{\omega}_{\cdot \mathbf{ab}}^{\mathbf{c} \cdot\cdot}$ are called the *connection coefficients* of the connection $\boldsymbol{D}$ in the $\{\boldsymbol{e}_{\mathbf{a}}\}$ basis and the objects $\boldsymbol{\omega}_{\cdot \mathbf{b}}^{\mathbf{a} \cdot} \in \sec T^* M$ defined by $\boldsymbol{\omega}_{\cdot \mathbf{b}}^{\mathbf{a} \cdot} := \boldsymbol{\omega}_{\cdot \mathbf{kb}}^{\mathbf{a} \cdot\cdot} \mathfrak{g}^{\mathbf{k}}$ are called the *connection* 1-forms in the $\{\boldsymbol{e}_{\mathbf{a}}\}$ basis. The *connection coefficients* $\boldsymbol{\varpi}_{\mathbf{ac}}^{\mathbf{b}}$ of $\boldsymbol{\nabla}$ and the connection 1-forms of $\boldsymbol{\nabla}$ in the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$ are null according to the second line of Eq. (11.1) and thus the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$ is called *teleparallel and* the connection $\boldsymbol{\nabla}$ defines an *absolute parallelism* on M. Of course, as it is well known the Riemann curvature tensor of the Levi-Civita connection $\boldsymbol{D}$ of $\boldsymbol{g}$, is in general non null in all points of M, but the torsion tensor of $\boldsymbol{D}$ is zero in all points of M. On the other hand the Riemann curvature tensor of $\boldsymbol{\nabla}$ is null in all points of M, whereas the torsion tensor of $\boldsymbol{\nabla}$ is non null in all points of M.

We recall from Chap. 4 that for a general connection, say $\mathbf{D}$ on M (not necessarily metric compatible) the *torsion and curvature operators* and the *torsion* and *curvature tensors* are respectively the mappings:

$$
\begin{aligned}
\boldsymbol{\rho} &: \sec TM \otimes TM \otimes TM \to \sec TM, \\
\boldsymbol{\rho}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \mathbf{D}_{\boldsymbol{u}} \mathbf{D}_{\boldsymbol{v}} \boldsymbol{w} - \mathbf{D}_{\boldsymbol{v}} \mathbf{D}_{\boldsymbol{u}} \boldsymbol{w} - \mathbf{D}_{[\boldsymbol{u}, \boldsymbol{v}]} \boldsymbol{w}, \\
\boldsymbol{\tau} &: \sec TM \otimes TM \to \sec TM, \\
\boldsymbol{\tau}(\boldsymbol{u}, \boldsymbol{v}) &= \mathbf{D}_{\boldsymbol{u}} \boldsymbol{v} - \mathbf{D}_{\boldsymbol{v}} \boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}].
\end{aligned} \tag{11.2}
$$

It is usual to write [5] $\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}$ and $\Theta(\alpha,\boldsymbol{u},\boldsymbol{v}) = \alpha(\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{v}))$ and $\mathbf{R}(\mathrm{w},\alpha,\boldsymbol{u},\boldsymbol{v}) = \alpha(\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w})$, for every $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$. In particular we write $T^{\mathbf{a}\cdot}_{\cdot\mathbf{bc}} := \Theta(\mathfrak{g}^{\mathbf{a}},\boldsymbol{e}_{\mathbf{b}},\boldsymbol{e}_{\mathbf{c}})$ and $R^{\mathbf{b}\cdots}_{\cdot\mathbf{acd}} := \mathbf{R}(\mathfrak{g}^{\mathbf{b}},\boldsymbol{e}_{\mathbf{a}},\boldsymbol{e}_{\mathbf{c}},\boldsymbol{e}_{\mathbf{d}})$, and define the Ricci tensor by *Ricci* $:= R_{\mathbf{ac}}\mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{c}}$ with $R_{\mathbf{ac}} := R^{\mathbf{b}\cdots}_{\cdot\mathbf{acb}} = R_{\mathbf{ca}}$. We take as in previous chapters $\bigwedge T^*M = \bigoplus_{r=0}^{4}\bigwedge^r T^*M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$.

Given that we introduced two different connections $\boldsymbol{D}$ and $\boldsymbol{\nabla}$ defined in the manifold M we can write *two different pairs* of Cartan's structure equations. Those pairs describe respectively the geometry of the structures $(M,\boldsymbol{D},\,\boldsymbol{g},\tau_{\boldsymbol{g}},\uparrow)$ and $(M,\boldsymbol{\nabla},\boldsymbol{g},\tau_{\boldsymbol{g}},\uparrow)$ called respectively a *Lorentzian spacetime* and a *teleparallel spacetime.* In the case $(M,\boldsymbol{D},\,\boldsymbol{g},\tau_{\boldsymbol{g}},\uparrow)$ we write

$$\Theta^{\mathbf{a}} := d\mathfrak{g}^{a} + \boldsymbol{\omega}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = 0, \quad \mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} := d\boldsymbol{\omega}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \boldsymbol{\omega}^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \boldsymbol{\omega}^{\mathbf{c}\cdot}_{\cdot\mathbf{b}},$$

where the $\Theta^{\mathbf{a}} \in \sec\bigwedge^2 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$, $\mathbf{a} = 0,1,2,3$ and the $\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \in \sec\bigwedge^2 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$, $\mathbf{a},\mathbf{b} = 0,1,2,3$ are respectively the torsion and the curvature 2-forms of $\boldsymbol{D}$ with

$$\Theta^{\mathbf{a}} = \frac{1}{2}T^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bc}}\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{c}},\ \mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = \frac{1}{2}R^{\mathbf{a}\cdots}_{\cdot\mathbf{bcd}}\mathfrak{g}^{\mathbf{c}} \wedge \mathfrak{g}^{\mathbf{d}}. \tag{11.3}$$

In the case of $(M,\boldsymbol{\nabla},\boldsymbol{g},\tau_{\boldsymbol{g}},\uparrow)$ since $\boldsymbol{\varpi}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = 0$ we have

$$\mathcal{F}^{\mathbf{a}} := d\mathfrak{g}^{a} + \boldsymbol{\varpi}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = d\mathfrak{g}^{a}, \quad \overset{\boldsymbol{\varpi}}{\mathcal{R}}{}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} := d\boldsymbol{\varpi}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \boldsymbol{\varpi}^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \boldsymbol{\varpi}^{\mathbf{c}\cdot}_{\cdot\mathbf{b}} = 0, \tag{11.4}$$

where the $\mathcal{F}^{\mathbf{a}} \in \sec\bigwedge^2 T^*M$, $\mathbf{a} = 0,1,2,3$ and the $\overset{\boldsymbol{\varpi}}{\mathcal{R}}{}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \in \sec\bigwedge^2 T^*M$, $\mathbf{a},\mathbf{b} = 0,1,2,3$ are respectively the torsion and the curvature 2-forms of $\boldsymbol{\nabla}$ given by formulas analogous to the ones in Eq. (11.3).

We next postulate that the $\{\mathfrak{g}^{\mathbf{a}}\}$ are the basic variables representing the gravitational field, and moreover postulate that the $\{\mathfrak{g}^{a}\}$ interacts with the matter fields through the following *Lagrangian density*

$$\mathcal{L} = \mathcal{L}_{g} + \mathcal{L}_{m}, \tag{11.5}$$

where $\mathcal{L}_m$ is the matter Lagrangian density and

$$\mathcal{L}_{g} = -\frac{1}{2}d\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}} + \frac{1}{2}\underset{g}{\delta}\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_{\mathbf{a}} + \frac{1}{4}\left(d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}\right) \wedge \underset{g}{\star}\left(d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}\right), \tag{11.6}$$

The form of this Lagrangian is notable, the first term is Yang-Mills like, the second one is a kind of gauge fixing term and the third term is an auto-interaction term describing the interaction of the "vorticities" of the potentials (or if you prefer, the interaction between *Chern-Simons* terms $d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}$). Before proceeding we observe that this Lagrangian is not invariant under arbitrary point dependent Lorentz rotations of the basic cotetrad fields. In fact, if $\mathfrak{g}^{\mathbf{a}} \mapsto \mathfrak{g}'^{\mathbf{a}} = \Lambda^{\mathbf{a}}_{\mathbf{b}}\mathfrak{g}^{\mathbf{b}} = R\mathfrak{g}^{\mathbf{a}}\tilde{R}$ (where for each $x \in M$, $\Lambda^{\mathbf{a}}_{\mathbf{b}}(x) \in L^{\uparrow}_{+}$, the homogeneous and orthochronous Lorentz group and $R(x) \in Spin_{1,3} \subset \mathbb{R}_{1,3}$) we get that

$$\mathcal{L}'_g = -\frac{1}{2} d\mathfrak{g}'^{\mathbf{a}} \wedge \underset{g}{\star} d\mathfrak{g}'_{\mathbf{a}} + \frac{1}{2} \underset{g}{\delta} \mathfrak{g}'^{\mathbf{a}} \wedge \underset{g}{\star} \underset{g}{\delta} \mathfrak{g}'_{\mathbf{a}} + \frac{1}{4} \left(d\mathfrak{g}'^{\mathbf{a}} \wedge \mathfrak{g}'_{\mathbf{a}}\right) \wedge \underset{g}{\star} \left(d\mathfrak{g}'^{\mathbf{b}} \wedge \mathfrak{g}'_{\mathbf{b}}\right), \tag{11.7}$$

differs from $\mathcal{L}_g$ by an exact differential. So, the field equations derived by the variational principle results invariant under a change of gauge and we can always choose a gauge such that $\underset{g}{\delta} \mathfrak{g}_{\mathbf{a}} = 0$.

Now, a derivation of the field equations directly from Eq. (11.6) using constrained variations of the $\mathfrak{g}^{\mathbf{a}}$ (i.e., variations induced by point dependent Lorentz rotations) that do not change the metric field $\boldsymbol{g}$ has been given in Chap. 9. The result is:

$$-d \underset{g}{\star} \mathcal{S}_{\mathbf{d}} - \underset{g}{\star} t_{\mathbf{d}} = \underset{g}{\star} \mathcal{T}_{\mathbf{d}}, \tag{11.8}$$

with

$$\begin{aligned}
\underset{g}{\star} t_{\mathbf{d}} := \frac{\partial \mathcal{L}_g}{\partial \mathfrak{g}^{\mathbf{d}}} &= \frac{1}{2}[(\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathbf{a}}) \wedge \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}} - d\mathfrak{g}^{\mathbf{a}} \wedge (\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}})] \\
&+ \frac{1}{2} d(\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathbf{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathbf{a}} + \frac{1}{2} (\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathbf{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathbf{a}} + \frac{1}{2} d\mathfrak{g}_{\mathbf{d}} \wedge \underset{g}{\star} (d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}) \\
&- \frac{1}{4} d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}} \wedge \left[\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} (d\mathfrak{g}^{\mathbf{c}} \wedge \mathfrak{g}_{\mathbf{c}})\right] - \frac{1}{4} \left[\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} (d\mathfrak{g}^{\mathbf{c}} \wedge \mathfrak{g}_{\mathbf{c}})\right] \wedge \underset{g}{\star} (d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}),
\end{aligned} \tag{11.9}$$

$$\underset{g}{\star} \mathcal{S}_{\mathbf{d}} := \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^{\mathbf{d}}} = -\underset{g}{\star} d\mathfrak{g}_{\mathbf{d}} - (\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathbf{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathbf{a}} + \frac{1}{2} \mathfrak{g}_{\mathbf{d}} \wedge \underset{g}{\star} (d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}). \tag{11.10}$$

and the[8]

$$\underset{g}{\star} \mathcal{T}_{\mathbf{d}} := \frac{\partial \mathcal{L}_m}{\partial \mathfrak{g}^{\mathbf{d}}} = -\underset{g}{\star} T_{\mathbf{d}} \tag{11.11}$$

with the $\underset{g}{\star} T_{\mathbf{d}}$ the energy-momentum 3-forms of the matter fields.

Recalling that from Eq. (11.4) it is $\mathcal{F}^{\mathbf{a}} := d\mathfrak{g}^{a}$, it is, of course, $d\mathcal{F}^{\mathbf{a}} = 0$ and the field equations (Eq. (11.8)) can be written (recalling Eq. (11.10)) as

$$d \underset{g}{\star} \mathcal{F}_{\mathbf{d}} = -\underset{g}{\star} \mathcal{T}_{\mathbf{d}} - \underset{g}{\star} t_{\mathbf{d}} - \underset{g}{\star} \mathfrak{h}_{d}, \tag{11.12}$$

[8] We suppose that $\mathcal{L}_m$ does not depend explicitly on the $d\mathfrak{g}^{\mathbf{a}}$.

where

$$\mathfrak{h}_{\mathbf{d}} = d\left[(\mathfrak{g}_{\mathbf{d}} \underset{g}{\lrcorner} \underset{g}{\star} \mathfrak{g}^{\mathbf{a}}) \wedge \underset{g}{\star} d \underset{g}{\star} \mathfrak{g}_{\mathbf{a}} - \frac{1}{2}\mathfrak{g}_{\mathbf{d}} \wedge \underset{g}{\star}(\mathcal{F}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}})\right]. \tag{11.13}$$

Recalling the definition of the Hodge coderivative operator acting on sections of $\bigwedge^r T^*M$ we can write Eq. (11.12) as

$$\underset{g}{\delta}\mathcal{F}^{\mathbf{d}} = -(\mathcal{T}^{\mathbf{d}} + \mathbf{t}^{\mathbf{d}}), \tag{11.14}$$

with the $\mathbf{t}^{\mathbf{d}} \in \sec \bigwedge^1 T^*M$ given by

$$\mathbf{t}^{\mathbf{d}} := t^{\mathbf{d}} + \mathfrak{h}^{\mathbf{d}}, \tag{11.15}$$

which are legitimate energy-momentum[9] 1-form fields for the gravitational field. Note that the *total energy-momentum* tensor of matter plus the gravitational field is trivially conserved in our theory, i.e.,

$$\underset{g}{\delta}(\underset{g}{\star}\mathcal{T}^{\mathbf{d}} + \mathbf{t}^{\mathbf{d}}) = 0. \tag{11.16}$$

Remark 11.1 Recalling Eqs. (11.9) and (11.13) the formula for the $\mathbf{t}^{\mathbf{d}}$ in Eq. (11.15) cannot be, of course, the nice and short formula we promised to present in the introduction. However, it is equivalent to the nice formula as shown in Sect. 11.3.

Exercise 11.2 Show that in a 2-dimensional spacetime the Einstein-Hilbert Lagrangian density is an exact differential (see, e.g.,[8]).

Recall the similarity of the equations satisfied by the gravitational field to Maxwell equations. Indeed, in electromagnetic theory on a Lorentzian spacetime we have only one potential $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ and the field equations are

$$dF = 0, \qquad \underset{g}{\delta}F = -J, \tag{11.17}$$

where $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is the electromagnetic field and $J \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is the electric current. As well known the two equations in Eq. (11.17) can be written (if you do not mind in introducing the connection $\boldsymbol{D}$ in the game as a mathematical tool to simplify some formulas) as a single equation using the Clifford bundle formalism, namely

$$\boldsymbol{\partial} F = J. \tag{11.18}$$

[9]This will become evident after we present below the nice formula for the $\mathbf{t}^{\mathbf{d}}$.

where we can write $\boldsymbol{\partial} = d - \underset{g}{\delta} = \mathfrak{g}^{\mathbf{a}} \boldsymbol{D}_{e_{\mathbf{a}}}$, where $\boldsymbol{\partial}$ is the Dirac operator (acting on sections of $\mathcal{C}\ell(M, \mathtt{g})$).

Now, if you feel uncomfortable in needing four distinct potentials $g^{\mathbf{a}}$ for describing the gravitational field you can put them together defining a vector valued differential form[10]

$$\mathfrak{g} = \mathfrak{g}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}} \in \sec \bigwedge^1 T^*M \otimes TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}) \otimes TM \tag{11.19}$$

and in this case the gravitational field equations are

$$d\mathcal{F} = 0, \quad \underset{g}{\delta}\mathcal{F} = -(\mathcal{T} + \mathbf{t}), \tag{11.20}$$

where $\mathcal{F} = \mathcal{F}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}, \mathcal{T} = T^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}, \mathbf{t} = \mathbf{t}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}$. Again, if you do not mind in introducing the connection $\boldsymbol{D}$ in the game) by considering the bundle $\mathcal{C}\ell(M, \mathtt{g}) \otimes TM$ we can write the two equations in Eq. (11.20) as a single equation, i.e.,

$$\boldsymbol{\partial}\mathcal{F} = \mathcal{T} + \mathbf{t}. \tag{11.21}$$

At this point you may be asking: which is the relation of the theory just presented with Einstein's GRT? The answer is that recalling (See Exercise 9.16) that the connection 1-forms $\boldsymbol{\omega}^{\mathbf{cd}}$ of $\boldsymbol{D}$ are given by

$$\boldsymbol{\omega}^{\mathbf{cd}} = \frac{1}{2}\left[\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathbf{c}} - \mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathbf{d}} + \mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} (\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} d\mathfrak{g}_{\mathbf{a}})\mathfrak{g}^{\mathbf{a}}\right] \tag{11.22}$$

we already showed in Chap. 9 that the Lagrangian density $\mathcal{L}_g$ becomes

$$\mathcal{L}_g = d(\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}}) + \mathcal{L}_{EH}, \tag{11.23}$$

where

$$\mathcal{L}_{EH} = \frac{1}{2}\mathcal{R}_{\mathbf{cd}} \wedge \underset{g}{\star}(\mathfrak{g}^{\mathbf{c}} \wedge \mathfrak{g}^{\mathbf{d}}) = -\frac{1}{2} \underset{g}{\star} R \tag{11.24}$$

(with $\mathcal{R}_{\mathbf{cd}}$ given by Eq. (11.3)) is the Einstein-Hilbert Lagrangian density. This, as we know from Chap. 9 permits (with some algebra) to show that Eq. (11.8) are indeed equivalent to the usual Einstein equation.

Before ending this section we recall from Chap. 9 that from Eq. (11.8) we can also define for our theory a meaningful energy-momentum for the gravitational plus matter fields Indeed, using Stokes theorem for a 'certain 3-dimensional volume',

[10]Recall that $\mathfrak{g} = \mathfrak{g}^{\mathbf{a}} \otimes \boldsymbol{e}_{\mathbf{a}}$ is the identity operator in $\bigwedge^1 T^*M$.

say a ball B we immediately get

$$P^{\mathbf{a}} := -\frac{1}{8\pi}\int_B \underset{g}{\star}(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = \frac{1}{8\pi}\int_{\partial B} \underset{g}{\star}\mathcal{S}^{\mathbf{a}}. \tag{11.25}$$

11.3 Comment on Einstein Most Happy Though

The exercises presented above indicate that a particular geometrical interpretation for the gravitational field is no more than an option among many ones. Indeed, it is not necessary to introduce any connection $\boldsymbol{D}$ or ∇ on M to have a perfectly well defined theory for the gravitational field whose field equations are (in a precise mathematical sense) equivalent to the Einstein field equation. Note that we have not given until now details on the *global topology* of the world manifold M, except that since we admitted that M carries four global (not all closed) 1-form fields $\mathfrak{g}^{\mathbf{a}}$ which defines the object $\boldsymbol{g}$, it follows as we know from Chap. 7 that $(M, \boldsymbol{D}, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ is a *spin manifold* [12, 13], i.e., it admits spinor fields. This, of course, is necessary if the theory is to be useful in the real world since fundamental matter fields are spinor fields. The most simple spin manifold is clearly Minkowski spacetime which is represented by a structure $(M = \mathbb{R}^4, D, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$ where $\boldsymbol{D}$ is the Levi-Civita connection of the Minkowski metric $\boldsymbol{\eta}$. In that case it is possible to interpret the gravitational field as a $(1,1)$-extensor field $\boldsymbol{h}$ which is a field in the Faraday sense living in $(M, D, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$. The field $\boldsymbol{h}$ (as we know from Eq. (2.121)) is a kind of square of $\boldsymbol{g}$ which has been called in [11] the plastic distortion field of the Lorentz vacuum. In that theory the potentials $\mathfrak{g}^{\mathbf{a}} = \boldsymbol{h}(\boldsymbol{\gamma}^{\mathbf{a}})$ where $\boldsymbol{\gamma}^{\mathbf{a}} = \delta^{\mathbf{a}}_{\mu} d\mathrm{x}^{\mu}$, with $\{\mathrm{x}^{\mu}\}$ being global naturally adapted coordinates (in Einstein-Lorentz-Poincaré gauge) to the *inertial reference frame* $\mathbf{I} = \partial/\partial\mathrm{x}^0$ according to the structure $(M = \mathbb{R}^4, D, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$, i.e., $D\mathbf{I} = 0$. In [11] we give the dynamics and coupling of $\boldsymbol{h}$ to the matter fields.

We want also to comment that, as well known, in Einstein's GRT one can easily distinguish in any *real physical laboratory,* i.e., not one modelled by a time like worldline (despite some claims on the contrary) [29] a true gravitational field from an acceleration field of a given reference frame in Minkowski spacetime. This is because in GRT the *mark* of a real gravitational field is the non null Riemann curvature tensor of $\boldsymbol{D}$, and the Riemann curvature tensor of the Levi-Civita connection of $\boldsymbol{D}$ (present in the definition of Minkowski spacetime) is null. However if we interpret a gravitational field as the torsion 2-forms on the structure $(M, \nabla, \boldsymbol{g}, \tau_{\boldsymbol{g}}, \uparrow)$ viewed as a kind of deformation of Minkowski spacetime then one can also interpret an acceleration field of an accelerated reference frame in Minkowski spacetime as generating an effective teleparallel spacetime $(M, \overset{e}{\nabla}, \eta, \tau_\eta, \uparrow)$. This can be done as follows. Let $\boldsymbol{Z} \in \sec TU$, $U \subset M$ with $\boldsymbol{\eta}(\boldsymbol{Z}, \boldsymbol{Z}) = 1$ an *accelerated reference frame* on Minkowski spacetime. This means as we know from Chap. 5 that $\boldsymbol{a} = \boldsymbol{D}_{\boldsymbol{Z}}\boldsymbol{Z} \neq 0$. Put $e_{\mathbf{0}} = \boldsymbol{Z}$ and define an accelerated reference frame as *non* trivial if $\vartheta^{\mathbf{0}} = \eta(e_0,)$ is not an exact differential. Next recall that in $U \subset M$ there always exist [5] three other $\boldsymbol{\eta}$-orthonormal vector fields $e_{\mathbf{i}}$, $\mathbf{i} = 1, 2, 3$ such that $\{e_a\}$ is an $\boldsymbol{\eta}$-orthonormal basis for TU, i.e., $\boldsymbol{\eta} = \eta_{\mathbf{ab}}\vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}}$, where $\{\vartheta^{\mathbf{a}}\}$ is the dual basis[11] of $\{e_a\}$. We then have, $\boldsymbol{D}_{e_{\mathbf{a}}}e_{\mathbf{b}} = \boldsymbol{\omega}^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}e_{\mathbf{c}}, \boldsymbol{D}_{e_{\mathbf{a}}}\vartheta^{b} = -\boldsymbol{\omega}^{\mathbf{b}\cdot\cdot}_{\cdot\mathbf{ac}}\vartheta^{\mathbf{c}}$.

What remains in order to be possible to interpret an acceleration field as a kind of 'gravitational field' is to introduce on M a $\boldsymbol{\eta}$-metric compatible connection $\overset{e}{\nabla}$ such that the $\{e_a\}$ is teleparallel according to it, i.e., $\overset{e}{\nabla}_{e_{\mathbf{a}}}e_{\mathbf{b}} = 0, \overset{e}{\nabla}_{e_{\mathbf{a}}}\vartheta^{\mathbf{b}} = 0$. Indeed, with this connection the structure $(M \simeq \mathbb{R}^4, \overset{e}{\nabla}, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$ has null Riemann curvature tensor but a non null torsion tensor, whose components are related with the components of the acceleration $\boldsymbol{a}$ and with the other coefficients $\boldsymbol{\omega}^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ab}}$ of the connection $\boldsymbol{D}$, which describe the motion on Minkowski spacetime of a *grid* represented by the orthonormal frame $\{e_a\}$. Schücking [35] thinks that such a description of the gravitational field makes Einstein most happy though, i.e., the equivalence principle (understood as equivalence between acceleration and gravitational field) a legitimate mathematical idea. However, a *true* gravitational field must satisfy (at least with good approximation) Eq. (11.12), whereas there is no single reason for an acceleration field to satisfy that equation.

11.4 Field Theory for the Gravitational Field in Minkowski Spacetime

Since the structure of the Lagrangian density for the gravitational field and the resulting field equations do not use any connection we can assume the gravitational field represented by the potentials $\mathfrak{g}^{\mathbf{a}}$ is a field in Faraday sense, i.e., it leaves in the Minkowski spacetime structure $(M = \mathbb{R}^4, D, \boldsymbol{\eta}, \tau_{\boldsymbol{\eta}}, \uparrow)$. All geometrical like objects like $\boldsymbol{g}, \boldsymbol{D}, \nabla$ introduced above are then to be understood as no more than auxiliary mathematical devices to present formulas and to suggest possible geometrical interpretations for the gravitational field. With this advise in mind we show next that we can give a very nice and compact formula for the energy-momentum tensor of the gravitational field.

11.4.1 Legitimate Energy-Momentum Tensor of the Gravitational Field. The Nice Formula

Taking into account that $\mathcal{F}^{\mathbf{d}} = d\mathfrak{g}^{d} = \boldsymbol{\partial} \wedge \mathfrak{g}^{d}$ we return to Eq. (11.14) and write it as

$$\boldsymbol{\partial}^2\mathfrak{g}^{d} = \mathcal{T}^{\mathbf{d}} + \mathfrak{t}^{\mathbf{d}}, \tag{11.26}$$

[11] In general we will also have that $d\vartheta^{\mathbf{i}} \neq 0, \mathbf{i} = 1, 2, 3$.

with $\mathfrak{t}^{\mathbf{d}} = \mathbf{t}^{\mathbf{d}} - d\delta\mathfrak{g}^{\mathbf{d}}$. Next we recall from Chap. 4 that the operator $\boldsymbol{\partial}^2$ (the Hodge D'Alembertian) has two distinct decompositions, namely, for each $M \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ we have

$$\begin{aligned} \boldsymbol{\partial}^2 M &= -(d\underset{g}{\delta} + \underset{g}{\delta} d)M \\ &= \boldsymbol{\partial} \wedge \boldsymbol{\partial}\, M + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, M \end{aligned} \tag{11.27}$$

where $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ is the Ricci operator and $\boldsymbol{\partial} \cdot \boldsymbol{\partial}$ is the covariant D'Alembertian operator. We have

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} = \mathcal{R}^{\mathbf{d}}, \tag{11.28}$$

where the $\mathcal{R}^{\mathbf{d}} = R^{\mathbf{d}}_{\mathbf{a}}\mathfrak{g}^{\mathbf{d}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ (with $R^{\mathbf{d}}_{\mathbf{a}}$ the components of the Ricci tensor) are the Ricci 1-form fields. Then we can write Eq. (11.26) as

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} = \mathcal{T}^{\mathbf{d}} + \mathfrak{t}^{\mathbf{d}}, \tag{11.29}$$

or

$$\mathcal{R}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} = \mathcal{T}^{\mathbf{d}} + \mathfrak{t}^{\mathbf{d}}. \tag{11.30}$$

Now, we recall that Einstein equation in components form is

$$R^{\mathbf{a}}_{\mathbf{d}} - \frac{1}{2}\delta^{\mathbf{a}}_{\mathbf{d}} R = -T^{\mathbf{a}}_{\mathbf{d}} \tag{11.31}$$

from where it follows immediately that

$$\mathcal{R}^{\mathbf{d}} - \frac{1}{2}R\mathfrak{g}^{\mathbf{d}} = -T^{\mathbf{d}} = \mathcal{T}^{\mathbf{d}}. \tag{11.32}$$

Then

$$\mathcal{R}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} = \mathcal{T}^{\mathbf{d}} + \frac{1}{2}R\mathfrak{g}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}}, \tag{11.33}$$

and comparing Eq. (11.29) with Eq. (11.33) we get

$$\mathfrak{t}^{\mathbf{d}} = \frac{1}{2}R\mathfrak{g}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}}, \tag{11.34}$$

and

$$\mathbf{t}^{\mathbf{d}} = \frac{1}{2}R\mathfrak{g}^{\mathbf{d}} + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, \mathfrak{g}^{\mathbf{d}} + d\delta\mathfrak{g}^{\mathbf{d}} \tag{11.35}$$

the nice formula promised and that clearly demonstrates that the objects $\mathbf{t_{da}} = \eta_{\mathbf{ac}}\eta_{\mathbf{dl}}\mathbf{t^c} \underset{g}{\lrcorner} \mathfrak{g}^{\mathbf{l}}$ are components of a legitimate gravitational energy-momentum tensor field $\mathbf{t} = \mathbf{t_{da}}\mathfrak{g}^{\mathbf{d}} \otimes \mathfrak{g}^{\mathbf{a}} \in \sec T_0^2 M$. We observe moreover that

$$\mathbf{t^{da}} - \mathbf{t^{ad}} = 2\boldsymbol{\partial} \cdot \boldsymbol{\partial}\ \mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} \mathfrak{g}^{\mathbf{a}}, \tag{11.36}$$

i.e., the energy-momentum tensor of the gravitational field in not symmetric. As shown in [11] this is important in order to have a total angular momentum conservation law for the system consisting of the gravitational plus the matter fields. At least observe that $\mathbf{t^d} = \mathfrak{t}^{\mathbf{d}}$ when the potentials are chosen in the Lorenz gauge.

11.5 Hamilton Formalism

If we define as usual the canonical momenta associated to the potentials $\{\mathfrak{g}^a\}$ by $\mathfrak{p}_{\mathbf{a}} = \partial \mathcal{L}_g / \partial d\mathfrak{g}^{\mathbf{a}} = \underset{g}{\star} \mathcal{S}_{\mathbf{a}}$ and suppose that this equation can be solved for the $d\mathfrak{g}^{\mathbf{a}}$ as function of the $\mathfrak{p}_{\mathbf{a}}$ we can introduce a Legendre transformation with respect to the fields $d\mathfrak{g}^{\mathbf{a}}$ by

$$\mathbf{L} : (\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha}) \mapsto \mathbf{L}(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha}) = d\mathfrak{g}^{\alpha} \wedge p_{\alpha} - \mathcal{L}_g(\mathfrak{g}^{\alpha}, d\mathfrak{g}^{\alpha}(\mathfrak{p}_{\alpha})) \tag{11.37}$$

We write in what follows $\mathfrak{L}_g(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha}) := \mathcal{L}_g(\mathfrak{g}^{\alpha}, d\mathfrak{g}^{\alpha}(\mathfrak{p}_{\alpha}))$ and observe that defining[12]

$$\frac{\delta \mathfrak{L}_g(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha})}{\delta \mathfrak{g}^{\alpha}} := d\mathfrak{p}_{\alpha} - \frac{\partial \mathbf{L}}{\partial \mathfrak{g}^{\alpha}}, \qquad \frac{\delta \mathfrak{L}_g(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha})}{\delta \mathfrak{p}_{\alpha}} := d\mathfrak{g}^{\alpha} - \frac{\partial \mathbf{L}}{\partial \mathfrak{p}_{\alpha}}$$

we can obtain (see details in [11])

$$\boldsymbol{\delta}\mathfrak{g}^{\alpha} \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^{\alpha}, d\mathfrak{g}^{\alpha})}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}} = \boldsymbol{\delta}\mathfrak{g}^{\alpha} \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha})}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}}\right) + \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha})}{\boldsymbol{\delta}\mathfrak{p}_{\alpha}}\right) \wedge \boldsymbol{\delta}\mathfrak{p}_{\alpha}. \tag{11.38}$$

Exercise 11.3 Prove Eq. (11.38).

To define the Hamiltonian form, we need something to act the role of time for our manifold, and we choose this ‘time’ to be given by the flow of an arbitrary timelike vector field $\boldsymbol{Z} \in \sec TM$ such that $\boldsymbol{g}(\boldsymbol{Z}, \boldsymbol{Z}) = 1$. Moreover, we define $Z = \boldsymbol{g}(\boldsymbol{Z},) \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$. With this choice, the variation $\boldsymbol{\delta}$ is generated by the Lie

[12] We use only constrained variations of the $\mathfrak{g}^{\mathbf{a}}$, which as already recalled in Sect. 11.1 do not change the metric field $\boldsymbol{g}$.

derivative $\pounds_Z$. Using Cartan's 'magical formula', we have

$$-\boldsymbol{\delta}\mathfrak{L}_g = \pounds_Z \mathfrak{L}_g = d(Z \lrcorner \mathfrak{L}_g) + Z \lrcorner d\mathfrak{L}_g = d(Z \lrcorner \mathfrak{L}_g). \tag{11.39}$$

and after some algebra we get

$$d(Z \lrcorner \mathfrak{L}_g) = d(\pounds_Z \mathfrak{g}^{\alpha} \wedge \mathfrak{p}_{\alpha}) + \pounds_Z \mathfrak{g}^{\alpha} \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}} + \pounds_Z \mathfrak{p}_{\alpha} \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}}{\boldsymbol{\delta}\mathfrak{p}_{\alpha}}\right) \tag{11.40}$$

and also

$$d(\pounds_Z \mathfrak{g}^{\alpha} \wedge \mathfrak{p}_{\alpha} - Z \lrcorner \mathfrak{L}_g) = -\pounds_Z \mathfrak{g}^{\alpha} \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}}. \tag{11.41}$$

Now, we define the *Hamiltonian* 3-form by

$$\mathcal{H}(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha}) := \pounds_Z \mathfrak{g}^{\alpha} \wedge \mathfrak{p}_{\alpha} - Z \lrcorner \mathfrak{L}_g. \tag{11.42}$$

We immediately have taking into account Eq. (11.41) that, when the field equations for the *free* gravitational field are satisfied (i.e., when the Euler-Lagrange functional is null, $\boldsymbol{\delta}\mathcal{L}_g/\boldsymbol{\delta}\mathfrak{g}^{\alpha} = 0$) that

$$d\mathcal{H} = 0. \tag{11.43}$$

Thus $\mathcal{H}$ is a *conserved* Noether current. We next write

$$\mathcal{H} = Z^{\alpha}\mathcal{H}_{\alpha} + dB. \tag{11.44}$$

We can show (details in [11]) that $\mathcal{H}_{\alpha} = -\boldsymbol{\delta}\mathcal{L}_g/\boldsymbol{\delta}\mathfrak{g}^{\alpha}$ and $B = Z^{\mathbf{a}}\mathfrak{p}_{\mathbf{a}}$ and now we investigate the meaning of the boundary term[13] B. Consider an arbitrary spacelike hypersurface σ. Then, we define

$$\mathbf{H} = \int_{\sigma} (Z^{\alpha}\mathcal{H}_{\alpha} + dB) = \int_{\sigma} Z^{\alpha}\mathcal{H}_{\alpha} + \int_{\partial\sigma} B.$$

If we recall that $\mathcal{H}_{\alpha} = -\boldsymbol{\delta}\mathcal{L}_g/\boldsymbol{\delta}\mathfrak{g}^{\alpha}$ we see that the first term in the above equation is null when the field equations (for the free gravitational field) are satisfied and we are thus left with

$$E = \int_{\partial\sigma} B, \tag{11.45}$$

which is called the quasi local energy [39].

[13]More details on possible choices of the boundary term for different physical situations may be found in [23].

Now, if $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\}$ is the dual basis of $\{\mathfrak{g}^{\alpha}\}$ we have $\mathfrak{g}^{\mathbf{0}}(\boldsymbol{e}_{\mathbf{i}}) = 0$, $\mathbf{i} = 1,2,3$ and if we take $\mathbf{Z} = \boldsymbol{e}_{\mathbf{0}}$ orthogonal to the hypersurface σ, such that for each $p \in \sigma$, $T\sigma_p$ is generated by $\{\boldsymbol{e}_{\mathbf{i}}\}$ we get recalling that $\mathfrak{p}_{\boldsymbol{\alpha}} = \underset{g}{\star}\mathcal{S}_{\boldsymbol{\alpha}}$ that

$$E = \int_{\partial\sigma} \underset{g}{\star}\, \mathcal{S}_0, \tag{11.46}$$

which we recognize (a constant factor apart) as being the same conserved quantity as the one defined by Eq. (11.25).

The relation of the energy defined by Eq. (11.46) with the energy concept defined in ADM formalism [3] can be seen as follows [42]. Instead of choosing an arbitrary unit timelike vector field $\mathbf{Z}$, start with a global timelike vector field $\mathbf{n} \in \sec TM$ such that $n = \boldsymbol{g}(\mathbf{n},\ ) = N^2 dt \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$, with $N : \mathbb{R} \supset \mathbb{I} \to \mathbb{R}$, a positive function called the lapse function of M. Then $n \wedge dn = 0$ and according to Frobenius theorem, n induces a foliation of M, i.e., topologically it is $M = \mathbb{I}\times\sigma_t$, where σ_t is a spacelike hypersurface with normal given by $\mathbf{n}$. Now, we can decompose any $A \in \sec \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$ into a tangent component $\underline{A}$ to σ_t and an orthogonal component ${}^{\perp}A$ to σ_t by

$$A = \underline{A} + {}^{\perp}A, \tag{11.47}$$

where

$$\underline{A} := n \lrcorner (dt \wedge A), \quad {}^{\perp}A = dt \wedge A_{\perp}, \quad A_{\perp} := n \lrcorner A. \tag{11.48}$$

Introduce also the parallel component $\underline{d}$ of the differential operator d by:

$$\underline{d}A := n \lrcorner (dt \wedge dA) \tag{11.49}$$

from where it follows (taking into account Cartan's magical formula) that

$$dA = dt \wedge (\pounds_{\mathbf{n}}\underline{A} - \underline{d}A_{\perp}) + \underline{dA}. \tag{11.50}$$

Call

$$\boldsymbol{m} := -\boldsymbol{g} + \boldsymbol{n} \otimes \boldsymbol{n} = \underline{\mathfrak{g}}^{i} \otimes \underline{\mathfrak{g}}_{i},$$

(where $\boldsymbol{n} = n/N$) the first fundamental form on σ_t and next introduce the Hodge dual operator associated to $\boldsymbol{m}$, acting on the (horizontal forms) forms $\underline{A}$ by

$$\underset{m}{\star}\underline{A} := \underset{g}{\star}\left(\frac{n}{N} \wedge \underline{A}\right). \tag{11.51}$$

At this point, we come back to the Lagrangian density Eq. (11.42) and, proceeding like above, but now leaving $\boldsymbol{\delta} n^{\mathbf{a}}$ to be non null, we eventually arrive at the following Hamiltonian density

$$\mathcal{H}(\underline{\mathfrak{g}}^i, \underline{\mathfrak{p}}_i) = \pounds_{\boldsymbol{n}} \underline{\mathfrak{g}}^i \wedge \underset{\boldsymbol{m}}{\star} \underline{\mathfrak{p}}_i - \mathcal{K}_{\boldsymbol{g}}, \tag{11.52}$$

where

$$\mathfrak{g}^i - \underline{\mathfrak{g}}^i = dt \wedge (n \lrcorner \mathfrak{g}^i) = n^i dt, \tag{11.53}$$

and where $\mathcal{K}_{\boldsymbol{g}}$ depends on $(n, \underline{d}n, \underline{\mathfrak{g}}^i, \underline{d}\underline{\mathfrak{g}}^i, \pounds_{\mathbf{n}} \underline{\mathfrak{g}}^i)$. We can show (after some tedious but straightforward algebra that $\mathcal{H}(\underline{\mathfrak{g}}^i, \underline{\mathfrak{p}}_i)$ can be put into the form

$$\mathcal{H} = n^i \mathcal{H}_i + \underline{d}B', \tag{11.54}$$

with as before $\mathcal{H}_i = -\boldsymbol{\delta}\mathcal{L}_{\boldsymbol{g}}/\boldsymbol{\delta}\mathfrak{g}^i = -\boldsymbol{\delta}\mathcal{K}_{\boldsymbol{g}}/\boldsymbol{\delta}n^i$ and

$$B' = -N \underline{\mathfrak{g}}_i \wedge \underset{\boldsymbol{m}}{\star} \underline{d}\underline{\mathfrak{g}}^i \tag{11.55}$$

Then, on shell, i.e., when the field equations are satisfied we get

$$E' = -\int_{\partial\sigma_t} N \underline{\mathfrak{g}}_i \wedge \underset{\boldsymbol{m}}{\star} \underline{d}\underline{\mathfrak{g}}^i \tag{11.56}$$

which is exactly the ADM energy, as can be seen if we take into account that taking $\partial\sigma_t$ as a two-sphere at infinity, we have (using coordinates in the Einstein-Lorentz-Poincaré gauge) $\underline{\mathfrak{g}}_i = h_{ij}\underline{d}\mathrm{x}^j$ and $h_{ij}, N \to 1$. Then

$$\underline{\mathfrak{g}}_i \wedge \underset{\boldsymbol{m}}{\star} \underline{d}\underline{\mathfrak{g}}^i = h^{ij}\left(\frac{\partial h_{ij}}{\partial \mathrm{x}^k} - \frac{\partial h_{ik}}{\partial \mathrm{x}^j}\right) \underset{\boldsymbol{m}}{\star} \underline{\mathfrak{g}}^k \tag{11.57}$$

and under the above conditions we have the ADM formula

$$E' = \int_{\partial\sigma_t} \left(\frac{\partial h_{ik}}{\partial \mathrm{x}^i} - \frac{\partial h_{ik}}{\partial \mathrm{x}^k}\right) \underset{\boldsymbol{m}}{\star} \underline{\mathfrak{g}}^k. \tag{11.58}$$

which, as is well known , is positive definite.[14] If we choose $n = \mathfrak{g}^0$ it may happen that $\mathfrak{g}^0 \wedge d\mathfrak{g}^0 \neq 0$ and thus it does not determine a spacelike hypersurface σ_t. However all algebraic calculations above up to Eq. (11.55) are valid (and of course, $\mathfrak{g}^k = \underline{\mathfrak{g}}^k$). So, if we take a spacelike hypersurface σ such that at spatial infinity the $\mathfrak{e}_i$ $(\mathfrak{g}^k(\mathfrak{e}_i) = \delta_i^k)$ are tangent to σ, and $\mathfrak{e}_0 \to \partial/\partial t$ is orthogonal to σ, then we have

[14] See a nice proof in [42].

$E = E'$ since in this case $-N\underline{\mathfrak{g}}_i \wedge \underset{m}{\star} d\underline{\mathfrak{g}}^i \to -\mathfrak{g}_i \wedge \underset{m}{\star}(\mathfrak{g}^0 \wedge \underset{m}{\star} d\mathfrak{g}^i)$ which as can be easily verified (see Eq. (11.10)) is the asymptotic value of $\underset{g}{\star}\mathcal{S}^0$ (taking into account that at spatial infinity $d\mathfrak{g}^0 \to 0$).

11.6 Mass of the Graviton

In the Lagrangian given by Eq. (11.6) the mass of the graviton is supposed to be zero. A non null mass m requires an extra term in the Lagrangian. As an example, consider the Lagrangian density

$$\mathcal{L}'_g = -\frac{1}{2} d\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}} + \frac{1}{2} \underset{g}{\delta}\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} \underset{g}{\delta}\mathfrak{g}_{\mathbf{a}} + \frac{1}{4}(d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}) \wedge \underset{g}{\star}\left(d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}\right) + \frac{1}{2} m^2 \mathfrak{g}_{\mathbf{a}} \wedge \underset{g}{\star} \mathfrak{g}^{\mathbf{a}} \tag{11.59}$$

With the extra term the equations for the gravitational field, for the $\mathcal{S}^{\mathbf{a}}$ result in

$$-d \underset{g}{\star} \mathcal{S}^{\mathbf{a}} = \underset{g}{\star}\mathcal{T}^{\mathbf{a}} + \underset{g}{\star} t^{\mathbf{a}} + m^2 \underset{g}{\star} \mathfrak{g}^{\mathbf{a}}, \tag{11.60}$$

from where we get

$$\underset{g}{\delta}(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = -m^2 \underset{g}{\delta}\mathfrak{g}^{\mathbf{a}} \tag{11.61}$$

If we impose the gauge $\underset{g}{\delta}\mathfrak{g}^{\mathbf{a}} = 0$, which is analogous to the Lorenz gauge in electrodynamics, Eq. (11.61) becomes

$$\underset{g}{\delta}(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = 0, \tag{11.62}$$

which is the same equation valid in the case $m = 0$!

There are other possibilities of having a non null graviton mass, as, e.g., in Logunov's theory [20, 21], which we do not discuss here.[15]

11.7 Comment on the Teleparallel Equivalent of GR

We observe that some people [10] think to have find a valid way of formulating a genuine energy-momentum conservation law in the teleparallel equivalent to general relativity. In that theory, as we already know (see also [22]), spacetime is teleparallel (a.k.a. Weintzenböck [25]), i.e., has a metric compatible connection with non zero

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[15] We only observe that Lagrangian density of Logunov's theory when written in terms of differential forms is not a very elegant expression.

torsion and with null curvature.[16] However, the claim of [10] must be qualified. Indeed, we have two important comments (a) and (b) concerning this issue.

(a) First, it must be clear that the structure of the teleparallel equivalent of GRT as formulated, e.g., by Maluf [22] or Andrade et al. [10] consists in nothing more than a trivial introduction of: (1) a bilinear form (a deformed metric tensor) $\boldsymbol{g} = \eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ and (2) a teleparallel connection in a manifold $M \simeq \mathbb{R}^4$ (part of the structure defining a Minkowski spacetime). Indeed, taking advantage of the discussion of the previous sections, we can present that theory with a cosmological constant term as follows. Start with $\mathcal{L}'_g$ (Eq. (11.59)) and write it (after some algebraic manipulations) as

$$\begin{aligned}\mathcal{L}'_g &= -\frac{1}{2}d\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star}\left[d\mathfrak{g}_{\mathbf{a}} - \mathfrak{g}_{\mathbf{a}} \wedge (\mathfrak{g}_{\mathbf{b}} \lrcorner d\mathfrak{g}_{\mathbf{b}}) + \frac{1}{2}\underset{g}{\star}(\mathfrak{g}_{\mathbf{a}} \wedge \star(d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}))\right] \\ &\quad + \frac{1}{2}m^2\mathfrak{g}_{\mathbf{a}} \wedge \underset{g}{\star}\mathfrak{g}^{\mathbf{a}} \\ &= -\frac{1}{2}d\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star}({}^{(1)}d\mathfrak{g}_{\mathbf{a}} - 2^{(2)}d\mathfrak{g}_{\mathbf{a}} - \frac{1}{2}\,{}^{(3)}d\mathfrak{g}_{\mathbf{a}}) + \frac{1}{2}m^2\mathfrak{g}_{\mathbf{a}} \wedge \underset{g}{\star}\mathfrak{g}^{\mathbf{a}}, \end{aligned} \tag{11.63}$$

where

$$\begin{aligned} d\mathfrak{g}^{\mathbf{a}} &= {}^{(1)}d\mathfrak{g}^{\mathbf{a}} + {}^{(2)}d\mathfrak{g}^{\mathbf{a}} + {}^{(3)}d\mathfrak{g}^{\mathbf{a}}, \\ {}^{(1)}d\mathfrak{g}^{\mathbf{a}} &= d\mathfrak{g}^{\mathbf{a}} - {}^{(2)}d\mathfrak{g}^{\mathbf{a}} - {}^{(3)}d\mathfrak{g}^{\mathbf{a}}, \\ {}^{(2)}d\mathfrak{g}^{\mathbf{a}} &= \frac{1}{3}\mathfrak{g}^{\mathbf{b}} \wedge (\mathfrak{g}_{\mathbf{b}} \underset{g}{\lrcorner} d\mathfrak{g}_{\mathbf{b}}), \\ {}^{(3)}d\mathfrak{g}^{\mathbf{a}} &= -\frac{1}{3}\underset{g}{\star}(\mathfrak{g}^{\mathbf{b}} \wedge \underset{g}{\star}(d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}). \end{aligned} \tag{11.64}$$

Next introduce a teleparallel connection by declaring that the cobasis $\{\mathfrak{g}^{\mathbf{a}}\}$ fixes the parallelism, i.e., we define the torsion 2-forms by

$$\Theta^{\mathbf{a}} := d\mathfrak{g}^{\mathbf{a}}, \tag{11.65}$$

and $\mathcal{L}'_g$ becomes

$$\mathcal{L}'_g = -\frac{1}{2}\Theta^{\mathbf{a}} \wedge \underset{g}{\star}\left({}^{(1)}\Theta^{\mathbf{a}} - 2^{(2)}\Theta^{\mathbf{a}} - \frac{1}{2}\,{}^{(3)}\Theta^{\mathbf{a}}\right) + \frac{1}{2}m^2\mathfrak{g}_{\mathbf{a}} \wedge \underset{g}{\star}\mathfrak{g}^{\mathbf{a}}, \tag{11.66}$$

[16] In fact, formulation of teleparallel equivalence of GRT is a subject with a old history. See, e.g., [18].

where ${}^{(1)}\Theta^{\mathbf{a}} = {}^{(1)} d\mathfrak{g}^{\mathbf{a}}$, ${}^{(2)}\Theta^{\mathbf{a}} = {}^{(31)} d\mathfrak{g}^{\mathbf{a}}$ and ${}^{(3)}\Theta^{\mathbf{a}} = {}^{(31)} d\mathfrak{g}^{\mathbf{a}}$, called *tractor* (four components), *axitor* (four components) and *tentor* (sixteen components) are the irreducible components of the tensor torsion under the action of $\mathrm{SO}^{e}_{1,3}$.

(b) Recalling the results of Chap. 9 we now show that even if the metric of a given teleparallel spacetime has some Killing vector fields there are genuine conservation laws involving only the energy-momentum and angular momentum tensors of *matter* only if some additional condition is satisfied. Indeed, in the teleparallel basis where $\boldsymbol{\nabla}_{\boldsymbol{e}_{\mathbf{a}}}\boldsymbol{e}_{\mathbf{b}} = 0$ and $[\boldsymbol{e}_{\mathbf{m}}, \boldsymbol{e}_{\mathbf{n}}] = c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{mn}}\boldsymbol{e}_{\mathbf{a}}$ we have that the torsion 2-forms satisfy

$$\Theta^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}} = -\frac{1}{2}c^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{mn}}\mathfrak{g}^{\mathbf{m}} \wedge \mathfrak{g}^{\mathbf{n}} = \frac{1}{2}T^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{mn}}\mathfrak{g}^{\mathbf{m}} \wedge \mathfrak{g}^{\mathbf{n}}. \tag{11.67}$$

Then, recalling once again that $\pounds_{\xi}(d\mathfrak{g}^{\mathbf{a}}) = d(\pounds_{\xi}\mathfrak{g}^{\mathbf{a}}) = d(\varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\mathfrak{g}^{\mathbf{b}})$ and Eq. (9.62) we can use Eq. (9.65) (which express the condition $\pounds_{\xi}\Theta = \mathbf{0}$) to write

$$d(\varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}\mathfrak{g}^{\mathbf{b}}) = \varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}d\mathfrak{g}^{\mathbf{b}}, \tag{11.68}$$

which implies

$$d\varkappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = 0. \tag{11.69}$$

Then, Eq. (11.69) is satisfied only if the torsion tensor of the teleparallel spacetime satisfy the following differential equation:

$$T^{\mathbf{m}\cdot\cdot}_{\cdot\mathbf{bd}}\,\boldsymbol{e}_{\mathbf{m}}(\xi^{\mathbf{a}}) + \boldsymbol{e}_{\mathbf{d}}(\xi^{\mathbf{m}}T^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bm}}) - \boldsymbol{e}_{\mathbf{b}}(\xi^{\mathbf{m}}T^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{dm}}) = 0. \tag{11.70}$$

Of course, Eq. (11.70) is in general *not* satisfied for a vector field ξ that is simply a Killing vector of $\boldsymbol{g}$. This means that in the teleparallel equivalent of GRT even if there are Killing vector fields, this in general do not warrant that there are conservation laws as in Eq. (9.59) involving *only* the energy and angular momentum tensors of *matter*.

Next, we remark that from $\mathcal{L}'_{g}$ we get as field equations (in an arbitrary basis, not necessarily the teleparallel one) satisfied by the gravitational field the Eq. (11.60), i.e.,

$$-d\underset{g}{\star}\mathcal{S}^{\mathbf{a}} = \underset{g}{\star}\mathcal{T}^{\mathbf{a}} + \underset{g}{\star}\mathfrak{t}^{\mathbf{a}}, \tag{11.71}$$

with

$$\underset{g}{\star}\mathfrak{t}^{\mathbf{a}} = \underset{g}{\star}t^{\mathbf{a}} + m^{2}\underset{g}{\star}\mathfrak{g}^{\mathbf{a}} \tag{11.72}$$

and $\mathcal{S}^{\mathbf{a}}$ and $t^{\mathbf{a}}$ given in Eq. (9.84) where it must also be taken into account that in the teleparallel equivalent of GRT and using the teleparallel basis the Levi-Civita connection 1-forms $\omega^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ there must be substituted by $-\kappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$, with

$$\kappa^{\mathbf{cd}} = -\frac{1}{2}\left[\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathbf{c}} - \mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} d\mathfrak{g}^{\mathbf{d}} + \left(\mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} (\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} d\mathfrak{g}_{\mathbf{a}})\right)\mathfrak{g}^{\mathbf{a}}\right]$$
$$= -\frac{1}{2}\left[\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} \Theta^{\mathbf{c}} - \mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} \Theta^{\mathbf{d}} + \left(\mathfrak{g}^{\mathbf{c}} \underset{g}{\lrcorner} (\mathfrak{g}^{\mathbf{d}} \underset{g}{\lrcorner} \Theta_{\mathbf{a}})\right)\mathfrak{g}^{\mathbf{a}}\right], \tag{11.73}$$

where $\kappa^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = K^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bc}}\mathfrak{g}^{\mathbf{c}}$, with $K^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bc}}$ the components of the so called contorsion tensor.[17] We have,

$$\star\, t^{\mathbf{c}} = \frac{1}{2}\kappa_{\mathbf{ab}} \wedge [\kappa^{\mathbf{c}\cdot}_{\cdot\mathbf{d}} \wedge \star(\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{c}}) + \kappa^{\mathbf{b}\cdot}_{\cdot\mathbf{d}} \wedge \star(\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{c}})]. \tag{11.74}$$

Under a change of gauge, $\mathfrak{g}^{\mathbf{a}} \mapsto \mathfrak{g}'^{\mathbf{a}} = U\mathfrak{g}^{\mathbf{a}}U = \Lambda^{\mathbf{a}}_{\mathbf{b}}\mathfrak{g}^{\mathbf{b}}$ ($U \in \sec \mathrm{Spin}^{e}_{1,3}(M) \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, $\Lambda^{\mathbf{a}}_{\mathbf{b}}(x) \in \mathrm{SO}^{e}_{1,3}$, $\forall\, x \in M$), we have that $\Theta^{\mathbf{a}} \mapsto \Theta'^{\mathbf{a}} = \Lambda^{\mathbf{a}}_{\mathbf{b}}\Theta^{\mathbf{b}}$. It follows that the $\mathfrak{t}^{\mathbf{a}}_{\mathbf{b}}$, which are the components of the energy-momentum 1-forms $\mathfrak{t}^{\mathbf{a}} = \mathfrak{t}^{\mathbf{a}}_{\mathbf{b}}\mathfrak{g}^{\mathbf{b}}$ defines a tensor field.

We then conclude that for each gravitational field modelled by a particular teleparallel spacetime, if the cosmological term is null or not there is a conservation law of energy-momentum for the coupled system of the matter field and the gravitational field which is represented by that *particular* teleparallel spacetime. Although the existence of such a conservation law in the teleparallel spacetime is a satisfactory fact with respect of the usual formulation of the gravitational theory where gravitational fields are modelled by Lorentzian spacetimes and where genuine conservation laws (in general) do not exist because in that theory the components of $t^{\mathbf{a}}$ defines only a pseudo-tensor, we cannot forget observation (a): the teleparallel equivalent of GRT as formulated, e.g., by Maluf [22] or Andrade et al. [10] consists in nothing more than a trivial introduction of: (1) a bilinear form (a deformed metric tensor) $\boldsymbol{g} = \eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ and (2) a teleparallel connection in the manifold $M \simeq \mathbb{R}^4$ of Minkowski spacetime structure. The crucial ingredient is still the old and good Einstein-Hilbert Lagrangian density.

Finally we must remark that if we insist in working with a teleparallel spacetime we lose in general the other six genuine angular momentum conservation laws which always hold in Minkowski spacetime. Indeed, we do not obtain in general even the chart dependent angular momentum 'conservation' law of GRT. The reason is that if we write the equivalent of Eq. (11.60) in a chart (U, φ) with coordinates $\{x^{\mu}\}$ for $U \subset M$ we did not get in general that $dx^{\mu} \wedge \star t^{\nu} = dx^{\nu} \wedge \star t^{\mu}$, which as well known is necessary in order to have a chart dependent angular momentum conservation law [40].

[17] See, Eq. (4.197) with $Q_{\alpha\beta\gamma} = 0$.

11.8 On Clifford's Little Hills

Even if we leave clear that there are at present time not a single indication that the topology of our spacetime is different from $\mathbb{R}^4$we cannot leave out the possibility that its topology is more complicated, in particular that it is modelled by a Lorentzian brane (See Chap. 5) living in a large dimensional manifold. One interesting aspect of this possibility is to transform the "wood" part of Einstein equation in "marble". Let us see how to do this applying the notable formula

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\ (v) = \mathcal{R}(v) = -\mathbf{S}^2(v)$$

(see Eq. (5.9)) of brane theory to General Relativity. As we will see this permits to give a mathematical formalization to Clifford's intuition[18] presented in [6], namely that:

(1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.
(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.
(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

To proceed. let $(M, \boldsymbol{g}, D, \tau_g, \uparrow)$ be a model of a gravitational field generated by an energy momentum tensor $\boldsymbol{T} := T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}$ describing all matter of the universe according to General Relativity theory. As well already know Einstein equation can be written as

$$\boldsymbol{\partial} \wedge \partial\ \theta^{\mathbf{a}} = -T^{\mathbf{a}} + \frac{1}{2} T\theta^{\mathbf{a}}, \tag{11.75}$$

where $T^{\mathbf{a}} := T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$ and $T = T^{\mathbf{a}}_{\mathbf{a}}$, with $T^{\mathbf{a}}_{\mathbf{b}}$. If we suppose that the structure $(M, \boldsymbol{g})$ is a submanifold of $(\mathring{M} \simeq \mathbb{R}^n, \mathring{\boldsymbol{g}})$ for n large enough as discussed in the beginning of Sect. 5.3 we can write Eq. (11.75) taking into account Eq. (5.146) as

$$\mathbf{S}^2(\theta^{\mathbf{a}}) = T^{\mathbf{a}} - \frac{1}{2} T\theta^{\mathbf{a}}. \tag{11.76}$$

Thus, in a region where there is no matter $\mathbf{S}^2(\theta^{\mathbf{a}}) = 0$, despite the fact that $\mathbf{S}(\theta^{\mathbf{a}}) = \mathcal{S}(\theta^{\mathbf{a}})$ may be *non* null. So, a being living in the hyperspace $\mathbb{R}^n$ and looking at our brane world will see the little hills (i.e., "matter") are special shapes in M, places

[18]Taking into account, of course, that differently from Clifford's idea, instead of a space theory of matter, we must talk about a spacetime theory of matter.

where the $\mathbf{S}^2(\theta^{\mathbf{a}}) \neq 0$ which act as sources for $\mathbf{P}(\mathring{\boldsymbol{\partial}} \lrcorner \mathcal{S}(\theta^{\mathbf{a}}))$ since $\mathbf{P}(\mathring{\boldsymbol{\partial}} \lrcorner \mathcal{S}(\theta^{\mathbf{a}})) = -\mathbf{S}^2(\theta^{\mathbf{a}})$.

Remark 11.4 To properly appreciate the above argument one must take in mind that the shape extensor depends for its definition on the metric $\mathring{\boldsymbol{g}}$ and the Levi-Civita $\mathring{D}$ connection of $\mathring{\boldsymbol{g}}$ used in $\mathring{M}$. So, a different choice of metric in $\mathring{M}$ will imply in Clifford's little hills to be represented by different shape extensors. Despite this fact, it seems to us that shape is most appealing than the curvature biform $\mathfrak{R}$[19] or the Ricci 1-form fields $\mathcal{R}^{\mathbf{a}} = \boldsymbol{\partial} \wedge \boldsymbol{\partial}\ \theta^{\mathbf{a}}$ as indicator of the presence of matter as distortions in the world brane M. Indeed, inner observers living in M in general may not have enough skills and technology to discover the topology of M and so cannot know if their brane world is a bended surface in the hyperspace (i.e., $\mathring{M}$) or even if a open set $U \subset M$ is a part of an hyperplane or not. Moreover, those inner observers that have learned a little bit of differential geometry know that they cannot say that their manifold is curved based on the fact that the curvature biform is non null, for they know that the curvature biform is a property of the connection (parallelism rule) that they decide to use by convention in M and not an intrinsic property of M. They know that if they choose a different connection it may happen that its curvature biform may be null and their connection (not their manifold) may have torsion and even a non null nonmetricity tensor[20] So, with their knowledge of differential geometry they infer that little hills (as seems for beings living in $\mathring{M}$) can only be associated to the shape extensor if they use Levi-Civita connection of g in M.

11.9 A Maxwell Like Equation for a Brane World with a Killing Vector Field

When $(M, \boldsymbol{g})$ admits a Killing vector field[21] $\boldsymbol{A} \in \sec TM$ then it follows [30] that $\delta A = 0$, where $A = \boldsymbol{g}(\boldsymbol{A},\ ) \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. In this case we can show that the Ricci operator applied to A is equal to the covariant D'Alembertian operator applied to A, i.e.,

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A = \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, A \tag{11.77}$$

Now, recalling Eq. (5.7) that the square of the Dirac operator $\boldsymbol{\partial}^2$ can be decomposed in two ways, i.e.,

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\, A + \boldsymbol{\partial} \cdot \boldsymbol{\partial}\, A = \boldsymbol{\partial}^2 A = -d\delta A - \delta d A \tag{11.78}$$

[19] Recall that $\mathfrak{R}$ is in general non null even in vacuum.

[20] Details about these possibilities are discussed in [11] where a theory of the gravitational field on a brane diffeomorphic to R^4 is discussed.

[21] See more details in Chap. 15.

we have writing $F = dA$ and taking into account that $\delta A = 0$ that Einstein equation can be written as

$$\delta F = 2\mathbf{S}^2(A) \tag{11.79}$$

and since $dF = ddA = 0$ we can write Einstein equation as:

$$\boldsymbol{\partial} F = -2\mathbf{S}^2(A). \tag{11.80}$$

Equation (11.80) shows that in a Lorentzian brane M of dim 4 which contains a Killing vector field $\boldsymbol{A}$, Einstein equation is encoded in an "electromagnetic like field" F having as source a current $J = 2\mathbf{S}^2(A) \in \sec \mathcal{C}\ell(M, \mathtt{g})$.

Exercise 11.5 Prove Eq. (11.77).

11.10 Conclusions

This is a good point to end this chapter. Our intention in this book was only to present some critical aspects of GRT theory (mainly using the Clifford bundle formalism when convenient) and to discuss matters of *principle*, in particular to let the reader aware of some very controversial issues concerning the orthodox interpretation of Einstein's theory, as, e.g., the case of the energy momentum 'conservation'. We showed that this particular problem can be solved if we interpret the gravitational field as a physical field living in Minkowski spacetime. The gravitational field, when the graviton mass is zero, creates an effective Lorentzian geometry where probe particles and probe fields move. In such theory there are no exotic topologies, black holes,[22] worm holes, no possibility for time-machines,[23]

[22]On this issue recall Sect. 6.9 keeping in mind that there are articles criticizing the notion that black holes are predictions of GRT due mainly to some mathematical misunderstandings as, e.g.[1, 4, 36] and/or physical grounds. When thinking on this issue take also into account the 'pasticcio' concerning the black hole information 'paradox' (see, [15, 17]) and its possible resolutions with the suggested existence of a "complementarity principle" [37] or existence of firewalls [2] as an indication that the foundations of GRT and its relation to other theories of Physics are not well understood as some people would like us to think. Recently adding stuff to the "pasticcio" Hawking [16] is claiming that "The absence of event horizons mean that there are no black holes—in the sense of regimes from which light can't escape to infinity". But this statement seems to be already an old idea. More information at http://asymptotia.com/2014/01/30/hawking-an-old-idea/ and http://www.physics.ohio-state.edu/~mathur/.

[23]The possibility for time machines arises when closed timelike curves exist in a Lorentzian manifold. Such exotic configurations, it is *said*, already appears in Gödel's universe model. However, a recent thoughtful analysis by Cooperstock and Tieu (which we endorse) shows that the old claim is wrong. Authors like, e.g, Davies [9] (which are proposing to build time machines even at home), Gott [14] and Novikov [28] are invited to read [7] and find a error in the argument of those authors.

etc., which according to our opinion are pure science fiction objects. Eventually, many will not like the viewpoint just presented,[24] but we feel that many will become interested in exploiting new ideas, which may be more close to the way Nature operates. We hope that our study clarifies the real difference between mathematical models and physical reality and leads people to think about the real physical nature of the gravitational field (and also of the electromagnetic field[25]). We briefly discussed also an Hamiltonian formalism for our theory and the concept of energy defined by Eq. (11.25) and the one given by the ADM formalism, which are shown to coincide.

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[24] For those people in that class we offer Chap. 12.

[25] As suggested, e.g., by the works of Laughlin [19] and Volikov [41]. Of course,, it may be necessary to explore also other ideas, like e.g., existence of Lorentzian branes in string theory or generalizations.

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Chapter 12
On the Many Faces of Einstein Equations

Abstract This chapter gives a Clifford bundle approach to a theory of the gravitational field where this field is interpreted as the curvature of a $\mathrm{Sl}(2,\mathbb{C})$ gauge theory. The proposal of this presentation is to emphasize the many faces of Einstein equation and in the development of the theory a collection of some non trivial and useful formulas is derived and some misconceptions in presentations of the theory of the gravitational field as a $\mathrm{Sl}(2,\mathbb{C})$ gauge theory is discussed and fixed.

12.1 Introduction

Even if we leave clear in the previous chapters our opinion that the gravitational field must be interpreted as a physical field in Minkowski spacetime, we decided to present here how the orthodox Einstein theory can be formulated in a way that *resembles* the gauge theories of particle physics, in particular a gauge theory with gauge group $\mathrm{Sl}(2,\mathbb{C})$. This exercise will reveal yet another face of Einstein's equations, besides the ones already discussed in previous chapters.[1] For our presentation we introduce mathematical objects called Clifford valued differential forms (*cliforms*) and a new operator $\mathcal{D}$ called the *fake exterior covariant differential*[2] (FECD) acting on them. Moreover, with our formalism we show that Einstein's equations can be put in a form that *apparently* resembles Maxwell equations written in coordinates and using the covariant of a Lorentzian spacetime structure $(\boldsymbol{M}, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$[3] and which is very different from Eq. (11.80) of Chap. 11 which

[1]Of course, this will not exhaust the possible faces of Einstein's equations, for finding new faces depends mainly on authors mathematical knowledge and imagination. Also, the reader is here advised that this chapter do not intend to be a presentation of the many developments that goes under the name of gauge theories of gravitation. The interested reader on this issue may consult, e.g., [9, 12].

[2]The name is due to the fact that $\mathcal{D}$ does not always satisfy a Leibniz type rule when applied to the $\otimes_\wedge$product of *arbitrary* cliforms.

[3]Such form of Einstein's equations leaded some people *equivocally* [17, 18] to think that they achieved an unified theory of the gravitational and electromagnetic field. A discussion of that issue may be found in [6, 15].

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really encodes Einstein equation in a Maxwell like equation when $(M, \boldsymbol{g})$ admits a Killing vector field.[4]

12.2 Preliminaries

Let $(M, \boldsymbol{g}, \boldsymbol{\nabla}, \tau_{\boldsymbol{g}}, \uparrow)$ be a Riemann-Cartan spacetime. We already introduced the Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathtt{g})$. To present from where 'Sl$(2, \mathbb{C})$ Gauge Theories of Gravitation' come from it will be useful to introduce Clifford valued differential forms. For presenting these objects we need to introduce besides the Clifford bundle of multiforms $\mathcal{C}\ell(M, \mathtt{g})$, also the Clifford bundle of multivector fields that will be denoted in what follows by $\mathcal{C}\ell(M, \boldsymbol{g})$

$$\mathcal{C}\ell(M, \boldsymbol{g}) = \bigcup_{x \in M} \mathcal{C}\ell(T_x M, \boldsymbol{g}). \tag{12.1}$$

This bundle has, of course, an analogous structure as the bundle $\mathcal{C}\ell(M, \mathtt{g})$ of multiform fields and all products, contractions, etc., are defined in analogous way, and we shall use the same symbols for them, since this procedure produces no confusion. If $\{\boldsymbol{e}_{\mathbf{a}}\} \in \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ is an orthonormal basis for $TM = \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g})$ we have

$$\boldsymbol{e}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{b}} + \boldsymbol{e}_{\mathbf{b}} \boldsymbol{e}_{\mathbf{a}} = 2\eta_{\mathbf{ab}}. \tag{12.2}$$

Moreover, we introduce $\{\boldsymbol{e}^{\mathbf{a}}\}$ as the *reciprocal basis* of $\{\boldsymbol{e}_{\mathbf{a}}\}$, i.e., $\boldsymbol{e}_{\mathbf{a}} \cdot \boldsymbol{e}^{\mathbf{b}} = \delta^{\mathbf{b}}_{\mathbf{a}}$. A general section $\boldsymbol{m} \in \sec \mathcal{C}\ell(M, \boldsymbol{g})$, called a (nonhomogeneous) multivector field has, e.g., the expansion

$$\boldsymbol{m} = s + v_{\mathbf{i}} \boldsymbol{e}^{\mathbf{i}} + \frac{1}{2!} b_{\mathbf{ij}} \boldsymbol{e}^{\mathbf{i}} \boldsymbol{e}^{\mathbf{j}} + \frac{1}{3!} t_{\mathbf{ijk}} \boldsymbol{e}^{\mathbf{i}} \boldsymbol{e}^{\mathbf{j}} \boldsymbol{e}^{\mathbf{k}} + p \boldsymbol{e}^{\mathbf{5}}, \tag{12.3}$$

where $\boldsymbol{e}^{\mathbf{5}} = \boldsymbol{e}^{\mathbf{0}} \boldsymbol{e}^{\mathbf{1}} \boldsymbol{e}^{\mathbf{2}} \boldsymbol{e}^{3}$ is the pseudoscalar generator and

$$s, v_{\mathbf{i}}, b_{\mathbf{ij}}, t_{\mathbf{ijk}}, p \in \sec \bigwedge^0 TM \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}). \tag{12.4}$$

To motivate the introduction of Clifford valued differential forms we shall need to recall some results from the general theory of connections (see Appendix) adapted for the case of the spacetime structure $(M, \boldsymbol{g}, \boldsymbol{\nabla}, \tau_{\boldsymbol{g}}, \uparrow)$.

In the Appendix we learned that a connection (a gauge potential) is a 1-form in the cotangent space of a principal bundle, with values in the Lie algebra of a gauge

[4] See also Chap. 15.

group. For a gauge theory of the gravitational field it seems natural (at least at first sight[5]) to take as gauge potential the linear connection[6]

$$\overset{\blacktriangle}{\boldsymbol{\omega}} \in \sec T^{*}\mathbf{P}_{\mathrm{SO}_{1,3}^{e}}(M) \otimes \mathrm{sl}(2,\mathbb{C}), \tag{12.5}$$

which determines *exterior* covariant derivative operators acting on sections of associated vector bundles to the principal bundle $\mathbf{P}_{\mathrm{SO}_{1,3}^{e}}(M)$. However, a moment of reflection shows that $\overset{\blacktriangle}{\boldsymbol{\omega}}$ cannot be the unique ingredient of our theory, since, if our objective is to end with a geometrical spacetime structure $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ modeling a gravitational field we needs to be able to reproduce the well known results obtained with the usual covariant derivative of tensor fields in the base manifold. For that we need besides $\overset{\blacktriangle}{\boldsymbol{\omega}}$ also the *soldering* form

$$\overset{\blacktriangle}{\boldsymbol{\theta}} \in \sec T^{*}\mathbf{P}_{\mathrm{so}_{1,3}^{e}}(M) \otimes \mathbb{R}^{1,3}. \tag{12.6}$$

Let be $U \subset M$ and let $\varsigma : U \to \varsigma(U) \subset \mathbf{P}_{\mathrm{so}_{1,3}^{e}}(M)$. We are here interested in the pullbacks $\varsigma^{*}\overset{\blacktriangle}{\boldsymbol{\omega}}$ and $\varsigma^{*}\overset{\blacktriangle}{\boldsymbol{\theta}}$ once we give a local trivialization of the respective bundles. As it is well known [5, 10], in local coordinates $\{x^{\mu}\}$ covering U and with $\{e_{\mu} = \partial_{\mu},\}$ a basis for TU, $\varsigma^{*}\overset{\blacktriangle}{\boldsymbol{\theta}}$ uniquely determines the tensor field[7]

$$\boldsymbol{\theta} = e_{\mu} \otimes dx^{\mu} \equiv e_{\mu}dx^{\mu} \in \sec TM \otimes \bigwedge^{1} T^{*}M. \tag{12.7}$$

Remark 12.1 In Eq. (12.7) and other formulas involving Clifford valued differential forms we will omit the symbol $\otimes$ when no confusion arises.

Now, if we give: (1) the Clifford algebra structure to the tangent bundle, thus generating the Clifford bundle $\mathcal{C}\ell(M, \boldsymbol{g})$ and; (2) recall moreover (see Sect. 3.3.4) that for each $x \in U \subset M$ the bivectors of $\mathcal{C}\ell(T_xM, \mathtt{g}_x)$ generate under the product defined by the commutator, the Lie algebra [11] $\mathrm{spin}_{1,3}^{e} \simeq \mathrm{sl}(2,\mathbb{C})$, we are naturally lead to define the representatives in $\mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge T^{*}M$ for $\boldsymbol{\theta}$ and for the pullback

[5] We are not going to discuss this issue here. Such a deficiency may be supplied by a careful and lucid analysis of the problem of formulating GRT as a possible gauge theory with gauge groups $\mathrm{Sl}(2,\mathbb{C})$ or T(4) as done by Wallner [20].

[6] In words, $\overset{\blacktriangle}{\boldsymbol{\omega}}$ is a 1-form in the cotangent space of the bundle of orthonormal frames with values in the Lie algebra $\mathrm{so}_{1,3}^{e} \simeq \mathrm{sl}(2,\mathbb{C})$ of the group $\mathrm{SO}_{1,3}^{e}$.

[7] Note that this tensor is the identity tensor acting on the space of vector fields on $U \subset M$. We denoted the identity tensor by $\mathtt{g}$ in Chap. 11.

$\boldsymbol{\omega} = \varsigma^* \overset{\blacktriangle}{\boldsymbol{\omega}}$ of the connection $\overset{\blacktriangle}{\boldsymbol{\omega}}$ *in a given gauge*.

$$\begin{aligned}
\boldsymbol{\theta} &= e_\mu dx^\mu = \boldsymbol{e}_{\mathbf{a}}\theta^{\mathbf{a}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^1 T^*M, \\
\boldsymbol{\omega} &= \frac{1}{2}\omega_{\cdot\mathbf{a}\cdot}^{\mathbf{b}\cdot\mathbf{c}} \boldsymbol{e}_{\mathbf{b}}\boldsymbol{e}_{\mathbf{c}}\theta^{\mathbf{a}} \\
&= \frac{1}{2}(\omega_{\cdot\mathbf{a}\cdot}^{\mathbf{b}\cdot\mathbf{c}} \boldsymbol{e}_{\mathbf{b}} \wedge \boldsymbol{e}_{\mathbf{c}}) \otimes \theta^{\mathbf{a}} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^1 T^*M.
\end{aligned} \tag{12.8}$$

From Appendix A.4 we recall that whereas $\boldsymbol{\theta}$ is a true tensor, $\boldsymbol{\omega}$ is not a true tensor, since its 'components' do not have the appropriate tensor transformation properties. Indeed, the $\omega_{\cdot\mathbf{a}\cdot}^{\mathbf{b}\cdot\mathbf{c}}$ are the 'components' of the connection relative to the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$. They are defined by

$$\nabla_{\boldsymbol{e}_{\mathbf{a}}} \boldsymbol{e}^{\mathbf{b}} = -\omega_{\cdot\mathbf{ac}}^{\mathbf{b}\cdot\cdot} \boldsymbol{e}^{\mathbf{c}}, \quad \omega_{\mathbf{abc}} = -\omega_{\mathbf{cba}} = \eta_{\mathbf{ad}}\omega_{\cdot\mathbf{bc}}^{\mathbf{d}\cdot\cdot}, \tag{12.9}$$

where $\nabla_{\boldsymbol{e}_{\mathbf{a}}}$ is a metric compatible covariant derivative operator defined on the tensor bundle and that acts naturally on $\mathcal{C}\ell(M, \boldsymbol{g})$ as we already learned in previous chapters. Objects like $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ will be called Clifford valued differential forms (or Clifford valued forms, or yet *cliforms*, for short), and below we give a detailed account of the algebra and calculus of that objects. Let us now recall some additional concepts of the theory of linear connections in the form appropriated for our enterprise.

12.2.1 Exterior Covariant Differential

To achieve our objectives of presenting a 'gauge like' formulation of the theory of the gravitational field we need to find an operator which acts naturally on sections of cliforms and which *mimics* the action of the pullback of the exterior covariant derivative operator acting on sections of a vector bundle associated with the principal bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$, once a linear metric compatible connection is given. This operator is introduced below and called fake exterior differential operator (FECD). It is used in the calculations of curvature bivectors, Bianchi identities, etc. With the FECD $\mathcal{D}$ and its associated operator $\mathcal{D}_{\boldsymbol{e}_{\mathbf{r}}}$ we can formulate Einstein's theory in such a way that the resulting equations looks like the equations for a gauge theory with $\mathrm{Sl}(2, \mathbb{C})$ as the gauge group. Before introducing $\mathcal{D}$ we first recall the concept of the exterior covariant derivative (differential) acting on arbitrary sections of a vector bundle $E(M)$ associated with $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ (having as typical fiber a l-dimensional real vector space) and on $\mathrm{End}E(M) = E(M) \otimes E^*(M)$, the bundle of endomorphisms of $E(M)$ introduced in Appendix A.5 and next the concept of absolute differential acting on sections of the tensor bundle, for the particular case of $\bigwedge^l TM$. In what follows we use a simplified notation in relation to the general one in Appendix A.5

Now, from Appendix A.5.4 we know that the exterior covariant differential operator $\mathbf{d}^E$ acting on sections of $E(M)$ and End$E(M)$ is the mapping

$$\mathbf{d}^E : \sec E(M) \to \sec E(M) \otimes \bigwedge^1 T^*M, \tag{12.10}$$

such that for any differentiable function $f : M \to \mathbb{R}$, $A \in \sec E(M)$ and any $F \in \sec \mathrm{End}E(M) \otimes \bigwedge^p T^*M)$, $G \in \sec \mathrm{End}E(M) \otimes \bigwedge^q T^*M$ we have[8]:

$$\begin{aligned}
\mathbf{d}^E(fA) &= df \otimes A + f\mathbf{d}^E A,\\
\mathbf{d}^E(F \otimes_\wedge A) &= \mathbf{d}^E F \otimes_\wedge A + (-1)^p F \otimes_\wedge \mathbf{d}^E A,\\
\mathbf{d}^E(F \otimes_\wedge G) &= \mathbf{d}^E F \otimes_\wedge G + (-1)^p F \otimes_\wedge \mathbf{d}^E G.
\end{aligned} \tag{12.11}$$

In Eq. (12.11), writing $F = F^a \otimes f_a^{(p)}$, $G = G^b \otimes g_b^{(q)}$ where F^a, $G^b \in \sec \mathrm{End}E(M)$, $f_a^{(p)} \in \sec \bigwedge^p T^*M$ and $g_b^{(q)} \in \sec \bigwedge^q T^*M$ we have

$$\begin{aligned}
F \otimes_\wedge A &= \left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge A,\\
F \otimes_\wedge G &= \left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge G^b \otimes g_b^{(q)},
\end{aligned} \tag{12.12}$$

with the product $\otimes_\wedge$ given through Definition A.68 in Appendix A.5.4. In order to simplify even more the notation we eventually use when there is no possibility of confusion, the simplified (sloppy) notation

$$\begin{aligned}
(F^a A) \otimes f_a^{(p)} &\equiv (F^a A) f_a^{(p)},\\
\left(F^a \otimes f_a^{(p)}\right) \otimes_\wedge G^b \otimes g_b^{(q)} &= \left(F^a G^b\right) f_a^{(p)} \wedge g_b^{(q)},
\end{aligned} \tag{12.13}$$

where $F^a A \in \sec E(M)$ and $F^a G^b$ means the composition of the respective endomorphisms.

Let $U \subset M$ be an open subset of M, $\{\mathbf{x}^\mu\}$ coordinate functions of a maximal atlas of M, $\{e_\mu\}$ a coordinate basis of $TU \subset TM$ and $\{s_\mathbf{K}\}$, $\mathbf{K} = 1, 2, \ldots l$ a basis for any $\sec E(U) \subset \sec E(M)$. Then, a basis for any section of $E(M) \otimes \bigwedge^1 T^*M$ is given by $\{s_\mathbf{K} \otimes dx^\mu\}$.

Recall also, that the covariant derivative operator $\nabla_{e_\mu} : \sec E(M) \to \sec E(M)$ is given by

$$\mathbf{d}^E A := \left(\nabla_{e_\mu} A\right) \otimes dx^\mu, \tag{12.14}$$

where, writing $A = A^\mathbf{K} \otimes s_\mathbf{K}$ we have

$$\nabla_{e_\mu} A = \partial_\mu A^\mathbf{K} \otimes s_\mathbf{K} + A^\mathbf{K} \otimes \nabla_{e_\mu} s_\mathbf{K}. \tag{12.15}$$

[8]Recall that the product $\otimes_\wedge$ is given in Definition A.68.

Now, let examine the case where $E(M) = TM \equiv \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g})$. Let $\{\boldsymbol{e}_\mathbf{j}\}$, be an orthonormal basis of TM. Then, using Eqs. (12.14) and (12.9)

$$\begin{aligned} \mathbf{d}^E \boldsymbol{e}_\mathbf{j} &= (\nabla_{\boldsymbol{e}_\mathbf{k}} \boldsymbol{e}_\mathbf{j}) \otimes \theta^\mathbf{k} \equiv \boldsymbol{e}_\mathbf{k} \otimes \boldsymbol{\omega}_{\cdot\mathbf{j}}^{\mathbf{k}\cdot}, \\ \boldsymbol{\omega}_{\cdot\mathbf{j}}^{\mathbf{k}\cdot} &= \omega_{\cdot\mathbf{rj}}^{\mathbf{k}\cdot\cdot} \theta^\mathbf{r}, \end{aligned} \tag{12.16}$$

where the $\boldsymbol{\omega}_{\cdot\mathbf{j}}^{\mathbf{k}\cdot} \in \sec \bigwedge^1 T^*M$ are the connection 1-forms.

Also, for $\mathbf{v} = v^\mathbf{i} \boldsymbol{e}_\mathbf{i} \in \sec TM$, we have

$$\begin{aligned} \mathbf{d}^E \mathbf{v} &= \nabla_{\boldsymbol{e}_\mathbf{i}} \mathbf{v} \otimes \theta^\mathbf{i} = \boldsymbol{e}_\mathbf{i} \otimes \mathbf{d}^E v^\mathbf{i}, \\ \mathbf{d}^E v^\mathbf{i} &= dv^\mathbf{i} + \boldsymbol{\omega}_{\cdot\mathbf{k}}^{\mathbf{i}\cdot} v^\mathbf{k}. \end{aligned} \tag{12.17}$$

12.2.2 Absolute Differential

Now, let $E(M) = \bigwedge^l TM \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g})$. In this case, $\mathbf{d}^E$ is the usual *absolute differential* ∇ of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, i.e., it is a mapping (see, e.g., [5])

$$\nabla : \sec \bigwedge^l TM \to \sec \bigwedge^l TM \otimes \bigwedge^1 T^*M, \tag{12.18}$$

such that for any differentiable $A \in \sec \bigwedge^l TM$ we have

$$\nabla A = (\nabla_{\boldsymbol{e}_\mathbf{i}} A) \otimes \theta^i, \tag{12.19}$$

where $\nabla_{e_\mathbf{i}} A$ is the standard covariant derivative of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Also, for any differentiable function $f : M \to \mathbb{R}$, and differentiable $A \in \sec \bigwedge^l TM$ we have

$$\nabla(fA) = df \otimes A + f\nabla A. \tag{12.20}$$

Now, if we suppose that the orthonormal basis $\{\boldsymbol{e}_\mathbf{j}\}$ of TM is such that each $\boldsymbol{e}_\mathbf{j} \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g})$, we can easily find using the Clifford algebra structure of the space of multivectors that we can write:

$$\begin{aligned} \nabla \boldsymbol{e}_\mathbf{j} &= (\nabla_{\boldsymbol{e}_\mathbf{k}} \boldsymbol{e}_\mathbf{j}) \otimes \theta^\mathbf{k} = \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{e}_\mathbf{j}] = -\boldsymbol{e}_\mathbf{j} \lrcorner \boldsymbol{\omega} \\ \boldsymbol{\omega} &= \frac{1}{2} \omega_{\cdot\mathbf{k}\cdot}^{\mathbf{a}\cdot\mathbf{b}} \boldsymbol{e}_\mathbf{a} \wedge \boldsymbol{e}_\mathbf{b} \otimes \theta^\mathbf{k} \\ &\equiv \frac{1}{2} \omega_{\cdot\mathbf{k}\cdot}^{\mathbf{a}\cdot\mathbf{b}} \boldsymbol{e}_\mathbf{a} \boldsymbol{e}_\mathbf{b} \otimes \theta^\mathbf{k} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^1 T^*M, \end{aligned} \tag{12.21}$$

where $\boldsymbol{\omega}$ is the *representative* of the connection in the given gauge.

The general case is given by the following proposition:

Proposition 12.2 *For* $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell\,(M, \boldsymbol{g})$ *we have*

$$\nabla A = dA + \frac{1}{2}[\boldsymbol{\omega}, A]. \tag{12.22}$$

Proof Left to the reader. Note that A is to be considered as a $\bigwedge^l TM$-valued 0-form. ■

Equation (12.22) can now be extended by linearity for an arbitrary nonhomogeneous multivector $A \in \sec \mathcal{C}\ell\,(TM, \mathtt{g})$.

We proceed now to find an appropriate *exterior* covariant differential which acts naturally on Clifford valued differential forms, i.e., objects that are sections of $\bigwedge T^*M \otimes \mathcal{C}\ell(M, \boldsymbol{g})$ ($\equiv \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge T^*M$) (see next section). Note that we cannot simply use the above definition putting $E\,(M) = \mathcal{C}\ell(M, \boldsymbol{g})$ and $\mathrm{End}E(M) = \mathrm{End}\mathcal{C}\ell(M, \boldsymbol{g})$, because $\mathrm{End}\mathcal{C}\ell(M, \boldsymbol{g}) \neq \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge T^*M$. Instead, we must find inspiration in the general theory in order to find an appropriate definition. Let us see how this can be done.

12.3 Clifford Valued Differential Forms

Definition 12.3 A homogeneous multivector valued differential form of type (l, p) is a section of $\bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(TM, \boldsymbol{g}) \otimes \bigwedge T^*M$, for $0 \leq l \leq 4$, $0 \leq p \leq 4$. A section of $\mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge T^*M$ such that the multivector part is non homogeneous is called a Clifford valued differential form (cliform).

We recall, that any $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^p T^*M$ can always be written as

$$\begin{aligned}
A = m_{(l)} \otimes \psi^{(p)} &\equiv \frac{1}{l!} m_{(l)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \boldsymbol{e}_{\mathbf{i}_1} \cdots \boldsymbol{e}_{\mathbf{i}_l} \otimes \psi^{(p)} \\
&= \frac{1}{p!} m_{(l)} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \cdots \wedge \theta^{\mathbf{j}_p} \\
&= \frac{1}{l!p!} m_{(l)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \boldsymbol{e}_{\mathbf{i}_1} \ldots \boldsymbol{e}_{\mathbf{i}_l} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \cdots \wedge \theta^{\mathbf{j}_p} \\
&= \frac{1}{l!p!} A^{\mathbf{i}_1 \ldots \mathbf{i}_l}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \boldsymbol{e}_{\mathbf{i}_1} \ldots \boldsymbol{e}_{\mathbf{i}_l} \otimes \theta^{\mathbf{j}_1} \wedge \cdots \wedge \theta^{\mathbf{j}_p}.
\end{aligned} \tag{12.23}$$

Moreover, recall that (See Definition A.68) the $\otimes_\wedge$ product of $A = \overset{m}{A} \otimes \psi^{(p)} \in \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^p T^*M$ and $B = \overset{m}{B} \otimes \chi^{(q)} \in \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^q T^*M$ is the mapping[9]:

$$\otimes_\wedge : \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^p T^*M \times \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^q T^*M$$
$$\to \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^{p+q} T^*M,$$
$$A \otimes_\wedge B = \overset{m}{A}\overset{m}{B} \otimes \psi^{(p)} \wedge \chi^{(q)}. \tag{12.24}$$

Definition 12.4 The commutator $[A,B]$ of $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^p T^*M$ and $B \in \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^q T^*M$ is the mapping:

$$[\quad,\quad] : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \times \sec \bigwedge^m TM \otimes \bigwedge^q T^*M$$
$$\to \sec \sum_{k=0}^{\frac{1}{2}[l+m-|l-m|]} \bigwedge^{|l-m|+2k} TM \otimes \bigwedge^{p+q} T^*M,$$
$$[A,B] = A \otimes_\wedge B - (-1)^{pq} B \otimes_\wedge A \tag{12.25}$$

Writing $A = \frac{1}{l!} A^{\mathbf{j}_1 \ldots \mathbf{j}_l} \boldsymbol{e}_{\mathbf{j}_1} \ldots \boldsymbol{e}_{\mathbf{j}_l} \psi^{(p)}$, $B = \frac{1}{m!} B^{\mathbf{i}_1 \ldots \mathbf{i}_m} \boldsymbol{e}_{\mathbf{i}_1} \ldots \boldsymbol{e}_{\mathbf{i}_m} \chi^{(q)}$, with $\psi^{(p)} \in \sec \bigwedge^p T^*M$ and $\chi^{(q)} \in \sec \bigwedge^q T^*M$, we have

$$[A,B] = \frac{1}{l!m!} A^{\mathbf{j}_1 \ldots \mathbf{j}_l} B^{\mathbf{i}_1 \ldots \mathbf{i}_m} \left[\boldsymbol{e}_{\mathbf{j}_1} \cdots \boldsymbol{e}_{\mathbf{j}_l}, \boldsymbol{e}_{\mathbf{i}_1} \cdots \boldsymbol{e}_{\mathbf{i}_m} \right] \psi^{(p)} \wedge \chi^{(q)}. \tag{12.26}$$

The definition of the commutator is extended by linearity to arbitrary sections of $\mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge T^*M$.

Now, we have the following proposition:

Proposition 12.5 *Let* $A \in \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^p T^*M$, $B \in \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^q T^*M$, $C \in A \in \sec \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^r T^*M$. *Then,*

$$[A,B] = (-1)^{1+pq}[B,A], \tag{12.27}$$

and

$$(-1)^{pr} \left[[A,B], C \right] + (-1)^{qp} \left[[B,C], A \right] + (-1)^{rq} \left[[C,A], B \right] = 0. \tag{12.28}$$

Proof It follows directly from a simple calculation, left to the reader. ■

[9]Note that $\overset{m}{A}$ and $\overset{m}{B}$ are general non-homogeneous multivector fields.

Equation (12.28) which is analogous to Eq. (A.45) may be called the *graded Jacobi identity* [3].

Corollary 12.6 *Let* $A^{(2)} \in \sec \bigwedge^2 TM \otimes \bigwedge^p T^*M$ *and* $B \in \sec \bigwedge^r TM \otimes \bigwedge^q T^*M$. *Then,*

$$[A^{(2)}, B] = C, \tag{12.29}$$

where $C \in \sec \bigwedge^r TM \otimes \bigwedge^{p+q} T^*M$.

Proof It follows from a direct calculation, left to the reader. ■

Proposition 12.7 *Let* $\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M$, $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M$, $B \in \sec \bigwedge^m TM \otimes \bigwedge^q T^*M$. *Then, we have*

$$[\boldsymbol{\omega}, A \otimes_\wedge B] = [\boldsymbol{\omega}, A] \otimes_\wedge B + (-1)^p A \otimes_\wedge [\boldsymbol{\omega}, B]. \tag{12.30}$$

Proof Using the definition of the commutator we can write

$$\begin{aligned}
[\boldsymbol{\omega}, A] \otimes_\wedge B &= (\boldsymbol{\omega} \otimes_\wedge A - (-1)^p A \otimes_\wedge \boldsymbol{\omega}) \otimes_\wedge B \\
&= (\boldsymbol{\omega} \otimes_\wedge A \otimes_\wedge B - (-1)^{p+q} A \otimes_\wedge B \otimes_\wedge \boldsymbol{\omega}) \\
&\quad + (-1)^{p+q} A \otimes_\wedge B \otimes_\wedge \boldsymbol{\omega} - (-1)^p A \otimes_\wedge \boldsymbol{\omega} \otimes_\wedge B \\
&= [\boldsymbol{\omega}, A \otimes_\wedge B] - (-1)^p A \otimes_\wedge [\boldsymbol{\omega}, B],
\end{aligned}$$

from where the desired result follows. ■

From Eq. (12.30) we have also[10]

$$\begin{aligned}
&(p+q)[\boldsymbol{\omega}, A \otimes_\wedge B] \\
&\quad = p[\boldsymbol{\omega}, A] \otimes_\wedge B + (-1)^p q A \otimes_\wedge [\boldsymbol{\omega}, B] \\
&\qquad + q[\boldsymbol{\omega}, A] \otimes_\wedge B + (-1)^p p A \otimes_\wedge [\boldsymbol{\omega}, B].
\end{aligned}$$

Definition 12.8 The action of the differential operator d acting on

$$A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g}) \otimes \bigwedge^p T^*M,$$

is given by:

$$\begin{aligned}
dA &:= \boldsymbol{e}_{\mathbf{j}_1} \cdots \boldsymbol{e}_{\mathbf{j}_l} \otimes dA^{\mathbf{j}_1 \ldots \mathbf{j}_l} \\
&= \boldsymbol{e}_{\mathbf{j}_1} \cdots \boldsymbol{e}_{\mathbf{j}_l} \otimes d\frac{1}{p!} A^{\mathbf{j}_1 \ldots \mathbf{j}_l}_{\mathbf{i}_1 \ldots \mathbf{i}_p} \theta^{\mathbf{i}_1} \wedge \cdots \wedge \theta^{\mathbf{i}_p}.
\end{aligned} \tag{12.31}$$

[10] The result printed in [15] is wrong.

We have the important

Proposition 12.9 *Let* $A \in \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^p T^*M$ *and* $B \in \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^q T^*M$. *Then,*

$$d[A, B] = [dA, B] + (-1)^p [A, dB]. \tag{12.32}$$

Proof The proof is a simple calculation, left to the reader. ■

We now define the fake exterior covariant differential operator (FECD) $\mathcal{D}$ and also an associated operator $\mathcal{D}_{\boldsymbol{e}_{\mathbf{r}}}$ acting on a cliform $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^p T^*M$, $(l, p \geq 1)$ as follows.

12.4 Fake Exterior Covariant Differential of Cliforms

Definition 12.10 The fake exterior covariant differential *(FECD)* of $\mathcal{A}$ is the mapping:

$$\begin{aligned} \mathcal{D} : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M &\to \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \otimes_\wedge \bigwedge^1 T^*M \\ &\subset \sec \bigwedge^l TM \otimes \bigwedge^{p+1} T^*M, \end{aligned}$$

$$\mathcal{D}\mathcal{A} = d\mathcal{A} + \frac{p}{2}[\boldsymbol{\omega}, \mathcal{A}], \text{ if } \mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M, \; l \geq 0, \;\; p \geq 1. \tag{12.33}$$

where $\boldsymbol{\omega}$ is given by Eq. (12.21).

Remark 12.11 For $p = 0$ we identify $\bigwedge^l TM \otimes \bigwedge^0 T^*M = \bigwedge^l TM$ and thus in order to have an agreement with Eq. (12.22) we put $\mathcal{D}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}, \mathcal{A}]$. When $l = 0$ we have $\mathcal{D}\mathcal{A} = d\mathcal{A}$.

Proposition 12.12 *Let* $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell\,(M, \mathtt{g}) \otimes \bigwedge^p T^*M$, $\mathcal{B} \in \sec \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^q T^*M$. *Then, the* FECD *satisfies*

$$\begin{aligned} \mathcal{D}(\mathcal{A} \otimes_\wedge \mathcal{B}) = \mathcal{D}\mathcal{A} \otimes_\wedge \mathcal{B} + (-1)^p \mathcal{A} \otimes_\wedge \mathcal{D}\mathcal{B} \\ + q[\boldsymbol{\omega}, \mathcal{A}] \otimes_\wedge \mathcal{B} + (-1)^p p \mathcal{A} \otimes_\wedge [\boldsymbol{\omega}, \mathcal{B}]. \end{aligned} \tag{12.34}$$

Proof It follows directly from the definition if we take into account the properties of the product $\otimes_\wedge$ and Eq. (12.30). ■

Remark 12.13 According to the above proposition $\mathcal{D}$ does not satisfy Leibniz's rule when applied to the product $\mathcal{A} \otimes_\wedge \mathcal{B}$. This may seems strange, but we recall that there are some important derivatives operators as, e.g., the Dirac operator acting on sections of a Clifford bundle of multiforms that also does not satisfy Leibniz's rule.

12.4.1 The Operator $\mathcal{D}_{\mathbf{e_r}}$

Definition 12.14 The operator $\mathcal{D}_{\boldsymbol{e}_\mathbf{r}}$ is the mapping

$$\mathcal{D}_{\boldsymbol{e}_\mathbf{r}} : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \to \sec \bigwedge^l TM \otimes \bigwedge^p T^*M,$$

such that for any $\mathcal{A} = L^{(l)} \otimes P^{(q)} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^p T^*M$, $l, p \geq 1$, we have

$$(\mathcal{D}_{\boldsymbol{e}_\mathbf{r}} \mathcal{A}) \otimes_\wedge \theta^r = \mathcal{D}\mathcal{A}. \tag{12.35}$$

We have immediately

$$\mathcal{D}_{\boldsymbol{e}_\mathbf{r}} \mathcal{A} = \partial_{e_\mathbf{r}} \mathcal{A} + \frac{p}{2}[\boldsymbol{\omega}_\mathbf{r}, \mathcal{A}], \tag{12.36}$$

where $\partial_{e_\mathbf{r}} \mathcal{A} := L^{(l)} \otimes \frac{1}{p|} \boldsymbol{e}_r(P_{\mathbf{i}_1 \ldots \mathbf{i}_p}) \theta^{\mathbf{i}_1 \ldots \mathbf{i}_p}$ $\boldsymbol{\omega}_\mathbf{r} = \boldsymbol{\omega}(\boldsymbol{e}_\mathbf{r})$ and, of course, in general[11]

$$\mathcal{D}_{\boldsymbol{e}_\mathbf{r}} \mathcal{A} \neq \nabla_{e_\mathbf{r}} \mathcal{A}. \tag{12.37}$$

Let us examine some important cases which will appear latter.

Case $p = 1$

Let $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^1 T^*M$. Then,

$$\mathcal{D}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}, \mathcal{A}], \tag{12.38}$$

or

$$\mathcal{D}_{e_\mathbf{k}} \mathcal{A} = \partial_{e_\mathbf{r}} \mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}_\mathbf{k}, \mathcal{A}]. \tag{12.39}$$

Case $p = 2$

Let $\mathcal{F} \in \sec \bigwedge^l TM \otimes \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^2 T^*M$. Then,

$$\mathcal{D}\mathcal{F} = d\mathcal{F} + [\boldsymbol{\omega}, \mathcal{F}], \tag{12.40}$$

$$\mathcal{D}_{e_\mathbf{r}} \mathcal{F} = \partial_{e_\mathbf{r}} \mathcal{F} + [\boldsymbol{\omega}_\mathbf{r}, \mathcal{F}]. \tag{12.41}$$

[11]For a Clifford algebra formula for the calculation of $\nabla_{e_\mathbf{r}} \mathcal{A}$, $\mathcal{A} \in \sec \bigwedge^p T^*M$ recall Eq. (7.44).

12.4.2 Cartan Exterior Differential of Vector Valued Forms

Recall that [7] *Cartan* defined the exterior covariant differential of $\mathfrak{C} = e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}} \in \sec \bigwedge^1 TM \otimes \bigwedge^p T^*M$ as a mapping

$$\nabla^c : \bigwedge^1 TM \otimes \bigwedge^p T^*M \longrightarrow \bigwedge^1 TM \otimes \bigwedge^{p+1} T^*M,$$
$$\nabla^c \mathfrak{C} = \nabla^c (e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}}) = e_{\mathbf{i}} \otimes d\mathfrak{C}^{\mathbf{i}} + \nabla^c e_{\mathbf{i}} \wedge \mathfrak{C}^{\mathbf{i}}, \tag{12.42}$$
$$\nabla^c e_{\mathbf{j}} = (\nabla_{e_{\mathbf{k}}} e_{\mathbf{j}}) \theta^{\mathbf{k}}$$

which in view of Eqs. (12.31) and (12.21) can be written as

$$\nabla^c \mathfrak{C} = \nabla^c (e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}}) = d\mathfrak{C} + \frac{1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{12.43}$$

So, we have, for $p > 1$, the following relation between the FECD $\mathcal{D}$ and Cartan's exterior differential ($p > 1$)

$$\mathcal{D}\mathfrak{C} = \nabla^c \mathfrak{C} + \frac{p-1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{12.44}$$

Note moreover that when $\mathfrak{C}^{(1)} = e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M$, we have

$$\mathcal{D}\mathfrak{C}^{(1)} = \nabla^c \mathfrak{C}^{(1)}. \tag{12.45}$$

We end this section with two observations:

(1) We emphasize that to the concept of the fake exterior covariant differential just introduced is different from the usual definition of the exterior covariant differential acting on sections of a vector bundle $E(M) \otimes \bigwedge^p T^*M$ and also in sections of End$E(M) \otimes \bigwedge^p T^*M$, as discussed in the Appendix and as appear in well know texts, e.g., [1, 2, 7, 8, 13, 14, 19]. Our intention in defining the fake exterior covariant differential of Clifford valued forms was to find an object that could mimic coherently the pullback under a local section of the covariant differential acting on sections of vector bundles associated with a given principal bundle as used in *gauge* theories. The consistence of our definition will be checked in several situations below.
(2) We note also that the fake exterior covariant differential is different from the concept of *exterior covariant derivative* of indexed p-forms[12] which are objects like the curvature 2-forms (see below) or the connection 1-forms introduced above.

[12]These objects have been introduced in Definition 4.89.

12.4.3 Torsion and Curvature Once Again

Let $\boldsymbol{\theta} = e_\mu dx^\mu = \boldsymbol{e}_{\mathbf{a}}\theta^{\mathbf{a}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^1 T^*M$ and $\boldsymbol{\omega} = \frac{1}{2}\left(\omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot}\boldsymbol{e}_{\mathbf{b}} \wedge \boldsymbol{e}_{\mathbf{c}}\right) \otimes \theta^{\mathbf{a}} \equiv \frac{1}{2}\omega^{\mathbf{b}\cdot\mathbf{c}}_{\cdot\mathbf{a}\cdot}\boldsymbol{e}_{\mathbf{b}}\boldsymbol{e}_{\mathbf{c}}\theta^{\mathbf{a}} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM,\boldsymbol{g}) \otimes \bigwedge^1 T^*M$ be respectively the *representatives* of a soldering form and a connection (in a fixed gauge) on the *basis manifold*. Then, following the standard procedure [10], the *torsion* of the connection and the *curvature* of the connection on the basis manifold are defined by

$$\boldsymbol{\Theta} = \mathcal{D}\boldsymbol{\theta} \in \sec \bigwedge^1 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^2 T^*M, \tag{12.46}$$

and

$$\mathcal{R} = \mathcal{D}\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}) \otimes \bigwedge^2 T^*M. \tag{12.47}$$

We now calculate $\boldsymbol{\Theta}$ and then $\mathcal{D}\mathcal{R}$. We have,

$$\mathcal{D}\boldsymbol{\theta} = \mathcal{D}(\boldsymbol{e}_{\mathbf{a}}\theta^{\mathbf{a}}) = \boldsymbol{e}_{\mathbf{a}}d\theta^{\mathbf{a}} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{b}}]\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \tag{12.48}$$

and since $\frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}}, \boldsymbol{e}_{\mathbf{d}}] = -\boldsymbol{e}_{\mathbf{d}} \lrcorner \boldsymbol{\omega}_{\mathbf{a}} = \omega^{\mathbf{c}\cdot\cdot}_{\cdot\mathbf{ad}}\boldsymbol{e}_{\mathbf{c}}$ we have

$$\mathcal{D}(\boldsymbol{e}_{\mathbf{a}}\theta^{\mathbf{a}}) = \boldsymbol{e}_{\mathbf{a}}[d\theta^{\mathbf{a}} + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bd}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}] = \boldsymbol{e}_{\mathbf{a}}\Theta^{\mathbf{a}}, \tag{12.49}$$

and we recognize

$$\Theta^{\mathbf{a}} = d\theta^{\mathbf{a}} + \omega^{\mathbf{a}\cdot\cdot}_{\cdot\mathbf{bd}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}, \tag{12.50}$$

as Cartan's first structure equation.

12.4.4 FECD and Levi-Civita Connections

For a torsion free connection, the torsion 2-forms $\Theta^{\mathbf{a}} = 0$, and it follows that $\boldsymbol{\Theta} = 0$. We recall that a metric compatible connection $\boldsymbol{\omega}$ (for which $D_{\boldsymbol{e}_{\mathbf{a}}}\boldsymbol{g} = 0, \mathbf{a} = 0,1,2,3$) satisfying $\Theta^{\mathbf{a}} = 0$ is called a Levi-Civita connection. In the remaining of this Chapter we *restrict* ourself to that case.

To start recall that from Eq. (12.33) we have,

$$\mathcal{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}]. \tag{12.51}$$

Then, taking into account that

$$\mathcal{R} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}], \tag{12.52}$$

and that from Eqs. (12.27), (12.28) and (12.32) it follows that

$$\begin{aligned} d[\boldsymbol{\omega},\boldsymbol{\omega}] &= [d\boldsymbol{\omega},\boldsymbol{\omega}] - [\boldsymbol{\omega},d\boldsymbol{\omega}], \\ [d\boldsymbol{\omega},\boldsymbol{\omega}] &= -[\boldsymbol{\omega},d\boldsymbol{\omega}], \\ [[\boldsymbol{\omega},\boldsymbol{\omega}],\boldsymbol{\omega}] &= 0, \end{aligned} \tag{12.53}$$

and it results that

$$\mathcal{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega},\mathcal{R}] = 0. \tag{12.54}$$

which is known as the *Bianchi identity*.

Note that, since $\{\boldsymbol{e}_{\mathbf{a}}\}$ is an orthonormal frame we can write ($R_{\cdot\mu\alpha\cdot}^{\mathbf{a}\cdot\cdot\mathbf{b}} \equiv R_{\mu\nu}^{\mathbf{ab}}$)

$$\begin{aligned} \mathcal{R} &= \frac{1}{4}R_{\mu\nu}^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}} \wedge \boldsymbol{e}_{\mathbf{b}} \otimes (dx^{\mu} \wedge dx^{\nu}) \\ &\equiv \frac{1}{4}\mathcal{R}_{\mathbf{cd}}^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}} \otimes \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}} = \frac{1}{4}R_{\rho\sigma}^{\alpha\beta}e_{\boldsymbol{\alpha}}e_{\beta} \otimes dx^{\rho} \wedge dx^{\boldsymbol{\sigma}} \\ &= \frac{1}{4}R_{\mu\nu\rho\sigma}e^{\mu}e^{\nu} \otimes dx^{\rho} \wedge dx^{\boldsymbol{\sigma}}, \end{aligned} \tag{12.55}$$

where $R_{\mu\nu\rho\sigma}$ are the components of the curvature tensor. We recall from Chap. 5 the well known symmetries

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu}. \end{aligned} \tag{12.56}$$

We can also write Eq. (12.55) as

$$\begin{aligned} \mathcal{R} &= \frac{1}{4}R_{\mathbf{cd}}^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}} \otimes (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) = \frac{1}{2}\mathbf{R}_{\mu\nu}dx^{\mu} \wedge dx^{\boldsymbol{\nu}} \\ &= \frac{1}{2}\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}^{\mathbf{b}}, \end{aligned} \tag{12.57}$$

with

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\boldsymbol{e}_{\mathbf{a}} \wedge \boldsymbol{e}_{\mathbf{b}} \in \sec \bigwedge^{2} TM \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}), \\ \mathcal{R}^{\mathbf{ab}} &= \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}dx^{\mu} \wedge dx^{\boldsymbol{\nu}} \in \sec \bigwedge^{2} T^{*}M, \end{aligned} \tag{12.58}$$

where $\mathbf{R}_{\mu\nu}$ is called *curvature bivectors* and we recognize from Chap. 4 the $\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ as the Cartan curvature 2-forms, which satisfy *Cartan's second structure equation*

$$\mathcal{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} = d\boldsymbol{\omega}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}} + \boldsymbol{\omega}^{\mathbf{a}\cdot}_{\cdot\mathbf{c}} \wedge \boldsymbol{\omega}^{\mathbf{c}\cdot}_{\cdot\mathbf{d}}, \tag{12.59}$$

a result that follows immediately calculating $d\mathcal{R}$ from Eq. (12.52). Now, we can also write,

$$\begin{aligned}
\mathcal{D}\mathcal{R} &= d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}] \\
&= \frac{1}{2}\{d(\frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}}dx^{\mu} \wedge dx^{\nu}) + [\boldsymbol{\omega}_{\rho}, \mathbf{R}_{\mu\nu}]dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu}\} \\
&= \frac{1}{2}\{\partial_{\rho}\mathbf{R}_{\mu\nu} + [\boldsymbol{\omega}_{\rho}, \mathbf{R}_{\mu\nu}]\}dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu} \\
&= \frac{1}{2}\mathcal{D}_{e_{\rho}}\mathbf{R}_{\mu\nu}dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu} \\
&= \frac{1}{3!}\left(\mathcal{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} + \mathcal{D}_{e_{\mu}}\mathbf{R}_{\nu\rho} + \mathcal{D}_{e_{\nu}}\mathbf{R}_{\rho\mu}\right)dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu} = 0,
\end{aligned} \tag{12.60}$$

from where it follows that

$$\mathcal{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} + \mathcal{D}_{e_{\mu}}\mathbf{R}_{\nu\rho} + \mathcal{D}_{e_{\nu}}\mathbf{R}_{\rho\mu} = 0. \tag{12.61}$$

Remark 12.15 Equation (12.61) is called in Physics textbooks on gauge theories (see, e.g., [13, 16]) Bianchi identities. Physicists call the operator

$$\mathcal{D}_{e_{\rho}} \equiv \mathcal{D}_{\rho} = \partial_{\rho} + [\boldsymbol{\omega}_{\rho},], \tag{12.62}$$

acting on the curvature bivectors as the '*covariant derivative*'. Note however that, as detailed above, this operator is not the usual covariant derivative operator $D_{\boldsymbol{e}_a}$ acting on sections of the tensor bundle, and not realizing this fact may give rise to a real confusion.

We now find the explicit expression for the curvature bivectors $\mathbf{R}_{\mu\nu}$ in terms of the connections bivectors $\boldsymbol{\omega}_{\mu} = \boldsymbol{\omega}(e_{\mu})$, which will be used latter. First recall that by definition[13]

$$\mathbf{R}_{\mu\nu} = \mathcal{R} \cdot (\theta_{\mu} \wedge \theta_{\nu}) = -\mathcal{R} \cdot (\theta_{\nu} \wedge \theta_{\mu}) = -\mathbf{R}_{\nu\mu}. \tag{12.63}$$

[13] $\{\theta^{\mu}\}$ is the dual basis of $\{e_{\mu}\}$ and the scalar product in Eqs. (12.63) and (12.64) refers to the scalar product of the form factors of $\mathcal{R}$.

Now, observe that using Eqs. (12.27), (12.28) and (12.32) we can easily show that

$$[\boldsymbol{\omega},\boldsymbol{\omega}]\cdot(\theta_\mu\wedge\theta_\nu) = \boldsymbol{\omega}\otimes_\wedge\boldsymbol{\omega}\cdot(\theta_\mu\wedge\theta_\nu) = 2[\boldsymbol{\omega}(e_\mu),\boldsymbol{\omega}(e_\nu)]$$
$$= 2[\boldsymbol{\omega}_\mu,\boldsymbol{\omega}_\nu]. \tag{12.64}$$

Using Eqs. (12.52), (12.63) and (12.64) we get

$$\mathbf{R}_{\mu\nu} = \partial_\mu\boldsymbol{\omega}_\nu - \partial_\nu\boldsymbol{\omega}_\mu + [\boldsymbol{\omega}_\mu,\boldsymbol{\omega}_\nu]. \tag{12.65}$$

which is to be compared with Eq. (5.44) of Chap. 5.

Exercise 12.16 Let $A \in \sec \bigwedge^p TM \hookrightarrow \sec \mathcal{C}\ell(M,\boldsymbol{g})$ and $\mathcal{R}$ the curvature of the connection as defined in Eq. (12.47). Show that,[14]

$$\mathcal{D}^2 A = \frac{1}{4}[\mathcal{R},A] + \frac{1}{4}[d\boldsymbol{\omega},A]. \tag{12.66}$$

12.5 General Relativity as a Sl(2, ℂ) Gauge Theory

12.5.1 The Non-homogeneous Field Equations

The analogy of the fields $\mathbf{R}_{\mu\nu} = \frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\boldsymbol{e}_\mathbf{a}\boldsymbol{e}_\mathbf{b} = \frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\boldsymbol{e}_\mathbf{a}\wedge\boldsymbol{e}_\mathbf{b} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g})$ with the gauge fields of particle fields is so appealing that it is irresistible to propose some kind of a $\mathrm{Sl}(2,\mathbb{C})$ formulation for the gravitational field. And indeed this has already been done, and the interested reader may consult, e.g., [4, 12] for details. Here, we observe that despite the similarities, the gauge theories of particle physics are in general formulated in flat Minkowski spacetime and the theory here must be for a field on a general Lorentzian spacetime. This introduces additional complications, but it is not our purpose to discuss that issue with all attention it deserves here. We only wants to discuss some issues related to the formalism just introduced above.

To start, recall that in gauge theories besides the homogeneous field equations given by Bianchi identities, we also have a nonhomogeneous field equation. This equation which is to be the analog of the nonhomogeneous equation for the electromagnetic field is written here as

$$\mathcal{D}\star\mathcal{R} = d\star\mathcal{R} + [\boldsymbol{\omega},\star\mathcal{R}] = -\star\mathcal{J}, \tag{12.67}$$

[14] The value obtained for $\mathcal{D}^2 A$ in [15] is wrong.

where the $\mathcal{J} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^1 T^*M$ is a 'current', which, if the theory is to be one equivalent to GRT, must be in some way related with the energy momentum tensor in Einstein theory. In order to write from this equation an equation for the curvature bivectors, it is very useful to imagine that $\bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$, the Clifford bundle of differential forms, for in that case the powerful calculus used in previous chapters can be used. So, we write:

$$\begin{aligned}
&\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \\
&\quad \hookrightarrow \mathcal{C}\ell(TM, \boldsymbol{g}) \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \mathcal{C}\ell(T^*M, \mathtt{g}), \\
&\mathcal{R} = \mathcal{D}\boldsymbol{\omega} \in \sec \bigwedge^2 TM \otimes \bigwedge^2 T^*M \\
&\quad \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \mathcal{C}\ell(T^*M, \mathtt{g}) \\
&\mathcal{J} = \mathbf{J}_{\boldsymbol{\nu}} \otimes \theta^\nu \equiv \mathbf{J}_\nu \theta^\nu \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \\
&\quad \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g}) \otimes \mathcal{C}\ell(T^*M, \mathtt{g}).
\end{aligned} \tag{12.68}$$

Now, using Definition 2.27 of the Hodge star operator and recalling that for a torsionless connection it is $d = \boldsymbol{\partial} \wedge$ and $\delta = -\boldsymbol{\partial} \lrcorner$ we can write

$$d \star \mathcal{R} = -\theta^5(-\boldsymbol{\partial} \lrcorner \mathcal{R}) = - \star (\boldsymbol{\partial} \lrcorner \mathcal{R}) = - \star ((\partial_\mu \mathbf{R}^\mu_\nu)\theta^\nu). \tag{12.69}$$

Also,

$$\begin{aligned}
[\boldsymbol{\omega}, \star\mathcal{R}] &= [\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \star(\theta^\alpha \wedge \theta^\beta) \\
&= -[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \theta^5(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \theta^5(\theta^\alpha \wedge \theta^\beta) + \theta^5(\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \frac{\theta^5}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu(\theta^\alpha \wedge \theta^\beta) - (\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \theta^5[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \lrcorner(\theta^\alpha \wedge \theta^\beta) \\
&= -2 \star ([\boldsymbol{\omega}_\mu, \mathbf{R}^\mu_\beta]\theta^\beta.
\end{aligned} \tag{12.70}$$

Using Eqs. (12.67)–(12.70) we get[15]

$$\partial_\mu \mathbf{R}^{\mu\cdot}_{\cdot\nu} + 2[\boldsymbol{\omega}_\mu, \mathbf{R}^{\mu\cdot}_{\cdot\nu}] = \mathcal{D}_{e_\mu} \mathbf{R}^{\mu\cdot}_{\cdot\nu} = \mathbf{J}_\nu. \tag{12.71}$$

[15] Recall that $\mathbf{J}_\nu \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}\ell(M, \boldsymbol{g})$.

So, the gauge theory of gravitation has as field equations the Eq. (12.71), the nonhomogeneous field equations, and Eq. (12.61) the homogeneous field equations (which is Bianchi identities). We summarize that equations, as

$$\mathcal{D}_{e_\mu}\mathbf{R}^{\mu\cdot}_{\cdot\nu} = \mathbf{J}_\nu, \qquad \mathcal{D}_{e_\rho}\mathbf{R}_{\mu\nu} + \mathcal{D}_{e_\mu}\mathbf{R}_{\nu\rho} + \mathcal{D}_{e_\nu}\mathbf{R}_{\rho\mu} = 0. \tag{12.72}$$

Equation (12.72) which looks like Maxwell equations, must, of course, be compatible with Einstein's equations, which may be eventually used to determine $\mathbf{R}^{\mu\cdot}_{\cdot\nu}$, $\boldsymbol{\omega}_\mu$ and $\mathbf{J}_\nu$, an exercise left to the interested reader.

12.6 Another Set of Maxwell-Like Equations for Einstein Theory

We now show, e.g., how a special combination of the $\mathbf{R}^{\mathbf{a}\cdot}_{\cdot\mathbf{b}}$ are directly related with a combination of products of the energy-momentum 1-vectors $\mathsf{T}_\mathbf{a}$ and the tetrad fields $\boldsymbol{e}_\mathbf{a}$ (see Eq. (12.75) below) in Einstein theory. In order to do that, we recall from previous chapters that Einstein's equations can be written in components in an orthonormal basis as

$$R_{\mathbf{ab}} - \frac{1}{2}\eta_{\mathbf{ab}}R = -T_{\mathbf{ab}}, \tag{12.73}$$

where $R_{\mathbf{ab}} = R_{\mathbf{ba}}$ are the components of the Ricci tensor ($R_{\mathbf{ab}} = R^{\mathbf{c}\cdots}_{\cdot\mathbf{abc}}$), $T_{\mathbf{ab}}$ are the components of the energy-momentum tensor of matter fields and $R = \eta_{\mathbf{ab}}R^{\mathbf{ab}}$ is the curvature scalar. We next introduce as (in analogy to what has been done in Chap. 9 for form fields) the *Ricci 1-vectors* and the *energy-momentum 1-vectors* by

$$R_\mathbf{a} = R_{\mathbf{ab}}\boldsymbol{e}^\mathbf{b} \in \sec \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}), \tag{12.74}$$

$$\mathsf{T}_\mathbf{a} = T_{\mathbf{ab}}\boldsymbol{e}^\mathbf{b} \in \sec \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(M,\boldsymbol{g}). \tag{12.75}$$

We have that

$$R_\mathbf{a} = -\boldsymbol{e}^\mathbf{b} \lrcorner \mathbf{R}_{\mathbf{ab}}. \tag{12.76}$$

Now, multiplying Eq. (12.73) on the right by $\boldsymbol{e}^\mathbf{b}$ we get

$$R_\mathbf{a} - \frac{1}{2}R\boldsymbol{e}_\mathbf{a} = -\mathsf{T}_\mathbf{a}. \tag{12.77}$$

Multiplying Eq. (12.77) first on the right by $\boldsymbol{e}_\mathbf{b}$ and then on the left by $\boldsymbol{e}_\mathbf{b}$ and making the difference of the resulting equations we get

$$(-\boldsymbol{e}^\mathbf{c} \lrcorner \mathbf{R}_{\mathbf{ac}})\,\boldsymbol{e}_\mathbf{b} - \boldsymbol{e}_\mathbf{b}\,(-\boldsymbol{e}^\mathbf{c} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2}R(\boldsymbol{e}_\mathbf{a}\boldsymbol{e}_\mathbf{b} - \boldsymbol{e}_\mathbf{b}\boldsymbol{e}_\mathbf{a}) = (\boldsymbol{e}_\mathbf{b}\mathsf{T}_\mathbf{a} - \mathsf{T}_\mathbf{a}\boldsymbol{e}_\mathbf{b}). \tag{12.78}$$

Defining

$$\begin{aligned}\mathcal{F}_{\mathbf{ab}} &= (-\boldsymbol{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\, \boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}}\, (-\boldsymbol{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2} R(\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}}\boldsymbol{e}_{\mathbf{a}}) \\ &= \frac{1}{2}(R_{\mathbf{ac}}\boldsymbol{e}^{\mathbf{c}}\boldsymbol{e}_{\mathbf{b}} + \boldsymbol{e}_{\mathbf{b}}\boldsymbol{e}^{\mathbf{c}}R_{\mathbf{ac}} - \boldsymbol{e}^{\mathbf{c}}R_{\mathbf{ac}}\boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}}R_{\mathbf{ac}}\boldsymbol{e}^{\mathbf{c}}) - \frac{1}{2}R(\boldsymbol{e}_{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}}\boldsymbol{e}_{\mathbf{a}}) \end{aligned} \tag{12.79}$$

and

$$\mathcal{J}_{\mathbf{b}} = \mathcal{D}_{e_{\mathbf{a}}}(\boldsymbol{e}_{\mathbf{b}}\mathsf{T}^{\mathbf{a}} - \mathsf{T}^{\mathbf{a}}\boldsymbol{e}_{\mathbf{b}}), \tag{12.80}$$

we have[16]

$$\mathcal{D}_{e_{\mathbf{a}}}\mathcal{F}^{\mathbf{a}}_{\mathbf{b}} = \mathcal{J}_{\mathbf{b}}. \tag{12.81}$$

It is quite obvious that with coordinates $\{x^{\mu}\}$ covering an open set $U \subset M$ we can write

$$\mathcal{D}_{e_{\rho}}\mathcal{F}^{\rho}_{\beta} = \mathcal{J}_{\beta}, \tag{12.82}$$

with $\mathcal{F}^{\rho}_{\beta} = g^{\rho\alpha}\mathcal{F}_{\alpha\beta}$ and

$$\mathcal{F}_{\alpha\beta} = \left(-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma}\right) e_{\beta} - e_{\beta}\left(-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma}\right) - \frac{1}{2}R(e_{\alpha}e_{\beta} - e_{\beta}e_{\alpha}), \tag{12.83}$$

$$\mathcal{J}_{\beta} = \mathcal{D}_{e_{\rho}}(e^{\rho}\mathsf{T}_{\beta} - \mathsf{T}^{\rho}e_{\beta}). \tag{12.84}$$

Remark 12.17 Equation (12.81) (or Eq. (12.82)) is a set of Maxwell-like nonhomogeneous equations. It looks like the nonhomogeneous classical Maxwell equations when that equations are written in components, but keep in mind that Eq. (12.82) is only a new way of writing the equation of the nonhomogeneous field equations in the $\mathrm{Sl}(2,\mathbb{C})$ like gauge theory version of Einstein's theory, discussed in the previous section. In particular, recall that any one of the six $\mathcal{F}^{\rho}_{\beta} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g})$. Or, in words, each one of the $\mathcal{F}^{\rho}_{\beta}$ is a bivector field, *not* a set of scalars which are components of a 2-form, as is the case in Maxwell theory. Also, recall that according to Eq. (12.84) each one of the four $\mathcal{J}_{\beta} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(M, \boldsymbol{g})$.

From Eq. (12.81) it is not obvious that in vacuum we must have $\mathcal{F}_{\mathbf{ab}} = 0$. However that is exactly what happens if we take into account Eq. (12.79) which defines that object. Moreover, $\mathcal{F}_{\mathbf{ab}} = 0$ does not imply that the curvature bivectors

[16]Note that we could also produce another Maxwell-like equation, by using the usual Levi-Civita covariant derivative operator D in the definition of the current, i.e., we can put $\mathcal{J}_{\mathbf{b}} = D_{\mathbf{e_a}}(\mathsf{T}^{\mathbf{a}}\mathbf{e_b} - \mathbf{e_b}\mathsf{T}^{\mathbf{a}})$, and in that case we obtain $D_{\mathbf{e_a}}\mathcal{F}^{\mathbf{a}}_{\mathbf{b}} = \mathcal{J}_{\mathbf{b}}$. An equation of this form appears in [17, 18].

$\mathbf{R_{ab}}$ are null in vacuum. Indeed, in that case, Eq. (12.79) implies only the identity (valid *only* in vacuum)

$$(\boldsymbol{e}^{\mathbf{c}} \lrcorner \mathbf{R_{ac}})\, \boldsymbol{e}_{\mathbf{b}} = (\boldsymbol{e}^{\mathbf{c}} \lrcorner \mathbf{R_{bc}})\, \boldsymbol{e}_{\mathbf{a}}. \tag{12.85}$$

Moreover, recalling definition (Eq. (12.58)) we have

$$\mathbf{R_{ab}} = \frac{1}{2} R_{\mathbf{abcd}} \boldsymbol{e}^{\mathbf{c}} \boldsymbol{e}^{\mathbf{d}}, \tag{12.86}$$

and we see that the $\mathbf{R_{ab}}$ are zero only if the Riemann tensor is null which is not the case in any non trivial general relativistic model.

The important fact that we want to emphasize here is that although eventually interesting, Eq. (12.81) does not seem (according to our opinion) to contain anything new in it. More precisely, all information given by that equation is already contained in the original Einstein's equation, for indeed it has been obtained from it by simple algebraic manipulations. We state again: According to our view terms like

$$\begin{aligned}
\mathcal{F}_{\mathbf{ab}} &= \frac{1}{2}(R_{\mathbf{ac}} \boldsymbol{e}^{\mathbf{c}} \boldsymbol{e}_{\mathbf{b}} + \boldsymbol{e}_{\mathbf{b}} \boldsymbol{e}^{\mathbf{c}} R_{\mathbf{ac}} - \boldsymbol{e}^{\mathbf{c}} R_{\mathbf{ac}} \boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}} R_{\mathbf{ac}} \boldsymbol{e}^{\mathbf{c}}) - \frac{1}{2} R(\boldsymbol{e}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}} \boldsymbol{e}_{\mathbf{a}}),\\
\mathfrak{R}_{\mathbf{ab}} &= (\boldsymbol{e}_{\mathbf{b}} \mathsf{T}_{\mathbf{a}} - \mathsf{T}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{b}}) - \frac{1}{2} R(\boldsymbol{e}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}} \boldsymbol{e}_{\mathbf{a}}),\\
\mathbf{F_{ab}} &= \frac{1}{2} R(\boldsymbol{e}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{b}} - \boldsymbol{e}_{\mathbf{b}} \boldsymbol{e}_{\mathbf{a}}),
\end{aligned} \tag{12.87}$$

are pure gravitational object and there is *no* relationship of any one of these objects with the ones appearing in Maxwell theory. Of course, these objects may eventually be used to formulate 'interesting' equations, like Eq. (12.81) which are equivalent to Einstein's field equations, but this fact does not seem to us to point to any new Physics. Even more, from the mathematical point of view, to find solutions to the new Eq. (12.81) is certainly as hard as to find solutions to the original Einstein equations.

In [17, 18] there appear equations resembling the above ones using *paravector* fields $\boldsymbol{q}_{\mathbf{a}} = \boldsymbol{e}_{\mathbf{a}} \boldsymbol{e}_{\mathbf{0}} \in \sec \mathcal{C}\ell^{(0)}(M, \boldsymbol{g})$. However, in those papers, the author mislead the equations that he obtained with the Maxwell equations describing the electromagnetic field, and claimed to have obtained an unified field theory for the gravitational plus electromagnetic field. According to our discussion, such claims are unfortunately equivocated. More details on this issue may be found in [6, 15].

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Chapter 13
Maxwell, Dirac and Seiberg-Witten Equations

Abstract In this Chapter we discuss three important issues. The first is how $\mathrm{i} = \sqrt{-1}$ makes its appearance in classical electrodynamics and in Dirac theory. This issue is important because if someone did not really know the real meaning uncovered by $\mathrm{i} = \sqrt{-1}$ in these theories he may infers nonsequitur results.[1] After that we present some 'Dirac like' representations of Maxwell equations. Within the Clifford bundle it becomes obvious why there are so many 'Dirac like' representations of Maxwell equations. The third issue discussed in this chapter are the *mathematical* Maxwell-Dirac equivalences of the first and second kinds and the relation of these mathematical equivalences with Seiberg-Witten equations in Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ which is the arena where we suppose physical phenomena to take place in this chapter. We denote by $\{x^\mu\}$ coordinates in Einstein-Lorentz-Poincaré gauge associated to an inertial reference frame $\boldsymbol{e}_0 \in \sec TM$. Moreover $\{\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu}\} \in \sec TM$, $(\mu = 0, 1, 2, 3)$ is an orthonormal basis, with $\boldsymbol{\eta}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $\{\gamma^\nu = dx^\nu\} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ is the dual basis of $\{\boldsymbol{e}_\mu\}$.

13.1 Dirac-Hestenes Equation and $\mathbf{i} = \sqrt{-1}$

We already learned in Chap. 7 that we can write Dirac equation in interaction with the electromagnetic field in the spin-Clifford bundle $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \eta)$, as

$$\boldsymbol{\partial}^s \boldsymbol{\psi} \boldsymbol{e}^{\mathbf{21}} - m\boldsymbol{\psi}\boldsymbol{e}^{\mathbf{0}} - qA\boldsymbol{\psi} = 0 \tag{13.1}$$

where $\boldsymbol{\psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \eta)$ is a DHSF and the $\boldsymbol{e}^{\mathbf{a}} \in \mathbb{R}_{1,3}$ are such that $\boldsymbol{e}^{\mathbf{a}}\boldsymbol{e}^{\mathbf{b}} + \boldsymbol{e}^{\mathbf{b}}\boldsymbol{e}^{\mathbf{a}} = 2\eta^{\mathbf{ab}}$ and $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$. Equation (13.1) that we named DE$\mathcal{C}\ell^l$ does *not* involve complex numbers. We also learned that once we fix a spin frame the DH$\mathcal{C}\ell^l$ has a representative in the Clifford bundle that we called the Dirac-

[1] A clear example of the veracity of that statement is discussed in Sect. 13.2.1 using material published in the literature.

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Hestenes equation,[2]

$$\boldsymbol{\partial}^{(s)}\psi_\Xi\gamma^{21} + m\psi_\Xi\gamma^0 - qA\psi_\Xi = 0, \tag{13.2}$$

where $\psi_\Xi \in \sec \mathcal{C}\ell(M,\eta)$ is the representative of $\boldsymbol{\psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\eta)$ in the spin coframe $\Xi \in P_{\mathrm{Spin}^e_{1,3}}(M)$. Again this equation does not contain any complex number in its formulation, and we already know from previous Chaps. 3 and 7 the important geometrical meaning of representatives of DHSF.

Now, we recall that $\mathrm{i} = \sqrt{-1}$ enters Dirac theory once we multiply Eq. (7.80) on the right by the idempotent $\mathbf{f} = \frac{1}{2}(1 + \boldsymbol{e}^{\mathbf{0}})\frac{1}{2}(1 + \mathrm{i}\boldsymbol{e}^{\mathbf{2}}\boldsymbol{e}^{\mathbf{1}}) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$. We get after some simple algebraic manipulations the following equation for the (complex) ideal left spin-Clifford field $\Psi\mathbf{f} = \Psi \in \sec \mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$,

$$\mathrm{i}\boldsymbol{\partial}^s\Psi - m\Psi - qA\Psi = 0. \tag{13.3}$$

From that equation we get using the standard matrix representation of the $\{\gamma^\mu\}$ the standard Dirac equation for column spinor fields (c.r., Eq. (7.77)). We would like to emphasize here that this introduction of $\mathrm{i} = \sqrt{-1}$ in (first quantization) Dirac theory is superfluous from any fundamental point of view, although it may be useful in calculations. However it must be clear to the reader that Eq. (13.3) hides the crucial geometrical information that γ^{21} refers to the spin plane.

13.2 How $\mathrm{i} = \sqrt{-1}$ Enters Maxwell Theory

The plane wave solutions (PWS) of ME are usually obtained by looking for solutions to these equations such that the potential $A = A_\mu\gamma^\mu$ is in the *Coulomb* (or *radiation*) gauge, i.e.,

$$A_0 = 0,\ \partial_i A^i := \nabla \cdot \vec{A} = 0 \tag{13.4}$$

Equation (13.4), of course, implies that $\boldsymbol{\partial} \cdot A = -\delta A = 0$, i.e., the Lorenz gauge condition is also satisfied. Now, in the absence of sources, A and $F = \boldsymbol{\partial} \wedge A = dA$ satisfy, respectively the homogeneous wave equation (HWE) and the free ME

$$\Box A = 0, \tag{13.5}$$

$$\boldsymbol{\partial} F = 0 \tag{13.6}$$

The PWS can be obtained directly from the free ME ($\partial F = 0$) once we suppose that F is a null field, i.e., $F^2 = 0$ (see Exercise 13.1). However, for the purposes we

[2]Recall that $\boldsymbol{\partial}^{(s)} = \boldsymbol{\varepsilon}^{\mathbf{a}}\boldsymbol{\nabla}^{(s)}_{\mathbf{e_a}}$ is the representative of the spin-Dirac operator in the Clifford bundle.

have in mind we think more interesting to find these solutions by solving $\Box A = 0$ with the subsidiary condition given by Eq. (13.4).

In order to do that we introduce besides $\{x^\mu\}$ another set of coordinate $\{x'^\mu\}$ also in the Einstein-Lorentz-Poincaré gauge which are also a $(nacs|\partial/\partial x^0)$, such that , $x'^0 = x^0$, $x'^i = R^i_j x^j$. Putting $\varepsilon^\mu = dx'^\mu$ we then have

$$\varepsilon^\mu = R\gamma^\mu \tilde{R};\ \varepsilon^0 = \gamma^0, \varepsilon^i \neq \gamma^i, \tag{13.7}$$

where the (constant) $R \in \sec \mathrm{Spin}_3(M)$ generates a global rotation of the space axes. We also write

$$\varepsilon_i\varepsilon_j \equiv \varepsilon_{ij}, \quad \varepsilon_i e_0 = \vec{e}_i,, \quad \vec{e}_i\vec{e}_j \equiv e_{ij} = -\varepsilon_{ij}, \quad \mathbf{i} = \vec{e}_1\vec{e}_2\vec{e}_3, i,j = 1,2,3 \text{ and } i \neq j. \tag{13.8}$$

We consider next the following two linearly independent monochromatic plane solutions $A^{(i)}$, $i = 1, 2$, of Eq. (13.5) satisfying the subsidiary condition giving by Eq. (13.4) and moving in the $\vec{e}_3$ direction,

$$\begin{aligned} A^{(i)} &= \exp\left[\frac{(-1)^{i+1}e_{21}\bar{\phi}_i}{2}\right] e_1 \exp\left[\frac{-(-1)^{i+1}e_{21}\bar{\phi}_i}{2}\right] = \exp\left[e_{21}(-1)^{i+1}\bar{\phi}_i\right]\varepsilon_1, \\ \bar{\phi}_i &: M \to \mathbb{R}, x \mapsto \bar{\phi}_i(x) = k'_\mu x'^\mu + \bar{\varphi}_i = \omega t - \vec{k}' \cdot \vec{x}' + \bar{\varphi}_i, \omega = |\vec{k}'|, \\ \vec{k}' &= \omega \vec{e}_3. \end{aligned} \tag{13.9}$$

where the $\bar{\varphi}_i$ are real constants, called the initial phase. Since $A_0^{(i)} = 0$ we write,

$$\vec{A}^{(i)} = A^{(i)}\varepsilon_0 = \exp\left[-(-1)^{i+1}e_{21}\bar{\phi}_i\right]\vec{e}_1 = \vec{e}_1 \exp\left[(-1)^{i+1}e_{21}\bar{\phi}_i\right], \tag{13.10}$$

Now,

$$\begin{aligned} F^{(i)} &= \partial \wedge A^{(i)} = \partial \wedge A^{(i)} = \boldsymbol{\partial}\varepsilon_0\varepsilon_0 A^{(i)} \\ &= (\partial_t - \nabla)(-\vec{A}^{(i)}). \end{aligned} \tag{13.11}$$

13.2.1 From Spatial Rotation to Duality Rotation

We calculate in details $F^{(1)}$ in order for the reader to see explicitly how $\mathbf{i} = \vec{e}_1\vec{e}_2\vec{e}_3$ enters in the classical formulation of the electromagnetic field. We have,

$$\begin{aligned} F^{(1)} &= -\omega\left[\vec{e}_1\vec{e}_2\vec{e}_1 \exp(e_{21}\bar{\phi}_1) - \vec{e}_3\vec{e}_1\vec{e}_2\vec{e}_1 \exp(e_{21}\bar{\phi}_1)\right] \\ &= \omega\left[\vec{e}_2 - \mathbf{i}\vec{e}_1\right]\exp(-e_{21}\bar{\phi}_1) \end{aligned}$$

$$= \omega \left[\vec{e}_1 + \mathbf{i}\vec{e}_2\right] \exp(-\mathbf{i}\phi_1)$$

$$\phi_1 = \omega t - \vec{k}' \cdot \vec{x}' + \varphi_1, \varphi_1 = \bar{\varphi}_1 - \frac{\pi}{2} \tag{13.12}$$

This formula shows how a *spatial* rotation turns up into a *duality* rotation, a really non trivial result.

13.2.2 Polarization and Stokes Parameters

Now, a monochromatic plane wave can be written as a linear combination of $F^{(1)}$ and $F^{(2)}$, which have, of course, opposite *helicities*,[3] denoted from now on respectively by $+$ and $-$. We write

$$\frac{F}{\omega} = (\mathbf{i}\vec{e}_1 - \vec{e}_2) \left\{a_+ e^{)[-\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}' + \varphi_+]} - a_- e^{[\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}' + \varphi_-)]}\right\} \tag{13.13}$$

where $a_\pm$ are real constants. Now, observe that F can be written as the product of two (different) 2-dimensional matrix fields with values in the Clifford bundle $\mathcal{I}(M, \eta) = \mathcal{C}\ell^0(M, \eta)\mathfrak{e}$,where

$$\mathfrak{e} = \frac{(1 - \vec{e}_3)}{2} \in \sec \mathcal{C}\ell^0(M, \eta) \tag{13.14}$$

is an idempotent field.[4] We write

$$F = \sqrt{2}\mathbf{i}\vec{e}_1 \left[\mathfrak{e}e^{-\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}')}, \mathfrak{e}e^{\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}')}\right] \begin{bmatrix} \sqrt{2}\omega a_+ e^{-\mathbf{i}\varphi_+} \mathfrak{e} \\ -\sqrt{2}\omega a_- e^{+\mathbf{i}\varphi_-} \mathfrak{e} \end{bmatrix} \tag{13.15}$$

This means that each line of , e.g., the 2-dimensional matrix field

$$\mathbf{\Omega} = \begin{bmatrix} \sqrt{2}\omega a_+ e^{-\mathbf{i}\varphi_+} \mathfrak{e} \\ -\sqrt{2}\omega a_- e^{+\mathbf{i}\varphi_-} \mathfrak{e} \end{bmatrix} \tag{13.16}$$

is an *amorphous* spinor field (recall Sect. 7.10).

Writing

$$F = \vec{E} + \mathbf{i}\vec{B}, F^{-} = \vec{E} - \mathbf{i}\vec{B},$$

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[3]For the moment different helicities means that the vectors $\vec{A}^{(i)}$ have opposite sense of rotation. We will be more precise later.

[4]Recall that $\mathcal{I}(M, \eta)$ is a bundle of amorphous spinor fields and it is not to be confused with the bundle $I(M, \eta)$ (Definition 7.16) of algebraic spinor fields.

we recall that the polarization of a given wave is in general specified by $\vec{E} = (F + F^{-})/2$, the electric field. Now, for every formally complex number or formally complex Euclidian vector field we define in an obvious way the operators Re and Im, such that, e.g.,

$$\vec{E} = \operatorname{Re} F = \operatorname{Re} \mathfrak{F} = \operatorname{Re}\left[\vec{\mathfrak{E}} e^{-\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}')}\right], \tag{13.17}$$

$$\vec{\mathfrak{E}} = \frac{\mathbf{i}}{\sqrt{2}}\left[\vec{e}_1 + \mathbf{i}\vec{e}_2, \vec{e}_1 - \mathbf{i}\vec{e}_2\right]\begin{bmatrix} \omega\sqrt{2}a_+ e^{-\mathbf{i}\varphi_+} \\ \omega\sqrt{2}a_- e^{\mathbf{i}\varphi_-} \end{bmatrix}. \tag{13.18}$$

The Clifford valued two dimensional complex vector field

$$E = \begin{bmatrix} \omega\sqrt{2}a_+ e^{-\mathbf{i}\varphi_+} \\ \omega\sqrt{2}a_- e^{\mathbf{i}\varphi_-} \end{bmatrix} \tag{13.19}$$

plays an important role since it enters in the definition of the *coherence* density matrix.

We observe that $\vec{\mathfrak{E}}$ looks like the *complex* electric field used in electrodynamic text books, (with the substitution $\mathbf{i} \mapsto \mathrm{i} = \sqrt{-1}$) as, e.g., in the books by Jackson [19] and Landau and Lifshitz [22] . However, it is necessary to emphasize once more that $\sqrt{-1}$ has no meaning in *classical* electromagnetic field and its use in general occults the fundamental geometrical meaning of $\mathbf{i}$.

To continue we put

$$a_1 = a_+ e^{-\mathbf{i}\varphi_+}, \quad a_2 = a_- e^{\mathbf{i}\varphi_-}, \tag{13.20}$$

which are formally complex numbers. Now,

$$\begin{aligned} u + \vec{S} &= 2FF^{-} \\ &= 4\omega^2(|a_1|^2 + e^{2\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}')} a_1^+ a_2 + e^{-2\mathbf{i}(\omega t - \vec{k}' \cdot \vec{x}')} a_1 a_2^+ + |a_2|^2)(1 + \vec{e}_3), \end{aligned} \tag{13.21}$$

where u and $\vec{S}$ are respectively the energy density and the Poynting vector. The mean value of this equation over the rapid oscillations of the field gives

$$\overline{(u + \vec{S})} = 4\omega^2(|a_1|^2 + |a_2|^2)(1 + \vec{e}_3). \tag{13.22}$$

Now we calculate the product EE^{+}. This gives,

$$\begin{aligned} \frac{1}{2\omega^2} EE^{+} &= \mathfrak{e}\begin{bmatrix} |a_1|^2 & a_1 a_2^+ \\ a_1^+ a_2 & |a_2|^2 \end{bmatrix} \\ &= \mathfrak{e}\left(\rho^0 + \rho^1\sigma_1 + \rho^2\sigma_2 + \rho^3\sigma_3\right). \end{aligned} \tag{13.23}$$

In Eq. (13.23) the σ_i, $i = 1, 2, 3$ are the Pauli spin matrices and the matrix $\boldsymbol{\rho}$,

$$\boldsymbol{\rho} = \rho_0 + \rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3\sigma_3,$$
$$\rho_0 = \frac{|a_1|^2 + |a_2|^2}{2}, \quad \rho_1 = \mathrm{Re}\, a_1^+ a_2,\ \rho_2 = \mathrm{Im} a_1^+ a_2,\ \rho_3 = \frac{|a_1|^2 - |a_2|^2}{2}. \tag{13.24}$$

is called the Cartesian representation of the *coherence density* ρ. The ρ^i are called the *Stokes parameters*. The expression for EE^+ suggest to us to define the fake 'spinor' field $\Upsilon = \rho \otimes \mathfrak{e} \in \sec \mathcal{C}\ell^0(M, \eta) \otimes \mathcal{I}(M, \eta) \subset \sec \mathcal{C}\ell(M, \eta) \otimes \mathcal{C}\ell(M, \eta)$, where ρ (the *coherence density matrix*) is given by,

$$\rho = \rho_0 + \rho_1\vec{e}_1 + \rho_2\vec{e}_2 + \rho_3\vec{e}_3 = \rho_0 + \vec{\rho}. \tag{13.25}$$

Now, consider the action of[5] $\mathrm{Spin}_3(M) \otimes \mathrm{Spin}_3(M)$ on the sections of $\sec \mathcal{C}l^0(M, \eta) \otimes \mathcal{I}(M, \eta) \subset \sec \mathcal{C}l(M, \eta) \otimes \mathcal{C}\ell(M, \eta)$ defined by

$$\Upsilon \mapsto \Upsilon' = U\rho U^{-1} \otimes U^{-1}\mathfrak{e},\ U \in \sec \mathrm{Spin}_3(M). \tag{13.26}$$

Since the information contained in Υ and Υ' are the same we see that $U\rho U^{-1}$has also the same information carried by ρ. The matrix representation of $U\rho U^{-1}$ is given by

$$\mathbf{U}\boldsymbol{\rho}\mathbf{U}^{-1} \tag{13.27}$$

where for any $x \in M$, $\mathbf{U}(x) \in SU(2)$ is the *matrix* representation of $U \in \sec \mathrm{Spin}_3(M)$ and $\boldsymbol{\rho}$ the matrix representation of ρ. For example, for

$$\mathbf{U} = \frac{1}{2}\begin{bmatrix} \mathrm{i} & \mathrm{i} \\ -1 & 1 \end{bmatrix} \tag{13.28}$$

we have the so called *Jones* representation. Writing

$$\Upsilon = \rho^0 \otimes \mathfrak{e} + \vec{\rho} \otimes \mathfrak{e} \tag{13.29}$$

we see that the coherence matrix, defines an unitary[6] vector $\vec{\rho} \otimes \mathfrak{e}$ in an *internal* space. This means that any tentative of associating $\vec{\rho}$ or better, $\rho\gamma_0 = \rho_0\gamma^0 - \rho_i\gamma^i$

[5] $\mathrm{Spin}_3 \simeq \mathrm{SU}(2)$.

[6] Recall that $\vec{\rho}$ is an unitary vector.

with a *spacetime* vector field will produce nonsense.[7] Recall the ratios

$$\tau_L = \sqrt{\frac{{\rho_1}^2 + {\rho_3}^2}{{\rho_0}^2}},\ \tau_C = \frac{\rho_2}{\rho_0} \tag{13.30}$$

which are called respectively the degree of linear polarization and the degree of circular polarization. We can write recalling Eqs. (13.24) and (13.20)

$$\begin{aligned}\tau_C &= \frac{|a_+|^2 - |a_-|^2}{|a_+|^2 + |a_-|^2} \\ &= \mathbf{i}\vec{e}_3 \cdot \frac{\mathfrak{F} \times \mathfrak{F}^+}{\mathfrak{F} \cdot \mathfrak{F}^+}.\end{aligned} \tag{13.31}$$

This last formula for τ_C appears in electrodynamics and optics books written in terms of a complex Euclidean vector field $\vec{\mathcal{E}}$ which has the same form as the biform field $\mathfrak{F} \in \sec \mathcal{C}\ell^0(M, \eta)$ (with the substitution $\mathbf{i} \mapsto \mathrm{i} = \sqrt{-1}$). Explicitly, we find, e.g., in Silverman's book [42],

$$\tau_C = \mathrm{i}\vec{e}_3 \cdot \frac{\vec{\mathcal{E}} \times \vec{\mathcal{E}}^+}{\vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^+}, \quad \mathrm{i} = \sqrt{-1}. \tag{13.32}$$

Now, in [9–15] it is defined

$$\vec{B}^{(3)} = \frac{-\mathrm{i}e}{\omega} \vec{\mathcal{E}} \times \vec{\mathcal{E}}^+ \tag{13.33}$$

and it is claimed that this phaseless field $\vec{B}^{(3)}$ is a fundamental longitudinal magnetic field with is an integral part of the plane wave field configurations. Obviously, this is sheer nonsense, and Silverman's in his wonderful book [42] writes in this respect:

"Expression (13.32)[8] is specially interesting, for it is not, in my experience, a particularly well-known relation. Indeed, it is sufficiently obscure that in recent years an extensive scientific literature has developed examining in minute detail the far reaching electrodynamic, quantum, and cosmological implications of a "new" nonlinear light interaction proportional to $\vec{\mathcal{E}} \times \vec{\mathcal{E}}^+$ (deduced by analogy to the Poynting vector $\vec{S} \propto \vec{\mathcal{E}} \times \vec{\mathcal{H}}^+$) and interpreted as a "longitudinal magnetic field" carried by the photon. Several books have been written on the subject. Were any of this true, such a radical revision of Maxwellian electrodynamics would of course be highly exciting, but it is regrettably the chimerical product of

[7] This is just what happened with the misleading $\vec{B}^{(3)}$ theory presented in a series of books and inumerous articles, see e.g., [9–15].

[8] In Silverman's book his Eq. (34), pp.167 is the one that corresponds to our Eq. (13.32).

self-delusion—just like the "discovery" of N-Rays in the early 1900s.(During the period 1903–1906 some 120 trained scientists published almost 300 papers on the origins and characteristics of a totally spurious radiation first purported by a French scientist, René Blondlot)." [9]

Of course, the real meaning of the right hand side of Eq. (13.31) (or Eq. (13.32)) is that it is a generalization of the concept of *helicity* which is defined for a single photon in quantum theory (see, e.g., [20, 42]). We shall not give the details here, we only quote that, e.g., for a right circularly polarized plane wave (helicity -1), $a_+ = 0, a_- = 1$ and $\tau_C = -1$.

Exercise 13.1 Find *PWS* of the free Maxwell equations $\boldsymbol{\partial} F = 0$ directly.

Solution As we know ME in vacuum can be written as

$$\boldsymbol{\partial} F = 0, \tag{13.34}$$

where $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$. The well known PWS of Eq. (13.34) are obtained as follows. In a given Lorentzian chart $\{x^\mu\}$ of the maximal atlas of M naturally adapted to the inertial reference frame $\partial/\partial x^0$ a PWS moving in the z-direction is written as

$$F = fe^{\gamma_5 k \cdot x}, \tag{13.35}$$

$$k = k^\mu \gamma_\mu,\ k^1 = k^2 = 0,\ x = x^\mu \gamma_\mu, \tag{13.36}$$

where $k, x \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ and where f is a constant 2-form. From Eqs. (13.34) and (13.35) it follows that

$$kF = 0\,. \tag{13.37}$$

Multiplying Eq. (13.37) by k we get

$$k^2 F = 0 \tag{13.38}$$

and since $k \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ then

$$k^2 = 0 \ \leftrightarrow\ k_0 = \pm|\vec{k}| = k^3, \tag{13.39}$$

i.e., the propagation vector is light-like. Also

$$F^2 = -F \cdot F + F \wedge F = 0 \tag{13.40}$$

[9]Of course, Silverman is referring to papers like, e.g., [9–15] and hundred of others by one author and his many associates. Some of the absurdities of those papers are discussed in [6].

as can be easily seen by multiplying both members of Eq. (13.37) by F and taking into account that $k \neq 0$. Equation (13.40) says that the field invariants are null for PWS. Such fields are known as *null fields*.

As in the previous section we emphasize again the fundamental role of the volume element γ_5 (duality operator) in electromagnetic theory. In particular since $e^{\gamma_5 k\cdot x} = \cos k\cdot x + \gamma_5 \sin k\cdot x$, we see that

$$F = f\cos k\cdot x + \gamma_5 f\sin k\cdot x. \tag{13.41}$$

Writing $F = \vec{E} + \mathbf{i}\vec{B}$, with $\mathbf{i} \equiv \gamma_5$ and choosing $f = \vec{e}_1 + \mathbf{i}\vec{e}_2$, $\vec{e}_1 \cdot \vec{e}_2 = 0$, $\vec{e}_1, \vec{e}_2$ constant vectors in the Pauli subbundle sense, Eq. (13.35) becomes

$$(\vec{E} + \mathbf{i}\vec{B}) = \vec{e}_1 \cos kx - \vec{e}_2 \sin k\cdot x + \mathbf{i}(\vec{e}_1 \sin k\cdot x + \vec{e}_2 \cos k\cdot x). \tag{13.42}$$

This equation is important because it shows once again that we must take care with the $\mathrm{i} = \sqrt{-1}$ that appears in usual formulations of Maxwell theory using complex electric and magnetic fields. The $\mathrm{i} = \sqrt{-1}$ in many cases unfolds a secret that can only be known through Eq. (13.42). From Eq. (13.37) we can also easily show that $\vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{B} = 0$, i.e., PWS of ME are *transverse waves*. However, contrary to common belief, the *free* Maxwell equations possess also solutions that are not transverse waves and for which $F^2 \neq 0$. Those (extraordinary) solutions [32, 33, 35] and their properties will be fully investigated in a forthcoming book [8].

We can rewrite Eq. (13.37) as

$$k\gamma_0\gamma_0 F\gamma_0 = 0 \tag{13.43}$$

and since $k\gamma_0 = k_0 + \vec{k}$, $\gamma_0 F\gamma_0 = -\vec{E} + \mathbf{i}\vec{B}$ we have

$$\vec{k}f = k_0 f. \tag{13.44}$$

Now, we recall that in $\mathcal{C}\ell^0(M,\eta)$ (where the typical fiber is isomorphic to the Pauli algebra $\mathbb{R}_{3,0}$) we can introduce the operator of space conjugation [18] denoted by $*$ such that writing $f = \vec{e} + \mathbf{i}\vec{b}$ we have

$$f^* = -\vec{e} + \mathbf{i}\vec{b} \ ; \ k_0^* = k_0 \ ; \ \vec{k}^* = -\vec{k}. \tag{13.45}$$

We can now interpret the two solutions of $k^2 = 0$, i.e., $k_0 = |\vec{k}|$ and $k_0 = -|\vec{k}|$ as corresponding to the solutions $k_0 f = \vec{k}f$ and $k_0 f^* = -\vec{k}f^*$; f and f^* correspond in quantum theory to "photons" which are of positive or negative helicities. We can interpret $k_0 = |\vec{k}|$ as a particle and $k_0 = -|\vec{k}|$ as an antiparticle.

Summarizing we have the following important facts concerning PWS of ME: (1) the propagation vector is light-like, $k^2 = 0$; (2) the field invariants are null, $F^2 = 0$; (3) the PWS are transverse waves, i.e., $\vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{B} = 0$.

13.3 'Dirac Like' Representations of ME

We exhibit in this section some *faces* (i.e., different mathematical representations) of ME where in any case the Maxwell field is represented by an *amorphous* spinor field (see Sect. 7.8.) satisfying a 'Dirac like' equation. In the next section we show how, given a spin coframe, we can associate a representative $\psi_\Xi \in \sec \mathcal{C}\ell(M,\eta)$ of a Dirac-Hestenes spinor field to the Faraday field $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. The equation satisfied by ψ_Ξ gives a *mathematical* Maxwell-Dirac equivalence of the first kind.

We start from the Maxwell equation written in the Clifford bundle, i.e.,

$$\boldsymbol{\partial} F = J, \tag{13.46}$$

where $F = \frac{1}{2}F^{\mu\nu}\gamma_\mu\gamma_\nu \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ and $J \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. Moreover, we recall that

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{13.47}$$

and that we can write

$$F = \vec{E} + \mathbf{i}\vec{B} \tag{13.48}$$

with $\vec{E} = E^i\vec{\sigma}_i$, $\vec{B} = B^i\vec{\sigma}_i$, $\vec{\sigma}_i = \gamma_I\gamma_0$ as discussed in the previous section.

13.3.1 'Dirac Like' Representation of ME on $\mathcal{I}'(M,\eta)$

Now, consider the bundle of amorphous spinor fields $\mathcal{I}'(M,\eta) = \mathcal{C}\ell^0(M,\eta)\mathrm{e}$, where

$$\mathrm{e} = \frac{1}{2}(1 - \gamma^3\gamma^0) = \frac{1}{2}(1 + \vec{\sigma}_3) \in \sec \mathcal{C}\ell^0(M,\eta) \tag{13.49}$$

is a primitive idempotent field.

We can perform the following algebraic manipulations in ME.

$$\boldsymbol{\partial} F = J \Rightarrow \boldsymbol{\partial}\gamma^0\gamma^0 F\gamma^0 = J\gamma^0. \tag{13.50}$$

Also,

$$\begin{gathered} \boldsymbol{\partial}\gamma^0 = \vec{\sigma}^\mu\partial_\mu, \vec{\sigma}^0 \equiv 1, \vec{\sigma}^i = -\vec{\sigma}_i, \\ \gamma^0 F\gamma^0 = -\vec{E} + \mathbf{i}\vec{B},\ J\gamma^0 = J_\mu\vec{\sigma}^\mu. \end{gathered} \tag{13.51}$$

Writing $F^{+} = \vec{E} - \mathbf{i}\vec{B}$, $\varphi = F^{+}\mathrm{e}$, $\chi = -J_{\mu}\vec{\sigma}^{\mu}\,\mathrm{e} \in \sec \mathcal{I}'(M,\eta)$, we have,

$$\vec{\sigma}^{\mu}\partial_{\mu}\varphi = \chi. \tag{13.52}$$

As the reader can easily verify recalling the results of Chap. 3, Eq. (13.52) has a two dimensional matrix representation, namely

$$\sigma^{\mu}\partial_{\mu}\boldsymbol{\varphi} = \boldsymbol{\chi} \tag{13.53}$$

with

$$\boldsymbol{\varphi} = \begin{bmatrix} -E_3 + \mathrm{i}B_3 & 0 \\ -E_1 + \mathrm{i}E_2 + \mathrm{i}B_1 - B_2 & 0 \end{bmatrix}, \qquad \boldsymbol{\chi} = \begin{bmatrix} -J_0 + J_3 & 0 \\ J_1 + \mathrm{i}J_2 & 0 \end{bmatrix}. \tag{13.54}$$

13.3.2 Sachs ‘Dirac Like’ Representation of ME

Consider the equation $\boldsymbol{\partial}\psi = \varkappa$, $\psi = F\mathrm{e}''$ and $\varkappa = J\mathrm{e}''$, where the idempotent field $\mathrm{e}'' = \frac{1}{2}(1 + \gamma^{3}\gamma^{0}) \in \sec \mathcal{C}\ell^{0}(M,\eta)$ generates the bundle of amorphous (Weyl) spinor fields $\mathcal{I}''(M,\eta) = \mathcal{C}\ell^{0}(M,\eta)\mathrm{e}''$. The objects living in $\mathcal{I}''(M,\eta)$ have a 4×4 matrix representation, as the reader may easily verify using the results of Chap. 3. We have,

$$\boldsymbol{\psi} = \begin{bmatrix} -E_3 + \mathrm{i}B_3 & 0 & 0 & 0 \\ -E_1 - \mathrm{i}E_2 + \mathrm{i}B_1 - B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_1 - \mathrm{i}E_2 + \mathrm{i}B_1 + B_2 \\ 0 & 0 & 0 & -E_3 - \mathrm{i}B_3 \end{bmatrix},$$

$$\boldsymbol{\chi} = \begin{bmatrix} 0 & 0 & 0 & -J_1 + \mathrm{i}J_2 \\ 0 & 0 & 0 & -J_0 + J_3 \\ -J_0 + J_3 & 0 & 0 & 0 \\ J_1 + \mathrm{i}J_2 & 0 & 0 & 0 \end{bmatrix}. \tag{13.55}$$

Putting now,

$$\hat{\boldsymbol{\varphi}}_1 = \begin{bmatrix} -E_3 + \mathrm{i}B_3 \\ -E_1 - \mathrm{i}E_2 + \mathrm{i}B_1 - B_2 \end{bmatrix}, \qquad \hat{\boldsymbol{\varkappa}}_1 = \begin{bmatrix} J_0 + J_3 \\ J_1 + \mathrm{i}J_2 \end{bmatrix}, \tag{13.56}$$

$$\hat{\boldsymbol{\varphi}}_2 = \begin{bmatrix} E_1 - \mathrm{i}E_2 + \mathrm{i}B_1 + B_2 \\ -E_3 - \mathrm{i}B_3 \end{bmatrix}, \qquad \hat{\boldsymbol{\varkappa}}_2 = \begin{bmatrix} -J_1 + \mathrm{i}J_2 \\ J_0 + J_3 \end{bmatrix},$$

the matrix representation of the equation $\boldsymbol{\partial} F\mathrm{e}'' = \chi \mathrm{e}''$ decouples in two differential equations for matrix representation of the amorphous (Weyl) spinor fields namely

$$\hat{\boldsymbol{\sigma}}^{\mu}\partial_{\mu}\hat{\boldsymbol{\phi}}_{a}\hat{\boldsymbol{\varphi}}_{a} = \hat{\boldsymbol{\xi}}_{a}, \quad a = 1, 2. \tag{13.57}$$

Equation (13.57) has been first obtained by Sachs [37] in an ad hoc way starting from a non covariant equation. Of course, both equations in Eq. (13.57) carries the same information, exactly the same information carried by Eq. (13.53).[10]

13.3.3 Sallhöfer 'Dirac Like' Representation of ME

Consider now the idempotent $\mathrm{e} = \frac{1}{2}(1 + \gamma^0) \in \sec \mathcal{C}\ell(M, \eta)$. The 4×4 matrix representations of the amorphous spinor fields $\phi = F\mathrm{e}$, $\xi = J\mathrm{e} \in \sec \mathcal{C}\ell(M, \eta)\mathrm{e}$ are easily found as

$$\boldsymbol{\phi} = \begin{bmatrix} \mathrm{i}B_3 & \mathrm{i}B_1 + B_2 & 0 & 0 \\ -\mathrm{i}B_1 + B_2 & -\mathrm{i}B_3 & 0 & 0 \\ E_3 & E_1 - \mathrm{i}E_2 & 0 & 0 \\ E_1 + \mathrm{i}E_2 & -E_3 & 0 & 0 \end{bmatrix}, \tag{13.58}$$

$$\boldsymbol{\xi} = \begin{bmatrix} J_0 & 0 & 0 & 0 \\ 0 & J_0 & 0 & 0 \\ J_3 & J_1 - \mathrm{i}J_2 & 0 & 0 \\ J_1 + \mathrm{i}J_2 & -J_3 & 0 & 0 \end{bmatrix}. \tag{13.59}$$

We can easily verify that each one of the non null columns of $\boldsymbol{\phi}$ satisfies a 'Dirac-like' equation. In particular, taking into account that the *ME* are invariant under the substitutions $\vec{B} \mapsto \vec{E}, \vec{E} \mapsto -\vec{B}$ and considering a medium with dielectric susceptibility ϵ and magnetic permeability μ (both possibly, spacetime functions) and making the substitution $\vec{B} \mapsto \vec{H}$ in Eq. (13.58), writing moreover c for the speed of light in vacuum,[11] we get that the free electromagnetic fields in the medium satisfies

$$\left\{ \underline{\vec{\gamma}} \cdot \boldsymbol{\nabla} - \begin{bmatrix} \epsilon \mathbf{I}_2 & \\ & -\mu \mathbf{I}_2 \end{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} \right\} \boldsymbol{\phi} = 0, \tag{13.60}$$

[10] The pair of Eq. (13.57) suggest the existence of new invariants for the electromagnetic fields, and indeed Sachs made interesting use of them in [37] .

[11] We are using s system of units such that $c = 1$.

which is an equation originally obtained by Sallhöfer [38–41] in an ad hoc way. Recently Smulik [44] obtained several interesting solutions for ME from known solutions of Eq. (13.60) for some ϵ and μ functions.

13.3.4 A Three Dimensional Representation of the Free ME

Here we derive a three dimensional representation of the free ME, first presented by Majorana [25] (see also [21]), but obtained in a completely different way from the one given below. Our starting point is Eq. (13.50) (obviously equivalent to ME) which since it is also satisfied by $\gamma_5 F$ can be rewritten (in the case $J = 0$) in the following equivalent ways in $\mathcal{C}\ell^0(M,\eta) \subset \mathcal{C}\ell(M,\eta)$.

$$\begin{aligned} \boldsymbol{\partial}(\gamma_5 F) &= 0, \\ \mathbf{i}\vec{\sigma}^{\mu}\partial_{\mu}F &= 0, \\ (\mathbf{i}\frac{\vec{\sigma}^0}{2})\frac{\partial}{\partial t}F &= -\mathbf{i}\frac{\vec{\sigma}^i}{2}\partial_i F. \end{aligned} \tag{13.61}$$

Now, we recall that $\left[\vec{\sigma}^i/2, \vec{\sigma}^j/2\right] = \mathbf{i}\varepsilon_k^{ij}\vec{\sigma}^k/2$, i.e., the set $\{\vec{\sigma}^i/2\}$ is a basis for any $x \in M$ of the Lie algebra su(2) of SU(2), the universal covering group of SO_3, the special rotation group in three dimensions. A three dimensional representation of su(2) is given by the Hermitian matrices

$$\mathbf{K}^p = \begin{bmatrix} 0 & -\mathrm{i}\delta^{p3} & \mathrm{i}\delta^{p2} \\ \mathrm{i}\delta^{p3} & 0 & -\mathrm{i}\delta^{p1} \\ -\mathrm{i}\delta^{p2} & \mathrm{i}\delta^{p1} & 0 \end{bmatrix} \tag{13.62}$$

and

$$[\mathbf{K}^p, \mathbf{K}^q] = \mathrm{i}\epsilon_r^{pq}\mathbf{K}^r. \tag{13.63}$$

Writing moreover $\mathbf{K}^0 = \mathbf{I}_3$ for the three dimensional unitary matrix and defining

$$|\mathcal{F}\rangle = \begin{bmatrix} E_1 + \mathrm{i}B_1 \\ E_2 + \mathrm{i}B_2 \\ E_3 + \mathrm{i}B_3 \end{bmatrix}, \tag{13.64}$$

we can obtain ME in three dimensional form with the substitutions

$$\frac{1}{2}\vec{\sigma}^{\mu} \mapsto \mathbf{K}^{\mu}, \mathbf{i} \mapsto \mathrm{i}, F \mapsto |\mathcal{F}\rangle. \tag{13.65}$$

in Eq. (13.61). We get

$$\mathrm{i}\frac{\partial}{\partial t}|\mathcal{F}\rangle = -\mathrm{i}\mathbf{K}\cdot\nabla|\mathcal{F}\rangle \tag{13.66}$$

Note the *doubling* of the representative of the unity element of $\mathcal{C}\ell^0(M,\eta)$ when going to the three dimensional representation. This corresponds to the fact that in relativistic quantum field theory, the 2×2 matrix representation of Eq. (13.61) (projected in the idempotent $\frac{1}{2}(1+\vec{\sigma}^3)$) represents[12] the wave equation for a single quantum of a massless spin 1/2 field, whereas Eq. (13.66) represents the wave equation for a single quantum of a massless spin 1 field.[13]

13.4 Mathematical Maxwell-Dirac Equivalence

In [1–5] using standard covariant spinor fields Campolattaro proposed that Maxwell equations are equivalent to a *non* linear Dirac like equation. The subject has been further developed in [31, 36, 47, 48] using the Clifford bundle formalism. The crucial point in proving the mentioned equivalence (abbreviated as MDE in what follows, when no confusion arises), starts once we observe that to any given *representative* $\psi \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M) \hookrightarrow \sec\mathcal{C}\ell(M,\eta)$ of a Dirac-Hestenes spinor field in a given spin coframe there is associated an electromagnetic field $F \in \sec\bigwedge^2 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\eta)$, ($F^2 \neq 0$) through the Rainich-Misner theorem, i.e., we have [30, 31, 36, 47, 48]

$$F = \psi\gamma_{21}\tilde{\psi}. \tag{13.67}$$

Before proceeding we recall that for null fields, i.e., $F^2 = 0$, the spinor associated with F through Eq. (13.67) must be a Majorana spinor field (Sect. 3.3). Now, since an electromagnetic field F, with $F^2 \neq 0$ satisfying Maxwell equation[14] has six degrees of freedom and a Dirac-Hestenes spinor field has eight (real) degrees of freedom some authors felt uncomfortable with the approach used in [31, 36, 47, 48] where some *gauge conditions* have been imposed on a *nonlinear* equation

[12]Of course, it is necessary for the quantum mechanical interpretation to multiply both sides of Eq. (13.66) by $\hbar$, the Planck constant.

[13]Indeed in quantum mechanics the Pauli matrices $\boldsymbol{\sigma}_i$ and the matrices $\mathbf{K}_i$ are the quantum mechanical spin operators and

$$\sum_{i=1}^{3}(\boldsymbol{\sigma}_i)^2 = \frac{1}{2}(1+\frac{1}{2}) = \frac{3}{4},\ \sum_{i=1}^{3}(\mathbf{K}_i)^2 = 1.(1+1) = 2.$$

.

[14]Such solutions exist [32, 33, 35] and are investigated in details in a forthcoming book [8].

(equivalent to Maxwell equation), thereby transforming it into an usual *linear* Dirac equation (the *Dirac-Hestenes equation* in the Clifford bundle formalism). The claim, e.g., in [16] is that the MDE found in [31, 36, 47, 48] cannot be general. The argument there is that the imposition of *gauge conditions* implies that a ψ satisfying Eq. (13.67) can have only six (real) degrees of freedom, and this implies that the Dirac-Hestenes equation corresponding to Maxwell equation can be only satisfied by a restricted class of Dirac-Hestenes spinor fields, namely the ones that have six degrees of freedom.

Incidentally, in [16] it is also claimed that the generalized Maxwell equation

$$\partial F = J_e + \gamma_5 J_m \tag{13.68}$$

(where $J_e, J_m \in \sec \bigwedge^1 T^*M$) describing the electromagnetic field generated by charges and monopoles [24] cannot hold in the Clifford bundle formalism, because according to that author the formalism implies that $J_m = 0$.

In what follows we analyze those claims of [16] and prove that they are equivocated. The reason for our enterprise is that as will become clear in what follows, understanding of Eqs. (13.67) and (13.68) together with some reasonable hypothesis permit a derivation and eventually even a possible physical interpretation of the famous Seiberg-Witten monopole equations [27, 28, 43]. So, our plan is the following: first we prove that given F in Eq. (13.67) we can solve that equation for ψ, and we find that ψ has eight degrees of freedom, two of them being undetermined, the indetermination being related to the elements of the *stability group* of the spin plane γ_{21}. This is a non trivial and beautiful result which can be called *inversion* formula. Next, we introduce a *generalized* Maxwell equation and the *generalized* Hertz equation. After that we prove a *mathematical* Dirac-Maxwell equivalence of the *first kind* [31, 36, 47, 48], thereby deriving a Dirac-Hestenes equation from the free Maxwell equations. Moreover, we introduce a new form of a *mathematical* Maxwell-Dirac equivalence, called MDE of the second kind. This new MDE of the second kind suggests that the electron is a 'composite' system. To prove the Maxwell-Dirac equivalence of the *second kind* we decompose a Dirac-Hestenes spinor field satisfying a Dirac-Hestenes equation in such a way that it results in a nonlinear generalized Maxwell (like) equation satisfied by a certain Hertz potential field, mathematically represented by an object of the same mathematical nature as an electromagnetic field, i.e., $\Pi \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$. This new equivalence is very suggestive in view of the fact that there are recent (wild) speculations that the electron can be splitted in two components [26] (see also [7]). If this fantastic claim announced by Maris [26] is true, it is necessary to understand what is going on. The new Maxwell-Dirac equivalence of the second kind may eventually be useful to understand the mechanism behind the "electron splitting" into electrinos, but we are not going to discuss these ideas here. Instead, we concentrate our attention in showing that (the *analogous* on Minkowski spacetime) of the famous Seiberg-Witten monopole equations arises naturally from the MDE of the first kind once a reasonable hypothesis is imposed. We also present a possible coherent interpretation of that equations. Indeed, we prove that when the Dirac-Hestenes spinor field satisfying the first of Seiberg-Witten equations is an eigenvector of the parity operator, then that equation describe a pair of massless 'monopoles' of opposite 'magnetic' like charges, coupled together by its interaction electromagnetic field.

13.4.1 Solution of $F = \psi\gamma_{21}\tilde{\psi}$

We now want solve Eq. (13.67) for ψ. Before proceeding we observe that on Euclidian spacetime this equation has been solved using Clifford algebra methods in [46]. Also, on Minkowski spacetime a *particular* solution of an equivalent equation (written in terms of biquaternions) appear in [17]. We are going to show that contrary to the claims of [16] a general solution for ψ has indeed eight degrees of freedom, although two of them are *arbitrary*, i.e., not fixed by F alone. Once we give a solution of Eq. (13.67) for ψ, the reason for the indetermination of two of the degrees of freedom will become clear. This involves the Fierz identities, the boomerangs and the general theorem permitting the reconstruction of spinors from their bilinear invariants discussed in Chap. 3.

We start by observing that from Eq. (13.67) and from Eq. (3.92) valid for invertible representatives of DHSF we can write

$$F = \rho e^{\beta\gamma_5} R\gamma_{21}\tilde{R}. \tag{13.69}$$

Then, defining $f = F/\rho e^{\beta\gamma_5}$ it follows that

$$f = R\gamma_{21}\tilde{R}, \tag{13.70}$$

$$f^2 = -1. \tag{13.71}$$

Now, since all objects in Eqs. (13.69) and (13.70) are even we can take advantage of the isomorphism $\mathbb{R}_{3,0} \equiv \mathbb{R}^0_{1,3}$ and making the calculations when convenient in the Pauli algebra. To this end we recall Eq. (13.47) for the components of F, where (E^1, E^2, E^3) and (B^1, B^2, B^3) are respectively the Cartesian components of the electric and magnetic fields.

We now write as already done above F in $\mathcal{C}\ell^0(M, \eta)$, the even subbundle of $\mathcal{C}\ell(M, \eta)$.

$$F = \vec{E} + \mathbf{i}\vec{B}, \tag{13.72}$$

with $\vec{E} = E^i\vec{\sigma}_i$, $\vec{B} = B^j\vec{\sigma}_j$, $i, j = 1, 2, 3$. We can write an analogous equation for f,

$$f = \vec{e} + \mathbf{i}\vec{b}. \tag{13.73}$$

Now, since $F^2 \neq 0$ and

$$\begin{aligned} F^2 &= F \lrcorner F + F \wedge F \\ &= (\vec{E}^2 - \vec{B}^2) + 2\mathbf{i}(\vec{E} \cdot \vec{B}) \end{aligned} \tag{13.74}$$

the above equations give (in the more general case where both $I_1 = (\vec{E}^2 - \vec{B}^2) \neq 0$ and $I_2 = (\vec{E} \cdot \vec{B}) \neq 0$):

$$\rho = \frac{\sqrt{\vec{E}^2 - \vec{B}^2}}{\cos[arctg 2\beta]}, \qquad \beta = \frac{1}{2} \arctan\left(\frac{2(\vec{E} \cdot \vec{B})}{\vec{E}^2 - \vec{B}^2}\right). \tag{13.75}$$

Also,

$$\vec{e} = \frac{1}{\rho}[(\vec{E} \cos\beta + \vec{B} \sin\beta)], \qquad \vec{b} = \frac{1}{\rho}[(\vec{B} \cos\beta - \vec{E} \sin\beta)]. \tag{13.76}$$

13.4.2 A Particular Solution

Now, we can verify that

$$L = \frac{\gamma_{21} + f}{\sqrt{2(1 - \gamma_5 \mathfrak{I})}} = \frac{\vec{\sigma}_3 - \mathbf{i}\vec{f}}{\mathbf{i}\sqrt{2(1 - \mathbf{i}(\vec{f} \cdot \vec{\sigma}_3)}}, \tag{13.77}$$

$$\mathfrak{I} = f^{03} - \gamma_5 f^{12} \equiv \vec{f} \cdot \vec{\sigma}_3. \tag{13.78}$$

is a Lorentz transformation, i.e., $L\tilde{L} = \tilde{L}L = 1$. Moreover, L is a particular solution of Eq. (13.70). Indeed,

$$\frac{\gamma_{21} + f}{\sqrt{2(1 - \gamma_5 \mathfrak{I})}} \gamma_{21} \frac{\gamma_{12} - f}{\sqrt{2(1 - \gamma_5 \mathfrak{I})}} = \frac{f[2(1 - \gamma_5 \mathfrak{I})]}{2(1 - \gamma_5 \mathfrak{I})} = f. \tag{13.79}$$

Of course, since $f^2 = -1$, $\vec{e}^2 = \vec{b}^2 - 1$and $\vec{e} \cdot \vec{b} = 0$, there are only four real degrees of freedom in the Lorentz transformation L. From this result in [16] it is concluded that the solution of the Eq. (13.67) is the Dirac-Hestenes spinor field

$$\phi = \sqrt{\rho} e^{\gamma_5 \beta} L, \tag{13.80}$$

which has only *six* degrees of freedom and thus is not equivalent to a general Dirac-Hestenes spinor field (the spinor field that must appears in the Dirac-Hestenes equation), which has *eight* degrees of freedom. In this way it is stated in [16] that a the MDE of first kind proposed in [31, 36, 47, 48] cannot hold. Well, although it is *true* that Eq. (13.80) is a solution of Eq. (13.67) it is not a *general* solution, but only a *particular* solution.

13.4.3 The General Solution

The general solution R of Eq. (13.67) is trivially found. It is

$$R = LS, \tag{13.81}$$

where L is the particular solution just found and S is any member of the *stability group* of γ_{21}, i.e.,

$$S\gamma_{21}\tilde{S} = \gamma_{21},\ S\tilde{S} = \tilde{S}S = 1. \tag{13.82}$$

It is trivial to find that we can parametrize the elements of the stability group as

$$S = \exp(\gamma_{03}\nu)\exp(\gamma_{21}\varphi), \tag{13.83}$$

with $0 \leq \nu < \infty$ and $0 \leq \varphi < \infty$. This shows that the most *general* Dirac-Hestenes spinor field that solves Eq. (13.67) has indeed eight degrees of freedom (as it must be the case, if the claims of [31, 36, 47, 48] are to make sense), although two degrees of freedom are arbitrary, i.e., they are like *hidden variables*!

Now, the reason for the *indetermination* of two degrees of freedom has to do with a fundamental mathematical result: the fact that a spinor can only be reconstruct through the knowledge of its bilinear invariants and the Fierz identities as discussed in Chaps. 3 and 7.

13.4.4 The Generalized Maxwell Equation

To comment on the basic error in [16] concerning the Clifford bundle formulation of the generalized Maxwell equation we recall the following.

The generalized Maxwell equation [24] which describes the electromagnetic field generated by charges and monopoles, can be written in the Cartan bundle (of the oriented manifold M) as

$$dF = K_m, \qquad dG = K_e \tag{13.84}$$

where $F, G \in \bigwedge^2 T^*M$ and $K_m, K_e \in \bigwedge^3 T^*M$, i.e., all these objects are taken to be *pair* forms (see Sect. 4.9.5)

These equations are independent of *any* metric structure defined on the world manifold. When a metric is given and the Hodge dual operator $\star$ is introduced it is supposed that in vacuum we have $G = \star F$. In this case putting $K_e = - \star J_e$ and $K_m = \star J_m$, with $J_e, J_m \in \sec \bigwedge^1 T^*M$, we can write the following equivalent set of equations

$$dF = - \star J_m, d \star F = - \star J_e, \quad (13.85)$$

$$\delta \star F = J_m, \delta F = -J_e \quad (13.86)$$

$$dF = - \star J_m, \delta F = -J_e. \quad (13.87)$$

Now, supposing that any $\sec \bigwedge^j T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ $(j = 0, 1, 2, 3, 4)$ we get from the above equations

$$(d - \delta)F = J_e + K_m \text{ or } \qquad (d - \delta) \star F = -J_m + K_e, \quad (13.88)$$

or equivalently

$$\boldsymbol{\partial} F = J_e + \gamma_5 J_m \text{ or } \qquad \boldsymbol{\partial}(-\gamma_5 F) = -J_m + \gamma_5 J_e. \quad (13.89)$$

Now, writing

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \gamma_\nu, \ \star F = \frac{1}{2} (^\star F^{\mu\nu}) \gamma_\mu \gamma_\nu, \quad (13.90)$$

then generalized Maxwell equations in the form given by Eq. (13.86) can be written in components (in a Lorentz coordinate chart) as

$$\partial_\mu F^{\mu\nu} = J_e^\mu, \partial_\mu (^\star F^{\mu\nu}) = -J_m^\mu. \quad (13.91)$$

Now, assuming as in Eq. (13.67) that $F = \psi \gamma_{21} \tilde{\psi}$ and taking into account the relation between ψ and the representation of the standard Dirac spinor field Ψ_D we can write Eq. (13.91) as

$$\partial_\mu \bar{\Psi}_D \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu \right] \Psi_D = 2 J_e^\mu, \qquad \partial_\mu \bar{\Psi}_D \hat{\gamma}_5 \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu \right] \Psi_D = -2 J_m^\mu,$$
$$F^{\mu\nu} = \frac{1}{2} \bar{\Psi}_D \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu \right] \Psi, (^\star F^{\mu\nu}) = \frac{1}{2} \bar{\Psi}_D \hat{\gamma}_5 \left[\hat{\gamma}_\mu, \hat{\gamma}_\nu \right] \Psi_D. \quad (13.92)$$

The reverse of the first formula in Eq. (13.89) reads

$$\boldsymbol{\partial} F = J_e - K_m. \quad (13.93)$$

First summing, and then subtracting Eq. (13.68) with Eq. (13.84) we get the following equations for $F = \psi\gamma_{21}\tilde{\psi}$:

$$\boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi} + (\widetilde{\boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi}}) = 2J_e, \qquad \boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi} - (\widetilde{\boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi}}) = 2K_m, \tag{13.94}$$

which is equivalent to Eq. (3.76) in [16] (where $\mathcal{G}$ is used for the three form of monopolar current). There, it is observed that J_e is even under reversion and K_m is odd. Then, it is claimed that "since reversion is a purely algebraic operation without any particular physical meaning, the monopolar current K_m is necessarily zero if the Clifford formalism is assumed to provide a representation of Maxwell's equation where the source currents J_e and K_m correspond to fundamental physical fields." It is also stated that Eqs. (13.92) and (13.94) imposes different constrains on the monopolar currents J_e and K_m.

It now is clear that those arguments of [16] are fallacious. Indeed, it is obvious that if any *comparison* is to be made, it must be done between J_e and J_m or between K_e and K_m. In this case, it is obvious that both pairs of currents have the same behavior under reversion. This kind of confusion is widespread in the literature, mainly by people that works with the generalized Maxwell equation(s) in component form (Eq. (13.91)).

It seems that experimentally $J_m = 0$ and the following question suggests itself: is there any real physical field governed by a equation of the type of the generalized Maxwell equation (Eq. (13.68))? The answer is *yes*.

13.4.5 The Generalized Hertz Potential Equation

In what follows we accept that $J_m = 0$ and take Maxwell equations for the electromagnetic field $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C\ell}(M,\eta)$ and a current $J_e \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C\ell}(M,\eta)$ as

$$\boldsymbol{\partial} F = J_e. \tag{13.95}$$

Let $\Pi = \frac{1}{2}\Pi^{\mu\nu}\gamma_\mu\gamma_\nu = \vec{\Pi}_e + \mathbf{i}\vec{\Pi}_m \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C\ell}(M,\eta)$ be the so called *Hertz potential* [6, 35]. We write

$$[\Pi^{\mu\nu}] = \begin{bmatrix} 0 & -\Pi_e^1 & -\Pi_e^2 & -\Pi_e^3 \\ \Pi_e^1 & 0 & -\Pi_m^3 & \Pi_m^2 \\ \Pi_e^2 & \Pi_m^3 & 0 & -\Pi_m^1 \\ \Pi_e^3 & -\Pi_m^2 & \Pi_m^1 & 0 \end{bmatrix} \tag{13.96}$$

and define the *electromagnetic potential* by

$$A = -\delta\Pi \in \sec \Lambda^1 T^\star M \hookrightarrow \sec \mathcal{C\ell}(M,\eta), \tag{13.97}$$

Since $\delta^2 = 0$ it is clear that A satisfies the Lorenz gauge condition, i.e.,

$$\delta A = 0. \tag{13.98}$$

Also, let

$$\gamma^5 S = d\Pi \in \sec \bigwedge{}^3 T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \eta), \tag{13.99}$$

and call S, the *Stratton potential*.[15] It follows also that

$$d\left(\gamma^5 S\right) = d^2 \Pi = 0. \tag{13.100}$$

But $d(\gamma^5 S) = \gamma^5 \delta S$ from which we get, taking into account Eq. (13.100),

$$\delta S = 0. \tag{13.101}$$

We can put Eqs. (13.97) and (13.99) in the form of a *single* generalized Maxwell like equation, i.e.,

$$\boldsymbol{\partial} \Pi = (d - \delta)\Pi = A + \gamma^5 S = \mathcal{A}. \tag{13.102}$$

Equation (13.102) is the equation we were looking for. It is a legitimate physical equation. We also have,

$$\Diamond\, \Pi = (d - \delta)^2 \Pi = dA + \gamma_5 dS. \tag{13.103}$$

Next, we define the electromagnetic field by

$$F = \boldsymbol{\partial} \mathcal{A} = \Diamond \Pi = dA + \gamma_5 dS = F_e + \gamma_5 F_m. \tag{13.104}$$

We observe that,

$$\Diamond\, \Pi = 0 \Rightarrow F_e = -\gamma_5 F_m. \tag{13.105}$$

Now, let us calculate $\boldsymbol{\partial} F$. We have,

$$\begin{aligned} \boldsymbol{\partial} F &= (d - \delta)F \\ &= d^2 A + d(\gamma^5 dS) - \delta(dA) - \delta(\gamma^5 dS). \end{aligned} \tag{13.106}$$

[15] This object first appears for the best of our knowledge in [45].

The first and last terms in the second line of Eq. (13.106) are obviously null. Writing,

$$J_e = -\delta dA, \text{ and } \gamma^5 J_m = -d(\gamma^5 dS), \tag{13.107}$$

we get Maxwell equation

$$\boldsymbol{\partial} F = (d - \delta)F = J_e, \tag{13.108}$$

if and only if the magnetic current $\gamma^5 J_m = 0$, i.e.,

$$\delta dS = 0. \tag{13.109}$$

a condition that we suppose to be satisfied in what follows. Then,

$$\begin{aligned} \Diamond A &= J_e = -\delta dA, \\ \Diamond S &= 0. \end{aligned} \tag{13.110}$$

Now, we define,

$$F_e = dA = \vec{E}_e + \mathbf{i}\vec{B}_e, \tag{13.111}$$

$$F_m = dS = \vec{B}_m + \mathbf{i}\vec{E}_m. \tag{13.112}$$

and also

$$F = F_e + \gamma_5 F_m = \vec{E} + \mathbf{i}\vec{B} = (\vec{E}_e - \vec{E}_m) + \mathbf{i}(\vec{B}_e + \vec{B}_m). \tag{13.113}$$

Then, we get

$$\Diamond\, \vec{\Pi}_e = \vec{E}, \qquad \Diamond \vec{\Pi}_m = \vec{B}. \tag{13.114}$$

It is important to keep in mind that:

$$\Diamond\, \Pi = 0 \Rightarrow \vec{E} = 0, \text{ and } \vec{B} = 0. \tag{13.115}$$

Nevertheless, despite this result we have,

Proposition 13.2 (Hertz Theorem)

$$\Diamond\, \Pi = 0 \implies \boldsymbol{\partial} F_e = 0. \tag{13.116}$$

Proof We have immediately from the above equations that

$$\boldsymbol{\partial} F_e = -d(\gamma_5 dS) + \delta(\gamma_5 dS) = \gamma_5 d^2 S - \gamma_5 \delta dS = 0,$$

which proves the proposition. ∎

We remark that Eq. (13.116) has been called the Hertz theorem in [6, 35]. Hertz theorem has been used to find nontrivial *subluminal* and *superluminal* solutions of the free Maxwell equation in [32, 33]. See also our forthcoming book [8].

13.4.6 Maxwell Dirac Equivalence of First Kind

Let us consider a *generalized* Maxwell equation

$$\boldsymbol{\partial} F = \mathcal{J}\,, \tag{13.117}$$

where $\mathcal{J}$ is the generalized electromagnetic current (an electric current J_e plus a magnetic monopole current $-\gamma_5 J_m$, where $J_e, J_m \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M,\eta)$). We proved in a previous section that if $F^2 \neq 0$, then we can write

$$F = \psi\gamma_{21}\tilde{\psi}\,, \tag{13.118}$$

where $\psi \in \sec \mathcal{C}\ell^0(M,\eta)$ is a representative of a Dirac-Hestenes field. If we use Eq. (13.118) in Eq. (13.117) we get

$$\boldsymbol{\partial}(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu\partial_\mu(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu(\partial_\mu\psi\gamma_{21}\tilde{\psi} + \psi\gamma_{21}\partial_\mu\tilde{\psi}) = \mathcal{J}. \tag{13.119}$$

from where it follows that

$$2\gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_2 = \mathcal{J}, \tag{13.120}$$

Consider the identity

$$\gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_2 = \boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi} - \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_0 - \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_4, \tag{13.121}$$

and define moreover the covector fields

$$j = \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_0, \tag{13.122}$$

$$g = \gamma^\mu\langle\partial_\mu\psi\gamma_5\gamma_{21}\tilde{\psi}\rangle_0. \tag{13.123}$$

Taking into account Eqs. (13.120)–(13.123), we can rewrite Eq. (13.119) as

$$\boldsymbol{\partial}\psi\gamma_{21}\tilde{\psi} = \left[\frac{1}{2}\mathcal{J} + (j + \gamma_5 g)\right]. \tag{13.124}$$

Equation (13.124) yields in the case where ψ is non-singular (which corresponds to non-null electromagnetic fields) a representation of Maxwell equation satisfied by a representative of a DHSF. Indeed, we have

$$\boldsymbol{\partial}\psi\gamma_{21} = \frac{e^{\gamma_5\beta}}{\rho}\left[\frac{1}{2}\mathcal{J} + (j + \gamma_5 g)\right]\psi. \tag{13.125}$$

The Eq. (13.125) representing Maxwell equation, written in that form, does not appear to have any relationship with the Dirac-Hestenes equation (Eq. (7.97)). However, we shall make some *algebraic* modifications on it in such a way as to put it in a form that suggests a very interesting and *intriguing relationship* between them, and eventually a possible hidden connection between electromagnetism and quantum mechanics.

Since ψ is supposed to be non-singular ($F^2 \neq 0$) we can use the canonical decomposition of ψ and write $\psi = \rho^{1/2}e^{\beta\gamma_5/2}R$, with $\rho,\ \beta \in \sec\bigwedge^0 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\eta)$ and $R \in \sec\mathrm{Spin}^e_{1,3}(M)$. Then

$$\partial_\mu\psi = \frac{1}{2}(\partial_\mu \ln\rho + \gamma_5\partial_\mu\beta + \Omega_\mu)\psi, \tag{13.126}$$

where we define the 2-form

$$\Omega_\mu = 2(\partial_\mu R)\tilde{R}. \tag{13.127}$$

Using this expression for $\partial_\mu\psi$ into the definitions of the covectors j and g (Eqs. (13.122) and (13.123)) we obtain that

$$j = \gamma^\mu(\Omega_\mu \cdot S)\rho\cos\beta + \gamma_\mu[\Omega_\mu \cdot (\gamma_5 S)]\rho\sin\beta, \tag{13.128}$$

$$g = [\Omega_\mu \cdot (\gamma_5 S)]\rho\cos\beta - \gamma_\mu(\Omega_\mu \cdot S)\rho\sin\beta, \tag{13.129}$$

where we define the *spin* 2-form S by

$$S = \frac{1}{2}\psi\gamma_{21}\psi^{-1} = \frac{1}{2}R\gamma_{21}\tilde{R}. \tag{13.130}$$

We define moreover

$$J = \psi\gamma_0\tilde{\psi} = \rho v = \rho R\gamma^0 R^{-1}, \tag{13.131}$$

where v is a '*velocity' field* for the system. To continue, we define also the 2-form $\Omega = v^\mu\Omega_\mu$ and the scalars Λ and K by

$$\Lambda = \Omega \cdot S, \tag{13.132}$$

$$K = \Omega \cdot (\gamma_5 S). \tag{13.133}$$

Using these definitions we have that

$$\Omega_\mu \cdot S = \Lambda v_\mu, \tag{13.134}$$

$$\Omega_\mu \cdot (\gamma_5 S) = K v_\mu, \tag{13.135}$$

and the vectors j and g can be written as

$$j = \Lambda v\rho\cos\beta + Kv\rho\sin\beta = \lambda\rho v, \tag{13.136}$$

$$g = Kv\rho\cos\beta - \Lambda v\rho\sin\beta = \kappa\rho v, \tag{13.137}$$

where we putted

$$\lambda = \Lambda\cos\beta + K\sin\beta, \tag{13.138}$$

$$\kappa = K\cos\beta - \Lambda\sin\beta. \tag{13.139}$$

The spinorial representation of Maxwell equation is written now as

$$\boldsymbol{\partial}\psi\gamma_{21} = \frac{e^{\gamma_5\beta}}{2\rho}\mathcal{J}\psi + \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0. \tag{13.140}$$

As we already mentioned [32, 33, 35] there are infinite families of non trivial solutions of Maxwell equations such that $F^2 \neq 0$ (which correspond to *subluminal* and *superluminal free* boundary solutions of Maxwell equation). Then, it is opportune to consider the case $\mathcal{J} = 0$. We have,

$$\boldsymbol{\partial}\psi\gamma_{21} = \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0, \tag{13.141}$$

which is *very* similar to the Dirac-Hestenes equation.

Constraining the Degrees of Freedom of ψ

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has *six* degrees of freedom, while a Dirac-Hestenes spinor field has *eight* degrees of freedom and as proved above two of those degrees of freedom are *hidden* variables. We are free therefore to impose two constraints on ψ if it is to represent an electromagnetic field. We choose as constraints the following equations saying that the "currents" j and g are conserved

$$\boldsymbol{\partial}\cdot j = 0 \text{ and } \boldsymbol{\partial}\cdot g = 0. \tag{13.142}$$

Using Eqs. (13.136) and (13.137) these two constraints become

$$\boldsymbol{\partial}\cdot j = \rho\dot{\lambda} + \lambda\boldsymbol{\partial}\cdot J = 0, \tag{13.143}$$

$$\boldsymbol{\partial} \cdot g = \rho \dot{\kappa} + k \boldsymbol{\partial} \cdot J = 0, \tag{13.144}$$

where $J = \rho v$ and $\dot{\lambda} = (v \cdot \boldsymbol{\partial})\lambda$, $\dot{k} = (v \cdot \boldsymbol{\partial})k$. These conditions imply that

$$\kappa \lambda = \lambda \kappa \tag{13.145}$$

which gives ($\lambda \neq 0$):

$$\frac{\kappa}{\lambda} = \text{constant} = -\tan \beta_0, \tag{13.146}$$

or from Eqs. (13.138) and (13.139):

$$\frac{K}{\Lambda} = \tan(\beta - \beta_0). \tag{13.147}$$

Now we observe that β is the angle of the duality rotation from F to $F' = e^{\gamma_5 \beta} F$. If we perform another duality rotation by β_0 we have $F \mapsto e^{\gamma_5(\beta + \beta_0)} F$, and for the Takabayashi angle $\beta \mapsto \beta + \beta_0$. If we work therefore with an electromagnetic field duality rotated by an additional angle β_0, the above relationship becomes

$$\frac{K}{\Lambda} = \tan \beta. \tag{13.148}$$

This is, of course, just a way to say that we can *choose* the constant β_0 in Eq. (13.146) to be zero. Now, this expression gives

$$\lambda = \Lambda \cos \beta + \Lambda \tan \beta \sin \beta = \frac{\Lambda}{\cos \beta}, \tag{13.149}$$

$$\kappa = \Lambda \tan \beta \cos \beta - \Lambda \sin \beta = 0, \tag{13.150}$$

and the spinorial representation of the Maxwell equation (Eq. (13.141)) becomes

$$\boldsymbol{\partial} \psi \gamma_{21} - \lambda \psi \gamma_0 = 0 \tag{13.151}$$

Note that λ is such that

$$\rho \dot{\lambda} = -\lambda \boldsymbol{\partial} \cdot J. \tag{13.152}$$

The current $J = \psi \gamma_0 \tilde{\psi}$ is not conserved unless λ is constant. So, if we suppose also that

$$\boldsymbol{\partial} \cdot J = 0 \tag{13.153}$$

we must have

$$\lambda = \text{constant}.$$

Now, throughout these calculations we have assumed $\hbar = c = 1$. We observe that in Eq. (13.151) λ has the units of $(\text{length})^{-1}$, and if we introduce the constants $\hbar$ and c we have to introduce another constant with unit of mass. If we denote this constant by m such that

$$\lambda = \frac{mc}{\hbar}, \tag{13.154}$$

then Eq. (13.151) assumes a form which is identical to Dirac-Hestenes equation:

$$\boldsymbol{\partial}\psi\gamma_{21} - \frac{mc}{\hbar}\psi\gamma_0 = 0. \tag{13.155}$$

Remark 13.3 It is true that we did not prove that Eq. (13.155) is really the Dirac-Hestenes equation since the constant m has to be identified in this case with the electron's mass, and we do not have *any* good physical argument to make that identification, until now. In resume, Eq. (13.155) has been obtained from Maxwell equation by imposing some gauge conditions allowed by the hidden parameters in the solution of Eq. (13.67) for ψ in terms of F. In view of that, it is certainly more appropriate instead of using the term *mathematical* Maxwell-Dirac equivalence of first kind to talk about a correspondence between that equations *under* which two degrees of freedom of the Dirac-Hestenes spinor field are treated as hidden variables.

We end this section with the observation that it is to earlier to know if the above results are of some physical value or only a mathematical curiosity. Let us wait...

13.4.7 Maxwell-Dirac Equivalence of Second Kind

We now look for a Hertz potential field $\Pi \in \sec \bigwedge^2 T^*M$ satisfying the following equation

$$\boldsymbol{\partial}\Pi = (\boldsymbol{\partial}\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_1) + \gamma_5(\boldsymbol{\partial}\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3) \tag{13.156}$$

where $\mathfrak{G}, \mathfrak{P} \in \sec \bigwedge^0 T^*M$, and m is a constant. According to previous results the electromagnetic and Stratton potentials are

$$A = \boldsymbol{\partial}\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_1, \tag{13.157}$$

$$\gamma_5 S = \gamma_5(\boldsymbol{\partial}\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3), \tag{13.158}$$

and must satisfy the following subsidiary conditions,

$$\Diamond(\boldsymbol{\partial}\mathfrak{G} + m\mathfrak{P}\gamma_3 + m\langle\Pi\gamma_{012}\rangle_1) = J_e, \tag{13.159}$$

$$\Diamond(\gamma_5(\boldsymbol{\partial}\mathfrak{P} + m\mathfrak{G}\gamma_3 - \gamma_5\langle m\Pi\gamma_{012}\rangle_3)) = 0, \tag{13.160}$$

$$\Diamond\mathfrak{G} + m\boldsymbol{\partial}\cdot\langle\Pi\gamma_{012}\rangle_1 = 0, \tag{13.161}$$

$$\Diamond\mathfrak{P} - m\boldsymbol{\partial}\cdot(\gamma_5\langle\Pi\gamma_{012}\rangle_3) = 0. \tag{13.162}$$

Now, in the Clifford bundle formalism, as we already explained above, the following sum is a legitimate operation

$$\psi = -\mathfrak{G} + \Pi + \gamma_5\mathfrak{P} \tag{13.163}$$

and according to previous results Eq. (13.163) defines ψ as a(representative of some Dirac-Hestenes spinor field. Now, we can verify that ψ satisfies the equation

$$\boldsymbol{\partial}\psi\gamma_{21} - m\psi\gamma_0 = 0 \tag{13.164}$$

which is as we already know a *representative* of the standard Dirac equation (for a free electron) in the Clifford bundle.

The above developments suggest (consistently with the spirit of the generalized Hertz potential theory developed above) the following interpretation. The Hertz potential field Π generates the real electromagnetic field of the electron.[16] Moreover, the above developments suggest that the electron is "composed" of two "fundamental" currents, one of *electric* type and the *other* of magnetic type circulating at the ultra microscopic level, which generate the observed electric charge and magnetic moment of the electron. Then, it may be the case, as speculated by Maris [34], that the electromagnetic field of the electron can be spliced into two parts, each corresponding to a new kind of subelectron type particle, the *electrino*. Of course, the above developments leaves open the possibility to generate electrinos of fractional charges. Well, it is time to stop speculations on this issue.

13.5 Seiberg-Witten Equations

As it is well known, the original Seiberg-Witten (monopole) equations have been written in Euclidean "spacetime" and for the *self dual* part of the field F. However, on Minkowski spacetime, of course, there are *no* self dual electromagnetic fields. Indeed, the equation $\star F = F$ implies that the unique solution (on Minkowski spacetime) is $F = 0$. This is the main reason for the difficulties in interpreting that equations in this case, and indeed in [46] it was attempted an interpretation of that equations only for the case of Euclidean manifolds. Here we derive and

[16]The question of the physical dimensions of the Dirac-Hestenes and Maxwell fields is discussed in [36].

give a possible interpretation to that equations on Minkowski spacetime based on a reasonable assumption.

To start we recall that the *analogous* of Seiberg-Witten monopole equations *read* in the Clifford bundle formalism and on Minkowski spacetime as

$$\begin{cases} \boldsymbol{\partial}\psi\gamma_{21} - A\psi = 0, \\ F = \frac{1}{2}\psi\gamma_{21}\tilde{\psi}, \\ F = dA, \end{cases} \tag{13.165}$$

where $\psi \in \sec \mathcal{C}\ell^0(M,\eta)$ is a representative of a Dirac-Hestenes spinor field in a given spin coframe, $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ is an electromagnetic vector potential and $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ is an electromagnetic field.

Our intention in this section is:

(a) To use the Maxwell Dirac-Equivalence of the first kind (proved above) and an additional hypothesis to be discussed below to derive the Seiberg-Witten equations on Minkowski spacetime.
(b) to give a (possible) physical interpretation for that equations.

13.5.1 Derivation of Seiberg-Witten Equations

- **Step 1.** Assume that the electromagnetic field F appearing in the second of the Seiberg-Witten equations satisfy the free Maxwell equation, i.e., $\partial F = 0$.
- **Step 2.** Use the Maxwell-Dirac equivalence of the first kind proved above to obtain Eq. (13.151),

$$\boldsymbol{\partial}\psi\gamma_{21} - \lambda\psi\gamma_0 = 0. \tag{13.166}$$

- **Step 3.** Introduce the *ansatz*

$$A = \lambda\psi\gamma_0\psi^{-1}. \tag{13.167}$$

This means that the electromagnetic potential (in our geometrical units) is identified with a multiply of the velocity field defined through Eq. (13.131). Under this condition Eq. (13.166) becomes

$$\boldsymbol{\partial}\psi\gamma_{21} - A\psi = 0, \tag{13.168}$$

which is the first Seiberg-Witten equation!

13.5.2 A Possible Interpretation of the Seiberg-Witten Equations

Well, it is time to find an interpretation for Eq. (13.168). In order to do that we recall that (as it is well known) if $\psi_\pm$ are Weyl spinor fields (recall Eq. (3.103)), then $\psi_\pm$ satisfy a Weyl equation, i.e.,

$$\boldsymbol{\partial}\psi_\pm = 0. \tag{13.169}$$

Consider now, the equation for ψ_+ coupled with an electromagnetic field $A = gB \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$, i.e.,

$$\boldsymbol{\partial}\psi_+\gamma_{21} + gB\psi_+ = 0. \tag{13.170}$$

This equation is invariant under the gauge transformations

$$\psi_+ \mapsto \psi_+ e^{g\gamma_5\theta}; B \mapsto B + \partial\theta. \tag{13.171}$$

Also, the equation for ψ_- coupled with an electromagnetic field gB is

$$\boldsymbol{\partial}\psi_-\gamma_{21} + gB\psi_- = 0. \tag{13.172}$$

which is invariant under the gauge transformations

$$\psi_- \mapsto \psi_- e^{g\gamma_5\theta}; B \mapsto B - \partial\theta. \tag{13.173}$$

showing clearly that the fields ψ_+ and ψ_- carry *opposite* 'charges'. Consider now the Dirac-Hestenes spinor fields $\psi^\uparrow, \psi^\downarrow$ (recall Eq. (3.107)) which are eigenvectors of the *parity* operator (recall Eq. (3.106)) and look for solutions of Eq. (13.168) such that $\psi = \psi^\uparrow$. We have,

$$\boldsymbol{\partial}\psi^\uparrow\gamma_{21} + gB\psi^\uparrow = 0 \tag{13.174}$$

which separates in two equations,

$$\boldsymbol{\partial}\psi_+^\uparrow + g\gamma_5 B\psi_+^\uparrow = 0, \qquad \boldsymbol{\partial}\psi_-^\uparrow - g\gamma_5 B\psi_-^\uparrow = 0. \tag{13.175}$$

These results show that when a Dirac-Hestenes spinor field associated with the first of the Seiberg-Witten equations is in an eigenstate of the parity operator, that spinor field describes a *pair* of particles with opposite 'charges'. We interpret these particles (following Lochak[17] [23]) as *massless* 'monopoles' in *auto-interaction*.

[17]Lochak suggested that an equation equivalent to Eq. (13.174) describe massless monopoles of opposite 'charges'.

Observe that our proposed interaction is also consistent with the third of Seiberg-Witten equations, for $F = dA$ implies a *null* magnetic current.

It is now well known that Seiberg-Witten equations have non trivial solutions on Minkowski manifolds (see [29]). From the above results, in particular, taking into account the inversion formula (Eq. (13.79)) it seems to be possible to find whole family of solutions for the Seiberg-Witten equations, which has been here derived from a Maxwell-Dirac equivalence of first kind with the additional hypothesis that electromagnetic potential A is parallel to the velocity field v (Eq. (13.131)) of the system described by Eq. (13.172). We conclude that a consistent set of Seiberg-Witten equations on Minkowski spacetime must be

$$\begin{cases} \boldsymbol{\partial}\psi\gamma_{21} - A\psi = 0, \\ F = \frac{1}{2}\psi\gamma_{21}\tilde{\psi}, \\ F = dA, \\ A = \lambda\psi\gamma_0\psi^{-1}. \end{cases} \tag{13.176}$$

We end this long chapter recalling that we exhibit two different kinds of possible Maxwell-Dirac equivalences. Many will find the ideas presented above speculative from the physical point of view, but we think that all will agree that the mathematical coincidences found deserves more careful investigation. We think it is really provocative that the MDE of the second kind reveal an unsuspected possible interpretation of the Dirac equation, namely that the electron seems to be a composed system build up from the self interaction of two currents of the 'electrical' and 'magnetic' types. Of course, it is to earlier to say if this discovery has any physical significance. We showed also, that by using the MDE of the first kind together with a reasonable hypothesis we can shed some light on the meaning of Seiberg-Witten monopole equations on Minkowski spacetime. We hope that the results just described may be an indication that Seiberg-Witten equations (which are a fundamental key in the study of the topology of four manifolds equipped with an *Euclidean* metric tensor), may play an important role in Physics, whose arena where phenomena occur is a *Lorentzian* manifold.

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Chapter 14
Superparticles and Superfields

Abstract This chapter shows very clearly that the Clifford and spin-Clifford bundle formalism (introduced in previous chapters) offers a very simple way to write the equation of motion of a massive spinning particle. Indeed this equation is immediately derived from Frenet equations for a *moving frame*. Moreover, with addition of very a reasonable hypothesis the deduced spinor equation for the spinning particle leads directly to a classical (nonlinear) Dirac-Hestenes equation. An additional hypothesis leads to a linear Dirac-Hestenes equation and suggests automatically a probability interpretation for the Dirac-Hestenes wave function. We also show how the Clifford and spin-Clifford bundle formalism permit the introduction of multiform valued Lagrangians and a simple interpretation of the so-called superparticle theory. Moreover, the Berezin differential and integral calculus is shown to be no more than the result of contractions in an appropriate Clifford algebra. Also, the nature of superfields is clearly disclosed.

14.1 Spinor Fields and Classical Spinning Particles

We supposed in what follows that spacetime $(M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ is a spin manifold, which implies, as we learned in Chap. 7 the existence of a global tetrad frame $\{\boldsymbol{e}_{\mathbf{a}}\}$ $\in \sec \mathbf{P}_{\mathrm{SO}^{e}_{1,3}}(M)$. Let $\{\gamma^{\mathbf{a}}\}, \gamma^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ be the dual frame of $\{\boldsymbol{e}_{\mathbf{a}}\}$. Also, let $\{\gamma_{\mathbf{a}}\}$ be the reciprocal frame of $\{\gamma^{\mathbf{a}}\}$, i.e., $\gamma^{\mathbf{a}} \cdot \gamma_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$. Suppose moreover that the *reference frame* (Definition 6.9) defined by $\boldsymbol{e}_{\mathbf{0}}$ is in free fall, i.e., $D_{\boldsymbol{e}_{\mathbf{0}}} \boldsymbol{e}_{\mathbf{0}} = 0$ and that the spatial axis along each one of the integral lines of $\boldsymbol{e}_{\mathbf{0}}$ have been constructed by Fermi transport of spinning gyroscopes. This is translated by the requirement that $D_{\boldsymbol{e}_{\mathbf{0}}} \boldsymbol{e}_{\mathbf{i}} = 0$, $\mathbf{i} = 1, 2, 3$, and we have, equivalently $D_{\boldsymbol{e}_0} \gamma^{\mathbf{a}} = 0$. We introduce a spin coframe $\Xi \in P_{\mathrm{Spin}^{e}_{1,3}}(M)$ by the method described in Chap. 7 such that $s(\pm\Xi) = \{\gamma^{\mathbf{a}}\}$. Now, let Ψ be the representative of an invertible Dirac-Hestenes spinor field over σ (the world line of a spinning particle) in the spin coframe Ξ. Let moreover $\{f_{\mathbf{a}}\}$ be a Frenet frame over σ such that $f_{\mathbf{0}} = \gamma_0|_{\sigma}$ and $\mathtt{g}(f_{\mathbf{0}}, \,) = \boldsymbol{e}_{\mathbf{0}}|_{\sigma}$. Then, as we know from Chap. 6 we can write

$$f_{\mathbf{a}} = \Psi \gamma_{\mathbf{a}} \Psi^{-1}. \tag{14.1}$$

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Now, the general form of a representative of an invertible Dirac-Hestenes over σ is $\Psi = \rho^{\frac{1}{2}} e^{\frac{\beta\gamma^5}{2}} R$, where $\rho, \beta : \sigma(I) \to \mathbb{R}$ and $R : \sigma(I) \to \mathrm{Spin}^e_{1,3} \subset \mathbb{R}^0_{1,3}$. Recalling Eq. (6.21) we get using Eq. (14.1) that

$$D_{e_0} R = \frac{1}{2} \Omega_D R, \tag{14.2}$$

which may be called the *spinor equation of motion* of a classical spinning particle.

14.1.1 Spinor Equation of Motion for a Classical Particle on Minkowski Spacetime

Now, let us analyze the spinor equation of motion of a *free* particle in *Minkowski spacetime*. In that case, recalling Eq. (6.24) we have $\Omega_D = \Omega_{\mathbf{S}}$. We can trivially redefine the Frenet frame in such a way as to have $\kappa_3 = 0$ (recall Sect. 6.1.3). Indeed, this can be done by rotating the original frame with $U = e^{\mathbf{f_3 f_1}\frac{\alpha}{2}}$ and choosing $\alpha = \arctan\left(-\frac{\kappa_3}{\kappa_1}\right)$. So, in what follows we suppose that this choice has already been done. We are interested in the case where κ_2 is a real constant. Then, Eq. (14.2) becomes

$$\boldsymbol{D}_{\boldsymbol{e_0}} R = \frac{1}{2} \kappa_{\mathbf{2}} \boldsymbol{f^2 f^1} R. \tag{14.3}$$

The solution of Eq. (14.3) is

$$R = \exp\left(\frac{\kappa_2}{2} \gamma^{\mathbf{2}} \gamma^{\mathbf{1}} t\right). \tag{14.4}$$

Suppose next the existence of a covector field $V \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and of an *unitary* Dirac-Hestenes spinor field with representative $\phi \in \mathcal{C}\ell(M, \mathtt{g})$ ($\phi\tilde{\phi} = 1$) in the spin coframe Ξ such that $V|_\sigma = v$ and $\phi|_\sigma = R$. Without any loss of generality we can choose a global tetrad field such that $\gamma^{\mathbf{a}} \mapsto \gamma^\mu = dx^\mu$ (where $\{x^\mu\}$ are coordinates in Einstein-Lorentz-Poincaré gauge for the Minkowski spacetime). Then, under all these conditions we can rewrite Eq. (14.3) (taking into account the definition of the spin-Dirac operator and its action on representatives of Dirac-Hestenes spinor fields) as:

$$\boldsymbol{D}_{\boldsymbol{e_0}} R = \gamma^0 \cdot \boldsymbol{\partial} \phi = \frac{1}{2} \kappa_2 \phi \gamma^2 \gamma^1. \tag{14.5}$$

Defining

$$p = \frac{\kappa_2}{2} \gamma^0, x = x^\mu \gamma_\mu, \tag{14.6}$$

we have that

$$\phi = e^{p \cdot x \gamma^2 \gamma^1}. \tag{14.7}$$

Now,

$$\gamma^0 \cdot \boldsymbol{\partial} \phi = \gamma^0 \boldsymbol{\partial} \phi = (\phi \gamma^0 \phi^{-1}) \boldsymbol{\partial} \phi, \tag{14.8}$$

and substituting this result in Eq. (14.5) we get

$$\boldsymbol{\partial} \phi \gamma^2 \gamma^1 + \frac{\kappa_2}{2} \phi \gamma^0 = 0. \tag{14.9}$$

Put $m = -\frac{\kappa_2}{2}$ and end with

$$\boldsymbol{\partial} \phi \gamma^2 \gamma^1 - m \phi \gamma^0 = 0. \tag{14.10}$$

Equation (14.10) is formally *identical* to the Dirac-Hestenes equation for a unitary spinor. Note that the signal of κ_2 merely defines the sense of rotation in the $\boldsymbol{e}_2 \wedge \boldsymbol{e}_1$ plane.

Recalling the various spin operators introduced in Chap. 7 we can call the bilinear invariant $\Omega_{\mathcal{S}} \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$

$$\Omega_{\mathcal{S}} = k \phi \gamma^2 \gamma^1 \tilde{\phi} \tag{14.11}$$

the spin biform. Note that for our example $\mathcal{S} = \star \Omega_{\mathcal{S}} \llcorner v = k \phi \gamma^3 \tilde{\phi} \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ and may be called the spin covector.

Remark 14.1 The results just obtained shows that a natural interpretation suggests itself for the plane 'wave function' ϕ in the theory just presented. It describes a 'spinning fluid' (i.e., determines a velocity field V) such that the particle follows one of its streamlines. It is not a physical field in any sense.[1] This interpretation is reinforced by the derivation in the next section of a classical non linear Dirac-Hestenes equation for a charged spinning particle moving under the action of an electromagnetic field.

14.1.2 Classical (Nonlinear) Dirac-Hestenes Equation

We start from the classical Lorentz force law, which as well known, describes the motion of a classical particle of mass m and charge e following a worldline σ :

[1] Of course, the question of renormalization of the wave function ϕ is to be deal in the same way as in standard quantum mechanics.

$\mathbb{R} \supset I \to M$ under the action of an electromagnetic field. Let σ_* be the velocity of the particle and let $\boldsymbol{v} = \boldsymbol{g}(\sigma_*,)$ be the *physically equivalent* 1-form. From now one we suppose, as in the previous section that $\boldsymbol{v} \in \sec T^*_\sigma M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Let $F' \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ be the total electromagnetic field acting on the charged particle, i.e., the sum of external electromagnetic field F and the self field F_s. Let $A' = A + A_s \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ be a potential generating F. Then we have

$$m\dot{\boldsymbol{v}} = e\boldsymbol{v}\lrcorner F' = e\boldsymbol{v}\lrcorner(dA'). \tag{14.12}$$

To continue, we suppose the existence of a 1-form field $V \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ such that its restriction over σ is $\boldsymbol{v}$, i.e., $V_{|\sigma} = \boldsymbol{v}$. Also we impose that $V^2 = 1$. Next we observe that

$$\dot{\boldsymbol{v}} = V\lrcorner\boldsymbol{\partial} V = V\lrcorner(\boldsymbol{\partial}\wedge V) + \frac{1}{2}\boldsymbol{\partial} V^2 = V\lrcorner(\boldsymbol{\partial}\wedge V) = V\lrcorner(dV) = -(dV)\llcorner V, \tag{14.13}$$

Using this result in the first member of Eq. (14.12) we get

$$[d(mV - eA - eA_s]\lrcorner V = 0. \tag{14.14}$$

A sufficient condition for Eq. (14.14) to hold is the existence of a 0-form field χ such that

$$mV - eA - eA_s = d\chi = \boldsymbol{\partial}\chi. \tag{14.15}$$

Equation (14.15) is more conveniently written as

$$d\chi + e(A + A_s) = mV. \tag{14.16}$$

Squaring this equation we get putting $A' = A + A_s$

$$(d\chi + eA')^2 = m^2 \tag{14.17}$$

which we recognize as the classical relativistic *Hamilton-Jacobi* equation, if we ignore the self field A_s.

As in the previous section, let $\psi = \rho^{\frac{1}{2}} e^{\frac{\beta}{2}\gamma^5} R \in \sec \mathcal{C}\ell^0(M, \mathtt{g})$, be the representative (in the spin coframe Ξ) of a *particular* invertible Dirac-Hestenes spinor field such that

$$\psi\tilde{\psi} \neq 0 \tag{14.18}$$

and since $\psi = \rho^{\frac{1}{2}} e^{\frac{\beta}{2}\gamma^5} R$ we have

$$V = \psi\gamma^0\psi^{-1} = e^{\beta\gamma^5} R\gamma^0 R^{-1}. \tag{14.19}$$

Remark 14.2 Observe that since $V \in \sec \bigwedge^1 T^*M$ we must necessarily have that $\beta = 0$ or $\beta = \pi$. These values correspond to charges of opposite signs.

Substituting Eq. (14.19) in Eq. (14.16) after multiplying both members by ψ and taking $\beta = 0$ we get

$$(\boldsymbol{\partial}\chi + eA')\psi = m\psi\gamma^0. \tag{14.20}$$

We now write

$$\psi = \psi_0 e^{-\gamma_{21}\chi},\ \gamma_{21} = \gamma_2\gamma_1 \tag{14.21}$$

where ψ_0 is also a representative of some invertible Dirac-Hestenes spinor field that determines the same *current* $J = \psi_0\gamma^0\tilde{\psi}_0$ as the one determined by ψ.

We observe moreover that

$$\boldsymbol{\partial}\chi\psi = \boldsymbol{\partial}\psi\gamma_{21} + \boldsymbol{\partial}\psi_0 e^{-\gamma_{21}\chi}\gamma_{12}. \tag{14.22}$$

Using this result in Eq. (14.20) and observing that $e^{-\gamma_{21}\chi} = \psi_0^{-1}\psi$ we arrive at the non linear equation

$$\boldsymbol{\partial}\psi\gamma_{21} + eA\psi + eA_s\psi - m\psi\gamma_0 - \boldsymbol{\partial}(\ln\psi_0)\psi\gamma_{21} = 0. \tag{14.23}$$

This result is to be compared with the *quantum* Dirac-Hestenes equation (presented in Chap. 76) for an electron interacting with a electromagnetic field A which is

$$\boldsymbol{\partial}\Psi\gamma_{21} + eA\Psi - m\Psi\gamma_0 = 0, \tag{14.24}$$

where $\Psi \in \sec \mathcal{C}\ell^0(M, \mathtt{g})$, but with $\Psi\tilde{\Psi} = \rho e^{\beta\gamma^5}$ with $\beta = 0$ or $\beta = \pi$.

Comparison of (14.23) and (14.24) shows that besides the difference in normalization there is a nonlinear term in Eq. (14.23) namely $\boldsymbol{\partial}(\ln\psi_0)\psi\gamma_{21}$ and that the quantum Dirac-Hestenes equation does not include the self interaction term $eA_s\psi$. The term $\boldsymbol{\partial}(\ln\psi_0)\psi\gamma_{21}$ is identical to Bohm's *quantum potential* (see, e.g., [5]).

Our exercise, may eventually serve as a *prelude* for a interpretation of quantum theory, since it is clear that ψ must be thought as a kind of probability amplitude defining a probability current distribution through the bilinear invariant $J = e\psi\gamma^0\tilde{\psi}$ (and a probability spin distribution biform $S = \frac{\hbar}{2}\psi\gamma^{21}\tilde{\psi}$). In this way, as it was the case with the discussion on the previous section, the 'wave function' ψ is not to be interpreted as a real field in any sense, at least in the theory here presented.

14.1.3 Digression

Despite this fact, a 'fanatic' field theorist may argue that 'the field *ate* the particle', i.e., the electron is indeed a field, not a particle and that ψ satisfying Eq. (14.23) is a real field living in Minkowski spacetime. If that is the case, it seems tempting to that field theorist to present the following

Conjecture 14.3

$$eA_s = \frac{\hbar}{2}\boldsymbol{\partial}(\ln\psi_0)\psi\gamma_{21}\psi^{-1}. \tag{14.25}$$

Indeed, if Eq. (14.25) holds, then Eq. (14.23) becomes

$$\boldsymbol{\partial}\psi\gamma_{21} + eA\psi - m\psi\gamma_0 = 0. \tag{14.26}$$

If we left $V = e^{-\beta\gamma^5}\psi\gamma^0\psi^{-1}$ and do not use Remark 14.2 then under the same hypothesis as above we would arrive at

$$\boldsymbol{\partial}\psi\gamma_{21} + eA\psi - m\psi\gamma_0 e^{\beta\gamma^5} = 0. \tag{14.27}$$

Well, we stop just here to conjecture and end this section observing that Eq. (14.27), which will be called hereafter the non linear Dirac-Hestenes equation (NLDHE), has been extensively studied by Daviau [2, 3]. He showed that it gives very good results for the hydrogen spectrum. Also, Eq. (14.27) equation possess very nice properties not possessed by the DHE and it is surprisingly connected in an intriguing way with the free Maxwell equation as discussed in [6]. Moreover, the nonlinear term $e^{\beta\gamma^5}$ causes no difficulty concerning the superposition principle, since it has been shown in [7] that we can superpose only 'wave functions' having the same fixed value of the Takabaysi angle β.

Remark 14.4 It is worth to observe that Eq. (14.24) can be derived heuristically by a very simple argument. Indeed, in Hamiltonian mechanics the canonical momentum Π of particle in interaction with an electromagnetic A potential and following a world line σ such that $v = \sigma_*$ satisfies

$$\Pi = p - eA \tag{14.28}$$

Then if Ψ is a Dirac-Hestenes spinor field such that $\Psi\gamma^0\Psi^{-1}\big|_\sigma = v$ we can write Eq. (14.28) as

$$\Pi\Psi + eA\Psi = m\Psi\gamma^0 \tag{14.29}$$

and thus heuristically postulating that in relativistic quantum mechanics $\Pi\Psi \mapsto \boldsymbol{\partial}\Psi\gamma^2\gamma^1$ we get Eq. (14.24).[2]

14.2 The Superparticle

The results of the previous sections showed that the Frenet equations and a Dirac-Hestenes like equation are (when properly interpreted) appropriate equations describing important aspects of the motion of classical spinning particle. In what follows we propose to develop a Lagrangian and Hamiltonian formalism for Frenet equations using the multiform calculus developed in Chap. 2. For reasons that will become clear in a while we call the classical spinning particle the *superparticle.* For great generality we consider in what follows a n-dimensional flat Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$, where $(T_xM, \boldsymbol{\eta}_x) = \mathbb{R}^{1,n-1}$. The Clifford bundle of multiforms is $\mathcal{C}\ell(M, \eta)$ and we choose as before a basis generated by the $\{\gamma^\mu\}$, $\gamma^\mu \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$, such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1),$$
$$\mu, \nu = 0, 1, 2, \ldots, n. \tag{14.30}$$

We have

Definition 14.5 A superparticle is a pair (σ, X), where $\sigma : \mathbb{R} \supset I \to M$ is a time like curve and $X : \mathbb{R} \supset I \to \sec \mathcal{C}\ell(M, \eta)$ is a Clifford-field over σ (or a set of Clifford fields over σ).

Definition 14.6 A multiform Lagrangian is a mapping

$$L : (X(s), \dot{X}(s)) \mapsto L(X(s), \dot{X}(s)) \in \sec \mathcal{C}\ell(M, \eta) \tag{14.31}$$

where $X \in \sec \mathcal{C}\ell(M, \eta)$ and s is an invariant time parameter on σ.

Recalling Chap. 7, we see that L is a multiform functional, i.e., given a trivialization of $\mathcal{C}\ell(M, \eta)$ it has values in the Clifford algebra $\mathbb{R}_{1,n-1}$ for each τ.

The most general L can then be written as

$$L = \sum_k \langle L \rangle_k \equiv \sum_k L_k. \tag{14.32}$$

[2]This simple heuristic argument has been generalized in order to obtain the wave equation for a spin $1/2$ particle in [9].

To gain confidence in the sophisticated multiform derivative calculations that we shall need to do, we start by studying a simple case, namely, we choose $X = X_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ and $L = \langle L \rangle_0$, a scalar multiform functional.

We postulate next that the action for the superparticle is

$$\mathcal{A}(X_r) = \int_{\tau_1}^{\tau_2} ds\, L(X_r(s), \dot{X}_r(s)) \tag{14.33}$$

and that the equations of motion can be derived from the principle of stationary action (recall Chap. 7), that reads

$$\frac{d}{dt}\, \mathcal{A}(X_r + tA_r)|_{t=0} = A_r \cdot \partial_{X_r} \mathcal{A}(X_r) = 0, \tag{14.34}$$

where $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ is a Clifford field over σ such that

$$A_r(s_1) = A_r(s_2) = 0. \tag{14.35}$$

From Eq. (14.34) we get

$$\int_{s_1}^{s_2} ds\, [(A_r \cdot \partial_{X_r})L + \dot{A}_r \cdot \partial_{\dot{X}_r} L] = 0, \tag{14.36}$$

Since $L = \langle L \rangle_0$ and $X_r, A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ we have

$$\begin{aligned} A_r \cdot \partial_{X_r} \langle L \rangle_0 &= \langle A_r(\partial_{X_r} L)\rangle_0 = \tilde{A}_r \cdot (\partial_{X_r} L)_r, \\ \dot{A}_r \cdot \partial_{\dot{X}_r} \langle L \rangle_0 &= \langle A_r(\partial_{\dot{X}_r} L)\rangle_0 = \tilde{\dot{A}}_r \cdot (\partial_{\dot{X}_r} L)_r. \end{aligned} \tag{14.37}$$

Using Eq. (14.37) into Eq. (14.36) results

$$\int_{s_1}^{s_2} ds\, [(\tilde{A}_r \cdot (\partial_{X_r} L)_r - \tilde{A}_r \cdot \frac{d}{ds}(\partial_{\dot{X}_r} L)_r] = 0, \tag{14.38}$$

i.e.,

$$\tilde{A}_r \cdot \langle \partial_{X_r} L - \frac{d}{ds}(\partial_{\dot{X}_r} L)\rangle_r = 0. \tag{14.39}$$

Since for $L = \langle L \rangle_0$, we have $\partial_{X_r} L = \langle \partial_{X_r} L \rangle_r$ and $\partial_{\dot{X}_r} L = \langle \partial_{\dot{X}_r} L \rangle$ and since A_r is arbitrary then Eq. (14.39) implies

$$\langle \partial_{X_r} L - \frac{d}{ds}(\partial_{\dot{X}_r} L)\rangle_r = 0, \tag{14.40}$$

or

$$\partial_{X_r} L - \frac{d}{ds}(\partial_{\dot{X}_r} L) = 0, \tag{14.41}$$

that is the *Euler-Lagrange* equation. It is quite clear that the Eq. (14.41) holds for L being a functional of a general Clifford field X over σ.

Next we study the general case where L is given by Eq. (14.32). However, we restrict ourselves without loss of generality to the case where $X = X_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$

We define

$$\tau = \sum \langle \tau \rangle_k = \sum \tau_k, \tag{14.42}$$

where the $\tau_k \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ are constant multiforms.

Then,

$$\langle L\tilde{\tau}\rangle_0 = L \cdot \tau = \sum_k L_k \cdot \tau_k = \sum_k L_k \cdot \tau_k. \tag{14.43}$$

In this way $\langle L\tilde{\tau}\rangle_0$ has the role of a scalar valued Lagrangian and we define the action by

$$\mathcal{A}(X) = \int_{s_1}^{s_2} ds \langle L(X, \dot{X})\tilde{\tau}\rangle_0. \tag{14.44}$$

The principle of stationary action then gives

$$\begin{aligned} &\int_{s_1}^{s_2} ds[(A_r \cdot \partial_X)\langle L\, \tilde{\tau}\rangle_0 + (\dot{A}_r \cdot \partial_{\dot{X}})\langle L\tilde{\tau}\rangle_0] \\ &\quad = \sum_k \int_{s_1}^{s_2} ds[(A_r \cdot \partial_X)(L_k \cdot \tau_k) + (\dot{A}_r \cdot \partial_{\dot{X}})(L_k \cdot \tau_k)] = 0. \end{aligned} \tag{14.45}$$

Since $(A_r \cdot \partial_X)\tau_k = 0$, we have (recalling Eq. (2.177))

$$\begin{aligned} (A_r \cdot \partial_X)\ (L_k \cdot \tau_k) &= \langle A_r \cdot \partial_X\ L_k\rangle \cdot \tau_k \\ &= \langle A_r \cdot \partial_X L_k\rangle_k \cdot \tau_k = \langle A_r(\partial_X L_k)\rangle_k \cdot \tau_k. \end{aligned} \tag{14.46}$$

But, since $X = X_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$, then

$$\partial_X L_k = \langle \partial_X L_k\rangle_{|r-k|} + \langle \partial_X L_k\rangle_{|r-k|+2} + \cdots \langle \partial_X L_k\rangle_{r+k}, \tag{14.47}$$

and we can write,

$$
\begin{aligned}
(A_r \cdot \partial_X)\ (L_k \cdot \tau_k) &= \langle A_r(\partial_X L_k)_{|r-k}\rangle_k \cdot \tau_k + \langle A_r(\partial_X L)_{|r-k|+2}\rangle_k \cdot \tau_k \\
&\quad + \cdots + \ \langle A_r(\partial_X L_k)_{r+k}\rangle_k \cdot \tau_k \\
&= \sum_{\ell=0}^{\frac{1}{2}(r+k-|r-k|)} \langle A_r(\partial_X L_k)_{|r-k|+2\ell}\rangle \cdot \tau_k.
\end{aligned} \tag{14.48}
$$

Also,

$$
\begin{aligned}
(\dot{A}_r \cdot \partial_{\dot{X}})\ (L_k \cdot \tau_k) &= \langle \dot{A}_r(\partial_{\dot{X}} L_k)\rangle_k \cdot \tau_k \\
&= \sum_{\ell=0}^{\frac{1}{2}[r+k-|r-k|]} \langle \dot{A}_r(\partial_{\dot{X}} L_k)_{|r-k|+2\ell}\rangle_k \cdot \tau_k \\
&= \sum_{\ell=0}^{\frac{1}{2}[r+k-|r-k|]} \Bigg[\frac{d}{ds}\langle A_r(\partial_{\dot{X}} L_k)_{|r-k|+2\ell}\rangle_k \\
&\qquad -\langle A_r \frac{d}{ds}(\partial_{\dot{X}} L_k)_{|r-k|+2\ell}\rangle_k\Bigg] \cdot \tau_k.
\end{aligned} \tag{14.49}
$$

Using Eqs. (14.48) and (14.49) into Eq. (14.45) and taking into account that $A_r(s_1) = A_r(s_2) = 0$ we get

$$
\sum_k \int_{s_1}^{s_2} ds \sum_{\ell=0}^{\frac{1}{2}[r+k-|r-k|]} \langle A_r[(\partial_X L_k)_{|r-k|+2\ell} \frac{d}{ds}(\partial_{\dot{X}} L_k)_{|r-k|+2\ell}\rangle_k \cdot \tau_k = 0. \tag{14.50}
$$

Now, the τ_k are constant sections of $\sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. Then if $p = \binom{n}{k}$, it follows that τ_k is of the form $(\tau_k)^{\mu_1 \cdots \mu_p}\ \gamma_{\mu_1} \ldots \gamma_{\mu_p}$ where $(\tau_k)^{\mu_1 \cdots \mu_p}$ are arbitrary real constants. Also the term in the brackets in Eq. (14.50) is of the form $\langle\ \rangle_k = (\langle\ \rangle_k)^{\mu_1 \ldots \mu_p} \gamma_{\mu_1} \ldots \gamma_{\mu_p}$ and Eq. (14.50) results in a sum of terms of the form $(\tau)^{\mu_1 \ldots \mu_p}(\langle\ \rangle_k)\gamma_{\mu_1} \ldots \gamma_{\mu_p}$. Since the $(\tau_k)^{\mu_1 \cdots \mu_p}$ are arbitrary constants, Eq. (14.50) implies that for each k we must have

$$
\langle A_r[\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)]_{|r-k|} + \ldots [\partial_X L_k - \frac{d}{ds}(\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)]_{r+k}\rangle_k = 0, \tag{14.51}
$$

or

$$
\langle A_r[\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)\rangle_k = 0. \tag{14.52}
$$

Next we show that Eq. (14.52) implies the multiform Euler-Lagrange equation

$$\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k) = 0, \tag{14.53}$$

which means

$$\begin{aligned} &[\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)]_{|r-k|} = 0, \\ &[\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)]_{|r-k|+2} = 0, \\ &\qquad \vdots \\ &[\partial_X L_k - \frac{d}{ds}(\partial_{\dot{X}} L_k)]_{r+k} = 0. \end{aligned} \tag{14.54}$$

Observe that if $k = 0$, Eq. (14.53) implies $\binom{n}{k} = \binom{n}{0} = 1$ equation, whereas the variation of A_r implies $\binom{n}{r}$ arbitrary variations. Then $\binom{n}{r} \times 1 = \binom{n}{r}$ and we conclude the existence of $\binom{n}{r}$ Euler-Lagrange equations, namely, one for each of the $\binom{n}{r}$ components of $[\partial_{X_r} L_0 - \frac{d}{ds}(\partial_{\dot{X}_r} L_0)] \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. The same happens with $k \neq 0$ and $r = 0$ since in this case we have $\binom{n}{k} \times 1 = \binom{n}{k}$ Euler-Lagrange equations for $[\partial_{X_r} L_k - \frac{d}{ds}(\partial_{\dot{X}_r} L_k)] \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ which has $\binom{n}{k}$ components.

We now must extend the above reasoning for $k \neq 0, r \neq 0$. Observe that in this general case we need

$$\begin{aligned} \sum_{\ell=0}^{\frac{1}{2}[r+k-|r-k|]} \binom{n}{|r-k|+2\ell} &= \binom{n}{|r-k|} + \binom{n}{|r-k|+2} + \ldots + \binom{n}{r+k} \\ &\leq \binom{n}{r}\binom{n}{k} \qquad ; n,k \leq n\ ,\ \ r+k < n, \end{aligned} \tag{14.55}$$

in order to deduce from Eq. (14.52) the validity of Eq. (14.53). This happens because Eq. (14.53) is equivalent to Eq. (14.54) which is a set of $\sum_{\ell=0}^{\frac{1}{2}[r+k-|r-k|]}$ Euler-Lagrange like equations, and from Eq. (14.52) we can deduce only $\binom{n}{r}\binom{n}{k}$ equations of the Euler-Lagrange type.

Now, Eq. (14.55) has been tested in a computer program to be true, and we conclude for the validity of Eq. (14.53), the *multiform* Euler-Lagrange equation. Indeed, Eq. (14.53) is valid also if X is a general multivector field over σ, and we conclude that the principle of stationary action with $L = \sum_k L_k$ produces the general multivector Euler-Lagrange equation

$$\partial_X L - \frac{d}{ds}(\partial_{\dot{X}} L) = 0. \tag{14.56}$$

14.3 Superparticle in Minkowski Spacetime

We now postulate the following biform valued Lagrangian for a superparticle whose world line in 4-dimensional Minkowski spacetime is σ

$$L_S = \frac{1}{2}\dot{e}^{\mu} \wedge e_{\mu} - \frac{1}{2}\omega^{\mu\nu} e_{\mu} \wedge e_{\nu}, \tag{14.57}$$

where $\{e_{\mu}\}$, $\mu = 0, 1, 2, 3$ a comoving coframe for σ in Minkowski spacetime and where the $\omega_{\mu\nu}$ are functions over σ. From Eq. (14.57) we have four multiform Euler-Lagrange equations

$$\partial_{e_{\mu}} L_S - \frac{d}{ds}(\partial_{\dot{e}_{\mu}} L_S) = 0. \tag{14.58}$$

We have taking into account the results of Exercise 2.108 (Chap. 2):

$$\partial_{e_{\mu}}\left(\frac{1}{2}\dot{e}^{\mu} \wedge e_{\mu}\right) = -\frac{3}{2}\dot{e}^{\mu}, \quad \partial_{e_{\mu}}\left(\frac{1}{2}\omega^{\mu\nu} e_{\mu} \wedge e_{\nu}\right)$$
$$= 3\omega^{\mu\nu} e_{\nu}, \partial_{\dot{e}_{\mu}}\left(\frac{1}{2}\dot{e}^{\mu} \wedge e_{\mu}\right) = \frac{3}{2}\dot{e}^{\mu}.$$

Defining $\Omega_D = \frac{1}{2}\omega_{\mu\nu} e^{\mu} \wedge e^{\nu}$ we arrive at

$$\dot{e}_{\mu} = \Omega_D \llcorner e_{\mu} \tag{14.59}$$

which is similar to the equations of motion of a Frenet tetrad (see Eq. (6.21)). And, indeed, for particular values of $\omega_{\mu\nu}$ Eq. (14.59) may be identified with Frenet equations.

14.4 Superfields

Here we show the connection of the Clifford bundle formalism and the concept of multivector derivatives with the Berezin supercalculus.

In 1977 Berezin [1] introduced the following calculus now known today as supercalculus [4, 11].

Let ξ_i, $i = 1, \ldots, n$ be the generations of the Grassmann algebra $\mathcal{G}_n$. Then,

$$\{\xi_i, \xi_j\} = 0 \tag{14.60}$$

where $\{\ \}$ is the anticommutator.

Obviously, we have the following isomorphism

$$\xi_i \leftrightarrow e_i \ ; \ \xi_i\xi_j \leftrightarrow e_i \wedge e_j \tag{14.61}$$

where e_i are orthonormal covectors generating the Clifford algebra $\mathbb{R}_{p,q}$, $p+q=n$. (We leave p, q unspecified at this moment).

With that identification any combination of Grassmann variables is isomorphic to a certain multiform.

Berezin introduced the differentiation of functions of Grassmann variables by the rules :

$$\frac{\partial \xi_j}{\partial \xi_i} = \delta_{ij} \ ; \ \xi_j \frac{\overleftarrow{\partial}}{\partial \xi_i} = \delta_{ij}. \tag{14.62}$$

Introducing the reciprocal basis $\{e^i\}$ of $\mathbb{R}_{p,q},\ e^i \cdot e_j = \delta^i_j$ we have

$$\frac{\partial}{\partial \xi_i}(\quad \mapsto e^i \lrcorner\,, \tag{14.63}$$

$$)\frac{\overleftarrow{\partial}}{\partial \xi_j} \mapsto \llcorner e^j\ . \tag{14.64}$$

We immediately verify with the above identifications that differentiation in the Berezin calculus satisfies the so called graded Leibniz's rule [4, 11].

Then if $f(\xi) = f(\xi_1, \ldots \xi_n)$ is a general Grassmann function a Taylor expansion yields

$$f(\xi) = f_0 + f_i\xi^i + \frac{1}{2}f_{ij}\xi^i\xi 1 + \ldots \frac{1}{n!}f_{i_n \ldots i_n}\xi^{i_n} \ldots \xi^{i_n}. \tag{14.65}$$

Berezin defined integration by the rules

$$\begin{gathered} \int 1 d\xi_i = 0\ , \qquad \int \xi_i d\xi_i = 1, \forall i, \\ \int f(\xi_1 \ldots \xi_n) d\xi_n\ d\xi_{n-1} \ldots d\xi_1 = f(\xi) \frac{\overleftarrow{\partial}}{\partial \xi_n} \frac{\overleftarrow{\partial}}{\partial \xi_{n-1}} \ldots \frac{\overleftarrow{\partial}}{\partial \xi_1}. \end{gathered} \tag{14.66}$$

It is obvious that $f(\xi)$ is clearly isomorphic to a multiform $F \in \mathbb{R}_{pq}$ with the same coefficients as in Eqs. (14.65) and (14.65) is equivalent to

$$(\ldots((F \llcorner e^n) \llcorner e^{n-1}) \ldots) = \langle FE^n \rangle_0, \tag{14.67}$$

$$E^n = e^n \wedge e^{n-1} \ldots \wedge e^1. \tag{14.68}$$

With this identification almost all supercalculus as presented, e.g., in deWitt [4] reduces to elementary algebraic identities for multiform functions. This also shows that superfields, first introduced by Salam and Strathdee [10] are *isomorphic* to sections of the Clifford bundle. Indeed to the superfield $A : M \times \mathcal{G}_n \to \mathcal{G}_n$, which has the obvious Taylor expansion,

$$A(x,\xi) = A_0(x) + (A_1(x))_i \xi^i + \left(\frac{1}{2}A_2(x)\right)_{ij} \xi^i \xi^j \tag{14.69}$$

$$+ \ldots \left(\frac{1}{n!}A_n(x)\right)_{\mu_1 \ldots \mu_n} \xi^{\mu_1 \ldots \xi^{\mu_n}} \tag{14.70}$$

it corresponds $\mathcal{C}(x) \in \sec \mathcal{C}\ell(M,\eta)$ given by

$$\mathcal{C}(x) = A(x) + (A_1(x))_i e^i + \left(\frac{1}{2}A_2(x)\right)_{ij} e^i e^j +$$
$$\ldots + \left(\frac{1}{n!}A_n^{(x)}(x)\right)_{\mu_1 \ldots \mu_n} e^{\mu_1} \ldots e^{\mu_n}. \tag{14.71}$$

Remark 14.7 The amazing fact that we would like to emphasize here is that, as we learned in Chap. 7, any representative of a Dirac-Hestenes spinor field is an even section of the Clifford bundle and thus has the same structure of superfield. Moreover the generalized potential $\mathcal{A} = A + \gamma^5 S$, where A is the usual electromagnetic potential and S is the Stratton potential introduced in Sect. 13.4.5 is also a superfield.

To end this section we write a Berezin-Marinov's like Lagrangian [1] for a spinning particle in Minkowski spacetime as

$$L_{BM} = \frac{1}{2}\dot{\xi}^{\mu}\xi_{\mu} - \frac{1}{2}\omega_{\mu\nu}\xi^{\mu}\xi^{\nu} \tag{14.72}$$

where $\xi_\mu : t \mapsto \mathcal{G}_4$, $\mu = 0,1,2,3$ are elementary Grassmann fields over σ, and $\omega_{\mu\nu}$ are functions over σ, which in the original Berezin-Marinov model are *constant* functions.

With the isomorphism defined by Eq. (14.61), namely $\xi_\mu \mapsto e_\mu$ where $\{e_\mu\}$ is an orthonormal coframe over σ (introduce above) we get the isomorphism

$$L_{BM} \simeq L_S \tag{14.73}$$

where L_S is the biform Lagrangian defined by Eq. (14.57).

From the identification $L_{BM} \simeq L_S$ it becomes clear that in Berezin-Marinov Lagrangian in a 4-dimensional Minkowski spacetime can produce a (classical) Dirac-Hestenes equation as discussed in Sect. 14.1.[3]

In conclusion, we showed that Frenet equations are naturally appropriate equations of motion a classical spinning particle, and from the spinor form of Frenet equations we even obtain a 'classical' Dirac-Hestenes equation for a unitary representative of a Dirac-Hestenes spinor field and also a nonlinear Dirac-like equation for a general representative of a Dirac-Hestenes spinor field.

We succeeded also in giving a multiform Lagrangian formalism for Frenet equations and showed that it is isomorphic to a generalization in a 4-dimensional Minkowski spacetime of the Lagrangian of the famous Berezin-Marinov [1] model, thus eventually providing a satisfactory geometrical interpretation of that model a physical system living in usual Minkowski spacetime. Moreover, our developments also indicates strongly that eventually superfields may also have geometrical interpretation as Clifford fields.[4] This statement is based on the observation that any representative of a Dirac-Hestenes spinor field in a spin frame may be identified with a superfield!

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[3]Compare this with the original Berezin-Marinov model [1], where it is necessary to use a pentadimensional Grassmann algebra in order to obtain the Dirac equation after quantization.

[4]In [8] we showed that superfields as defined by Witten [12] may be identified with ideal sections of an hyperbolic Clifford bundle over Minkowski spacetime.

Chapter 15
Maxwell, Einstein, Dirac and Navier-Stokes Equations

Abstract In the previous chapters we exhibit several different faces of Maxwell, Einstein and Dirac equations. In this chapter we show that given certain conditions we can encode the contents of Einstein equation in Maxwell like equations for a field $F = dA \in \sec \bigwedge^2 T^*M$ (see below), whose contents can be also encoded in a Navier-Stokes equation. For the particular cases when it happens that $F^2 \neq 0$ we can also using the Maxwell-Dirac equivalence of the first kind discussed in Chap. 13 to encode the contents of the previous quoted equations in a Dirac-Hestenes equation for $\psi \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M)$ such that $F = \psi\gamma^{21}\tilde{\psi}$.

Specifically, we first show in Sect. 15.1 how each LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ which, as we already know, is a model of a gravitational field generated by $\mathbf{T} \in \sec T_2^0 M$ (the matter plus non gravitational fields energy-momentum tensor) in Einstein GRT is such that for any $\mathbf{K} \in \sec TM$ which is a vector field generating a one parameter group of diffeomorphisms of M we can encode Einstein equation in Maxwell like equations satisfied by $F = dK$ where $K = \boldsymbol{g}(\boldsymbol{K},\ )$ with a well determined current term named the *Komar current* $J_{\boldsymbol{K}}$, whose explicit form is given. Next we show in Sect. 15.2 that when $\mathbf{K} = \mathbf{A}$ is a Killing vector field, due to some noticeable results [Eqs. (15.28) and (15.29)] the Komar current acquires a very simple form and is then denoted $J_{\boldsymbol{A}}$. Next, interpreting, as in Chap. 11 the Lorentzian spacetime structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ as no more than an useful representation for the gravitational field represented by the gravitational potentials $\{\mathfrak{g}^a\}$ which live in Minkowski spacetime (here denoted by $(M = \mathbb{R}^4, \mathring{\boldsymbol{g}}, \mathring{D}, \tau_{\mathring{\boldsymbol{g}}}, \uparrow)$) we show in Sect. 15.3 that we can find a Navier-Stokes equation which encodes the contents of the Maxwell like equations (already encoding Einstein equations) once a proper identification is made between the variables entering the Navier-Stokes equations and the ones defining $\mathring{A} = \mathring{\boldsymbol{g}}(\boldsymbol{A},\ )$ and $\mathring{F} = d\mathring{A}$, objects clearly related [see Eq. (15.49)] to A and $F = dA$. We also explicitly determine also the constraints imposed by the nonhomogeneous Maxwell like equation $\underset{\boldsymbol{g}}{\delta} F = -J_{\boldsymbol{A}}$ on the variables entering the Navier-Stokes equations and the ones defining A (or $\mathring{A}$).

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15.1 The Conserved Komar Current

Let $\mathbf{K} \in \sec TM$ be the generator of a one parameter group of diffeomorphisms of M in the spacetime structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ which is a model of a gravitational field generated by $\mathbf{T} \in \sec T_2^0 M$ (the matter plus non gravitational fields energy-momentum tensor) in Einstein GRT. It is quite obvious that if we put $F := dK$, where $K = \boldsymbol{g}(\boldsymbol{K},) \in \sec \bigwedge^1 T^*M$ then we can define a current $J_{\boldsymbol{K}} \in \sec \bigwedge T^*M$ by

$$J_{\boldsymbol{K}} = -\underset{\boldsymbol{g}}{\delta} F \tag{15.1}$$

which, of course, is conserved, i.e.,

$$\underset{\boldsymbol{g}}{\delta} J_{\boldsymbol{K}} = 0. \tag{15.2}$$

Surprisingly such a trivial *mathematical* result seems to be very important by people working in GRT who call $J_{\boldsymbol{K}}$ the Komar current[1] [10]. Komar called[2]

$$\mathfrak{E} := -\frac{1}{8\pi}\int_V \underset{\boldsymbol{g}}{\star} J_{\boldsymbol{K}} = \frac{1}{8\pi}\int_{\partial V} \underset{\boldsymbol{g}}{\star} F \tag{15.3}$$

the *generalized energy.*

To understand why $J_{\boldsymbol{K}}$ is considered important write the action for the gravitational plus matter and non gravitational fields as

$$\mathcal{A} = \int \mathcal{L}_{EH} + \int \mathcal{L}_m = -\frac{1}{2}\int R\boldsymbol{\tau}_{\boldsymbol{g}} + \int \mathcal{L}_m. \tag{15.4}$$

Now, under the (infinitesimal) diffeomorphism $\mathtt{h} : M \to M$ generated by $\mathbf{K}$ we have that $\boldsymbol{g} \mapsto \boldsymbol{g}' = \mathtt{h}^* \boldsymbol{g} = \boldsymbol{g} + \boldsymbol{\delta g}$ where the variation $\boldsymbol{\delta g} = -\pounds_{\mathbf{K}} \boldsymbol{g}$ and taking into account Cartan's magical formula ($\pounds_{\mathbf{K}} P = K \underset{\boldsymbol{g}}{\lrcorner} dP + d(K \underset{\boldsymbol{g}}{\lrcorner} P)$, for any $P \in \sec \bigwedge T^*M$) we have

$$\begin{aligned}
\boldsymbol{\delta}\mathcal{A} &= \int \boldsymbol{\delta}\mathcal{L}_{EH} + \int \boldsymbol{\delta}\mathcal{L}_m \\
&= -\int \pounds_{\mathbf{K}} \mathcal{L}_{EH} - \int \pounds_{\mathbf{K}} \mathcal{L}_m \\
&= -\int d(K \lrcorner \mathcal{L}_{EH}) - \int d(K \lrcorner \mathcal{L}_m) \\
&:= \int d(\underset{\boldsymbol{g}}{\star} \mathcal{C})
\end{aligned} \tag{15.5}$$

[1] Komar called a *related* quantity the generalized flux.

[2] V denotes a spacelike hypersurface and $S = \partial V$ its boundary. Usually the integral $\mathfrak{E}$ is calculated at a constant x^0 time hypersurface and the limit is taken for S being the boundary at infinity.

Of course, if $\underset{g}{\star}\mathcal{C}$ is null on the boundary of the integration region we have $\boldsymbol{\delta}\mathcal{A} = 0$. On the other hand, recall that [11]

$$\mathcal{A} = -\frac{1}{2}\int R\sqrt{-\det \boldsymbol{g}}dx^0dx^1dx^2dx^3 + \int L_m\sqrt{-\det \boldsymbol{g}}dx^0dx^1dx^2dx^3. \tag{15.6}$$

Variation of $\mathcal{A}$ with respect to $\boldsymbol{g}$ gives as we know Einstein equation $G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -T^\mu_\nu$ with R^μ_ν and T^μ_ν the components of the Ricci and the energy-momentum tensor and R the scalar curvature. Now, recall that $\mathcal{G}^\mu := \mathcal{R}^\mu - \frac{1}{2}R\mathfrak{g}^\mu$ are the Einstein 1-form field and that $D_\mu G^{\mu\nu} = 0 = D_\mu T^{\mu\nu}$. Put $\mathrm{T}^\mu := T^\mu_\nu \mathfrak{g}_\nu$ and

$$\mathcal{E}^\mu := \mathcal{G}^\mu + \mathrm{T}^\mu, \;\; \mathcal{E}^\mu := E^{\mu\nu}\mathfrak{g}_\nu = (G^{\mu\nu} + T^{\mu\nu})\mathfrak{g}_\nu \tag{15.7}$$

to get

$$\begin{aligned}\boldsymbol{\delta}\mathcal{A} &= -\frac{1}{2}\int E^{\mu\nu}(\mathfrak{L}_{\mathbf{K}}\boldsymbol{g})_{\mu\nu}\sqrt{-\det \boldsymbol{g}}dx^0dx^1dx^2dx^3\\ &= -\int E^{\mu\nu}D_\mu K_\nu\sqrt{-\det \boldsymbol{g}}dx^0dx^1dx^2dx^3\\ &= -\int D_\mu(E^{\mu\nu}K_\nu)\sqrt{-\det \boldsymbol{g}}dx^0dx^1dx^2dx^3\\ &= -\int(\boldsymbol{\partial}\underset{g}{\lrcorner}\mathcal{E}^\nu K_\nu)\boldsymbol{\tau}_{\boldsymbol{g}}\\ &= \int \underset{g}{\star}\,\underset{g}{\delta}(\mathcal{E}^\nu K_\nu)\\ &= -\int d\underset{g}{\star}(\mathcal{E}^\nu K_\nu).\end{aligned} \tag{15.8}$$

From Eqs. (15.5) and (15.6) we have immediately that

$$\int d(\underset{g}{\star}\mathcal{E}^\nu K_\nu) + d(\underset{g}{\star}\mathcal{C}) = 0, \tag{15.9}$$

and thus

$$\underset{g}{\delta}(\mathcal{E}^\nu K_\nu) + \underset{g}{\delta}\mathcal{C} = 0. \tag{15.10}$$

It follows that the current $\mathcal{C} \in \sec \bigwedge^1 T^*M$ is conserved if the field equations $\mathcal{E}^\nu = 0$ are satisfied. An equation (in component form) equivalent to Eq. (15.10) already appears in [10] (and also previously in [1]) who took $\mathcal{C} = \mathcal{E}^\nu K_\nu + N$ where $\underset{g}{\delta}N = 0$.

Here, to continue we prefer to write an identity involving only $\boldsymbol{\delta}\mathcal{A}_g = \int \boldsymbol{\delta}\mathcal{L}_g$. Proceeding exactly as before we get putting $\mathcal{G}(K) = \mathcal{G}^\mu K_\mu$ that there exists $\mathcal{P} \in \sec \bigwedge^1 T^*M$ such that

$$\boldsymbol{\partial}\lrcorner\mathcal{G}(K) + \boldsymbol{\partial}\lrcorner\mathcal{P} = 0. \tag{15.11}$$

and we see that we can identify

$$\mathcal{P} := -\mathcal{G}^{\mu} K_{\mu} + L \tag{15.12}$$

where $\underset{g}{\delta} L = 0$. Now, we claim that we can find $L \in \sec \bigwedge^{1} T^{*}M$ such that

$$\mathcal{P} = -\mathcal{G}^{\mu} K_{\mu} + L = \underset{g}{\delta} dK. \tag{15.13}$$

Let us find such a L and investigate if we can give some nontrivial physical meaning to such $\mathcal{P} \in \sec \bigwedge^{1} T^{*}M$.

In order to prove our claim we observe that we can write (taking into account the identities of Chap. 4 involving the Ricci, covariant D'Alembertian operators and the square of the Dirac operator)

$$\begin{aligned}
\mathcal{G}^{\mu} K_{\mu} &= \mathcal{R}^{\mu} K_{\mu} - \frac{1}{2} RK \\
&= \boldsymbol{\partial} \wedge \boldsymbol{\partial} K - \frac{1}{2} RK \\
&= \boldsymbol{\partial} \wedge \boldsymbol{\partial} K + \boldsymbol{\partial} \cdot \boldsymbol{\partial} \boldsymbol{K} - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \\
&= \boldsymbol{\partial}^{2} K - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \\
&= -\underset{g}{\delta} dK - d \underset{g}{\delta} K - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K
\end{aligned} \tag{15.14}$$

Then we take

$$\mathcal{P} := -\mathcal{G}^{\mu} K_{\mu} - d \underset{g}{\delta} K - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K = \underset{g}{\delta} dK \tag{15.15}$$

and thus[3]

$$L = -d \underset{g}{\delta} K - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K. \tag{15.16}$$

Next, we recall the action of the extensor field $\boldsymbol{T} := T^{\mu}_{\nu} dx^{\nu} \otimes \partial_{\mu}$ on K is

$$\boldsymbol{T}(K) = \mathrm{T}^{\mu} K_{\mu}. \tag{15.17}$$

[3]Note that since $\underset{g}{\delta}(\mathcal{G}^{\mu} K_{\mu}) = 0$ it follows from Eq. (15.16) that indeed $\underset{g}{\delta} L = 0$.

Then, we can write Eq. (15.15) as

$$\underset{g}{\delta} dK = \boldsymbol{T}(\mathcal{K}) - d\underset{g}{\delta} K - \frac{1}{2} RK - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K. \tag{15.18}$$

We can write Eq. (15.18) taking into account that $R = T := T^{\mu}_{\mu}$ and putting $F := dK$ as

$$\underset{g}{\delta} F = -J_{\boldsymbol{K}}, \tag{15.19}$$

with

$$J_{\boldsymbol{K}} = -\boldsymbol{T}(K) + \frac{1}{2} TK + \ d\underset{g}{\delta} K + \boldsymbol{\partial} \cdot \boldsymbol{\partial} K. \tag{15.20}$$

Equation (15.20) gives the explicit form for the Komar current.[4] Moreover, since $\underset{g}{\delta} F = \underset{g}{\star} d \underset{g}{\star} F$, we have:

$$\begin{aligned} d \underset{g}{\star} F &= \underset{g}{\star}^{-1} \left(\boldsymbol{T}(K) - \frac{1}{2} TK - \ d\underset{g}{\delta} K - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \right) \\ &= \underset{g}{\star} \left(\boldsymbol{T}(K) - \frac{1}{2} TK - \ d\underset{g}{\delta} K - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \right) \end{aligned} \tag{15.21}$$

and thus taking into account Stokes theorem

$$\textstyle\int_V d \underset{g}{\star} F = \int_{\partial V} \underset{g}{\star} F \tag{15.22}$$

we arrive at the conclusion that the quantity

$$\begin{aligned} \mathfrak{E} &:= \frac{1}{8\pi} \int_{S=\partial V} \underset{g}{\star} F \\ &= \frac{1}{8\pi} \textstyle\int_V \star \left((\boldsymbol{T}(K) - \frac{1}{2} TK - \ d\underset{g}{\delta} K - \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \right) \end{aligned} \tag{15.23}$$

is a conserved one.

Remark 15.1 As already remarked an equation equivalent to Eq. (11.36) has already been obtained in [10] who called that quantity the conserved *generalized energy*. But according to our best knowledge Eq. (15.23) first appeared in [17] and it shows explicitly that all terms in the integrand are legitimate 3-form fields and thus the

[4] Something that is not given in [10].

value of the integral is, of course, independent of the coordinate system used to calculate it.

However, considering that for each $\boldsymbol{K} \in \sec TM$ that generates a one parameter group of diffeomorphisms of M we have a conserved quantity it is not in our opinion appropriate to think in this quantity as a generalized energy. Indeed, why should the energy depends on terms like $d\underset{\boldsymbol{g}}{\delta}K$ and $\boldsymbol{\partial} \cdot \boldsymbol{\partial} K$ if K is not a dynamical field?

We show in the next section that when $\boldsymbol{K} = \boldsymbol{A}$ is a Killing vector field, i.e., $\pounds_{\mathbf{A}}\boldsymbol{g} = 0$, we can write Eq. (15.23) as

$$\mathfrak{E} = \frac{1}{4\pi}\int_V \underset{\boldsymbol{g}}{\star} (\boldsymbol{T}(A) - \frac{1}{2}TA) \tag{15.24}$$

which is a conserved quantity[5] which relativistic physicists [18] think when $\boldsymbol{A}$ is a timelike Killing vector field to be more directly associated with the energy-momentum tensor of the matter plus non gravitational fields.[6] For a Schwarzschild spacetime, as well known, $\mathbf{A} = \partial/\partial t$ is a timelike Killing vector field and in his case since the components of $\mathbf{T}$ are $T^{\mu}_{\nu} = \frac{8\pi}{\sqrt{-\det \boldsymbol{g}}}\rho(r)v^{\mu}v_{\nu}$ and $v^i v_j = 0$ [since $v^{\mu} = \frac{1}{\sqrt{g_{00}}}(1, 0, 0, 0)$] we get $\mathcal{E} = m$.

Remark 15.2 Originally Komar obtained the same result directly from Eq. (15.23) supposing that the generator of the one parameter group of diffeomorphism was $\boldsymbol{A} = \partial/\partial t$, so he got $\mathcal{E} = m$ by pure chance. Had he picked another vector field generator of a one parameter group of diffeomorphisms $\boldsymbol{A} \neq \boldsymbol{\partial}/\boldsymbol{\partial}\ t$, he of course, would not obtained that result.

Remark 15.3 The previous remark shows clearly that the above approach does not to solve the energy-momentum conservation problem for a system consisting of the matter and non gravitational fields plus the gravitational field. It only gives a conserved energy for the matter plus non gravitational fields if the spacetime structure possess a timelike Killing vector field. To claim that a solution for total energy-momentum of the total system[7] problem exist it is necessary to find a way to define a total energy-momentum 1-form for the total system. This can only be done if the spacetime structure modeling a gravitational field (generated by the matter plus non gravitational fields energy-momentum tensor $\mathbf{T}$) possess additional structure, or if we interpret the gravitational field as a field in the Faraday sense living in Minkowski spacetime as did in Chap. 11. See also [7, 16].

[5]Observe that when $\mathbf{A}$ is a Killing vector field the quantities $\int_V \star \boldsymbol{T}(A)$ and $\int_V \frac{1}{2} \star TA$ are separately conserved as it is easily verified.

[6]An equivalent formula appears, e.g., as Eq. (11.2.10) in [23]. However, it is to be emphasized here the simplicity and transparency of our approach concerning traditional ones based on classical tensor calculus.

[7]The total system is the system consisting of the gravitational plus matter and non gravitational fields.

15.2 A Maxwell Like Equation Encoding Einstein Equation

We have seen above that if $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ is a model of a gravitational field generated by $\mathbf{T} \in \sec T_2^0 M$ (the matter plus non gravitational fields energy-momentum momentum tensor) in Einstein GRT and $\boldsymbol{K} \in \sec TM$ is a vector field generating a one parameter group of diffeomorphisms of M the we can encode Einstein equations in Maxwell like equations for $F = dK$ where $K = \boldsymbol{g}(\boldsymbol{K}, \ )$. Indeed, the encoding is given by

$$dF = 0, \quad \underset{\boldsymbol{g}}{\delta} F = -J_{\boldsymbol{K}}, \tag{15.25}$$

$$J_{\boldsymbol{K}} = -\boldsymbol{T}(K) + \frac{1}{2} T K + \ d \underset{\boldsymbol{g}}{\delta} K + \boldsymbol{\partial} \cdot \boldsymbol{\partial} K \tag{15.26}$$

Moreover, taking into account that the Dirac operator acting on sections of the Clifford bundle is $\boldsymbol{\partial} = d - \underset{\boldsymbol{g}}{\delta}$ we have the remarkable result that we can write a single Maxwell like equation encoding Einstein equations, namely

$$\boldsymbol{\partial} F = J_{\boldsymbol{K}}. \tag{15.27}$$

15.3 The Case When $\boldsymbol{K} = \boldsymbol{A}$ Is a Killing Vector Field

To proceed we suppose that $A = \boldsymbol{g}(\boldsymbol{A}, \ ) \in \sec \bigwedge^1 T^* M$, where $\boldsymbol{A}$ is a nontrivial Killing vector field in a LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ which represents a gravitational field generated by a given energy-momentum distribution $\mathbf{T} \in \sec T_2^0 M$ according to GRT. We will need two results that are presented in the form of exercises whose solutions are left to the reader.[8]

Exercise 15.4 Show that if $\boldsymbol{A} \in \sec TM$ is a Killing vector field in the *LSTS* $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$, then

$$\underset{\boldsymbol{g}}{\delta} A = 0, \tag{15.28}$$

where $A = \boldsymbol{g}(\boldsymbol{A},) = A_{\mathbf{a}} \mathfrak{g}^{\mathbf{a}} = A^{\mathbf{a}} \mathfrak{g}_{\mathbf{a}}$.

Remark 15.5 We recall now Exercise 11.5 (Chap. 11) that says that if $\boldsymbol{A} \in \sec TM$ is a Killing vector field in the *LSTS* $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$, then

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = \Box A = \mathcal{R}^{\mathbf{a}} A_{\mathbf{a}}, \tag{15.29}$$

[8] If you need help for the solution of the exercises, see, [15, 17].

where $\boldsymbol{\partial} = \mathfrak{g}^{\mathbf{a}} D_{e_{\mathbf{a}}}$ is the Dirac operator acting on the sections of the Clifford bundle $\mathcal{C}\ell(M, \mathfrak{g})$ and $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ is the Ricci operator acting on $\sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathfrak{g})$. Finally $\mathcal{R}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathfrak{g})$ are the Ricci 1-form fields, with $\mathcal{R}^{\mathbf{a}} = R^{\mathbf{a}}_{\mathbf{b}} \mathfrak{g}^{\mathbf{b}}$, where $R^{\mathbf{a}}_{\mathbf{b}}$ are the components of the Ricci tensor. Also $\boldsymbol{\partial} \cdot \boldsymbol{\partial} A = \Box A$ is the covariant D'Alembertian.

Using Eq. (14.4) we can immediately show taking into account the extensorial property of the Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ that Einstein equations $\mathcal{R}^{\mathbf{a}} - \frac{1}{2} R \boldsymbol{g}^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}$ after multiplying both members by $A_{\mathbf{a}}$ can be written as

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A - \frac{1}{2} RA = \boldsymbol{\partial} \cdot \boldsymbol{\partial} A - \frac{1}{2} RA = -\boldsymbol{T}(A), \tag{15.30}$$

from where the current in Eq. (15.26) now denoted $J_{\boldsymbol{A}}$ is given by

$$J_{\boldsymbol{A}} = RA - 2\boldsymbol{T}(A) \tag{15.31}$$

thus justifying Eq. (15.24) as a "generalized energy" in the case in which $\boldsymbol{A}$ is a Killing vector field.

In this case the Maxwell like equation written in terms of the Dirac operator or the field F associated to the Killing form A is simply

$$\boldsymbol{\partial} F = RA - 2\boldsymbol{T}(A). \tag{15.32}$$

Remark 15.6 In the theory presented in [15] the field A is supposed to be (up to a dimensional constant) the electromagnetic potential of a genuine electromagnetic field created by a given superconducting current and interacting with the gravitational field. Then, clearly, the field $F = dA$ in [15] is supposed to automatically satisfy Maxwell equations. In what follows we only suppose that $A = \boldsymbol{g}(\boldsymbol{A},)$, where $\boldsymbol{A}$ is a Killing vector field in the structure $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$. In this chapter, of course, it is not supposed that A is the electromagnetic potential of any electromagnetic field.

15.4 From Maxwell Equation to a Navier-Stokes Equation

In this section we obtain a Navier-Stokes equation following from the Maxwell like equation [Eq. (15.32)] that encodes Einstein equation once we identify appropriately the magnetic and electric like components of $F = dA$ with variables appearing in theory of the Navier-Stokes equation.

To appreciate what follows we recall that the original Navier-Stokes equation describes the non relativistic motion of a general fluid in Newtonian spacetime. It is not thus adequate to use—at least in principle—a general LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ to describe a fluid motion. In fact we want to describe a fluid motion in a background spacetime such that the fluid medium, together with its dynamics, is equivalent to a LSTS governed by Einstein equation in the sense described below.

Next we recall that the theory of the gravitational field in Minkowski spacetime presented in Chap. 11 interprets gravitation as a plastic distortion of the Lorentz vacuum.[9] In that theory the gravitational field is represented by a $(1,1)$-extensor field $\boldsymbol{h}:\ \sec\bigwedge^1 T^*M \to \sec\bigwedge^1 T^*M$ living in Minkowski spacetime structure. The field $\boldsymbol{h}$—generated by a given energy-momentum distribution in some region U of Minkowski spacetime—distorts the Lorentz vacuum described by the global cobasis[10] $\{\boldsymbol{\gamma}^\mu = d\mathrm{x}^\mu\}$, dual with respect to the basis $\langle \mathbf{e}_\mu = \partial/\partial \mathrm{x}^\mu\rangle$ of TM, thus generating the gravitational potentials $\mathfrak{g}^{\mathbf{a}} = \boldsymbol{h}(\delta^{\mathbf{a}}_\mu \boldsymbol{\gamma}^\mu)$.

Now, in the inertial reference frame e_0 (according to the Minkowski spacetime structure $(M = \mathbb{R}^4, \mathring{\boldsymbol{g}}, \mathring{D}, \tau_{\mathring{\boldsymbol{g}}}, \uparrow)$), we write using the global coordinate functions $\{\mathrm{x}^\mu\}$ for $M \simeq \mathbb{R}^4$,[11]

$$\boldsymbol{A} = \mathring{A}^\mu e_\mu := \left(\frac{1}{\sqrt{1-\boldsymbol{v}^2}} + V_0 + q\right) e_0 - v^i e_i = \mathring{A}_\mu e^\mu = \phi e^0 - v_i e^i, \qquad (15.33)$$

where the vector function $\boldsymbol{v} = (v_1, v_2, v_3)$ is to be identified with the 3-velocity of a Navier-Stokes fluid—in the inertial frame e_0 according to the conditions disclosed below. Also, V_0 denotes a scalar function representing an external potential acting on the fluid, and

$$q = \int_0^{(t,\mathbf{x})} \frac{dp}{\rho}, \qquad (15.34)$$

where the functions p and ρ are identified respectively with the pressure and density of the fluid and supposed functionally related, i.e., $dp \wedge dq = 0$. Furthermore $\boldsymbol{v}^2 := \sum_{i=1}^3 (v_i)^2$.

Observe that ϕ looks like the relativistic energy per unit mass of the fluid. Then we will write $\boldsymbol{A}$ as

$$\boldsymbol{A} = \left(\frac{1}{2}\boldsymbol{v}^2 + V + q\right) e_0 - v^i e_i = \phi e^0 - v_i e^i, \qquad (15.35)$$

where the new potential function V is the sum of V_0 with the sum of the Taylor expansion terms of $[(1-\boldsymbol{v}^2)^{-1/2} - \frac{1}{2}\boldsymbol{v}^2]$. Hence we have

$$\mathring{A} = \mathring{\boldsymbol{g}}(\boldsymbol{A},) = \mathring{A}_\mu \boldsymbol{\gamma}^\mu = \eta_{\mu\nu}\mathring{A}^\nu \boldsymbol{\gamma}^\mu = \mathring{A}^\mu \boldsymbol{\gamma}_\mu, \quad \mathring{F} = d\mathring{A} = \frac{1}{2}\mathring{F}_{\mu\nu}\boldsymbol{\gamma}^\mu \wedge \boldsymbol{\gamma}^\nu, \qquad (15.36)$$

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[9] More details may be found in [7].

[10] The $\{\mathrm{x}^\mu\}$ are global coordinate functions in Einstein-Lorentz Poincaré gauge for the Minkowski spacetime that are naturally adapted to an inertial reference frame $e_0 = \partial/\partial \mathrm{x}^0, \mathring{D}e_0 = 0$.

[11] The basis $\{e^\mu\}$ is the *reciprocal* basis of the basis $\{e_\mu\}$, i.e., $\mathring{\boldsymbol{g}}(e^\mu, e_\nu) = \delta^\mu_\nu$.

and

$$A = \boldsymbol{g}(\boldsymbol{A},) = A_\mu \boldsymbol{\gamma}^\mu = g_{\mu\nu}\mathring{A}^\nu \boldsymbol{\gamma}^\mu = A^\mu \boldsymbol{\gamma}_\mu, \quad F = dA = \frac{1}{2} F_{\mu\nu} \boldsymbol{\gamma}^\mu \wedge \boldsymbol{\gamma}^\nu. \tag{15.37}$$

15.4.1 Identification Postulate

We proceed by identifying the magnetic like and the electric like components of the field $\mathring{F}$. It seems natural to identify the magnetic like components of $\mathring{F}$ with the vorticity field, but we cannot identify electric like components of $\mathring{F}$ with the Lamb vector field. The correct identification is given as follows. Write $\mathring{F}_{\mu\nu} = (d\mathring{A})_{\mu\nu}$ as

$$\mathring{F}_{\mu\nu} = \begin{pmatrix} 0 & l_1 - d_1 & l_2 - d_2 & l_3 - d_3 \\ -l_1 + d_1 & 0 & -w_3 & w_2 \\ -l_2 + d_2 & w_3 & 0 & -w_1 \\ -l_3 + d_3 & -w_2 & w_1 & 0 \end{pmatrix} \tag{15.38}$$

were $\boldsymbol{w}$ is the *vorticity* of the velocity field

$$\boldsymbol{w} := \nabla \times \boldsymbol{v}, \tag{15.39}$$

and

$$\boldsymbol{l} := \boldsymbol{w} \times \boldsymbol{v}, \tag{15.40}$$

is the so called *Lamb* vector and moreover

$$\boldsymbol{d} = -\nabla \chi, \tag{15.41}$$

where χ is a smooth function.

Remark 15.7 We emphasize that the identification of the components of $\mathring{F}$ has been done in an arbitrary but fixed inertial frame $e_0 = \partial/\partial \mathrm{x}^0$ as introduced above.

Next we recall that the non relativistic Navier-Stokes equation for an *inviscid fluid* is given by Chorin and Marsden [5] and Flanders [8]

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla(V + q), \tag{15.42}$$

or using a well known vector identity,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{w} \times \boldsymbol{v} - \nabla \left(V + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{v}^2 \right). \tag{15.43}$$

By these identifications,[12] we get a Navier-Stokes like equation from the straightforward identification of $\boldsymbol{l} - \boldsymbol{d} = (\mathring{F}_{01}, \mathring{F}_{02}, \mathring{F}_{03})$ and $\boldsymbol{w} = (\mathring{F}_{32}, \mathring{F}_{13}, \mathring{F}_{21})$. Indeed, we have

$$\mathring{F}_{0i} = (\boldsymbol{w} \times \boldsymbol{v})_i - d_i = -\frac{\partial v_i}{\partial t} - \frac{\partial \phi}{\partial \mathrm{x}^i}, \tag{15.44}$$

$$\mathring{F}_{jk} = -\textstyle\sum_{i=1}^{3} \epsilon_{ijk} w_i, \tag{15.45}$$

where ϵ_{ijk} is the 3-dimensional Kronecker symbol. Equation (15.44) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{w} \times \boldsymbol{v} + \nabla \left(\frac{1}{2} \boldsymbol{v}^2 \right) = -\nabla \left(V + \frac{p}{\rho} \right) + d_i, \tag{15.46}$$

and since $\boldsymbol{d} = -\nabla \chi$ for some smooth function χ then Eq. (15.46) can be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{w} \times \boldsymbol{v} + \nabla \left(\frac{1}{2} \boldsymbol{v}^2 \right) = -\nabla \left(V + \chi + \frac{p}{\rho} \right) \tag{15.47}$$

which is now a Navier-Stokes like equation for a fluid moving in an external potential $V' = V + \chi$. Moreover, the homogeneous Maxwell equation $d\mathring{F} = 0$ is equivalent to

$$\begin{aligned} \nabla \times \boldsymbol{l} + \frac{\partial \boldsymbol{w}}{\partial t} &= 0, \\ \nabla \cdot \boldsymbol{w} &= 0, \end{aligned} \tag{15.48}$$

which express Helmholtz equation for conservation of vorticity.

Remark 15.8 Note that since $A = \boldsymbol{g}(\boldsymbol{A},) = A_\mu \boldsymbol{\gamma}^\mu = \mathring{A}^\mu g_{\mu\nu} \boldsymbol{\gamma}^\nu$, we can write[13]

$$A = g(\mathring{A}), \tag{15.49}$$

where the extensor field g is defined by $g(\boldsymbol{\gamma}_\mu) = g_{\mu\nu} \boldsymbol{\gamma}^\nu$. Thus, in general, we can write since $d(F - \mathring{F}) = 0$, $F = \mathring{F} + G$ where G is a closed 2-form field. So, $dF = d\mathring{F} = 0$ express the same content, namely Eq. (15.48), the Helmholtz equation for conservation of vorticity. Even more, taking into account that the Minkowski manifold is star shape we have that G is exact. Thus $F = \mathring{F} + dP$, for some smooth 1-form field P.

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[12]Other identifications of Navier-Stokes equation with Maxwell equations may be found in [21, 22].

[13]We have (details in [7]) $g = \boldsymbol{h}^\dagger \boldsymbol{h}$ and $\mathring{\boldsymbol{g}}(g(\boldsymbol{\gamma}_\mu), \boldsymbol{\gamma}_\nu) = g_{\mu\nu} = \mathring{\boldsymbol{g}}(\boldsymbol{h}(\boldsymbol{\gamma}_\mu), \boldsymbol{h}(\boldsymbol{\gamma}_\nu)) = \mathring{\boldsymbol{g}}(\mathfrak{g}_\mu, \mathfrak{g}_\nu)$.

15.4.2 A Realization of Eqs. (15.40) and (15.41)

The attentive reader will realize that what has been done until now will be not an empty exercise only in case there exist nontrivial solutions of Eqs. (15.40) and (15.41), i.e., $\overset{\circ}{F}_{0i} = (\boldsymbol{w} \times \boldsymbol{v})_i + d_i$ for at least a model of GRT given by the LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ modelling a gravitational field. That this is the case, is easily seem if we take, e.g., the Schwarzschild spacetime structure. As well known [12] the vector field

$$\boldsymbol{A} = \partial_{\varphi} = -\mathrm{x}^2 \partial_{\mathrm{x}^1} + \mathrm{x}^1 \partial_{\mathrm{x}^2} \tag{15.50}$$

is a Killing vector field for the Schwarzschild metric.[14] The 1-form field corresponding to it and living in Minkowski spacetime is $\overset{\circ}{A} = \mathrm{x}^2 d\mathrm{x}^1 - \mathrm{x}^1 d\mathrm{x}^2$. Thus $\phi = 0$ and $\boldsymbol{v} = (\mathrm{x}^2, -\mathrm{x}^1, 0)$. This gives $0 = \overset{\circ}{F}_{0i} = -d_i + (\boldsymbol{w} \times \boldsymbol{v})_i$, i.e.,

$$\boldsymbol{d} = -\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \nabla[(\mathrm{x}^1)^2 + (\mathrm{x}^2)^2], \tag{15.51}$$

and Eq. (15.47) holds. The reader is invited to find other examples for other solutions of Einstein equations.

Remark 15.9 For the example just given above we have simply $A = f\overset{\circ}{A}$ where $f = r^2 \cos^2 \theta$. Thus in this case, we have the simple expression

$$F = df \wedge \overset{\circ}{A} + f\overset{\circ}{F}. \tag{15.52}$$

There are many examples of Killing vector fields for which $A = f\overset{\circ}{A}$ and for such fields that developments given below in terms of A are easily translated in terms of $\overset{\circ}{A}$. In particular, when $A = f\overset{\circ}{A}$ we have the following identification of the components of the Lamb and vorticity vector fields with the components of F,

$$\begin{aligned} l_i &= (\boldsymbol{w} \times \boldsymbol{v})_i = \frac{1}{f} F_{0i} - (d \ln f \wedge \overset{\circ}{A})_{0i}, \\ \boldsymbol{w}_i &= -\frac{1}{2} \sum_{i=1}^{3} \epsilon_{ijk} \left[\frac{1}{f} F_{jk} - (d \ln f \wedge \overset{\circ}{A})_{jk} \right]. \end{aligned} \tag{15.53}$$

[14] The spherical coordinate functions are (r, θ, φ).

15.4.3 Constraints Imposed by the Nonhomogeneous Maxwell Like Equation

To continue, we recall that in order for the Navier-Stokes equation just obtained to be compatible with Einstein equation it is necessary yet to take into account the equation $\underset{\boldsymbol{g}}{\delta} F = -J_{\boldsymbol{A}}$ written in the LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ since that equation produce as we are going to see constraints among the several fields involved. We want now to express the constraints implicit in $\underset{\boldsymbol{g}}{\delta} F = -J_{\boldsymbol{A}}$ in terms of the objects defining the Minkowski spacetime structure $(M = \mathbb{R}^4, \mathring{\boldsymbol{g}}, \mathring{D}, \tau_{\mathring{\boldsymbol{g}}}, \uparrow)$.

Now, taking into account that $\mathring{D}\mathring{\boldsymbol{g}} = 0$, we have

$$D\mathring{\boldsymbol{g}} = \mathcal{A} \in \sec T_0^2 M \otimes \bigwedge\nolimits^1 T^*M, \tag{15.54}$$

where $\mathcal{A} \in \sec T_0^2 M \otimes \bigwedge^1 T^*M$ is the nonmetricity tensor of D with respect to $\mathring{\boldsymbol{g}}$. In the coordinates $\{\mathrm{x}^\mu\}$ introduced above it follows that

$$\mathcal{A} = Q_{\alpha\beta\sigma} \boldsymbol{\gamma}^\alpha \otimes \boldsymbol{\gamma}^\beta \otimes \boldsymbol{\gamma}^\sigma. \tag{15.55}$$

Then, as we recall from Chap. 4 the relation between the coefficients $\Gamma^{\nu\cdot\cdot}_{\cdot\mu\alpha}$ and $\mathring{\Gamma}^{\nu\cdot\cdot}_{\cdot\mu\alpha}$ associated to the connections D and $\mathring{D}$ ($D_{e_\mu}\mathfrak{g}^\nu = -\Gamma^{\nu\cdot\cdot}_{\cdot\mu\alpha}\mathfrak{g}^\alpha$, $\mathring{D}_{e_\mu}\mathfrak{g}^\nu = -\mathring{\Gamma}^{\nu\cdot\cdot}_{\cdot\mu\alpha}\mathfrak{g}^\alpha$) in an arbitrary coordinate vector $\{\frac{\partial}{\partial x^\mu}\}$ and covector $\langle\vartheta^\nu = dx^\nu\rangle$ bases—associated to arbitrary coordinate functions $\{x^\mu\}$ covering $U \subset M$—are given by[15]

$$\Gamma^{\nu\cdot\cdot}_{\cdot\mu\alpha} = \mathring{\Gamma}^{\nu\cdot\cdot}_{\cdot\mu\alpha} + \frac{1}{2} S^{\nu\cdot\cdot}_{\cdot\mu\alpha}, \tag{15.56}$$

where

$$S^{\rho\cdot\cdot}_{\cdot\alpha\beta} = \mathring{g}^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) \tag{15.57}$$

are the components of the so called *strain tensor* of the connection.

In the coordinate bases $\{\partial/\partial\mathrm{x}^\mu\}$ and $\{\boldsymbol{\gamma}^\mu = d\mathrm{x}^\mu\}$, associated to the coordinate functions $\{\mathrm{x}^\mu\}$, it follows that $\mathring{\Gamma}^{\nu\cdot\cdot}_{\cdot\mu\alpha} = 0$ and in addition the following relation for the Ricci tensor of D holds:

$$R_{\mu\nu} = J_{(\mu\nu)}.$$

[15] We use that $\mathring{\boldsymbol{g}} = \mathring{g}_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu = \mathring{g}^{\mu\nu}\vartheta_\mu \otimes \vartheta_\nu$, where $\{\vartheta_\mu\}$ is the *reciprocal* basis of $\{\vartheta^\mu\}$, namely $\vartheta_\mu = \mathring{g}_{\alpha\mu}\vartheta^\alpha$ and $\mathring{g}^{\mu\nu}\mathring{g}_{\mu\kappa} = \delta^\nu_\kappa$. In the bases associated to $\{\mathrm{x}^\mu\}$ it is $\mathring{\boldsymbol{g}} = \eta_{\mu\nu}\boldsymbol{\gamma}^\mu \otimes \boldsymbol{\gamma}^\nu = \eta^{\mu\nu}\boldsymbol{\gamma}_\mu \otimes \boldsymbol{\gamma}_\nu$.

Denoting $K_{\cdot\alpha\beta}^{\rho\cdot\cdot} = -\frac{1}{2}S_{\cdot\alpha\beta}^{\rho\cdot\cdot}$, the $J_{(\mu\nu)}$ is the symmetric part of

$$J_{\mu\alpha} = \mathring{D}_\alpha K_{\cdot\rho\mu}^{\rho\cdot\cdot} - \mathring{D}_\rho K_{\cdot\alpha\mu}^{\rho\cdot\cdot} + K_{\cdot\alpha\sigma}^{\rho\cdot\cdot} K_{\cdot\rho\mu}^{\sigma\cdot\cdot} - K_{\cdot\rho\sigma}^{\rho\cdot\cdot} K_{\cdot\alpha\mu}^{\sigma\cdot\cdot}.$$

Now, we introduce the Dirac operator associated to the Levi-Civita connection $\mathring{D}$ of $\mathring{\boldsymbol{g}}$ in our game,

$$\mathring{\boldsymbol{\partial}} := \vartheta^\mu \mathring{D}_{\partial/\partial x^\mu} = \gamma^\mu \mathring{D}_{\partial/\partial \mathrm{x}^\mu} \tag{15.58}$$

and recall Eq. (4.248) of Chap. 4, i.e.,

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = (\mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}})\check{A} + \boldsymbol{L}^\alpha \underset{\mathring{g}}{\cdot} \gamma_\alpha \check{A}, \tag{15.59}$$

where $A = A_\mu \boldsymbol{\gamma}^\mu$, $\check{A}_\kappa := \eta_{\beta\kappa} g^{\beta\sigma} A_\sigma$, and $\boldsymbol{L}^\alpha = \eta^{\alpha\beta} J_{\beta\sigma} \boldsymbol{\gamma}^\sigma$ and the symbol $\underset{\mathring{g}}{\cdot}$ denotes the scalar product accomplished with $\mathring{\boldsymbol{g}}$. Since $\mathring{\boldsymbol{\partial}} \wedge \mathring{\boldsymbol{\partial}} \check{A} = \mathring{\mathcal{R}}^\sigma \check{A}_\sigma = \mathring{R}_\alpha^\sigma \check{A}_\sigma \boldsymbol{\gamma}^\alpha = 0$, Eq. (15.59) reads

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = \eta^{\alpha\beta} J_{\beta\alpha} \check{A} = \eta^{\alpha\beta} J_{\beta\alpha} \eta_{\iota\kappa} g^{\iota\sigma} A_\sigma \boldsymbol{\gamma}^\kappa. \tag{15.60}$$

Recalling now Eqs. (15.29) and (15.30), Eq. (15.60) can be written as an algebraic constraint,[16] relating the components A_σ to the components of the energy-momentum tensor of matter and the components of the $\boldsymbol{g}$ field that is part of the original LSTS. We have,

$$\eta^{\alpha\beta} J_{\beta\alpha} \eta_{\iota\kappa} g^{\iota\sigma} A_\sigma = \frac{1}{2} g^{\mu\alpha} J_{(\mu\alpha)} A_\kappa - T_\kappa^\sigma A_\sigma. \tag{15.61}$$

Equation (15.61) are the constraints need to be satisfied by the variables of our theory in order for the Navier-Stokes equation to be compatible with the contents of Einstein equation.

15.5 Conclusions

Besides having determined the precise form of the so called Komar current we showed that for each LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ representing a gravitational field in GRT which contains an arbitrary Killing vector field $\boldsymbol{A}$, the field $F = dA$ [where

[16]Of course, it is a partial differential equation that needs to be satisfied by the components of the stress tensor of the connection.

$A = \boldsymbol{g}(\boldsymbol{A},\ )]$ satisfies Maxwell like equations $dF = 0$, $\underset{\boldsymbol{g}}{\delta} F = -J_{\boldsymbol{A}}$ with $J_{\boldsymbol{A}}$ given by Eq. (15.31). Moreover we showed that for Killing 1-form field $\overset{\circ}{A} = \overset{\circ}{\boldsymbol{g}}(\boldsymbol{A},)$, when some identifications of the components of $\overset{\circ}{A}$ and the variables entering the Navier-Stokes equation are accomplished and in particular when the postulated nontrivial conditions [Eq. (15.38)]—$\overset{\circ}{F}_{0i} = (d\overset{\circ}{A})_{0i} = l_i - d_i$—is satisfied, the Maxwell like equations for $\overset{\circ}{F}$ and thus the ones for F can be written as a Navier-Stokes equation representing an *inviscid fluid*. Thus, the Maxwell and Navier-Stokes like equations presented above[17] are almost directly obtained from Einstein equation through thoughtful identification of fields. All fields in our approach live in a 4-dimensional background spacetime, namely Minkowski spacetime and a LSTS $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ is considered only a (sometimes useful) description of a gravitational field, as discussed in detail in Chap. 11. We observe moreover that the results just presented are in contrast with the very interesting and important studies in, e.g., [4, 9, 14] where it is shown through some identifications that every solution for an incompressible Navier-Stokes equation in a $(p+1)$-dimensional spacetime gives rise to a solution of Einstein equation in $(p+2)$-dimensional spacetime.[18] It is worth also to quote [13] where it is also suggested an interesting relation between Einstein equations and the Navier-Stokes equation. Finally we remark that it is clear that we can find examples [19] of Lorentzian spacetimes that do not have any nontrivial Killing vector field[19] and of course, for such cases, it is not possible to find a Navier-Stokes equation encoding Einstein equation.

As a last remark, we observe that the Killing vector field of the Schwarzschild spacetime given by Eq. (15.50) is such that $\overset{\circ}{F} = d\overset{\circ}{A}$ is given by $\overset{\circ}{F} = 2\boldsymbol{\gamma}^{21}$ and thus $\overset{\circ}{F}{}^2 \neq 0$. Thus, for this case taking into account the MDE of first kind discussed in Chap. 13 we can also encode the contents of Einstein equation in Maxwell like, Navier-Stokes like and Dirac-Hestenes like equations. All the fields entering the original named equations are, of course, of very different nature, but it seems to the authors a noteworthy fact that for the case just studied the very different equations may have their contents encoded by equations resembling the most important equations of twentieth century Physics.

[17] In [20] a fluid satisfying a particular Navier-Stokes equation is also shown to be approximately equivalent to Einstein equation. The approach here which follows the one in [17] is completely different from the one in [20].

[18] For other papers relating Einstein equations and Navier-Stokes equations we quote also here that the authors in [2] show that cosmic censorship might be associated to global existence for Navier-Stokes or the scale separation characterizing turbulent flows, and in the context of black branes in AdS_5, Einstein equations are shown to be led to the nonlinear equations of boundary fluid dynamics [3]. In addition, gravity variables can provide a geometrical framework for investigating fluid dynamics, in a sense of a geometrization of turbulence [6].

[19] Although, as asserted in Weinberg [24], all Lorentzian spacetimes that represent gravitational fields of physical interest possess some Killing vector fields.

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Chapter 16
Magnetic Like Particles and Elko Spinor Fields

Abstract This chapter scrutinizes the theory of the so-called Elko spinor fields (in Minkowski spacetime) which always appears in pairs and which from the algebraic point of view are in class five in Lounesto classification of spinor fields. We show how these fields differs from Majorana fields (which also are in class five in Lounesto classification of spinor fields) and that Elko spinor fields (as it is the case for Majorana fields) do not satisfy the Dirac equation. We discuss the class of *generalized* Majorana spinor fields (objects that are spinor with "components" with take values in a Grassmann algebra) that satisfy Dirac equations, clarifying some obscure presentations of that theory appearing in the literature. More important, we show that the original presentation of the theory of Elko spinor fields as having mass dimension 1 leads to breakdown of Lorentz and rotational invariance by a simple choice of the spatial axes in an inertial reference frame. We then present a Lagrangian field theory for Elko spinor fields where these fields (as it is the case of Dirac spinor fields) have mass dimension $3/2$. We explicitly demonstrate that Elko spinor fields cannot couple to the electromagnetic field, that they describe pairs of "magnetic" like particles which are coupled to a short range su(2) gauge potential. Thus they eventually can serve to model dark matter. The causal propagator for the $3/2$ mass dimension Elko spinor is explicitly calculated with the Clifford bundle of (multivector) fields. Taking the opportunity given by the formalism developed in our theory we present a very nice representation of the parity operator acting on Dirac-Hestenes spinor fields.

16.1 Introduction

Modern Cosmology based on GRT implies that our universe is permeated by dark matter and dark energy. Elko spinor fields have been introduced in [2, 3] to supposedly describe dark matter. These objects are *dual helicity* eigenspinors of the charge conjugation operator satisfying Klein-Gordon equation and carrying according to the authors of [2, 3] mass dimension 1 instead of mass dimension $3/2$ carried by Dirac spinor fields. A considerable number of interesting papers have been published in the literature on these intriguing objects in the past few years. In particular, according to the theory in [2, 3] the anticommutator of a second quantized

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elko spinor field with its conjugate momentum is nonlocal and thus it is claimed that the theory possess an axis of locality which implies also that the theory of elko spinor fields break Lorentz invariance. We discuss this issue below (Sect. 16.7) which according our view is an odd and acceptable feature of the theory in [2, 3]. We recall in Sect. 16.2 that differently from the theory described in [2, 3] where a second quantized elko spinor field satisfies a Klein-Gordon equation (instead of a Dirac equation) the classical elko spinor fields of $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ types satisfy by their construction a *csfopde* that is Lorentz invariant. The *csfopde* once interacted leads to Klein-Gordon equations for the $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ type fields. However, since the *csfopde* is the basic one and since the Klein-Gordon equations for $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ possess solutions that are *not* solutions of the *csfopde* for $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ we think that it is not necessary to get the field equations for $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ from a Lagrangian where those fields have mass dimension 1 as in [2, 3]. Indeed, we claim that we can attribute mass dimension of $3/2$ for these fields as it is the case for Dirac spinor fields. A proof of this fact is offered by deriving in Sect. 16.3 the *csfopde for* $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ from a Lagrangian where these fields have mass dimension $3/2$. This, fact is to be contrasted with the quantum theory of these fields as presented in [2, 3, 5, 6, 11, 12, 14] (and references therein), namely that elko fields have mass dimension 1.

Taking seriously the view that elko spinor fields due to the special properties given by their bilinear invariants may be the description of some kind of particles in the real world a question then arises: what is the physical meaning of these fields?

In what follows we propose on Sect. 16.4 that the fields λ and ρ (the representatives in the Clifford bundle $\mathcal{C}\ell(M,\eta)$ of the covariant spinor fields $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$) serve the purpose of building Clifford valued multiform fields, i.e., $\mathcal{K} \in \mathcal{C}\ell^0(M,\eta) \otimes \mathbb{R}^0_{1,3}$ and $\mathcal{M} \in \sec \mathcal{C}\ell^0(M,\eta) \otimes \mathbb{R}^0_{1,3}$ [see Eq. (16.40)]. These fields are *electrically neutral* but carry *magnetic* like charges which permit that they couple to a su(2) $\simeq$ $\text{spin}_{3,0} \subset \mathbb{R}^0_{1,3}$ valued potential $\mathcal{A} \in \sec \bigwedge^1 T^*M \otimes \text{spin}_{3,0}$. If the field $\mathcal{A}$ is of short range the particles described by the $\mathcal{K}$ and $\mathcal{M}$ may be interacting and forming a system of spin zero particles with zero magnetic like charge and eventually form condensates something analogous to dark matter, in the sense that they do not couple with the electromagnetic field and are thus invisible.

In Sect. 16.5 we investigate the similarities and main difference between Majorana and elko spinor fields. We observe that elko and Majorana fields are in class 5 of Lounesto classification [22] and although an elko spinor field does *not* satisfy the Dirac equation as correctly claimed in [2, 3], a Majorana spinor field $\boldsymbol{\psi}_{\mathbf{M}} : M \to \mathbb{C}^4$ which is a dual helicity object according to some authors (see e.g., [23]) *does* satisfy the Dirac equation. However this statement is not correct. An operator (quantum) Majorana field $\boldsymbol{\psi}_{\mathbf{M}}$ can satisfy Dirac equation if it is not a dual helicity object (see Sect. 16.5.3). Also, even at a "classical level" a Majorana spinor field satisfies Dirac equation if for any $x \in M$ their components take values in a Grassmann algebra. Also, differently from the case of elko spinor fields some authors claim that Majorana fields are *not* dual helicities objects [2], a statement that is correct only for Majorana quantum fields as constructed in Sect. 16.5.3. For a Majorana field (even at "classical level") whose components take values in a Grassmann algebra the statement is not correct.

Finally, since according to our findings the elko spinor fields as well as the fields $\mathcal{K}$ and $\mathcal{M}$ are of mass dimension $3/2$ we show in Sect. 16.6 how to calculate the correct propagators for $\mathcal{K}$ and $\mathcal{M}$. We also show that the causal propagator for the covariant $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ fields (of mass dimension $3/2$) is simply the standard Feynman propagator of Dirac theory.

In presenting the above results we use the representation of spinor fields in the Clifford bundle formalism (CBF) as developed in Chap. 8. For the convenience of the reader the necessary results are summarized in Sect. 16.2 where a useful translation for the standard matrix formalism to the CBF is given. The CBF makes all calculations easy and transparent and in particular permits to infer [13] in a while that elko spinor fields are class 5 spinor fields in Lounesto classification [13, 22]. In Sect. 16.7 we present a note on the calculation of the anticommutator of elko spinor fields of mass dimension 1 as introduced originally in [2] which implies nonlocality and worse,we show that it leads to an odd (and unacceptable) inference, namely breakdown of rotation invariance and Lorentz by a simple choice by an observer of the (x, y, z) of his laboratory! For completeness of our study on Dirac, Majorana and elko spinor fields we present also in Sect. 16.8 a new representation for the parity operator acting on Dirac-Hestenes spinor fields which although not well known is really noteworthy. In Sect. 16.9 we present our conclusions.

Remark 16.1 The contents of this chapter has been first published in [26].

16.2 Dictionary Between Covariant and Dirac-Hestenes Spinor Fields Formalisms

Let $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}})$ be the Minkowski spacetime structure where $\boldsymbol{\eta} \in \sec T_2^0 M$ is Minkowski metric and D is the Levi-Civita connection of $\boldsymbol{\eta}$. Also, $\tau_{\boldsymbol{\eta}} \in \sec \bigwedge^4 T^*M$ defines an orientation. We denote by $\eta \in \sec T_0^2 M$ the metric of the cotangent bundle. It is defined as follows. Let $\{x^\mu\}$ be coordinates for M in the Einstein-Lorentz-Poincaré gauge. Let $\{\boldsymbol{e}_\mu = \partial/\partial x^\mu\}$ a basis for TM and $\{\gamma^\mu = dx^\mu\}$ the corresponding dual basis for T^*M, i.e., $\gamma^\mu(\boldsymbol{e}_\alpha) = \delta^\mu_\alpha$. Then, if $\boldsymbol{\eta} = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu$ then $\eta = \eta^{\mu\nu}\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu$, where the matrix with entries $\eta_{\mu\nu}$ and the one with entries $\eta^{\mu\nu}$ are the equal to the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$. If $a, b \in \sec \bigwedge^1 T^*M$ we write $a \cdot b = \eta(a, b)$. We also denote by $\{\gamma_\mu\}$ the reciprocal basis of $\{\gamma^\mu = dx^\mu\}$, which satisfies $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$.

We denote the Clifford bundle of differential forms on Minkowski spacetime by $\mathcal{C}\ell(M, \eta)$ and recall once more the fundamental relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \tag{16.1}$$

As we know from Chap. 3 (covariant) spinor fields carrying a $(1/2,0)\oplus(0,1/2)$ representation of $\mathrm{Spin}^0_{1,3} \simeq \mathrm{Sl}(2,\mathbb{C})$ belongs to one of the six Lounesto classes [22]. Moreover, we recall that a $(1/2,0)\oplus(0,1/2)$ spinor field in Minkowski spacetime is an equivalence class of triplets $(\boldsymbol{\psi},\Sigma,\Xi)$ where for each $x\in M$, $\boldsymbol{\psi}(x)\in\mathbb{C}^4$, Σ is an orthonormal coframe and $\Xi = u\in \mathrm{Spin}^0_{1,3}(M,\eta)\subset \mathcal{C}\ell(M,\eta)$ is a spinorial frame. If we fix a fiducial global coframe $\Sigma_0=\{\mathring{\gamma}^\mu\}$ and take, e.g., $\Xi_0 = u_0 = 1 \in \mathrm{Spin}^0_{1,3}(M,\eta)\subset\mathcal{C}\ell(M,\eta)$ the triplet $(\boldsymbol{\psi}_0,\Sigma_0,\Xi_0)$ is equivalent to (ψ,Σ,Ξ) if $\gamma^\mu = \Lambda^\mu_\nu\mathring{\gamma}^\nu = (\pm u)\gamma^\mu(\pm u^{-1})$ and $\boldsymbol{\psi}(x) = S(u)\boldsymbol{\psi}_0(x)$ where $S(u)$ is the standard $(1/2,0)\oplus(0,1/2)$ matrix representation of $\mathrm{Sl}(2,\mathbb{C})$. Dirac gamma matrices in standard and Weyl representations will be denoted in this chapter by $\boldsymbol{\gamma}^\mu$ and $\boldsymbol{\gamma}^{\mu'}$ and are not to be confused with the $\gamma^\mu\in\sec\bigwedge^1T^*M\hookrightarrow\mathcal{C}\ell(M,\eta)$. As well known the gamma matrices also satisfy $\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu+\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2\eta^{\mu\nu}$ and $\boldsymbol{\gamma}^{\mu'}\boldsymbol{\gamma}'^\nu+\boldsymbol{\gamma}'^\nu\boldsymbol{\gamma}'^\mu = 2\eta^{\mu\nu}$. The relation between the $\boldsymbol{\gamma}^\mu$ and the $\boldsymbol{\gamma}'^\mu$ is given by

$$\boldsymbol{\gamma}'^\mu = S\boldsymbol{\gamma}^\mu S^{-1} \tag{16.2}$$

where[1]

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & -\mathbf{1}\end{pmatrix}. \tag{16.3}$$

We recall also that a representation of a $(1/2,0)\oplus(0,1/2)$ spinor field in the Clifford bundle is an equivalence class of triplets (ψ,Σ,Ξ) where $\psi\in\sec\mathcal{C}\ell^0(M,\eta)$ (the even subbundle of $\sec\mathcal{C}\ell(M,\eta)$), Σ is an orthonormal coframe and $\Xi_u = u\in\mathrm{Spin}^0_{1,3}(M,\eta)\subset\mathcal{C}\ell(M,\eta)$ is a spinorial frame. If we fix a fiducial global coframe $\Sigma_0 = \{\Gamma^\mu\}$ and take $\Xi_{u_0} = u_0 = 1 \in \sec\mathrm{Spin}^0_{1,3}(M,\eta)\subset\sec\mathcal{C}\ell(M,\eta)$ the triplet (ψ_0,Σ_0,Ξ_0) is equivalent to[2] (ψ,Σ,Ξ_u) if $\gamma^\mu = \Lambda^\mu_\nu\Gamma^\nu = (u)\boldsymbol{\gamma}^\mu(u^{-1})$ and $\psi = \psi_0 u^{-1}$. Field ψ is called an operator spinor field and the operator spinor fields belonging to Lounesto classes 1, 2, 3 are also known as Dirac-Hestenes spinor fields.

If $\boldsymbol{\gamma}^\mu$, $\mu = 0,1,2,3$ are the Dirac gamma matrices in the *standard representation* and $\{\gamma_\mu\}$ are as introduced above, we define

$$\sigma_k := \gamma_k\gamma_0\in\sec\textstyle\bigwedge^2T^*M\hookrightarrow\sec\mathcal{C}\ell^0(M,\eta),\ k=1,2,3, \tag{16.4}$$

$$\mathbf{i} = \gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3\in\sec\textstyle\bigwedge^4T^*M\hookrightarrow\sec\mathcal{C}\ell(M,\eta), \tag{16.5}$$

$$\boldsymbol{\gamma}_5 := \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\in\mathrm{Mat}(4,\mathbb{C}). \tag{16.6}$$

[1] We will suppress the writing of the 4×4 and the 2×2 unity matrices when no confusion arises.

[2] Take notice that (ψ,Σ,Ξ_u) is not equivalent to (ψ,Σ,Ξ_{-u}) even if $(u)\boldsymbol{\gamma}^\mu(u^{-1}) = (-u)\boldsymbol{\gamma}^\mu(-u^{-1})$.

Then, to the covariant spinor $\boldsymbol{\psi} : M \to \mathbb{C}^4$ (in standard representation of the gamma matrices) where ($\mathrm{i} = \sqrt{-1}$, $\boldsymbol{\phi}, \boldsymbol{\varsigma} : M \to \mathbb{C}^2$)

$$\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varsigma} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m^0 + \mathrm{i}m^3 \\ -m^2 + \mathrm{i}m^1 \end{pmatrix} \\ \begin{pmatrix} n^0 + \mathrm{i}n^3 \\ -n^2 + \mathrm{i}n^1 \end{pmatrix} \end{pmatrix}, \tag{16.7}$$

there corresponds the operator spinor field $\psi \in \sec \mathcal{C}\ell^0(M, \eta)$ given by

$$\psi = \phi + \varsigma\sigma_3 = (m^0 + m^k \mathbf{i}\sigma_k) + (n^0 + n^k \mathbf{i}\sigma_k)\sigma_3. \tag{16.8}$$

We then have the useful formulas in Eq. (16.9) below that one can use to immediately translate results of the standard matrix formalism in the language of the Clifford bundle formalism and vice-versa

$$\begin{aligned} \boldsymbol{\gamma}_\mu \boldsymbol{\psi} &\leftrightarrow \gamma_\mu \psi \gamma_0, \\ \mathrm{i}\boldsymbol{\psi} &\leftrightarrow \psi\gamma_{21} = \psi \mathbf{i}\sigma_3, \\ \mathrm{i}\boldsymbol{\gamma}_5 \boldsymbol{\psi} &\leftrightarrow \psi\sigma_3 = \psi\gamma_3\gamma_0, \\ \bar{\boldsymbol{\psi}} &= \boldsymbol{\psi}^\dagger \boldsymbol{\gamma}^0 \leftrightarrow \tilde{\psi}, \\ \boldsymbol{\psi}^\dagger &\leftrightarrow \gamma_0 \tilde{\psi} \gamma_0, \\ \boldsymbol{\psi}^* &\leftrightarrow -\gamma_2 \psi \gamma_2. \end{aligned} \tag{16.9}$$

Remark 16.2 Note that $\boldsymbol{\gamma}_\mu, i\mathbf{1}_4$ and the operations and † are for each $x \in M$ mappings $\mathbb{C}^4 \to \mathbb{C}^4$. Then they are represented in the Clifford bundles formalism by extensor fields which maps $\mathcal{C}\ell^0(M, \eta) \to \mathcal{C}\ell^0(M, \eta)$. Thus, to the operator $\boldsymbol{\gamma}_\mu$ there corresponds an extensor field, call it $\underline{\boldsymbol{\gamma}}_\mu : \mathcal{C}\ell^0(M, \eta) \to \mathcal{C}\ell^0(M, \eta)$ such that $\underline{\boldsymbol{\gamma}}_\mu \psi = \gamma_\mu \psi \gamma_0$.

Using the above dictionary the standard Dirac equation for a Dirac spinor field $\boldsymbol{\psi} : M \to \mathbb{C}^4$

$$i\boldsymbol{\gamma}^\mu \partial_\mu \boldsymbol{\psi} - m\boldsymbol{\psi} = 0 \tag{16.10}$$

translates immediately in the so-called Dirac-Hestenes equation, i.e.,

$$\boldsymbol{\partial} \psi \gamma_{21} - m\psi\gamma_0 = 0, \tag{16.11}$$

where $\boldsymbol{\partial}$ is the Dirac operator acting on $\mathcal{C} \in \sec \mathcal{C}\ell(M, \eta)$, which using the basis introduced above is simply

$$\boldsymbol{\partial}\mathcal{C} := \gamma^{\mu} \lrcorner (\partial_{\mu}\mathcal{C}) + \gamma^{\mu} \wedge (\partial_{\mu}\mathcal{C}) \tag{16.12}$$

Remark 16.3 It is sometimes useful, in particular when studying in Sect. 16.6 solutions for the Dirac-Hestenes equation to consider besides the Clifford bundle of differential forms $\mathcal{C}\ell(M, \eta)$ also the Clifford bundle of multivector fields $\mathcal{C}\ell(M, \boldsymbol{\eta})$. We will write $\check{\psi} \in \sec \mathcal{C}\ell(M, \boldsymbol{\eta})$ for the sections of the $\mathcal{C}\ell(M, \boldsymbol{\eta})$ bundle. The Dirac-Hestenes equation in $\mathcal{C}\ell(M, \boldsymbol{\eta})$ is.

$$\check{\boldsymbol{\partial}} \check{\psi} \boldsymbol{e}_{21} - m\check{\psi} \boldsymbol{e}_0 = 0. \tag{16.13}$$

where $\boldsymbol{e}_{\mu}\boldsymbol{e}_{\nu} + \boldsymbol{e}_{\nu}\boldsymbol{e}_{\mu} = 2\eta_{\mu\nu}$ and $\check{\boldsymbol{\partial}} \boldsymbol{:=} \boldsymbol{e}^{\mu} \partial_{\mu}$ with $\boldsymbol{e}^{\mu} := \eta^{\mu\nu}$ and (when using the basis introduced above)

$$\check{\boldsymbol{\partial}}\check{\mathcal{C}} := \boldsymbol{e}^{\mu} \lrcorner (\partial_{\mu}\check{\mathcal{C}}) + \boldsymbol{e}^{\mu} \wedge (\partial_{\mu}\check{\mathcal{C}}), \tag{16.14}$$

for $\check{\mathcal{C}} \in \sec \mathcal{C}\ell(M, \boldsymbol{\eta})$. Keep in mind that in definition of $\check{\boldsymbol{\partial}}$ the $\boldsymbol{e}^{\mu}$ are not supposed to act as a derivatives operators, i.e., $\boldsymbol{e}^{\mu} \lrcorner (\partial_{\mu}\check{\mathcal{C}})$ (respectively $\boldsymbol{e}^{\mu} \wedge (\partial_{\mu}\check{\mathcal{C}})$) is the left contraction of $\boldsymbol{e}^{\mu}$ with $\partial_{\mu}\check{\mathcal{C}}$ (respectively, the exterior product of $\boldsymbol{e}^{\mu}$ with $\partial_{\mu}\check{\mathcal{C}}$).

The basic positive and negative energy solutions of Eq. (16.10) which are eigenspinors of the helicity operator are [31]

$$\mathbf{u}^{(1)}(\mathbf{p})e^{-ip_{\mu}x^{\mu}}, \quad \mathbf{u}^{(2)}(\mathbf{p})e^{-ip_{\mu}x^{\mu}}, \quad \mathbf{v}^{(1)}(\mathbf{p})e^{ip_{\mu}x^{\mu}}, \quad \mathbf{v}^{(2)}(\mathbf{p})e^{ip_{\mu}x^{\mu}}. \tag{16.15}$$

The $\mathbf{u}^{(\alpha)}(\mathbf{p})$ and $\mathbf{v}^{(\alpha)}(\mathbf{p})$ ($\alpha = 1, 2$) are eigenspinors of the parity operator[3] $\mathbf{P}$, i.e.,

$$\mathbf{P}\mathbf{u}^{(\alpha)}(\mathbf{p}) = \mathbf{u}^{(\alpha)}(\mathbf{p}), \quad \mathbf{P}\mathbf{v}^{(\alpha)}(\mathbf{p}) = \mathbf{v}^{(\alpha)}(\mathbf{p}), \tag{16.16}$$

which makes Dirac equation invariant under a parity transformation. This will be discussed below. These fields are represented in the Clifford bundle formalism by the following operator spinor fields,

$$u^{(r)}(\mathbf{p}) = L(\mathbf{p})\varkappa^{(r)}, \quad v^{(r)}(\mathbf{p}) = L(\mathbf{p})\varkappa^{(r)}\sigma_3, \tag{16.17}$$

[3]The parity operator acting on covariant spinor fields is defined as in [2], i.e., $\mathbf{P} = i\gamma^0\mathcal{R}$, where $\mathcal{R}$ changes $\mathbf{p} \mapsto -\mathbf{p}$ and changes the eingenvalues of the helicity operator. For other possibilities for the parity operator, see e.g., page 50 of [7].

where $\varkappa^{(r)} = \{1, -\mathbf{i}\sigma_2\}$ and $L(\mathbf{p})$ is the following boost operator[4]

$$L(\mathbf{p}) = \frac{p\gamma^0 + m}{\sqrt{2m(E+m)}}, \tag{16.18}$$

satisfying $L(\mathbf{p})\tilde{L}(\mathbf{p}) = \mathbf{1}$.

Remark 16.4 Recall that Dirac-Hestenes spinor fields couple to the electromagnetic potential $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ as

$$\boldsymbol{\partial}\psi\gamma_{21} - m\psi\gamma_0 + eA\psi = 0. \tag{16.19}$$

As it is well known this equation is invariant under a parity transformation of the fields A and ψ.

In [2] the following (covariant) self and anti-self dual elko spinor fields $\boldsymbol{\lambda}'^{s,a}_{\{+-\}}, \boldsymbol{\lambda}'^{s,a}_{\{-+\}}$ $\boldsymbol{\rho}'^{s,a}_{\{+-\}}, \boldsymbol{\rho}'^{s,a}_{\{-+\}}$: $M \to \mathbb{C}^4$ which are eigenspinors of the charge conjugation operator $(\mathbf{C})$[5] are defined using the Weyl (chiral) representation of the gamma matrices by

$$\boldsymbol{\lambda}'^{s}_{\{\mp\,\pm\}}(\mathbf{p}) = \begin{pmatrix} \boldsymbol{\sigma}_2[\boldsymbol{\phi}^{\pm}_L(\mathbf{p})]^* \\ \boldsymbol{\phi}_L\pm(\mathbf{p}) \end{pmatrix}, \quad \boldsymbol{\lambda}'^{a}_{\{\mp\,\pm\}}(\mathbf{p}) = \begin{pmatrix} -\boldsymbol{\sigma}_2[\boldsymbol{\phi}^{\pm}_L(\mathbf{p})]^* \\ \boldsymbol{\phi}_L\pm(\mathbf{p}) \end{pmatrix}, \tag{16.20}$$

$$\boldsymbol{\rho}'^{s}_{\{\pm\,\mp\}}(\mathbf{p}) = \begin{pmatrix} \boldsymbol{\phi}_R\pm(\mathbf{p}) \\ -\boldsymbol{\sigma}_2[\boldsymbol{\phi}^{\pm}_R(\mathbf{p})]^* \end{pmatrix}, \quad \boldsymbol{\rho}'^{a}_{\{\pm\,\mp\}}(\mathbf{p}) = \begin{pmatrix} \boldsymbol{\phi}_R\pm(\mathbf{p}) \\ \boldsymbol{\sigma}_2[\boldsymbol{\phi}^{\pm}_R(\mathbf{p})]^* \end{pmatrix}, \tag{16.21}$$

where the $\mathbf{C}\boldsymbol{\lambda}'^{s} = +\boldsymbol{\lambda}'^{s}$, $\mathbf{C}\boldsymbol{\lambda}'^{a} = -\boldsymbol{\lambda}'^{a}$ and the indices $\{+-\}, \{-+\}$ refers to the helicities of the upper and down components of the elko spinor fields, and where as in [2] we introduce the following helicity eigenstates,[6] $\boldsymbol{\phi}^{+}_L(\mathbf{0})$ and $\boldsymbol{\phi}^{-}_L(\mathbf{0})$ and $\boldsymbol{\phi}^{+}_R(\mathbf{0})$ and $\boldsymbol{\phi}^{-}_R(\mathbf{0})$ such that with $\hat{\mathbf{p}}\frac{\mathbf{p}}{|\mathbf{p}|}$ we have

$$\begin{aligned} &\boldsymbol{\sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\boldsymbol{\phi}^{\pm}_L(\mathbf{0}) := \pm\boldsymbol{\phi}^{\pm}_L(\mathbf{0}), \quad \boldsymbol{\sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}[\boldsymbol{\sigma}_2(\boldsymbol{\phi}^{\pm}_L(\mathbf{0}))^*] = \mp[\boldsymbol{\sigma}_2(\boldsymbol{\phi}^{\pm}_L(\mathbf{0}))^*], \\ &\boldsymbol{\sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\boldsymbol{\phi}^{\pm}_R(\mathbf{0}) := \pm\boldsymbol{\phi}^{\pm}_R(\mathbf{0}), \quad \boldsymbol{\sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}[-\boldsymbol{\sigma}_2(\boldsymbol{\phi}^{\pm}_R(\mathbf{0}))^*] = \mp[-\boldsymbol{\sigma}_2(\boldsymbol{\phi}^{\pm}_R(\mathbf{0}))^*]. \end{aligned} \tag{16.22}$$

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[4]Recall that $p\gamma^0 = p_\mu\gamma^\mu\gamma^0 = E + \mathbf{p}$.

[5]The conjugation operator used in [2] is $\mathbf{C}\boldsymbol{\psi} = -\gtrsim^2\boldsymbol{\psi}^*$. Using the dictionary given by Eq. (3.68) we find that in the Clifford bundle formalism we have $\mathbf{C}\psi = -\psi\gtrsim_{20}$.

[6]The indices L and R in $\boldsymbol{\phi}^{\pm}_L(\mathbf{p})$ and $\boldsymbol{\phi}^{\pm}_L(\mathbf{p})$ refer to the fact that these spinors fields transforms according to the basic non equivalent two dimensional representation of $Sl(2,\mathbb{C})$.

Also recall that being a *general* boost operator in the $D^{1/2,0} \oplus D^{0,1/2}$ representation of $Sl(2, \mathbb{C})$

$$\boldsymbol{K} = \boldsymbol{K}^{1/2,0} \oplus \boldsymbol{K}^{0,1/2} = e^{\frac{\sigma}{2}\cdot\boldsymbol{\alpha}} \oplus e^{-\frac{\sigma}{2}\cdot\boldsymbol{\alpha}} \tag{16.23}$$

we have, e.g., taking $\boldsymbol{\alpha} = \mathbf{p}$

$$\boldsymbol{\lambda}'^{s}_{\{-+\}}(\mathbf{p}) = \sqrt{\frac{E+m}{m}}\left(1 - \frac{|\mathbf{p}|}{E+m}\right)\boldsymbol{\lambda}'^{s}_{\{-+\}}(\mathbf{0}), \tag{16.24}$$

More details, if necessary, may be found in [2].

Remark 16.5 By dual helicity field we simply mean here that the formulas in Eq. (16.22) are satisfied. Note that the helicity operator (in both Weyl and standard representation of the gamma matrices) is

$$\boldsymbol{\Sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} = \begin{pmatrix} \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \end{pmatrix}. \tag{16.25}$$

$\mathbb{C}^4$-valued spinor fields depends for its definition of a choice of an inertial frame where the momentum of the particle is $(p_0, \mathbf{p})$. The operator $(\boldsymbol{K}^{1/2,0} \oplus \boldsymbol{K}^{0,1/2})$ commutes with $\boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}$ only if $\boldsymbol{\sigma} \cdot \boldsymbol{\alpha}$ is proportional to $\boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}$. So, the statement in [2] that the helicity operator commutes with the boost operator must be qualified. However, it remains true that $\boldsymbol{\sigma}_2[\boldsymbol{\phi}_l^+(\mathbf{p})]^*$ and $\boldsymbol{\phi}_l^+(\mathbf{p})$ have opposite helicities for any $\mathbf{p}$.

Remark 16.6 Recall that a $\mathbb{C}^4$-valued spinor field $\boldsymbol{\lambda}'^{s}_{\{-\,+\}}(\mathbf{p})$ given in the Weyl representation of the gamma matrices is represented by $\boldsymbol{\lambda}^{s}_{\{-\,+\}}(\mathbf{p})$ in the standard representation of the gamma matrices. We have

$$\begin{aligned}\boldsymbol{\lambda}^{s}_{\{-\,+\}}(\mathbf{p}) = S\boldsymbol{\lambda}'^{s}_{\{-\,+\}}(\mathbf{p}) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* \\ \boldsymbol{\phi}_L^+(\mathbf{p}) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* + \boldsymbol{\phi}_L^+(\mathbf{p}) \\ \boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* - \boldsymbol{\phi}_L^+(\mathbf{p}) \end{pmatrix}\end{aligned} \tag{16.26}$$

and then

$$\boldsymbol{\Sigma}' \cdot \frac{\mathbf{p}}{|\mathbf{p}|}\frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* + \boldsymbol{\phi}_L^+(\mathbf{p}) \\ \boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* - \boldsymbol{\phi}_L^+(\mathbf{p}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* + \boldsymbol{\phi}_L^+(\mathbf{p}) \\ -\boldsymbol{\sigma}_2[\boldsymbol{\phi}_L^+(\mathbf{p})]^* - \boldsymbol{\phi}_L^+(\mathbf{p}) \end{pmatrix}. \tag{16.27}$$

Equations (16.26) and (16.27) show that the labels $\{-\,+\}$ (and also $\{+\,-\}$) as defining the helicities of the upper and down $\mathbb{C}^2$-valued components of a $\boldsymbol{\lambda}$ type spinor field in the standard representation of the gamma matrices have no meaning at all.

Also, in [2] the following identifications are made:

$$\begin{aligned} \boldsymbol{\rho}^{s}_{\{+-\}}(\mathbf{p}) &= +i\boldsymbol{\lambda}^{a}_{\{+-\}}(\mathbf{p}), \quad \boldsymbol{\rho}^{s}_{\{-+\}}(\mathbf{p}) = -i\boldsymbol{\lambda}^{a}_{\{-+\}}(\mathbf{p}), \\ \boldsymbol{\rho}^{a}_{\{+-\}}(\mathbf{p}) &= -i\boldsymbol{\lambda}^{s}_{\{+-\}}(\mathbf{p}), \quad \boldsymbol{\rho}^{a}_{\{-+\}}(\mathbf{p}) = +i\boldsymbol{\lambda}^{s}_{\{-+\}}(\mathbf{p}). \end{aligned} \tag{16.28}$$

Moreover, we recall that the elko spinor fields are not eigenspinors of the parity operator and indeed (see Eqs. (4.14) and (4.15) in [2]),

$$\begin{aligned} \mathbf{P}\boldsymbol{\lambda}^{s}_{\{-+\}}(\mathbf{p}) &= +i\boldsymbol{\lambda}^{a}_{\{+-\}}(\mathbf{p}) = \boldsymbol{\rho}^{s}_{\{+-\}}(\mathbf{p}), \\ \mathbf{P}\boldsymbol{\lambda}^{s}_{\{+-\}}(\mathbf{p}) &= -i\boldsymbol{\lambda}^{a}_{\{-+\}}(\mathbf{p}) = \boldsymbol{\rho}^{s}_{\{-+\}}(\mathbf{p}), \\ \mathbf{P}\boldsymbol{\lambda}^{a}_{\{-+\}}(\mathbf{p}) &= -i\boldsymbol{\lambda}^{s}_{\{+-\}}(\mathbf{p}) = \boldsymbol{\rho}^{a}_{\{+-\}}(\mathbf{p}), \\ \mathbf{P}\boldsymbol{\lambda}^{a}_{\{+-\}}(\mathbf{p}) &= +i\boldsymbol{\lambda}^{s}_{\{-+\}}(\mathbf{p}) = \boldsymbol{\rho}^{a}_{\{-+\}}(\mathbf{p}). \end{aligned} \tag{16.29}$$

Then if $\boldsymbol{\lambda}^{s,a}(x) := \boldsymbol{\lambda}^{s,a}(\mathbf{p}) \exp(\epsilon^{s,a} i p_\mu x^\mu)$, with $\epsilon^s = -1$ and $\epsilon^a = +1$ we have due to their construction that the elko spinor fields must satisfy the following *csfopde*:

$$\begin{aligned} i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\lambda}^{s}_{\{-+\}} + m\boldsymbol{\rho}^{a}_{\{+-\}} &= 0, \quad i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\rho}^{a}_{\{-+\}} + m\boldsymbol{\lambda}^{s}_{\{+-\}} = 0, \\ i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\lambda}^{a}_{\{-+\}} - m\boldsymbol{\rho}^{s}_{\{+-\}} &= 0, \quad i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\rho}^{s}_{\{-+\}} - m\boldsymbol{\lambda}^{a}_{\{+-\}} = 0, \\ i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\lambda}^{s}_{\{+-\}} - m\boldsymbol{\rho}^{a}_{\{-+\}} &= 0, \quad i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\rho}^{a}_{\{+-\}} - m\boldsymbol{\lambda}^{s}_{\{-+\}} = 0, \\ i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\lambda}^{a}_{\{+-\}} + m\boldsymbol{\rho}^{s}_{\{-+\}} &= 0, \quad i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\rho}^{s}_{\{+-\}} + m\boldsymbol{\lambda}^{a}_{\{-+\}} = 0. \end{aligned} \tag{16.30}$$

If $\lambda^{s,a}_{\{+-\}}, \lambda^{s,a}_{\{-+\}}, \rho^{s,a}_{\{+-\}}, \rho^{s,a}_{\{-+\}} \in \sec \mathcal{C}\ell^{0}(M,\eta)$ are the representatives of the covariant spinors $\boldsymbol{\lambda}^{s,a}_{\{+-\}}, \boldsymbol{\lambda}^{s,a}_{\{-+\}}, \boldsymbol{\rho}^{s,a}_{\{+-\}}, \boldsymbol{\rho}^{s,a}_{\{-+\}} : M \to \mathbb{C}^4$ then they satisfy the *csfopde*:

$$\begin{aligned} \boldsymbol{\partial}\lambda^{s}_{\{-+\}}\gamma_{21} + m\rho^{a}_{\{+-\}}\gamma_{0} &= 0, \quad \boldsymbol{\partial}\rho^{a}_{\{-+\}}\gamma_{21} + m\lambda^{s}_{\{+-\}}\gamma_{0} = 0, \\ \boldsymbol{\partial}\lambda^{a}_{\{-+\}}\gamma_{21} - m\rho^{s}_{\{+-\}}\gamma_{0} &= 0, \quad \boldsymbol{\partial}\rho^{s}_{\{-+\}}\gamma_{21} - m\lambda^{a}_{\{+-\}}\gamma_{0} = 0, \\ \boldsymbol{\partial}\lambda^{s}_{\{+-\}}\gamma_{21} - m\rho^{a}_{\{-+\}}\gamma_{0} &= 0, \quad \boldsymbol{\partial}\rho^{a}_{\{+-\}}\gamma_{21} - m\lambda^{s}_{\{-+\}}\gamma_{0} = 0, \\ \boldsymbol{\partial}\lambda^{a}_{\{+-\}}\gamma_{21} + m\rho^{s}_{\{-+\}}\gamma_{0} &= 0, \quad \boldsymbol{\partial}\rho^{s}_{\{+-\}}\gamma_{21} + m\lambda^{a}_{\{-+\}}\gamma_{0} = 0. \end{aligned} \tag{16.31}$$

Remark 16.7 From Eq. (16.31) it follows trivially that the operator spinor fields $\lambda^{s,a}_{\{+-\}}, \lambda^{s,a}_{\{-+\}}, \rho^{s,a}_{\{+-\}}, \rho^{s,a}_{\{-+\}} \in \sec \mathcal{C}\ell^{0}(M,\eta)$ satisfy Klein-Gordon equations. However, e.g., the Klein-Gordon equations

$$\Box\lambda^{s}_{\{-+\}} + m^2\lambda^{s}_{\{-+\}} = 0, \quad \Box\rho^{a}_{\{+-\}} + m^2\rho^{a}_{\{+-\}} = 0, \tag{16.32}$$

possess (as it is trivial to verify) solutions that are not solutions of the csfopde satisfied $\lambda^s_{\{-+\}}$and $\rho^a_{\{+-\}}$. An immediate consequence of this observation is that attribution of mass dimension 1 to elko spinor fields seems equivocated. Elko spinor fields as Dirac spinor fields have mass dimension $3/2$, and the equation of motion for the elkos can be obtained from a Lagrangian (where the mass dimension of the fields are obvious) as we recall next.

16.3 The *CSFOPDE* Lagrangian for Elko Spinor Fields

A (multiform) Lagrangian that gives Eq. (16.31) for the operator elko spinor fields $\lambda^s_{\{-+\}}, \lambda^a_{\{-+\}}, \rho^a_{\{+-\}}, \rho^s_{\{-+\}} \in \sec \mathcal{C}\ell^0(M,\eta)$ having *mass dimension* $3/2$ is:

$$\mathcal{L} = \frac{1}{2}\left\{ \begin{array}{c} (\boldsymbol{\partial}\lambda^s_{\{+-\}}\mathbf{i}\gamma_3)\cdot\lambda^s_{\{+-\}} + (\boldsymbol{\partial}\lambda^a_{\{-+\}}\mathbf{i}\gamma_3)\cdot\lambda^a_{\{-+\}} + (\boldsymbol{\partial}\rho^a_{\{+-\}}\mathbf{i}\gamma_3)\cdot\rho^a_{\{+-\}} \\ +(\boldsymbol{\partial}\rho^s_{\{-+\}}\gamma\mathbf{i}\gamma_3)\cdot\rho^s_{\{-+\}} - 2m\lambda^s_{\{+-\}}\cdot\rho^a_{\{-+\}} + 2m\lambda^a_{\{-+\}}\cdot\rho^s_{\{+-\}} \end{array} \right\} \tag{16.33}$$

We know from Chap. 8 that the Euler-Lagrange equation obtained, from the variation, e.g., of the field $\lambda^S_{\{+-\}}$ is:

$$\partial_{\lambda^S_{\{+-\}}}\mathcal{L} - \boldsymbol{\partial}\left(\partial_{\boldsymbol{\partial}\lambda^s_{\{+-\}}}\mathcal{L}\right) = 0. \tag{16.34}$$

We have immediately[7]

$$\begin{aligned} \partial_{\lambda^s_{\{+-\}}}\mathcal{L} &= \frac{1}{2}\boldsymbol{\partial}\lambda^s_{\{+-\}}\mathbf{i}\gamma_3 - m\rho^a_{\{-+\}}, \\ \partial_{\boldsymbol{\partial}\lambda^s_{\{-+\}}}\mathcal{L} &= -\frac{1}{2}\partial_{\boldsymbol{\partial}\lambda^s_{\{+-\}}}\left(\boldsymbol{\partial}\lambda^s_{\{-+\}})\cdot\lambda^s_{\{+-\}}\mathbf{i}\gamma_3\right) = -\frac{1}{2}\lambda^s_{\{+-\}}\mathbf{i}\gamma_3, \\ -\boldsymbol{\partial}\left(\partial_{\boldsymbol{\partial}\lambda^s_{\{+-\}}}\mathcal{L}\right) &= +\frac{1}{2}\boldsymbol{\partial}\lambda^s_{\{+-\}}\mathbf{i}\gamma_3. \end{aligned} \tag{16.35}$$

Recalling that $\mathbf{i}\gamma_3 = -\gamma_0\gamma_1\gamma_2$ the resulting Euler-Lagrange equation is

$$\boldsymbol{\partial}\lambda^s_{\{+-\}}\gamma_{21} - m\rho^a_{\{-+\}}\gamma_0 = 0.$$

Remark 16.8 With this result we must say that the main claim concerning the attributes of elko spinor fields appearing in recent literature, i.e., that these objects are of mass dimension 1, seems to us not necessary if not equivocated and the

[7]In the second line of Eq. (16.35) we used the identity $(KL)\cdot M = K\cdot(M\tilde{L})$ for all $K, L, M \in \sec \mathcal{C}\ell(M,\eta)$.

question arises: which kind of particles are described by these fields and to which gauge field do they couple? This question is answered in the next section.

16.4 Coupling of the Elko Spinor Fields a su(2) $\simeq$ spin$_{3,0}$ Valued Potential $\mathcal{A}$

We start by introducing Clifford valued differential multiforms fields, i.e., the objects

$$\mathcal{K} = \lambda^s_{\{-+\}} \otimes 1 - \rho^a_{\{+-\}} \otimes \mathrm{i}\tau_2 \in \sec \mathcal{C}\ell^0(M,\eta) \otimes \mathbb{R}^0_{1,3}$$

$$\mathcal{M} = \lambda^s_{\{-+\}} \otimes 1 - \rho^a_{\{+-\}} \otimes \mathrm{i}\tau_2 \in \sec \mathcal{C}\ell^{0\subset}(M,\eta) \otimes \mathbb{R}^0_{1,3} \tag{16.36}$$

where $\tau_{\mathbf{1},}$ $\tau_{\mathbf{2},}$, $\tau_{\mathbf{3}}$ are the generators of the Pauli algebra $\mathbb{R}_{3,0} \simeq \mathbb{R}^0_{1,3}$ and $\mathrm{i} := \tau_1\tau_{\mathbf{2}}\tau_{\mathbf{3}}$. So, we have $\tau_{\mathbf{i}} := \Gamma_{\mathbf{i}}\Gamma_0$ where the Γ_μ are the generators of $\mathbb{R}_{1,3}$, i.e., $\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\eta_{\mu\nu}$. Also, $\mathrm{i} := \tau_1\tau_{\mathbf{2}}\tau_{\mathbf{3}} = \Gamma_{\mathbf{0}}\Gamma_{\mathbf{1}}\Gamma_{\mathbf{2}}\Gamma_{\mathbf{3}} =: \Gamma_{\mathbf{5}}$.

We define the reverse a general Clifford valued differential multiforms field

$$\mathcal{N} = \mathcal{N}^{\mathbf{0}} \otimes 1 + \mathcal{N}^{\mathbf{k}} \otimes \tau_{\mathbf{k},} + \frac{1}{2}\mathcal{N}^{\mathbf{k}} \otimes \tau_{\mathbf{i}}\tau_{\mathbf{j}} + \frac{1}{3!}\mathcal{N}^{\mathbf{ikj}}\tau_{\mathbf{i}}\tau_{\mathbf{k}}\tau_{\mathbf{j}} \in \sec \mathcal{C}\ell(M,\eta) \otimes \mathbb{R}^0_{1,3}, \tag{16.37}$$

where $\mathcal{N}^{\mathbf{0}}, \mathcal{N}^{\mathbf{k}}, \mathcal{N}^{\mathbf{k}}, \mathcal{N}^{\mathbf{ikj}} \in \sec \mathcal{C}\ell(M,\eta)$ by

$$\tilde{\mathcal{N}} = \tilde{\mathcal{N}}^{\mathbf{0}} \otimes 1 + \tilde{\mathcal{N}}^{\mathbf{k}} \otimes \tau_{\mathbf{k},} + \frac{1}{2}\tilde{\mathcal{N}}^{\mathbf{ij}} \otimes \tau_{\mathbf{j}}\tau_{\mathbf{i}} + \frac{1}{3!}\tilde{\mathcal{N}}^{\mathbf{ijk}}\tau_{\mathbf{k}}\tau_{\mathbf{j}}\tau_{\mathbf{i}} \tag{16.38}$$

Since, as well known the $\tau_{\mathbf{1},}$ $\tau_{\mathbf{2},}$, $\tau_{\mathbf{3}}$ have a matrix representation in $\mathbb{C}(2)$, namely $\boldsymbol{\tau}_{\mathbf{1},}$ $\boldsymbol{\tau}_{\mathbf{2},}$, $\boldsymbol{\tau}_{\mathbf{3}}$, a set of Pauli matrices, we have the correspondences

$$\mathcal{K} \leftrightarrow \begin{pmatrix} \lambda^s_{\{-+\}} & -\rho^a_{\{+-\}} \\ \rho^a_{\{+-\}} & \lambda^s_{\{-+\}} \end{pmatrix}, \quad \mathcal{M} \leftrightarrow \begin{pmatrix} \lambda^s_{\{+-\}} & -\rho^a_{\{-+\}} \\ \rho^a_{\{-+\}} & \lambda^s_{\{+-\}} \end{pmatrix} \tag{16.39}$$

We observe moreover that

$$\mathbf{K} = \mathcal{K}\frac{1}{2}(1+\tau_{\mathbf{3}}) \leftrightarrow \begin{pmatrix} \lambda^s_{\{-+\}} & 0 \\ \rho^a_{\{+-\}} & 0 \end{pmatrix}, \ \mathbf{M} = \frac{1}{2}\mathcal{K}\frac{1}{2}(1+\tau_{\mathbf{3}}) \leftrightarrow \begin{pmatrix} \lambda^s_{\{+-\}} & 0 \\ \rho^a_{\{-+\}} & 0 \end{pmatrix} \tag{16.40}$$

Then, from Eqs. (16.31) we can show that the $\mathcal{K}$ and $\mathcal{M}$ fields satisfy the following linear partial differential equations

$$\boldsymbol{\partial}\mathcal{K}\gamma_{21} - m\mathcal{K}\mathrm{i}\tau_{\mathbf{2}}\gamma_0 = 0, \tag{16.41}$$

$$\boldsymbol{\partial}\mathcal{M}\gamma_{21} + m\mathcal{M}\mathrm{i}\tau_{\mathbf{2}}\gamma_0 = 0. \tag{16.42}$$

Indeed, $\mathcal{K}\mathrm{i}\boldsymbol{\tau_2} = \mathrm{i}\boldsymbol{\tau_2}\mathcal{K} = \lambda^s_{\{-+\}} \otimes \mathrm{i}\tau_2 - \rho^a_{\{+-\}} \otimes 1$, $\mathcal{M}\mathrm{i}\boldsymbol{\tau_2} = \mathrm{i}\mathcal{M}\boldsymbol{\tau_2} = \lambda^s_{\{-+\}} \otimes \mathrm{i}\tau_2 - \rho^a_{\{-+\}} \otimes 1$ and we have the correspondences:

$$\mathcal{K} \leftrightarrow \begin{pmatrix} \lambda^s_{\{-+\}} & -\rho^a_{\{+-\}} \\ \rho^a_{\{+-\}} & \lambda^s_{\{-+\}} \end{pmatrix}, \quad \mathcal{M} \leftrightarrow \begin{pmatrix} \lambda^s_{\{+-\}} & -\rho^a_{\{-+\}} \\ \rho^a_{\{-+\}} & \lambda^s_{\{+-\}} \end{pmatrix},$$

$$\mathcal{K}\mathrm{i}\boldsymbol{\tau_2} \leftrightarrow \begin{pmatrix} \rho^a_{\{+-\}} & \lambda^s_{\{-+\}} \\ -\lambda^s_{\{-+\}} & \rho^a_{\{+-\}} \end{pmatrix}, \quad \mathcal{M}\mathrm{i}\boldsymbol{\tau_2} \leftrightarrow \begin{pmatrix} \rho^a_{\{-+\}} & \lambda^s_{\{+-\}} \\ -\lambda^s_{\{+-\}} & \rho^a_{\{-+\}} \end{pmatrix}. \tag{16.43}$$

Then, from Eqs. (16.41) and (16.42) we see that **K** and **M** satisfy the following linear partial differential equations

$$\boldsymbol{\partial}\mathbf{K}\gamma_{21} + \mathrm{i}m\boldsymbol{\tau_2}\mathbf{K}\gamma_0 = 0, \tag{16.44}$$

$$\boldsymbol{\partial}\mathbf{M}\gamma_{21} - \mathrm{i}m\boldsymbol{\tau_2}\mathbf{M}\gamma_0 = 0, \tag{16.45}$$

which, on taking the corresponding matrix representation gives the coupled equations for the pairs $(\lambda^s_{\{-+\}}, \rho^a_{\{+-\}})$ and $(\lambda^s_{\{+-\}}, \rho^a_{-+\}})$ appearing in Eq. (16.31).

Before proceeding we observe that the currents

$$\boldsymbol{J}_{\mathcal{K}} = \mathcal{K}\boldsymbol{\tau_1}\gamma_0\tilde{\mathcal{K}} \in \sec \bigwedge^1 T^*M \otimes \mathrm{spin}_{3,0} \hookrightarrow \sec \mathcal{C}\ell(M,\eta) \otimes \mathbb{R}^0_{1,3}, \tag{16.46}$$

$$\boldsymbol{J}_{\mathcal{M}} = \mathcal{M}\boldsymbol{\tau_1}\gamma_0\tilde{\mathcal{M}} \in \sec \bigwedge^1 T^*M \otimes \mathrm{spin}_{3,0} \hookrightarrow \sec \mathcal{C}\ell(M,\eta) \otimes \mathbb{R}^0_{1,3}, \tag{16.47}$$

are conserved, i.e.,

$$\boldsymbol{\partial} \lrcorner \boldsymbol{J}_{\mathcal{K}} = 0, \quad \boldsymbol{\partial} \lrcorner \boldsymbol{J}_{\mathcal{M}} = 0. \tag{16.48}$$

Indeed, let us show that $\boldsymbol{\partial} \lrcorner \boldsymbol{J}_{\mathcal{K}} = 0$. We have

$$\boldsymbol{\partial} \lrcorner \boldsymbol{J}_{\mathcal{K}} = \frac{1}{2}\left(\boldsymbol{\partial}\mathcal{K}\boldsymbol{\tau_1}\gamma_0\tilde{\mathcal{K}} + \mathcal{K}\boldsymbol{\tau_1}\gamma_0\tilde{\mathcal{K}}\overleftarrow{\boldsymbol{\partial}}\right) \tag{16.49}$$

From Eq. (16.41) we have

$$\boldsymbol{\partial}\mathcal{K} = \mathrm{i}m\mathcal{K}\boldsymbol{\tau_2}\gamma_{012}, \quad \tilde{\mathcal{K}}\overleftarrow{\boldsymbol{\partial}} = \partial_\mu\tilde{\mathcal{K}}\gamma^\mu = \mathrm{i}m\gamma_{012}\boldsymbol{\tau_2}\tilde{\mathcal{K}}. \tag{16.50}$$

Then,

$$\begin{aligned} \boldsymbol{\partial} \lrcorner \boldsymbol{J}_{\mathcal{K}} &= \frac{1}{2}(\mathrm{i}m\mathcal{K}\boldsymbol{\tau_2}\gamma_{012}\boldsymbol{\tau_1}\gamma_0\tilde{\mathcal{K}} + \mathrm{i}m\mathcal{K}\gamma_{12}\boldsymbol{\tau_1}\boldsymbol{\tau_2}\tilde{\mathcal{K}}) \\ &= \frac{\mathrm{i}m}{2}(\mathcal{K}(\boldsymbol{\tau_2}\boldsymbol{\tau_1} + \boldsymbol{\tau_1}\boldsymbol{\tau_2})\gamma_{12}\tilde{\mathcal{K}} = 0. \end{aligned}$$

The fields $\mathcal{K}$ and $\mathcal{M}$ are electrically neutral, but they can couple with an su(2) $\simeq$ $\text{spin}_{3,0} \subset \mathbb{R}_{3,0}$ valued potential

$$\mathcal{A} = A^{\mathbf{i}} \otimes \boldsymbol{\tau}_{\mathbf{i}} \in \sec \bigwedge\nolimits^1 T^*M \otimes \text{spin}_{3,0} \hookrightarrow \sec \mathcal{C}\ell(M,\eta) \otimes \mathbb{R}_{1,3}. \tag{16.51}$$

Indeed, we have taking into account that $\mathfrak{i} = \Gamma_{\mathbf{5}}$, $\tau_{\mathbf{i}} = \Gamma_{\mathbf{i}0}$ that the coupling is

$$\boldsymbol{\partial}\mathcal{K}\gamma_{21} - m\mathcal{K}\Gamma_5\Gamma_{20}\gamma_0 + q\Gamma_5\mathcal{A}\mathcal{K} = 0, \tag{16.52}$$

$$\boldsymbol{\partial}\mathcal{M}\gamma_{21} + m\mathcal{M}\Gamma_5\Gamma_{20}\gamma_0 + q\Gamma_5\mathcal{A}\mathcal{M} = 0. \tag{16.53}$$

Equations (16.52) and (16.53) are invariant under the following transformation of the fields and change of the basis of the $\text{spin}_{3,0} \subset \mathbb{R}_{1,3}^{00}$ algebra:

$$\mathcal{K} \mapsto \mathcal{K}' = e^{\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}\mathcal{K}, \quad \mathcal{M} \mapsto \mathcal{M}' = e^{\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}\mathcal{M},$$

$$\mathcal{A} \mapsto \mathcal{A}' = e^{\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}\mathcal{A}\, e^{-\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}, \quad \Gamma_{\mathbf{i}} \mapsto \Gamma'_{\mathbf{i}} = e^{\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}\Gamma_{\mathbf{i}}e^{-\Gamma_5 q\theta^{\mathbf{i}}\Gamma_{\mathbf{i}0}}. \tag{16.54}$$

With the above result we propose that elko spinor fields of the λ and ρ types,are the crucial ingredients permitting the existence of the $\mathcal{K}$ and $\mathcal{M}$ fields which do not carry electric charges but possess *magnetic*[8] like charges that couple to an $spin_{3,0} \subset \mathbb{R}_{1,3}^{00}$ valued potential $\mathcal{A}$.

16.5 Difference Between Elko and Majorana Spinor Fields

Here we recall that a Majorana field (also in class five in Lounesto classification[9] and supposedly describing a Majorana neutrino) differently from an elko spinor field is supposed in some textbooks to satisfy the Dirac equation (see, e.g., [23]), even if that equation cannot be derived from a Lagrangian (unless, as it is well known the components of Majorana fields for each $x \in M$ are Grassmann 'numbers'). The "proof" in [23] for the statement that a Majorana field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ satisfies the Dirac equation is as follows. That author writes that $\phi_r : M \to \mathbb{C}^2$ and $\phi_l : M \to \mathbb{C}^2$ belonging respectively to the carrier spaces of the representations $D^{0,1/2}$ and $D^{1/2,0}$

[8]The use of the term magnetic like charge here comes from the analogy to the possible coupling of Weyl fields describing massless magnetic monopoles with the electromagnetic potential $A \in \sec \bigwedge^1 T^*M$. See Chap. 13.

[9]We mention Dirac spinor fields are the real type fermion fields and that Majorana and Elko spinor fields are the imaginary type fermion fields according to Yang and Tiomno [34] classification of spinor fields according to their transformation laws under parity.

of $Sl(2,\mathbb{C})$ satisfy

$$\boldsymbol{\sigma}^{\mu} i\partial_{\mu}\boldsymbol{\phi}_{r} = m\boldsymbol{\phi}_{l}, \tag{16.55}$$

$$\breve{\boldsymbol{\sigma}}^{\mu} i\partial_{\mu}\boldsymbol{\phi}_{l} = m\boldsymbol{\phi}_{r}, \tag{16.56}$$

with $\boldsymbol{\sigma}^{\mu} = (\mathbf{1},\boldsymbol{\sigma}^{i})$ and $\breve{\boldsymbol{\sigma}}^{\mu} = (\mathbf{1},-\boldsymbol{\sigma}^{i})$ where $\boldsymbol{\sigma}^{i}(=\boldsymbol{\sigma}_{i})$ are the Pauli matrices. From this we can see that we can write:

$$i\begin{pmatrix} \mathbf{0} & \breve{\boldsymbol{\sigma}}^{\mu} \\ \boldsymbol{\sigma}^{\mu} & \mathbf{0} \end{pmatrix}\partial_{\mu}\begin{pmatrix} \boldsymbol{\phi}_{r} \\ \boldsymbol{\phi}_{l} \end{pmatrix} = m\begin{pmatrix} \boldsymbol{\phi}_{r} \\ \boldsymbol{\phi}_{l} \end{pmatrix}. \tag{16.57}$$

The set of matrices $\boldsymbol{\gamma}'^{\mu} := \begin{pmatrix} \mathbf{0} & \breve{\boldsymbol{\sigma}}^{\mu} \\ \boldsymbol{\sigma}^{\mu} & \mathbf{0} \end{pmatrix}$ is (as well known) a representation of Dirac matrices in Weyl representation,. It follows that $\boldsymbol{\psi}'$ satisfy the Dirac equation, i.e.,

$$i\boldsymbol{\gamma}'^{\mu}\partial_{\mu}\boldsymbol{\psi}' - m\boldsymbol{\psi}' = 0. \tag{16.58}$$

Have saying that, Maggiore [23] defines a Majorana field (in Weyl representation) by

$$\boldsymbol{\psi}'_{\mathbf{M}} = \begin{pmatrix} \boldsymbol{\phi}_{l} \\ \boldsymbol{\phi}_{r} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}_{l} \\ i\boldsymbol{\sigma}^{2}\boldsymbol{\phi}_{l}^{*} \end{pmatrix}, \tag{16.59}$$

and write

$$i\boldsymbol{\gamma}'^{\mu}\partial_{\mu}\boldsymbol{\psi}'_{\mathbf{M}} - m\boldsymbol{\psi}'_{\mathbf{M}} = 0, \tag{16.60}$$

concluding his “proof”.

Now, let us investigate more deeply that “proof”. First recall that writing

$$\boldsymbol{\phi}_{r}(x) = \boldsymbol{\phi}_{r}(\mathbf{p})e^{\mp ip_{\mu}x^{\mu}}, \quad \boldsymbol{\phi}_{l}(x) = \boldsymbol{\phi}_{l}(\mathbf{p})e^{\mp ip_{\mu}x^{\mu}}, \tag{16.61}$$

we have from Eqs. (16.55) and (16.56) that

$$(p_0 - \boldsymbol{\sigma}\cdot\mathbf{p})\boldsymbol{\phi}_{r}(\mathbf{p}) = \pm m\boldsymbol{\phi}_{l}(\mathbf{p}), \tag{16.62}$$

$$(p_0 + \boldsymbol{\sigma}\cdot\mathbf{p})\boldsymbol{\phi}_{l}(\mathbf{p}) = \pm m\boldsymbol{\phi}_{r}(\mathbf{p}). \tag{16.63}$$

However, if $\boldsymbol{\phi}_{l}(\mathbf{0})$ and $\boldsymbol{\phi}_{r}(\mathbf{0})$ are the zero momentum fields we have (with $\varkappa$ being the boost parameter, i.e., $\sinh\varkappa/2 = \sqrt{(\gamma-1)/2)}$ with $\gamma = 1/\sqrt{1-v^2}$ and $\mathbf{n}$ the direction of motion) by definition:

$$\begin{aligned}\boldsymbol{\phi}_{r}(\mathbf{p}) &:= e^{\frac{1}{2}\varkappa\cdot\boldsymbol{\sigma}}\boldsymbol{\phi}_{r}(\mathbf{0}) = (\cosh\varkappa/2 + \boldsymbol{\sigma}\cdot\mathbf{n}\sinh\varkappa/2)\boldsymbol{\phi}_{l}(\mathbf{0}) \\ &= \frac{p_0 + m + \boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p_0+m)]^{1/2}}\boldsymbol{\phi}_{r}(\mathbf{0}),\end{aligned} \tag{16.64}$$

$$\boldsymbol{\phi}_l(\mathbf{p}) := e^{-\frac{1}{2}\varkappa\cdot\boldsymbol{\sigma}}\boldsymbol{\phi}_l(\mathbf{0}) = (\cosh\varkappa/2 - \boldsymbol{\sigma}\cdot\mathbf{n}\sinh\varkappa/2)\boldsymbol{\phi}_l(\mathbf{0})$$
$$= \frac{p_0 + m - \boldsymbol{\sigma}\cdot\mathbf{p}}{[2m(p_0+m)]^{1/2}}\boldsymbol{\phi}_l(\mathbf{0}). \tag{16.65}$$

We can now verify that Eqs. (16.64) and (16.65) only imply Eqs. (16.62) and (16.63) if[10]

$$\boldsymbol{\phi}_l(\mathbf{0}) = \pm\boldsymbol{\phi}_r(\mathbf{0}). \tag{16.66}$$

But this condition cannot be satisfied by a Majorana field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ as defined by Maggiore [23] where $\boldsymbol{\phi}_r(\mathbf{0}) = i\boldsymbol{\sigma}^2\boldsymbol{\phi}_l^*(\mathbf{0})$. Indeed, writing $\boldsymbol{\phi}_l^t(\mathbf{0}) = (\nu, w)$ with $v, w \in \mathbb{C}$ we see that to have $\boldsymbol{\phi}_l(\mathbf{0}) = \pm\boldsymbol{\phi}_r(\mathbf{0})$ we need $\nu = \omega^*$ and $\omega = -\nu^*$, i.e., $\nu = \omega = 0$. We conclude that a Majorana field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ *cannot* satisfy the Dirac equation.

16.5.1 Some Majorana Fields Are Dual Helicities Objects

Before continuing we recall also that it is a well known fact (see, e.g.,[18]) that the Dirac Hamiltonian *commutes* with the operator $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$ given by Eq. (16.25). Thus any $\Psi : M \to \mathbb{C}^4$ satisfying Dirac equation which is an eigenspinor of the Dirac Hamiltonian may be constructed such that $\boldsymbol{\phi}_l$ and $\boldsymbol{\phi}_r$ have the same helicity. Since a Majorana spinor field $\boldsymbol{\psi}'_M : M \to \mathbb{C}^4$ as defined by Maggiore [23] does not satisfy Dirac equation we may suspect that it is not an eigenspinor of the of the operator $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$. And indeed this is the case, for we now show $\boldsymbol{\phi}_l(\mathbf{0})$ and $\boldsymbol{\phi}_r(\mathbf{0})$ in a Majorana field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ are not equal. Taking the momentum (without loss of generality) in the direction of the z-axis (of an inertial frame) and $\boldsymbol{\phi}_l^t(\mathbf{0}) = (1, 0)$ we have

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\boldsymbol{\phi}_l(\mathbf{0}) = -\boldsymbol{\phi}_l(\mathbf{0}), \quad \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}(i\boldsymbol{\sigma}^{\mathbf{2}}\boldsymbol{\phi}_l(\mathbf{0})) = -i\boldsymbol{\sigma}^2\boldsymbol{\phi}_l(\mathbf{0}), \tag{16.67}$$

and as the elko spinor fields they are also dual helicities objects.

Remark 16.9 Keep also in mind that as well known even if a Majorana field is described by a field [24] $\boldsymbol{\varphi} : M \to \mathbb{C}^2$ carrying the $D^{1/2,0}$ (or $D^{0,1/2}$) representation of $Sl(2, \mathbb{C})$ the value of the helicity obviously depends on the inertial reference frame where the measurement is done [7, 27] because the helicity is invariant only under

[10] That $\boldsymbol{\phi}_l(\mathbf{0}) = \pm\boldsymbol{\phi}_r(\mathbf{0})$ is a necessary condition for a spinor field $\boldsymbol{\psi} : M \to \mathbb{C}^4$ to satisfy Dirac equation can bee seem, e.g., from Eqs. (2.85) and (2.86) in Ryder's book [30]. However, Ryder misses the possible solution $\boldsymbol{\phi}_l(\mathbf{0}) = -\boldsymbol{\phi}_r(\mathbf{0})$. This has been pointed by Ahluwalia [1] in his review of Ryder's book.

those Lorentz transformations which did not alter the direction of $\mathbf{p}$ along which the angular momentum component is taken.

16.5.2 The Majorana Currents $J_{\mathbf{M}}$ and $J^{5}_{\mathbf{M}}$

We observe moreover that if a Majorana field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ should satisfy the Dirac equation then Eq. (16.60) should translate in the Clifford bundle formalism as

$$\boldsymbol{\partial}\psi_{\mathbf{M}}\gamma_{21} - m\psi_{\mathbf{M}}\gamma_0 = 0, \tag{16.68}$$

where $\psi_{\mathbf{M}} \in \sec \mathcal{C}\ell^0(M,\eta)$. Then, current $\boldsymbol{J}_{\mathbf{M}} = \psi_{\mathbf{M}}\gamma_0\tilde{\psi}_{\mathbf{M}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ is conserved as it is trivial to verify. Moreover, it is *lightlike* (since for a class five spinor field $\tilde{\psi}_{\mathbf{M}}\psi_{\mathbf{M}} = 0$ and thus $\boldsymbol{J}_{\mathbf{M}} \cdot \boldsymbol{J}_{\mathbf{M}} = \psi_{\mathbf{M}}\gamma_0(\tilde{\psi}_{\mathbf{M}}\psi_{\mathbf{M}})\gamma_0\tilde{\psi}_{\mathbf{M}} = 0$) but it is a non null covector field if the components of the spinor field $\boldsymbol{\psi}'_M : M \to \mathbb{C}^4$ have values in $\mathbb{C}^2$. Indeed writing $\boldsymbol{J}_{\mathbf{M}} = \psi_{\mathbf{M}}\gamma_0\tilde{\psi}_{\mathbf{M}} = J^{\mu}_{\mathbf{M}}\gamma_\mu$ we see immediately that

$$J^0_{\mathbf{M}} = \bar{\boldsymbol{\psi}}'_M \boldsymbol{\gamma}'^0 \boldsymbol{\psi}'_M \neq 0. \tag{16.69}$$

Also, the current

$$\boldsymbol{J}^5_{\mathbf{M}} := \psi_{\mathbf{M}}\gamma_3\tilde{\psi}_{\mathbf{M}} = (\bar{\boldsymbol{\psi}}'_M \, \boldsymbol{\gamma}'^5 \boldsymbol{\gamma}'^\mu \boldsymbol{\psi}'_M)\gamma_\mu \tag{16.70}$$

is non null as it is easy to verify, and is also lightlike. If the Majorana spinor field was to satisfy the Dirac equation the current $\boldsymbol{J}^5_{\mathbf{M}}$ would be also conserved, i.e., $\boldsymbol{\partial} \lrcorner \boldsymbol{J}^5_{\mathbf{M}} = 0$. In that case we would have a subtle question to answer: how can a massive particle have associated to it currents $\boldsymbol{J}_{\mathbf{M}}$ and $\boldsymbol{J}^5_{\mathbf{M}}$ that are lightlike? What is the meaning of these currents?

Remark 16.10 Of course, the answer to the above question from the point of view of a first quantized theory is that a Majorana field cannot carry any electric or magnetic charge, i.e., the physical currents $e_{\mathbf{M}}\boldsymbol{J}_{\mathbf{M}}$ and $q_{\mathbf{M}}\boldsymbol{J}^5_{\mathbf{M}}$ are null because $e_{\mathbf{M}} = q_{\mathbf{M}} = 0$.

16.5.3 Making of Majorana Fields That Satisfy the Dirac Equation

Is it possible to construct a Majorana spinor field that satisfies the Dirac equation?

There are two possibilities of answering yes for the above question.

First Possibility: As, e.g., in [20] and [9] we consider ab initio a Majorana field as a quantum field and which is *not* a dual helicity object. Indeed, define a Majorana *quantum field* as $\boldsymbol{\psi}'_M$ as an operator valued field satisfying Majorana condition $\boldsymbol{\psi}'^{c}_{M} := -\boldsymbol{\gamma}'^{2}\bar{\boldsymbol{\psi}}'^{\star}_{M} = \boldsymbol{\psi}'_{M}$. This condition can be satisfied if we define[11]

$$\boldsymbol{\psi}'_{M}(x) := \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \sum_{s} (u(\mathbf{p},s)a(\mathbf{p},s)e^{ip_\mu x^\mu} + v(\mathbf{p},s)a^{\dagger}(\mathbf{p},s)e^{-ip_\mu x^\mu}), \tag{16.71}$$

with

$$\{a(\mathbf{p},s), a^{\dagger}(\mathbf{p}',s')\} = \delta_{ss'}\delta(\mathbf{p}-\mathbf{p}'), \quad \{a(\mathbf{p},s), a(\mathbf{p}',s')\} = 0 \tag{16.72}$$

and where the zero momentum spinors $u(\mathbf{0},s)$ and $v(\mathbf{0},s)$ are

$$u(\mathbf{0},1/2) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u(\mathbf{0},-1/2) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v(\mathbf{0},1/2) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \quad v(\mathbf{0},-1/2) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \tag{16.73}$$

satisfying

$$\boldsymbol{\gamma}'_0 u(\mathbf{0},s) = u(\mathbf{0},s), \quad \boldsymbol{\gamma}'_0 v(\mathbf{0},s) = -v(\mathbf{0},s) \tag{16.74}$$

Indeed, we can verify by explicit calculation that

$$u(\mathbf{p},s) = \frac{m + p_\mu \boldsymbol{\gamma}'^{\mu}\boldsymbol{\gamma}'^{0}}{\sqrt{2p_0(p_0+m)}} u(\mathbf{0},s), \quad u(\mathbf{p},s) = \frac{m - p_\mu \boldsymbol{\gamma}'^{\mu}\boldsymbol{\gamma}'^{0}}{\sqrt{2p_0(p_0+m)}} v(\mathbf{0},s) \tag{16.75}$$

and taking into account Eq. (16.74) we see that

$$(p_\mu \boldsymbol{\gamma}'^{\mu} - m)u(\mathbf{p},s) = 0, \quad (p_\mu \boldsymbol{\gamma}'^{\mu} + m)v(\mathbf{p},s) = 0. \tag{16.76}$$

With this results we can immediately verify that the quantum Majorana field $\boldsymbol{\psi}'_{M}(x)$ satisfy the Dirac equation,

$$(i\boldsymbol{\gamma}'^{\mu}\partial_\mu - m)\boldsymbol{\psi}'_{M}(x) = 0. \tag{16.77}$$

[11]Here $\boldsymbol{\psi}'^{\star}_{M} := \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \sum_s (u^*(\mathbf{p},s)a^{\dagger}(\mathbf{p},s)e^{-ip_\mu x^\mu} + v^*(\mathbf{p},s)a(\mathbf{p},s)e^{ip_\mu x^\mu})$.

For that Majorana field that is not a dual helicity object we can construct in the canonical way the causal propagator that is nothing more than the standard Feynman propagator for the Dirac equation (see, e.g.,[20]).

Remark 16.11 In a second quantized theory the currents $\boldsymbol{J}_{\mathbf{M}}$ and $\boldsymbol{J}_{\mathbf{M}}^{5}$ are given by the normal product of the field operators and in this case the current: $\bar{\boldsymbol{\psi}}'_M \boldsymbol{\gamma}'^0 \boldsymbol{\psi}'_M$: as well known is null, but : $\bar{\boldsymbol{\psi}}'_M\, \boldsymbol{\gamma}'^5 \boldsymbol{\gamma}'^\mu \boldsymbol{\psi}'_M$: is non null. For a proof of these statements, see, e.g., [20] where in particular a consistent quantum field theory for Majorana fields satisfying the Dirac equation is presented, showing in particular that the causal propagator for that field is the standard Feynman propagator of Dirac theory.

Second Possibility: In several treatises, e.g., [28, 33] even at the "classical level" it is supposed that any Fermi field must be a Grassmann valued spinor field, i.e., an object where $\phi_L^t = (\nu\ \omega)$ and $v, \omega : M \to \mathcal{G}$, with $\mathcal{G}$ a Grassmann algebra [8, 15], i.e., $v(x)$ and $w(x)$ are Grassmann elements of a Grassmann algebra for all $x \in M$.

In this case it is possible to show that the Majorana field defined, e.g., in [28] by

$$\boldsymbol{\Psi}'^{\mathbf{M}} = \begin{pmatrix} \phi_L \\ -\boldsymbol{\sigma}_2 \phi_L^* \end{pmatrix} \tag{16.78}$$

does satisfy the Dirac equation.

To prove that statement write $\phi_L^t = (\nu\ \omega)$ where for any $x \in M$, $v(x)$ and $\omega(x)$ take values in a *Grassmann algebra.* If $\boldsymbol{\Psi}'^{\mathbf{M}}$ does satisfy Dirac equation we must have for $\boldsymbol{\Psi}'^{\mathbf{M}}(\mathbf{p})$ at $\mathbf{p} = \mathbf{0}$ that

$$m \begin{pmatrix} -\boldsymbol{\sigma}_2 \phi_L^* \\ \phi_L \end{pmatrix} - m \begin{pmatrix} \phi_L \\ -\boldsymbol{\sigma}_2 \phi_L^* \end{pmatrix} = 0. \tag{16.79}$$

Then we need simultaneously to satisfy the equations

$$\phi_L = -\boldsymbol{\sigma}_2 \phi_L^* \text{ and } \phi_L = \boldsymbol{\sigma}_2 \phi_L^*, \tag{16.80}$$

which at first sight seems to be incompatible, but are not. Indeed, from $\phi_L = -\boldsymbol{\sigma}_2 \phi_L^*$ we obtain

$$\nu = i\omega^* \text{ and } \omega = -i\nu^* \tag{16.81}$$

and from $\phi_L = \boldsymbol{\sigma}_2 \phi_L^*$ we obtain

$$\nu = -i\omega^* \text{ and } \omega = i\nu^*. \tag{16.82}$$

So, if we *understand* the symbol $*$ as denoting the involution defined by Berizin[12] Eq. (16.81) is consistent if we take

$$\nu = \nu^* \text{ and } \omega = \omega^*. \tag{16.83}$$

But since for any $c \in \mathbb{C}$ and $\varphi \in \mathcal{G}$ it is $(c\varphi)^* = c^*\varphi^*$ the equation $\nu = i\omega^*$ implies

$$v^* = (i\omega^*)^* = -i\omega^{**} = -i\omega \tag{16.84}$$

and since $\nu = \nu^*$ and $\omega = \omega^*$ Eq. (16.80) implies $\nu = -i\omega^*$. Thus, surprisingly as it may be at first sight Eq. (16.81) is compatible with Eq. (16.82).

Claim 16.12 *We may then claim that a Majorana field whose components take values in a Grassmann algebra satisfies the Dirac equation. This is consistent with the fact that Dirac equation under these conditions may be derived from a Lagrangian [28, 33]. We can also verify that for such a Majorana field the current* $J_{\mathbf{M}} = 0$.

Remark 16.13 In resume, from the algebraic point of view there is no difference between elko spinor fields $\boldsymbol{\lambda}, \boldsymbol{\rho} : M \to \mathbb{C}^4$and Majorana spinor fields $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$. However have in mind that the Majorana field defined in [28] (Eq. (16.78) above) looks like an elko spinor field, but, of course, is not the same object, since the components of an elko spinor fields are for any $x \in M$ complex numbers but the components of $\boldsymbol{\Psi}'^{\mathbf{M}}$ in [28] take values in a Grassmann algebra.

Of course, if we recall that in building a quantum field theory for elkos make automatically the components of elko spinor fields objects taking values in a Grassmann algebra we cannot see any good reason for the building of a theory like in [2]. Instead we think that elko spinor fields are worth objects of study because they permit the construction of the $\mathcal{K}$ and $\mathcal{M}$ fields introduced above which may describe possible of "magnetic like" particles.

16.6 The Causal Propagator for the $\mathcal{K}$ and $\mathcal{M}$ Fields

We now calculate the causal propagator $\mathcal{S}_F(x-x')$ for, e.g., the $\check{\mathcal{K}} \in \sec \mathcal{C}\ell^0(M, \boldsymbol{\eta}) \otimes \mathbb{R}^0_{1,3}$ field. Recall from Remark 16.3 that the $\check{\mathcal{K}}$ field must satisfy

$$\check{\boldsymbol{\partial}}\check{\mathcal{K}}\boldsymbol{e}_{21} - m\check{\mathcal{K}}\Gamma_5\Gamma_{20}\boldsymbol{e}_0 + \Gamma_5 q\check{\mathcal{A}}\check{\mathcal{K}} = 0. \tag{16.85}$$

If $\check{\mathcal{K}}_i(x)$ is a solution of the homogeneous equation

$$\check{\boldsymbol{\partial}}\check{\mathcal{K}}_i\boldsymbol{e}_{21} - m\check{\mathcal{K}}_i\Gamma_5\Gamma_{20}\boldsymbol{e}_0 = 0,$$

[12] See pages 66 and of [8].

we can rewrite Eq. (16.85) as an integral equation

$$\check{\mathcal{K}}(x) = \check{\mathcal{K}}_i(x) + q\int d^4y\mathcal{S}_F(x,y)\check{\mathcal{A}}(y)\check{\mathcal{K}}(y)\Gamma_5\Gamma_{20}\Gamma_5. \tag{16.86}$$

Putting Eq. (16.86) in Eq. (16.85) we see that $\mathcal{S}_F(x,y)$ must satisfy for an arbitrary $\check{\mathcal{P}} \in \sec \mathcal{C}\ell(\mathcal{M},\boldsymbol{\eta}) \otimes \mathbb{R}^0_{1,3}$

$$\check{\boldsymbol{\partial}}\mathcal{S}_F(x-y)\check{\mathcal{P}}(y)\boldsymbol{e}_{21} - m\mathcal{S}_F(x-y)\check{\mathcal{P}}(y)\boldsymbol{e}_0 = \delta^4(x-y)\check{\mathcal{P}}(y) \tag{16.87}$$

whose solution as it is easy to verify is [19]

$$\mathcal{S}_F(x-y)\check{\mathcal{P}}(y) = \frac{1}{(2\pi)^4}\int d^4p\frac{\check{p}\check{\mathcal{P}}(y) + m\check{\mathcal{P}}(y)\boldsymbol{e}_0}{\check{p}^2 - m^2}e^{-ip_\mu(x^\mu - y^\mu)}. \tag{16.88}$$

For the causal Feynman propagator we get with $E = p_0 = \sqrt{\mathbf{p}^2 + m^2}$

$$\begin{aligned}\mathcal{S}_F(x-y)\check{\mathcal{K}}(x) = &\frac{-1}{2(2\pi)^3}\theta(t-t')\int d^3p\frac{(\check{p}\check{\mathcal{K}}(y) + m\check{\mathcal{K}}(y)\boldsymbol{e}_0)\boldsymbol{e}_{21}}{E}e^{-ip_\mu(x^\mu - y^\mu)} \\ &+ \frac{1}{2(2\pi)^3}\theta(t-t')\int d^3p\frac{(\check{p}\check{\mathcal{K}}(y) - m\check{\mathcal{K}}(y)\boldsymbol{e}_0)\boldsymbol{e}_{21}}{E}e^{-ip_\mu(x^\mu - y^\mu)}.\end{aligned} \tag{16.89}$$

For a scattering problem defining $\check{\mathcal{K}}_s = \check{\mathcal{K}} - \check{\mathcal{K}}_i$ with $\check{\mathcal{K}}_i$ an asymptotic in-state we get when $t \to \infty$

$$\check{\mathcal{K}}_s(x) = q\int d^4y\int d^3p\frac{(\check{p}\check{\mathcal{A}}(y)\check{\mathcal{K}}(y) + m\check{\mathcal{A}}(y)\check{\mathcal{K}}(y)\boldsymbol{e}_0)\boldsymbol{e}_{21}}{2E}e^{-ip_\mu(x^\mu - y^\mu)}. \tag{16.90}$$

This permits to define a set of final states $\check{\mathcal{K}}_f$ given by

$$\check{\mathcal{K}}_f(x) = q\int d^4y\frac{(\check{p}_f\check{\mathcal{A}}(y)\check{\mathcal{K}}(y) + m\check{\mathcal{A}}(y)\check{\mathcal{K}}(y)\boldsymbol{e}_0)\boldsymbol{e}_{21}}{2E_f}e^{-ip_\mu(x^\mu - y^\mu)} \tag{16.91}$$

which are plane waves solutions to the free field Dirac-Hestenes equation with momentum $\check{p}_f$. Equipped with the $\check{\mathcal{K}}_i(x)$ and $\check{\mathcal{K}}_f(x)$ we can proceed to calculate the scattering matrix elements, Feynman rules and all that (see details if necessary in [19]).

For the covariant $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ fields the causal propagator is the standard Dirac propagator $S_F(x-x')$. Indeed, it can be used to solve, e.g., the *csfopde*

$$i\gamma^\mu\partial_\mu\boldsymbol{\lambda}^s_{\{-+\}}\gamma_{21} + m\boldsymbol{\rho}^a_{\{+-\}} = 0, \quad i\gamma^\mu\partial_\mu\boldsymbol{\rho}^a_{\{+-\}} - m\boldsymbol{\lambda}^s_{\{-+\}} = 0 \tag{16.92}$$

once appropriate initial conditions are given. To see this it is only necessary to rewrite the formulas in Eq. (16.92) as

$$i\gamma^{\mu}\partial_{\mu}\boldsymbol{\lambda}^{s}_{\{-+\}} - m\boldsymbol{\lambda}^{s}_{\{-+\}} = -m(\boldsymbol{\lambda}^{s}_{\{-+\}} + \boldsymbol{\rho}^{a}_{\{+-\}}) = \chi, \tag{16.93}$$

$$i\gamma^{\mu}\partial_{\mu}\boldsymbol{\rho}^{a}_{\{+-\}} - m\boldsymbol{\rho}^{a}_{\{+-\}} = m(\boldsymbol{\lambda}^{s}_{\{-+\}} + \boldsymbol{\rho}^{a}_{\{+-\}}) = \varkappa. \tag{16.94}$$

Equations (16.93) and (16.94) have solutions

$$\lambda^{s}_{\{-+\}}(x) = \int d^4y S_F(x-y)\chi, \tag{16.95}$$

$$\boldsymbol{\rho}^{a}_{\{+-\}}(x) = \int d^4y S_F(x-y)\varkappa \tag{16.96}$$

once we recall that

$$(i\gamma^{\mu}\partial_{\mu} - m)S_F(x-y) = \delta^4(x-y). \tag{16.97}$$

16.7 A Note on the Anticommutator of Mass Dimension 1 Elkos According to [2]

According to the theory of elko spinor fields as originally developed in [2] (see also [1, 3–6] the evaluation of the anticommutator of an elko spinor field with its canonical momentum gives

$$\{\Lambda(\mathbf{x},t), \Pi(\mathbf{x},t\} = i\delta(\mathbf{x}-\mathbf{x}')\mathbb{I} \; + i\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}), \tag{16.98}$$

with

$$\mathcal{G}(\mathbf{p}) := \gamma^5\gamma^{\mu}n_{\mu}(\mathbf{p}), \tag{16.99}$$

where the spacelike $\mathbf{p}$-dependent field $n = n_{\mu}(\mathbf{p})\mathbf{e}_{\mu}$ is

$$\begin{aligned} &(n_0(\mathbf{p}), n_1(\mathbf{p}), n_2(\mathbf{p}), n_3(\mathbf{p})) := (0, \mathbf{n}(\mathbf{p})), \\ &\mathbf{p} = (p\cos\theta, p\sin\theta\cos\varphi, p\sin\theta\sin\varphi) \end{aligned} \tag{16.100}$$

$$\begin{aligned} \mathbf{n}(\mathbf{p}) :&= \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right) = (-\sin\varphi, \cos\varphi, 0) \\ &= \left(-\tau(1+\tau^2)^{-1/2}, \tau(1+\tau^2)^{-1/2}, 0\right), \quad \tau = p_y/p_x. \end{aligned} \tag{16.101}$$

Putting $\boldsymbol{\Delta} = \mathbf{x} - \mathbf{x}'$ it is[13]

$$\hat{\mathcal{G}}(\boldsymbol{\Delta}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \mathcal{G}(\mathbf{p})$$
$$= -\gamma^5\gamma^1 P(\boldsymbol{\Delta}) + \gamma^5\gamma^2 Q(\boldsymbol{\Delta}) \tag{16.102}$$

with

$$P(\boldsymbol{\Delta}) = -\frac{i}{2\pi}\delta(\Delta_z)\frac{\Delta_y}{(\Delta_x^2+\Delta_y^2)^{\frac{3}{2}}}, \quad Q(\boldsymbol{\Delta}) = \frac{i}{2\pi}\delta(\Delta_z)\frac{\Delta_x}{(\Delta_x^2+\Delta_y^2)^{\frac{3}{2}}}. \tag{16.103}$$

Remark 16.14 In [25] the integral in Eq. (16.102) has been evaluated for the case when $\boldsymbol{\Delta}$ lies in the direction of *one* of the spatial axes $\mathbf{e}_i = \partial/\partial x^i$ of an arbitrary inertial reference frame $\mathbf{e}_0 = \partial/\partial x^0$ (where x^μ, $\mu = 0,1,2,3$) are coordinates in Einstein-Lorentz-Poincaré gauge naturally adapted to $\mathbf{e}_0$).We note that the evaluation of each one of the integrals in [25] is correct, but they *do not* express the values of the Fourier transform $\hat{\mathcal{G}}(\mathbf{x}-\mathbf{x}')$ for the particular values of $\boldsymbol{\Delta}$ used in the calculations of those integrals. It is not licit to fix a priori two of the components of $\boldsymbol{\Delta}$ as being null to calculate the integral $(2\pi)^{-3}\int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p})$ for this procedure excludes the singular behavior in the sense of distributions of the Fourier integral. So, it is wrong the statement in [25] that elko theory as constructed originally in [2] is local. However, let us show now that this nonlocality is really odd.

16.7.1 Plane of Nonlocality and Breakdown of Lorentz Invariance

When $\Delta_z \neq 0$, $\hat{\mathcal{G}}(\mathbf{x}-\mathbf{x}')$ is null the anticommutator is *local* and thus there exists in the elko theory as constructed in [2, 5] an infinity number of "locality directions". On the other hand $\hat{\mathcal{G}}(\mathbf{x}-\mathbf{x}')$ is a distribution with support in $\Delta_z = 0$. So,the directions $\boldsymbol{\Delta} = (\Delta_x, \Delta_y, 0)$ are nonlocal in each *arbitrary* inertial reference frame $\mathbf{e}_0$ chosen to evaluate $\hat{\mathcal{G}}(\mathbf{x}-\mathbf{x}')$. Recall that given an inertial (coordinate) reference frame $\mathbf{e}_0 = \partial/\partial x^0$ in Minkowski spacetime there exists [10] and infinity of triples of vector fields $\{\mathbf{e}_1^{(k)} = \partial/\partial x^1_{(k)}, \mathbf{e}_1^{(k)} = \partial/\partial x^2_{(k)}, \mathbf{e}_1^{(k)} = \partial/\partial x^3_{(k)}\}$ (with $x^0, x^1_{(k)}, x^2_{(k)}, x^3_{(k)}$), coordinates in Einstein-Lorentz-Poincaré gauge naturally adapted to $\mathbf{e}_0$ differing by a spatial rotation) which constitutes a global section of the frame bundle. So, the labels x, y and z directions in inertial reference frame $\mathbf{e}_0$ are no more than a mere convention and thus without *any* physical significance. This means that the theory as

[13]This result has been also found by Ahluwalia and Grumiller. We find the above result without knowing their calculations. See erratum to [25] at *Phys. Rev. D* **88**, 129901 (2013).

constructed in [2] breaks in each inertial reference frame rotational invariance by a subjective choice of an observer and since in different inertial references frames there are different (x, y) planes the theory breaks also Lorentz invariance. This physically unacceptable feature of the theory of elko spinor fields as constructed originally in [2] was eventually the main reason that lead us to present investigation.

16.8 A New Representation of the Parity Operator Acting on Dirac Spinor Fields

Let as before $\{\mathring{\boldsymbol{e}}_\mu = \frac{\partial}{\partial \mathrm{x}^\mu}\}$ and $\{\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu}\}$ be two arbitrary orthonormal frames for TM and let $\Sigma_0 = \{\Gamma^\mu = d\mathrm{x}^\mu\}$ and $\Sigma = \{\gamma^\mu = dx^\mu\}$ be the respective dual frames. Of course, $\mathring{\boldsymbol{e}}_0$ and $\boldsymbol{e}_0$ are inertial reference frames and we suppose now that $\boldsymbol{e}_0$ is moving relative to $\mathring{\boldsymbol{e}}_0$ with 3-velocity $\mathbf{v} = (\mathbf{v}^1, v^2, v^3)$, i.e.,

$$\boldsymbol{e}_0 = \frac{1}{\sqrt{1-v^2}}\mathring{\boldsymbol{e}}_0 - \sum_{i=1}^{3}\frac{v^i}{\sqrt{1-v^2}}\mathring{\boldsymbol{e}}_i \tag{16.104}$$

Let Ξ_{u_0} and Ξ_u be the spinorial frames associated with Σ_0 and Σ. Consider a Dirac particle at rest in the inertial frame $\mathring{\boldsymbol{e}}_0$ (take as a fiducial frame). The triplet $(\psi_0, \Sigma_0, \Xi_0)$ is the representative of the wave function of our particle in (Σ_0, Ξ_0) and of course, its representative in (Σ, Ξ) is (ψ, Σ, Ξ). Now,

$$\psi = u\psi_0 \tag{16.105}$$

where u describes in the spinor space the boost sending Γ^μ to γ^μ, i.e.,$\gamma^\mu = u\,\Gamma^\mu u^{-1} = \Lambda^\mu_\nu \Gamma^\nu$. Now, the representative of the parity operator in (Σ_0, Ξ_0) is $\mathcal{P}_{u_0}$ and in (Σ, Ξ) is $\mathcal{P}_u$; We have according to our dictionary [Eq. (3.68)] that

$$\mathcal{P}_u\psi = \gamma^0\psi\gamma^0, \quad \mathcal{P}_{u_0}\psi_0 = \Gamma^0\psi_0\Gamma^0, \tag{16.106}$$

or

$$\mathcal{P}_u\boldsymbol{\Psi} = \gamma^0\mathcal{R}\boldsymbol{\Psi}, \quad \mathcal{P}_{u_0}\boldsymbol{\Psi}_0 = \Gamma^0\mathcal{R}\boldsymbol{\Psi}_0, \tag{16.107}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_0$ are Dirac *ideal* real spinor fields. From Chap. 3 we have

$$\boldsymbol{\Psi} = \psi\frac{1}{2}(1+\gamma^0), \quad \boldsymbol{\Psi}_0 = \psi_0\frac{1}{2}(1+\Gamma^0), \tag{16.108}$$

and if the momentum of our particle is the covector field $\boldsymbol{p} = \mathring{p}_\mu \Gamma^\mu = p_\mu \gamma^\mu$ with $(\mathring{p}_0, \mathring{p}_1, \mathring{p}_2, \mathring{p}_3) := (m, \mathbf{0})$ and $(p_0, p_1, p\mathring{p}_2, p_3) := (E, \mathbf{p})$ (and of course $p_\mu = \Lambda_\mu^\nu \mathring{p}_\mu = \Lambda_\mu^0 \mathring{p}_0$) $\mathcal{R}$ an the operator such that if $\psi = \phi(\mathbf{p}) e^{i\boldsymbol{px}}$ then

$$\mathcal{R}\psi = \phi(\mathbf{p}) e^{-i\boldsymbol{p}_\mu \boldsymbol{x}^\mu} = \phi(\mathbf{p}) e^{-i(\boldsymbol{p}_0 \boldsymbol{x}^0 - p_i x^i)}. \tag{16.109}$$

Also $u\mathcal{R} = \mathcal{R}u$ and clearly $\mathcal{R}\psi_0 = \psi_0$. Now,

$$u\mathcal{P}_{u_0} u^{-1} u \boldsymbol{\Psi}_0 = u\Gamma^0 \mathcal{R} \boldsymbol{\Psi}_0 = u\Gamma^0 u^{-1} \mathcal{R} u \boldsymbol{\Psi}_0 = \gamma^0 \mathcal{R} \boldsymbol{\Psi}, \tag{16.110}$$

from where it follows that

$$\mathcal{P}_u = u\mathcal{P}_{u_0} u^{-1}. \tag{16.111}$$

Now we rewrite $\mathcal{P}_u \boldsymbol{\Psi} = \gamma^0 \mathcal{R} \boldsymbol{\Psi}$ as

$$\begin{aligned} \mathcal{P}_u \boldsymbol{\Psi} &= \frac{\mathring{p}_0}{m} u \Gamma^0 \mathcal{R} \boldsymbol{\Psi}_0 = \frac{\mathring{p}_0}{m} u \Gamma^0 u^{-1} u \boldsymbol{\Psi}_0, \\ &= \frac{\mathring{p}_0}{m} \Lambda_\mu^0 \gamma^\mu \boldsymbol{\Psi} = \frac{1}{m} p_\mu \gamma^\mu \boldsymbol{\Psi}. \end{aligned} \tag{16.112}$$

We conclude that the parity operator in an arbitrary orthonormal and spin frames (Σ, Ξ) acting on a Dirac ideal spinor field $\boldsymbol{\psi}$ is

$$\mathcal{P} = \mathcal{P}_u = \frac{1}{m} p_\mu \gamma^\mu. \tag{16.113}$$

Of course, when applied to covariant spinor fields $\boldsymbol{\psi} : M \to \mathbb{C}^4$ the operator $\mathcal{P}$ is represented by

$$\boldsymbol{P} = \frac{1}{m} p_\mu \boldsymbol{\gamma}^\mu. \tag{16.114}$$

A derivation of this result using covariant spinor fields (and which can be easy generalized for arbitrary higher spin fields) has been obtained in [32].

16.9 Conclusions

In Chap. 13 (see also [29]) it was shown that the massless Dirac-Hestenes equation decouples in a pair of operator Weyl spinor fields, each one carrying opposite magnetic like charges that couple to the electromagnetic potential $A \in \sec \bigwedge^1 T^*M$

in a non standard way.[14] Here we proposed that the fields $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ serves the purpose of building the fields $\mathcal{K}, \mathcal{M} \in \sec \mathcal{C}\ell(M,\eta) \otimes \mathbb{R}^0_{1,3}$. These fields are electrically neutral but carry *magnetic* like charges which permit them to couple to a $\text{spin}_{3,0}$ valued potential $\mathcal{A} \in \sec \bigwedge^1 T^*M \otimes \text{spin}_{3,0}$. If the field $\mathcal{A}$ is of short range the particles described by the $\mathcal{K}$ and $\mathcal{M}$ may interact forming something analogous to dark matter, in the sense that they may form a condensate of spin zero particles with zero total magnetic like charges that do not couple with the electromagnetic field and are thus invisible.

We obtained also the causal propagators for the $\mathcal{K}$ and $\mathcal{M}$ fields, which can be used to calculate scattering matrix elements, Feynman rules, etc.[15]

Before closing this last chapter of our book we observe yet that elko spinor fields already appeared in the literature before the publication of [2]. A history about these objects may be found in [16, 17]. In those papers a Lagrangian equivalent to Eq. (3.79) written for the covariant spinor fields $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ is given. However, the author of those papers did not comment that since the basic *csfopde* satisfied by the elko spinor fields is by construction the ones given in Eq. (3.74) and as a consequence these fields, contrary to the claim of[2], must have mass dimension $3/2$ and not 1.

We recalled also that as claimed in [2] an elko spinor field (of class five in Lounesto classification) does *not* satisfy the Dirac equation. According to some claims in the literature (see, e.g., [23]) a Majorana spinor field $\boldsymbol{\psi}'_{\mathbf{M}} : M \to \mathbb{C}^4$ and which is a dual helicity object (that also belongs to class five in Lounesto classification) *does* satisfy the Dirac equation. However we showed that this claim is equivocated. At "classical level" a Majorana spinor field can satisfy Dirac equation only if its components for any $x \in M$ take values in a Grassmann algebra.

It is important to emphasize in order to avoid misunderstandings that the theory presented in this paper is an alternative theory to the one originally built in [2] and developed in a series of interesting and challenging papers (see references). It differs drastically from that theory. The main differences are that the equations satisfied by our elko spinor fields of mass dimension $3/2$ [see Eq. (16.31)] and their solutions are trivially Lorentz invariant. In the theory in [2] the elko spinor fields are of mass dimension 1 and that theory breaks Lorentz invariance as shown in Sect. 16.7. Also our theory gives a prediction of a new type of particle that is electrically and magnetically neutral but has a magnetic like charge which can couple with an $\text{spin}_{3,0}$ valued gauge field. The other theory (for the best of our understanding) does not fix the nature of the field that intermediates the interaction of the particles described by their elko spinor fields of mass dimension 1.

[14] In [21] it is proposed that the massless Dirac equation describe (massless) neutrinos which carry pair of opposite magnetic charges.

[15] At least, we can say that now we have all the ingredients to formulate a quantum field theory for the $\mathcal{K}$ and $\mathcal{M}$ objects if one wish to do so.

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Appendix A
Principal Bundles, Vector Bundles and Connections

Abstract The appendix defines fiber bundles, principal bundles and their associate vector bundles, recall the definitions of frame bundles, the orthonormal frame bundle, jet bundles, product bundles and the Whitney sums of bundles. Next, equivalent definitions of connections in principal bundles and in their associate vector bundles are presented and it is shown how these concepts are related to the concept of a covariant derivative in the base manifold of the bundle. Also, the concept of exterior covariant derivatives (crucial for the formulation of gauge theories) and the meaning of a curvature and torsion of a linear connection in a manifold is recalled. The concept of covariant derivative in vector bundles is also analyzed in details in a way which, in particular, is necessary for the presentation of the theory in Chap. 12. Propositions are in general presented without proofs, which can be found, e.g., in Choquet-Bruhat et al. (Analysis, Manifolds and Physics. North-Holland, Amsterdam, 1982), Frankel (The Geometry of Physics. Cambridge University Press, Cambridge, 1997), Kobayashi and Nomizu (Foundations of Differential Geometry. Interscience Publishers, New York, 1963), Naber (Topology, Geometry and Gauge Fields. Interactions. Applied Mathematical Sciences. Springer, New York, 2000), Nash and Sen (Topology and Geometry for Physicists. Academic, London, 1983), Nicolescu (Notes on Seiberg-Witten Theory. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000), Osborn (Vector Bundles. Academic, New York, 1982), and Palais (The Geometrization of Physics. Lecture Notes from a Course at the National Tsing Hua University, Hsinchu, 1981).

A.1 Fiber Bundles

Definition A.1 A fiber bundle over M with Lie group G will be denoted by $\mathcal{E} = (E, M, \pi, G, F)$. E is a topological space called the total space of the bundle, $\pi : E \to M$ is a continuous surjective map, called the canonical projection and F is the typical fiber. The following conditions must be satisfied:

(a) $\pi^{-1}(x)$, the fiber over x, is homeomorphic to F.

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(b) Let $\{U_i,\ i \in \mathfrak{I}\}$, where $\mathfrak{I}$ is an index set, be a covering of M, such that:

- Locally a fiber bundle E is trivial, i.e., it is diffeomorphic to a product bundle, i.e., $\boldsymbol{\pi}^{-1}(U_i) \simeq U_i \times F$ for all $i \in \mathfrak{I}$.
- The diffeomorphisms $\Phi_i : \boldsymbol{\pi}^{-1}(U_i) \to U_i \times F$ have the form

$$\Phi_i(p) = (\pi(p), \phi_{i,x}(p)), \tag{A.1}$$

$$\phi_i|_{\pi^{-1}(x)} \equiv \phi_{i,x} : \boldsymbol{\pi}^{-1}(x) \to F \text{ is onto.} \tag{A.2}$$

The collection $\{(U_i, \Phi_i)\}$, $i \in \mathfrak{I}$, are said to be a family of local trivializations for E.

- The group G acts on the typical fiber. Let $x \in U_i \cap U_j$. Then,

$$\phi_{j,x} \circ \phi_{i,x}^{-1} : F \to F \tag{A.3}$$

must coincide with the action of an element of G for all $x \in U_i \cap U_j$ and $i, j \in \mathfrak{I}$.

- We call transition functions of the bundle the continuous induced mappings

$$g_{ij} : U_i \cap U_j \to G, \text{ where } g_{ij}(x) = \phi_{i,x} \circ \phi_{j,x}^{-1}. \tag{A.4}$$

For consistence of the theory the transition functions must satisfy the cocycle condition (Fig. A.1)

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x). \tag{A.5}$$

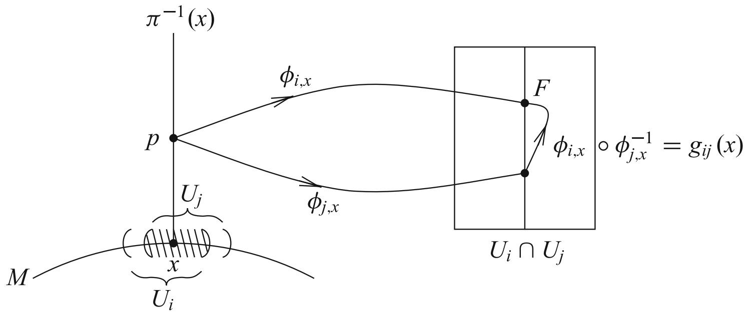

Fig. A.1 Transition functions on a fiber bundle

Transition functions on a fiber bundle

Definition A.2 $(P, M, \boldsymbol{\pi}, G, F \equiv G) \equiv (P, M, \boldsymbol{\pi}, G)$ is called a principal fiber bundle (*PFB*) if all conditions in Definition A.1 are fulfilled and moreover, if there is a right action of G on elements $p \in P$, such that:

(a) the mapping (defining the right action) $P \times G \ni (p, g) \mapsto pg \in P$ is continuous.
(b) given $g, g' \in G$ and $\forall p \in P$, $(pg)g' = p(gg')$.
(c) $\forall x \in M, \boldsymbol{\pi}^{-1}(x)$ is invariant under the action of G, i.e., each element of $p \in \boldsymbol{\pi}^{-1}(x)$ is mapped into $pg \in \boldsymbol{\pi}^{-1}(x)$, i.e., it is mapped into an element of the same fiber.
(d) G acts free and transitively on each fiber $\boldsymbol{\pi}^{-1}(x)$, which means that all elements within $\boldsymbol{\pi}^{-1}(x)$ are obtained by the action of all the elements of G on any given element of the fiber $\boldsymbol{\pi}^{-1}(x)$. This condition is, of course necessary for the identification of the typical fiber with G.

Definition A.3 A bundle $(E, M, \boldsymbol{\pi}_1, G = Gl(m, \mathbb{F}), F = \boldsymbol{V})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (respectively the real and complex fields), $Gl(m, \mathbb{F})$ is the linear group, and $\boldsymbol{V}$ is an m-dimensional vector space over $\mathbb{F}$ is called a vector bundle.

Definition A.4 A vector bundle $(E, M, \boldsymbol{\pi}, G, F)$ denoted $E = P \times_\rho F$ is said to be associated to a *PFB* bundle $(P, M, \boldsymbol{\pi}, G)$ by the linear representation ρ of G in $F = \mathbf{V}$ (a linear space of finite dimension over an appropriate field, which is called the *carrier space* of the representation) if its transition functions are the images under ρ of the corresponding transition functions of the *PFB* $(P, M, \boldsymbol{\pi}, G)$. This means the following: consider the following local trivializations of P and E respectively

$$\Phi_i : \boldsymbol{\pi}^{-1}(U_i) \to U_i \times G, \tag{A.6}$$

$$\Xi_i : \boldsymbol{\pi}_1^{-1}(U_i) \to U_i \times F, \tag{A.7}$$

$$\Xi_i(q) = (\boldsymbol{\pi}_1(q), \chi_i(q)) = (x, \chi_i(q)), \tag{A.8}$$

$$\chi_i|_{\boldsymbol{\pi}_1^{-1}(x)} \equiv \chi_{i,x} : \boldsymbol{\pi}_1^{-1}(x) \to F, \tag{A.9}$$

where $\boldsymbol{\pi}_1 : P \times_\rho F \to M$ is the projection of the bundle associated to $(P, M, \boldsymbol{\pi}, G)$. Then, for all $x \in U_i \cap U_j$, $i, j \in \mathfrak{I}$, we have

$$\chi_{j,x} \circ \chi_{i,x}^{-1} = \rho(\phi_{j,x} \circ \phi_{i,x}^{-1}). \tag{A.10}$$

In addition, the fibers $\boldsymbol{\pi}^{-1}(x)$ are vector spaces isomorphic to the representation space V.

Definition A.5 Let $\mathcal{E} = (E, M, \boldsymbol{\pi}, G, F)$ be a fiber bundle and $U \subset M$ an open set. A local (cross) section of the fiber bundle $(E, M, \boldsymbol{\pi}, G, F)$ on U is a mapping

$$s : U \to E \quad \text{such that} \quad \pi \circ s = \mathrm{Id}_U. \tag{A.11}$$

A *global section* s is one for which $U = M$. Not all bundles admit global sections (see below).

Notation A.6 Let $U \subset M$. We will say that $\rho \in \sec \mathcal{E}|_U$ (or $\rho \in \sec E|_U$) if there exists a local section $s|_\rho : U \to E, x \mapsto (x, \rho(x))$, with $\rho : U \to F$.

Remark A.7 There is a relation between sections and local trivializations for principal bundles. Indeed, each local section s, (on $U_i \subset M$) for a principal bundle $(P, M, \boldsymbol{\pi}, G)$ determines a local trivialization $\Phi_i : \boldsymbol{\pi}^{-1}(U) \to U \times G$, of P by setting

$$\Phi_i^{-1}(x, g) = s(x)g = pg = R_g p. \tag{A.12}$$

Conversely, Φ_i determines s since

$$s(x) = \Phi_i^{-1}(x, e). \tag{A.13}$$

Proposition A.8 *A principal bundle is trivial, if and only if, it has a global cross section.*

Proposition A.9 *A vector bundle is trivial, if and only if, its associated principal bundle is trivial.*

Proposition A.10 *Any fiber bundle* $(E, M, \boldsymbol{\pi}, G, F)$ *such that* M *is contractible to a point is trivial.*

Proposition A.11 *Any fiber bundle* $(E, M, \boldsymbol{\pi}, G, F)$ *such that* M *is paracompact and the fiber* F *is a vector space admits a local section.*

Remark A.12 Take notice that any vector bundle admits a global section, namely the zero section. However, it admits a global nonvanishing global section if its Euler class is zero.

Definition A.13 The structure group G of a fiber bundle $(E, M, \boldsymbol{\pi}, G, F)$ is said to be reducible to G' if the bundle admits an equivalent structure defined with a subgroup G' of the structure group G. More precisely, this means that the fiber bundle admits a family of local trivializations such that the transition functions takes values in G', i.e., $g_{ij} : U_i \cap U_j \to G'$.

A.1.1 Frame Bundle

The tangent bundle TM to a differentiable n-dimensional manifold M is an associated bundle to a principal bundle called the frame bundle $F(M) = \bigcup_{x \in M} F_x M$, where $F_x M$ is the set of frames at $x \in M$. Let $\{x^i\}$ be coordinates associated to a local chart (U_i, φ_i) of the maximal atlas of M. Then, $T_x M$ has a natural basis $\{ \frac{\partial}{\partial x^i}\big|_x \}$ on $U_i \subset M$.

Definition A.14 A frame at T_xM is a set $\mathbf{\Sigma}_x = \{e_1|_x, \ldots, e_n|_x\}$ of linearly independent vectors such that

$$e_i|_x = F_i^j \left.\frac{\partial}{\partial x^j}\right|_x, \tag{A.14}$$

and where the matrix (F_i^j) with entries $A_i^j \in \mathbb{R}$, belongs to the real general linear group in n dimensions $Gl(n, \mathbb{R})$. We write $(F_i^j) \in Gl(n, \mathbb{R})$.

A local trivialization $\phi_i : \pi^{-1}(U_i) \to U_i \times Gl(n, \mathbb{R})$ is defined by

$$\phi_i(f) = (x, \mathbf{\Sigma}_x), \pi(f) = x. \tag{A.15}$$

The action of $a = (a_i^j) \in Gl(n, \mathbb{R})$ on a frame $f \in F(U)$ is given by $(f, a) \to fa$, where the new frame $fa \in F(U)$ is defined by $\phi_i(fa) = (x, \mathbf{\Sigma}'_x)$, $\pi(f) = x$, and

$$\begin{aligned} \mathbf{\Sigma}'_x &= \{e'_1\big|_x, \ldots, e'_n\big|_x\}, \\ e'_i\big|_x &= e_j\big|_x a_i^j. \end{aligned} \tag{A.16}$$

Conversely, given frames $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}'_x$ there exists $a = (a_i^j) \in Gl(n, \mathbb{R})$ such that Eq. (A.16) is satisfied, which means that $Gl(n, \mathbb{R})$ acts on $F(M)$ actively.

Let $\{x^i\}$ and $\{\bar{x}^i\}$be the coordinates associated to the local chart (U_i, φ_i) (U'_i, φ'_i) and of the maximal atlas of M. If $x \in U_i \cap U_j$ we have

$$\begin{aligned} e_i|_x = F_i^j \left.\frac{\partial}{\partial x^j}\right|_x &= \bar{F}_i^j \left.\frac{\partial}{\partial \bar{x}^j}\right|_x, \\ (F_i^j), (\bar{F}_i^j) &\in Gl(n, \mathbb{R}). \end{aligned} \tag{A.17}$$

Since $F_i^j = \bar{F}_k^j \left.\left(\frac{\partial x^k}{\partial \bar{x}^i}\right)\right|_x$ we have that the transition functions are

$$g_i^k(x) = \left.\left(\frac{\partial x^k}{\partial \bar{x}^i}\right)\right|_x \in Gl(n, \mathbb{R}). \tag{A.18}$$

Remark A.15 Given $U \subset M$ we shall denote by $\mathbf{\Sigma} \in \sec F(U)$ a section of $F(U) \subset F(M)$. This means that given a local trivialization $\phi : \pi^{-1}(U) \to U \times Gl(n, \mathbb{R})$, $\phi(\mathbf{\Sigma}) = (x, \mathbf{\Sigma}_x)$, $\pi(f) = x$. Sometimes, we also use the sloppy notation $\{e_i\} \in \sec F(U)$ or even $\{e_i\} \in \sec F(M)$ when the context is clear.

A.1.2 Orthonormal Frame Bundle

Suppose that the manifold M is equipped with a metric field $\boldsymbol{g} \in \sec T_2^0 M$ of signature (p,q), $p+q=n$. Then, we can introduce *orthonormal* frames in each $T_x U$. In this case we denote an orthonormal frame by $\boldsymbol{\Sigma}_x = \{\boldsymbol{e}_1|_x, \ldots, \boldsymbol{e}_n|_x\}$ and

$$\boldsymbol{e}_i|_x = h_i^j \left.\frac{\partial}{\partial x^j}\right|_x, \tag{A.19}$$

$$\boldsymbol{g}(\boldsymbol{e}_i|_x, \boldsymbol{e}_j\big|_x)\big|_x = \operatorname{diag}(1, 1, \ldots 1, -1, \ldots, -1) \tag{A.20}$$

with $(h_i^j) \in \mathrm{O}_{p,q}$, the real orthogonal group in n dimensions. In this case we say that the frame bundle has been reduced to the *orthonormal frame bundle*, which will be denoted by $\mathbf{P}_{\mathrm{O}n}(M)$. A section $\Sigma \in \sec \mathbf{P}_{\mathrm{O}n}(U)$ is called a *vierbein*.

Remark A.16 The principal bundle of orthonormal frames $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ over a Lorentzian manifold modelling spacetime and its covering bundle called spin bundle $\mathbf{P}_{\mathrm{Spin}_{1,3}^e}(M)$ discussed in Chap. 6 play an important role in this book. Also, vector bundles associated these bundles are very important. Associated to $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ we have the tensor bundle, the exterior bundle and the Clifford bundle. Associated to $\mathbf{P}_{\mathrm{Spin}_{1,3}^e}(M)$ we have several spinor bundles, in particular the spin-Clifford bundle, whose sections are the Dirac-Hestenes spinor fields. All those bundles and their relationship are studied in Chap. 7.

Remark A.17 In complete analogy to the construction of orthonormal frame bundle we may define an orthonormal coframe bundle that may be denoted by $P_{\mathrm{O}n}(M)$. Since to each given frame $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{O}n}(M)$ there is a natural coframe field $\Sigma \in \sec P_{\mathrm{O}n}(M)$, the one where the covectors are the duals of the vectors of the frame. It follows that $P_{\mathrm{O}n}(M) \simeq \mathbf{P}_{\mathrm{O}n}(M)$. In particular $P_{\mathrm{SO}_{1,3}^e}(M) \simeq \mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$.

A.1.3 Jet Bundles

Let $\mathcal{E} = (E, M, \boldsymbol{\pi}, G, \boldsymbol{F})$ be a fiber bundle. Let $x \in M$ and let $U_x \subset M$ be a neighborhood of x. Let $\sec \mathcal{E}|_{U_X}$ be the set of all local sections of $\mathcal{E}$ defined in U_x. Let $r \in \mathbb{Z}^+$, $r \geq 1$. Next define in $\sec \mathcal{E}|_{U_X}$ and equivalence relation $\sim_x^r$ such that if $s|_\kappa, s|_\lambda : U_x \to E$ we say that

$$s|_\kappa \sim_x^r s|_\lambda \tag{A.21}$$

if for any smooth function $f : E \to \mathbb{R}$ and any smooth curve $\sigma : \mathbb{R} \to M$, $t \mapsto \sigma(t)$, with $\sigma(0) = x$ we have that the mappings $f \circ \kappa \circ \sigma$ and $f \circ \lambda \circ \sigma$ have the same r-order Taylor expansion at $t = 0$.

If we introduce local coordinates for $\mathcal{E}|_{U_X}$ we immediately see that the equivalence relation $\sim_x^r$ reduces to the requirement that the local expressions of κ and λ have the same r-order Taylor expansion at x. We denote by $j_x^r\kappa$ the equivalence class identified by the representative $\kappa \in \sec\, \mathcal{E}|_{U_X}$.

Definition A.18 Let $\boldsymbol{j}_x^r(\mathcal{E})$ be the quotient space $\sec \mathcal{E}|_{U_X} / \sim_x^r$. We call the the r-jet bundle of $\mathcal{E}$ the disjoint union of all $\boldsymbol{j}_x^r(\mathcal{E})$, i.e., the bundle $\boldsymbol{J}^r(\mathcal{E}) = \bigcup_{x\in M} \boldsymbol{j}_x^r(\mathcal{E})$. We moreover define the maps π^r and π_0^r such that

$$\pi^r : \boldsymbol{J}^r(\mathcal{E}) \to M, \quad \pi^r(j_x^r\kappa) = x,$$
$$\pi_0^r : (\boldsymbol{J}^r(\mathcal{E})) \to E, \quad \pi_0^r((j_x^r\kappa)) = s|_\kappa\,.$$

More details on jet bundles, a natural setup for rigorous formulation of field theories may be found, e.g., in [2].

A.2 Product Bundles and Whitney Sum

Given two vector bundles $(E, M, \boldsymbol{\pi}, G, \mathbf{V})$ and $(E', M', \boldsymbol{\pi}', G', \mathbf{V}')$ we have

Definition A.19 The product bundle $E \times E'$ is a fiber bundle whose basis space is $M \times M'$, the typical fiber is $\mathbf{V} \oplus \mathbf{V}'$, the structural group of $E \times E'$ acts separately as G and G' in each one of the components of $\mathbf{V} \oplus \mathbf{V}'$ and the projection $\boldsymbol{\pi} \times \boldsymbol{\pi}'$ is such that $E \times E' \overset{\boldsymbol{\pi}\times\boldsymbol{\pi}'}{\to} M \times M'$.

Definition A.20 Let $(E, M, \boldsymbol{\pi}, G, \mathbf{V})$ and $(E', M, \boldsymbol{\pi}', G', \mathbf{V}')$ be vector bundles over the same basis space. The Whitney sum bundle $E \oplus E'$ is the pullback of $E \times E'$ by $h : M \to M \times M$, $h(p) = (p, p)$.

Definition A.21 Let $(E, M, \boldsymbol{\pi}, G, \mathbf{V})$ and $(E', M, \boldsymbol{\pi}', G', \mathbf{V}')$ be vector bundles over the same basis space. The tensor product bundle $E \otimes E'$ is the bundle obtained from E and E' by assigning the tensor product of fibers $\boldsymbol{\pi}_x^{-1} \otimes \boldsymbol{\pi}_x'^{-1}$ for all $x \in M$.

Remark A.22 With the above definitions we can easily show that given three vector bundles, say, E, E', E'' we have

$$E \oplus (E' \otimes E'') = (E \otimes E') \oplus (E \otimes E''). \tag{A.22}$$

A.3 Connections

A.3.1 Equivalent Definitions of a Connection in Principal Bundles

To define the concept of a *connection* on a PFB $(P, M, \boldsymbol{\pi}, G)$, we recall that since $\dim(M) = m$, if $\dim(G) = n$, then $\dim(P) = n + m$. Obviously, for all $x \in M$, $\boldsymbol{\pi}^{-1}(x)$ is an n-dimensional submanifold of P diffeomorphic to the structure group G and $\boldsymbol{\pi}$ is a submersion, $\boldsymbol{\pi}^{-1}(x)$ is a closed submanifold of P for all $x \in M$.

The tangent space T_pP, $p \in \boldsymbol{\pi}^{-1}(x)$, is an $(n + m)$ -dimensional vector space and the tangent space $V_pP \equiv T_p(\boldsymbol{\pi}^{-1}(x))$ to the fiber over x at the same point $p \in \boldsymbol{\pi}^{-1}(x)$ is an n-dimensional linear subspace of T_pP called the *vertical subspace* of T_pP.[1]

Now, roughly speaking a connection on P is a rule that makes possible a *correspondence* between any two fibers along a curve $\sigma : \mathbb{R} \supseteq I \to M, t \mapsto \sigma(t)$. If p_0 belongs to the fiber over the point $\sigma(t_0) \in \sigma$, we say that p_0 is parallel translated along σ by means of this *correspondence*.

Definition A.23 A horizontal lift of σ is a curve $\hat{\sigma} : \mathbb{R} \supseteq I \to P$ (described by the parallel transport of p).

It is intuitive that such a transport takes place in P along directions specified by vectors in T_pP, which do not lie within the vertical space V_pP. Since the tangent vectors to the paths of the basic manifold passing through a given $x \in M$ span the entire tangent space T_xM, the corresponding vectors $Y_p \in T_pP$ (in whose direction parallel transport can generally take place in P) span a n-dimensional linear subspace of T_pP called the *horizontal space* of T_pP and denoted by H_pP. Now, the mathematical concept of a connection can be presented. This is done through three equivalent definitions given below which encode rigorously the intuitive discussion given above. We have,

Definition A.24 A connection on a *PFB* $(P, M, \boldsymbol{\pi}, G)$ is an assignment to each $p \in P$ of a subspace $H_pP \subset T_pP$, called the horizontal subspace for that connection, such that H_pP depends smoothly on p and the following conditions hold:

(i) $\boldsymbol{\pi}_* : H_pP \to T_xM$, $x = \boldsymbol{\pi}(p)$, is an isomorphism.
(ii) H_pP depends smoothly on p.
(iii) $(R_g)_* H_pP = H_{pg}P, \forall g \in G,\ \forall p \in P$.

Here we denote by $\boldsymbol{\pi}_*$ the *differential* of the mapping $\boldsymbol{\pi}$ and by $(R_g)_*$ the differential of the mapping $R_g : P \to P$ (the right action) defined by $R_g(p) = pg$.

---

[1]Here we may be tempted to realize that as it is possible to construct the vertical space for all $p \in P$ then we can define a horizontal space as the complement of this space in respect to T_pP. Unfortunately this is not so, because we need a smoothly association of a horizontal space in every point. This is possible only by means of a connection.

Since $x = \pi(\hat{\sigma}(t))$ for any curve in P such that $\hat{\sigma}(t) \in \pi^{-1}(x)$ and $\hat{\sigma}(0) = p_0$, we conclude that π_* maps all vertical vectors in the zero vector in T_xM, i.e., $\pi_*(V_pP) = 0$ and we have,[2]

$$T_pP = H_pP \oplus V_pP. \tag{A.23}$$

Then every $Y_p \in T_pP$ can be written as

$$\mathbf{Y}_p = \mathbf{Y}_p^h + \mathbf{Y}_p^v, \qquad \mathbf{Y}_p^h \in H_pP, \qquad \mathbf{Y}_p^v \in V_pP. \tag{A.24}$$

Therefore, given a vector field $\boldsymbol{Y}$ over M it is possible to lift it to a horizontal vector field over P, i.e., $\pi_*(Y_p) = \pi_*(Y_p^h) = \boldsymbol{Y}_x \in T_xM$ for all $p \in P$ with $\pi(p) = x$. In this case, we call Y_p^h the *horizontal lift* of $\boldsymbol{Y}_x$. We say moreover that Y is a horizontal vector field over P if $Y^h = Y$.

Definition A.25 A *connection* on a *PFB* (P, M, π, G) is a mapping $\Gamma_p : T_xM \to T_pP$, such that $\forall p \in P$ and $x = \pi(p)$ the following conditions hold:

(i) Γ_p is linear.
(ii) $\pi_* \circ \Gamma_p = Id_{T_xM}$.
(iii) the mapping $p \mapsto \Gamma_p$ is differentiable.
(iv) $\Gamma_{R_gp} = (R_g)_*\Gamma_p$, for all $g \in G$.

We need also the concept of parallel transport. It is given by,

Definition A.26 Let $\sigma : \mathbb{R} \supset I \to M,\ t \mapsto \sigma(t)$ with $x_0 = \sigma(0) \in M$, be a curve in M and let $p_0 \in P$ such that $\pi(p_0) = x_0$. The parallel transport of p_0 along σ is given by the curve $\hat{\sigma} : \mathbb{R} \supset I \to P, t \mapsto \hat{\sigma}(t)$ defined by

$$\frac{d}{dt}\hat{\sigma}(t) = \Gamma_p(\frac{d}{dt}\sigma(t)), \tag{A.25}$$

with $p_0 = \hat{\sigma}(0)$ and $\hat{\sigma}(t) = p_{\|t}$, $\pi(p_{\|t}) = x$.

In order to present yet a *third* definition of a connection we need to know more about the nature of the vertical space V_pP. For this, let $\mathfrak{Y} \in T_eG = \mathfrak{G}$ be an element of the Lie algebra $\mathfrak{G}$ of G. The vector $\mathfrak{Y}$ is the tangent to the curve produced by the exponential map

$$\mathfrak{Y} = \left.\frac{d}{dt}\left(\exp(t\mathfrak{Y})\right)\right|_{t=0}. \tag{A.26}$$

Then, for every $p \in P$ we can attach to each $\mathfrak{Y} \in T_eG = \mathfrak{G}$ a unique element $Y_p^v \in V_pP$ as follows: let $f : (-\varepsilon, \varepsilon) \to P,\ t \mapsto p \exp t\mathfrak{Y}$ be a curve on P. Observe

[2] We also write $TP = HP \oplus VP$.

that it is obtained by right translation and then $\pi(p) = \pi(p \exp t\mathfrak{Y}) = x$ and so the curve lies in $\pi^{-1}(x)$, the fiber over $x \in M$. Next let $\mathfrak{F} : P \to \mathbb{R}$ be a smooth function. Then we define

$$\mathbf{Y}_p^v \mathfrak{F}(p) = \frac{d}{dt}\mathfrak{F} \circ f(t)\bigg|_{t=0} = \frac{d}{dt}\mathfrak{F}\,(p\exp(t\mathfrak{Y}))\bigg|_{t=0}. \tag{A.27}$$

By this construction we attach to each $\mathfrak{Y} \in T_eG = \mathfrak{G}$ a unique vector field over P, called the fundamental field corresponding to this element. We then have the canonical isomorphism

$$\mathbf{Y}_p^v \leftrightarrow \mathfrak{Y}, \quad \mathbf{Y}_p^v \in V_pP, \quad \mathfrak{Y} \in T_eG = \mathfrak{G} \tag{A.28}$$

from which we get

$$V_pP \simeq \mathfrak{G}. \tag{A.29}$$

Definition A.27 A connection on a *PFB* $(P, M, \boldsymbol{\pi}, G)$ is a 1-form field $\boldsymbol{\omega}$ on P with values in the Lie algebra $\mathfrak{G} = T_eG$ such that $\forall p \in P$ we have,

(i) $\boldsymbol{\omega}_p(Y_p^v) = \mathfrak{Y}$ and $Y_p^v \leftrightarrow \mathfrak{Y}$, where $Y_p^v \in V_pP$ and $\mathfrak{Y} \in T_eG = \mathfrak{G}$.
(ii) $\boldsymbol{\omega}_p$ depends smoothly on p.
(iii) $\boldsymbol{\omega}_p[(R_g)_* Y_p] = (Ad_{g^{-1}}\boldsymbol{\omega}_p)(Y_p)$, where $Ad\boldsymbol{\omega}_p = g^{-1}\boldsymbol{\omega}_p g$.

It follows that if $\{\mathcal{G}_a\}$ is a basis of $\mathfrak{G}$ and $\{\theta^i\}$ is a basis for T^*P then

$$\boldsymbol{\omega}_p = \omega_p^a \otimes \mathcal{G}_a = \omega^{a\cdot}_{\cdot i}(p)\theta_p^i \otimes \mathcal{G}_a, \tag{A.30}$$

where ω^a are 1-forms on P.

Then the horizontal spaces can be defined by

$$H_pP = \ker(\boldsymbol{\omega}_p), \tag{A.31}$$

which shows the equivalence between the definitions.

A.3.2 The Connection on the Base Manifold

Definition A.28 Let $U \subset M$ and

$$s : U \to \boldsymbol{\pi}^{-1}(U) \subset P, \qquad \boldsymbol{\pi} \circ s = Id_U, \tag{A.32}$$

be a local section of the *PFB* $(P, M, \boldsymbol{\pi}, G)$.

Definition A.29 Let $\boldsymbol{\omega}$ be a connection on P. The 1-form $s^*\boldsymbol{\omega}$ (the pullback of $\boldsymbol{\omega}$ under s) given by

$$(s^*\boldsymbol{\omega})_x(Y_x) = \boldsymbol{\omega}_{s(x)}(s_* Y_x),\ Y_x \in T_x U,\ s_* Y_x \in T_p P, \quad p = s(x), \tag{A.33}$$

is called the *local gauge potential*.

It is quite clear that $s^*\boldsymbol{\omega} \in \sec T^*U \otimes \mathfrak{G}$. This object differs from the *gauge field* used by physicists by numerical constants (with units). Conversely we have the following

Proposition A.30 *Given $\omega \in \sec T^*U \otimes \mathfrak{G}$ and a differentiable section of $\boldsymbol{\pi}^{-1}(U) \subset P$, $U \subset M$, there exists one and only one connection $\boldsymbol{\omega}$ on $\boldsymbol{\pi}^{-1}(U)$ such that $s^*\boldsymbol{\omega} = \omega$.*

Consider now

$$\omega \in T^*U \otimes \mathfrak{G}, \quad \omega = (\Phi^{-1}(x,e))^*\boldsymbol{\omega} = s^*\boldsymbol{\omega}, \quad s(x) = \Phi^{-1}(x,e),$$

$$\omega' \in T^*U' \otimes \mathfrak{G}, \quad \omega' = (\Phi'^{-1}(x,e))^*\boldsymbol{\omega} = s'^*\boldsymbol{\omega}, \quad s'(x) = \Phi'^{-1}(x,e). \tag{A.34}$$

Then we can write, for each $p \in P$ ($\boldsymbol{\pi}(p) = x$), parameterized by the local trivializations Φ and Φ' respectively as (x,g) and (x,g') with $x \in U \cap U'$, that

$$\boldsymbol{\omega}_p = g^{-1}dg + g^{-1}\omega_x g = g'^{-1}dg' + g'^{-1}\omega'_x g'. \tag{A.35}$$

Now, if

$$g' = hg, \tag{A.36}$$

we immediately get from Eq. (A.35) that

$$\bar{\boldsymbol{\omega}}'_x = hdh^{-1} + h\bar{\boldsymbol{\omega}}_x h^{-1}, \tag{A.37}$$

which can be called the *transformation law* for the gauge fields.

A.4 Exterior Covariant Derivatives

Let $\bigwedge^k(P,\mathfrak{G}) = \bigwedge^k T^*P \otimes \mathfrak{G}, 0 \leq k \leq n$, be the set of all k-form fields over P with values in the Lie algebra $\mathfrak{G}$ of the gauge group G (and, of course, the connection $\boldsymbol{\omega} \in \sec \bigwedge^1(P,\mathfrak{G})$).

Definition A.31 For each $\boldsymbol{\varphi} \in \sec \bigwedge^k (P, \mathfrak{G})$ we define the so called *horizontal form* $\boldsymbol{\varphi}^h \in \sec \bigwedge^k (P, \mathfrak{G})$ by

$$\boldsymbol{\varphi}_p^h(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) = \boldsymbol{\varphi}(\mathbf{X}_1^h, \mathbf{X}_2^h, \ldots, \mathbf{X}_k^h), \tag{A.38}$$

where $\mathbf{X}_i \in T_pP$, $i = 1, 2, .., k$.

Notice that $\boldsymbol{\varphi}_p^h(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) = 0$ if one (or more) of the $\mathbf{X}_i \in T_pP$ are vertical.

Definition A.32 $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ (where $\mathbf{V}$ is a vector space) is said to be horizontal if $\boldsymbol{\varphi}_p(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) = 0$, implies that at least one of the $\mathbf{X}_i \in T_pP$, $i = 1, 2, \ldots, n$ is vertical.

Definition A.33 $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ is said to be of type $(\rho, \mathbf{V})$ if $\forall g \in G$ we have

$$R_g^* \boldsymbol{\varphi} = \rho(g^{-1}) \boldsymbol{\varphi}. \tag{A.39}$$

Definition A.34 Let $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ be horizontal. Then, $\boldsymbol{\varphi}$ is said to be tensorial of type $(\rho, \mathbf{V})$.

Definition A.35 The exterior covariant derivative of $\boldsymbol{\varphi} \in \sec \bigwedge^k (P, \mathfrak{G})$ in relation to the connection $\boldsymbol{\omega}$ is

$$D^{\omega} \boldsymbol{\varphi} = (d\boldsymbol{\varphi})^h \in \sec \bigwedge^k (P, \mathfrak{G}), \tag{A.40}$$

where $D^{\omega} \boldsymbol{\varphi}_p(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k, \mathbf{X}_{k+1}) = d\boldsymbol{\varphi}_p(\mathbf{X}_1^h, \mathbf{X}_2^h, \ldots, \mathbf{X}_k^h, \mathbf{X}_{k+1}^h)$. Notice that $d\boldsymbol{\varphi} = d\boldsymbol{\varphi}^a \otimes \mathcal{G}_a$ where $\boldsymbol{\varphi}^a \in \sec \bigwedge^k T^*P$, $a = 1, 2, \ldots, n$.

Definition A.36 The commutator of $\boldsymbol{\varphi} \in \sec \bigwedge^i (P, \mathfrak{G})$ and $\boldsymbol{\psi} \in \sec \bigwedge^j (P, \mathfrak{G})$, $0 \leq i, j \leq n$, denoted by $[\boldsymbol{\varphi}, \boldsymbol{\psi}] \in \sec \bigwedge^{i+j} (P, \mathfrak{G})$ such that if $\mathbf{X}_1, \ldots, \mathbf{X}_{i+j} \in \sec TP$, then

$$\begin{aligned} &[\boldsymbol{\varphi}, \boldsymbol{\psi}](\mathbf{X}_1, \ldots, \mathbf{X}_{i+j}) \\ &\quad = \frac{1}{i!j!} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} [\boldsymbol{\varphi}(\mathbf{X}_{\iota(1)}, \ldots, \mathbf{X}_{\iota(i)}), \boldsymbol{\psi}(\mathbf{X}_{\iota(i+1)}, \ldots, \mathbf{X}_{\iota(i+j)})], \end{aligned} \tag{A.41}$$

where $\mathcal{S}_n$ is the permutation group of n elements and $(-1)^{\sigma} = \pm 1$ is the sign of the permutation. The brackets [,] in the second member of Eq. (A.41) are the Lie brackets in $\mathfrak{G}$.

Writing

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\psi} = \boldsymbol{\psi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\varphi}^a \in \sec \bigwedge^i (T^*P), \quad \boldsymbol{\psi}^a \in \sec \bigwedge^j (T^*P), \tag{A.42}$$

we have

$$\begin{aligned}[\boldsymbol{\varphi}, \boldsymbol{\psi}] &= \boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b \otimes [\mathcal{G}_a, \mathcal{G}_b] \\ &= f_{\cdot ab}^{c\cdot\cdot}(\boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b) \otimes \mathcal{G}_c\end{aligned} \tag{A.43}$$

where $f_{\cdot ab}^{c\cdot\cdot}$ are the structure constants of the Lie algebra.

Remark A.37 In this section we are using (in order to get formulas as the ones printed in main textbooks) the exterior product as given in Remark 2.24 of Chap. 2 (there denoted $\dot{\wedge}$). We hope this will cause no misunderstanding.

With Eq. (A.43) we can prove *easily* the following important properties involving commutators:

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = (-1)^{1+ij}[\boldsymbol{\psi}, \boldsymbol{\varphi}], \tag{A.44}$$

$$(-1)^{ik}[[\boldsymbol{\varphi}, \boldsymbol{\psi}], \boldsymbol{\tau}] + (-1)^{ji}[[\boldsymbol{\psi}, \boldsymbol{\tau}], \boldsymbol{\varphi}] + (-1)^{kj}[[\boldsymbol{\tau}, \boldsymbol{\varphi}], \boldsymbol{\psi}] = 0, \tag{A.45}$$

$$d[\boldsymbol{\varphi}, \boldsymbol{\psi}] = [d\boldsymbol{\varphi}, \boldsymbol{\psi}] + (-1)^i[\boldsymbol{\varphi}, d\boldsymbol{\psi}], \tag{A.46}$$

for $\boldsymbol{\varphi} \in \sec \bigwedge^i(P, \mathfrak{G})$, $\boldsymbol{\psi} \in \sec \bigwedge^j(P, \mathfrak{G})$, $\boldsymbol{\tau} \in \sec \bigwedge^k(P, \mathfrak{G})$.

We shall also need the following identity

$$[\boldsymbol{\omega}, \boldsymbol{\omega}](\mathbf{X}_1, \mathbf{X}_2) = 2[\boldsymbol{\omega}(\mathbf{X}_1), \boldsymbol{\omega}(\mathbf{X}_2)]. \tag{A.47}$$

The proof of Eq. (A.47) is as follows:

(i) Recall that

$$[\boldsymbol{\omega}, \boldsymbol{\omega}] = (\boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b) \otimes [\mathcal{G}_a, \mathcal{G}_b]. \tag{A.48}$$

(ii) Let $\mathbf{X}_1, \mathbf{X}_2 \in \sec TP$ (i.e., $\mathbf{X}_1$ and $\mathbf{X}_2$ are vector fields on P). Then,

$$\begin{aligned}[\boldsymbol{\omega}, \boldsymbol{\omega}](\mathbf{X}_1, \mathbf{X}_2) &= (\boldsymbol{\omega}^a(\mathbf{X}_1)\boldsymbol{\omega}^b(\mathbf{X}_2) - \boldsymbol{\omega}^b(\mathbf{X}_2)\boldsymbol{\omega}^a(\mathbf{X}_1))[\mathcal{G}_a, \mathcal{G}_b] \\ &= 2[\boldsymbol{\omega}(\mathbf{X}_1), \boldsymbol{\omega}(\mathbf{X}_2)].\end{aligned} \tag{A.49}$$

Definition A.38 The curvature of the connection $\boldsymbol{\omega} \in \sec \bigwedge^1(P, \mathfrak{G})$ is $\boldsymbol{\Omega}^{\boldsymbol{\omega}} \in \sec \bigwedge^2(P, \mathfrak{G})$ defined by

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega}. \tag{A.50}$$

Definition A.39 The connection $\boldsymbol{\omega}$ is said to be flat if $\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0$.

Proposition A.40

$$D^{\boldsymbol{\omega}}\boldsymbol{\omega}(\mathbf{X_1},\mathbf{X_2}) = d\boldsymbol{\omega}(\mathbf{X}_1,\mathbf{X}_2) + [\boldsymbol{\omega}(\mathbf{X}_1),\boldsymbol{\omega}(\mathbf{X}_2)]. \tag{A.51}$$

Eq. (A.51) can be written using Eq. (A.49) (and recalling that $\boldsymbol{\omega}(\mathbf{X}) = \omega^a(\mathbf{X})\mathcal{G}_a$) as

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]. \tag{A.52}$$

Proof See, e.g., [1]. ■

Proposition A.41 (Bianchi Identity)

$$D\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0. \tag{A.53}$$

Proof

(i) Let us calculate $d\boldsymbol{\Omega}^{\boldsymbol{\omega}}$. We have,

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = d\left(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]\right). \tag{A.54}$$

We now take into account that $d^2\boldsymbol{\omega} = 0$ and that from the properties of the commutators given by Eqs. (A.44)–(A.46) above, we have

$$\begin{aligned} d[\boldsymbol{\omega},\boldsymbol{\omega})] &= [d\boldsymbol{\omega},\boldsymbol{\omega}] - [\boldsymbol{\omega},d\boldsymbol{\omega}],\\ [d\boldsymbol{\omega},\boldsymbol{\omega}] &= -[\boldsymbol{\omega},d\boldsymbol{\omega}],\\ [[\boldsymbol{\omega},\boldsymbol{\omega}],\boldsymbol{\omega}] &= 0. \end{aligned} \tag{A.55}$$

By using Eq. (A.55) in Eq. (A.54) gives

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = [d\boldsymbol{\omega},\boldsymbol{\omega}]. \tag{A.56}$$

(ii) In Eq. (A.56) use Eq. (A.52) and the last equation in Eq. (A.55) to obtain

$$d\boldsymbol{\Omega}^{\boldsymbol{\omega}} = [\boldsymbol{\Omega}^{\boldsymbol{\omega}},\boldsymbol{\omega}]. \tag{A.57}$$

(iii) Use now the definition of the exterior covariant derivative [Eq. (A.40)] together with the fact that $\boldsymbol{\omega}(\mathbf{X^h}) = 0$, for all $\mathbf{X} \in T_pP$ to obtain

$$D^{\boldsymbol{\omega}}\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0,$$

which is the result we wanted to prove. ■

We can then write the very important formula (known as the Bianchi identity),

$$D^{\boldsymbol{\omega}} \boldsymbol{\Omega}^{\boldsymbol{\omega}} = d\boldsymbol{\Omega}^{\boldsymbol{\omega}} + [\boldsymbol{\omega}, \boldsymbol{\Omega}^{\boldsymbol{\omega}}] = 0. \tag{A.58}$$

A.4.1 Local Curvature in the Base Manifold M

Let (U, Φ) be a local trivialization of $\boldsymbol{\pi}^{-1}(U)$ and s the associated cross section as defined above. Then, $s^*\boldsymbol{\Omega}^{\boldsymbol{\omega}} := \Omega^{\omega}$ (the pull back of $\boldsymbol{\Omega}^{\boldsymbol{\omega}}$) is a well defined 2-form field on U which takes values in the Lie algebra $\mathfrak{G}$. Let $\omega = s^*\boldsymbol{\omega}$ (see Eq. (A.34)). If we recall that the differential operator d commutes with the pullback, we immediately get

$$\Omega^{\omega} = s^* D^{\boldsymbol{\omega}} \boldsymbol{\omega} = d\omega + \frac{1}{2}[\omega, \omega]. \tag{A.59}$$

It is convenient to define the symbols

$$\mathbf{D}\omega := s^* D^{\boldsymbol{\omega}} \boldsymbol{\omega}, \tag{A.60}$$

$$\mathbf{D}\Omega^{\omega} := s^* D^{\boldsymbol{\omega}} \boldsymbol{\Omega}^{\boldsymbol{\omega}} \tag{A.61}$$

and to write

$$\begin{aligned} \mathbf{D}\Omega^{\omega} &= 0, \\ \mathbf{D}\Omega^{\omega} &= d\Omega^{\omega} + [\omega, \Omega^{\omega}] = 0. \end{aligned} \tag{A.62}$$

Eq. (A.62) is *also* known as Bianchi identity.

Remark A.42 In gauge theories (Yang-Mills theories) Ω^{ω} is (except for numerical factors with physical units) called a *field strength in the gauge* Φ.

Remark A.43 When G is a matrix group, as is the case in the presentation of gauge theories by physicists, Definition A.36 of the commutator $[\boldsymbol{\varphi}, \boldsymbol{\psi}] \in \sec \bigwedge^{i+j}(P, \mathfrak{G})$ ($\boldsymbol{\varphi} \in \sec \bigwedge^{i}(P, \mathfrak{G})$, $\boldsymbol{\psi} \in \sec \bigwedge^{j}(P, \mathfrak{G})$) gives

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = \boldsymbol{\varphi} \wedge \boldsymbol{\psi} - (-1)^{ij} \boldsymbol{\psi} \wedge \boldsymbol{\varphi}, \tag{A.63}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are considered as matrices of forms with values in $\mathfrak{G}$ and $\boldsymbol{\varphi} \wedge \boldsymbol{\psi}$ stands for the usual matrix multiplication where the entries are combined via the exterior product. Then, when G is a matrix group, we can write Eqs. (A.52) and (A.59) as

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}} \boldsymbol{\omega} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}, \tag{A.64}$$

$$\Omega^{\omega} := \mathbf{D}\omega = d\omega + \omega \wedge \omega. \tag{A.65}$$

A.4.2 Transformation of the Field Strengths Under a Change of Gauge

Consider two local trivializations (U, Φ) and (U', Φ') of P such that $p \in \boldsymbol{\pi}^{-1}(U \cap U')$ has (x, g) and (x, g') as images in $(U \cap U') \times G$, where $x \in U \cap U'$. Let s, s' be the associated cross sections to Φ and Φ' respectively. By writing $s'^{*}\boldsymbol{\Omega}^{\boldsymbol{\omega}} = \Omega^{\omega\prime}$, we have the following relation for the local curvature in the two different gauges such that $g' = hg$

$$\Omega^{\omega\prime} = h\Omega^{\omega}h^{-1}, \quad \forall x \in U \cap U'. \tag{A.66}$$

We now give the *coordinate expressions* for the potential and field strengths in the trivialization Φ. Let $\{x^{\mu}\}$ be coordinates for a local chart for $U \subset M$ and let $\left\{\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}\right\}$ and $\{dx^{\mu}\}$, $\mu = 0, 1, 2, 3$, be (dual) bases of TU and $T^{*}U$ respectively. Then,

$$\omega = \omega^{a} \otimes \mathcal{G}_{a} = \omega^{a\cdot}_{\cdot\mu} dx^{\mu} \otimes \mathcal{G}_{a}, \tag{A.67}$$

$$\Omega^{\omega} = (\Omega^{\omega})^{a} \otimes \mathcal{G}_{a} = \frac{1}{2}\Omega^{a\cdot\cdot}_{\cdot\mu\nu} dx^{\mu} \wedge dx^{\nu} \otimes \mathcal{G}_{a}. \tag{A.68}$$

where $\omega^{a\cdot}_{\cdot\mu}, \Omega^{a\cdot\cdot}_{\cdot\mu\nu} : M \supset U \to \mathbb{R}$ (or $\mathbb{C}$) and we get

$$\Omega^{a\cdot\cdot}_{\cdot\mu\nu} = \partial_{\mu}\omega^{a\cdot}_{\cdot\nu} - \partial_{\nu}\omega^{a\cdot}_{\cdot\mu} + f^{a\cdot\cdot}_{\cdot bc}\omega^{b}_{\mu}\omega^{c}_{\nu}, \tag{A.69}$$

with $f^{c\cdot\cdot}_{\cdot ab}$ the structure constants of $\mathfrak{G}$, i.e., $[\mathcal{G}_{a}, \mathcal{G}_{b}] = f^{c\cdot\cdot}_{\cdot ab}\mathcal{G}_{c}$.

The following objects appear frequently in the presentation of gauge theories by physicists.

$$(\Omega^{\omega})^{a} = \frac{1}{2}\Omega^{a\cdot\cdot}_{\cdot\mu\nu} dx^{\mu} \wedge dx^{\nu} = d\omega^{a} + \frac{1}{2} f^{a\cdot\cdot}_{\cdot bc}\omega^{b} \wedge \omega^{c}, \tag{A.70}$$

$$\Omega^{\omega}_{\mu\nu} = \Omega^{a\cdot\cdot}_{\cdot\mu\nu}\mathcal{G}_{a} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} + [\omega_{\mu}, \omega_{\nu}], \tag{A.71}$$

$$\omega_{\mu} = \omega^{a\cdot}_{\cdot\mu}\mathcal{G}_{a}. \tag{A.72}$$

Next we give the local expression of Bianchi identity. Using Eqs. (A.62) and (A.70) we have

$$\mathbf{D}\Omega^{\omega} := \frac{1}{2}(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} dx^{\rho} \wedge dx^{\mu} \wedge dx^{\nu} = 0. \tag{A.73}$$

Putting

$$(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} := \mathbf{D}_{\rho}\Omega^{\omega}_{\mu\nu} \tag{A.74}$$

we have

$$\mathbf{D}_\rho \overset{\omega}{\Omega}_{\mu\nu} = \partial_\rho \overset{\omega}{\Omega}_{\mu\nu} + [\omega_\rho, \overset{\omega}{\Omega}_{\mu\nu}], \tag{A.75}$$

and

$$\mathbf{D}_\rho \overset{\omega}{\Omega}_{\mu\nu} + \mathbf{D}_\mu \overset{\omega}{\Omega}_{\nu\rho} + \mathbf{D}_\nu \overset{\omega}{\Omega}_{\rho\mu} = 0. \tag{A.76}$$

Physicists call the operator

$$\mathbf{D}_\rho := \partial_\rho + [\omega_\rho,]. \tag{A.77}$$

the *covariant derivative*. The reason for this name will be given below.

A.4.3 Induced Connections

Let $(P_1, M_1, \boldsymbol{\pi}_1, G_1)$ and $(P_2, M_2, \boldsymbol{\pi}_2, G_2)$ be two principal bundles and let $\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism, i.e., $\mathcal{F}$ is fiber preserving, it induces a diffeomorphism $f : M_1 \to M_2$ and there exists a homomorphism $\lambda : G_1 \to G_2$ such that for $g_1 \in G_1, p_1 \in P_1$ we have

$$\mathcal{F}(p_1 g_1) = R_{\lambda(g_1)} \mathcal{F}(p_1). \tag{A.78}$$

Proposition A.44 *$\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism. Then a connection $\boldsymbol{\omega}_1$ on P_1determines a unique connection om P_2.*

Remark A.45 Let $(P, M, \boldsymbol{\pi}', \mathrm{O}_{p,q}) = \mathbf{P}_{\mathrm{O}p,q}(M)$ be the orthonormal frame bundle, which is as explained above reduction of the frame bundle $F(M)$. Then, a connection on $\mathbf{P}_{\mathrm{O}p,q}(M)$ determines a unique connection on $F(M)$. This is a very important result that has been used implicitly in Sect. 4.9.9 and the solution of Exercise 4.150.

Proposition A.46 *Let $F(M)$ be the frame bundle of a paracompact manifold M. Then, $F(M)$ can be reduced to a principal bundle with structure group $\mathrm{O}_{p,q}$, and to each reduction there corresponds a Riemannian metric field on M.*

Remark A.47 If M has dimension 4, and we substitute $\mathrm{O}_{p,q} \mapsto \mathrm{SO}^e_{1,3}$ then to each reduction of $F(M)$ there corresponds a Lorentzian metric field on M.

A.4.4 Linear Connections on a Manifold M

Definition A.48 A linear connection on a smooth manifold M is a connection $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n, \mathbb{R})$.

Remark A.49 Given a Riemannian (Lorentzian) manifold $(M, \boldsymbol{g})$ a connection on $F(M)$ which is determined by a connection on the orthonormal frame bundle $\mathbf{P}_{\mathrm{O}p,q}(M)$ $(\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M))$ is called a metric connection. After introducing the concept of covariant derivatives on vector bundles, we can show that the covariant derivative of the metric tensor with respect to a metric connection is null.

Consider the mapping $f|_p : T_x(M) \to \mathbb{R}^n$ (with $p = (x, \Sigma_x)$ in a given trivialization) which sends $\mathbf{v} \in T_xM$ into its components relative to the frame $\Sigma_x = \{e_1|_x, \ldots, e_n|_x\}$. Let $\{\theta^j\big|_x\}$ be the dual basis of $\{e_i|_x\}$. We write

$$f|_p(\mathbf{v}) = (\theta^j\big|_x(\mathbf{v})). \tag{A.79}$$

Definition A.50 The canonical soldering form of M is the 1-form $\boldsymbol{\theta} \in \sec T^*F(M) \otimes \mathbb{R}^n$ such that for any $v \in \sec T_pF(M)$ such that $\mathbf{v} = \pi_* v$ we have

$$\begin{aligned}(\boldsymbol{\theta}(v)) &:= \boldsymbol{\theta}^a|_p(v)\mathbf{e}_a \\ &= \theta^a|_x(\mathbf{v})\mathbf{e}_a,\end{aligned} \tag{A.80}$$

where $\{\mathbf{e}_a\}$ is the canonical basis of $\mathbb{R}^n$ and $\{\boldsymbol{\theta}^a\}$ is a basis of $T^*F(M)$, with $\boldsymbol{\theta}^a = \pi^*\theta^a$, $\boldsymbol{\theta}^a|_p(v) = \theta^a|_x(\mathbf{v})$.

Definition A.51 The torsion of a linear connection $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n, \mathbb{R})$ is the 2-form $D\boldsymbol{\theta} = \boldsymbol{\Theta} \in \sec \bigwedge^2 T^*F(M) \otimes \mathbb{R}^n$.

As it is easy to verify, the soldering form $\boldsymbol{\theta}$ and the torsion 2-form $\boldsymbol{\Theta}$ are tensorial of type $(\rho, \mathbb{R}^n)$, where $\rho(u) = u$, $u \in Gl(n, \mathbb{R})$.

Using the same techniques employed in the calculation of $D^{\boldsymbol{\omega}}\boldsymbol{\omega}(\mathbf{X}_1, \mathbf{X}_2)$ [Eq. (A.51)] it can be shown that

$$\boldsymbol{\Theta} = d\boldsymbol{\theta} + [\boldsymbol{\omega}, \boldsymbol{\theta}], \tag{A.81}$$

where [,] is the commutator product in the Lie algebra of the *affine* group $A(n, \mathbb{R}) = Gl(n, \mathbb{R}) \boxtimes \mathbb{R}^n$, where $\boxtimes$ means the *semi-direct* product. Suppose that $(\boldsymbol{e}^b_a, \boldsymbol{e}_c)$ is the canonical basis of $a(n, \mathbb{R})$, the Lie algebra of $A(n, \mathbb{R})$. Recalling that

$$\boldsymbol{\omega}(v) = \boldsymbol{\omega}^a_b(v)\boldsymbol{e}^b_a, \tag{A.82}$$

$$\boldsymbol{\theta}(v) = \boldsymbol{\theta}^a(v)\boldsymbol{e}_a, \tag{A.83}$$

we can show without difficulties that

$$D^{\boldsymbol{\omega}}\boldsymbol{\Theta} = [\boldsymbol{\Omega}, \boldsymbol{\theta}]. \tag{A.84}$$

A.4.5 Torsion and Curvature on M

Let $\{x^i\}$ be the coordinates associated to a local chart (U, φ) of the maximal atlas of M. Let $\boldsymbol{\Sigma} \in \sec F(U)$ with $\boldsymbol{e}_i = F_i^j \frac{\partial}{\partial x^j}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^a \boldsymbol{e}_a$. Take $\boldsymbol{\pi}_* v = \boldsymbol{v}$. Then

$$(\boldsymbol{\theta}_p(v)) = f|_p(\boldsymbol{v}) = f|_p(dx^j(\boldsymbol{v})\boldsymbol{\partial}_j) = f|_p(dx^j(\boldsymbol{v})(F_j^k)^{-1}\boldsymbol{e}_k)$$
$$= ((F_j^k)^{-1}dx^j(\pi_* v)). \tag{A.85}$$

With this result it is quite obvious that given any $\mathbf{w} \in \mathbb{R}^n$, $\boldsymbol{\theta}$ determines a horizontal field $v_{\mathbf{w}} \in secTF(M)$ by

$$(\boldsymbol{\theta}(v_{\mathbf{w}}(p))) = \mathbf{w}. \tag{A.86}$$

With these preliminaries we have the

Proposition A.52 *There is a bijective correspondence between sections of $T^*M \otimes T_s^r M$ and sections of $T^*F(M) \otimes \mathbb{R}^{n_q}$, the space of tensorial forms of the type $(\rho, \mathbb{R}^{n_q})$ in $F(M)$, with ρ and q being determined by $T_s^r M$.*

Using the above proposition and recalling that the soldering form is tensorial of type $(\rho(u), \mathbb{R}^n)$, $\rho(u) = u$, we see that it determines on M a vector valued differential 1-form[3] $\theta = \boldsymbol{e}_a \otimes \theta^a \in \sec TM \otimes \bigwedge^1 T^*M$. Also, the torsion $\boldsymbol{\Theta}$ is tensorial of type $(\rho(u), \mathbb{R}^n)$, $\rho(u) = u$ and thus define a vector valued 2-form on M, $\Theta = \boldsymbol{e}_a \otimes \Theta^a \in \sec TM \otimes \bigwedge^2 T^*M$. We can show from Eq. (A.81) that given $u, w \in T_pF(M)$,

$$\Theta^a(\boldsymbol{\pi}_* u, \boldsymbol{\pi}_* w) = d\theta^a(\boldsymbol{\pi}_* u, \pi_* w) + \omega_b^a(\boldsymbol{\pi}_* u)\theta^b(\boldsymbol{\pi}_* w) - \omega_b^a(\boldsymbol{\pi}_* w)\theta^b(\boldsymbol{\pi}_* u). \tag{A.87}$$

On the basis manifold this equation is often written:

$$\Theta = \mathbf{D}\theta = \boldsymbol{e}_a \otimes (\mathbf{D}\theta^a)$$
$$= \boldsymbol{e}_a \otimes (d\theta^a + \omega_{\cdot b}^{a\cdot} \wedge \theta^b), \tag{A.88}$$

where we recognize $\mathbf{D}\theta^a$ as the exterior covariant derivative of index forms introduced in Sect. 3.3.4.[4]

---

[3] θ is clearly the identity operator in the space of vector fields.

[4] Rigorously speaking, if Eq. (A.88) is to agree with Eq. (A.87) we must have $\bar{\boldsymbol{\omega}}_b^a \wedge \bar{\boldsymbol{\theta}}^b = \left(\bar{\boldsymbol{\omega}}_b^a \otimes \bar{\boldsymbol{\theta}}^b - \bar{\boldsymbol{\theta}}^b \otimes \bar{\boldsymbol{\omega}}_b^a\right)$, i.e., as already observed $\wedge$ must be understood as $\dot{\wedge}$ given in Remark 2.24. However, this cause no troubles in the calculations we done using the Clifford bundle formalism.

Remark A.53 Also, the curvature $\boldsymbol{\Omega}^{\boldsymbol{\omega}}$ is tensorial of type $(\mathrm{Ad}, \mathbb{R}^{n^2})$. It then defines $\Omega = \boldsymbol{e}_a \otimes \theta^b \otimes \mathcal{R}^{a\cdot}_{\cdot b} \in \sec T^1_1 M \otimes \bigwedge^2 T^*M$ which we easily find to be given by

$$\Omega = \boldsymbol{e}_a \otimes \theta^b \otimes \mathcal{R}^a_b$$
$$= \boldsymbol{e}_a \otimes \theta^b \otimes (d\omega^{a\cdot}_{\cdot b} + \omega^{a\cdot}_{\cdot c} \wedge \omega^{c\cdot}_{\cdot b}), \tag{A.89}$$

where the $\mathcal{R}^a_b \in \sec \bigwedge^2 T^*M$ are the curvature 2-forms introduced in Chap. 4, explicitly given by

$$\mathcal{R}^{a\cdot}_{\cdot b} = d\omega^{a\cdot}_{\cdot b} + \omega^{a\cdot}_{\cdot c} \wedge \omega^{c\cdot}_{\cdot b}. \tag{A.90}$$

Note that sometimes the symbol $\mathbf{D}\omega^{a\cdot}_{\cdot b}$ such that $\mathcal{R}^{a\cdot}_{\cdot b} := \mathbf{D}\omega^a_b$ is introduced in some texts. Of course, the symbol $\mathbf{D}$ cannot be interpreted in this case as the exterior covariant derivative of index forms.[5] This is expected since $\boldsymbol{\omega} \in \sec \bigwedge^1 T^*P \otimes gl(n, \mathbb{R})$ is *not* tensorial.

A.5 Covariant Derivatives on Vector Bundles

Consider a vector bundle $(E, M, \boldsymbol{\pi}_1, G, \mathbf{V})$[6] *associated* to a *PFB* bundle $(P, M, \boldsymbol{\pi}, G)$ by the linear representation ρ of G in the vector space $\mathbf{V}$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Also, let $\dim_{\mathbb{F}} \mathbf{V} = m$. Consider again the trivializations of P and E given by Eqs. (A.7)–(A.9). Then, we have the

Definition A.54 The *parallel transport* of $\boldsymbol{\Psi}_{\mathbf{0}} \in E$, $\boldsymbol{\pi}_1(\boldsymbol{\Psi}_{\mathbf{0}}) = x_0$, along the curve $\sigma : \mathbb{R} \ni I \to M$, $t \mapsto \sigma(t)$ from $x_0 = \sigma(0) \in M$ to $x = \sigma(t)$ is the element $\boldsymbol{\Psi}_{\parallel t} \in E$ such that:

(i) $\boldsymbol{\pi}_1(\boldsymbol{\Psi}_{\parallel t}) = x$,
(ii) $\chi_i(\boldsymbol{\Psi}_{\parallel t}) = \rho(\varphi_i(p_{\parallel t}) \circ \varphi_i^{-1}(p_0))\chi_i(\boldsymbol{\Psi}_{\mathbf{0}})$.
(iii) $p_{\parallel t} \in P$ is the parallel transport of $p_0 \in P$ along σ from x_0 to x as defined in Eq. (A.25) above.

Definition A.55 Let $\mathbf{Y}$ be a vector at x_0 tangent to the curve σ (as defined above). The covariant derivative of $\boldsymbol{\Psi} \in \sec E$ in the direction of $\mathbf{Y}$ is denoted $(D^E_{\mathbf{Y}}\boldsymbol{\Psi})_{x_0} \in \sec E$ and

$$(D^E_{\mathbf{Y}}\boldsymbol{\Psi})(x_0) \equiv (D^E_{\mathbf{Y}}\boldsymbol{\Psi})_{x_0} = \lim_{t\to 0} \frac{1}{t}(\boldsymbol{\Psi}^0_{\parallel t} - \boldsymbol{\Psi}_{\mathbf{0}}), \tag{A.91}$$

[5]See Sect. A.3.

[6]Also denoted $E = P \times_\rho \mathbf{V}$.

where $\boldsymbol{\Psi}^0_{\|t}$ is the "vector" $\boldsymbol{\Psi}_t \equiv \boldsymbol{\Psi}(\sigma(t))$ of a section $\boldsymbol{\Psi} \in \sec E$ parallel transported along σ from $\sigma(t)$ to x_0, the only requirement on σ being

$$\left.\frac{d}{dt}\sigma(t)\right|_{t=0} = \mathbf{Y}. \tag{A.92}$$

In the local trivialization (U_i, Ξ_i) of E (see Eqs. (A.7)–(A.9)) if Ψ_t is the element in $\mathbf{V}$ representing $\boldsymbol{\Psi}_t$, we have

$$\chi_i(\Psi^0_{\|t}) = \rho(g_0 g_t^{-1})\chi_{i|\sigma(t)}(\Psi_t). \tag{A.93}$$

By choosing p_0 such that $g_0 = e$ we can compute Eq. (A.91):

$$\begin{aligned}(D^E_{\mathbf{Y}}\boldsymbol{\Psi})_{x_0} &= \left.\frac{d}{dt}\rho(g^{-1}(t)\Psi_t)\right|_{t=0} \\ &= \left.\frac{d\Psi_t}{dt}\right|_{t=0} - \left(\rho'(e)\left.\frac{dg(t)}{dt}\right|_{t=0}\right)(\Psi_0).\end{aligned} \tag{A.94}$$

This formula is trivially generalized for the covariant derivative in the direction of an arbitrary vector field $\boldsymbol{Y} \in \sec TM$.

With the aid of Eq. (A.94) we can calculate, e.g., the covariant derivative of $\boldsymbol{\Psi} \in \sec E$ in the direction of the vector field $\boldsymbol{Y} = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu$. This covariant derivative is denoted $D^E_{\partial_\mu}\boldsymbol{\Psi}$.

We need now to calculate $\left.\frac{dg(t)}{dt}\right|_{t=0}$. In order to do that, recall that if $\frac{d}{dt}$ is a tangent to the curve σ in M, then $s_*\left(\frac{d}{dt}\right)$ is a tangent to $\hat{\sigma}$, the horizontal lift of σ, i.e., $s_*\left(\frac{d}{dt}\right) \in HP \subset TP$. As defined before $s = \Phi_i^{-1}(x, e)$ is the cross section associated to the trivialization Φ_i of P [see Eq. (A.6)]. Then, as g is a mapping $U \to G$ we can write

$$\left[s_*(\frac{d}{dt})\right](g) = \frac{d}{dt}(g \circ \sigma). \tag{A.95}$$

To simplify the notation, introduce local coordinates $\{x^\mu, g\}$ in $\pi^{-1}(U)$ and write $\sigma(t) = (x^\mu(t))$ and $\hat{\sigma}(t) = (x^\mu(t), g(t))$. Then,

$$s_*\left(\frac{d}{dt}\right) = \dot{x}^\mu(t)\frac{\partial}{\partial x^\mu} + \dot{g}(t)\frac{\partial}{\partial g}, \tag{A.96}$$

in the local coordinate basis of $T(\boldsymbol{\pi}^{-1}(U))$. An expression like the second member of Eq. (A.96) defines in general a vector tangent to P but, according to its definition,

$s_*\left(\frac{d}{dt}\right)$ is in fact horizontal. We must then impose that

$$s_*\left(\frac{d}{dt}\right) = \dot{x}^\mu(t)\frac{\partial}{\partial x^\mu} + \dot{g}(t)\frac{\partial}{\partial g} = \alpha^\mu\left(\frac{\partial}{\partial x^\mu} + \omega^{a\cdot}_{\cdot\mu}\mathcal{G}_a g\frac{\partial}{\partial g}\right), \tag{A.97}$$

for some α^μ.

We used the fact that $\frac{\partial}{\partial x^\mu} + \omega^{a\cdot}_{\cdot\mu}\mathcal{G}_a g\frac{\partial}{\partial g}$ is a basis for HP, as can easily be verified from the condition that $\boldsymbol{\omega}(Y^h) = 0$, for all $Y\in HP$. We immediately get that

$$\alpha^\mu = \dot{x}^\mu(t), \tag{A.98}$$

and

$$\frac{dg(t)}{dt} = \dot{g}(t) = -\dot{x}^\mu(t)\omega^{a\cdot}_{\cdot\mu}\mathcal{G}_a g, \tag{A.99}$$

$$\left.\frac{dg(t)}{dt}\right|_{t=0} = -\dot{x}^\mu(0)\omega^a_\mu\mathcal{G}_a. \tag{A.100}$$

With this result we can rewrite Eq. (A.94) as

$$(D^E_{\mathbf{Y}}\boldsymbol{\Psi})_{x_0} = \left.\frac{d\Psi_t}{dt}\right|_{t=0} - \rho'(e)\omega(\mathbf{Y})(\Psi_0), \qquad \mathbf{Y} = \left.\frac{d\sigma}{dt}\right|_{t=0}. \tag{A.101}$$

which generalizes trivially for the covariant derivative along a vector field $\boldsymbol{Y} \in \sec TM$.

Remark A.56 Many texts introduce the covariant derivative operator D^E_Y acting on sections of the vector bundle E as follows.

Definition A.57 A connection D^E on M is a mapping

$$\begin{aligned} D^E &: \sec TM \times \sec E \to \sec E, \\ (\boldsymbol{X}, \boldsymbol{\Psi}) &\mapsto D^E_{\boldsymbol{X}}\boldsymbol{\Psi}. \end{aligned} \tag{A.102}$$

such that $D^E_{\boldsymbol{X}} : \sec E \to \sec E$ satisfies the following properties:

$$\begin{aligned} &(i) && D^E_{\boldsymbol{X}}(a\boldsymbol{\Psi}) = aD^E_{\boldsymbol{X}}\boldsymbol{\Psi}, \\ &(ii) && D^E_{\boldsymbol{X}}(\boldsymbol{\Psi} + \boldsymbol{\Phi}) = D^E_{\boldsymbol{X}}\boldsymbol{\Psi} + D^E_{\boldsymbol{X}}\boldsymbol{\Phi}, \\ &(iii) && D^E_{\boldsymbol{X}}(f\boldsymbol{\Psi}) = \boldsymbol{X}(f) + fD^E_{\boldsymbol{X}}\boldsymbol{\Psi}, \\ &(iv) && D^E_{\boldsymbol{X}+\boldsymbol{Y}}\boldsymbol{\Psi} = D^E_{\boldsymbol{X}}\boldsymbol{\Psi} + D^E_{\boldsymbol{Y}}\boldsymbol{\Psi}, \\ &(v) && D^E_{f\boldsymbol{X}}\boldsymbol{\Psi} = fD^E_{\boldsymbol{X}}\boldsymbol{\Psi}, \end{aligned} \tag{A.103}$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \sec TM$, $\boldsymbol{\Psi}, \boldsymbol{\Phi} \in \sec E$, $\forall a \in \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (the field of scalars entering the definition of the vector space $\mathbf{V}$ of the bundle) $\forall f \in C^{\infty}(M)$, where $C^{\infty}(M)$ is the set of smooth functions with values in $\mathbb{F}$.

Of course, all properties in Eq. (A.103) follows directly from Eq. (A.101). However, the point of view encoded in Definition A.57 may be appealing to physicists. To see, recall that in general in physical theories $E = P \times_{\rho} \mathbf{V}$ where ϱ stands for the representation of G in the vector space $\mathbf{V}$.

Definition A.58 The dual bundle of E is the bundle $E^* = P \times_{\rho^*} \mathbf{V}^*$, where $\mathbf{V}^*$ is the dual space of $\mathbf{V}$ and ϱ^* is the representation of G in the vector space $\mathbf{V}^*$.

Example A.59 As examples of bundles of the kind $E = P \times_{\rho} \mathbf{V}$ we have the tangent bundle which is $TM = F(M) \times_{\rho} \mathbb{R}^n$ where $\rho : Gl(n, \mathbb{R}) \to Gl(n, \mathbb{R})$ denotes the standard representation and $T^*M = F(M) \times_{\rho^*} (\mathbb{R}^n)^*$ where the dual representation ρ^* satisfies $\rho^*(g) = \rho(g^{-1})^t$. Other important examples are the tensor bundle of tensors of type (r, s), the bundle of homogenous k-vectors and the bundle of homogeneous k-forms, respectively:

$$
\begin{aligned}
T_s^r M &= \bigotimes_s^r TM = F(M) \times_{\otimes_s^r} (\bigotimes_s^r \mathbb{R}^n), \\
\bigwedge^k TM &= F(M) \times_{\bigwedge_\rho^k} \bigwedge^k \mathbb{R}^n, \\
\bigwedge^k T^*M &= F(M) \times_{\bigwedge_{\rho^*}^k} \bigwedge^k \mathbb{R}^n,
\end{aligned} \tag{A.104}
$$

where $\otimes_s^r$, $\bigwedge_\rho^k$ and $\bigwedge_{\rho^*}^k$ are the induced tensor product and exterior powers representations.

Definition A.60 The bundle $E \otimes E^*$ is called the bundle of endomorphisms of E and will be denoted by $\mathrm{End}(E)$.

Definition A.61 A connection D^{E^*} *acting* on E^* is defined by

$$
(D_X^{E^*} \boldsymbol{\Xi}^*)(\boldsymbol{\Psi}) = \boldsymbol{X}\left(\boldsymbol{\Xi}^*(\boldsymbol{\Psi})\right) - \boldsymbol{\Xi}^*(D_X^E \boldsymbol{\Psi}), \tag{A.105}
$$

for $\forall \boldsymbol{\Xi}^* \in \sec E^*$, $\forall \boldsymbol{\Psi} \in \sec E$ and $\forall \boldsymbol{X} \in \sec TM$.

Definition A.62 A connection $D^{E \otimes E^*}$ acting on sections of $E \otimes E^*$ is defined for $\forall \boldsymbol{\Xi}^* \in \sec E^*$, $\forall \boldsymbol{\Psi} \in \sec E$ and $\forall X \in \sec TM$ by

$$
D_X^{E \otimes E^*} \boldsymbol{\Xi}^* \otimes \boldsymbol{\Psi} = D_X^{E^*} \boldsymbol{\Xi}^* \otimes \boldsymbol{\Psi} + \boldsymbol{\Xi}^* \otimes D_X^E \boldsymbol{\Psi}. \tag{A.106}
$$

We shall abbreviate $D^{E \otimes E^*}$ by $D^{\mathrm{End}E}$. Equation (A.106) may be generalized in an obvious way in order to define a connection on arbitrary tensor products of bundles $E \otimes E' \otimes \ldots E'^{\cdots\prime}$. Finally, we recall for completeness that given two bundles, say

E and E' and given connections D^E and $D^{E'}$ there is an obvious connection $D^{E\oplus E'}$ defined in the Whitney bundle $E \oplus E'$ (recall Definition A.20). It is given by

$$D_{\boldsymbol{X}}^{E\oplus E'}(\boldsymbol{\Psi} \oplus \boldsymbol{\Psi}') = D_{\boldsymbol{X}}^{E}\boldsymbol{\Psi} \oplus D_{\boldsymbol{X}}^{E'}\boldsymbol{\Psi}', \tag{A.107}$$

for $\forall \boldsymbol{\Psi} \in \sec E$, $\forall \boldsymbol{\Psi}' \in \sec E'$ and $\forall \boldsymbol{X} \in \sec TM$.

A.5.1 Connections on E over a Lorentzian Manifold

In what follows we suppose that $(M, \boldsymbol{g})$ is a Lorentzian manifold (Definition 4.73). We recall that the manifold M in a Lorentzian structure is supposed paracompact. Then, according to Proposition A.10 the bundles E, E^*, $T_s^r M$ and End$E$ admit cross sections.

We then write for the covariant derivative of $\boldsymbol{\Psi} \in \sec E$ and $\boldsymbol{X} \in \sec TM$,

$$D_X^E \boldsymbol{\Psi} = D_X^{0E} \boldsymbol{\Psi} \ + \ \mathcal{W}(\boldsymbol{X})\boldsymbol{\Psi}, \tag{A.108}$$

where $\mathcal{W} \in \sec \mathrm{End}E \otimes T^*M$ will be called *connection* 1-form (or *potential*) for D_X^E and D_X^{0E} is a well defined connection on E, that we are going to determine.

Consider then a open set $U \subset M$ and a trivialization of E in U. Such a trivialization is said to be a *choice of a gauge*.

Let $\{\mathbf{e}_i\}$ be the canonical basis of $\mathbf{V}$. Let $\boldsymbol{\Psi}|_U$ be a section of the bundle E. Consider the trivialization $\Xi : \boldsymbol{\pi}^{-1}(U) \to U \times \mathbf{V}$, $\Xi(\boldsymbol{\Psi}) = (\boldsymbol{\pi}(\boldsymbol{\Psi}), \chi(\boldsymbol{\Psi})) = (x, \chi(\boldsymbol{\Psi}))$. In this trivialization we write

$$\boldsymbol{\Psi}|_U := (x, \Psi(x)), \tag{A.109}$$

$\Psi(x) \in \mathbf{V}$, $\forall x \in U$, with $\Psi : U \to \mathbf{V}$ a smooth function. Let $\{s_i\} \in \sec E|_U$, $s_i = \chi^{-1}(e_i)$ $i = 1, 2, \ldots, m$ be a basis of sections of $E|_U$ and $\{e_\mu\} \in \sec F(U)$, $\mu = 0, 1, 2, 3$ a basis for TU. Let also $\{\varepsilon^\nu\}$, $\varepsilon^\nu \in \sec T^*U$, be the dual basis of $\{e_\mu\}$ and $\{s^{*i}\} \in \sec E^*|_U$, be a basis of sections of $E^*|_U$ dual to the basis $\{s_i\}$.

We define the connection coefficients in the chosen gauge by

$$D_{e_\mu}^E s_i = \mathcal{W}_{\cdot\mu i}^{j\cdot\cdot} s_j. \tag{A.110}$$

Then, if $\Psi = \Psi^i s_i$ and $\boldsymbol{X} = X^\mu e_\mu$

$$\begin{aligned} D_X^E \Psi &= X^\mu D_{e_\mu}^E (\Psi^i s_i) \\ &= X^\mu \left[e_\mu(\Psi^i) + \mathcal{W}_{\cdot\mu j}^{i\cdot\cdot} \Psi^j \right] s_i. \end{aligned} \tag{A.111}$$

Now, let us concentrate on the term $X^{\mu}\mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}\Psi^{j}s_{i}$. It is, of course a new section $\boldsymbol{F} := (x, X^{\mu}\mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}\Psi^{j}s_{i})$ of $E|_{U}$ and $X^{\mu}\mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}\Psi^{j}s_{i}$ is linear in both X and Ψ.

This observation shows that $\mathcal{W}^{U} \in \sec(\mathrm{End}\, E|_{U} \otimes T^{*}U)$, such that in the trivialization introduced above is given by

$$\mathcal{W}^{U} = \mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}s_{i} \otimes s^{*j} \otimes \varepsilon^{\mu} \tag{A.112}$$

is the representative of $\mathcal{W}$ in the chosen gauge.

Note that if $\boldsymbol{X} \in \sec TU$ and $\boldsymbol{\Psi} := (x, \Psi(x))$ a section of $E|_{U}$ we have

$$\begin{aligned}\omega^{U}(\boldsymbol{X}) &:= \omega^{U}_{\boldsymbol{X}} = X^{\mu}\mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}s_{i} \otimes s^{*j},\\ \omega^{U}_{\boldsymbol{X}}(\Psi) &= X^{\mu}\mathcal{W}^{i\cdot\cdot}_{\cdot\mu j}\Psi^{j}s_{i}.\end{aligned} \tag{A.113}$$

We can then write

$$D^{E}_{\boldsymbol{X}}\Psi = X(\Psi) + \omega^{U}_{\boldsymbol{X}}(\Psi), \tag{A.114}$$

thereby identifying $D^{0E}_{\boldsymbol{X}}\Psi = \boldsymbol{X}(\Psi)$. In this case $D^{0E}_{\boldsymbol{X}}$ is called the standard *flat* connection.

Now, we can state a very important result which has been used in Chap. 2 to write the different decompositions of Riemann-Cartan connections.

Proposition A.63 *Let D^{0E} and D^{E} be arbitrary connections on E then there exists $\bar{\mathcal{W}} \in \sec \mathrm{End} E \otimes T^{*}M$ such that for any $\boldsymbol{\Psi} \in \sec E$ and $\boldsymbol{X} \in \sec TM$,*

$$D^{E}_{\boldsymbol{X}}\boldsymbol{\Psi} = D^{0E}_{\boldsymbol{X}}\boldsymbol{\Psi} + \bar{\mathcal{W}}(\boldsymbol{X})\boldsymbol{\Psi}. \tag{A.115}$$

A.5.2 Gauge Covariant Connections

Definition A.64 A connection D^{E} on E is said to be a G-connection if for any $u \in G$ and any $\boldsymbol{\Psi} \in \sec E$ there exists a connection D'^{E} on E such that for any $\boldsymbol{X} \in \sec TM$

$$D'^{E}_{\boldsymbol{X}}(\rho(u)\boldsymbol{\Psi}) = \rho(u)D^{E}_{\boldsymbol{X}}\boldsymbol{\Psi}. \tag{d11}$$

Proposition A.65 *If $D^{E}_{\boldsymbol{X}}\boldsymbol{\Psi} = D^{0E}_{\boldsymbol{X}}\boldsymbol{\Psi} + \bar{\mathcal{W}}(\boldsymbol{X})\boldsymbol{\Psi}$ for $\boldsymbol{\Psi} \in \sec E$ and $\boldsymbol{X} \in \sec TM$, then $D'^{E}_{\boldsymbol{X}}\boldsymbol{\Psi} = D^{0E}_{\boldsymbol{X}}\boldsymbol{\Psi} + \bar{\mathcal{W}}'(\boldsymbol{X})\boldsymbol{\Psi}$ with*

$$\bar{\mathcal{W}}'(\boldsymbol{X}) = u\bar{\mathcal{W}}(\boldsymbol{X})u^{-1} + udu^{-1}. \tag{A.116}$$

Suppose that the vector bundle E has the same structural group as the orthonormal frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$, which as we know is a reduction of the frame bundle $F(M)$. In this case we give the

Definition A.66 A connection $\mathbf{D}^E$ on E is said to be a generalized G-connection if for any $u \in G$ and any $\boldsymbol{\Psi} \in \sec E$ there exists a connection $\mathbf{D}'^E$ on E such that for any $X \in \sec TM$, $TM = \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M) \times_{\rho^{TM}} \mathbb{R}^4$

$$D'^E_{\boldsymbol{X}'}\left(\rho(u)\boldsymbol{\Psi}\right) = \rho(u) D^E_{\boldsymbol{X}}\boldsymbol{\Psi}, \tag{A.117}$$

where $\boldsymbol{X}' = \rho^{TM}\boldsymbol{X} \in \sec TM$.

A.5.3 Curvature Again

Definition A.67 Let D^E be a G-connection on E. The curvature operator $\mathbf{R}^E \in \sec \bigwedge^2 T^*M \otimes \mathrm{End}E$ of D^E is the mapping

$$\mathbf{R}^E : \sec TM \otimes TM \otimes E \to E, \tag{A.118}$$

$$\mathbf{R}^E(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\Psi} = D^E_{\boldsymbol{X}} D^E_{\boldsymbol{Y}}\boldsymbol{\Psi} - D^E_{\boldsymbol{Y}} D^E_{\boldsymbol{X}}\boldsymbol{\Psi} - D^E_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{\Psi}$$

$$\mathbf{R}^E(\boldsymbol{X},\boldsymbol{Y}) = D^E_{\boldsymbol{X}} D^E_{\boldsymbol{Y}} - D^E_{\boldsymbol{Y}} D^E_{\boldsymbol{X}} - D^E_{[\boldsymbol{X},\boldsymbol{Y}]}, \tag{d14}$$

for any $\boldsymbol{\Psi} \in \sec E$ and $\boldsymbol{X},\boldsymbol{Y} \in \sec TM$.

If $\boldsymbol{X} = \partial_\mu, \boldsymbol{Y} = \partial_\nu \in \sec TU$ are coordinate basis vectors associated to the coordinate functions $\{x^\mu\}$ we have

$$\mathbf{R}^E(\partial_\mu,\partial_\nu) := \mathbf{R}^E_{\mu\nu} = \left[D^E_{\partial_\mu}, D^E_{\partial_\nu}\right]. \tag{A.119}$$

In a local basis $\{s_i \otimes s^{*j}\}$ of $\mathrm{End}E$ we have under the local trivialization used above

$$\begin{aligned}
\mathbf{R}^E_{\mu\nu} &= \mathbf{R}^{a\cdots}_{\cdot\mu b\nu} s_a \otimes s^{*b},\\
\mathbf{R}^{a\cdots}_{\cdot\mu\nu b} &= \partial_\mu \mathcal{W}^{a\cdot\cdot}_{\cdot\nu b} - \partial_\nu \mathcal{W}^{a\cdot\cdot}_{\cdot\mu b} + \mathcal{W}^{a\cdot\cdot}_{\cdot\mu c}\mathcal{W}^{c\cdot\cdot}_{\cdot\nu b} - \mathcal{W}^{a\cdot\cdot}_{\cdot\nu c}\mathcal{W}^{c\cdot\cdot}_{\cdot\mu b}.
\end{aligned} \tag{A.120}$$

Equation (A.120) can also be written

$$\mathbf{R}^E_{\mu\nu} = \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + \left[\mathcal{W}_\mu, \mathcal{W}_\nu\right]. \tag{A.121}$$

A.5.4 Exterior Covariant Derivative Again

Definition A.68 Consider $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$ and $B_s \in \bigwedge^s T^*M$. We define $(\Psi \otimes A_r) \otimes_\wedge B_s$ by

$$(\Psi \otimes A_r) \otimes_\wedge B_s = \Psi \otimes (A_r \wedge B_s). \tag{A.122}$$

Definition A.69 Let $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$ and $\mathbf{\Pi} \otimes B_s \in \sec \operatorname{End} E \otimes \bigwedge^s T^*M$. We define $(\Pi \otimes B_s) \otimes_\wedge (\Psi \otimes A_r)$ *by*

$$(\Pi \otimes B_s) \otimes_\wedge (\Psi \otimes A_r) = \Pi(\Psi) \otimes (B_s \wedge A_r). \tag{A.123}$$

Definition A.70 Given a connection D^E acting on E, the exterior covariant derivative $\mathbf{d}^{D^E}$ acting on sections of $E \otimes \bigwedge^r T^*M$ and the exterior covariant derivative $\mathbf{d}^{D^{\mathrm{EndE}}}$ acting on sections of $\operatorname{End} E \otimes \bigwedge^s T^*M$ $(r, s = 0, 1, 2, 3, 4)$ is given by

(i) if $\boldsymbol{\Psi} \in \sec E$ then for any $\boldsymbol{X} \in \sec TM$

$$\mathbf{d}^{D^E} \Psi(X) = D^E \Psi, \tag{A.124}$$

(ii) For any $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$

$$\mathbf{d}^{D^E} (\Psi \otimes A_r) = \mathbf{d}^{D^E} \Psi \otimes \wedge A_r + \Psi \otimes dA_r, \tag{A.125}$$

(iii) For any $\mathbf{\Pi} \otimes B_s \in \sec \operatorname{End} E \otimes \bigwedge^s T^*M$

$$\mathbf{d}^{D^{\mathrm{EndE}}} (\Pi \otimes B_s) = \mathbf{d}^{D^{\mathrm{EndE}}} \Pi \otimes_\wedge B_s + \Pi \otimes dB_s, \tag{A.126}$$

Proposition A.71 *Consider the bundle product* $\mathfrak{E} = (\operatorname{End} E \otimes \bigwedge^s T^*M) \otimes_\wedge (E \otimes \bigwedge^r T^*M)$. *Let* $\mathbf{\Pi} = \Pi \otimes B_s \in \sec \operatorname{End} E \otimes \bigwedge^s T^*M$ *and* $\boldsymbol{\Psi} = \Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$. *Then the exterior covariant derivative* $\mathbf{d}^{D^{\mathfrak{E}}}$ *acting on sections of* $\mathfrak{E}$ *satisfies*

$$\mathbf{d}^{D^{\mathfrak{E}}} (\mathbf{\Pi} \otimes_\wedge \boldsymbol{\Psi}) = (\mathbf{d}^{D^{\mathrm{EndE}}} \mathbf{\Pi}) \otimes_\wedge \boldsymbol{\Psi} + (-1)^s \mathbf{\Pi} \otimes_\wedge \mathbf{d}^{D^E} \boldsymbol{\Psi}. \tag{A.127}$$

Exercise A.72 The reader can now show several interesting results, which make contact with results obtained earlier when we analyzed the connections and curvatures on principal bundles and which allowed us sometimes the use of sloppy notations in the main text:

(i) Suppose that the bundle admits a flat connection D^{0E}. We put $\mathbf{d}^{D^{0E}} = d$. Then, if $\chi \in \sec E \otimes \bigwedge^r T^*M$ we have

$$\mathbf{d}^{D^{0E}} \chi = \mathbf{d}\chi + \mathcal{W} \otimes_\wedge \chi.$$

(ii) If $\chi \in \sec E \otimes \bigwedge^r T^*M$ we have

$$(\mathbf{d}^{D^E})^2 \chi = \mathbf{R}^E \otimes_\wedge \chi. \tag{A.128}$$

(iii) If $\chi \in \sec E \otimes \bigwedge^r T^*M$ we have

$$(\mathbf{d}^{D^E})^3 \chi = \mathbf{R}^E \otimes_\wedge \mathbf{d}^{D^E} \chi. \tag{A.129}$$

(iii) Suppose that the bundle admits a flat connection D^{0E}. We put $\mathbf{d}^{D^{0E}} = d$. Then, if

(iv) $\mathbf{\Pi} \in \sec \mathrm{End} E \otimes \bigwedge^s T^*M$

$$\mathbf{d}^{D^{\mathrm{End}E}} \mathbf{\Pi} = \mathbf{d}\mathbf{\Pi} + [\mathcal{W}, \mathbf{\Pi}]. \tag{A.130}$$

(v)

$$\mathbf{d}^{D^{\mathrm{End}E}} \mathbf{R}^E = 0. \tag{A.131}$$

(vi)

$$\mathbf{R}^E = d\mathcal{W} + \mathcal{W} \otimes_\wedge \mathcal{W}. \tag{A.132}$$

Remark A.73 Note that $\mathbf{R}^E \neq \mathbf{d}^{D^{\mathrm{End}E}} \mathcal{W}$.

We end here this long appendix, hopping that the material presented be enough to permit our reader to follow the more difficult parts of the text and in particular to see the reason for our use of many eventual sloppy notations.

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Acronyms and Abbreviations

ADM	Arnowitt-Deser-Misner
CBF	Clifford Bundle Formalism
GRT	General Relativity Theory
GSS	Geometrical Space Structure
MCGSS	Metrical Compatible Geometrical Space Structure
MCPS	Metrical Compatible Parallelism Structure
DHSF	Dirac-Hestenes Spinor Field
MECM	Multiform and Extensor Calculus on Manifolds
iff	If and Only If
i.e.	Id Est
e.g.	Exempli Gratia

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List of Symbols

$\mathbf{V}$	Vector Space	21
$\mathbf{V}^*$	Dual Space of $\mathbf{V}$	21
dim	Dimension	21
$\mathbb{N}$	Set of Natural Numbers	21
$T_k\mathbf{V}$	Space of k-Cotensors	22
$\oplus$	Direct Sum	22
$\langle\rangle_k$	k-Part Operator	22
$\otimes$	Tensor Product	23
$\bigoplus_{k=0}^{\infty}$	See Definition 2.3	24
$T_s^r\mathbf{V}$	Space of r-Contravariant, s-Covariant Tensors	23
$\mathcal{T}(\mathbf{V})$	General Tensor Algebra of $\mathbf{V}$	24
$T^k\mathbf{V}$	Space of Contravariant Multitensors	24
$\wedge$	Main Involution	24
$\sim$	Reversion	24
$-$	Conjugation	24
$\mathbf{g}$	Metric Tensor of $\mathbf{V}$	25
$\cdot$	Scalar Product	25
$\{\mathbf{e}_k\}$	Basis of $\mathbf{V}$	25
$\{\varepsilon^k\}$	Dual Basis of $\{\mathbf{e}_k\}$	25
$\{\boldsymbol{e}^k\}$	Reciprocal Basis $\{\boldsymbol{e}_k\}$	25
$\{\varepsilon_k\}$	Reciprocal Basis of $\{\varepsilon^k\}$	25
$\bigwedge\mathbf{V}$	Exterior Algebra	26
$\wedge$	Exterior Product	26
A	Antisymmetrization Operator	27
$\epsilon_{i_1\ldots i_k}$	Permutation Symbol of Order k	27
$\mathring{\boldsymbol{g}}$	Fiducial Metric Tensor of $\mathbf{V}$	28
$\mathring{g}$	Fiducial Metric Tensor of $\mathbf{V}^*$	28

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$(\mathbf{V}^*, \mathring{g})$	Metric Vector Space	28
$\star$	Hodge Star Operator	29
$\underset{\mathring{\mathbf{g}}}{\star}$	Hodge Operator Associated with $\mathring{\mathbf{g}}$	29
$\llcorner$	Left Contraction	30
$\lrcorner$	Right Contraction	30
$\underset{\mathbf{g}}{\llcorner}$	Left Contraction Associated with $\mathbf{g}$	30
$\underset{\mathbf{g}}{\lrcorner}$	Right Contraction Associated with $\mathbf{g}$	30
$(\bigwedge \boldsymbol{V}, \mathring{g})$	Grassmann Algebra	31
$\mathcal{C}\ell(\boldsymbol{V}, \mathring{g})$	Clifford Algebra of $(\boldsymbol{V}, \mathring{g})$	32
AB	Clifford Product of A and B	32
$\mathbb{R}_{p,q}$	Vector Space $\mathbb{R}^n$ Equipped with Metric of Signature (p, q)	35
$\mathbb{R}^{p,q}$	Clifford Algebra of $\mathbb{R}^{p,q}$	35
$A \hookrightarrow B$	A is Embedded in B and $A \subseteq B$	35
$ext\text{-}(\boldsymbol{V})$	Space of Extensors	38
$ext\text{-}(\bigwedge^p \boldsymbol{V}, \bigwedge^q \boldsymbol{V})$	Space of (p, q) Extensors	38
$\dagger$	Adjoint Operator	39
$\bullet$	Scalar Product	39
$\underline{t}$	Exterior Power Extension Operator	40
tr	Trace Operator	41
$\mathfrak{det}$	Determinant Operator of a Extensor	42
$bif(t)$	Biform of t	43
g	Endomorphism Associated with g	49
$S_{[ab]}$	Skew-Symmetric Part of S_{ab}	50
$\mathbb{R}^{1,3}$	Minkowski Vector Space	52
$\uparrow$	Time Orientation	54
$\mathbf{L}$	Lorentz Transformation	55
$\mathrm{O}_{1,3}$	Lorentz Group	55
$\mathrm{SO}_{1,3}$	Proper Lorentz Group	55
$\mathrm{SO}^e_{1,3} \equiv \mathcal{L}^{\uparrow}_{+}$	Proper Orthochronous Lorentz Group	55
$\boxtimes$	Semi-Direct Product	56
$\mathcal{P}$	Poincaré Group	55
$\circledast$	Any One of the Products $\wedge$, $\cdot$, $\llcorner$, $\lrcorner$ or Clifford Product	56
$\lVert\mathbf{v}\rVert$	Norm of $\mathbf{v}$	56
$A \cdot \partial_Y F(y)$	Directional Derivative	58
$\partial_Y \circledast F(Y)$	One of the Derivative Operators Acting on $F(Y)$	59
$\mathcal{A}$	Associative algebra	69
$\mathbb{D}$	Division Algebra	69
$\mathbb{D}(m)$	$m \times n$ Matrix Algebra	73
$\mathcal{C}\ell_{p+q}$	Complex Clifford Algebra	73
$\mathbb{R}^0_{p,q}$	Even Subalgebra of $\mathbb{R}_{p,q}$	74
$\mathbb{R}^1_{p,q}$	Set of Odd Elements of $\mathbb{R}_{p,q}$	74

$\mathbb{R}_{1,3}$	Spacetime Algebra	74
$\mathbb{R}_{3,1}$	Majorana Algebra	74
$\mathbb{R}_{4,1}$	Dirac Algebra	74
e_{pq}	Idempotent of $\mathbb{R}_{p,q}$	74
$\hat{\mathrm{A}}\mathrm{d}$	Twisted Adjoint Representation	77
Ad	Adjoint Representation	77
$\Gamma_{p,q}$	Clifford-Lipschitz Group	77
$\mathbb{R}^{\star}_{p,q}$	Invertible Elements of $\mathbb{R}_{p,q}$	77
$\Gamma^{0}_{p,q}$	Special Clifford-Lipschitz Group	78
$\mathrm{Pin}_{p,q}$	Pinor Group	78
$\mathrm{Spin}_{p,q}$	Spin Group	78
$\mathrm{Spin}_{p,q}$	Special Spin Group	78
$\mathrm{O}_{p,q}$	Pseudo Orthogonal Group of $\mathbb{R}^{p,q}$	78
$\mathrm{SO}_{p,q}$	Special Proper Pseudo Orthogonal Group of $\mathbb{R}^{p,q}$	79
$\mathrm{SO}^{e}_{p,q}$	Orthochronous Pseudo Orthogonal Group of $\mathbb{R}^{p,q}$	79
β	Takabayasi Angle	92
$\underline{\gamma}_{\mu}$	Standard Dirac γ Matrices	94
$\mathfrak{B}$	Boomerang	94
ϕ_{Ξ_u}	Representative of Dirac-Hestenes Spinor	95
$\mathbb{H}$	Quaternion Skew Field	99
$\mathbb{C}$	Complex Field	99
$\mathbb{H}$	Quaternion Skew Field	99
$\boxtimes$	Kronecker Product of Matrices	103
$\boldsymbol{x}^{i}(x), \boldsymbol{x}^{\mu}(x)$	Coordinate Functions	109
$T^{r}_{s}M$	Bundle of r-Covariant, s-Covariant Tensors	113
$\sec TM$	Section of the Tangent Bundle of M	113
$\sec T^{*}M$	Section of the Cotangent Bundle of M	113
ϕ_{*}	Derivative Mapping	113
$\phi^{*}f$	Pullback Mapping	113
$\bigwedge T^{*}M$	Cartan (Exterior) Bundle	122
TM	Tangent Bundle of M	122
$T^{*}M$	Cotangent Bundle of M	122
$\bigwedge^{r} T^{*}M$	Bundle of r-Forms	122
d	Exterior Derivative	122
$\mathbf{i}_{v}\alpha$	Interior Product of Vector and Form Fields	123
$\{\mathbf{e}_{j}\}$	Basis for Sections of $TU \subset TM$	123
$\{\theta^{i}\}$	Basis for Section of $T^{*}U \subset T^{*}M$	123
$H^{r}(M)$	r-de Rham Cohomology Group	125
$H_{r}(M)$	r Homology Group	125
$\mathbf{g}$	Metric of TM	132
$(M, \mathbf{g})$	Riemannian or Lorentzian Manifold	132
$\underset{g}{\cdot}$	Scalar Product Induced by $\boldsymbol{g}$	133
$\tau_{\mathbf{g}}$	Metric Volume Element	133

$\bigwedge(M)$	Hodge Bundle	134
$\delta, \underset{\mathbf{g}}{\delta}$	Hodge Codifferential Associated with **g**	134
$\mathcal{T}\mathbf{M}$	Tensor Bundle	144
$\star, \underset{\mathbf{g}}{\star}$	Hodge Star Operator Associated with **g**	145
$[\![\alpha, \beta]\!]$	Standard Dirac Commutator	149
$\{\alpha, \beta\}$	Standard Dirac Anticommutator	149
$\pounds_{\mathbf{v}}$	Lie Derivative	151
$\boldsymbol{\partial}\lrcorner$	See Eq. (4.170)	155
$\boldsymbol{\partial}\wedge$	See Eq. (4.170)	155
$\boldsymbol{\partial}\vee$	See Eq. (4.185)	158
$[\alpha, \beta]$	Dirac Commutator	158
$\{\!\{\alpha, \beta\}\!\}$	Dirac Anticommutator	158
Ricci	Ricci Tensor	161
$\underset{\mathbf{g}}{\diamond}$	Hodge Laplacian	164
$\square = \boldsymbol{\partial}\cdot\boldsymbol{\partial}$	Covariant D'Alembertian Associated with $\mathring{D}$ and $\mathcal{C}\updownarrow(M, \mathring{g})$	166
$\boldsymbol{\partial}\wedge\boldsymbol{\partial}$	Ricci Operator Associated with $\mathring{D}$ and $\mathcal{C}\updownarrow(M, \mathring{g})$	167
$\mathring{\mathcal{R}}^{\mu}$	Ricci 1-Forms Associated with $\mathring{D}$ and $\mathcal{C}\updownarrow(M, \mathring{g})$	167
$\mathring{\mathcal{G}}^{\mu}$	Einstein 1-Forms Associated with $\mathring{D}$ and $\mathcal{C}\updownarrow(M, \mathring{g})$	169
$\blacksquare$	Einstein Operator	169
$\boldsymbol{\partial}\cdot\boldsymbol{\partial}$	See Eq. (4.238)	170
$\boldsymbol{\partial}\wedge\boldsymbol{\partial}$	See Eq. (4.238)	170
Θ	Torsion Tensor	172
$\{\theta^{\mathbf{a}}\}$	Orthonormal Cobasis for Sections of $T^*U \subset T^*M$	172
$\mathrm{i} = \sqrt{-1}$	Imaginary Unity	177
$\mathbf{R}$	Curvature (Riemann) Tensor	179
$\boldsymbol{\nabla}^{+}$	See Eq. (4.291)	183
$\boldsymbol{\nabla}^{-}$	See Eq. (4.292)	183
$\mathring{\partial}$	See Eq. (5.6)	191
$\mathcal{S}$	Shape Biform	191
∂_{v}	Pfaff Derivative	193
$[\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}]$	Commutator of Vector Fields	193
ω^{ρ}_{β}	Connection 1-Forms	193
Θ^{ρ}	Torsion 2-Forms	194
$\mathcal{R}^{\rho}_{\mu}$	Curvature 2-Forms	196
I_{m}	Global Volume Element	204
$\mathbf{P}$	Projection Operator	204
$\mathbf{S}$	Shape Operator	205
$\mathfrak{R}$	Curvature Biform	216
$\mathbf{P}_{u}$	Covariant Derivative of the Projection Operator	222
$(\mathrm{nacs}\vert\mathbf{Q})$	Naturally Adapted Coordinate Chart with **Q**	229
$\{\boldsymbol{\epsilon}_{\mathbf{a}}\}$	Moving Orthonormal Frame Over σ	230

$\{\epsilon_{\mathbf{a}}\}$	Moving Orthonormal Coframe Over σ	232
Ω_{D}	Darboux Biform	232
$\Omega_{\mathbf{S}}$	Angular Velocity of Fermi Transported Coframe	233
$\mathcal{F}$	Fermi-Walker Connection	236
$\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$	Principal Bundle of Oriented Lorentz Tetrads	292
$P_{\mathrm{SO}^e_{1,3}}(M)$	Principal Bundle of Oriented Lorentz Cotetrads	292
$\bigwedge^k V$	Space of k-Forms	293
$\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M)$	Spin Frame Bundle	294
$P_{\mathrm{Spin}^e_{1,3}}(M)$	Spin Coframe Bundle	294
$S(M, \mathtt{g})$	Real (Left) Spinor Bundle	297
$S^\star(M, \mathtt{g})$	Dual Real Spinor Bundle	297
$\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$	Complexified Left Real Spinor Bundles	299
$\mathbb{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$	Complexified Right Real Spinor Bundles	299
$\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$	Left Real Spinor Bundles	300
$\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$	Right Real Spinor Bundle	300
$\mathring{\mathbf{g}}$	Another (Fiducial) Metric for TM	303
$\mathtt{g}$	Metric for T^*M	303
$\mathbf{1}^l_i(x)$	Unity Section in $\mathcal{C}\ell^l_{Spin^e_{1,3}}(M, g)$	306
$\mathbf{1}^r_i(x)$	Unity Section in $\mathcal{C}\ell^r_{Spin^e_{1,3}}(M, g)$	306
$\boldsymbol{\nabla}^s_V$	Spinor Covariant Derivative	307
$\boldsymbol{\partial}^s$	Spin-Dirac Operator	315
$\boldsymbol{\nabla}^{(s)}_V$	Representative of $\boldsymbol{\nabla}^s_V$ in $\mathcal{C}\ell(M, \mathtt{g})$	318
$\boldsymbol{\partial}^{(s)}$	Representative of $\boldsymbol{\partial}^s$ in $\mathcal{C}\ell(M, \mathtt{g})$	318
$\boldsymbol{\partial}_x$	See Eq. (8.15)	335
$\mathbf{J}_{\circledast}(\mathcal{C}\ell(M, \eta))$	Clifford Jet Bundle in Minkowski Spacetime	335
$\mathcal{L}(x, Y, \boldsymbol{\partial}_x \circledast Y)$	Lagrangian Density	336
$\mathfrak{L}(x, Y, \boldsymbol{\partial}_x \circledast Y)$	Lagrangian	336
$\boldsymbol{\delta}_v$	Vertical Variation	336
$\boldsymbol{\delta}_h$	Horizontal Variation	337
$\boldsymbol{\delta}$	Total Variation	337
$T^{\mathbf{a}} = -\mathcal{T}^{\mathbf{a}}$	Energy-Momentum 1-Forms	342
$T(n)$	Canonical Energy-Momentum Extensor	342
$\mathbf{J}_X$	Canonical Angular Momentum Extensor	349
$\mathbf{L}^\dagger_X$	Orbital Angular Momentum Extensor	349
$\mathbf{S}^\dagger_X$	Spin Extensor	352
$J^1[(\bigwedge T^*M)^{n+2}]$	See Eq. (9.2)	360
$\mathcal{L}_m$	Lagrangian Mapping for Matter Fields	360
$\star\boldsymbol{\Sigma}(\phi)$	Euler-Lagrange Functional	363
$\mathcal{L}_g$	Gravitational Lagrangian Density	374
$\star t^{\mathbf{c}}, \star\mathcal{S}^{\mathbf{c}}$	See Eq. (9.84)	375
$\mathcal{L}_{EH}$	Einstein-Hilbert Lagrangian Density	386
$\mathring{D}$	Levi-Civita Covariant Derivative of $\mathring{\mathbf{g}}$	407

$\mathfrak{M}$	Lorentzian Spacetime	430
$\mathcal{M}$	Minkowski Spacetime	430
∇	General Covariant Derivative	432
$\mathbf{D}$	Exterior Covariant Differential of Indexed Forms	432
$\mathbf{d}^E$	Exterior Covariant Differential Operator	433
$\mathscr{D}$	Fake Exterior Covariant Differential	438
D	Levi-Civita Covariant Derivative of $\mathbf{g}$	441
$\mathbf{R}_{\mu\nu}$	Curvature Bivector	443
$\mathring{\boldsymbol{\partial}}$	Dirac Operator Associated with $\mathring{D}$ and $\mathcal{C}\ell(M, \mathring{\mathtt{g}})$	452
τ_L	Degree of Linear Polarization	457
τ_C	Degree of Circular Polarization	457
Π	Hertz Potential	470
S	Stratton Potential	471
$\mathsf{T}_{\mathbf{a}}$	Energy Momentum 1-vector Fields	501
$\boldsymbol{\partial}$	Dirac Operator Associated with ∇ and $\mathcal{C}\ell(M, \mathtt{g})$	514
(E, M, π, G, F)	Fiber Bundle	546
$F(M)$	Frame Bundle	548
$\mathring{\mathtt{g}}$	Metric Field for T^*M	550
$\{\mathbf{e_i}\}$	Orthonormal Basis for Sections of $TU \subset TM$	550
$\mathcal{C}\ell(M, \mathtt{g})$	Clifford Bundle of $(M, \mathtt{g})$	550
$\sim^r_x$	Equivalence Relation on a Jet Bundle	551
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