Chapter 8

Applications of the Feynman Calculus

Introduction

This chapter is devoted to a few applications of the Feynman operator calculus. We first consider the theory of linear evolution equations and provide a unified approach to a class of time-dependent parabolic and hyperbolic equations.

We then show that $KS^2[\mathbb{R}^3]$ allows us to construct the elementary path integral in the manner intended by Feynman. We also use the sum over paths theory of the last chapter along with time-ordering to extend the Feynman path integral to a very general setting. We then prove an extended version of the Feynman–Kac theorem. Finally, we prove the last remaining Dyson conjecture concerning the foundations for quantum electrodynamics.

8.1. Evolution Equations

As our first application, we provide a unified approach to a class of time-dependent parabolic and hyperbolic evolution equations. We restrict ourselves to first and second order initial-value problems

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$\dot{u}(t) = A(t)u(t)$, $u(0) = u_0$, or $\ddot{v}(t) = B^2(t)v(t)$, $v(0) = v_1$ and $\dot{v}(0) = v_2$. In each case, we assume that $A(t)$, $B(t)$ generates a C_0-semigroup for each $t \in I$.

For second order equations, let

$$u(t) = \begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix}, \quad u_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & I \\ B^2(t) & 0 \end{pmatrix}.$$

We now define a norm on $\mathcal{X} = \mathcal{H} \times \mathcal{H}$ by

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathcal{X}} = \|f\|_{\mathcal{H}} + \|g\|_{\mathcal{H}}.$$

This makes $\mathcal{X}$ a Hilbert space. It follows that the second order equation on $\mathcal{H}$ becomes the first order equation on $\mathcal{X}$: $\dot{u}(t) = A(t)u(t)$, $u(0) = u_0$. Thus, it suffices to study first order equations. For additional details on this approach, see Yosida [YS] or Goldstein [GS].

In order to prove existence and uniqueness for the initial-value (Cauchy) problem a number of conditions are imposed (see Pazy [PZ], in Chap. 7). The important assumption for the time-ordered theory is a weak continuity condition. (In the following, let $\mathcal{H}$ be a Hilbert space.)

8.2. Parabolic Equations

In the abstract parabolic problem, it is assumed that, on $\mathcal{H}$, the family $A(t)$, $t \in I$, satisfies:

(1) For each $t \in I$, $A(t)$ is densely defined, $R(\lambda;\ A(t))$ exists in a sector $\Sigma = \Sigma(\phi + \pi/2)$ for some ϕ, $0 < \phi < \pi/2$ and a constant ϕ independent of t, such that

$$\|R(\lambda;\ A(t))\| \leq 1/|\lambda| \ \mathit{for}\ \lambda \in \Sigma,\ t \in I.$$

(2) The function $A^{-1}(t)$ is continuously differentiable on I.

(3) There are constants $C_1 > 0$ and $\rho:\ 0 < \rho < 1$, such that, for each $\lambda \in \Sigma$ and every $t \in I$, we have

$$\|D_t R(\lambda;\ A(t))\| \leq C_1/|\lambda|^{1-\rho}.$$

(4) The function $DA^{-1}(t)$ is Holder continuous in $\mathcal{H}$ and there are positive constants $C_2,\ \alpha$ such that

$$\left\|DA^{-1}(t) - DA^{-1}(s)\right\| \leq C_2 \left|t - s\right|^{\alpha},\ s,\ t \in I.$$

The first condition states that $A(t)$ generates an analytic contraction semigroup for each $t \in I$. The four conditions are required to prove the following theorem.

Theorem 8.1. *Let the family $A(t)$, $t \in I$, have a common dense domain and satisfy assumptions (1)–(4). Then the problem*

$$\frac{\partial u(t)}{\partial t} = A(t)u(t), \quad u(a) = u_a,$$

has a unique solution $u(t) = V(t,s)u_a$, for $t,\ s \in I$. Furthermore,

(1) *$V(t,s)$ is strongly continuous on I and continuously differentiable (in the norm of $\mathcal{H}$) with respect to both s and $t \in I$,*
(2) *$V(t,s)\mathcal{H} \subset D(A(t))$,*
(3) *$A(t)V(t,s)$ and $V(t,s)A(s)$ are bounded,*
(4) *$D_tV(t,s) = A(t)V(t,s)$, $D_sV(t,s) = -\overline{V(t,s)A(s)}$, and*
(5) *for $t,\ s \in I$,*

$$\|D_tV(t,s)\| \le C/(t-s), \quad \|D_sV(t,s)\| \le C/(t-s).$$

In the proof of this result takes seven pages plus five pages of preparatory work (see page 397). (In Pazy [PZ], in Chap. 7, the proof takes 17 pages.)

Example 8.2. *Let the family of operators $A(t)$, $t \in I = [0,1]$, be defined on $\mathcal{H} = L^2(0,1)$ by:*

$$A(t)u(x) = -\frac{1}{(t-x)^2}u(x).$$

It is easy to see that each $A(t)$ is self-adjoint and $(A(t)u,u) \le -\|u\|_{\mathcal{H}}^2$ for $u \in D(A(t))$. It follows that the spectrum of $A(t)$, $\sigma(A(t)) \subset (-\infty,-1]$, for $t \in [0,1]$. The first condition is satisfied for any $\phi \in (0,\pi/2)$, while the second condition is clear and makes the fourth condition obvious. For $\lambda \notin (-\infty,-1]$, we have

$$R(\lambda;\ A(t))u(x) = \frac{(t-x)^2}{\lambda(t-x)^2+1}u(x),$$

so that

$$\|R(\lambda;\ A(t))u(x)\|_{\mathcal{H}}^2 = \int_0^1 \frac{(t-x)^4}{[\lambda(t-x)^2+1]^2}u^2(x)dx \le \frac{1}{|\lambda|^2}\|u\|_{\mathcal{H}}^2 .$$

It is now clear that each $A(t)$ generates a contraction semigroup and

$$D_t R(\lambda;\ A(t))u(x) = \frac{2(t-x)}{[\lambda(t-x)^2+1]^2}u(x).$$

From here, an easy estimation shows that, for $\lambda \in \Sigma$,

$$\|D_t R(\lambda;\ A(t))\|_{\mathcal{H}} \le \frac{C}{|\lambda|^{1/2}},$$

so that the third condition follows. The theorem would follow if there was a common dense domain. However, it is not hard to see that $\bigcap_{t\in I} D(A(t)) = \{0\}$.

We now notice that

$$(A(t)-A(s))\,A(\tau)^{-1} = \left[\frac{(\tau-x)^2}{(s-x)} + \frac{(\tau-x)^2}{(t-x)}\right](s-t),$$

so that, for some constants $C > 0,\ 0 < \beta \le 1$, we have

$$\left\|(A(t)-A(s))\,A(\tau)^{-1}\right\| \leqslant C\,|t-s|^{\beta}\ \ (a.s)\ \ \textit{for all}\, t,\ s,\ \tau \in [0,1].$$

It follows that the family $A(t),\ t \in [0,1]$, is strongly continuous and hence satisfies (7.3). Thus, the time-ordered integral exists and generates a contraction semigroup. It is now an exercise to prove that the semigroup is also analytic in the same sector, Σ.

Returning to the abstract parabolic problem, the conditions used by Pazy [PZ], in Chap. 7, make it easy to see that the $A(t),\ t \in I$, is strongly continuous in general:

(1) For each $t \in I,\ A(t)$ generates an analytic C_0-semigroup with domains $D(A(t)) = D$ independent of t.

(2) For each $t \in I,\ R(\lambda, A(t))$ exists for all λ such that $\operatorname{Re}\lambda \leqslant 0$, and there is an $M > 0$ such that:

$$\|R(\lambda, A(t))\| \leqslant M/[|\lambda|+1].$$

(3) There exist constants L and $0 < \alpha \leqslant 1$ such that

$$\left\|(A(t)-A(s))\,A(\tau)^{-1}\right\| \leqslant L\,|t-s|^{\alpha}\ \ \text{for all}\ t,\ s,\ \tau \in I.$$

From (3), for $\varphi \in D$, we have

$$\begin{aligned}\|[A(t)-A(s)]\,\varphi\| &= \left\|\left[(A(t)-A(s))\,A^{-1}(\tau)\right]A(\tau)\varphi\right\| \\ &\leqslant \left\|(A(t)-A(s))\,A^{-1}(\tau)\right\|\|A(\tau)\varphi\| \leqslant L\,|t-s|^{\alpha}\,\|A(\tau)\varphi\|.\end{aligned}$$

Thus, the family $A(t),\ t \in I$, is strongly continuous on D. For comparison with the time-ordered approach, we have:

Theorem 8.3. *Let the family $A(t)$, $t \in I$ be weakly continuous on $\mathcal{H}$ satisfying:*

(1) *For any complete orthonormal basis $\{e^i\}$, for $\mathcal{H}$ and any partition $\mathcal{P}_n$, of I with mesh μ, there is a number δ, with $0 < \delta < 1$ such that:*

$$\sum_{k=1}^{n} \Delta t_k \left\| A(s_k)e^i - \left\langle A(s_k)e^i, e^i \right\rangle e^i \right\|^2 \leqslant C\mu_n^{\delta-1} \tag{8.1}$$

(2) *For each $t \in I$, $A(t)$ generates an analytic C_0-semigroup with dense domains $D(A(t)) = D(t) \subset \mathcal{H}$.*

(3) *For each $t \in I$, $R(\lambda, A(t))$ exists for all λ such that $\operatorname{Re}\lambda \leqslant 0$, and there is an $M(t) > 0$, $t \in I$ such that:*

$$\|R(\lambda, A(t))\| \leqslant M(t)/[|\lambda| + 1],$$

with $\sup_{t\in I} M(t) < \infty$.

Then, for each $\phi \in \mathcal{H}$ the time-ordered family $\mathcal{A}(t)$, $t \in I$ has a strong Riemann integral on $D_0 = \otimes_{t\in I} D(t) \cap \mathcal{H}^2_{\otimes}(\Phi)$, which generates an analytic C_0-semigroup on $\mathcal{H}^2_{\otimes}(\Phi)$, where $\Phi = \otimes_{t\in I}\phi_t$, $\phi_t = \phi$ for all $t \in I$.

Remark 8.4. The left-hand side of Eq. (8.1) could diverge as $\mu \to 0$, but remains finite if the family $A(t)$, $t \in I$ is strongly continuous. If the family $A(t)$, $t \in I$ is not strongly continuous, Eq. (8.1) ensures that weak continuity on $\mathcal{H}$ is sufficient in order for the time-ordered family $\mathcal{A}(t)$, $t \in I$ to have a strong Riemann integral on $\mathcal{H}^2_{\otimes}(\Phi)$, for each Φ. (We do not require a common dense domain.)

8.3. Hyperbolic Equations

In the abstract approach to hyperbolic evolution equations, it is assumed that:

(1) For each $t \in I$, $A(t)$ generates a C_0-semigroup.

(2) For each $t \in I$, $A(t)$ is stable with constants $(M, 0)$ and the resolvent set $\rho(A(t)) \supset (0, \infty)$, $t \in I$, such that:

$$\left\| \prod_{j=1}^{k} \exp\{\tau_j A(t_j)\} \right\| \leqslant M.$$

(3) There exists a Hilbert space $\mathcal{Y}$ densely and continuously embedded in $\mathcal{H}$ such that, for each $t \in I$, $D(A(t)) \supset \mathcal{Y}$ and $A(t) \in L[\mathcal{Y}, \mathcal{H}]$ (i.e., $A(t)$ is bounded as a mapping from $\mathcal{Y} \to \mathcal{H}$), and the function $g(t) = \|A(t)\|_{\mathcal{Y}\to\mathcal{H}}$ is continuous.

(4) The space $\mathcal{Y}$ is an invariant subspace for each semigroup $S_t(\tau) = \exp\{\tau A(t)\}$ and $S_t(\tau)$ is a stable C_0-semigroup on $\mathcal{Y}$ with the same stability constants.

This case is not as easily analyzed as the parabolic case, so we need the following:

Lemma 8.5. *Suppose conditions (3) and (4) above are satisfied with* $\|\varphi\|_{\mathcal{H}} \leqslant \|\varphi\|_{\mathcal{Y}}$. *Then the family* $A(t)$, $t \in I$, *is strongly continuous on* $\mathcal{H}$ *(a.e.) for* $t \in I$.

Proof. Let $\varepsilon > 0$ be given and, without loss, assume that $\|\varphi\|_{\mathcal{H}} \leqslant 1$. Set $c = \|\varphi\|_{\mathcal{Y}}/\|\varphi\|_{\mathcal{H}}$, so that $1 \leqslant c < \infty$. Now

$$\begin{aligned}\|[A(t+h) - A(t)]\,\varphi\|_{\mathcal{H}} &\leqslant \left\{\|[A(t+h) - A(t)]\,\varphi\|_{\mathcal{H}}/\|\varphi\|_{\mathcal{Y}}\right\}\left[\|\varphi\|_{\mathcal{Y}}/\|\varphi\|_{\mathcal{H}}\right] \\ &\leqslant c\,\|A(t+h) - A(t)\|_{\mathcal{Y}\to\mathcal{H}}\,.\end{aligned}$$

Choose $\delta > 0$ such that $|h| < \delta$ implies $\|A(t+h) - A(t)\|_{\mathcal{Y}\to\mathcal{H}} < \varepsilon/c$, which completes the proof. □

Remark 8.6. The important point of this section is that once we know that $A(t)$ generates a semigroup for each t, the only other conditions required are that the family $\{A(t) : t \in I\}$ is weakly continuous and satisfies the growth condition (8.1). However, when the family $\{A(t) : t \in I\}$ is strongly continuous, the growth condition (8.1) is automatically satisfied.

8.4. Path Integrals I: Elementary Theory

Introduction

In this and the next section, we will obtain a general theory for path integrals in exactly the manner envisioned by Feynman. Our approach is distinct from the methods of functional integration, so we do not discuss that subject directly. However, since functional integration represents an important approach to path integrals, a few brief remarks are in order. The methods of functional differentiation and integration were major tools for the Schwinger program in quantum electrodynamics, which was developed in parallel with the Feynman theory (see [DY], in Chap. 7). Thus, these methods were not developed for the study of path integrals. However, historically, path integrals have been studied from the functional integration point of view, and many authors have sought to restrict consideration to the space of continuous functions or related function spaces in their definition of the path integral. The best known example is undoubtedly the Wiener integral [WSRM]. However, from the time-ordering point of view, such a restriction is not natural nor desirable. Thus, our approach does not depend on formulations with countably additive measures. In fact, we take the view that integration theory, as contrasted with measure theory, is the appropriate vehicle for path integrals. Indeed, as shown in [GZ1], there is a one-to-one mapping between path integrals and semigroups of operators that have a kernel representation. In this case, the semigroup operation generates the reproducing property of the kernel.

In their recent (2000) review of functional integration, Cartier and DeWitt-Morette [CDM1] discuss three of the most fruitful and important applications of functional integration to the construction of path integrals. In 1995, the *Journal of Mathematical Physics* devoted a special issue to this subject, Vol. 36, No. 5 (edited by Cartier and DeWitt-Morette). Thus, those with interest in the functional integration approach will find ample material in the above references (see also the book [CDM2]). Both the review and book are excellent on many levels, in addition to the historical information that could only come from one with first-hand information on the evolution of the subject.

8.4.1. Summary. In this section, we restrict our discussion to kernel representations for an interesting class of solutions to partial differential equations. In each case, a path integral representation is fairly straightforward.

We begin with the path integral as first introduced by Feynman [FY1]. The purpose is to show that the simplicity of his original approach becomes possible when the problem is considered on $KS^2[\mathbb{R}^3]$.

Recall that, in elementary quantum theory, the (simplest) problem to solve in $\mathbb{R}^3$ is:

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}-\frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t)=0,\ \psi(\mathbf{x},s)=\delta(\mathbf{x}-\mathbf{y}), \tag{8.2}$$

with solution

$$\psi(\mathbf{x},t) = K\left[\mathbf{x},\ \ t;\ \mathbf{y},\ \ s\right] = \left[\frac{2\pi i\hbar(t-s)}{m}\right]^{-3/2} \exp\left[\frac{im}{2\hbar}\frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)}\right].$$

In his formulation of quantum theory, Feynman wrote the solution to Eq. (8.2) as

$$K\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right] = \int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x} \mathcal{D}\mathbf{x}(\tau) \exp\left\{\frac{im}{2\hbar}\int_s^t \ \left|\frac{d\mathbf{x}}{dt}\right|^2 d\tau\right\}, \tag{8.3}$$

where

$$\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x} \mathcal{D}\mathbf{x}(\tau) \exp\left\{\frac{im}{2\hbar}\int_s^t \left|\frac{d\mathbf{x}}{dt}\right|^2 d\tau\right\} =: \\ \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\varepsilon(N)}\right]^{3N/2} \int_{\mathbb{R}^3} \prod_{j=1}^{N} d\mathbf{x}_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\varepsilon(N)}\left(\mathbf{x}_j-\mathbf{x}_{j-1}\right)^2\right]\right\}, \tag{8.4}$$

with $\varepsilon(N) = (t-s)/N$.

Equation (8.4) represents an attempt to define an integral on the space of continuous paths with values in $\mathbb{R}^3$ (i.e., $\mathbb{C}\left([s,\ t]:\ \mathbb{R}^3\right)$). This approach has a number of well-known mathematical problems:

- The kernel $K\left[\mathbf{x},\ \ t;\ \mathbf{y},\ \ s\right]$ and $\delta(\mathbf{x})$ are not in $L^2[\mathbb{R}^3]$, the standard space for quantum theory.
- The kernel $K\left[\mathbf{x},\ \ t;\ \mathbf{y},\ \ s\right]$ cannot be used to define a measure.

Notwithstanding these problems, the physics community has continued to make progress using this integral and have consistently obtained correct answers, which have been verified whenever independent (rigorous) methods were possible.

In response, the mathematics community has developed a large variety of indirect methods to justify the integral. The recent book by Johnson and Lapidus [JL] discusses many important contributions from the literature.

If we want to retain the approach used by Feynman, the problems identified above must be faced directly. Thus, the natural question is: Does there exist a separable Hilbert space containing $\mathbf{K}\left[\mathbf{x},\ t;\ \mathbf{y},\ s\right]$ and $\delta(\mathbf{x})$? A positive answer is required if the limit in Eq. (8.4) is to make sense. If we also want a space that allows us to include the Feynman, Heisenberg, and Schrödinger representations, we must require that the convolution and Fourier transform exist on the space as bounded linear operators. This requirement is necessary, since the convolution operator is needed for the path integral and the position and momentum operators, $\mathbf{x}, \mathbf{p}$, are canonically conjugate variables (i.e., Fourier transform pairs).

8.4.2. Background. The properties of $KS^2[\mathbb{R}^n]$ derived in Chap. 3 suggest that it is a perfect choice for the Feynman formulation. It is easy to see that both the position and momentum operators have closed, densely defined extensions to $KS^2[\mathbb{R}^n]$. A full theory requires that the Fourier transform, $\mathfrak{F}$, and the convolution operator $\mathfrak{C}$ (as bounded linear operators) have extensions $KS^2[\mathbb{R}^n]$ in order to ensure that both the Schrödinger and Heisenberg theories have faithful representations on $KS^2[\mathbb{R}^n]$. For this, we restate Theorem 5.15 as it applies to $KS^2[\mathbb{R}^n]$.

Theorem 8.7. *Let A be a bounded linear operator on a Banach space $\mathcal{B} \subset KS^2$. If $\mathcal{B}' \subset KS^2$, then A has a bounded extension to $L[KS^2]$, with $\|A\|_{KS^2} \leq k\,\|A\|_{\mathcal{B}}$ with k constant.*

We can now use Theorem 8.7 to prove that $\mathfrak{F}$ and $\mathfrak{C}$, the Fourier (transform) operator and the convolution operator respectively, defined on $L^1[\mathbb{R}^n]$, have bounded extensions to $KS^2[\mathbb{R}^n]$.

Theorem 8.8. *Both $\mathfrak{F}$ and $\mathfrak{C}$ extend to bounded linear operators on $KS^2[\mathbb{R}^n]$.*

Proof. To prove our result, first note that $C_0[\mathbb{R}^n]$, the bounded continuous functions on $\mathbf{R}^n$ which vanish at infinity, is contained in $KS^2[\mathbb{R}^n]$. Now $\mathfrak{F}$ is a bounded linear operator from $L^1[\mathbb{R}^n]$ to $C_0[\mathbb{R}^n]$, so we can consider it as a bounded linear operator from $L^1[\mathbb{R}^n]$ to $KS^2[\mathbb{R}^n]$. Since $\mathbf{L}^1[\mathbf{R}^n]$ is dense in $KS^2[\mathbb{R}^n]$ and $L^\infty[\mathbf{R}^n] \subset KS^2[\mathbb{R}^n]$, by Theorem 8.7, $\mathfrak{F}$ extends to a bounded linear operator on $KS^2[\mathbb{R}^n]$.

To prove that $\mathfrak{C}$ has a bounded extension, fix g in $L^1[\mathbb{R}^n]$ and define $\mathfrak{C}_g$ on $L^1[\mathbb{R}^n]$ by:

$$\mathfrak{C}_g(f)(\mathbf{x}) = \int g(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Once again, since $\mathfrak{C}_g$ is bounded on $L^1[\mathbb{R}^n]$ and $L^1[\mathbb{R}^n]$ is dense in $KS^2[\mathbb{R}^n]$, by Theorem 8.7 it extends to a bounded linear operator on $KS^2[\mathbb{R}^n]$. Now use the fact that convolution is commutative to get that $\mathfrak{C}_f$ is a bounded linear operator on $L^1[\mathbb{R}^n]$ for all $f \in KS^2[\mathbb{R}^n]$. Another application of Theorem 8.7 completes the proof. $\square$

We now return to $\mathfrak{M}[\mathbb{R}^n]$, the space of bounded finitely additive measures on $\mathbb{R}^n$, that are absolutely continuous with respect to Lebesgue measure.

Definition 8.9. A uniformly bounded sequence $\{\mu_k\} \subset \mathfrak{M}[\mathbb{R}^n]$ is said to converge weakly to μ ($\mu_n \xrightarrow{w} \mu$), if, for every bounded uniformly continuous function $h(\mathbf{x})$,

$$\int_{\mathbb{R}^n} h(\mathbf{x}) d\mu_n \to \int_{\mathbb{R}^n} h(\mathbf{x}) d\mu.$$

Theorem 8.10. *If $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbb{R}^n]$, then $\mu_n \xrightarrow{s} \mu$ (strongly) in $KS^p[\mathbb{R}^n]$.*

Proof. Since the characteristic function of a closed cube is a bounded uniformly continuous function, $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbb{R}^n]$ implies that

$$\int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) d\mu_n \to \int_{\mathbb{R}^n} \mathcal{E}_m(\mathbf{x}) d\mu$$

for each m, so that $\lim_{n\to\infty} \|\mu_n - \mu\|_{KS^p} = 0$. □

A little reflection gives:

Theorem 8.11. *The space $KS^2[\mathbb{R}^n]$ is a commutative Banach algebra with unit.*

Since $KS^2[\mathbb{R}^n]$ contains the space of measures, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $KS^2[\mathbb{R}^n]$ topology. For example, $[\sin(\lambda \cdot \mathbf{x})/(\lambda \cdot \mathbf{x})] \in KS^2[\mathbb{R}^n]$ and converges strongly to $\delta(\mathbf{x})$. On the other hand, the function $e^{-2\pi i \mathbf{z}(\mathbf{x}-\mathbf{y})} \in KS^2[\mathbb{R}^n]$, so we can define the delta function directly:

$$\delta(\mathbf{x}-\mathbf{y}) = \int_{\mathbb{R}^n} e^{-2\pi i \mathbf{z}(\mathbf{x}-\mathbf{y})} d\lambda_n(\mathbf{z}),$$

as an HK-integral.

It is easy to see that the Feynman kernel [FH], defined by (with $m = 1$ and $\hbar = 1$):

$$\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B] = \int_B (2\pi i(t-s))^{-n/2} \exp\{i|\mathbf{x}-\mathbf{y}|^2 \big/ 2(t-s)\} d\mathbf{y},$$

is in $KS^2[\mathbb{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B]\|_{\mathrm{KS}} \leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B]\|_{\mathfrak{M}} = \infty$ (the variation norm). Furthermore,

$$\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B] = \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, \tau, d\mathbf{z}] \mathbb{K}_{\mathbf{f}}[\tau, \mathbf{z}\,;\, s, B], \text{ (HK-integral).}$$

Remark 8.12. It is not hard to show that $\mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, B]$ generates a finitely additive set function defined on the algebra of sets B, such that $\mathcal{E}_B(|\mathbf{y}|)$ is of bounded variation in the $|\mathbf{y}|$ variable.

Definition 8.13. Let $\mathbf{P}_k = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_k, t_k\}$ be a HK-partition of the interval $[0, t]$ for each k, with $\lim_{k\to\infty} \mu_k = 0$ (mesh). Set $\Delta t_j = t_j - t_{j-1}, \tau_0 = 0$ and, for $\psi \in KS^2[\mathbb{R}^n]$, define

$$\int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{0})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)] = e^{-\lambda t} \sum_{k=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j)\,;\, t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\begin{aligned} \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{0})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)]\psi[\mathbf{x}(0)] &= \lim_{\lambda\to\infty} \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{0})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda \mathbf{x}(\tau)]\psi[\mathbf{x}(0)] \\ &= \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{k=0}^{[\![\lambda t]\!]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{f}}[t_j, \mathbf{x}(\tau_j)\,;\, t_{j-1}, d\mathbf{x}(\tau_{j-1})]\psi[\mathbf{x}(0)] \right\}, \end{aligned} \tag{8.5}$$

whenever the limit exists.

It is easy to see that the limit exists in $KS^2[\mathbb{R}^n]$, whenever we have a reproducing kernel.

Theorem 8.14. *The function* $\psi(\mathbf{x}) \equiv 1 \in KS^2[\mathbb{R}^n]$ *and*

$$\int_{\mathbf{x}(\tau)=\mathbf{y}(\mathbf{s})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)]\psi[\mathbf{y}(s)] = \mathbb{K}_{\mathbf{f}}[t, \mathbf{x}\,;\, s, \mathbf{y}] = \tfrac{1}{\sqrt{[2\pi i(t-s)]^n}} \exp\{i|\mathbf{x}-\mathbf{y}|^2/2(t-s)\}.$$

The above result is what Feynman was trying to obtain (in this simple case).

8.5. Examples and Extensions

In this section, we provide a few interesting examples. Those with broader interest should consult the references below.

Independent of the mathematical theory, the practical development and use of path integral methods has proceeded at a continuous rate. At this time, it would be impossible to give a survey of the many different types of path integrals and the problems that they have been used to solve. It would be a separate task to provide a reasonable set of references on the subject. However, the following books are suggested for both the material they cover and the references contained in them: Albeverio and H∅egh-Krohn [AH], Cartier and Dewitt-Morette [CDM2], Feynman and Hibbs [FH], Grosche and Steiner [GS], and Kleniert [KL].

8.5.1. The Diffusion Problem. For our first example, let $\mathcal{H} = L^2[\mathbb{R}^3, d\mu]$, where $d\mu = e^{-\pi|\mathbf{x}|^2} d\lambda_3(\mathbf{x})$. The form is nonstandard, but has advantages as discussed in Chap. 2. Consider the problem:

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) - \mathbf{x} \cdot \nabla \mathbf{u}(\mathbf{x}, t), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

This is the Ornstein–Uhlenbeck equation, with solution $(T(t)u_0)(\mathbf{x}) = u(t, \mathbf{x})$, where:

$$(T(t)\mathbf{u}_0)(\mathbf{x}) = \frac{1}{\sqrt{[(1 - e^{-t})]^3}} \int_{\mathbb{R}^3} exp\left\{-\pi \frac{\left(e^{-t/2}\mathbf{x} - \mathbf{y}\right)^2}{(1 - e^{-t})}\right\} \mathbf{u}_0(\mathbf{y}) d\lambda_3(\mathbf{y}).$$

The operator $T(t)$ is a (analytic) contraction semigroup, with generator $D^2 = \Delta - \mathbf{x} \cdot \nabla$. It follows that the kernel is given by

$$\mathbb{K}_{\mathbf{f}}\left[t, \mathbf{x}\,; 0, d\mathbf{y}\right] = \frac{1}{\sqrt{[(1 - e^{-t})]^3}} \exp\left\{-\pi \frac{\left(e^{-t/2}\mathbf{x} - \mathbf{y}\right)^2}{(1 - e^{-t})}\right\} d\lambda_3(\mathbf{y}).$$

By Theorem 8.7 $T(t)$ can be extended to $KS^2(\mathbb{R}^3)$, as a C_0-contraction semigroup. It now follows that

$$u(t, \mathbf{x}) = \int_{\mathbf{x}(\tau)=\mathbf{y}(\mathbf{0})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}\left[\mathcal{D}\mathbf{x}(\tau)\right] u[\mathbf{y}(\mathbf{0})].$$

For a more interesting example, let B be a nondegenerate 3×3 matrix with eigenvalues λ such that $Re(\lambda) < 0$, with Q strictly positive definite and set

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds.$$

In this case, $\mathcal{H} = L^2[\mathbb{R}^3, d\mu]$, with

$$\mu(d\mathbf{x}) = \frac{1}{\sqrt{(\det Q_\infty}} \exp\left\{-\pi \left\langle Q_\infty^{-1}\mathbf{x}, \mathbf{x}\right\rangle\right\} d\lambda_3(\mathbf{x}),$$

and we consider the problem:

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}) - B\mathbf{x} \cdot \nabla \mathbf{u}(\mathbf{x}, t), \quad u(0, \mathbf{x}) = u_0(\mathbf{x}).$$

This is also a version of the Ornstein–Uhlenbeck equation. (However, since we don't assume that B is symmetric, $A = \Delta - B\mathbf{x} \cdot \nabla$ need not be self-adjoint.) The explicit solution is generated by the contraction semigroup $(T(t)$, where:

$$(T(t)u_0(\mathbf{x}) = \frac{1}{\sqrt{\det Q_t}} \int_{\mathbb{R}^3} \exp\left\{-\pi \left\langle Q_t^{-1}\left(e^{tB}\mathbf{x} - \mathbf{y}\right), e^{tB}\mathbf{x} - \mathbf{y}\right\rangle\right\} u_0(\mathbf{y}) d\lambda_3(\mathbf{y}).$$

It follows that

$$\mathbb{K}_{\mathbf{f}}\left[t, \mathbf{x}\,;\, 0, d\mathbf{y}\right] = \frac{1}{\sqrt{\det Q_t}} \exp\left\{-\pi \left\langle Q_t^{-1}\left(e^{tB}\mathbf{x} - \mathbf{y}\right), e^{tB}\mathbf{x} - \mathbf{y}\right\rangle\right\} d\lambda_3(\mathbf{y}).$$

For this equation, we can also replace $\mathbb{R}^3$ by a separable Hilbert space $\mathcal{H}$ and λ_3 by cylindrical Gaussian measure μ. In this case, B is a symmetric bounded linear operator with spectrum $\sigma(B) < 0$ and $0 < Q_\infty < \infty$ is strictly positive definite. Those with interest in this subject can consult Lorenzi and Bertoldi [LB] for the finite-dimensional case and De Prato [DP] for Hilbert space. In either case, the path integral representation is defined on $KS^2[\mathbb{R}^3]$ or $KS^2[\mathcal{H}]$.

8.5.2. Wave Equation. For this example, write the standard wave equation as

$$\frac{\partial^2\psi}{\partial t^2} - \mathbf{I}c^2\Delta\psi = \frac{1}{\hbar^2}\left[i\hbar\frac{\partial}{\partial t} + \beta\sqrt{-c^2\hbar^2\Delta}\right]\left[-i\hbar\frac{\partial}{\partial t} + \beta\sqrt{-c^2\hbar^2\Delta}\right]\psi = 0.$$

In electromagnetic theory, we only see the wave equation on the left and assume that $\mathbf{I} = 1$. On the right, the β matrix can be of any finite order. Thus, the above equation introduces a rather interesting relationship between quantum theory and the classical wave equation, namely the massless square root equation for any spin. In order to solve this equation, we follow Lieb and Loss [LL], in Chap. 3, and use imaginary time to get:

$$\psi(\mathbf{x}, t) = \frac{it\beta}{\pi^2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{\psi_0(\mathbf{y})d\mathbf{y}}{\left[\left|\mathbf{x}-\mathbf{y}\right|^2 - t^2\right]^2 + \varepsilon^2} = U(t)\psi_0(\mathbf{x}), \qquad (8.6)$$

where $\psi_0(\mathbf{x})$ is the given initial data at time $t = 0$. The convergence factor is necessary for the integral representation because of the light cone problem (in the Lebesgue sense). This is not necessary if we interpret it in the Henstock–Kurzweil sense. We could also compute the solution directly by using the fact that the square root operator is a self-adjoint generator of a unitary group. However, extra work would still be required to obtain the integral representation.

We can now use (8.6) to provide a new representation for the solution of the wave equation. Assume that $\psi(\mathbf{x}, t)\,|_{t=0} = \psi_0(\mathbf{x})$ and $\dot{\psi}(\mathbf{x}, t)\,|_{t=0} = \dot{\psi}_0(\mathbf{x})$ are given (smooth) initial data. Let $A = \beta\sqrt{-c^2\hbar^2\Delta}$ and $\varphi(\mathbf{x}, t) = (-i\hbar\partial_t + A)\,\psi(\mathbf{x}, t)$. It follows from this that

$$\varphi(\mathbf{x}, 0) = \varphi_0 = i\hbar\dot{\psi}_0(\mathbf{x}) + A\psi_0(\mathbf{x}).$$

We must now solve:

$$(i\hbar\partial_t + A)\,\varphi(\mathbf{x},t) = 0,\;\; \varphi(\mathbf{x},0) = \varphi_0.$$

The solution to this problem is easily seen to be (8.6), with t replaced by $-t$, so that $\varphi(\mathbf{x},t) = U(-t)\varphi_0$. Using this result, we can now get our new representation. The solution to the wave equation has been reduced to solving:

$$(-i\hbar\partial_t + A)\,\psi(\mathbf{x},t) = U(-t)\varphi_0.$$

Using the method of variation of constants, we have: (see Sell and You [SY], p. 7).

$$\psi(\mathbf{x},t) = U(t)\psi_0 + \int_0^t U(t-s)U(-s)\varphi_0(\mathbf{x})ds.$$

Combining terms, we have:

$$\psi(\mathbf{x},t) = U(t)\psi_0 + \int_0^t U(t-2s)\varphi_0(\mathbf{x})ds. \tag{8.7}$$

It is now easy to check that $\psi(\mathbf{x},0) = \psi_0(\mathbf{x})$ and that $\dot{\psi}(\mathbf{x},0) = \dot{\psi}_0(\mathbf{x})$. We can now use Eq. (8.6) to obtain the explicit representation for a general solution to the wave equation:

$$\begin{aligned}\psi(\mathbf{x},t) &= -\frac{it\beta}{\pi^2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{\psi_0(\mathbf{y})d\mathbf{y}}{\left[|\mathbf{x}-\mathbf{y}|^2 - t^2\right]^2 + \varepsilon^2} \\ &+ \int_0^t\left\{\frac{i(t-2s)\beta}{\pi^2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{\varphi_0(\mathbf{y})d\mathbf{y}}{\left[|\mathbf{x}-\mathbf{y}|^2 - (t-2s)^2\right]^2 + \varepsilon^2}\right\}ds.\end{aligned} \tag{8.8}$$

We have only worked in $\mathbb{R}^3$. For n arbitrary, the only change (other than initial data) is the kernel. In the general case, we must replace Eq. (8.6) by

$$\psi(\mathbf{x},t) = \frac{it\beta\Gamma\left[\frac{n+1}{2}\right]}{\pi^{(n+1)/2}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\frac{\psi_0(\mathbf{y})d\mathbf{y}}{\left[|\mathbf{x}-\mathbf{y}|^2 - t^2\right]^{\frac{(n+1)}{2}} + \varepsilon^2}.$$

Thus, the method is quite general. Recall that the standard approach is based on the method of spherical means (see Evans [EV]). This approach requires different computations depending on the dimension (even or odd). It follows that our approach has some advantages. The path integral representation is straightforward.

8.5.3. The Square-Root Klein–Gordon Equation. The fourth example is taken from [GZ4] and provides another example that is not directly related to a Gaussian kernel. It is shown that if the vector potential $\mathbf{A}$ is constant, $\mu = mc/\hbar$, and $\boldsymbol{\beta}$ is the standard beta matrix, $(I, O:\ O, -I)$, then the solution to the equation for a spin 1/2 particle in square-root form,

$$i\hbar\partial\psi(\mathbf{x},t)/\partial t = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p}-\tfrac{e}{c}\mathbf{A}\right)^2 + m^2c^4}\right\}\psi(\mathbf{x},t),\ \ \psi(\mathbf{x},0)=\psi_0(\mathbf{x}),$$

is given by:

$$\psi(\mathbf{x},t) = \mathbf{U}[t,0]\psi_0(\mathbf{x}) = \int\limits_{\mathbf{R}^3} \exp\left\{\frac{ie}{2\hbar c}(\mathbf{x}-\mathbf{y})\cdot\mathbf{A}\right\}\mathbb{K}_{\mathbf{f}}\left[\mathbf{x},t\,;\,\mathbf{y},0\right]\psi_0(\mathbf{y})d\mathbf{y},$$

where

$$\mathbb{K}_{\mathbf{f}}\left[\mathbf{x},t\,;\,\mathbf{y},0\right] = \frac{ct\mu^2\beta}{4\pi}\begin{cases} \dfrac{-H_2^{(1)}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]}{\left[c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right]}, & ct < -\left\|\mathbf{x}-\mathbf{y}\right\|, \\ \dfrac{-2iK_2\left[\mu\left(\|\mathbf{x}-\mathbf{y}\|^2-c^2t^2\right)^{1/2}\right]}{\pi\left[\|\mathbf{x}-\mathbf{y}\|^2-c^2t^2\right]}, & c\left|t\right| < \left\|\mathbf{x}-\mathbf{y}\right\|, \\ \dfrac{H_2^{(2)}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]}{\left[c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right]}, & ct > \left\|\mathbf{x}-\mathbf{y}\right\|. \end{cases}$$

The function $K_2(\,\cdot\,)$ is a modified Bessel function of the third kind of second order, while $H_2^{(1)}$, $H_2^{(2)}$ are the Hankel functions (see Gradshteyn and Ryzhik [GRRZ]). Thus, we have a kernel that is far from standard. To our knowledge, this representation is new.

8.5.4. Semigroups, Kernels, and Pseudodifferential Operators. In this section, we investigate the general question of the existence of relations of the form:

$$U(t)\phi(\mathbf{x}) = \int_{\mathbb{R}^{\mathbf{n}}} \mathbf{K}(\mathbf{x},\mathbf{t}:\mathbf{y},\mathbf{0})\phi_{\mathbf{0}}(\mathbf{y})\mathbf{dy}, \tag{8.9}$$

between a semigroup of operators $U(t)$, $t \in \mathbb{R}$ and a kernel K. We observe that if a kernel exists, then the semigroup property automatically induces the reproducing property of the kernel and vice versa. Equation (8.9) also leads to a discussion of the close relationship between kernels and the theory of pseudodifferential operators. In this section we show how to associate a reproducing kernel with a large class of semigroups $U(t)$. A more detail discussion of pseudodifferential operators can be found in Treves [TR], Kumano-go [KG], Taylor [TA], Cordes [CO], or Shubin [SHB].

Pseudodifferential operators are a natural extension of linear partial differential operators and interest in them grew out of the study of singular integral operators like the one induce by the square-root operator. The basic idea is that the use of pseudodifferential operators allows one to convert the theory of partial differential equations into an algebraic theory for the characteristic polynomials, or symbols, of the differential operators by means of Fourier transforms.

We begin our study with the definition of hypoelliptic pseudodifferential operators of class $S^m_{\rho,\delta}$ and investigate their basic properties. As noted above, we confine our discussion to Euclidean spaces, $\mathbb{R}^n$, and only consider those parts that pertain to the construction of kernel representations. (Readers interested in more general treatments can consult the cited references.)

Definition 8.15. Recall that a complex-valued function f defined on $\mathbb{R}^n$ is a Schwartz function ($f \in S(\mathbb{R}^n)$ or S) if, for all multi-indices α and β, there exist positive constants $C_{\alpha,\beta}$ such that

$$\sup_{\mathbf{x}\in\mathbb{R}^n} \left|\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})\right| = C_{\alpha,\beta} < \infty.$$

In what follows, $\mathbb{R}^n_{\mathbf{x}}$ denotes n-dimensional space in the $\mathbf{x}$ variable. For continuity with the literature, we keep the standard notation, where one works on the tangent space of a differential manifold.

Definition 8.16. Let $p(\mathbf{x},\eta)$ be a C^∞ function on $\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}^n_\eta$.

(1) We say that $p(\mathbf{x},\eta)$ is a symbol of class $S^m_{\rho,\delta}$ ($n \in \mathbb{N},\ 0 \le \delta \le \rho \le 1,\ \delta < 1$) if, for any multi-indices $\alpha,\ \beta$, there exists a constant $C_{\alpha,\beta}$ such that

$$\left|p^{(\alpha)}_{(\beta)}(\mathbf{x},\eta)\right| = C_{\alpha,\beta}\, \langle\eta\rangle^{m+\delta|\beta|-\rho|\alpha|},$$

where

$$p^{(\alpha)}_{(\beta)}(\mathbf{x},\eta) = \partial^\alpha_\eta D^\beta_{\mathbf{x}} p(\mathbf{x},\eta),\ \ \langle\eta\rangle = \sqrt{1+\|\eta\|^2},\ \ |\alpha| = \sum\nolimits_{i=1}^n \alpha_i,$$

$$\partial^\alpha_\eta = \partial^{\alpha_1}_{\eta_1}\cdots\partial^{\alpha_n}_{\eta_n},\ \ D^\beta_{\mathbf{x}} = D^{\beta_1}_{x_1}\cdots D^{\beta_n}_{x_n} \text{ and } D_{x_j} = -i\frac{\partial}{\partial x_j}.$$

Also, we set

$$S^{-\infty}_{\rho,\delta} = \bigcap_{m=1}^{\infty} S^m_{\rho,\delta} \text{ and } S^{\infty}_{\rho,\delta} = \bigcup_{m=1}^{\infty} S^m_{\rho,\delta}.$$

(2) A linear operator $P : S(\mathbb{R}^n_{\mathbf{x}}) \to S(\mathbb{R}^n_{\mathbf{x}})$ is said to be a pseudodifferential operator with symbol $p(\mathbf{x}, \eta)$ of class $S^m_{\rho,\delta}$ if, for $u \in S(\mathbb{R}^n_{\mathbf{x}})$, we can write $Pu(\mathbf{x})$ as

$$Pu(\mathbf{x}) = \int_{\mathbb{R}^n} e^{\pi i \mathbf{x}\cdot\eta} p(\mathbf{x}, \eta)\hat{u}(\eta)d\eta,$$

where

$$\hat{u}(\eta) = \mathfrak{F}(u)(\eta) = \int_{\mathbb{R}^n} e^{-\pi i \mathbf{x}\cdot\eta} u(\mathbf{x})d\mathbf{x}$$

is the Fourier transform of $u(\mathbf{x})$.

Whenever $m \leq m'$, $\rho' \leq \rho$, $\delta \leq \delta'$, for any ρ and δ, we have $S^m (= S^m_{1,0}) \subset S^m_{\rho,\delta} \subset S^{m'}_{\rho',\delta'}$. It follows that

$$\bigcap_{m=1}^{\infty} S^m_{\rho,\delta} = \bigcap_{m=1}^{\infty} S^m_{1,0},$$

so that $S^{-\infty} = \bigcap_{m=1}^{\infty} S^m_{\rho,\delta}$. For $p(\mathbf{x}, \eta) \in S^m_{\rho,\delta}$ we define the family of seminorms $|p|_l^{(m)}$, $l = 0, 1, \ldots$ by

$$|p|_l^{(m)} = \max_{|\alpha+\beta|=l} \sup_{\mathbb{R}^n_{\mathbf{x}}\times\mathbb{R}^n_{\eta}} \left\{ \left| p^{(\alpha)}_{(\beta)}(\mathbf{x}, \eta) \right| \langle \eta \rangle^{(m+\delta|\beta|-\rho|\alpha|)} \right\}.$$

Then $S^m_{\rho,\delta}$ is a Fréchet space with these seminorms, and we have, for any $p(\mathbf{x}, \eta) \in S^m_{\rho,\delta}$:

$$\left| p^{(\alpha)}_{(\beta)}(\mathbf{x}, \eta) \right| \leqslant |p|^{(m)}_{|\alpha+\beta|} \langle \eta \rangle^{(m+\delta|\beta|-\rho|\alpha|)}.$$

We say that a set $B \subset S^m_{\rho,\delta}$ is bounded in $S^m_{\rho,\delta}$ if $\sup_{p\in B} \left\{ |p|_l^{(m)} \right\} < \infty$.

For $p(\mathbf{x}, \eta) \in S^m_{\rho,\delta}$ we can represent $Pu(\mathbf{x})$, $u \in S(\mathbb{R}^n)$, in terms of oscillatory integrals. These are integrals of the form:

$$Af(\mathbf{x}) = \int_{\mathbb{R}^n} e^{\pi i s(\mathbf{x},\eta)} a(\mathbf{x}, \eta)\hat{f}(\eta)d\eta,$$

where $s(\mathbf{x}, \eta)$ is called the phase function and $a(\mathbf{x}, \eta)$ is called the amplitude function. These functions were first introduced by Lax [LX1] and used to construct asymptotic solutions of hyperbolic differential equations. (In the hands of Hörmander [HO], this later led to the (related) theory of Fourier integral operators.)

We are interested in a restricted class of these integrals.

Definition 8.17. We say that a C^∞ function $a(\zeta, \mathbf{y})$, $(\zeta, \mathbf{y}) \in \mathbb{R}^{2n}_{\zeta,\mathbf{y}} = \mathbb{R}^n_\zeta \times \mathbb{R}^n_{\mathbf{y}}$ belongs to the class $\mathfrak{A}^m_{\delta,\tau}$ $(m \in \mathbb{N},\ 0 \le \delta < 1,\ 0 \le \tau)$ if, for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|\partial^\alpha_\zeta \partial^\beta_{\mathbf{y}} a(\zeta, \mathbf{y})\right| \leqslant C_{\alpha,\beta} \langle\zeta\rangle^{(m+\delta|\beta|)} \langle\mathbf{y}\rangle^\tau, \quad l = |\alpha + \beta|.$$

We set

$$\mathfrak{A} = \bigcup_{0\leqslant\delta<1} \bigcup_{m=-\infty}^{\infty} \bigcup_{0\leqslant\tau} \mathfrak{A}^m_{\delta,\tau}.$$

Then we have

Theorem 8.18. *For $a(\zeta, \mathbf{y}) \in \mathfrak{A}^m_{\delta,\tau}$, we define the seminorms $|a|_l$, $l = 0, 1, \ldots,$ by*

$$|a|_l = \max_{|\alpha+\beta|\leqslant l} \sup_{\zeta,\mathbf{y}} \left\{\left|\partial^\alpha_\zeta \partial^\beta_{\mathbf{y}} a(\zeta, \mathbf{y})\right| \langle\zeta\rangle^{-(m+\delta|\beta|)} \langle\mathbf{y}\rangle^\tau\right\}.$$

(1) *Then $\mathfrak{A}^m_{\delta,\tau}$ is a Frechet space and for $a(\zeta, \mathbf{y}) \in \mathfrak{A}^m_{\delta,\tau}$ we have*

$$\left|\partial^\alpha_\zeta \partial^\beta_{\mathbf{y}} a(\zeta, \mathbf{y})\right| \leqslant |a|_l \langle\zeta\rangle^{(m+\delta|\beta|)} \langle\mathbf{y}\rangle^\tau, \quad l = |\alpha + \beta|.$$

(2) *If $a, a_1, a_2 \in \mathfrak{A}$, then $\partial^\alpha_\eta \partial^\beta_{\mathbf{y}} a$, $a_1 + a_2$, $a_1 a_2 \in \mathfrak{A}$.*

Definition 8.19. We say that $B \subset \mathfrak{A}$ is a bounded subset of $\mathfrak{A}$ if there exists $\mathfrak{A}^m_{\delta,\tau}$ such that

$$B \subset \mathfrak{A}^m_{\delta,\tau} \quad \text{and} \quad \sup_{a\in B} \{|a|_l\} < \infty$$

for $l \in \{0\} \cup \mathbb{N}$.

Definition 8.20. For $a(\zeta, \mathbf{y}) \in \mathfrak{A}$, we define the oscillatory integral $O_s(e^{-\pi i \mathbf{y}\dot{\eta}} a) =: O_s$ by

$$O_s = \lim_{\varepsilon\to 0} \iint_{\mathbb{R}^{2n}} e^{-\pi i \mathbf{y}\cdot\eta} \chi(\varepsilon\eta, \varepsilon\mathbf{y}) a(\eta, \mathbf{y}) d\mathbf{y} d\eta,$$

where $\chi(\eta, \mathbf{y}) \in S(\mathbb{R}^{2n}_{\eta,\mathbf{y}})$ and $\chi(0, 0) = 1$.

It is shown in Kumano-go ([KG], p. 47) that O_s is well defined and independent of the choice of $\chi(\eta, \mathbf{y}) \in S(\mathbb{R}^{2n}_{\eta,\mathbf{y}})$ satisfying $\chi(0, 0) = 1$. We note that when $a(\eta, \mathbf{y}) \in L^1(\mathbb{R}^{2n}_{\eta,\mathbf{y}})$, the Lebesgue dominated convergence theorem shows that O_s coincides with the Lebesgue integral $\iint e^{-\pi i \mathbf{y}\cdot\eta} a(\eta, \mathbf{y}) d\mathbf{y} d\eta$.

A fundamental question is: under what general conditions can we expect a given (time-independent) generator of a semigroup to have an associated kernel? Here, we discuss a class of general conditions for unitary groups. It will be clear that the results of this section carry over to semigroups with minor changes.

Let $A(\mathbf{x},\ \mathbf{p})$ denote a $k \times k$ matrix operator $[A_{ij}(\mathbf{x},\ \mathbf{p})]$, $i,\ j = 1, 2, \cdots, k$, whose components are pseudodifferential operators with symbols $a_{ij}(\mathbf{x},\ \boldsymbol{\eta}) \in \mathbb{C}^{\infty}(\mathbf{R^n} \times \mathbf{R^n})$, and we have, for any multi-indices α and β,

$$\left| a_{ij(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta}) \right| \leqslant C_{\alpha\beta}(1 + |\boldsymbol{\eta}|)^{m - \xi|\alpha| + \delta|\beta|}, \tag{8.10}$$

where

$$a_{ij(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta}) = \partial^{\alpha} \mathbf{p}^{\beta} a_{ij}(\mathbf{x}, \boldsymbol{\eta}),$$

with $\partial_l = \partial/\partial\eta_l$, and $p_l = (1/i)(\partial/\partial x_l)$. The multi-indices are defined in the usual manner by $\alpha = (\alpha_1, \cdots, \alpha_n)$ for integers $\alpha_j \geq 0$, and $|\alpha| = \sum_{j=1}^{n} \alpha_j$, with similar definitions for β. The notation for derivatives is $\partial^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $\mathbf{p}^{\beta} = p_1^{\beta_1} \cdots p_n^{\beta_n}$. Here, m, β, and δ are real numbers satisfying $0 \leq \delta < \xi$. Equation (8.10) states that each $a_{ij}(\mathbf{x},\ \boldsymbol{\eta})$ belongs to the symbol class $S_{\xi,\delta}^{m}$ (see [SH]).

Let $a(\mathbf{x}, \boldsymbol{\eta}) = [a_{ij}(\mathbf{x}, \boldsymbol{\eta})]$ be the matrix-valued symbol for $A(\mathbf{x}, \boldsymbol{\eta})$, and let $\lambda_1(\mathbf{x}, \boldsymbol{\eta}) \cdots \lambda_k(\mathbf{x}, \boldsymbol{\eta})$ be its eigenvalues. If $|\cdot|$ is the norm in the space of $k \times k$ matrices, we assume that the following conditions are satisfied by $a(\mathbf{x}, \boldsymbol{\eta})$. For $0 < c_0 < |\boldsymbol{\eta}|$ and $\mathbf{x} \in \mathbb{R}^n$ we have

(1) $\left| a_{(\beta)}^{(\alpha)}(\mathbf{x}, \boldsymbol{\eta}) \right| \leq C_{\alpha\beta} \left| a(\mathbf{x}, \boldsymbol{\eta}) \right| (1 + |\boldsymbol{\eta}|)^{-\xi|\alpha| + \delta|\beta|}$ (hypoellipticity),

(2) $\lambda_0(\mathbf{x}, \boldsymbol{\eta}) = \max_{1 \leqslant j \leqslant k} \operatorname{Re} \lambda_j(\mathbf{x}, \boldsymbol{\eta}) < 0$,

(3) $\frac{|a(\mathbf{x}, \boldsymbol{\eta})|}{|\lambda_0(\mathbf{x}, \boldsymbol{\eta})|} = O\left[(1 + |\boldsymbol{\eta}|)^{(\xi - \delta)/(2k - \varepsilon)}\right],\ \varepsilon > 0.$

We assume that $A(\mathbf{x}, \mathbf{p})$ is a self-adjoint generator of an unitary group $U(t, 0)$, so that

$$U(t, 0)\psi_0(\mathbf{x}) = exp[(i/\hbar)tA(\mathbf{x}, \mathbf{p})]\psi_0(\mathbf{x}) = \psi(\mathbf{x}, t)$$

solves the Cauchy problem

$$(i/\hbar)\partial\psi(\mathbf{x}, t)/\partial t = A(\mathbf{x}, \mathbf{p})\psi(\mathbf{x}, t), \quad \psi(\mathbf{x}, t) = \psi_0(\mathbf{x}). \tag{8.11}$$

Definition 8.21. We say that $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ is a symbol for the Cauchy problem (8.11) if $\psi(\mathbf{x}, t)$ has a representation of the form

$$\psi(\mathbf{x}, t) = \int_{\mathbb{R}^n} e^{\pi i(\mathbf{x},\boldsymbol{\eta})} Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)\hat{\psi}_0(\boldsymbol{\eta}) d\boldsymbol{\eta}. \tag{8.12}$$

It is sufficient that ψ_0 belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, which is contained in the domain of $A(\mathbf{x}, \mathbf{p})$, in order that (8.12) makes sense.

Following Shishmarev [SH], and using the theory of Fourier integral operators, we can define an operator-valued kernel for $U(t, 0)$ by

$$K(\mathbf{x}, t\,; \mathbf{y}, 0) = \int_{\mathbb{R}^n} e^{\pi i(\mathbf{x}-\mathbf{y},\boldsymbol{\eta})} Q(\mathbf{x}, t, \boldsymbol{\eta}, 0) d\boldsymbol{\eta},$$

so that

$$\psi(\mathbf{x}, t) = U(t, 0)\psi_0(\mathbf{x}) = \int_{\mathbb{R}^n} K(\mathbf{x}, t\,; \mathbf{y}, 0)\psi_0(\mathbf{y}) d\mathbf{y}. \tag{8.13}$$

The following results are due to Shishmarev [SH].

Theorem 8.22. *If $A(\mathbf{x}, \mathbf{p})$ is a self-adjoint generator of a strongly continuous unitary group with domain D, $\mathcal{S}(\mathbb{R}^n) \subset D$ in $L^2(\mathbb{R}^n)$, such that conditions (1)–(3) are satisfied, then there exists precisely one symbol $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ for the Cauchy problem (8.11).*

Theorem 8.23. *If we replace condition (3) in Theorem 8.22 by the stronger condition*

$$(3') \quad \frac{|a(\mathbf{x},\boldsymbol{\eta})|}{|\lambda_0(\mathbf{x},\boldsymbol{\eta})|} = O\left[(1 + |\boldsymbol{\eta}|)^{(\xi-\delta)/(3k-1-\varepsilon)}\right],\ \varepsilon > 0, |\boldsymbol{\eta}| > c_0,$$

then the symbol $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ of the Cauchy problem (8.11) has the asymptotic behavior near $t = 0$:

$$Q(\mathbf{x}, t, \boldsymbol{\eta}, 0) = exp[-(i/\hbar)ta(\mathbf{x}, \boldsymbol{\eta})] + o(1),$$

uniformly for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$.

Now, using Theorem 8.23 we see that, under the stronger condition (3′), the kernel $K(\mathbf{x}, t\,; \mathbf{y}, 0)$ satisfies

$$K(\mathbf{x}, t; \mathbf{y}, 0) = \int_{\mathbb{R}^n} \exp[\pi i(\mathbf{x} - \mathbf{y}, \boldsymbol{\eta}) - (i/\hbar)ta(\mathbf{x}, \boldsymbol{\eta})] d\boldsymbol{\eta} + \int_{\mathbb{R}^n} \exp[\pi i(\mathbf{x} - \mathbf{y}, \boldsymbol{\eta})] o(1) d\boldsymbol{\eta}.$$

From the extension theory of Chap. 5, we see that $A(\mathbf{x}, \mathbf{p})$ has a self-adjoint extension to $KS^2(\mathbb{R}^n)$, which also generates a unitary group.

This means that we can construct a path integral in the same (identical) way as was done for the free-particle propagator (i.e., for all Hamiltonians with symbols in $\mathcal{S}^m_{\alpha,\delta}$). Furthermore, it follows that the same comment applies to any Hamiltonian that has a kernel representation, independent of its symbol class. This partially proves a conjecture made in [GZ3], to the effect that there is a kernel for every "physically generated" semigroup.

8.6. Path Integrals II: Time-Ordered Theory

If we want to consider perturbations of the Hamiltonians with various potentials, the normal analytical problems arise. In this case, we must resort to the limited number of Trotter–Kato type results that may apply on $KS^2(\mathbb{R}^n)$. The general question is, "Under what conditions can we expect a path integral to exist?"

8.6.1. Time-Ordered Path Integrals. The results of the last section have direct extensions to time-dependent Hamiltonians, but the operators need not commute. Thus, in order to construct general path integrals, we must use the full power of the time-ordered operator theory in Chap. 7. In this section, we show that the path integral is a special case of the time-ordered operator theory as suggested by Feynman and automatically leads to a generalization and extension of Feynman–Kac theory.

Before proceeding, we should briefly pause for a few words about progress on the development of the Feynman–Kac theory as it relates to nonautonomous systems, evolution processes or time-dependent propagators and their relationship to path integrals and quantum field theory. The major developments in these areas along with many interesting applications can be found in the relatively recent books by: Jefferies [JE], Lorinczi [LO], Gulishashvili and Van Casteren [GC], and Del Moral [MO].

Let $U[t,a]$ be an evolution operator on $KS^2(\mathbb{R}^3)$, with time-dependent generator $A(t)$, which has a kernel $\mathbf{K}[\mathbf{x}(t), t\,; \mathbf{x}(s), s]$ such that:

$$\mathbf{K}\left[\mathbf{x}(t),\, t;\, \mathbf{x}(s),\, s\right] = \int_{\mathbf{R}^3} \mathbf{K}\left[\mathbf{x}(t),\, t;\, d\mathbf{x}(\tau),\, \tau\right]\mathbf{K}\left[\mathbf{x}(\tau),\, \tau;\, \mathbf{x}(s),\, s\right],$$

$$U[t,s]\varphi(s) = \int_{\mathbf{R}^3} \mathbf{K}\left[\mathbf{x}(t),\, t;\, d\mathbf{x}(s),\, s\right]\varphi(s).$$

Now let $\mathbf{U}[t,s]$ be the corresponding time-ordered version defined on $\mathcal{FD}^2_\otimes \subset \mathcal{H}^2_\otimes$, with kernel $\mathbb{K}_\mathbf{f}\,[\mathbf{x}(t),\, t;\, \mathbf{x}(s),\, s]$. Since $\mathbf{U}[t,\tau]\mathbf{U}[\tau,s] = \mathbf{U}[t,s]$, we have:

$$\mathbb{K}_\mathbf{f}\,[\mathbf{x}(t),\, t;\, \mathbf{x}(s),\, s] = \int_{\mathbf{R}^3} \mathbb{K}_\mathbf{f}\,[\mathbf{x}(t),\, t;\, d\mathbf{x}(\tau),\, \tau]\mathbb{K}_\mathbf{f}\,[\mathbf{x}(\tau),\, \tau;\, \mathbf{x}(s),\, s]\,.$$

From our sum over paths representation for $\mathbf{U}[t,s]$, we have:

$$\begin{aligned}\mathbf{U}[t,s]\Phi(s) &= \lim_{\lambda\to\infty} \mathbf{U}_\lambda[t,s]\Phi(s)\\ &= \lim\nolimits_{\lambda\to\infty} \mathrm{e}^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{[\lambda\,(t-s)]^k}{k!}\mathbf{U}_k[t,s]\Phi(s),\end{aligned}$$

where

$$\mathbf{U}_k[t,s]\Phi(s) = \exp\left\{(-i/\hbar)\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} \mathbf{E}[(j/\lambda),\tau]\mathcal{A}(\tau)d\tau\right\}\Phi(s).$$

As in Sect. 8.1, we define $\mathbb{K}_\mathbf{f}[\mathcal{D}_\lambda \mathbf{x}(\tau)]$ by:

$$\begin{aligned}&\int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{s})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_\mathbf{f}[\mathcal{D}_\lambda \mathbf{x}(\tau)]\\ &=: \mathrm{e}^{-\lambda(t-s)}\sum_{k=0}^{n}\frac{[\lambda(t-s)]^k}{k!}\left\{\prod_{j=1}^{k}\int_{\mathbf{R}^3} \mathbb{K}_\mathbf{f}[t_j, \mathbf{x}(t_j)\,;\, d\mathbf{x}(t_{j-1}), t_{j-1}]\big|^{(j/\lambda)}\right\},\end{aligned}$$

where $n = [|\lambda(t-s)|]$, the greatest integer in $\lambda(t-s)$, and $|^{(j/\lambda)}$ denotes that the integration is performed in the time-slot (j/λ).

Definition 8.24. We define the Feynman path integral associated with $\mathbf{U}[t,s]$ by:

$$\mathbf{U}[t,s]\Phi(s) = \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{s})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_\mathbf{f}[\mathcal{D}\mathbf{x}(\tau)]\Phi(s) = \lim_{\lambda\to\infty}\int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{s})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_\mathbf{f}[\mathcal{D}_\lambda\mathbf{x}(\tau)]\Phi(s).$$

Theorem 8.25. *For the time-ordered theory, whenever a kernel exists, we have that:*

$$\lim_{\lambda\to\infty}\mathbf{U}_\lambda[t,s]\Phi(s) = \mathbf{U}[t,s]\Phi(s) = \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{s})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_\mathbf{f}[\mathcal{D}_\lambda\mathbf{x}(\tau)]\Phi[\mathbf{x}(s)],$$

and the limit is independent of the space of continuous functions.

Let us assume that $A_0(t)$ and $A_1(t)$ are strongly continuous generators of C_0-contraction semigroups, with a common dense domain $D(t)$, for each $t \in E = [a,b]$, and let $\mathcal{A}_{1,\rho}(t) = \rho\mathcal{A}_1(t)\mathbf{R}(\rho,\mathcal{A}_1(t))$ be the Yosida approximator for the time-ordered version of $A_1(t)$, with dense domain $D = \mathcal{F}\mathcal{D}^2_{\otimes} \cap \otimes_{t \in I} D(t)$. Define $\mathbf{U}^{\rho}[t,a]$ and $\mathbf{U}^0[t,a]$ by:

$$\mathbf{U}^{\rho}[t,a] = \exp\{(-i/\hbar)\int_a^t [\mathcal{A}_0(s) + \mathcal{A}_{1,\rho}(s)]ds\},$$

$$\mathbf{U}^0[t,a] = \exp\{(-i/\hbar)\int_a^t \mathcal{A}_0(s)ds\}.$$

Since $\mathcal{A}_{1,\rho}(s)$ is bounded, $\mathcal{A}_0(s) + \mathcal{A}_{1,\rho}(s)$ is a generator of a C_0-contraction semigroup for $s \in E$ and finite ρ. Now assume that $\mathbf{U}^0[t,a]$ has an associated kernel, so that $\mathbf{U}^0[t,a] = \int_{\mathbb{R}^{3[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(a)]$. We now have the following general result, which is independent of the space of continuous functions.

Theorem 8.26. *(Feynman–Kac)* If $\mathcal{A}_0(s) \oplus \mathcal{A}_1(s)$ is a generator of a C_0-contraction semigroup, then*

$$\lim_{\rho\to\infty} \mathbf{U}^{\rho}[t,a]\Phi(a) = \mathbf{U}[t,a]\Phi(a)$$
$$= \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{a})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)] \exp\{(-i/\hbar)\int_a^{\tau} \mathcal{A}_1(s)ds]\}\Phi[\mathbf{x}(a)].$$

Proof. The fact that $\mathbf{U}^{\rho}[t,a]\Phi(a) \to \mathbf{U}[t,a]\Phi(a)$ is clear. To prove that

$$\mathbf{U}[t,a]\Phi(a) = \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{a})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)] \exp\{(-i/\hbar)\int_a^t \mathcal{A}_1(s)ds\}\Phi(a),$$

first note that, since the time-ordered integral exists and we are only interested in the limit, we can write for each k

$$U_k^{\rho}[t,a]\Phi(a)$$
$$= \exp\left\{(-i/\hbar)\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} \left[\mathbf{E}[\tau_j,s]\mathcal{A}_0(s) + \mathbf{E}[\tau_j',s]\mathcal{A}_{1,\rho}(s)\right] ds\right\}\Phi(a),$$

where τ_j and τ_j' are distinct points in the interval (t_{j-1}, t_j). Thus, we can also write $U_k^{\rho}[t,a]\Phi(a)$ as

$$\begin{aligned}
&\mathbf{U}_k^{\rho}[t,a] \\
&= exp\left\{\tfrac{-i}{\hbar}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\mathbf{E}[\tau_j,s]\mathcal{A}_0(s)ds\right\} exp\left\{\tfrac{-i}{\hbar}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\mathbf{E}[\tau_j',s]\mathcal{A}_{1,\rho}(s)ds\right\} \\
&= \prod_{j=1}^{k} exp\left\{\tfrac{-i}{\hbar}\int_{t_{j-1}}^{t_j}\mathbf{E}[\tau_j,s]\mathcal{A}_0(s)ds\right\} exp\left\{\tfrac{-i}{\hbar}\sum_{j=1}^{k}\int_{t_{j-1}}^{tj}\mathbf{E}[\tau_j',s]\mathcal{A}_{1,\rho}(s)ds\right\} \\
&= \prod_{j=1}^{k}\int_{\mathbb{R}^3}\mathbb{K}_{\mathbf{f}}\left[t_j,\mathbf{x}(t_j)\ ;\ t_{j-1},d\mathbf{x}(t_{j-1})\right] exp\left\{\tfrac{-i}{\hbar}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\mathbf{E}[\tau_j',s]\mathcal{A}_{1,\rho}(s)ds\right\}.
\end{aligned}$$

If we put this in our experimental evolution operator $\mathbf{U}_{\lambda}^{\rho}[t,a]\Phi(a)$ and compute the limit, we have:

$$\begin{aligned}
&\mathbf{U}^{\rho}[t,a]\Phi(a) \\
&= \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{a})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})}\mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)]\exp\left\{(-i/\hbar)\int_a^t\mathcal{A}_{1,\rho}(s)ds\right\}\Phi(a).
\end{aligned}$$

Since the limit as $\rho\to\infty$ on the left exists, it defines the limit on the right. □

8.6.2. Examples. In this section, we pause to discuss a few examples. Theorem 8.26 is somewhat abstract, so it may not be clear as to its application. Our first example is a direct application of this theorem, which covers all of nonrelativistic quantum theory (i.e., the Feynman formulation of quantum theory).

Let $\triangle$ be the Laplacian on $\mathbf{R}^n$ and let V be any potential such that $A=(-\hbar^2/2)\triangle+V$ generates an unitary group. Then the problem:

$$(i\hbar)\partial\psi(\mathbf{x},t)/\partial t = A\psi(\mathbf{x},t),\ \ \psi(\mathbf{x},0)=\psi_0(\mathbf{x}),$$

has a solution with a Feynman–Kac representation.

Our second example is more specific and is due to Albeverio and Mazzucchi [AM]. Their paper provides an excellent view of the power of the approach first introduced by Albeverio and H∅egh-Krohn [AH]. Let $\mathbb{C}$ be a completely symmetric positive definite fourth-order covariant tensor on $\mathbb{R}^n$, let Ω be a symmetric positive definite $n\times n$ matrix, and let λ be a nonnegative constant. It is known [RS1] that the operator

$$\bar{A} = -\tfrac{\hbar^2}{2}\Delta + \tfrac{1}{2}\mathbf{x}\Omega^2\mathbf{x} + \lambda\mathbb{C}[\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}]$$

is a densely defined self-adjoint generator of an unitary group on $L^2[\mathbb{R}^n]$. Using a substantial amount of elegant analysis, Albeverio and Mazzucchi [AM] prove that $\bar{A}$ has a path integral representation as the analytic continuation (in the parameter λ) of an infinite dimensional generalized oscillatory integral.

Our approach to the same problem is both simple and direct using the results of the previous sections. First, since $\bar{A} = \bar{A}^*$ is densely defined on $L^2[\mathbb{R}^n]$, $\bar{A}$ has a closed densely defined self-adjoint extension A to $KS^2[\mathbb{R}^n]$, which generates a unitary group. If we set $V = \frac{1}{2}\mathbf{x}\Omega^2\mathbf{x} + \lambda\mathbb{C}[\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}]$ and $V_\rho = V(I+\rho V^2)^{-1/2}$, $\rho > 0$, it is easy to see that V_ρ is a bounded self-adjoint operator which converges to V on $D(V)$. (This follows from the fact that a bounded (self-adjoint) perturbation of an unbounded self-adjoint operator is self-adjoint.) Now, since $-\frac{\hbar^2}{2}\Delta$ generates a unitary group, $A_\rho = -\frac{\hbar^2}{2}\Delta + V_\rho$ also generates one and converges to A on $D(A)$. Let

$$\mathcal{A}(\tau) = (\hat{\otimes}_{t \geqslant s > \tau} \mathrm{I}_s) \otimes A \otimes (\otimes_{\tau > s \geqslant 0} \mathrm{I}_s),$$

then $\mathcal{A}(t)$ generates a unitary group for each t and $\mathcal{A}_\rho(t)$ converges to $\mathcal{A}(t)$ on $D[\mathcal{A}(t)] \subset \mathcal{FD}^2_\otimes$. We can now apply Theorem 8.26 to get that:

$$\begin{aligned}
&\mathbf{U}[t,a]\Phi \\
&= \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{a})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)] \exp\{-(i/\hbar)\int_a^\tau V(s)ds\}\Phi \\
&= \lim_{\rho\to 0} \int_{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{a})}^{\mathbf{x}(\tau)=\mathbf{x}(\mathbf{t})} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)] \exp\{-(i/\hbar)\int_a^\tau V_\rho(s)ds\}\Phi.
\end{aligned}$$

Under additional assumptions, Albeverio and Mazzucchi are able to prove Borel summability of the solution in power series of the coupling constant. With Theorem 7.25 of Chap. 7, we get the Dyson expansion to any order with remainder.

8.7. Dyson's First Conjecture

This section is the last one in the book for two reasons. First, our original objective, leading to most of the work in the book, was to provide an answer this conjecture. The second reason is that this section does not provide any additional mathematics. It essentially gives a physical reinterpretation of the mathematics developed earlier.

At the end of his second paper on the relationship between the Feynman and Schwinger–Tomonaga theories, Dyson explored the difference between the divergent Hamiltonian formalism that one must begin with and the finite S-matrix that results from renormalization (see [DY2]). He takes the view that it is a contrast between a real observer and a fictitious (ideal) observer. The real observer can only determine particle positions with limited accuracy and always gets finite results from his measurements. Dyson then suggests that "... The ideal observer, however, using nonatomic apparatus whose location in space and time is known with infinite precision, is imagined to be able to disentangle a single field from its interactions with others, and to measure the interaction. In conformity with the Heisenberg uncertainty principle, it can perhaps be considered a physical consequence of the infinitely precise knowledge of (particle) location allowed to the ideal observer, that the value obtained when he measures (the interaction) is infinite." He goes on to remark that if his analysis is correct, the problem of divergences is attributable to an idealized concept of measurability. The purpose of this section is to develop the conceptual and technical framework that will allow us to discuss this conjecture.

8.7.1. The S-Matrix. The objective of this section is to provide a formulation of the S-matrix that will allow us to investigate the mathematical sense in which we can believe Dyson's conjecture.

In order to explore this idea, we work in the interaction representation with obvious notation. Replace the interval $[t, 0]$ by $[T, -T]$, $H(t)$ by $\frac{-i}{\hbar} H_I(t)$, and our experimental evolution operator $\mathbf{U}_\lambda[T, -T]\Phi$ by the experimental scattering operator (or S-matrix) $\mathbf{S}_\lambda[T, -T]\Phi$, where

$$\mathbf{S}_\lambda[T, -T]\Phi = \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp\left[-2\lambda T\right] \mathbf{S}_n[T, -T]\Phi, \tag{8.14}$$

$$\mathbf{S}_n[T, -T]\Phi = \exp\left\{ \frac{-i}{\hbar} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E[\tau_j, s] H_I(s) ds \right\} \Phi, \tag{8.15}$$

and $H_I(t) = \int_{\mathbb{R}^3} H_I(\mathbf{x}(t), t) d\mathbf{x}(t)$ is the interaction energy. We now give a physical interpretation of our formalism. Rewrite Eq. (8.14) as

$$\mathbf{S}_\lambda[T, -T]\Phi$$
$$= \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp\left\{ \frac{-i}{\hbar} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[E[\tau_j, s] H_I(s) - i\lambda\hbar \mathrm{I}_\otimes\right] ds \right\} \Phi. \tag{8.16}$$

In this form, it is clear that the term $-i\lambda\hbar \mathrm{I}_{\otimes}$ has a physical interpretation as the absorption of photon energy of amount $\lambda\hbar$ in each subinterval $[t_{j-1}, t_j]$ (cf. Mott and Massey [MM]). When we compute the limit, we get the standard S-matrix (on $[T, -T]$). It follows that we must add an infinite amount of photon energy to the mathematical description of the experimental picture (at each point in time) in order to obtain the standard scattering operator. This is the ultraviolet divergence and shows explicitly that the transition from the experimental to the ideal scattering operator requires that we illuminate the particle throughout its entire path. Thus, it appears that we have, indeed, violated the uncertainty relation. This is further supported if we look at the form of the standard S-matrix:

$$\mathbf{S}[T, -T]\Phi = \exp\left\{(-i/\hbar)\int_{-T}^{T} H_I(s)ds\right\}\Phi \tag{8.17}$$

and note that the differential ds in the exponent implies perfect infinitesimal time knowledge at each point, strongly suggesting that the energy should be totally undetermined. If violation of the Heisenberg uncertainty relation is the cause for the ultraviolet divergence, then as it is a variance relation, it will not appear in first order (perturbation) but should show up in all higher-order terms. On the other hand, if we eliminate the divergent terms in second order, we would expect our method to prevent them from appearing in any higher order term of the expansion. The fact that this is precisely the case in quantum electrodynamics is a clear verification of Dyson's conjecture.

If we allow T to become infinite, we once again introduce an infinite amount of energy into the mathematical description of the experimental picture, as this is also equivalent to precise time knowledge (at infinity). Of course, this is the well-known infrared divergence and can be eliminated by keeping T finite (see Dahmen et al. [DA]) or introducing a small mass for the photon (see Feynman [FY3]). If we hold λ fixed while letting T become infinite, the experimental S-matrix takes the form:

$$\begin{gathered}\mathbf{S}_\lambda[\infty, -\infty]\Phi = \exp\left\{(-i/\hbar)\sum_{j=1}^{\infty}\int_{t_{j-1}}^{t_j} E[\tau_j, s]H_I(s)ds\right\}\Phi, \\ \bigcup_{j=1}^{\infty}[t_{j-1}, t_j] = (-\infty, \infty)\,, \ \& \ \Delta t_j = \lambda^{-1}.\end{gathered} \tag{8.18}$$

This form is interesting since it shows how a minimal time eliminates the ultraviolet divergence. Of course, this is not unexpected, and has been known at least since Heisenberg [HE] introduced his fundamental length as a way around the divergences. This was a prelude to the various lattice approximation methods. The review by Lee [LE] is interesting in this regard. In closing this section, we record our exact expansion for the S-matrix to any finite order. Let $\mathcal{H}_k = H_I(s_1)\cdots H_I(s_k)$ and let $\Phi \in D\left[(\mathbf{Q}[\infty,-\infty])^{n+1}\right]$, we have

$$\begin{aligned}\mathbf{S}[\infty,-\infty]\Phi(-\infty) &= \sum_{k=0}^{n}\left(\tfrac{-i}{\hbar}\right)^k \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{s_{k-1}} ds_k \mathcal{H}_k \Phi \\ &+ \left(\tfrac{-i}{\hbar}\right)^{n+1} \int_0^1 (1-\xi)^n d\xi \int_{-\infty}^{\infty} ds_1 \cdots \int_{-\infty}^{s_n} ds_{n+1}\mathcal{H}_{n+1}\mathbf{S}^{\xi}[s_{n+1},-\infty]\Phi.\end{aligned} \tag{8.19}$$

It follows that (in a theoretical sense) we can consider the standard S-matrix expansion to be exact, when truncated at any order, by adding the last term of Eq. (8.19) to give the remainder. This result also means that whenever we can construct an exact nonperturbative solution, it always implies the existence of a perturbative solution valid to any order. However, in general, only in particular cases can we know if the series at some n (without the remainder) approximates the solution.

In this section we have provided a precise formulation and proof of Dyson's conjecture that the ultraviolet divergence is caused by a violation of the Heisenberg uncertainty relation at each point in time.

In closing, since the time of Dyson's original work, a large amount of progress has been made in understanding the mathematical and physical foundations of relativistic quantum theory. (For a brief discussion including references for further reading, see Gill and Zachary [GZ] and [GZ1].) However, many of the problems encountered by the earlier workers are still with us in one form or another.

References

[AH] S. Albeverio, R. H∅egh-Krohn, *Mathematical Theory of Feynman Path Integrals*. Lecture Notes in Mathematics, vol. 523 (Springer, Berlin, 1976)

[AM] S. Albeverio, S. Mazzucchi, Feynman path integrals for polynomially growing potentials. J. Funct. Anal. **221**, 83–121 (2005)

[CDM1] P. Cartier, C. Dewitt-Morette, Functional integration. J. Math. Phys. **41**, 4154–4185 (2000)

[CDM2] P. Cartier, C. Dewitt-Morette, *Functional Integration, Action and Symmetries* (Cambridge University Press, New York, 2006)

[CO] H.O. Cordes, *The Technique of Pseudodifferential Operators* (Cambridge University Press, Cambridge, 1995)

[DA] H.D. Dahmen, B. Scholz, F. Steiner, Infrared dynamics of quantum electrodynamics and the asymptotic behavior of the electron form factor. Nucl. Phys. B **202**, 365 (1982)

[DP] G. Da Prato, *Kolmogorov Equations for Stochastic PDEs*. Advanced Courses in Mathematics CRM Barcelona (Birkhäuser, Boston, 2004)

[MO] P. Del Moral, *Feynman-Kac Formulae: Genealogical and Interacting Particle Systems with Applications* (Springer, New York, 2004)

[DY2] F. Dyson, The S-matrix in quantum electrodynamics. Phys. Rev. **75**, 1736–1755 (1949)

[EV] L.C. Evans, *Partial Differential Equations*. AMS Graduate Studies in Mathematics, vol. 18 (American Mathematical Society, Providence, 1998)

[FY1] R.P. Feynman, The principle of least action in quantum mechanics, Ph.D. dissertation, Physics, Princeton University, 1942 (Available from University Microfilms Publications, No. 2948, Ann Arbor, MI)

[FY3] R.P. Feynman, *Quantum Electrodynamics* (Benjamin, New York, 1964)

[FH] R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965)

[GZ3] T.L. Gill, W.W. Zachary, Time-ordered operators and Feynman-Dyson algebras. J. Math. Phys. **28**, 1459–1470 (1987)

[GZ] T.L. Gill, W.W. Zachary, Foundations for relativistic quantum theory I: Feynman's operator calculus and the Dyson conjectures. J. Math. Phys. **43**, 69–93 (2002)

[GZ4] T.L. Gill, W.W. Zachary, Analytic representation of the square-root operator. J. Phys. A Math. Gen. **38**, 2479–2496 (2005)

[GZ1] T.L. Gill, W.W. Zachary, Feynman operator calculus: the constructive theory. Expo. Math. **29**, 165–203 (2011)

[GRRZ] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965)

[GS] C. Grosche, F. Steiner, *Handbook of Feynman Path Integrals*. Springer Tracts in Modern Physics, vol. 145 (Springer, New York, 1998)

[GC] A. Gulishashvili, J.A. Van Casteren, *Non-autonomous Kato Classes and Feynman-Kac Propagators* (Springer, New York, 2006)

[HE] W. Heisenberg, Die "beobachtbaren Grossen" in der theorie der Elementarteilchen II. Z. Phys. **120**, 673–702 (1943)

[HO] L. Hörmander, Fourier integral operators I. Acta Math. **127**, 79–183 (1971)

[KL] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, 3rd edn. (World Scientific, Singapore, 2004)

[KG] H. Kumano-go, *Pseudo-Differential Operators* (The MIT Press, Cambridge/London, 1982)

[JE] B. Jefferies, *Evolution Processes and the Feynman-Kac Formulas* (Springer, New York, 2010)

[JL] G.W. Johnson, M.L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus* (Oxford University Press, New York, 2000)

[LX1] P. Lax, *The L^2 Operator Calculus of Mikhlin, Calderón and Zygmund* (Lectures Notes). Courant Institute of Mathematical Sciences (New York University, New York, 1963)

[LE] T.D. Lee, in *The Lesson of Quantum Theory*, ed. by J. de Boer, E. Dal, O. Ulfbeck (Elsevier, Amsterdam, 1986), p. 181

[LB] L. Lorenzi, M. Bertoldi, *Analytical Methods for Markov Semigroups*. Pure and Applied Mathematics (Chapman & Hall/CRC, New York, 2007)

[LO] J. Lorinczi, *Feynman-Kac-type Theorems and Gibbs Measures on Path Space: With Applications to Rigorous Quantum Field Theory*. De Gruyter Studies in Mathematics (De Gruyter, New York, 2008)

[M] V.P. Maslov, *Operational Methods* (Mir, Moscow, 1973). English translation 1976 (revised from the Russian edition)

[MM] N.F. Mott, H.S.W. Massey, *The Theory of Atomic Collisions*. The International Series of Monographs on Physics (Clarendon Press/Oxford University Press, Oxford/New York, 1933)

[RS1] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic, New York, 1972)

[SY] G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*. Applied Mathematical Sciences, vol. 143 (Springer, New York, 2002)

[SH] I.A. Shishmarev, On the Cauchy problem and T-products for hypoelliptic systems. Math. USSR Izvestiya **20**, 577–609 (1983)

[SHB] M.A. Shubin, *Pseudodifferential Operators and Spectral Theory* (Nauka, Moscow, 1978) (Russian)

[TA] M.E. Taylor, *Pseudodifferential Operators* (Princeton University Press, Princeton, 1981)

[TR] F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators.* Pseudodifferential Operators, vol. 1. Fourier Integral Operators, vol. 2 (Plenum Press, New York/London, 1980)

[WSRM] N. Wiener, A. Siegel, B. Rankin, W.T. Martin, *Differential Space, Quantum Systems, and Prediction* (The MIT Press, Cambridge, 1966)

[YS] K. Yosida, *Functional Analysis*, 2nd edn. (Springer, New York, 1968)