*Chapter 5*

# Operators on Banach Space

The Feynman operator calculus and the Feynman path integral develop naturally on Hilbert space. In this chapter we develop the theory of semigroups of operators, which is the central tool for both. In order to extend the theory to other areas of interest, we begin with a new approach to operator theory on Banach spaces. We first show that the structure of the bounded linear operators on Banach space with an S-basis is much closer to that for the same operators on Hilbert space. We will exploit this new relationship to transfer the theory of semigroups of operators developed for Hilbert spaces to Banach spaces. The results are complete for uniformly convex Banach spaces, so we restrict our presentation to that case, with one exception. In the Appendix (Sect. 5.3), we show that all of the results in Chap. 4 have natural analogues for uniformly convex Banach spaces.

## 5.1. Preliminaries

Let $\mathcal{B}$ be a uniformly convex Banach space with an S-basis. Let $\mathcal{C}[\mathcal{B}]$ be the set of closed densely defined linear operators and let $L[\mathcal{B}]$ be the set of bounded linear operators on $\mathcal{B}$.




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**Definition 5.1.** A duality map $\mathcal{J} : \mathcal{B} \mapsto \mathcal{B}'$ is a set

$$\mathcal{J}(u) = \left\{ u^* \in \mathcal{B}' \,\middle|\, \langle u, u^* \rangle = \|u\|_{\mathcal{B}}^2 = \|u^*\|_{\mathcal{B}'}^2 \right\}, \ \forall u \in \mathcal{B}.$$

**Example 5.2.** *Let* $\Omega$ *be a bounded open subset of* $\mathbb{R}^n$, $n \in \mathbb{N}$. *If* $u \in L^p[\Omega] = \mathcal{B}$, $1 < p < \infty$, *then*

$$\mathcal{J}(u)(x) = \|u\|_p^{2-p} |u(x)|^{p-2} u(x) = u^* \in L^q[\Omega], \ \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{5.1}$$

*Furthermore,*

$$\langle u, u^* \rangle = \|u\|_p^{2-p} \int_\Omega |u(x)|^p \, d\lambda_n(x) = \|u\|_p^2 = \|u^*\|_q^2$$

It can be shown that $L^p[\Omega]$ is uniformly convex and that $u^* = \mathcal{J}(u)$ is uniquely defined for each $u \in \mathcal{B}$. Thus, if $\{u_n\}$ is an S-basis for $L^p[\Omega]$, then the family vectors $\{u_n^*\}$ is an S-basis for $L^q[\Omega] = (L^p[\Omega])'$. The relationship between $u$ and $u^*$ is nonlinear [see Eq. (5.1)]. In the next section we prove the remarkable result that there is another representation of $\mathcal{B}'$, with $u^* = \mathbf{J}_{\mathcal{B}}(u)$ linear, for each $u \in \mathcal{B}$. (However, $u^*$ is no longer a duality mapping.)

**5.1.1. The Natural Hilbert Space for a Uniformly Convex Banach Space.** We follow the same ideas used in Chap. 3 to embed $L^2$ in $KS^2$. However, we take a restricted approach that applies to all uniformly convex Banach spaces with an S-basis. Fix $\mathcal{B}$ and let $\{\mathcal{E}_n\}$ be an S-basis for $\mathcal{B}$. For each $n$, let $t_n = 2^{-n}$ and for each $\mathcal{E}_n$, let $\mathcal{E}_n^*$ be the corresponding dual vector in $\mathcal{B}'$. For each pair of functions $u, v$ on $\mathcal{B}$, define an inner product by:

$$(u, v) = \sum_{n=1}^{\infty} t_n \langle \mathcal{E}_n^*, u \rangle \overline{\langle \mathcal{E}_n^*, v \rangle}.$$

we let $\mathcal{H}$ be the completion of $\mathcal{B}$ in the induced norm. It is clear that $\mathcal{B} \subset \mathcal{H}$ densely and

$$\begin{aligned} \|u\|_{\mathcal{H}} &= \left[ \sum_{n=1}^{\infty} t_n |\langle \mathcal{E}_n^*, u \rangle|^2 \right]^{1/2} \\ &\leqslant \sup_n |\langle \mathcal{E}_n^*, u \rangle| \\ &\leqslant \sup_{\|\mathcal{E}^*\|_{\mathcal{B}'} \leqslant 1} |\langle \mathcal{E}^*, u \rangle| = \|u\|_{\mathcal{B}}, \end{aligned} \tag{5.2}$$

so that the embedding is both dense and continuous. It is clear that $\mathcal{H}$ is unique up to a change of S-basis.

**Definition 5.3.** If $\mathcal{B}$ be a Banach space, we say that $\mathcal{B}'$ has a Hilbert space representation if there exists a Hilbert space $\mathcal{H}$, with $\mathcal{B} \subset \mathcal{H}$ as a continuous dense embedding and for each $u^* \in \mathcal{B}'$, $u^* = (\cdot, u)_{\mathcal{H}}$ for some $u \in \mathcal{B}$.

**Theorem 5.4.** *If $\mathcal{B}$ be a uniformly convex Banach space with an S-basis, then $\mathcal{B}'$ has a Hilbert space representation.*

**Proof.** Let $\mathcal{H}$ be the natural Hilbert space for $\mathcal{B}$ and let $\mathbf{J}$ be the natural linear mapping from $\mathcal{H} \to \mathcal{H}'$, defined by

$$\langle v, \mathbf{J}(u)\rangle = (v, u)_{\mathcal{H}}, \text{ for all } u, v \in \mathcal{H}.$$

It is easy to see that $\mathbf{J}$ is bijective and $\mathbf{J}^* = \mathbf{J}$. First, we note that the restriction of $\mathbf{J}$ to $\mathcal{B}$, $\mathbf{J}_{\mathcal{B}}$, maps $\mathcal{B}$ to a unique subset of linear functionals $\{\mathbf{J}_{\mathcal{B}}(u),\ u \in \mathcal{B}\}$ and, $\mathbf{J}_{\mathcal{B}}(u+v) = \mathbf{J}_{\mathcal{B}}(u) + \mathbf{J}_{\mathcal{B}}(v)$, for each $u, v \in \mathcal{B}$. We are done if we can prove that $\{\mathbf{J}_{\mathcal{B}}(u),\ u \in \mathcal{B}\} = \mathcal{B}'$. For this, it suffices to show that $\mathbf{J}_{\mathcal{B}}(u)$ is bounded for each $u \in \mathcal{B}$. Since $\mathcal{B}$ is dense in $\mathcal{H}$, from equation (5.2) we have:

$$\|\mathbf{J}_{\mathcal{B}}(u)\|_{\mathcal{B}'} = \sup_{v \in \mathcal{B}} \frac{\langle v, \mathbf{J}_{\mathcal{B}}(u)\rangle}{\|v\|_{\mathcal{B}}} \leqslant \sup_{v \in \mathcal{B}} \frac{\langle v, \mathbf{J}_{\mathcal{B}}(u)\rangle}{\|v\|_{\mathcal{H}}} = \|u\|_{\mathcal{H}} \leqslant \|u\|_{\mathcal{B}}.$$

Thus, $\{\mathbf{J}_{\mathcal{B}}(u),\ u \in \mathcal{B}\} \subset \mathcal{B}'$. Since $\mathcal{B}$ is uniformly convex, there is a (unique) one-to-one relationship between $\mathcal{B}$ and $\mathcal{B}'$, so that $\{\mathbf{J}_{\mathcal{B}}(u),\ u \in \mathcal{B}\} = \mathcal{B}'$. □

**5.1.2. Construction of the Adjoint on $\mathcal{B}$.** We can now show that if $\mathcal{B}'$ has a Hilbert space representation, then each closed densely linear operator on $\mathcal{B}$ has a natural adjoint defined on $\mathcal{B}$.

**Theorem 5.5.** *Let $\mathcal{B}$ be a uniformly convex Banach space with an S-basis. If $\mathbb{C}[\mathcal{B}]$ denotes the closed densely linear operators on $\mathcal{B}$ and $L[\mathcal{B}]$ denotes the bounded linear operators, then every $A \in \mathcal{C}[\mathcal{B}]$ has a well-defined adjoint $A^* \in \mathcal{C}[\mathcal{B}]$. Furthermore, if $A \in L[\mathcal{B}]$, then $A^* \in L[\mathcal{B}]$ with:*

(1) $(aA)^* = \bar{a}A^*$,

(2) $A^{**} = A$,

(3) $(A^* + B^*) = A^* + B^*$

(4) $(AB)^* = B^*A^*$ *and*

(5) $\|A^*A\|_{\mathcal{B}} \leq \|A\|_{\mathcal{B}}^2$.

*Thus, $L[\mathcal{B}]$ is a $^*$algebra.*

**Proof.** Let $\mathbf{J}$ be the natural linear mapping from $\mathcal{H} \to \mathcal{H}'$ and let $\mathbf{J}_{\mathcal{B}}$ be the restriction of $\mathbf{J}$ to $\mathcal{B}$. If $A \in \mathcal{C}[\mathcal{B}]$, then $A'\mathbf{J}_{\mathcal{B}} : \mathcal{B}' \to \mathcal{B}'$. Since $A'$ is closed and densely defined, it follows that $\mathbf{J}_{\mathcal{B}}^{-1}A'\mathbf{J}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a closed and densely defined linear operator. We define $A^* = [\mathbf{J}_{\mathcal{B}}^{-1}A'\mathbf{J}_{\mathcal{B}}] \in \mathcal{C}[\mathcal{B}]$. If $A \in L[\mathcal{B}]$, $A^* = \mathbf{J}_{\mathcal{B}}^{-1}A'\mathbf{J}_{\mathcal{B}}$ is defined on all of $\mathcal{B}$. By the Closed Graph Theorem, $A^* \in L[\mathcal{B}]$. The proofs of (1)–(3) are straightforward. To prove (4),

$$\begin{aligned}(BA)^* &= \mathbf{J}_{\mathcal{B}}^{-1}(BA)'\mathbf{J}_{\mathcal{B}} = \mathbf{J}_{\mathcal{B}}^{-1}A'B'\mathbf{J}_{\mathcal{B}} \\ &= \left[\mathbf{J}_{\mathcal{B}}^{-1}A'\mathbf{J}_{\mathcal{B}}\right]\left[\mathbf{J}_{\mathcal{B}}^{-1}B'\mathbf{J}_{\mathcal{B}}\right] = A^*B^*.\end{aligned} \tag{5.3}$$

If we replace $B$ by $A^*$ in Eq. (5.3), noting that $A^{**} = A$, we also see that $(A^*A)^* = A^*A$. To prove (5), we first see that:

$$\langle A^*Av, \mathbf{J}_{\mathcal{B}}(u)\rangle = (A^*Av, u)_{\mathcal{H}} = (v, A^*Au)_{\mathcal{H}},$$

so that $A^*A$ is symmetric. Thus, by Lax's Theorem, $A^*A$ has a bounded extension to $\mathcal{H}$ and $\|A^*A\|_{\mathcal{H}} \leqslant k\,\|A^*A\|_{\mathcal{B}}$, where $k$ is a positive constant. We also have that

$$\|A^*A\|_{\mathcal{B}} \leqslant \|A^*\|_{\mathcal{B}}\|A\|_{\mathcal{B}} \leqslant \|A\|_{\mathcal{B}}^2. \tag{5.4}$$

It follows that $\|A^*A\|_{\mathcal{B}} \leq \|A\|_{\mathcal{B}}^2$. If equality holds in (5.4), for all $A \in L[\mathcal{B}]$, then it is a $C^*$-algebra. This is true if and only if $\mathcal{B}$ is a Hilbert space. Thus, in general the inequality in (5.4) is strict. □

5.1.2.1. **Example: Differential Operators**. Let $A$ be a closed densely defined linear operator defined on $L^p[\mathbb{R}^n]$, $1 < p < \infty$, and let $A'$ be the dual defined on $L^q[\mathbb{R}^n]$, $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to show that if $A'$ is densely defined on $L^p[\mathbb{R}^n]$, it has a closed extension to $L^p[\mathbb{R}^n]$ (without using $\mathcal{H}_2 = KS^2[\mathbb{R}^n]$).

**Example 5.6.** *Let $A$ be a second order differential operator on $L^p[\mathbb{R}^n]$ of the form*

$$A = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} x_i b_{ij}(\mathbf{x})\frac{\partial}{\partial x_j},$$

*where $\mathbf{a}(\mathbf{x}) = [\![a_{ij}(\mathbf{x})]\!]$ and $\mathbf{b}(\mathbf{x}) = [\![b_{ij}(\mathbf{x})]\!]$ are matrix-valued functions in $\mathbb{C}_c^\infty[\mathbb{R}^n \times \mathbb{R}^n]$ (infinitely differentiable functions with compact support). We also assume that for all $\mathbf{x} \in \mathbb{R}^n$ det $[\![a_{ij}(\mathbf{x})]\!] > \varepsilon$ and the imaginary part of the eigenvalues of $\mathbf{b}(\mathbf{x})$ are bounded above by $-\varepsilon$, for some $\varepsilon > 0$. Note, since we don't require $\mathbf{a}$ or $\mathbf{b}$ to be symmetric, $A \neq A'$.*

*It is well known that $\mathbb{C}_c^\infty[\mathbb{R}^n] \subset L^p[\mathbb{R}^n] \cap L^q[\mathbb{R}^n]$ for all $1 < p \leq q < \infty$. Furthermore, since $A'$ is invariant on $\mathbb{C}_c^\infty[\mathbb{R}^n]$,*

$$A' : \mathbb{C}_c^\infty\left[\mathbb{R}^n\right] \subset L^p\left[\mathbb{R}^n\right] \to \mathbb{C}_c^\infty\left[\mathbb{R}^n\right] \subset L^p\left[\mathbb{R}^n\right].$$

*It follows that $A'$ has a closed extension to $L^p[\mathbb{R}^n]$. (In this case, we do not need $\mathcal{H}_2$ directly, we can identify $\mathbf{J}_2$ with the identity on $\mathcal{H}_2$ and $A^*$ with $A'$.)*

**Remark 5.7.** For a general $A$, which is closed and densely defined on $L^p[\mathbb{R}^n]$, we know that it is densely defined on $KS^2[\mathbb{R}^n]$. Thus, it has a well-defined adjoint $A^*$ on $KS^2[\mathbb{R}^n]$. By Theorem 5.5, we can take the restriction of $A^*$ from $KS^2[\mathbb{R}^n]$ to obtain our adjoint on $L^p[\mathbb{R}^n]$.

5.1.2.2. **Example: Integral Operators**. In one dimension, the Hilbert transform can be defined on $L^2[\mathbb{R}]$ via its Fourier transform:

$$\widehat{H(f)} = -i \operatorname{sgn} x\, \hat{f}.$$

It can also be defined directly as principal-value integral:

$$(Hf)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y| \geqslant \varepsilon} \frac{f(y)}{x-y} dy.$$

For a proof of the following results see Grafakos [GRA, Chap. 4].

**Theorem 5.8.** *The Hilbert transform on $L^2[\mathbb{R}]$ satisfies:*

(1) *$H$ is an isometry, $\|H(f)\|_2 = \|f\|_2$ and $H^* = -H$.*
(2) *For $f \in L^p[\mathbb{R}]$, $1 < p < \infty$, there exists a constant $C_p > 0$ such that,*

$$\|H(f)\|_p \leq C_p \|f\|_p. \tag{5.5}$$

The next result is technically obvious, but conceptually nontrivial.

**Corollary 5.9.** *The adjoint of $H$, $H^*$ defines a bounded linear operator on $L^p[\mathbb{R}]$ for $1 < p < \infty$, and $H^*$ satisfies Eq. (5.5) for the same constant $C_p$.*

The Riesz transform, $\mathbf{R}$, is the $n$-dimensional analogue of the Hilbert transform and its $j$th component is defined for $f \in L^p[\mathbb{R}^n]$, $1 < p < \infty$, by:

$$R_j(f) = c_n \lim_{\varepsilon \to 0} \int_{|\mathbf{y}-\mathbf{x}| \geqslant \varepsilon} \frac{y_j - x_j}{|\mathbf{y}-\mathbf{x}|^{n+1}} f(\mathbf{y}) d\mathbf{y}, \quad c_n = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{(n+1)/2}}.$$

**Definition 5.10.** Let $\Omega$ be defined on the unit sphere $S^{n-1}$ in $R^n$.

(1) The function $\Omega(x)$ is said to be homogeneous of degree $n$ if $\Omega(tx) = t^n\Omega(x)$.

(2) The function $\Omega(x)$ is said to have the cancellation property if

$$\int_{S^{n-1}} \Omega(\mathbf{y})d\sigma(\mathbf{y}) = 0, \text{ where } d\sigma \text{ is the induced}$$

Lebesgue measure on $S^{n-1}$.

(3) The function $\Omega(x)$ is said to have the Dini-type condition if

$$\sup_{\substack{|\mathbf{x}-\mathbf{y}|\leqslant\delta \\ |\mathbf{x}|=|\mathbf{y}|=1}} |\Omega(\mathbf{x}) - \Omega(\mathbf{y})| \leqslant \omega(\delta) \Rightarrow \quad \int_0^1 \frac{\omega(\delta)d\delta}{\delta} < \infty.$$

A proof of the following theorem can be found in Stein [STE] (see p. 39).

**Theorem 5.11.** *Suppose that* $\Omega$ *is homogeneous of degree* 0*, satisfying both the cancellation property and the Dini-type condition. If* $f \in L^p[\mathbb{R}^n],\ 1 < p < \infty$ *and*

$$T_\varepsilon(f)(\mathbf{x}) = \int_{|\mathbf{y}-\mathbf{x}|\geqslant\varepsilon} \frac{\Omega(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^n} f(\mathbf{y})d\mathbf{y}.$$

*Then*

(1) *There exists a constant* $A_p$*, independent of both* $f$ *and* $\varepsilon$ *such that*

$$\|T_\varepsilon(f)\|_p \leqslant A_p\|f\|_p.$$

(2) *Furthermore,* $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ *exists in the* $L^p$ *norm and*

$$\|T(f)\|_p \leqslant A_p\|f\|_p. \tag{5.6}$$

Treating $T_\varepsilon(f)$ as a special case of the Henstock–Kurzweil integral, conditions (1) and (2) are automatically satisfied and we can write the integral as

$$T(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Omega(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^n} f(\mathbf{y})d\mathbf{y}.$$

For $g \in L^q,\ \frac{1}{p} + \frac{1}{q} = 1$, we have $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$. Using Fubini's Theorem for the Henstock–Kurzweil integral (see [HS]), we have that

**Corollary 5.12.** *The adjoint of* $T$*,* $T^* = -T$ *is defined on* $L^p$ *and satisfies Eq. (5.6)*

It is easy to see that the Riesz transform is a special case of the above theorem and corollary.

Another closely related integral operator is the Riesz potential, $I_\alpha(f)(\mathbf{x}) = (-\Delta)^{-\alpha/2} f(\mathbf{x}),\ 0 < \alpha < n$, is defined on $L^p[\mathbb{R}^n],\ 1 < p < \infty$, by (see Stein [STE], p. 117):

$$I_\alpha(f)(\mathbf{x}) = \gamma^{-1}(\alpha) \int_{\mathbb{R}^n} \frac{f(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}}, \quad \text{and } \gamma(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Since the kernel is symmetric, application of Fubini's Theorem shows that the adjoint $I_\alpha^* = I_\alpha$ is also defined on $L^p[\mathbb{R}^n]$. Since $(-\Delta)^{-1}$ is not bounded, we cannot obtain $L^p$ bounds for $I_\alpha(f)(\mathbf{x})$. However, if $1/q = 1/p - \alpha/n$, we have the following (see Stein [STE], p. 119)

**Theorem 5.13.** *If $f \in L^p[\mathbb{R}^n]$ and $0 < \alpha < n,\ 1 < p < q < \infty,\ 1/q = 1/p - \alpha/n$, then the integral defining $I_\alpha(f)$ converges absolutely for almost all $\mathbf{x}$. Furthermore, there is a constant $A_{p,q}$, such that*

$$\|I_\alpha(f)\|_q \leqslant A_{p,q}\|f\|_p. \tag{5.7}$$

**5.1.3. Extension of the Adjoint.** In this section we discuss an extension of the adjoint for a Banach space $\mathcal{B}$, which need not be uniformly convex. If $\mathcal{B}$ is not uniformly convex, Theorem 5.5 no longer holds and we need $\mathcal{H}_1$. The next theorem shows that, for $A$ bounded, we can always define a reasonable version of the adjoint $A^*$, which has many of the essential properties that we find for a Hilbert space.

**Theorem 5.14.** *Let $A$ be a bounded linear operator on $\mathcal{B}$. Then $A$ has a well-defined adjoint $A^*$ defined on $\mathcal{B}$ such that:*

(1) *the operator $A^*A \geq 0$ (accretive),*

(2) *$(A^*A)^* = A^*A$ (naturally self-adjoint), and*

(3) *$I + A^*A$ has a bounded inverse.*

**Proof.** For $i = 1, 2$, let $\mathbf{J}_i : \mathcal{H}_i \to \mathcal{H}_i'$. As in Theorem 5.5, $\mathbf{J}_i^* = \mathbf{J}_i$. Now, let $A_1 = A_{|\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_2$, and $A'_1 : \mathcal{H}_2' \to \mathcal{H}_1'$.

It follows that $A'_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_1'$ and $\mathbf{J}_1^{-1}A'_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_1 \subset \mathcal{B}$ so that, if we define $A^* = [\mathbf{J}_1^{-1}A'_1\mathbf{J}_2]_{\mathcal{B}}$, then $A^* : \mathcal{B} \to \mathcal{B}$ (i.e., $A^* \in L[\mathcal{B}]$).

To prove (1), let $g \in \mathcal{B}$, then $(A^*Ag, g)_{\mathcal{H}_2} \geq 0$ for all $g \in \mathcal{B}$. Hence $\langle A^*Ag, g^*\rangle \geq 0$ for all $g^* \in J(g)$ (the duality map of $g$), so that $A^*A$ is accretive.

To prove (2), we have for $g \in \mathcal{H}_1$,

$$\begin{aligned}(A^*A)^*g &= (\{\mathbf{J}_1^{-1}[\{[\mathbf{J}_1^{-1}A'{}_1\mathbf{J}_2]|_{\mathcal{B}}A\}_1]'\mathbf{J}_2\}|_{\mathcal{B}})g \\ &= (\{\mathbf{J}_1^{-1}[\{A'{}_1[\mathbf{J}_2A_1\mathbf{J}_1^{-1}]|_{\mathcal{B}}\}]\mathbf{J}_2\}|_{\mathcal{B}})g \\ &= A^*Ag.\end{aligned}$$

It follows that the same result holds on all of $\mathcal{B}$.

The proof of (3), that $I + A^*A$ is invertible, follows the same lines as in von Neumann's theorem. □

Since $A^*A$ is self-adjoint on $\mathcal{B}$ (in the sense of (2) above), it is natural to expect that the same is true on $\mathcal{H}_2$. However, this need not be the case. To obtain a simple counterexample, recall that, in standard notation, the simplest class of bounded linear operators on $\mathcal{B}$ is $\mathcal{B} \otimes \mathcal{B}'$, in the sense that:

$$\mathcal{B} \otimes \mathcal{B}' : \mathcal{B} \to \mathcal{B}, \text{ by } Au = (b \otimes l_{b'}(\cdot))u = \left\langle b', u \right\rangle b.$$

Thus, if $l_{b'}(\cdot) \in \mathcal{B}' \backslash \mathcal{H}_2'$, then $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(u)\}$ is not in $\mathcal{H}_2'$, so that $A^*A$ is not defined as an operator on all of $\mathcal{H}_2$ and thus cannot have a bounded extension.

We now provide the correct extension of Lax's Theorem.

**Theorem 5.15.** *Let $A$ be a bounded linear operator on $\mathcal{B}$. If $\mathcal{B}' \subset \mathcal{H}_2$, then $A$ has a bounded extension to $L[\mathcal{H}_2]$, with $\|A\|_{\mathcal{H}_2} \le k \|A\|_{\mathcal{B}}$ (for some positive $k$).*

**Proof.** We first note that if $g, h \in \mathcal{B}$, then $\mathbf{J}_1^{-1}\mathbf{J}_2(g) = g$ and $(A_1')'h = Ah$. Now let $T = A^*A$, then

$$\begin{aligned}&(Tg, h)_{\mathcal{H}_2} = \langle Tg, \mathbf{J}_2(h)\rangle \\ &= \langle A^*Ag, \mathbf{J}_2(h)\rangle = \left\langle \mathbf{J}_1^{-1}A_1'\mathbf{J}_2(Ag), \mathbf{J}_2(h)\right\rangle \\ &= \left\langle A_1'\mathbf{J}_2(Ag), h\right\rangle = \left\langle \mathbf{J}_2(Ag), (A_1')'h\right\rangle \\ &= \langle Ag, \mathbf{J}_2(Ah)\rangle = \left\langle g, (A_1')\mathbf{J}_2(Ah)\right\rangle \\ &= \left\langle \mathbf{J}_1^{-1}\mathbf{J}_2(g), (A_1')\mathbf{J}_2(Ah)\right\rangle = \left\langle \mathbf{J}_2(g), \mathbf{J}_1^{-1}(A_1')\mathbf{J}_2(Ah)\right\rangle \\ &= (g, Th)_{\mathcal{H}_2}\end{aligned}$$

We can now apply Lax's Theorem to see that, for some $k$, $\|T\|_{\mathcal{H}_2} = \|A\|_{\mathcal{H}_2}^2 \le k^2 \|A\|_{\mathcal{B}}^2$. □

**Remark 5.16.** Thus, the algebra $L[\mathcal{B}]$ also has a $^*$operation for all Banach spaces with an S-basis and $\mathcal{B}' \subset \mathcal{H}_2$. However, if $\mathcal{B}$ is not uniformly convex and $A \neq B$, $B'$ then, unless

$$(AB\,|_{\mathcal{H}_1})' = (B\,|_{\mathcal{H}_1})'\,(A\,|_{\mathcal{H}_1})', \quad (AB)^* \neq A^*B^*.$$

A natural question is "which Banach spaces with an S-basis have the property that, $\mathcal{B}' \subset \mathcal{H}_2$"? This question has no general answer. However, if $\mathcal{B}$ is one of the following classical Banach spaces and $\mathcal{H}_2 = KS^2[\mathbb{R}^n]$, then $\mathcal{B}' \subset \mathcal{H}_2$ ($\mathcal{H}_1 = GS^2[\mathbb{R}^n]$). A few of the spaces below are not separable (do not have an S-basis).

(1) $\mathbb{C}_b[\mathbb{R}^n]$, the bounded continuous functions on $\mathbb{R}^n$.
(2) $\mathbb{C}_u[\mathbb{R}^n]$, the bounded uniformly continuous functions on $\mathbb{R}^n$.
(3) $\mathbb{C}_0^k[\mathbb{R}^n]$, the continuous functions on $\mathbb{R}^n$, with $k$ derivatives that vanish at infinity.
(4) $L^p[\mathbb{R}^n]$, $1 \leq p \leq \infty$, the Lebesgue integrable functions on $\mathbb{R}^n$ of order $p$.
(5) $\mathfrak{M}[\mathbb{R}^n]$, the space of finitely additive set functions (measures) on $\mathbb{R}^n$.

We note that both $\mathbb{C}_b[\mathbb{R}^n]$ and $L^\infty[\mathbb{R}^n]$ are nonseparable Banach spaces, with the same dual space $\mathfrak{M}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ and, the dual space of $\mathbb{C}_u[\mathbb{R}^n]$, $\mathbb{C}_u'[\mathbb{R}^n] \subset \mathfrak{M}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. In each case, we can use Theorem 5.15.

## 5.2. Semigroups of Operators

**Introduction.** Semigroups of operators form the basis for both the Feynman operator calculus and path integral theory of Chaps. 7 and 8. We have restricted our presentation to those aspects that are absolutely necessary and should even be reviewed those with some training in the subject. We provide all of the basic results along with proofs, for those without prior background.

The theory of semigroups of operators is a fairly mature field of study, which has continued to attract the interest of those in analysis, probability theory, partial differential equations, dynamical systems, and quantum theory, in addition to the many areas of applied mathematics. This continued interest is expected because of the simple (conceptual) framework provided, the robustness of the technical methodology, and the wealth of problems and new applications.

Those interested in the finer details are encouraged to seek out the wealth of interesting material by consulting some of the major works in the field. See the standards by Hille and Phillips [HP], Yosida [YS], Kato [K], Pazy [PZ], Goldstein [GS] and the recent ones by Engel and Nagel [EN] and Vrabie [VR]. The book by Vrabie [VR] offers a number of new and interesting applications.

We develop most of the theory for a fixed separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and will assume when convenient that $\mathcal{H} = KS^2[\mathbb{R}^n]$. However, we begin with the general theory on a Banach space $\mathcal{B}$.

**Definition 5.17.** A family of linear operators $\{S(t), 0 \le t < \infty\}$ (not necessarily bounded), defined on $\mathcal{D} \subset \mathcal{B}$, is a semigroup if

(1) $S(t+s)f = S(t)S(s)f$ for $f \in \mathcal{D}$, the domain of the semigroup.
(2) The semigroup is said to be strongly continuous if $\lim_{\tau \to 0} S(t+\tau)f = S(t)f$ for all $f \in \mathcal{D},\ t > 0$.
(3) It is said to be a $C_0$-semigroup if it is strongly continuous, $S(0) = I,\ \mathcal{D} = \mathcal{B}$ and $\lim_{t \to 0} S(t)f = f$ for all $f \in \mathcal{B}$.
(4) $S(t)$ is a $C_0$-contraction semigroup if $\|S(t)\|_{\mathcal{B}} \leqslant 1$.
(5) $S(t)$ is a $C_0$-unitary group if $S^*(t)$ exists and $S(t)S(t)^* = S(t)^*S(t) = I$, and $\|S(t)\|_{\mathcal{B}} = 1$.

**Definition 5.18.** For a $C_0$-semigroup $S(t)$, the linear operator $A$ defined by

$$D(A) = \left\{ f \in \mathcal{B} \,\middle|\, \lim_{t \downarrow 0} \tfrac{1}{t}[S(t)f - f] \text{ exists} \right\} \quad \textit{and}$$

$$Af = \lim_{t \downarrow 0} \tfrac{1}{t}[S(t)f - f] = \left. \frac{d^+ S(t)f}{dt} \right|_{t=0} \quad \text{for} \quad f \in D(A)$$

is the infinitesimal generator of the semigroup $S(t)$ and $D(A)$ is the domain of $A$.

**Lemma 5.19.** *Let $S(t)$ be a $C_0$-semigroup. Then there exist constants $\omega \ge 0$ and $M \ge 1$ such that:*

$$\|S(t)\|_{\mathcal{H}} \leqslant M e^{\omega t}, \quad \text{for} \quad 0 \leqslant t < \infty.$$

**Proof.** If $\|S(t)\|_{\mathcal{B}}$ is not bounded in any interval $0 \le t \le m,\ m > 0$, then there is a nonnegative sequence $t_n$ such that $\lim_{n\to\infty} t_n = 0$ and $\|S(t_n)\|_{\mathcal{B}} \geqslant n$. By the uniform boundedness theorem it follows that, for some $f$, $S(t)f$ is unbounded. But then $S(t)$ is not strongly continuous (see (3) above). Thus $\|S(t)\|_{\mathcal{B}} \leqslant M$ for $0 \leq t \leq m$. From $\|S(0)\|_{\mathcal{B}} = 1$ and $M \geqslant 1$, we can choose $\omega = m^{-1} log M$. Let $t \geq 0$ be given, then $t = nm + \delta$, where $0 \leq \delta < m$, so, by the semigroup property of $S(t)$, we have:

$$\|S(t)\|_{\mathcal{B}} = \|S(\delta)S(m)^n\|_{\mathcal{H}} \leqslant M^{n+1} \leqslant MM^{t/m} = Me^{\omega t}.$$

□

**Theorem 5.20.** *Let $S(t)$ be a $C_0$ contraction semigroup and let $A$ be its infinitesimal generator. Then*

(1) *For all $f \in \mathcal{B}$, we have*

$$\lim_{h \to 0} \tfrac{1}{h} \int_t^{t+h} S(u) f du = S(t) f.$$

(2) *For all $f \in \mathcal{B}$, $\int_0^t S(u) f du \in D(A)$ and,*

$$A \int_0^t S(u) f du = S(t) f - f.$$

(3) *For all $f \in D(A)$,*

$$\frac{d}{dt} S(t) f = AS(t) f = S(t) Af.$$

(4) *For all $f \in D(A)$,*

$$S(t) f - S(u) f = \int_u^t AS(\tau) f d\tau = \int_u^t S(\tau) Af d\tau.$$

(5) *$A$ is closed and $\overline{D(A)} = \mathcal{B}$.*

(6) *The resolvent set $\rho(A)$ of $A$ contains $R^+$ and, for every $\lambda > 0$,*

$$\|R(\lambda, A)\|_{\mathcal{B}} \leqslant \frac{1}{\lambda}.$$

**Proof.** The proof of (1) follows from the strong continuity of $S(t)$. To prove (2), let $f \in \mathcal{B}$ and suppose that $h > 0$. Then

$$\frac{S(h) - I}{h} \int_0^t S(u) f du = \tfrac{1}{h} \int_0^t (S(u+h) f - S(u) f) du$$
$$= \tfrac{1}{h} \int_t^{t+h} S(u) f du - \tfrac{1}{h} \int_0^h S(u) f du$$

and, as $h \searrow 0$, the right-hand side tends to $S(t)f - f$. To prove (3), if $f \in D(A)$ and $h > 0$, we have

$$\frac{S(h)-I}{h} S(t)f = S(t)\left(\frac{S(h)-I}{h}\right) f \xrightarrow{h \to 0} S(t)Af.$$

It follows that $S(t)f \in D(A)$ and $S(t)Af = AS(t)f$. This also means that

$$\frac{d^+}{dt} S(t)f = AS(t)f = S(t)Af.$$

To complete our proof, we need to show that, for $t > 0$, the left-hand derivative exists and is equal to $S(t)Af$. To prove this, note that

$$\begin{aligned} &\lim_{h \searrow 0} \left[\frac{S(t)f - S(t-h)f}{h} - S(t)Af\right] \\ &= \lim_{h \searrow 0} S(t-h)\left(\frac{S(h)f - f}{h} - Af\right) + \lim_{h \searrow 0} \left(S(t-h)Af - S(t)Af\right). \end{aligned}$$

We are done since the limit of both terms on the right is zero. To prove (4), we need to only look at the integral of $\frac{d}{dt} S(t)f = AS(t)f = S(t)Af$. To prove (5), for each $f \in \mathcal{B}$ set $f_h = \frac{1}{h}\int_0^h S(u)f du$. By (2), $f_h \in D(A)$ and, by (1), $f_h \to f$, so that $\overline{D(A)} = \mathcal{B}$. To prove that $A$ is closed, let $f_n \in D(A)$, $f_n \to f$ and $Af_n \to g$ (as $n \to \infty$). From (4), we have that

$$S(t)f_n - f_n = \int_0^t S(u)Af_n du \to S(t)f - f = \int_0^t S(u)Ag du.$$

If we divide the last integral by $t$ and let $t \searrow 0$, we see from (1) that $f \in D(A)$ and $Af = g$. The proof of (6) requires a little additional work. If $f \in \mathcal{H}$ and $\lambda > 0$, define a bounded linear operator $R(\lambda, A)$ by (the Laplace transform of $S(t)$):

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} S(t)f dt.$$

Since the function $t \to S(t)f$ is continuous and uniformly bounded, the integral exists and provides a well-defined linear operator with

$$\|R(\lambda, A)f\|_{\mathcal{B}} \leqslant \int_0^\infty e^{-\lambda t} \|S(t)f\|_{\mathcal{B}} dt \leqslant \tfrac{1}{\lambda} \|f\|_{\mathcal{B}}.$$

For $h > 0$,

$$\begin{aligned} &\frac{S(h)-I}{h} R(\lambda, A)f = \tfrac{1}{h}\int_0^\infty e^{-\lambda t}\left(S(t-h)f - S(t)f\right) \\ &= \frac{e^{-\lambda h}-1}{h}\left(\int_0^\infty e^{-\lambda t} S(t)f dt\right) - \frac{e^{-\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)f dt \underset{h \searrow 0}{\to} \lambda R(\lambda, A)f - f. \end{aligned}$$

Thus, we see that, for every $\lambda > 0$ and $f \in \mathcal{B}$, $R(\lambda, A)f \in D(A)$ and $AR(\lambda, A)f = \lambda R(\lambda, A)f - f \Rightarrow (\lambda I - A)R(\lambda, A)f = f$. We also have that, for $f \in D(A)$,

$$R(\lambda, A)Af = \int_0^\infty e^{-\lambda t} S(t)Af dt = \int_0^\infty e^{-\lambda t} AS(t)f dt$$
$$= A\left[\int_0^\infty e^{-\lambda t} S(t)f dt\right] = AR(\lambda, A)f.$$

It now follows that $R(\lambda, A)(\lambda I - A)f = f$ for each $f \in D(A)$, so that $R(\lambda, A)$ is the inverse of $(\lambda I - A)$ for all $\lambda > 0$ and

$$\|R(\lambda, A)f\|_{\mathcal{B}} \leqslant \tfrac{1}{\lambda} \|f\|_{\mathcal{B}}.$$

□

**Lemma 5.21.** *Suppose that $R(\lambda, A) = (\lambda I - A)^{-1}$, where $A$ is a linear operator such that:*

(1) *$A$ is closed and $\overline{D(A)} = \mathcal{B}$.*

(2) *The resolvent set $\rho(A)$ of $A$ contains $R^+$ and, for every $\lambda > 0$,*

$$\|R(\lambda, A)\|_{\mathcal{B}} \leqslant 1/\lambda.$$

*Then $\lim_{\lambda \to \infty} \lambda R(\lambda, A)f = f$ for all $f \in \mathcal{B}$.*

**Proof.** For each $f \in D(A)$, we have that

$$\|\lambda R(\lambda, A)f - f\|_{\mathcal{B}} = \|AR(\lambda, A)f\|_{\mathcal{B}} = \|R(\lambda, A)Af\|_{\mathcal{B}} \leqslant \tfrac{1}{\lambda} \|Af\|_{\mathcal{B}} \xrightarrow{\lambda \to \infty} 0.$$

Since $D(A)$ is dense and $\|\lambda R(\lambda, A)\|_{\mathcal{B}} \leqslant 1$, as $\lambda \to \infty$, $\lambda R(\lambda, A)f \to f$ for each $f \in \mathcal{B}$. □

**5.2.1. Hilbert Space.** We now look at the case when $\mathcal{B} = \mathcal{H}$ is a Hilbert space.

**Definition 5.22.** For each $\lambda > 0$, we define the Yosida approximator by: $A_\lambda = \lambda AR(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I$.

The next result is due to Yosida and applies to generators of strongly continuous semigroups defined on $[0, \infty)$. We will prove a generalized version of the theorem, which applies to strongly continuous semigroups $(0, \infty)$.

**Theorem 5.23.** *(Yosida) Let $A$ be a closed linear operator with $\overline{D(A)} = \mathcal{H}$. If the resolvent set $\rho(A)$ of $A$ contains $R^+$ and, for every $\lambda > 0$, $\|R(\lambda, A)\|_{\mathcal{H}} \leqslant \lambda^{-1}$. Then*

(1) $\lim_{\lambda \to \infty} A_\lambda f = Af \quad \text{for } f \in D(A)$.

(2) $A_\lambda$ *is a bounded generator of a contraction semigroup and, for each* $f \in \mathcal{H}$, $\lambda$, $\mu > 0$, *we have:*

$$\left\| e^{tA_\lambda} f - e^{tA_\mu} f \right\|_{\mathcal{H}} \leqslant t \left\| A_\lambda f - A_\mu f \right\|_{\mathcal{H}}.$$

If all we know is that $A$ is the generator of a strongly continuous semigroup $S(t) = \exp(tA)$ for $t > 0$, the above result is not enough. Unfortunately, for general strongly continuous semigroups, $A$ may not have a bounded resolvent. The following (artificial example) shows what can (and will) happen in some real cases.

**Example 5.24.** *Let* $\mathcal{H} = \mathbf{H}_0(\mathbb{R}^n)$ *be the Hilbert space* (over $\mathbb{R}$) *of functions mapping* $\mathbb{R}^n$ *to itself, which vanish at infinity. Consider the Cauchy problem:*

$$\frac{d}{dt}\mathbf{u}(\mathbf{x},t) = a\,|\mathbf{x}|\,\mathbf{u}(\mathbf{x},t),\ \ \mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}),$$

*where* $a = \prod_{i=1}^{n} sign(x_i)$. *Let* $S(t)\mathbf{f}(\mathbf{x}) = \mathrm{e}^{ta|\mathbf{x}|}\,\mathbf{f}(\mathbf{x})$, *where* $\mathbf{x} = [x_1, \cdots, x_n]^t$. *It is easy to see that* $S(t)$ *is a semigroup on* $\mathcal{H}$ *with generator* $A$ *such that* $A\mathbf{f}(\mathbf{x}) = a\,|\mathbf{x}|\,\mathbf{f}(\mathbf{x})$. *It follows that* $u(\mathbf{x},t) = S(t)\mathbf{f}(\mathbf{x})$ *solves the above initial-value problem. If we compute the resolvent, we get that:*

$$R(\lambda, A)\mathbf{f}(\mathbf{x}) = \int_0^\infty e^{-\lambda t} \exp\{-t\,|\mathbf{x}|\}\mathbf{f}(\mathbf{x})dt = \frac{1}{\lambda - a\,|\mathbf{x}|}\mathbf{f}(\mathbf{x}).$$

*It is clear that the spectrum of* $A$ *is the real line, so that* $R(\lambda, A)$ *is an unbounded operator for all real* $\lambda$. *However, it can be checked that the bounded linear operator*

$$A_\lambda = a\lambda|\mathbf{x}|/[\lambda + |\mathbf{x}|]$$

*converges strongly to* $A$ *(on* $D(A)$*) as* $\lambda \to \infty$, *and*

$$\lim_{\lambda \to 0} S_\lambda(t)\mathbf{f}(\mathbf{x}) = S(t)\mathbf{f}(\mathbf{x}).$$

As an application of the polar decomposition, the next result shows that the Yosida approach can be generalized in such a way as to give a contractive approximator for all strongly continuous semigroups of operators on $\mathcal{H}$.

For any closed densely defined linear operator $A$ on $\mathcal{H}$, let $T = -[A^*A]^{1/2}$, $\bar{T} = -[AA^*]^{1/2}$. Since $T(\bar{T})$ is m-dissipative, it generates a contraction semigroup. We can now write $A$ as $A = VT$, where $V = -U$ is the unique partial isometry of Chap. 4. Define $A_\lambda$ by $A_\lambda = \lambda AR(\lambda, T)$. Note that $A_\lambda = \lambda UTR(\lambda, T) = \lambda^2 UR(\lambda, T) - \lambda U$ and, although $A$ does not commute with $R(\lambda, T)$, we have $\lambda AR(\lambda, T) = \lambda R(\lambda, \bar{T})A$.

**Theorem 5.25.** *(Generalized Yosida) Let $A$ be a closed densely defined linear operator on $\mathcal{H}$. Then*

(1) $A_\lambda = \lambda AR(\lambda, T)$ *is a bounded linear operator and* $\lim_{\lambda\to\infty} A_\lambda f = Af$*, for all* $f \in D(A)$*,*

(2) $\exp[tA_\lambda]$ *is a bounded contraction for* $t > 0$*, and*

(3) *if* $S(t) = \exp[tA]$ *is defined on* $\mathcal{D}$*,* $D(A) \subset \mathcal{D}$*, then for* $t > 0$*,* $f \in \mathcal{D}$*,* $\lim_{\lambda\to\infty} \|\exp[tA_\lambda]f - \exp[tA]f\|_{\mathcal{H}} = 0$*.*

**Proof.** To prove (1), let $f \in D(A)$. Now use the fact that

$$\lim_{\lambda\to\infty} \lambda R(\lambda, \bar{T})f = f$$

and $A_\lambda f = \lambda R(\lambda, \bar{T})Af$. To prove (2), use

$$A_\lambda = \lambda^2 UR(\lambda, T) - \lambda U$$

with $\|\lambda R(\lambda, T)\|_{\mathcal{H}} = 1$, and $\|U\|_{\mathcal{H}} = 1$ to get that

$$\|\exp[t\lambda^2 UR(\lambda, T) - t\lambda U]\|_{\mathcal{H}} \le \exp[-t\lambda\|U\|_{\mathcal{H}}]\exp[t\lambda\|U\|_{\mathcal{H}}\|\lambda R(\lambda, T)\|_{\mathcal{H}}] \le 1.$$

To prove (3), let $t > 0$ and $f \in D(A)$. Then

$$\begin{aligned} \|\exp[tA]f - \exp[tA_\lambda]f\|_{\mathcal{H}} &= \|\int_0^t \frac{d}{ds}[e^{(t-s)A_\lambda}e^{sA}]f ds\|_{\mathcal{H}} \\ &\le \int_0^t \|[e^{(t-s)A_\lambda}(A - A_\lambda)e^{sA}f]\|_{\mathcal{H}} \\ &\le \int_0^t \|[(A - A_\lambda)e^{sA}f]\|_{\mathcal{H}} ds. \end{aligned}$$

Now use

$$\|[A_\lambda e^{sA}f]\|_{\mathcal{H}} = \|[\lambda R(\lambda, \bar{T})e^{sA}Af]\|_{\mathcal{H}} \le \|[e^{sA}Af]\|_{\mathcal{H}},$$

to get

$$\|[(A - A_\lambda)e^{sA}f]\|_{\mathcal{H}} \le 2\|[e^{sA}Af]\|_{\mathcal{H}}.$$

Since $\|[e^{sA}Af]\|_{\mathcal{H}}$ is continuous, by the bounded convergence theorem we have

$$\lim_{\lambda\to\infty} \|\exp[tA]f - \exp[tA_\lambda]f\|_{\mathcal{H}} \le \int_0^t \lim_{\lambda\to\infty} \|[(A - A_\lambda)e^{sA}f]\|_{\mathcal{H}} ds = 0.$$

Thus, $S(t)f$ exists and the convergence is uniform on bounded intervals for $t > 0$ and all $f \in D(A)$. Since $D(A)$ is dense in $\mathcal{D}$, $S(t)$ can be extended to all of $\mathcal{D}$. $\square$

**Remark 5.26.** The first result (1) provides an independent proof that every closed densely defined linear operator on a Hilbert space is of first Baire class (may be approximated by bounded linear operators on its domain).

We now turn to the main theorem for semigroups of linear operators.

**Theorem 5.27.** (Hille–Yosida Theorem) *A linear operator $A$ is the generator of a $C_0$-semigroup of contractions $S(t)$, $t \geq 0$, if and only if $A$ is closed, densely defined, $\mathbb{R}^+ \subset \rho(A)$ and, for every $\lambda > 0$, $\|R(\lambda, A)\|_{\mathcal{H}} \leq \lambda^{-1}$.*

**Proof.** The necessity is shown in Theorem 5.23. To prove sufficiency, from Theorem 5.25, we see that, if $A$ is closed and densely defined, with

$$\|R(\lambda, A)\|_{\mathcal{H}} \leq \frac{1}{\lambda}$$

for $\lambda > 0$, then, for $\mu > 0$ we have

$$\left\|e^{tA_\lambda} f - e^{tA_\mu} f\right\|_{\mathcal{H}} \leqslant t\,\|A_\lambda f - A_\mu f\|_{\mathcal{H}} \leq t\,\|A_\lambda f - Af\|_{\mathcal{H}} + t\,\|Af - A_\mu f\|_{\mathcal{H}}\,.$$

It follows that for $f \in D(A)$, $e^{tA_\lambda} f$ converges as $\lambda \to \infty$ and the convergence is uniform on bounded intervals. Since $\left\|e^{tA_\lambda} f\right\|_{\mathcal{H}} \leq 1$, it follows that $e^{tA_\lambda} f \to S(t)$ for every $f \in \mathcal{H}$. It is clear that $S(t)$ is a semigroup and that $\left\|e^{tA}\right\|_{\mathcal{H}} \leq 1$, with $S(0) = 1$. Thus, $S(t)$ is a $C_0$-semigroup, since it is strongly continuous. Finally,

$$e^{tA_\lambda} f - f = \int_0^t e^{sA_\lambda} A_\lambda f ds \to \int_0^t e^{sA} Af ds = e^{tA} f - f,$$

so that $A$ is the generator. $\square$

**5.2.2. Lumer–Phillips Theory.** We now discuss the characterization of an infinitesimal generator of a $C_0$-semigroup of contractions, due to Lumer and Phillips [LP].

**Definition 5.28.** Let $A$ be a linear operator on $\mathcal{H}$. $A$ is said to be dissipative if

$$\operatorname{Re}\langle Af, f\rangle \leqslant 0 \text{ for all } f \in D(A).$$

**Theorem 5.29.** (Lumer–Phillips) *Let $A$ be a linear operator on $\mathcal{H}$; then*

(1) *$A$ is dissipative if and only if*

$$\|(\lambda I - A) f\|_{\mathcal{H}} \geqslant \lambda \|f\|_{\mathcal{H}} \quad \text{for all } f \in D(A) \quad \text{and all } \lambda > 0.$$

(2) *If $D(A)$ is dense in $\mathcal{H}$ and there is a $\lambda_0$ such that $Ran(\lambda_0 I - A) = \mathcal{H}$, then $A$ is the generator of a $C_0$ semigroup of contractions.*

(3) *If $A$ is the generator of a $C_0$ semigroup of contractions on $\mathcal{H}$, then $Ran(\lambda I - A) = \mathcal{H}$ for all $\lambda > 0$ and $A$ is dissipative.*

**Remark 5.30.** We note that (2) implies that $A$ is m-dissipative, while (3) asserts that every generator of a contraction semigroup is m-dissipative.

**Proof.** To prove (1), let $A$ be dissipative, $f \in D(A)$ and $\lambda > 0$. If $\operatorname{Re}\langle Af, f\rangle \leq 0$ then:

$$\|(\lambda I - A) f\| \, \|_{\mathcal{H}} f\| \geq |\langle(\lambda I - A) f, f\rangle| \geq \operatorname{Re}\langle(\lambda I - A) f, f\rangle \geq \lambda \|f\|_{\mathcal{H}}^2 .$$

It follows that $\|(\lambda I - A) f\|_{\mathcal{H}} \geq \lambda \|f\|_{\mathcal{H}}$. Conversely, assume that $\lambda \|f\|_{\mathcal{H}} \leq \|(\lambda I - A) f\|_{\mathcal{H}}$ for $f \in D(A)$ and all $\lambda > 0$. If we square both sides, an easy calculation shows that

$$\|Af\|_{\mathcal{H}}^2 - 2\lambda \operatorname{Re}\langle Af, f\rangle \geq 0.$$

Since this is true for all $\lambda > 0$, we see that $\operatorname{Re}\langle Af, f\rangle \leq 0$. To prove (2), note that since $A$ is dissipative we can use (1) for $\lambda > 0$ to get that $\|(\lambda I - A) f\|_{\mathcal{H}} \geqslant \lambda \|f\|_{\mathcal{H}}$ for all $f \in D(A)$. Since $Ran(\lambda_0 I - A) = \mathcal{H}$, with $\lambda = \lambda_0$, it follows that $(\lambda_0 I - A)^{-1}$ is a bounded linear operator. But this means that it is a closed operator, so that $(\lambda_0 I - A)$ and hence $A$ is also a closed operator. Now note that if $Ran(\lambda I - A) = \mathcal{H}$ for every $\lambda > 0$, then $(0, \infty) \subset \rho(\lambda)$ and $\|R(\lambda, A)\|_{\mathcal{H}} \leq \lambda^{-1}$. It will then follow by Theorem 5.27 (Hille–Yosida) that $A$ is the generator of a $C_0$ contraction semigroup. Thus, we need to show that $Ran(\lambda I - A) = \mathcal{H}$ for every $\lambda > 0$. Let

$$\Lambda = \{\, \lambda : 0 < \lambda < \infty \} \quad \text{and} \quad Ran(\lambda I - A) = \mathcal{H}.$$

If $\lambda \in \Lambda$, $\lambda \in \rho(\lambda)$. As $\rho(\lambda)$ is an open set, there is a nonempty neighborhood of $\lambda \subset \rho(\lambda)$. It follows that the intersection of this neighborhood with $\mathbb{R}$ is in $\Lambda$, so that $\Lambda$ is an open set. If $\lambda_n \in \Lambda$, $\lambda_n \to \lambda > 0$, then, for every $g \in \mathcal{H}$, there exists a $f_n \in D(A)$ such that

$$\lambda_n f_n - A f_n = g. \tag{5.8}$$

Since $A$ is dissipative, we have that $\|f_n\|_{\mathcal{H}} \leq \lambda_n^{-1} \|g\|_{\mathcal{H}} \leq C$ for some $C > 0$. We also have that:

$$\begin{aligned}\lambda_m \|f_n - f_m\|_{\mathcal{H}} &\leq \|\lambda_m (f_n - f_m) - A(f_n - f_m)\|_{\mathcal{H}} \\ &= |\lambda_n - \lambda_m| \, \|f_n\|_{\mathcal{H}} \leq C \, |\lambda_n - \lambda_m| ,\end{aligned}$$

so that $\{f_n\}$ is a Cauchy sequence. If we let $f_n \to f$, we see from (5.5) that $Af_n \to \lambda f - g$. As $A$ is closed, $f \in D(A)$ and $\lambda f - Af = g$. It follows that $Ran(\lambda I - A) = \mathcal{H}$ and $\lambda \in \Lambda$ so that $\Lambda$ is also closed in $(0, \infty)$. Since $\lambda_0 \in \Lambda$, we see that $\Lambda \neq \emptyset$ and therefore $\Lambda = (0, \infty)$.

To prove (3), we first observe that if $A$ is the generator of a $C_0$ contraction semigroup $S(t)$ on $\mathcal{H}$, then it is closed and densely defined. Furthermore, by Theorem 5.27 (Hille–Yosida), $(0, \infty) \subset \rho(A)$ and $Ran(\lambda I - A) = \mathcal{H}$ for all $\lambda > 0$. If $f \in D(A)$ then

$$|\langle S(t)f, f\rangle| \leqslant \|S(t)f\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \leqslant \|f\|_{\mathcal{H}}^2$$

so that

$$\operatorname{Re}\langle S(t)f - f, f\rangle = \operatorname{Re}\langle S(t)f, f\rangle - \|f\|_{\mathcal{H}}^2 \leq 0.$$

If we divide the above equation by $t > 0$ and let $t \downarrow 0$, we get that:

$$\operatorname{Re}\langle Af, f\rangle \leqslant 0,$$

so that $A$ is dissipative. □

The next result follows from the Lumer–Phillips Theorem (see Remark 5.30).

**Theorem 5.31.** *Suppose $A$ is a densely defined m-dissipative operator. Then $A$ is the generator of a $C_0$ semigroup $S(t)$ of contraction operators on $\mathcal{H}$.*

**Theorem 5.32.** *If $A$ is closed and densely defined on $\mathcal{H}$, with both $A$ and $A^*$ dissipative, then $A$ is m-dissipative.*

**Proof.** It suffices show that $Ran(I - A) = \mathcal{H}$. Since $A$ is both closed and dissipative, $Ran(I - A)$ is closed in $\mathcal{H}$. If $Ran(I - A) \neq \mathcal{H}$ then there is a nonzero $g \in \mathcal{H}$ such that $(f - Af, g) = 0$ for all $f \in D(A)$. This implies that $(g, g - A^*g) = \|g\|^2 - (g, A^*g) = 0$, so that $g - A^*g = 0$. Since $A^*$ is dissipative, from part (1) of Theorem 5.29 (Lumer–Phillips), we must have that $g = 0$. But this is a contradiction since we assumed that $g \neq 0$. □

We now consider an important class of operators which generates $C_0$-contractions. The next result is due to Vrabie [VR].

**Theorem 5.33.** *Suppose $-A$ is a closed densely defined positive self-adjoint operator. Then $A$ is the generator of a $C_0$-contraction semigroup $S(t)$. Furthermore, if $f \in \mathcal{H}$ and $h(t) = S(t)f$, then the problem:*

$$h'(t) = Ah(t),\ h(0) = f, \tag{5.9}$$

*has an unique solution*

$$h \in D(A) \cap \mathbb{C}^1((0, \infty); \mathcal{H})$$

*and*

$$\|Ah(t)\|_{\mathcal{H}} \leq \tfrac{1}{2t}\|f\|_{\mathcal{H}}.$$

**Proof.** First, since $-A$ is a positive, self-adjoint, closed, and densely defined linear operator on $\mathcal{H}$, it follows that both $A$ and $A^* = A$ are dissipative. Hence, by Theorem 5.29, $A$ is m-dissipative so that A generates a $C_0$-contraction semigroup and for $Re(\lambda) > 0,\ \|R(\lambda, T)\|_{\mathcal{H}} \leq \frac{1}{Re(\lambda)}$.

It is clear that both $S(t)$ and $A$ determine each other uniquely on $D(A)$, so that, at least for $f \in D(A)$, the solution to (5.6) is unique. If $f \in D(A^2)$, we see that, since $(h''(t), h'(t)) = (Ah'(t), h'(t))$, the problem

$$h''(t) = Ah'(t),\ h(0) = f,$$

has an unique solution. Thus, with $(h''(t), h'(t)) = (Ah'(t), h'(t))$ and, for $0 \leq s \leq t$, we have

$$\frac{1}{2}\left\|h'(t)\right\|_{\mathcal{H}}^2 - \frac{1}{2}\left\|h'(s)\right\|_{\mathcal{H}}^2 = \int_s^t (Ah'(\tau), h'(\tau))d\tau \leq 0$$

(since $A$ is dissipative). This shows that $\|h'(t)\|_{\mathcal{H}}$ is a nonincreasing function. Furthermore,

$$\frac{d}{dt}\|h(t)\|_{\mathcal{H}}^2 = 2(Ah(t), h(t)) \tag{5.10}$$

and

$$\frac{d}{dt}(Ah(t), h(t)) = 2(Ah(t), Ah(t)) = 2\left\|h'(t)\right\|_{\mathcal{H}}^2 \geq 0. \tag{5.11}$$

It follows that $(Ah(t), h(t))$ is nondecreasing. If we integrate Eq. (5.7) from $0 \to t$, we have:

$$\|h(t)\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{H}}^2 = 2\int_0^t (Ah(\tau), h'(\tau))d\tau \leq 2t(Ah(t), h(t)),$$

$$\Rightarrow\ -t(Ah(t), h(t)) \leq -\int_0^t (Ah(\tau), h'(\tau))d\tau = -\frac{1}{2}\left\|h'(t)\right\|_{\mathcal{H}}^2 + \frac{1}{2}\|f\|_{\mathcal{H}}^2 \leq \frac{1}{2}\|f\|_{\mathcal{H}}^2.$$

Now recall that $\|h'(t)\|_{\mathcal{H}}$ is a nonincreasing function and integrate equation (5.8) from $0 \to t$ to get

$$(Ah(t), h(t)) - (Af, f) = 2\int_0^t \left\|h'(\tau)\right\|_{\mathcal{H}}^2 d\tau \geq 2t\left\|h'(t)\right\|_{\mathcal{H}}^2.$$

Since $(Ah(t), h(t)) \leq 0$, we see that $2t\,\|h'(t)\|_{\mathcal{H}}^2 \leq (-Af, f)$. If we now multiply both sides of Eq. (5.7) by $t$ and integrate, we see that

$$2t^2\left\|h'(t)\right\|_{\mathcal{H}}^2 \leq \int_0^t \tau(Ah(\tau), h'(\tau))d\tau = \int_0^t \tau\frac{d}{d\tau}(Ah(\tau), h(\tau))d\tau$$
$$= t(Ah(t), h(t)) - \int_0^t \tau(Ah(\tau), h(\tau))d\tau.$$

Since $t(Ah(t), h(t)) \leq 0$, we see from the inequality above and Eq. (5.7) that $4t^2\,\|Ah(t)\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}}^2$ so that

$$\|Ah(t)\|_{\mathcal{H}} \leq \frac{\|f\|_{\mathcal{H}}}{2t}.$$

□

The next result shows that we can recover the semigroup as the inverse Laplace transform of the resolvent. It will be important for our study of analytic semigroups in the next section.

**Theorem 5.34.** *Let $A$ be a closed densely defined dissipative linear operator on $\mathcal{H}$ satisfying:*

(1) *For some $0 < \delta < \pi/2$,*
$$\rho(A) \supset \Sigma_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}.$$
(2) *The resolvent of $A$ satisfies $\|R(\lambda, A)\| \leqslant 1/|\lambda|$, for each $\lambda \in \Sigma_\delta$, with $\lambda \neq 0$.*

*Then $A$ is the generator of a $C_0$-contraction semigroup $S(t)$, which can be represented as:*

$$S(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)d\lambda, \tag{5.12}$$

*where $\Gamma$ is a smooth curve in $\Sigma_\delta$ going from $\infty e^{-i\theta} \to \infty e^{i\theta}$, for $\pi/2 < \theta < \pi/2+\delta$ and the integral converges in the uniform topology for $t > 0$.*

**Proof.** Let

$$Z(t) = \frac{1}{2\pi i}\int_\Gamma e^{\mu t}R(\mu, A)d\mu. \tag{5.13}$$

Since $\|R(\mu, A)\| \leqslant 1/|\mu|$, we see from the definition of $\Sigma_\delta$ that, for $t > 0$, this integral converges in the uniform norm. In order to see that $Z(t)$

is a semigroup, suppose that $Z(s)$ also has the above representation, with another slightly shifted path $\Gamma'$ inside $\Sigma_\delta$. Then

$$Z(s)Z(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'}\int_{\Gamma} e^{\mu t}R(\mu, A)e^{\mu' t}R(\mu', A)d\mu d\mu'$$
$$= \left(\frac{1}{2\pi i}\right)^2 \left[\int_{\Gamma'} e^{\mu' s}R(\mu', A)d\mu' \int_{\Gamma} e^{\mu t}(\mu - \mu')^{-1}d\mu\right.$$
$$\left. - \int_{\Gamma} e^{\mu t}R(\mu, A)d\mu \int_{\Gamma'} e^{\mu' s}(\mu - \mu')^{-1}d\mu'\right],$$

where we have used the resolvent equation, $R(\mu', A)R(\mu, A) = (\mu - \mu')^{-1}R(\mu', A) - R(\mu, A)$, in the second line. If we now use the fact that:

$$\int_{\Gamma'} e^{\mu' s}(\mu - \mu')^{-1}d\mu' = 2\pi i e^{\mu s} \ \& \ \int_{\Gamma'} e^{\mu t}(\mu - \mu')^{-1}d\mu = 0,$$

we get that

$$Z(s)Z(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\mu(t+s)}R(\mu, A)d\mu = Z(t+s).$$

Since the resolvent uniquely determines the semigroup, we are done if we can show that $R(\lambda, A)$ is the resolvent of $Z(t)$. To do this, use the fact that $R(\lambda, A)$ is analytic in $\Sigma_\delta$, so that we can shift the path of integration to a new path $\Gamma_t$, still inside $\Sigma_\delta$. We choose $\Gamma_t = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 = \{re^{-i\theta} : t^{-1} \le r < \infty\}$, $\Gamma_2 = \{t^{-1}e^{i\phi} : -\theta \le \phi \le \theta\}$ and $\Gamma_3 = \{re^{i\theta} : t^{-1} \le r < \infty\}$ without changing the value of the integral. In this case, for the path $\Gamma_3$, we have

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_3} e^{\mu t}R(\mu, A)d\mu\right\|_{\mathcal{H}} \leqslant \frac{1}{2\pi}\int_{t^{-1}}^{\infty} e^{-rt\sin(\theta-\pi/2)}r^{-1}dr$$
$$= \frac{1}{2\pi}\int_{\sin(\theta-\pi/2)}^{\infty} e^{-s}s^{-1}ds \leqslant C_1.$$

For the path $\Gamma_2$, we see that

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_2} e^{\mu t}R(\mu, A)d\mu\right\|_{\mathcal{H}} \leqslant \frac{1}{2\pi}\int_{-\theta}^{\theta} e^{\cos(\phi)}d\phi \leqslant C_2.$$

The estimate for $\Gamma_1$ is like that of $\Gamma_3$. This shows that $Z(t)$ is bounded by some constant $K$ for $0 < t < \infty$. Now, if we multiply Eq. (5.10) by $e^{-\lambda t}$ and integrate from 0 to $T$, using Fubini's Theorem along with the residue theorem, we have

$$\int_0^T e^{-\lambda t} Z(t)dt = \frac{1}{2\pi i}\int_0^T e^{-\lambda t}\left[\int_\Gamma e^{\mu t}R(\mu, A)d\mu\right]dt$$
$$= \frac{1}{2\pi i}\int_\Gamma\left[\int_0^T e^{(\mu-\lambda)t}dt\right]R(\mu, A)d\mu = \frac{1}{2\pi i}\int_\Gamma \frac{\left(e^{(\mu-\lambda)T}-1\right)}{\mu-\lambda}R(\mu, A)d\mu$$
$$= R(\lambda, A) + \frac{1}{2\pi i}\int_\Gamma e^{(\mu-\lambda)T}\frac{R(\mu, A)}{\mu-\lambda}d\mu.$$

However, on $\Gamma$,

$$\left\|\frac{1}{2\pi i}\int_\Gamma e^{(\mu-\lambda)T}\frac{R(\mu, A)}{\mu-\lambda}d\mu\right\| \leqslant e^{-\lambda T}\int_\Gamma \frac{d\,|\mu|}{|\mu|\,|\mu-\lambda|} \to 0, \quad T\to\infty.$$

Thus, if we take the limit in our equation, we get

$$\int_0^\infty e^{-\lambda t}Z(t)dt = R(\lambda, A).$$

Since for $Re(\lambda) > 0$, $\frac{1}{|\lambda|} \leq \frac{1}{Re(\lambda)}$, we see that $Z(t) = S(t)$ is a contraction semigroup. □

**5.2.3. Analytic Semigroups.** Let $\Delta = \{w \in \mathbf{C} : \theta_1 < \arg w < \theta_2,\ \theta_1 < 0 < \theta_2\}$. For each $w \in \Delta$, let $S(w)$ be a bounded linear operator on $\mathcal{H}$.

**Definition 5.35.** The family $S(w)$ is said to be an analytic semigroup on $\mathcal{H}$, for $w \in \Delta$, if

(1) $S(w)f$ is an analytic function of $w \in \Delta$ for each $f$ in $\mathcal{H}$,
(2) $S(0) = I$ and $\lim_{w\to 0} S(w)f = f$ for every $f \in \mathcal{H}$,
(3) $S(w_1 + w_2) = S(w_1)S(w_2)$ for $w_1,\ w_2 \in \Delta$.

**Theorem 5.36.** *Let $S(t)$ be a $C_0$-contraction semigroup and let $A$ be the generator of $S(t)$, with $0 \in \rho(A)$. Suppose $A$ satisfies:*

(1) *For $0 < \delta < \pi/2$,*
$$\rho(A) \supset \Sigma_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}.$$
(2) $\|R(\lambda, A)\| \leqslant M/|\lambda|$ *for each $\lambda \in \Sigma_\delta$, with $\lambda \neq 0$.*

*Then the following are equivalent:*

(1) *$S(t)$ is differentiable for $t > 0$ and there is a constant $C$ such that*
$$\|AS(t)\|_{\mathcal{H}} \leq \frac{C}{t} \text{ for } t > 0.$$

(2) *For* $t > 0$ *and* $|z - t| \leqslant Kt$ *for some constant* $K$*, the series*

$$S(z+t) = S(t) + \sum_{n=1}^{\infty} (z^n/n!) S^{(n)}(t)$$

*converges uniformly in the above interval.*

(3) $S(t)$ *can be extended to a* $C_0$*-analytic semigroup* $S(z)$*, for* $z \in \bar{\Delta}_{\delta'}$*, with* $\bar{\Delta}_{\delta'} = \{z : |\arg z| \leqslant \delta' < \delta\}$.

**Proof.** From Eq. (5.9), $S(t) = (1/2\pi i) \int_\Gamma e^{\lambda t} R(\lambda, A) d\lambda$, where $\Gamma$ is a smooth curve in $\Sigma_\delta$ composed of two rays $\rho e^{i\theta}$ and $\rho e^{-i\theta}$, $0 < \rho < \infty$ and $\pi/2 < \theta < \pi/2 + \delta$ and $\Gamma$ is oriented so that $\mathrm{Im}(\lambda)$ increases along $\Gamma$. The integral converges in the uniform topology for $t > 0$. If we differentiate it formally, we see that:

$$S'(t) = \frac{1}{2\pi i} \int_\Gamma \lambda e^{\lambda t} R(\lambda, A) d\lambda.$$

However, this integral converges in $\mathcal{H}$ for all $t > 0$, since

$$\left\| S'(t) \right\| \leqslant (1/\pi) \int_0^\infty e^{-\rho \cos\theta t} d\rho = \frac{1}{\pi t \cos\theta} = \left( \frac{1}{\pi \cos\theta} \right) \frac{1}{t}. \quad (5.14)$$

Thus, the formal differentiation is justified for $t > 0$ and

$$\|AS(t)\|_{\mathcal{H}} \leq \frac{C}{t}, \text{ where } C = \frac{1}{\pi \cos\theta}.$$

We now prove that $S(t)$ has derivatives of any order, by induction. From above, we know it is true for $k = 1$. Suppose that it is true for $k = n$ and $t \geqslant s$, then

$$S^{(n)}(t) = (AS(t/n))^n = S(t-s)\,(AS(s/n))^n. \quad (5.15)$$

If we differentiate Eq. (5.12) with respect to $t$ we have

$$S^{(n+1)}(t) = (AS(t/n))^n = AS(t-s)\,(AS(s/n))^n.$$

Now set $s = nt/(n+1)$ to get $S^{(n+1)}(t) = [AS(t/(n+1)]^{n+1}$, so that $S(t)$ has derivatives of all orders. If we use this result in Eq. (5.11), and the fact that $n!e^n \geqslant n^n$, we get that:

$$\frac{1}{n!} \left\| S^{(n)}(t) \right\| \leqslant \left( \frac{Ce}{t} \right)^n.$$

Now, consider the power series

$$S(z) = S(t) + \sum_{n=1}^{\infty} \frac{S^{(n)}(t)}{n!} (z-t)^n.$$

The series converges uniformly in $L[\mathcal{H}]$ for $|z - t| \leqslant Kt$, where $K = k/eC$, $0 < k < 1$. Thus, $S(z)$ is analytic in $\Delta = \{z : |\arg z| <$

$\arctan K\}$ and hence extends $S(t)$. It is easy to check that $S(z)$ is a $C_0$-contraction semigroup in any closed subsector $\bar{\Delta}_\varepsilon = \{z : |\arg z| \leqslant \arctan M - \varepsilon\}$ of $\Delta$. □

**5.2.4. Perturbation Theory.** One of the major concerns for the theory of semigroups of operators is to identify conditions under which the sum of two generators is a generator (when properly understood). We restrict our attention to generators of analytic contraction semigroups. (In practice, by the use of an equivalent norm and a shift in the spectrum, most semigroups of interest can be reduced to contractions.) The next result shows when the sum of generators of analytic contraction semigroups generate an analytic contraction semigroup.

**Theorem 5.37.** *Let $A_0$ be an m-dissipative generator of an analytic $C_0$-semigroup and let $A_1$ be closed on $\mathcal{H}$, with $D(A_1) \supseteq D(A_0)$. Suppose and there are positive constants $0 \leq \alpha < 1$, $\beta \geq 0$ such that*

$$\|A_1\varphi\| \leqslant \alpha \|A_0\varphi\| + \beta \|\varphi\|, \ \varphi \in D(A_0). \tag{5.16}$$

*Then $A = A_0 + A_1$, with domain $D(A) = D(A_1)$, generates an analytic $C_0$ semigroup.*

**Remark 5.38.** We note that, by the Closed Graph Theorem, it suffices to assume that $A_1$ is dissipative and $D(A_1) \supseteq D(A_0)$ in order to find constants $0 \leq \alpha < 1$, $\beta \geq 0$ satisfying Eq. (5.13).

**Proof.** To prove our result, first use the fact that $A_0$ generates an analytic $C_0$-semigroup to find a sector $\Sigma$ in the complex plane, with $\rho(A_0) \supset \Sigma$ ($\Sigma = \{\lambda : \ |\arg \lambda| < \pi/2 + \delta'\}$, for some $\delta' > 0$), and for $\lambda \in \Sigma$, $\|R(\lambda :, A_0)\|_{\mathcal{H}} \leqslant |\lambda|^{-1}$. From (5.13), $A_1 R(\lambda, A_0)$ is a bounded operator and:

$$\begin{aligned}
\|A_1 R(\lambda, A_0)\varphi\|_{\mathcal{H}} &\leqslant \alpha \|A_0 R(\lambda, A_0)\varphi\|_{\mathcal{H}} + \beta \|R(\lambda, A_0)\varphi\|_{\mathcal{H}} \\
&\leqslant \alpha \|[R(\lambda, A_0) - I]\,\varphi\|_{\mathcal{H}} + \beta |\lambda|^{-1} \|\varphi\|_{\mathcal{H}} \\
&\leqslant 2\alpha \|\varphi\|_{\mathcal{H}} + \beta |\lambda|^{-1} \|\varphi\|_{\mathcal{H}}.
\end{aligned}$$

Thus, if we set $\alpha = 1/4$ and $|\lambda| > 2\beta$, we have $\|A_1 R(\lambda, A_0)\|_{\mathcal{H}} < 1$ and it follows that the operator $I - A_1 R(\lambda, A_0)$ is invertible. Now it is easy to see that:

$$(\lambda I - (A_0 + A_1))^{-1} = R(\lambda, A_0)\,(I - A_1 R(\lambda, A_0))^{-1}. \tag{5.17}$$

Using $|\lambda| > 2\beta$, with $|\arg\lambda| < \pi/2 + \delta''$ for some $\delta'' > 0$, and the fact that $A_0$ and $A_1$ are m-dissipative generators, we get from (5.14) that

$$\|R(\,\lambda\,,\,A_0 + A_1)\|_{\mathcal{B}} \leqslant |\lambda|^{-1}\,.$$

Thus, $A$ generates a $C_0$-analytic semigroup. Finally, we note that if $Re(\lambda) > 0$, then $\frac{1}{|\lambda|} \leq \frac{1}{Re(\lambda)}$, so that $A$ also generates a $C_0$-contraction semigroup. □

**Corollary 5.39.** *Let $A_0$ be the generator of an analytic $C_0$-semigroup and suppose that $A_1$ is bounded. Then $A_0 + A_1$ is the generator of an analytic $C_0$-semigroup on $\mathcal{H}$.*

**Corollary 5.40.** *Let $A$, $A_1$ be generators of $C_0$-contraction semigroups on $\mathcal{H}$ and assume that $A_1$ is bounded. Then $A + A_1$ is the generator of a $C_0$-contraction semigroup $S(t)$.*

Theorem 5.25 shows that all closed densely defined linear operators on $\mathcal{H}$ may be approximated by bounded generators of contraction semigroups. This leads to the following result, which shall prove quite useful later.

**Theorem 5.41.** *Let $A_0$, $A_1$* and *$A_0 + A_1$ be generators of contraction semigroups on $\mathcal{H}$, with a common dense domain. Then:*

$$\lim_{\lambda\to\infty} \exp\left\{(A_0 + A_{1,\lambda})\,t\right\}\varphi = \exp\left\{(A_0 + A_1)\,t\right\}\varphi \quad \text{for} \quad t > 0.$$

**Proof.** The proof is standard. Set $A = A_0 + A_1$, & $A_\lambda = A_0 + A_{1,\lambda}$; then, for $\varphi \in D(A_0) \cap D(A_1)$:

$$\begin{aligned}
\left\|\left(e^{tA_\lambda} - e^{tA}\right)\varphi\right\|_{\mathcal{H}} &= \left\|\int_0^1 \frac{d}{ds}\left[e^{tsA_\lambda}e^{t(1-s)A}\right]\varphi ds\right\|_{\mathcal{H}} \\
&= \left\|t\int_0^1 \left[e^{tsA_\lambda}A_\lambda e^{t(1-s)A} - e^{tsA_\lambda}Ae^{t(1-s)A}\right]\varphi ds\right\|_{\mathcal{H}} \\
&= \left\|t\int_0^1 \left[e^{tsA_\lambda}\left(A_\lambda - A\right)e^{t(1-s)A}\right]\varphi ds\right\|_{\mathcal{H}} \\
&\leqslant t\sup_{s\geq 0}\left\|\left(A_\lambda - A\right)e^{t(1-s)A}\varphi\right\|_{\mathcal{H}} = t\left\|\left(A_{1,\lambda} - A_1\right)e^{t(1-\bar{s})A}\varphi\right\|_{\mathcal{H}},
\end{aligned}$$

where $\bar{s}$ is the point in $[0, 1]$ where the sup is attained. The limit of this last term is clearly zero. (Note that $A_\lambda$ need not commute with $A$.) □

We reserve our proof of the next result until Chap. 7 (see [K1]). There, we will use it to provide a very general version of the Feynman–Kac formula.

**Theorem 5.42.** Trotter–Kato product formula *Suppose that* $A_0$, $A_1$ *and* $A=\overline{A_0+A_1}$ *are generators of* $C_0$*-contraction semigroups* $T_0(t)$, $T_1(t)$ *and* $T(t)$ *on* $\mathcal{H}$. *Then, for* $\varphi \in \mathcal{H}$, *we have*

$$\lim_{n\to\infty} \{T_0(\tfrac{t}{n})T_1(\tfrac{t}{n})\}^n\varphi = T(t)\varphi.$$

**5.2.5. Semigroups on Banach Spaces.** The purpose of this section is to show that the Hilbert space theory is sufficient for the theory on separable Banach spaces. We assume that "$\mathcal{B}$ is rigged," so that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings.

**Theorem 5.43.** *Suppose that* $A$ *generates a* $C_0$*-contraction semigroup* $T(t)$, *on* $\mathcal{B}$ *and* $\mathcal{B}' \subset \mathcal{H}_2$ *then:*

(1) $A$ *has a closed densely defined extension* $\bar{A}$ *to* $\mathcal{H}_2$, *which is also the generator of a* $C_0$*-contraction semigroup.*

(2) $\rho(\bar{A}) = \rho(A)$ *and* $\sigma(\bar{A}) = \sigma(A)$.

(3) *The adjoint of* $\bar{A}$, $\bar{A}^*$, *restricted to* $\mathcal{B}$, *is the adjoint* $A^*$ *of* $A$, *that is:*
   - *the operator* $A^*A \geqslant 0$,
   - $(A^*A)^* = A^*A$ *and*
   - $I + A^*A$ *has a bounded inverse.*

**Proof. Part I**

Let $T(t)$ be the $C_0$-contraction semigroup generated by $A$. By Theorem 5.15, $T(t)$ has a bounded extension $\bar{T}(t)$ to $\mathcal{H}_2$.

We prove that $\bar{T}(t)$ is a $C_0$-semigroup. (The fact that it is a contraction semigroup will follow later.) It is clear that $\bar{T}(t)$ has the semigroup property. To prove that it is strongly continuous, use the fact that $\mathcal{B}$ is dense in $\mathcal{H}_2$ so that, for each $g \in \mathcal{H}_2$, there is a sequence $\{g_n\}$ in $\mathcal{B}$ converging to $g$. We then have:

$$\begin{aligned}
&\lim_{t\to 0}\left\|\bar{T}(t)g - g\right\|_2 \leqslant \lim_{t\to 0}\left\{\left\|\bar{T}(t)g - \bar{T}(t)g_n\right\|_2 + \left\|\bar{T}(t)g_n - g_n\right\|_2\right\} + \|g_n - g\|_2 \\
&\leqslant k\|g - g_n\|_2 + \lim_{t\to 0}\left\|\bar{T}(t)g_n - g_n\right\|_2 + \|g_n - g\|_2 \\
&= (k+1)\|g - g_n\|_2 + \lim_{t\to 0}\|T(t)g_n - g_n\|_2 = (k+1)\|g - g_n\|_2,
\end{aligned}$$

where we have used the fact that $\bar{T}(t)g_n = T(t)g_n$ for $g_n \in \mathcal{B}$, and $k$ is the constant in Theorem 5.15. It is clear that we can make the last term on the right as small as we like by choosing $n$ large enough, so that $\bar{T}(t)$ is a $C_0$-semigroup.

To prove (2), note that if $\bar{A}$ is the extension of $A$, and $\lambda I - \bar{A}$ has an inverse, then $\lambda I - A$ also has one, so $\rho(\bar{A}) \subset \rho(A)$ and $Ran(\lambda I - A)_{\mathcal{B}} \subset Ran(\lambda I - \bar{A})_{\mathcal{H}_2} \subset \overline{Ran(\lambda I - A)}_{\mathcal{H}_2}$ for any $\lambda \in \mathbb{C}$. For the other direction, since $A$ generates a $C_0$-contraction semigroup, $\rho(A) \neq \emptyset$. Thus, if $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1}$ is a continuous mapping from $Ran(\lambda I - A)$ onto $D(A)$ and $Ran(\lambda I - A)$ is dense in $\mathcal{B}$. Let $g \in D(\bar{A})$, so that $(g, \bar{A}g) \in \hat{G}(A)$, the closure of the graph of $A$ in $\mathcal{H}_2$. Thus, there exists a sequence $\{g_n\} \subset D(A)$ such that $\|g - g_n\|_G = \|g - g_n\|_{\mathcal{H}_2} + \left\|\bar{A}g - \bar{A}g_n\right\|_{\mathcal{H}_2} \to 0$ as $n \to \infty$. Since $\bar{A}g_n = Ag_n$, it follows that $(\lambda I - \bar{A})g = \lim_{n\to\infty}(\lambda I - A)g_n$. However, by the boundedness of $(\lambda I - A)^{-1}$ on $Ran(\lambda I - A)$, we have that, for some $\delta > 0$,

$$\left\|(\lambda I - \bar{A})g\right\|_{\mathcal{H}_2} = \lim_{n\to\infty} \|(\lambda I - A)g_n\|_{\mathcal{H}_2} \geq \lim_{n\to\infty} \delta \|g_n\|_{\mathcal{H}_2} = \delta \|g\|_{\mathcal{H}_2}.$$

It follows that $\lambda I - \bar{A}$ has a bounded inverse and since $D(A) \subset D(\bar{A})$ implies that $Ran(\lambda I - A) \subset Ran(\lambda I - \bar{A})$, we see that $Ran(\lambda I - \bar{A})$ is dense in $\mathcal{H}_2$ so that $\lambda \in \rho(\bar{A})$ and hence $\rho(A) \subset \rho(\bar{A})$. It follows that $\rho(A) = \rho(\bar{A})$ and necessarily, $\sigma(A) = \sigma(\bar{A})$.

Since $A$ generates a $C_0$-contraction semigroup, it is m-dissipative. From the Lumer–Phillips Theorem, we have that $Ran(\lambda I - A) = \mathcal{B}$ for $\lambda > 0$. It follows that $\bar{A}$ is m-dissipative and $Ran(\lambda I - \bar{A}) = \mathcal{H}_2$. Thus, $\bar{T}(t)$ is a $C_0$-contraction semigroup.

We now observe that the same proof applies to $\bar{T}^*(t)$, so that $\bar{A}^*$ is also the generator of a $C_0$-contraction semigroup on $\mathcal{H}_2$.

Clearly $\bar{A}^*$ is the adjoint of $\bar{A}$ so that, from von Neumann's Theorem, $\bar{A}^*\bar{A}$ has the expected properties. $\bar{\mathbf{D}} = D(\bar{A}^*\bar{A})$ is a core for $\bar{A}$ (i.e., the set of elements $\{g, \bar{A}g\}$ is dense in the graph, $G[\bar{A}]$, of $\bar{A}$ for $g \in \bar{\mathbf{D}}$). From here, we see that the restriction $A^*$ of $\bar{A}^*$ to $\mathcal{B}$ is the generator of a $C_0$-contraction semigroup and $\mathbf{D} = D(A^*A)$ is a core for $A$. The proof of (3) for $A^*A$ now follows. □

**Remark 5.44.** Theorem 5.43 shows that all $C_0$-contraction semigroups defined on $\mathcal{B}$ have the same properties as its extension to $\mathcal{H}_2$. Thus, if $\mathcal{B}$ is reflexive or $\mathcal{B}' \subset \mathcal{H}_2$, then all the theorems on $\mathcal{H}_2$ apply to $\mathcal{B}$.

The next result implies that the generalized Yosida Approximation Theorem applies to $C_0$-semigroups on $\mathcal{B}$

**Theorem 5.45.** *Let $A \in \mathcal{C}[\mathcal{B}]$ be the generator of a $C_0$-contraction semigroup. Then there exists an m-accretive operator $T$ and a partial isometry $W$ such that $A = WT$ and $D(A) = D(T)$.*

**Proof.** The fact that $\mathcal{B}' \subset \mathcal{H}_2$ ensures that $A^*A$ is a closed self-adjoint operator on $\mathcal{B}$ by Theorem 5.40. Furthermore, both $A$ and $A^*$ have closed densely defined extensions $\bar{A}$ and $\bar{A}^*$ to $\mathcal{H}_2$. Thus, the operator $\hat{T} = [\bar{A}^*\bar{A}]^{1/2}$ is a well-defined m-accretive self-adjoint linear operator on $\mathcal{H}_2$, $\bar{A} = \bar{W}\bar{T}$ for some partial isometry $\bar{W}$ defined on $\mathcal{H}_2$, and $D(\bar{A}) = D(\bar{T})$. Our proof is complete when we notice that the restriction of $\bar{A}$ to $\mathcal{B}$ is $A$ and $\bar{T}^2$ restricted to $\mathcal{B}$ is $A^*A$, so that the restriction of $\bar{W}$ to $\mathcal{B}$ is well defined and must be a partial isometry. The equality of the domains is obvious. □

With respect to our definition of natural self-adjointness, the following related definition is due to Palmer [PL], where the operator is called symmetric. This is essentially the same as a Hermitian operator as defined by Lumer [LU].

**Definition 5.46.** A closed densely defined linear operator $A$ on $\mathcal{B}$ is called self-conjugate if both $iA$ and $-iA$ are dissipative.

**Theorem 5.47.** *(Vidav–Palmer) A linear operator $A$, defined on $\mathcal{B}$, is self-conjugate if and only if $iA$ and $-iA$ are generators of isometric semigroups.*

**Theorem 5.48.** *The operator $A$, defined on $\mathcal{B}$, is self-conjugate if and only if it is naturally self-adjoint.*

**Proof.** Let $\bar{A}$ and $\bar{A}^*$ be the closed densely defined extensions of $A$ and $A^*$ to $\mathcal{H}_2$. On $\mathcal{H}_2$, $\bar{A}$ is naturally self-adjoint if and only if $i\bar{A}$ generates a unitary group, if and only if it is self-conjugate. Thus, both definitions coincide on $\mathcal{H}_2$. It follows that the restrictions coincide on $\mathcal{B}$. □

Additional discussion of the adjoint for operators on Banach spaces can be found in the Appendix (Sect. 5.3).

## 5.3. Appendix

The appendix is devoted to a number of topics that are not directly related to our main direction, but have independent interest for functional analysis and operator theory. We first discuss the existence of an adjoint for spaces that are not uniformly convex. We then apply our results in subsequent sections to show that the spectral theory that is natural for Hilbert spaces and the Schatten theory of compact operators can also be partially extended to Banach spaces.

## 5.4. The Adjoint in the General Case

In this section we continue our discussion of the adjoint for an operator on Banach space with an S-basis $\mathcal{B}$, which is not uniformly convex.

**5.4.1. The General Case for Unbounded $A$.** A Banach space is said to have the approximation property if every compact operator is the limit of operators of finite rank. It is known that every classical Banach space has the approximation property. However, it is also known that there are separable Banach spaces without the approximation property (see Diestel [**DI**]). Theorem 5.15 tells us that if $\mathcal{B}' \subset \mathcal{H}_2$, then $L[\mathcal{B}] \subset L[\mathcal{H}_2]$ as a continuous embedding. (It's not hard to show that if $\mathcal{B}$ has the approximation property, the embedding is dense.)

Let $A \in \mathcal{C}[\mathcal{B}]$, the closed densely defined linear operators on $\mathcal{B}$. By definition, $A$ is of Baire class one if it can be approximated by a sequence, $\{A_n\}$, of bounded linear operators. In this case, it is natural to define $A^* = s\text{-}\lim A_n^*$ (see below). However, if $\mathcal{B}$ is not uniformly convex there may be operators $A \in \mathcal{C}[\mathcal{B}]$ that are not of Baire class one, so that it is not reasonable to expect Theorem 5.11 to hold for all of $\mathcal{C}[\mathcal{B}]$. First, we note that every uniformly convex Banach space is reflexive. In order to understand the problem, we need the following:

**Definition 5.49.** A Banach space $\mathcal{B}$ is said to be:

(1) quasi-reflexive if $\dim\{\mathcal{B}''/\mathcal{B}\} < \infty$, and
(2) nonquasi-reflexive if $\dim\{\mathcal{B}''/\mathcal{B}\} = \infty$.

A theorem by Vinokurov et al. [**VPP**] shows that, for every nonquasi-reflexive Banach space $\mathcal{B}$ (for example, $C[0;1]$ or $L^1[\mathbb{R}^n]$, $n \in \mathbb{N}$), there is at least one closed densely defined linear operator $A$, which is not of Baire class one. It can even be arranged so that $A^{-1}$ is a bounded linear injective operator (with a dense range). This means, in particular, that there does not exist a sequence of bounded linear operators $A_n \in L[\mathcal{B}]$ such that, for $g \in D(A)$, $A_n g \to Ag$, as $n \to \infty$. The following result shows that whenever $\mathcal{B}' \subset \mathcal{H}_2$, every operator of Baire class one has an adjoint.

**Theorem 5.50.** *If $A \in \mathcal{C}[\mathcal{B}]$ and $\mathcal{B}' \subset \mathcal{H}_2$, then $A$ is in the first Baire class if and only if it has an adjoint $A^* \in \mathcal{C}[\mathcal{B}]$.*

**Proof.** Let $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ and suppose that $A$ has an adjoint $A^* \in \mathcal{C}[\mathcal{B}]$. Let $T = [A^*A]^{1/2}$, $\bar{T} = [AA^*]^{1/2}$. Since $T$ is m-accretive and naturally self-adjoint, for all $\alpha > 0$, $I + \alpha T$ has a bounded inverse $S(\alpha) = (I + \alpha T)^{-1}$. It is easy to see that $AS(\alpha)$ is bounded and, for $g \in D(A)$, $AS(\alpha)g = \bar{S}(\alpha)Ag = (I + \alpha\bar{T})^{-1}Ag$. Using this result, we have:

$$\lim_{\alpha \to 0^+} AS(\alpha)g = \lim_{\alpha \to 0^+} \bar{S}(\alpha)Ag = Ag, \text{ for } g \in D(A).$$

It follows that $A$ is in the first Baire class.

To prove the converse suppose that $A \in \mathcal{C}[\mathcal{B}]$ is of first Baire class. If $\{A_n\}$ is a sequence of bounded linear operators with $A_n g \to Ag$, for all $g \in D(A)$, then each $A_n$ has an adjoint $A_n^*$. Since $\mathcal{B}' \subset \mathcal{H}_2$, each $A_n A_n^*$ has a bounded extension $\bar{A}_n \bar{A}_n^*$ to $\mathcal{H}_2$. Furthermore, since $A$ is densely defined, it has a closed densely defined extension $\bar{A}$ on $\mathcal{H}_2$. Let $\bar{A}^*$ be the adjoint of $\bar{A}$. Then, for all $g \in D(A)$, $h \in \mathcal{B}$, we have:

$$\begin{aligned}&\lim_{n\to\infty} (A_n g, h)_{\mathcal{H}_2} = \lim_{n\to\infty} (g, A_n^* h)_{\mathcal{H}_2} = (Ag, h)_{\mathcal{H}_2} \\ &= \left(\bar{A}g, h\right)_{\mathcal{H}_2}\end{aligned}$$

From here, we see that $A^* = \lim_{n\to\infty} A_n^*$ is a densely defined linear operator. If we let $D(A^*) \subset \mathcal{B}$ be the dense set, then for $h \in D(A^*)$

$$\lim_{n\to\infty} (g, A_n^* h)_{\mathcal{H}_2} = \lim_{n\to\infty} (g, A^* h)_{\mathcal{H}_2} = \left(g, \bar{A}^* h\right)_{\mathcal{H}_2},$$

so that $A^*$ is the restriction of $\bar{A}^*$ to $\mathcal{B}$. □

**Corollary 5.51.** *If $A \in \mathcal{C}[\mathcal{B}]$ is in the first Baire class and $\mathcal{B}' \subset \mathcal{H}_2$, then $A = WT$, where $W$ is a partial isometry and $T = [A^*A]^{1/2}$.*

5.4.1.1. **The Adjoint Is Not Unique.** In this section we show that if $A$ is defined on a fixed Banach space $\mathcal{B}$, then two different Hilbert space riggings can produce two different adjoints for $A$.

Recall that a regular $\sigma$-finite measure on the $\sigma$-algebra of Borel sets of a Hausdorff topological space is called a Radon measure, and a function $u$ is of bounded variation on $\Omega$, or $u \in BV[\Omega]$, if $u \in L^1[\Omega]$ and there is a Radon vector measure $Du$ such that

$$\int_\Omega u(x)\nabla\phi(x)dx = -\int_\Omega \phi(x)Du(x),$$

for all functions $\phi \in \mathbb{C}_c^\infty[\Omega, \mathbb{R}^n]$, the $\mathbb{R}^n$-valued infinitely differentiable functions on $\Omega$ with compact support. It is easy to see that $W_0^{1,1}[\Omega] \subset BV[\Omega]$. (In this case, we can show that $Du(x) = \nabla u(x)dx$.)

Let us return to the two pair of Hibert spaces $H_0^1[\Omega] \subset \mathbb{C}_0[\Omega] \subset H_0[\Omega]$ and $\mathcal{H}_1[\Omega] \subset \mathbb{C}_0[\Omega] \subset \mathcal{H}_2[\Omega]$ of Example 3.32 in Chap. 3.

Let $A = [-\Delta]$ be defined on $\mathbb{C}_0[\Omega]$, with domain:

$$D_c(A) = \{\Delta u \in \mathbb{C}_0[\Omega] \,|u = 0 \text{ on } \partial\Omega\}.$$

It is easy to see that $A$ extends to a self-adjoint operator on $H_0[\Omega]$, with domain

$$D_2(A) = \{\Delta u \in H_0[\Omega] \,|u=0 \text{ on } \partial\Omega \text{ and, } \nabla u \text{ is absolutely continuous}\}.$$

To begin, we first compute the adjoint $A^*$, of $A$ directly as an operator on $\mathbb{C}_0[\Omega]$. The dual space of $\mathbb{C}_0[\Omega]$ is $\mathbb{C}_0^*[\Omega] = rca[\Omega]$, the space of regular countable additive measures on $\Omega$.

It follows from

$$\langle Au, v\rangle = -\int_\Omega \Delta u(x)v(x)dx,$$

that

$$\langle u, A^*v\rangle = -\int_\Omega u(x)\Delta v(x)dx$$

and

$$D_c(A^*) = \{u : \Delta u \in BV[\Omega] \,|\, u = 0 \text{ on } \partial\Omega\},$$

so that $D_c(A) \subset D_c(A^*)$ (proper). Thus, if we restrict $A^*$ to $D_c(A)$ it becomes a self-adjoint operator on $\mathbb{C}_0[\Omega]$ without the rigging.

We now investigate the adjoint obtained from use of the first rigging, $H_0^1[\Omega] \subset \mathbb{C}_0[\Omega] \subset H_0[\Omega]$ (see Barbu [B], p. 4). In this case, $\mathbf{J}_1 = [-\Delta]$ and $\mathbf{J}_2 = \mathbf{I}_2$, the identity operator on $H_0[\Omega]$, so that

$$A_1^* = \mathbf{J}_1^{-1}A_1'\mathbf{J}_2, = \mathbf{I}_2.$$

In the second rigging, $\mathcal{H}_1[\Omega] \subset \mathbb{C}_0[\Omega] \subset \mathcal{H}_2[\Omega]$, constructed in Example 3.10 in Chap. 3, we have

$$A_2^* = \mathbf{J}_1^{-1}A_1'\mathbf{J}_2.$$

In this case,

$$\mathbf{J}_1(v) = \sum\nolimits_{n=1}^\infty t_n^{-1}(e_n, v)_2(\,\cdot, e_n)_2, \quad \mathbf{J}_2(v) = \sum\nolimits_{n=1}^\infty t_n\bar{F}_n(v)F_n(\,\cdot\,)$$

and

$$(e_n, v)_2 = \sum\nolimits_{k=1}^\infty t_k\bar{F}_k(v)F_k(e_n) = t_n\bar{F}_n(v),$$

so that $\mathbf{J}_1(v) = \sum_{n=1}^{\infty} \bar{F}_n(v)(\,\cdot\,, e_n)_2$. However,

$$(\,\cdot\,, e_n)_2 = \sum_{k=1}^{\infty} t_k \bar{F}_k(e_n) F_k(\,\cdot\,) = t_n F_n(\,\cdot\,), \text{ so that } \mathbf{J}_1 = \mathbf{J}_2.$$

It follows that $\mathbf{J}_2(A_2^* u) = \mathbf{J}_2(Au)$, so that $A_2^* = A = [-\Delta]$, with the same domains.

It follows that the natural adjoint obtained on $\mathbb{C}_0[\Omega]$ coincides with the adjoint constructed from our special rigging. On the other hand, we also see that different riggings can give distinct adjoints. (It is clear that the requirements of Theorem 5.5 are satisfied by both adjoints.)

**Definition 5.52.** We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ is an adjoint canonical pair for $\mathcal{B}$ if $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings and $\mathcal{B}' \subset \mathcal{H}_2$. In this case, when $A \in \mathcal{C}[\mathcal{B}]$, $A^*$ is called the canonical adjoint.

### 5.4.2. Operators on $\mathcal{B}$.

**Definition 5.53.** Let $\mathcal{B}$ have an S-basis, $U$ be bounded, $A \in \mathbb{C}[\mathcal{B}]$ and let $\mathcal{U}$, $\mathcal{V}$ be subspaces of $\mathcal{B}$. Then:

(1) $A$ is said to be naturally self-adjoint if $D(A) = D(A^*)$ and $A = A^*$.

(2) $A$ is said to be normal if $D(A) = D(A^*)$ and $AA^* = A^*A$.

(3) $U$ is unitary if $UU^* = U^*U = I$.

(4) The subspace $\mathcal{U}$ is $\perp$ to $\mathcal{V}$ if, for each $v \in \mathcal{V}$ and $\forall u \in \mathcal{U}$, $\langle v, J(u)\rangle = 0$ and, for each $u \in \mathcal{U}$ and $\forall v \in \mathcal{V}$, $\langle u, J(v)\rangle = 0$ ($J(u)$ respectively $J(v)$ may be multivalued).

The last definition is transparent since, for example,

$$\langle v, J(u)\rangle = 0 \Leftrightarrow \langle v, J_2(u)\rangle = (v, u)_2 = 0 \ \ \forall v \in \mathcal{V}.$$

Thus, orthogonal subspaces in $\mathcal{H}_2$ induce orthogonal subspaces in $\mathcal{B}$.

**Theorem 5.54.** *(Gram–Schmidt) For each fixed basis $\{\varphi_i,\ 1 \leqslant i < \infty\}$ of $\mathcal{B}$, there is at least one set of dual functionals $\{S_i\}$ such that $\{\{\psi_i\}, \{S_i\}, 1 \leqslant i < \infty\}$ is a biorthonormal set of vectors for $\mathcal{B}$, (i.e., $\langle \psi_i, S_j\rangle = \delta_{ij}$).*

**Proof.** Since each $\varphi_i$ is in $\mathcal{H}_2$, we can construct an orthogonal set of vectors $\{\phi_i,\ 1 \leqslant i < \infty\}$ in $\mathcal{H}_2$ by the standard Gram–Schmidt process. Set $\psi_i = \phi_i / \|\phi_i\|_{\mathcal{B}}$, choose $\hat{S}_i \in J(\psi_i)/\|\psi_i\|_{\mathcal{H}}^2$ and restrict it to the subspace $M_i = [\psi_i] \subset \mathcal{B}$. For each $i$, let $M_i^{\perp}$ be the subspace spanned by $\{\psi_j,\ i \neq j\}$. Now use the Hahn–Banach Theorem to extend $\hat{S}_i$ to $S_i$, defined on all of $\mathcal{B}$, with $S_i = 0$ on $M_i^{\perp}$ (see Theorem 1.47). From here, it is easy to check that $\{\{\psi_i\}, \{S_i\},\ 1 \leqslant i < \infty\}$ is a biorthonormal set. If $\mathcal{B}$ is reflexive, the family $\{S_i\}$ is unique. □

We close this section with the following observation about the use of $\mathcal{H}_2 = KS^2$, when $\mathcal{B}$ is one of the classical spaces. Let $A$ be any closed densely defined positive naturally self-adjoint linear operator on $\mathcal{B}$ with a discrete positive spectrum $\{\lambda_i\}$. In this case, $-A$ generates a $C_0$-contraction semigroup, so that it can be extended to $\mathcal{H}_2$ with the same properties. If we compute the ratio $\frac{\langle A\psi, S_\psi\rangle}{\langle \psi, S_\psi\rangle}$ in $\mathcal{B}$, it will be "close" to the value of $\frac{(\bar{A}\psi,\psi)_{\mathcal{H}_2}}{(\psi,\psi)_{\mathcal{H}_2}}$ in $\mathcal{H}_2$. On the other hand, note that we can use the min-max theorem on $\mathcal{H}_2$ to compute the eigenvalues and eigenfunctions of $A$ via $\bar{A}$ exactly on $\mathcal{H}_2$. Thus, in this sense, the min-max theorem holds on $\mathcal{B}$.

## 5.5. The Spectral Theorem

**5.5.1. Background.** Dunford and Schwartz define a spectral operator as one that has a spectral family similar to that defined in Theorem 5.29 of Chap. 4, for self-adjoint operators. (A spectral operator is an operator with countably additive spectral measure on the Borel sets of the complex plane.) Strauss and Trunk [**STT**] define a bounded linear operator $A$, on a Hilbert space $\mathcal{H}$, to be spectralizable if there exists a nonconstant polynomial $p$ such that the operator $p(A)$ is a scalar spectral operator (has a representation as in Eq. (4.27) in Chap. 4). Another interesting line of attack is represented in the book of Colojoară and Foiaş [**CF**], where they study the class of generalized spectral operators. Here, one is not opposed to allowing the spectral resolution to exist in a generalized sense, so as to include operators with spectral singularities.

The following theorem was proven by Helffer and Sjöstrand [HSJ] (see Proposition 7.2):

**Theorem 5.55.** *Let $g \in \mathcal{C}_0^\infty[\mathbb{R}]$ and let $\hat{g} \in \mathcal{C}_0^\infty[\mathbf{C}]$ be an extension of $g$, with $\frac{\partial \hat{g}}{\partial \bar{z}} = 0$ on $\mathbb{R}$. If $A$ is a self-adjoint operator on $\mathcal{H}$, then*

$$g(A) = -\frac{1}{\pi} \iint\limits_{\mathbf{C}} \frac{\partial \hat{g}}{\partial \bar{z}} (z - A)^{-1} \, dx dy.$$

This defines a functional calculus. Davies [DA] showed that the above formula can be used to define a functional calculus on Banach spaces for a closed densely defined linear operator $A$, provided $\rho(A) \cap \mathbb{R} = \emptyset$. In this program the objective is to construct a functional calculus pre-supposing that the operator of concern has a reasonable resolvent.

5.5.1.1. *Problem.* The basic problem that causes additional difficulty is the fact that many bounded linear operators are of the form $A = B + N$, where $B$ is normal and $N$ is nilpotent (i.e., there is a $k \in \mathbb{N}$, such that $N^{k+1} = 0,\ N^k \neq 0$). In this case, $A$ does not have a representation with a standard spectral measure. On the other hand, $T = [N^*N]^{1/2}$ is a self-adjoint operator, and there is a unique partial isometry $W$ such that $N = WT$. If $\mathbf{E}(\cdot)$ is the spectral measure associated with $T$, then $W\mathbf{E}(\Omega)x$ is not a spectral measure, but it is a measure of bounded variation. This idea was used in Chap. 4 (Theorem 4.57) to provide an alternate approach to the spectral theory. In this section, we consider the same possibly for operators on Banach spaces.

To begin, we note that in either of the Strauss and Trunk [**STT**], Helffer and Sjöstrand [**HSJ**], or Davies [**DA**] theory, the operator $A$ is in Baire class one. Thus, $A$ has an adjoint, so that, by Corollary 5.51 $A = WT$, where $W$ is a partial isometry and $T$ is a nonnegative self-adjoint linear operator.

### 5.5.2. Scalar Case.

**Theorem 5.56.** *If $\mathcal{B}' \subset \mathcal{H}_2$ and $A \in \mathcal{C}[\mathcal{B}]$ is an operator of Baire class one, then there exists a unique vector-valued function $\mathbf{F}_x(\lambda)$ of bounded variation such that, for each $x \in D(A)$, we have:*

(1) *$D(A)$ also satisfies*

$$D(A) = \left\{ x \in \mathcal{B} \mid \int_{|\sigma(A)|} \lambda^2 \langle d\mathbf{F}_x(\lambda), x^* \rangle < \infty \right\}$$

*for each $x^* \in J(x)$ and*

(2) $Ax = \lim_{n\to\infty} \int_0^n \lambda d\mathbf{F}_x(\lambda)$, for all $x \in D(A)$.

**Proof.** Let $A = WT$, where $W$ is the unique partial isometry and $T = [A^*A]^{1/2}$. Let $\bar{T}$ be the extension of $T$ to $\mathcal{H}_2$. It follows that there is a unique spectral measure $\bar{\mathbf{E}}(\Omega)$ such that for each $x \in D(\bar{T})$:

$$\bar{T}x = \lim_{n\to\infty} \int_0^n \lambda d\bar{\mathbf{E}}(d\lambda)x. \tag{5.18}$$

Furthermore, $\bar{\mathbf{E}}(\lambda)x$ is a vector-valued function of bounded variation and, if $\bar{W}$ is the extension of $W$, $\bar{W}\bar{\mathbf{E}}_x(\lambda)$ is of bounded variation, with $Var(\bar{W}\bar{\mathbf{E}}_x, \mathbb{R}) \leq Var(\bar{\mathbf{E}}_x, \mathbb{R})$. If we set $\bar{\mathbf{F}}_x(\lambda) = \bar{W}\bar{\mathbf{E}}_x(\lambda)$, for each interval $(a, b) \subset [0, \infty)$,

$$\left\{\bar{W} \int_a^b \lambda d\bar{\mathbf{E}}_x(\lambda)\right\} = \int_a^b \lambda d\bar{\mathbf{F}}_x(\lambda).$$

Since $\bar{A}x = \bar{W}\bar{T}x$ and the restriction of $\bar{A}$ to $\mathcal{B}$ is $A$, we have, for all $x \in D(A)$,

$$Ax = \lim_{n\to\infty} \int_0^n \lambda d\mathbf{F}_x(\lambda). \tag{5.19}$$

This proves (2). The proof of (1) follows from (1) in Theorem 4.61 of Chap. 4 and the definition of $x^*$. □

**5.5.3. General Case.** In this section, we assume that for each $i$, $1 \leq i \leq n$, $n \in \mathbb{N}$, $\mathcal{B}_i = \mathcal{B}$ is a fixed separable Banach space. We set $\mathfrak{B} = \times_{i=1}^n \mathcal{B}_i$, and represent a vector $\mathbf{x} \in \mathfrak{B}$ by $\mathbf{x}^t = [x_1,\ x_2,\ \cdots,\ x_n]$. An operator $\mathbf{A} = [A_{ij}] \in C[\mathfrak{B}]$ is defined whenever $A_{ij} : \mathcal{B} \to \mathcal{B}$, is in $\mathcal{C}[\mathcal{B}]$.

If $\mathcal{B}' \subset \mathcal{H}_2$ and $A_{ij}$ is of Baire class one, then by Theorem 5.54, there exists a unique vector-valued function $F_x^{ij}(\lambda)$ of bounded variation such that, for each $x \in D(A_{ij})$, we have:

(1) $D(A_{ij})$ also satisfies

$$D(A_{ij}) = \left\{ x \in \mathcal{B} \mid \int_0^\infty \lambda^2 \left\langle dF_x^{ij}(\lambda), x^* \right\rangle_{\mathcal{B}} < \infty \right\}$$

for all $x^* \in J(x)$ and

(2)

$$A_{ij}x = \lim_{n\to\infty} \int_0^n \lambda dF_x^{ij}(\lambda), \text{ for all } x \in D(A_{ij}).$$

If we let $d\mathcal{F}(\lambda) = [dF^{ij}(\lambda)]$, then we can represent $\mathbf{A}$ by:

$$\mathbf{Ax} = \lim_{n\to\infty} \int_0^n \lambda d\boldsymbol{\mathcal{F}}(\lambda)\mathbf{x}, \text{ for all } \mathbf{x} \in D(\mathbf{A}).$$

## 5.6. Schatten Classes on Banach Spaces

In this section, we show how our approach allows us to provide a natural definition for the Schatten class of operators on $\mathcal{B}$. Here, we assume that the reader has at least read the section concerning compact operators on Hilbert spaces in Chap. 4.

**5.6.1. Background: Compact Operators on Banach Spaces.** Let $\mathbb{K}(\mathcal{B})$ be the class of compact operators on $\mathcal{B}$ and let $\mathbb{F}(\mathcal{B})$ be the set of operators of finite rank. Recall that, for separable Banach spaces, $\mathbb{K}(\mathcal{B})$ is an ideal that need not be the maximal ideal in $L[\mathcal{B}]$. If $\mathbb{M}(\mathcal{B})$ is the set of weakly compact operators and $\mathbb{N}(\mathcal{B})$ is the set of operators that map weakly convergent sequences into strongly convergent sequences, it is known that both are closed two-sided ideals in the operator norm, and, in general, $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{M}(\mathcal{B})$ and $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$ (see part I of Dunford and Schwartz [DS], p. 553). For reflexive Banach spaces, $\mathbb{K}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$ and $\mathbb{M}(\mathcal{B}){=}L[\mathcal{B}]$. For the space of continuous functions $\mathbf{C}[\Omega]$ on a compact Hausdorff space $\Omega$, Grothendieck [GO] has shown that $\mathbb{M}(\mathcal{B}){=}\mathbb{N}(\mathcal{B})$. On the other hand, it was shown in part I of Dunford and Schwartz [DS] that for a positive measure space, $(\Omega, \Sigma, \mu)$, on $\mathbf{L}^1(\Omega, \Sigma, \mu)$, $\mathbb{M}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$.

**5.6.2. Uniformly Convex Spaces.** We assume that $\mathcal{B}$ is uniformly convex, with an S-basis. In operator theoretic language, the interpretation of our S-basis assumption is that the compact operators on $\mathcal{B}$ have the approximation property, namely that every compact operator can be approximated by operators of finite rank. In this section, we will show that, for the class of uniformly convex Banach spaces with an S-basis, $L[\mathcal{B}]$ almost has the same structure as that of $L[\mathcal{H}]$, when $\mathcal{H}$ is a Hilbert space. The difference being that $L[\mathcal{B}]$ is not a $C^*$-algebra (i.e., $\|A^*A\| = \|A\|^2$, for all $A \in L[\mathcal{B}]$).

In what follows, we fix $\mathcal{H}_2$. Let $A$ be a compact operator on $\mathcal{B}$ and let $\bar{A}$ be its extension to $\mathcal{H}_2$. For each compact operator $\bar{A}$ on $\mathcal{H}_2$, there exists an orthonormal set of functions $\{\bar{\varphi}_n \mid n \geqslant 1\}$ such that

$$\bar{A} = \sum\nolimits_{n=1}^{\infty} \mu_n(\bar{A})\,(\cdot\,,\bar{\varphi}_n)_2\,\bar{U}\bar{\varphi}_n.$$

Where the $\mu_n$ are the eigenvalues of $[\bar{A}^*\bar{A}]^{1/2} = \left|\bar{A}\right|$, counted by multiplicity and in decreasing order, and $\bar{U}$ is the partial isometry associated with the polar decomposition of $\bar{A} = \bar{U}\left|\bar{A}\right|$. Without loss, we can assume that the set of functions $\{\bar{\varphi}_n \mid n \geqslant 1\}$ is contained in $\mathcal{B}$

and $\{\varphi_n \,|n \geqslant 1\}$ is normalized version in $\mathcal{B}$. If $\mathbb{S}_p[\mathcal{H}_2]$ is the Schatten Class of order $p$ in $L[\mathcal{H}_2]$, it is well known that if $\bar{A} \in \mathbb{S}_p[\mathcal{H}_2]$, its norm can be represented as:

$$\left\|\bar{A}\right\|_p^{\mathcal{H}_2} = \left\{Tr\left[\bar{A}^*\bar{A}\right]^{p/2}\right\}^{1/p} = \left\{\sum_{n=1}^{\infty}\left(\bar{A}^*\bar{A}\bar{\varphi}_n, \bar{\varphi}_n\right)_{\mathcal{H}_2}^{p/2}\right\}^{1/p}$$
$$= \left\{\sum_{n=1}^{\infty}\left|\mu_n\left(\bar{A}\right)\right|^p\right\}^{1/p}.$$

**Definition 5.57.** We represent the Schatten Class of order $p$ in $L[\mathcal{B}]$ by:

$$\mathbb{S}_p[\mathcal{B}] = \mathbb{S}_p[\mathcal{H}_2]\,|_{\mathcal{B}}\,.$$

Since $\bar{A}$ is the extension of $A \in \mathbb{S}_p[\mathcal{B}]$, we can define $A$ on $\mathcal{B}$ by

$$A = \sum\nolimits_{n=1}^{\infty}\mu_n(A)\left\langle\cdot\,,\varphi_n^*\right\rangle U\varphi_n,$$

where $\varphi_n^*$ is the unique dual map in $\mathcal{B}'$ associated with $\varphi_n$ and $U$ is the restriction of $\bar{U}$ to $\mathcal{B}$. The corresponding norm of $A$ on $\mathbb{S}_p[\mathcal{B}]$ is defined by:

$$\|A\|_p^{\mathcal{B}} = \left\{\sum\nolimits_{n=1}^{\infty}\left\langle A^*A\varphi_n, \varphi_n^*\right\rangle^{p/2}\right\}^{1/p}.$$

**Theorem 5.58.** *Let* $A \in \mathbb{S}_p[\mathcal{B}]$*, then* $\|A\|_p^{\mathcal{B}} = \left\|\bar{A}\right\|_p^{\mathcal{H}_2}$.

**Proof.** It is clear that $\{\varphi_n \,|n \geqslant 1\}$ is a set of eigenfunctions for $A^*A$ on $\mathcal{B}$. Furthermore, by Theorem 5.11, $A^*A$ is naturally self-adjoint and, since every compact operator generates a $C_0$-semigroup, by Theorem 5.40, the spectrum of $A^*A$ is unchanged by its extension to $\mathcal{H}_2$. It follows that $A^*A\varphi_n = |\mu_n(A)|^2\,\varphi_n$, so that

$$\left\langle A^*A\varphi_n, \varphi_n^*\right\rangle = |\mu_n|^2\left\langle\varphi_n, \varphi_n^*\right\rangle = |\mu_n(A)|^2\,,$$

and

$$\|A\|_p^{\mathcal{B}} = \left\{\sum\nolimits_{n=1}^{\infty}\left\langle A^*A\varphi_n, \varphi_n^*\right\rangle^{p/2}\right\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty}|\mu_n(A)|^p\right\}^{1/p} = \left\|\bar{A}\right\|_p^{\mathcal{H}_2}.$$

□

It is clear that all of the theory of operator ideals on Hilbert spaces extend to uniformly convex Banach spaces with an S-basis in a straightforward way. We state a few of the more important results to give a sense of the power provided by the existence of adjoints. The first result extends theorems due to Weyl [WY], Horn [HO], Lalesco [LE] and Lidskii [LI]. The proofs are all straightforward, for a given

$A$ extend it to $\mathcal{H}_2$, use the Hilbert space result and then restrict back to $\mathcal{B}$.

**Theorem 5.59.** *Let $A \in \mathbb{K}(\mathcal{B})$, the set of compact operators on $\mathcal{B}$, and let $\{\lambda_n\}$ be the eigenvalues of $A$ counted up to algebraic multiplicity. If $\Phi$ is a mapping on $[0, \infty]$ which is nonnegative and monotone increasing, then we have:*

(1) *(Weyl)*

$$\sum\nolimits_{n=1}^{\infty} \Phi\left(|\lambda_n(A)|\right) \leqslant \sum\nolimits_{n=1}^{\infty} \Phi\left(\mu_n(A)\right)$$

*and*

(2) *(Horn) If $A_1,\ A_2 \in \mathbb{K}(\mathcal{B})$*

$$\sum\nolimits_{n=1}^{\infty} \Phi\left(|\lambda_n(A_1 A_2)|\right) \leqslant \sum\nolimits_{n=1}^{\infty} \Phi\left(\mu_n(A_1)\mu_n(A_2)\right).$$

*In case $A \in \mathbb{S}_1(\mathcal{B})$, we have:*

(3) *(Lalesco)*

$$\sum\nolimits_{n=1}^{\infty} |\lambda_n(A)| \leqslant \sum\nolimits_{n=1}^{\infty} \mu_n(A)$$

*and*

(4) *(Lidskii)*

$$\sum\nolimits_{n=1}^{\infty} \lambda_n(A) = Tr(A).$$

Simon [SI1] provides a very nice approach to infinite determinants and trace class operators on separable Hilbert spaces. He gives a comparative historical analysis of Fredholm theory, obtaining a new proof of Lidskii's Theorem as a side benefit and some new insights. A review of his paper shows that much of it can be directly extended to operator theory on separable reflexive Banach spaces.

**5.6.3. Discussion.** On a Hilbert space $\mathcal{H}$, the Schatten classes $\mathbb{S}_p(\mathcal{H})$ are the only ideals in $\mathbb{K}(\mathcal{H})$, and $\mathbb{S}_1(\mathcal{H})$ is minimal. In a general Banach space, this is far from true. A complete history of the subject can be found in the recent book by Pietsch [PI1] (see also Retherford [RE], for a nice review). We limit this discussion to a few major topics in the subject. First, Grothendieck [GO] defined an important class of nuclear operators as follows:

**Definition 5.60.** If $A \in \mathbb{F}(\mathcal{B})$ (the operators of finite rank), define the ideal $\mathbf{N}_1(\mathcal{B})$ by:

$$\mathbf{N}_1(\mathcal{B}) = \{A \in \mathbb{F}(\mathcal{B}) \mid \mathbf{N}_1(A) < \infty\},$$

where

$$\mathbf{N}_1(A) = \text{glb}\left\{\sum\nolimits_{n=1}^{m} \|f_n\| \, \|\phi_n\| \;\middle|\, f_n \in \mathcal{B}', \ \phi_n \in \mathcal{B}, \ A = \sum\nolimits_{n=1}^{m} \phi_n \left\langle \cdot \,, f_n \right\rangle \right\}$$

and the greatest lower bound is over all possible representations for $A$.

Grothendieck showed that $\mathbf{N}_1(\mathcal{B})$ is the completion of the finite rank operators and is a Banach space with norm $\mathbf{N}_1(\cdot)$. It is also a two-sided ideal in $\mathbb{K}(\mathcal{B})$. It is easy to show that:

**Corollary 5.61.** $\mathbb{M}(\mathcal{B}), \mathbb{N}(\mathcal{B})$ *and* $\mathbf{N}_1(\mathcal{B})$ *are two-sided *ideals.*

In order to compensate for the (apparent) lack of an adjoint for Banach spaces, Pietsch [PI2], [PI3] defined a number of classes of operator ideals for a given $\mathcal{B}$. Of particular importance for our discussion is the class $\mathbb{C}_p(\mathcal{B})$, defined by

$$\mathbb{C}_p(\mathcal{B}) = \left\{ A \in \mathbb{K}(\mathcal{B}) \;\middle|\, \mathbb{C}_p(A) = \sum\nolimits_{i=1}^{\infty} [s_i(A)]^p < \infty \right\},$$

where the singular numbers $s_n(A)$ are defined by:

$$s_n(A) = \inf\left\{ \|A - K\|_{\mathcal{B}} \mid \text{rank of } K \leqslant n \right\}.$$

Pietsch has shown that $\mathbb{C}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$, while Johnson et al. [JKMR] have shown that for each $A \in \mathbb{C}_1(\mathcal{B})$, $\sum_{n=1}^{\infty} |\lambda_n(A)| < \infty$. On the other hand, Grothendieck [GO] has provided an example of an operator $A$ in $\mathbf{N}_1(L^\infty[0,1])$ with $\sum_{n=1}^{\infty} |\lambda_n(A)| = \infty$ (see Simon [SI], p. 118). Thus, it follows that, in general, the containment is strict. It is known that if $\mathbb{C}_1(\mathcal{B}) = \mathbf{N}_1(\mathcal{B})$, then $\mathcal{B}$ is isomorphic to a Hilbert space (see Johnson et al.). It is clear from the above discussion that:

**Corollary 5.62.** $\mathbb{C}_p(\mathcal{B})$ *is a two-sided *ideal in* $\mathbb{K}(\mathcal{B})$*, and* $\mathbb{S}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$.

For a given Banach space, it is not clear how the spaces $\mathbb{C}_p(\mathcal{B})$ of Pietsch relate to our Schatten Classes $\mathbb{S}_p(\mathcal{B})$ (clearly $\mathbb{S}_p(\mathcal{B}) \subseteq \mathbb{C}_p(\mathcal{B})$). Thus, one question is that of the equality of $\mathbb{S}_p(\mathcal{B})$ and $\mathbb{C}_p(\mathcal{B})$. (We suspect that $\mathbb{S}_1(\mathcal{B}) = \mathbb{C}_1(\mathcal{B})$.)

**Remark 5.63.** In closing, we should point out that if $\mathcal{B}$ is not uniformly convex, then for a given $\phi \in \mathcal{B}$ the set $J(\phi) \in \mathcal{B}'$ can be multivalued and there is no unique way to define $\mathbb{S}_p(\mathcal{B})$ (i.e., to choose $\phi^* \in J(\phi)$). If $\mathcal{B}'$ is strictly convex, $J(\phi) \in \mathcal{B}'$ is uniquely defined (single-valued), so that all of our results still hold. However, to our knowledge, all known examples Banach spaces with $\mathcal{B}'$ strictly convex are uniformly convex.

*Conclusion.* The most interesting aspect of this section is the observation that the dual space of a Banach space can have more than one representation. It is well known that a given Banach space $\mathcal{B}$ can have many equivalent norms that generate the same topology. However, the geometric properties of the space depend on the norm used. We have shown that the properties of the linear operators on $\mathcal{B}$ depend on the family of linear functionals used to represent the dual space $\mathcal{B}'$. This approach offers an interesting tool for a closer study of the structure of bounded linear operators on $\mathcal{B}$.

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