

Quantum Thermodynamics: A Case Study for Emergent Behavior

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Abstract. Any attempt to provide a foundation of thermodynamics faces this central question: how come that a qualitatively different type of behavior emerges (as an effective description) from the underlying physical substrate?

Quantum thermodynamics is able to show that the partitioning of a closed quantum system into a smaller and a significantly larger part typically gives rise to thermal properties of the former, even though the system as a whole continues to exhibit unitary motion. Being based on entanglement, this feature may show up already in rather small total quantum systems, the dynamics of which can still be solved exactly. Furthermore, it allows for nano-thermodynamics, an entirely self-contradictory concept in the classical regime.

This picture differs substantially from the classical (statistical) description: It is not the system as such, which is thermal; rather it is made thermal by its environment. Thermal behavior is thus “apparent” only, i.e. dependent on the way the observer chooses to look. A much closer look would make the thermal properties disappear – just like a portrait will become unrecognizable after focusing on individual pixels. However, that very type of “looking” has to be included as part of the detailed modeling. Operational quantum thermodynamics establishes an intuitive link between the new quantum and the old classical description.

Keywords: Partitions, entanglement, typicality, Hilbert-space statistics, thermal attractor, quantum measurement

1 Introduction

1.1 Concepts

Compared to synergetics as a truly interdisciplinary field [1], quantum thermodynamics focusses on a much more restrictive theme – though fundamental in its own right.

Despite these obvious differences in scale a common set of concepts can be recognized to enter both fields: This set includes, in particular, hierarchical aspects (slow/fast variables, macro/micro-descriptions, relevant subspaces), interrelated

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with the extensive use of effective descriptions (coarse-graining, emergence, universality, control parameters).

It has long since been realized that physical phenomena are not just “out there” but depend on the appropriate level of description. Simplifications (based on relevance concepts) and models (declaring certain constraints as given) are the tools for intuitive understanding. These effective theories [2] try to capture what is relevant in a given physical domain.

Effective theories are “closed”, i.e., they do not require the explicit input of more detailed descriptions, except for the fit of some phenomenological parameters. This “closure” is quite remarkable and further underlines the usefulness of approaches on different hierarchical levels.

“Brute force” numerical studies may establish valuable tests and help to find convincing effective descriptions, but cannot be a substitute for “qualitative modeling”. In this context one may recall a comment by Picasso: “Art is a lie that helps us see the truth” [3].

1.2 Operationality

Information, quantum dynamics, and thermodynamics are the basic input for the study of quantum thermodynamic processes. Remarkably, these three fields are particularly suitable for an operational approach. In fact, the concept of information can hardly be introduced without reference to communicating agents. Observational quantum mechanics with its focus on measurement and statistical features differs substantially from “abstract” (isolated) quantum mechanics. Thermodynamics can be seen as a control theory, its main laws as statements about what can be done by agents equipped with limited resources. And operations/descriptions eventually provide links between the three fields.

1.3 Contextuality

A common thread running through both synergetics and quantum thermodynamics is contextuality.

In quantum physics contextuality has a rather dramatic significance: The Kochen-Specker theorem [4] demonstrates that we arrive at a paradox (within the rules of quantum mechanics), if we assume that all possible experimentally answerable questions that can be asked about a given quantum system have definite yes/no answers. As a consequence the answer to a certain question depends on the context (of other questions having been posed).

A somewhat different type of contextuality is related to the choice of partitions. While also operational in nature, this choice has nothing to do with actual measurement results as such, but rather concerns the mere possibility for selecting global or local questions, respectively. A bi-partite scenario underlies the emergence of local thermal behavior [5].

2 Hilbert-space

The state space of quantum physics is a complex vector-space, the so-called Hilbert-space \mathcal{H} (of dimension d). Each quantum system $\mu, \mu = A, B, \dots$ contributes its own Hilbert-space, $\mathcal{H}(\mu)$. A vector (“pure state”) in such a space is conveniently written in Dirac-notation as $\psi(\mu) \rightarrow |\psi(\mu)\rangle$ with the complex conjugate $\psi^*(\mu) \rightarrow \langle \psi(\mu)|$. The vectors are taken to be normalized, i.e., $\langle \psi(\mu)|\psi(\mu)\rangle = 1$. The Hilbert-space can be spanned by a set of orthogonal basis vectors, $|f^{(j)}(\mu)\rangle, j = 1, 2, \dots, d(\mu)$. In order to specify such a set (for Hilbert-space μ) consider the eigen-equation of some properly chosen operator $\hat{F}(\mu)$ with

$$\hat{F}(\mu)|f^{(j)}(\mu)\rangle = F_j(\mu)|f^{(j)}(\mu)\rangle . \quad (1)$$

This is still entirely abstract; we postpone the specification of model classes we have in mind.

2.1 Partitioning

A classical particle is described by $d = 6$ coordinates, 3 position and 3 momentum coordinates. For a composite system (N particles) the respective space is a direct sum, the so-called Γ -space. Its dimension is

$$D = \sum_{\mu=1}^N d(\mu) = 6N . \quad (2)$$

It scales linearly with N .

Definition 1 (Tensor space). *In quantum mechanics each subsystem μ brings its own Hilbert-space of dimension $d(\mu)$. For a composite system the total Hilbert-space is thus a direct product (or tensor-) space of dimension*

$$D = \prod_{\mu=1}^N d(\mu) . \quad (3)$$

For $d(\mu) = d$ one finds $D = d^N$, i.e., the total dimension scales exponentially with N . Typical Hilbert-spaces become incredibly huge indeed.

Definition 2 (Product representation). *A convenient set of basis vectors for a composite system are product states. For $N = 2$ and if $\hat{F}(A)$ and $\hat{F}(B)$ are complete operators within their respective subspace, A, B , the completeness relation reads*

$$\hat{1} = \hat{1}(A) \otimes \hat{1}(B) = \sum_{i=1}^{d(A)} \sum_{j=1}^{d(B)} |f^{(i)}(A), f^{(j)}(B)\rangle \langle f^{(i)}(A), f^{(j)}(B)| . \quad (4)$$

Any pure state for such a bi-partite system can thus be represented in terms of those product-states, $|f^{(i)}(A) \rangle \otimes |f^{(j)}(B) \rangle = |f^{(i)}(A), f^{(j)}(B) \rangle$, as

$$|\psi(A, B) \rangle = \sum_{i=1}^{d(A)} \sum_{j=1}^{d(B)} |f^{(i)}(A), f^{(j)}(B) \rangle \psi_{ij} ,$$

$$\psi_{ij} = \langle f^{(i)}(A), f^{(j)}(B) | \psi(A, B) \rangle . \quad (5)$$

This is easily generalized to $N = 3, 4, \dots$

The opposite to composition is partitioning:

Definition 3 (“Virtual” partitioning). Consider a total Hilbert-space of dimension d . If d is a prime number, the system is necessarily elementary. Otherwise it has a non-trivial prime-factorization[8]

$$d = \prod_{j=1}^r q_j^{n_j} . \quad (6)$$

The number of such elementary subsystems, each of some prime-factor dimension q_j , would be given by

$$N_d = \sum_{j=1}^r n_j . \quad (7)$$

Definition 4 (Operationally accessible partitioning). For a concrete system \hat{H} living in such a product-Hilbert-space of dimension d only a fraction of those elementary subsystems will be “real”, i.e. correspond to operationally well-defined and accessible subunits. Its number, $N \leq N_d$, and type is not an absolute property of the system but contextual. It defines a kind of reference frame. Each such unit is characterized by a (classical) index, $\nu = A, B, C, \dots$. The index serves as a kind of address.

For example, a Hilbert-space with $N_d = 3$ may be described as a single system, $N = 1$, (ABC) , as a bi-partite system, $N=2$, $(A)(BC)$ or $(AB)(C)$ or as $N = N_d$, $(A)(B)(C)$. This is more than a formal aspect. While all these decompositions are formally equivalent, i.e., span the same total space, they give rise to different phenomena in terms of inter-subsystem correlations (entanglement). Different decompositions also lead to different notations for states: $|k(ABC) \rangle$ or $|k(A); m(BC) \rangle$ or $|k(A); m(B); n(C) \rangle$, where the parameters k, m, n specify the respective (sub-) states.

2.2 Typicality

We first note that the study of thermodynamic behavior is intimately related to typicality [6]: We are not so much concerned with specific models and specific states, but rather with typical models and typical states. Indeed, we know that thermal properties are not guaranteed; nevertheless, we want to argue that

under given (rather weak) constraints thermal behavior will be “likely”. As a consequence, we do not need to run tests for each case in question (which would often be very hard to do) but instead base our judgement on mere probability of being correct (accepting occasional failures).

So what are “typical properties” of pure states? Underlying is subjective ignorance about the actual state, which is conveniently represented by an ensemble. Typical – in agreement with its colloquial meaning – are then properties, which occur for “almost all” ensemble members. In order to quantify such a behavior one needs to define a pure-state distribution function (cf. [5]). A convenient way to proceed is to introduce a specific parametrization for the pure states in question, with respect to a given basis set.

Definition 5 (Parametrization of pure states). *Any pure state $|\phi\rangle$ in a given d -dimensional Hilbert-space (for the moment we suppress index μ) can be written in that basis $|f^{(j)}\rangle$ as*

$$|\phi\rangle = \sum_{j=1}^d (\eta_j + i\xi_j) |f^{(j)}\rangle. \quad (8)$$

Here, η_j, ξ_j are $2d$ real parameters. The normalization condition reads

$$G(\eta_1, \xi_1, \eta_2, \xi_2, \dots, \eta_d, \xi_d) = \sum_{j=1}^d (\eta_j^2 + \xi_j^2) = 1. \quad (9)$$

The corresponding pure state distribution does not relate to a concrete physical scenario like a thermal equilibrium state. There should be no bias, except symmetry (see below).

Definition 6 (Pure-state distribution w). *The distribution function parametrized as $w(\eta_1, \xi_1, \dots, \eta_d, \xi_d)$ may now be considered a “prior” in the sense of Bayesian statistics. A simple way to arrive at this distribution is to apply the maximum entropy principle (MEP) subject to the single macro-constraint (norm)*

$$\overline{G} = \int d^d \eta d^d \xi w(\eta_1, \xi_1, \dots, \eta_d, \xi_d) \sum_{j=1}^d (\eta_j^2 + \xi_j^2) = \text{const}. \quad (10)$$

This procedure immediately leads to the Gaussian [7]

$$w(\eta_1, \xi_1, \dots, \eta_d, \xi_d) = \left(\frac{1}{\sqrt{\pi\gamma}} \right)^{2d} \exp(-G(\eta_1, \xi_1, \dots, \eta_d, \xi_d)/\gamma) \quad (11)$$

We note that this distribution fulfills the normalization condition on average only. This deficiency can be cured in the asymptotic limit for large d , if we set $\gamma = 1/d$. In this limit the fluctuations of G become negligible.

Alternatively, the above result can be obtained by requiring the distribution function $w(\eta_1, \xi_2, \dots, \eta_d, \xi_d)$ to be invariant under unitary transformations [5]. The choice of the underlying representation is arbitrary should thus not matter.

According to eq.(11) an unbiased ensemble of pure states is characterized by the distribution function $w(\eta_1, \xi_1, \dots, \eta_d, \xi_d)$. This function allows us to define Hilbert-space averages of any function $h(|\phi \rangle) = h(\eta_1, \xi_1, \dots, \eta_d, \xi_d)$ as an integral over the $2d$ - dimensional parameter-space:

$$\bar{h} = \int w(\eta_1, \xi_1, \dots, \eta_d, \xi_d) h(\eta_1, \xi_1, \dots, \eta_d, \xi_d) d^d \eta d^d \xi . \tag{12}$$

Likewise one can define higher moments. Recall that this Hilbert-space statistics reflects subjective ignorance (an entirely classical concept); it has nothing to do with the statistical features of quantum mechanics proper.

3 Hilbert-space average method (HAM)

3.1 Landscapes

Proposition 1 (Hilbert-space average method (HAM)).

The function $h(|\phi \rangle)$ can be visualized as a “landscape” over its $2d$ -dimensional parameter-space. We can approximate

$$\boxed{h(|\phi \rangle) = h(\eta_1, \xi_1, \dots, \eta_d, \xi_d) \approx \bar{h}} \tag{13}$$

provided that landscape of h is “flat” enough.

In particular let us consider some operator \hat{Y} acting on the Hilbert-space \mathcal{H} . Its eigen-representation reads

$$\hat{Y} = \sum_k Y_k |y^{(k)} \rangle \langle y^{(k)}| . \tag{14}$$

We are free to take these eigenfunctions as the basis for the ensemble of pure states $|\phi \rangle$ and their distribution w ,

$$|\phi \rangle = \sum_{j=1}^d (\eta_j + i\xi_j) |y^{(j)} \rangle . \tag{15}$$

The expectation value of \hat{Y} with respect to $|\phi \rangle$ is an example for $h(|\phi \rangle)$; its Hilbert-space average is

$$\bar{h} = \overline{\langle \phi | \hat{Y} | \phi \rangle} = \overline{\langle Y \rangle} = \sum_j Y_j (\overline{\eta_j^2} + \overline{\xi_j^2}) = \frac{1}{d} \text{Tr}\{\hat{Y}\} . \tag{16}$$

In the last step we have used that by symmetry and normalization $\overline{\eta_j^2} = \overline{\xi_j^2} = 1/(2d)$. Correspondingly one finds for

$$\overline{\langle Y \rangle^2} = \frac{1}{d(d+1)} \left(\text{Tr}\{\hat{Y}^2\} + \text{Tr}\{\hat{Y}\}^2 \right) \tag{17}$$

(for details see [5]), which allows us to introduce the Hilbert-space variance

$$\Delta^2(\langle Y \rangle) = \frac{1}{d+1} \left(\text{Tr}\{\hat{Y}^2\}/d - \text{Tr}\{\hat{Y}\}^2/d^2 \right). \quad (18)$$

HAM is thus justified for \hat{Y} , if

$$\Delta(\langle Y \rangle) \ll \overline{\langle Y \rangle}. \quad (19)$$

Definition 7 (Accessible region (AR)). *Unless the system is non-integrable, the Hilbert-space accessible from some initially prepared state may severely be constrained by various constants of motion. The system will never be able to leave the respective subspace.*

While the distinction between integrable and non-integrable models is often considered essential for the resulting statistical behavior, it is of minor concern here. We simply have to make sure that certain conditions of subsystem dimensions and weak inter-subsystem couplings are obeyed. The arguments in terms of HAM then go through.

3.2 Embedded system $A(B)$: thermalizing environment

We now turn to a bi-partite quantum system $A(B)$ with Hilbert-space dimension $d(A, B) = d_A d_B$. Here A is the embedded system of interest; to stress this functional asymmetry only B is written in parenthesis. Thermalizing environments B are the prime target of quantum thermodynamics: Here one is concerned with the question under which conditions an embedding quantum system is able to impart thermal properties on the embedded system A .

A simple argument can be based on Hilbert-space statistics (cf. [5]): Here we consider a closed bi-partite system $A(B)$ under energy-exchange; the subsystem A is taken to be a spin. We apply HAM for $\hat{Y} \rightarrow \hat{H}$, and accept that the Hamiltonian \hat{H} is not complete: degeneracies will show up.

Example 1 (Equilibrium via energy exchange). We take system A to be a two-level system with energy-splitting ΔE and B to be a multi-level system with states $|j, k(B)\rangle$, where k describes the degeneracies, $k = 1, 2, \dots, g_j$. (The degeneracy structure will turn out to be essential!) We single out two energy bands, $j = 0$ and $j+1 = 1$, and assume weak (resonant) interaction between A and B , $E_1 - E_0 = \Delta E$. The total energy of the system (A, B) is taken to be ΔE . Energy conservation then induces a correlation between allowed states in A and B , i.e., not all $d(A, B)$ states of the product-space are accessible. The accessible region AR can be projected out via the following two projectors,

$$\begin{aligned} \hat{P}_0 &= |0(A)\rangle\langle 0(A)| \otimes \sum_{k=1}^{g_1} |1, k(B)\rangle\langle 1, k(B)|, \\ \hat{P}_1 &= |1(A)\rangle\langle 1(A)| \otimes \sum_{m=1}^{g_0} |0, m(B)\rangle\langle 0, m(B)|. \end{aligned} \quad (20)$$

The accessible Hilbert-space has dimension $d_{acc}(A, B) = g_0 + g_1$. The respective occupation probabilities $\langle P_0 \rangle$, $\langle P_1 \rangle$, averaged according to HAM, eq.(16), are then found to be

$$\overline{\langle P_0 \rangle} = \frac{\text{Tr}\{\hat{P}_0\}}{d(B)} = \frac{g_1}{g_0 + g_1}, \quad (21)$$

$$\overline{\langle P_1 \rangle} = \frac{\text{Tr}\{\hat{P}_1\}}{d(B)} = \frac{g_0}{g_0 + g_1}. \quad (22)$$

The relative variance, $\Delta(\langle P_\mu \rangle)/\overline{\langle P_\mu \rangle}$ scales with $1/d_{acc}(B)$.

For $d_{acc}(B) \gg 1$ HAM is thus justified (i.e. the above properties are typical) and the expected equilibrium state of A reflects the degeneracy structure of the embedding subsystem B .

Definition 8 (Embedding temperature). *Even though the embedding system B is not a bath (i.e., is not in a stationary thermal state) the smaller subsystem A may be said to be in a thermal state with temperature $T(A)$,*

$$\frac{\Delta E}{k_B T(A)} = \ln \left(\frac{\overline{\langle P_0 \rangle}}{\overline{\langle P_1 \rangle}} \right) = \ln \left(\frac{g_1}{g_0} \right). \quad (23)$$

This effective temperature will be called “embedding temperature”, as it derives from the degeneracy structure of the quantum environment B with respect to the transition energy ΔE selected by A . In general, a different subsystem A (different transition energy) will have a different embedding temperature even within the same B .

On the other hand, different embeddings B will lead to different temperatures $T(A)$ for given A . In particular, for $g_0 = g_1$ one finds

$$\overline{\langle P_0 \rangle} = \overline{\langle P_1 \rangle} \approx 0,5. \quad (24)$$

so that $T(A) = \infty$. For $g_0 = 1 \ll g_1$

$$\overline{\langle P_0 \rangle} \approx 1, \quad \overline{\langle P_1 \rangle} \approx 0. \quad (25)$$

This would imply $T(A) = 0$, apparently giving a simple route to zero temperature.

Remark 1 (Embedding temperature and third law). For the weak coupling condition to be valid in the limit $T(A) \rightarrow 0$, the interaction energy must be small compared to the energy of subsystem A , which approaches zero. As the thermal relaxation time depends on the interaction strength, this essentially means that the time to reach thermal equilibrium would go to infinity. In this sense the third law is saved: There is no efficient process to reach the absolute zero of temperature.

Remark 2 (Negative temperature). Negative embedding temperatures obtain for $g_1 < g_0$. Such a degeneracy structure can be realized by means of a cluster of non-interacting (identical) spins in the high energy regime. For $g_1 \rightarrow 1 \ll g_0$ one would get $T(A) \rightarrow -0$ (total inversion).

For such special Hamilton models (with bounded spectrum) embedding temperatures can be positive or negative: they both characterize possible equilibrium states.

Proposition 2 (Embedding temperature versus bath temperature).

Consider a more general environment B with discrete energy-spectrum $\{E_j, j = 1, 2, \dots\}$ and respective degeneracies g_j . Furthermore, let this system be in a thermal state with partition sum $Z(B)$ and temperature $\beta(B) = 1/k_B T(B)$,

$$\hat{\rho}_{equ}(B) = \frac{1}{Z(B)} \sum_j \exp(-\beta(B)E_j) \sum_{k=1}^{g_j} |j, k(B)\rangle \langle j, k(B)|. \quad (26)$$

Then consider a two-level system A in its excited state $|1(A)\rangle$, which is weakly coupled to B . What will now be the effect of B on A ?

Energy conservation implies that the initial pure state $|1(A); j, m(B)\rangle$ with $1 \leq m \leq g_j$ would preferably be coupled to $|0(A); j+1, k(B)\rangle$ with $1 \leq k \leq g_{j+1}$, so that – under HAM – the equilibrium state is expected to be

$$\overline{\langle P_0 \rangle_j} = \frac{\text{Tr}\{\hat{P}_0\}}{d_j(C)} = \frac{g_{j+1}}{g_j + g_{j+1}}, \quad (27)$$

$$\overline{\langle P_1 \rangle_j} = \frac{\text{Tr}\{\hat{P}_1\}}{d_j(C)} = \frac{g_j}{g_j + g_{j+1}}. \quad (28)$$

But in addition to taking the quantum expectation value and to apply the HAM we have now, in a third step, to perform a thermal averaging over the environmental states j :

$$\hat{\rho}_{equ}(A) = \frac{1}{Z(B)} \sum_j g_j \exp(-\beta(B)E_j) \times \{\overline{\langle P_0 \rangle_j} |0(A)\rangle \langle 0(A)| + \overline{\langle P_1 \rangle_j} |1(A)\rangle \langle 1(A)|\}. \quad (29)$$

In general, this equation lacks a clear interpretation. For more transparent results specific models are needed about the environmental spectrum E_j and the degeneracy g_j .

For the following we assume for the environment B an equidistant spectrum,

$$E_j = j\Delta E. \quad (30)$$

All transitions between adjacent levels are thus resonant with the two-level system A . The following examples concern the degeneracy of B .

Example 2 (Exponential degeneracy). We assume

$$g_j = \exp(\gamma j \Delta E), \tag{31}$$

γ is a real positive parameter. Then

$$\overline{\langle P_0 \rangle} = \frac{1}{1 + \exp(-\gamma \Delta E)}, \tag{32}$$

$$\overline{\langle P_1 \rangle} = \frac{1}{1 + \exp(\gamma \Delta E)}, \tag{33}$$

independent of j . The resulting temperature is $\Delta E/k_B T(A) = \gamma \neq \beta(B)$.

Example 3 (Binomial degeneracy).

$$g_j = \binom{n}{j} = \frac{n!}{(n-j)!j!} \tag{34}$$

Here, n is an integer > 0 ; for an intuitive realization we could think of an environment consisting of n non-interacting spins with identical energy splitting ΔE . One then finds,

$$\overline{\langle P_0 \rangle}_j = \frac{n-j}{n+1}, \tag{35}$$

$$\overline{\langle P_1 \rangle}_j = \frac{j+1}{n+1}. \tag{36}$$

Here we have made use of the identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}. \tag{37}$$

After thermal averaging over the various bands j and with the partition sum

$$Z(B) = \sum_{j=0}^n \binom{n}{j} \exp(-\beta(B)j \Delta E) = (1 + \exp(-\beta(B)\Delta E))^n \tag{38}$$

one gets in the thermodynamic limit, $n \rightarrow \infty$,

$$\beta(A) \approx \beta(B). \tag{39}$$

In this case there is no conflict between embedding temperature and the bath temperature.

Both degeneracy models are idealized; deviations from the strict exponential form appear to be mandatory, though.

4 Observational quantum-thermodynamics

Quantum-thermodynamic settings do not include the observer: No means whatsoever are provided to transfer information to the outside world. In that sense the scenarios are similar to those studied in abstract quantum theory, i.e. based on isolated systems. It thus appears quite natural to add a section on observational quantum thermodynamics, just as we would have to add a section on observational quantum mechanics. In either case observation requires additional physical interactions: The act of observation tends to influence the observed.

In the following we will be concerned with a thermal system under permanent (“stroboscopic”) supervision [9].

4.1 Periodic measurements

We will be concerned with the following bi-partite system

$$\hat{H}(AB) = \frac{\delta(A)}{2} \hat{\sigma}_3(A) + \hat{H}_0(B) + \hat{V}(BB) + \lambda \hat{V}(AB). \quad (40)$$

The environment B consists of $n \gg 1$ spins. The corresponding product states

$$|\psi(B)\rangle = |m(1), m(2), \dots, m(n)\rangle \quad m(\mu) = \mp 1, \quad (41)$$

are eigenfunctions to $\hat{H}_0(B)$ with eigenvalues $E_k(B) = \delta(B)k$, where k denotes the number of spins in state $m = 1$. For given band-index k all the states, $|n_k\rangle, n_k = 1, 2, \dots, g_k$, have the same energy. Now focus on some index k_0 , the “working point” in energy-space. For $n > k_0 \gg 1$ the binomial degeneracies (cf. Example 3) can be approximated by

$$g_k \approx g_0 \exp(\beta(k_0)E_k), \quad (42)$$

$$\beta(k_0) \approx \frac{1}{\delta(B)} \ln\left(\frac{n}{k_0} - 1\right). \quad (43)$$

Due to the weak interaction $\hat{V}(BB)$, each degenerate energy-level k becomes a band of width $\Delta\epsilon_k \ll \delta(B)$.

The interaction between subsystem A and B is scaled by the strength factor λ and given by

$$\hat{V}(AB) = \hat{\sigma}_1(A) \otimes \hat{I}(AB), \quad (44)$$

$$\hat{I}(AB) = \sum_k \sum_{n_k, m_{k+1}} C_{k+1, k}(n_k, m_{k+1}) |n_k\rangle \langle m_{k+1}| + c.c. \quad (45)$$

Here c.c. means complex conjugate term to be added. Only transitions between next neighbor bands are allowed. The coupling parameters $C_{k+1, k}(n_k, m_{k+1})$ form a set of hermitian matrices $C_{i, j}(a, b)$. The respective entries are taken from a Gaussian distribution normalized to $\overline{|C_{i, j}(a, b)|^2} = \sqrt{(g_i g_j)}$.

Note that this random-matrix model (for the subsystem-interaction) is an essential ingredient: in this way the Hamilton-model represents a whole class

of models rather than one given variant. Indeed, one can show that different realizations typically lead to the same desired behavior, eq.(50). (This does not exclude the existence of very special interaction types which would violate this simple picture.)

The total state will be written as

$$\hat{\rho}(AB) = \hat{\rho}(A) \otimes \hat{\rho}(B) - \hat{C}(AB) , \quad (46)$$

$$\hat{\rho}(A) = \text{Tr}_B\{\hat{\rho}(AB)\} \quad \hat{\rho}(B) = \text{Tr}_A\{\hat{\rho}(AB)\} . \quad (47)$$

Here $\hat{C}(AB)$ specifies the correlation between A and B, i.e., the deviation from product-form. Let the initial state at time t_0 have zero correlation; in particular, we start from the special product-state

$$\hat{\rho}(AB; t_0) = \hat{\rho}(A; t_0) \otimes \hat{\rho}(B; k_0) , \quad (48)$$

$$\hat{\rho}(B; k_0) \equiv \frac{1}{g_{k_0}} \sum_{n_{k_0}=1}^{g_{k_0}} |n_{k_0} \rangle \langle n_{k_0}| . \quad (49)$$

Resonance will be assumed, i.e., $\delta(B) = \delta(A) = \delta$. Then the subsystem B acts as a thermalizing environment, correlation $\hat{C}(AB)$ builds up, and A relaxes to a state with temperature

$$\frac{1}{k_B T(A)} = \beta(A) = \beta(k_0) , \quad (50)$$

independent of the initial state of A. Up to this point this is just a typical scenario for quantum thermodynamics. The relaxation time scales with the interaction strength between system and environment.

Now we intend to retrieve information about A. This could be done in two different ways: Direct measurements on A or indirect measurements via B exploiting the correlation between A and B.

Measuring the energy of subsystem A directly would mean to find it in the ground- or excited state with probability $P_m(A)$ given by the thermal distribution. Immediately after measurement the subsystem would be found in the respective energy eigenstate $|m(A) \rangle$, $m \mp 1$. This momentary state and the associated information gain would not last for long, though: With the embedding still present system A would again relax to its equilibrium state.

We switch now to the indirect measurement scenario. We restrict ourselves to the measurement of the band-index, k i.e., to an incomplete measurement. The projection by $\hat{P}_{k_1}(B)$ at time $t_1 = t_0 + \Delta t$ also influences subsystem A (co-jump, cf. [10]):

$$\hat{\rho}'(A; t_1) = \frac{\text{Tr}_B\{\hat{P}_{k_1}(B)\hat{\rho}(AB; t_1)\}}{\text{Tr}_B\{\hat{P}_{k_1}(B)\hat{\rho}(B; t_1)\}} \quad (51)$$

$$= \hat{\rho}(A; t_1) + \frac{\text{Tr}_B\{\hat{P}_{k_1}(B)\hat{C}(AB; t_1)\}}{\text{Tr}_B\{\hat{P}_{k_1}(B)\hat{\rho}(B; t_1)\}} . \quad (52)$$

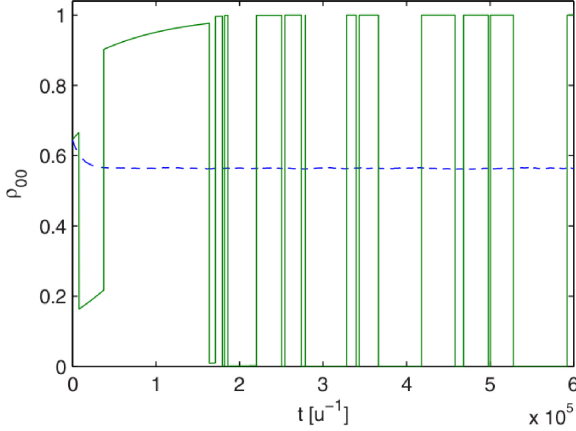


Fig. 1. Subsystem A subject to periodic measurements of the environment B [9]: single trajectory $\rho_{00}(t)$ (solid line) and ensemble average $\bar{\rho}_{00}(t)$ (broken line). Chosen parameters: $\delta = 0.8u$, $\beta = 0.75u^{-1}$, $\lambda = 4 \times 10^{-3}u$, $\Delta t = 2u^{-1}$, u is arbitrary energy unit. Initial state: unperturbed attractor.

After measurement we simplify the total state as the product state

$$\hat{\rho}(AB; t_1) \approx \hat{\rho}'(A; t_1) \otimes \hat{\rho}(B; k_1) . \quad (53)$$

This is an approximation as, due to the incomplete measurement, some correlations between A and B could still have survived. The whole process is now iterated: unitary evolution steps of duration Δt are interrupted by instantaneous measurement projections as described above. The result is a stochastic trajectory as shown in Fig. 1.

These trajectories correspond to the classical statistical idea that the thermal system is always in some well-defined state, but “fluctuates” between those such that the time-average is identical with the ensemble average. We now investigate the ensemble-average over such trajectories after a certain number of measurements (i.e. after the memory about the initial state has been lost). The result represents an attractor state, which can be expressed analytically as [9]

$$\bar{\rho}_{00}(\Delta t) = \frac{\exp(-\beta\delta/2) \sin^2 \delta\Delta t + \exp(\beta\delta/2) \delta^2(\Delta t)^2}{2 \cosh \beta\delta/2 (\sin^2 \delta\Delta t + \delta^2(\Delta t)^2)} . \quad (54)$$

The corresponding function is shown in Fig. 2.

This attractor state can alternatively be calculated as a time-average over a single trajectory; it has two remarkable bounds:

$$\lim_{\Delta t \rightarrow 0} \bar{\rho}_{00}(\Delta t) = 1/2 . \quad (55)$$

This lower bound means that for very rapid repetitions of measurements the system eventually heats up to $T(A) \rightarrow \infty$. Only apparently is this in conflict

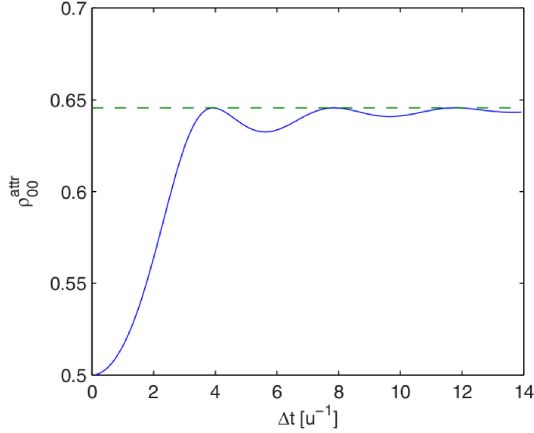


Fig. 2. Attractor for subsystem A depending on the time Δt between periodic measurements on B. Parameters as before. Broken line: upper bound and unperturbed attractor [9].

with the so-called Zeno effect, i.e. the freezing-in of the original state under “continuous” measurement: Indeed, for $\Delta t \rightarrow 0$ it would take infinitely long to reach that attractor state.

An upper bound for $\bar{\rho}_{00}$ is reached for

$$\Delta t = \frac{n\pi}{\delta} \quad n = 1, 2, \dots \quad (56)$$

This bound corresponds to $k_B T(A) = 1/\beta(k_0)$. Remarkably, this is the same temperature subsystem A would gain in the absence of any measurements. At these waiting times one finds a strict correlation between measuring the band-index k to have gone up by 1 (compared to its previous value) or down by 1 and the state of A: For “up” A is in the ground state, for “down” A is in the excited state. This is the strict “measurement logic”. It guarantees that our information retrieval “makes sense”.

For $\Delta t \rightarrow 0$, on the other hand, the correlation goes to zero: System B has no (useful) information about A. The measurements of the environment B only lead to an additional perturbation of A.

Observational quantum-thermodynamics thus provides an interesting link between the abstract thermal properties, which are stationary, and the measured thermal properties, which are fluctuating. The long-time average and the ensemble average of the latter agree with the former result, provided the measurement is run under optimal conditions, i.e. there is a clear measurement logic (in which case the information gained is “useful”). While measurement interactions mean perturbations – indeed, the observed local dynamics is stochastic, the unobserved dynamics stationary – the underlying thermal parameters turn out to be the same.

5 Summary

We have started from (abstract) Hilbert-space and used partitioning as a kind of external reference frame. This frame defines the notion of local versus global properties.

The parametrization of pure states (for the total system) allows to introduce functions on this parameter-space; examples are expectation-values of operators. These functions can be visualized as “landscapes”. Flat landscapes may well be approximated by the respective average value, the so-called Hilbert-space average (HAM). This average is then “typical”. Rarely will we find large deviations. A simple analogue would be the height within the Netherlands (average applicable) versus the height in Switzerland (average does not give a good predictor for one’s actual height).

Typicality can be extended to composite quantum systems: For bi-partite systems in a pure state the local entropy of the smaller subsystem is found to be typically maximum (confirming entanglement as a typical feature).

These features do not depend on details of the Hamilton-model considered. Such details constrain the motion in Hilbert-space: In fact, the accessible region is, in general, much different from the total region (due to conservation laws).

It turns out that for weak coupling and for the environment B being sufficiently larger than the embedded system A , the latter exhibits thermal behavior (canonical distribution).

The effective dynamics for the embedded system shows irreversibility (relaxation into an equilibrium state), even though the total dynamics continues to be unitary. This underlines the fact that the thermal properties of A are emergent and contextual; they are absent from the point of view of the unpartitioned description.

The equilibrium state is quasi-stationary (temporal fluctuations occur as finite size effects) and characterized by a temperature.

Periodic observation of the environment allows to re-interpret the equilibrium state of the embedded system in terms of a time average over measurement results. Under ideal conditions this time-average approaches the ensemble-average (ergodicity); the resulting value is the same as obtained without observation. There is a “peaceful coexistence” between the classical and the quantum picture.

This coexistence has remarkable consequences: Without it a quasi-classical understanding of thermal phenomena would not have been possible, thus blocking scientific progress in this field at a time when quantum mechanics was unknown yet. On the other hand, the discussion of “quantum corrections” proper – based on quantum thermodynamics – will not be straight-forward but will require detailed analysis based on the formulation of pertinent quantum models.

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