Chapter 9
Catastrophe Theory: Methodology, Epistemology, and Applications in Learning Science

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Introduction

Catastrophe theory is a mathematical theory that addresses discontinuities and qualitative changes in dynamical systems. It states that in a complex dynamical system changes could be smooth and linear, but that they could also be nonlinear, and contrary to the common sense anticipation, they might be surprisingly large even though the input is quite small. In reality, we observe that except human constructions, straight lines do not exist in nature neither in social and human experience. The assumption of linearity in social science research, in both qualitative and quantitative approaches, has been a philosophical convention, since it is the simplest one to examine, by the methodological tools available thus far. Moreover it facilitated the cause-and-effect notion of classical reductionistic interpretations. Catastrophe theory is acknowledged for its descriptive and interpretative modeling power and its uniqueness to be the most applicable methodological approach that infers nonlinearity from cross-sectional empirical data. This chapter begins with a brief history of its mathematical foundation and continues with the presentation of catastrophe theory in its deterministic and stochastic forms. Subsequently, all the current statistical methodologies are presented and the epistemology associated with catastrophe theory and nonlinear dynamics is extensively discussed. Finally, applications within the neo-Piagetian framework and science education research are presented.

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M. Koopmans, D. Stamovlasis (eds.), *Complex Dynamical Systems in Education*, DOI 10.1007/978-3-319-27577-2_9

A Brief History of Catastrophe Theory

The history of catastrophe theory begins in the decade of 1880s, when the famous French mathematician Henri Poincaré founded *bifurcation theory* while working on a qualitative analysis of systems by means of nonlinear differential equations. Poincaré was interested in answering questions concerning the structural stability of the solar system. The main question was whether the planets would escape to infinity or crash into each other if they experience an external shock. He found that small perturbations would either leave the system relatively unchanged or would cause it to move in a very different mode. This signified the onset of bifurcation theory, which led to *singularity theory*, as a special case of which catastrophe theory appeared decades later. A substantial contribution on the development of the above mathematical theories is credited also to the Russian mathematician Vladimir Arnol'd (1988, 1992). However, the basic notions and the formulation were Poincaré's work: Bifurcation theory considers a dynamical system described by ordinary differential equations. Certain points where the first derivative equals zero characterize equilibrium states. At a critical point, called a *singularity*, this set of equilibria bifurcates into separate branches. Note that such critical points are the degenerate ones and they are not associated extrema.[1] This splitting is a bifurcation of the degenerate equilibrium and since it concerns equilibrium solutions it makes the connection between the singularity of mapping and structural stability (Morse, 1931). A crucial step in the history of catastrophe theory was the invention that there were many types of such functions and two of them are stable in all their forms; later they become known as the fold and the cusp catastrophe (Whitney, 1955). The discovery of these two types of structurally stable singularities for differentiable mappings was the first element of *catastrophe theory*, even though at that time the emerging theory was not referred with this name.

In 1950s René Thom, a French mathematician, working on structural stability introduced the notion of *transversality* and stated the corresponding theorem in order to describe the transverse intersection properties of smooth maps (Thom, 1956). According to Thom's theorem any smooth map may be deformed by an arbitrary small amount into a map that is transverse to a given sub-manifold.[2] The *transversality theorem* facilitated the classification of singularities or elementary catastrophes and Thom (1972) managed to define seven types of singularities, which can be described by up to six dimensions and named them as the "elementary" catastrophes. For systems with dimensionality greater than eleven,

[1] In mathematics, a critical point of a differentiable function is a point where the derivative is zero (or undefined). Degeneracy refers to a property of a case in which an element of a class of objects is qualitatively different from the rest of the class belonging to a different, usually simpler, class. A singular point is a degenerate one and is not associated with usual non-degenerate extrema, maximum or minimum, where the first derivative is also zero.

[2] Transversality in Thom's theorem refers to a generic property of the maps according to which any smooth map $f: X \rightarrow Y$ may be deformed by an arbitrary small amount into a map that is transverse to a given $Z \subseteq Y$ sub-manifold (Arnol'd, 1988).

singularities were difficult to be classified, because, as it was shown later by Arnol'd and his coworkers, the number of categories becomes infinite (Arnol'd, Gusein-Zade, & Varchenko, 1985).

Thom specified the basic mathematical formulation of the elementary catastrophe theory considering the behavior of a *deterministic* dynamical system that is described by n state variables y_j and r control variables b_i. A potential function is assumed to be operating on this set of state and control variables so that for all y_j the first derivative is zero, while the set of points where the derivative equals zero constitutes the equilibrium manifold. The first catastrophe which attracted attention was the cusp catastrophe with a three-dimensional equilibrium surface described by one state variable as a function of two control variables. The topology suggests that when the control variables change, even slowly, the state variable adjusts quickly on the equilibrium manifold. The topological characteristics of the response surface of catastrophe model exhibit a number of features, such as hysteresis, bimodality, inaccessibility, sudden jumps, and divergence, which are presented in the following section.

While the formulation of catastrophe theory was being developed in the area of mathematics, in 1960s and 1970s, a number of applications appeared in the literature of economics, psychology, and other behavioral and social science (see Poston & Stewart, 1978; Woodcock & Davis, 1978). Most of the very early applications were with low-dimensional catastrophes, in the sense of having a few predictors, and their onset brought up methodological and epistemological considerations with a plethora of concerns. One fundamental question in a continuing debate over catastrophe theory was the existence of system's potential function. A potential function posits a symmetry condition that all cross-partial derivatives are equal, which again singularity theory does not require. Within the ongoing discussion it seemed also that the mathematics of Arnol'd had departed from Thom's original formulation and this became a further controversy which appeared in late 1970s. Another issue of debate that appeared during the early discussions was the issue of time. Since singularity theory is about mappings, unfolding in space, and it might not involve time at all, the question arises about whether catastrophe theory has to involve the time dimension. Thom strongly associated catastrophe theory with dynamical systems, where time might be explicit a dimension as well. Since years earlier, Thom had argued that an elementary catastrophe form might be embedded in a larger system, which incorporates time as variable. He stressed that the discussion concerns dynamical systems evolving in $S \times t$ space, where S is the structural characteristics and t is time. If the larger system is *transversal* to the catastrophe set in the enlarged space, then time could be control variable; however the argument was a theoretical one and hard to demonstrate in empirical applications. Moreover, some crucial details were brought up in the discussion when considering transitions between stable states. These are associated with the notion of *discontinuity* and led to various misconceptions and furthermore to criticism (Zahler & Sussman, 1977). On this matter, two conventions regarding the way that the system moves between multiple equilibria were stated: the *Maxwell* and the *delay convention*. Note that the choice of one or the other convention might exclude a range of applications (i.e., in real systems behavior). Thom (1972) clearly fostered Maxwell convention from the beginning, but this later proved to be a problem, mainly because it appeared to be a weak point concerning the definition of catastrophe theory itself. The intense dispute on this issue led Zeeman to state later "*there is strictly speaking no* '*catastrophe theory*, *but then this is more or less true for any non-axiomatic theory in mathematics that attempts to describe nature*" (Zeeman, 1974, p. 623). Obviously, until that period the foundation of catastrophe theory at mathematical level was not a completed issue; nevertheless, active researcher in other fields rather intuitively had acknowledged a merit to it.

Catastrophe theory started to become popular around 1970s and a plethora of applications appeared in many fields of research, which however followed a rather descriptive and qualitative approach. It seemed a fascinated premise that could aid to understand unforeseen changes in nature and society. The unstable sociocultural environment that existed during that period, with radical political movements, facilitated its dissemination and appeal among mainly intellectuals (Rosser, 2007). This explosion of popularity triggered criticism and counteractions against the emerging theory, which, at that time, existed only in its deterministic version. A lot of theoretical, epistemological, and ultimately methodological questions were raised.

Catastrophe theory faced a severe condemnation mainly by Kolata (1977), Zahler and Sussman (1977), and Sussman and Zahler (1978a, 1978b). Their criticism was centered on the mathematical formulation and indirectly on its epistemology, which was unclear at that period. The most striking points of criticism referred to (1) the descriptive and qualitative approaches that were implemented; (2) the incorrect ways of quantification; (3) the existence of potential function; (4) the exclusion of time as a control variable in many applications; (5) the limited set of possible elementary catastrophes; and (6) the incorrect verification of global forms from local estimates (i.e., any surface can be fit to a set of points). The criticizing group also focused the disapproval on basic mathematical concepts associated with nonlinear behavior. For example (7) they claim that no real discontinuous jumps exist and cusp or fold model could be inferred with a few points. Extrapolation tells nothing about predicted behavior and, due to observational error, any surface could be arbitrarily close to a surface that Thom's theorems examine; (8) they also criticized Zeeman for incorrectly using the concept of genericity in his frontier example; (9) the predictions based on catastrophe theory are not testable and are unverified expectations, while many underlying hypotheses are often ambiguous; (10) the cusp models used (e.g., Zeeman's) were based on hypotheses carefully chosen in order to facilitate it, and the critique was talking about "*mystifying*" terms.

Some of the points of criticism, such as those regarding the use of qualitative methods, made sense, because the way bifurcation theory was founded by Poincaré had a qualitative character. Quantitative deterministic models had been demonstrated in physical sciences, but they seemed inappropriate for the social (soft) sciences. Most points of criticism for inappropriate ways of model design and quantification challenged mainly Zeeman's work. Thom, who had already acknowledged Whitney's work on singularity theory, agreed to some extent with the criticism on the qualitative character of catastrophe theory. Interestingly, on this issue, Thom appeared to be more a theoretician and philosopher, rather than as a mathematician in his responses towards defending the emerging theory. Thom wrote:

> On the plane of philosophy properly speaking, of metaphysics, catastrophe theory cannot, to be sure, supply any answer to the great problems which torment mankind. But it favors a dialectical, Heraclitean view of the universe, of a world which is the continual theatre of the battle between 'logoi,' between archetypes . . . Just as the hero of the Iliad could go against the will of a God, such as Poseidon, only by invoking the power of an opposed divinity, such as Athena, so shall we be able to restrain the action of an archetype only by opposing to it an antagonistic archetype, in an ambiguous contest of uncertain outcome. Thom (1975, p. 384):

The above expresses Thom's intention to demonstrate metaphorically his *dialectical view* on uncertain outcomes upon the operation of two opponent processes or actions. Similarily to Hegel's dialectics, his position was that catastrophe theory was the means to demonstrate how qualitative changes could emerge from quantitative changes. Arnol'd referred to the "*mysticism*" of catastrophe theory showing his disagreement on Thom's "metaphysical" turn; however he admitted that "*in mathematics always there is an mysterious element: the astonishing concurrences and ties between objects and theories, which at first glance seem far apart*" (Arnol'd, 1992, p. 103).

The consequences of the criticism were *καταστροφικές* (disastrous) for catastrophe theory. Research showed a declined interest in applying catastrophe theory, and finally it became out of fashion for some years. Despite its temporary overthrow, catastrophe theory came back restored and more rigorous in 1980s, due to the work of Cobb (1978) in statistics, Oliva and Capdeville (1980) in economics, and Guastello (1981) in psychology. They defended the emergent theory by responding to the points of criticisms, while they made substantial contributions to the development of methodology for application of catastrophe theory in social sciences. With their pioneer work they maintained and showed that finally "*the baby was thrown out with the bathwater*" (Oliva & Capdeville, 1980). For more than a decade strong-minded scholars in various fields, who were convinced that catastrophe theory could become a valuable asset in research for social sciences (i.e., Cobb & Zacks, 1985; Cobb, Koppstein & Chen, 1983; Guastello, 2002; Lorenz, 1989; Puu, 1981; Rosser, 1991), worked for its development. Strong and clear responses to all points criticism were given also by van der Maas and Molenaar (1992); Wagenmakers, Grasman, and Molenaar (2005); and Wagenmakers, Molenaar, Grasman, Hartelman, and van der Maas (2005), while catastrophe theory has gained its reputation among scientist. Presently, it has been understood that the criticism was based mainly on confusions and conceptual misunderstanding of core ideas of the new theory and the only weak point, at the earliest times, which has now been overcome, was the lack of the proper statistical methodology applied to real-world research.

The return of catastrophe theory ensued in late 1980s where it stayed in the stage of social sciences with the development of its stochastic version, which permitted testing research hypotheses related to discontinuous changes in empirical data. Overviews of the theory and some applications across disciplines can be found in Arnol'd (1992), Castrigiano and Hayes (2004), Gilmore (1981), Saunders (1980), Poston and Stewart (1978), Thompson (1982), and Woodcock and Davis (1978).

Catastrophe Theory

Deterministic Catastrophe Theory

Catastrophe theory based on the initial work of Thom (1956, 1972, 1983) and Arnol'd (1988, 1992) is concerned with the classification of equilibrium behavior of systems in the neighborhood of singularities. The mathematical foundation of catastrophe theory includes the proof that the dynamics of systems in such singular points can be locally modeled by seven elementary catastrophes, which implement up to four independent variables. These elementary behaviors of systems in the neighborhood of singularities depend only on the number of predictors, the *control factors*. The seven elementary catastrophes are namely *fold catastrophe, cusp catastrophe, swallowtail catastrophe, butterfly catastrophe, elliptic umbilic catastrophe, hyperbolic umbilic catastrophe,* and *parabolic umbilic catastrophe*. The first four, known as cuspoids, have one behavioral axis while the last three have two behavioral axes; the formers are the most common and pertinent to social science. The fold, the cusp, the swallowtail, and the butterfly catastrophe have one, two, three, and four control variables, respectively. Each catastrophe is associated with a potential function in which the control parameters are represented as coefficients (a, b, c, or d), while one state variable, y, describes the behavior of the system. The behavior surface is the geometrical representation of all points where the first derivative of the potential function is zero (Zeeman, 1976). The cuspoids, which are the most applicable, are summarized in Table 9.1.

Deterministic catastrophe theory has been applied in physics and engineering for modeling various phenomena, such as the propagation of stock waves, the minimum area of surfaces, or nonlinear oscillations. Moreover interesting applications have been developed for conceptual formulation of thermodynamics, scattering, elasticity, and in the predictions of van der Waals equation in the transition between the liquid and the gaseous phase of matter using a cusp catastrophe, where temperature and pressure were implemented as two conflicting control factors, while density is the behavioral variable (Gilmore, 1981; Poston & Stewart, 1978). On the other hand, in social and human systems, where nonlinear effects and sudden changes are ubiquitous, with the development of its stochastic form, the perspectives for catastrophe theory became by far promising.

In order to attain a conceptual understanding of the core idea in catastrophe theory models, consider an analogy from a physical system that is moving toward an equilibrium state. The system in Fig. 9.1 comprises a hypothetical "one-dimensional surface" on which a sphere is moving driven by gravitational forces. The sphere represents the state of the system that can be at local minima or maxima, which are the equilibrium states. The minimum is the stable state where the system will stay or return when permutated by an external cause. The maximum is an unstable state, that is, small perturbations cause system's shift to another state. The above qualitative behaviors of changing states can be characterized according to the configuration of the corresponding positions, which are critical points, local

Table 9.1 The four *cuspoid* elementary catastrophes describing all possible discontinuities in phenomena controlled by no more than four factors

Catastrophe	Control dimensions	Potential function	First derivative
Fold	1 (a)	$U(y) = \frac{1}{3}y^3 - ay$	$\frac{\partial U}{\partial y} = y^2 - a$
Cusp	2 (a, b)	$U(y) = \frac{1}{4}y^4 - \frac{1}{2}by^2 - ay$	$\frac{\partial U(y)}{\partial y} = y^3 - by - a$
Swallowtail	3 (a, b, c)	$U(y) = \frac{1}{5}y^5 - \frac{1}{3}cy^3 - \frac{1}{2}by^2 - ay$	$\frac{\partial U(y)}{\partial y} = y^4 - cy^2 - by - a$
Butterfly	4 (a, b, c, d)	$U(y) = \frac{1}{6}y^6 - \frac{1}{4}dy^4 - \frac{1}{3}cy^3 - \frac{1}{2}by^2 - ay$	$\frac{\partial U(y)}{\partial y} = y^5 - dy^3 - cy^2 - by - a$

Each catastrophe is associated with a potential function in which the control parameters are represented as coefficients (a, b, c, or d), while one state variable, y, describes the behavior of the system. The behavior surface is the geometrical representation of all points where the first derivative of the potential function is zero (Zeeman, 1976)

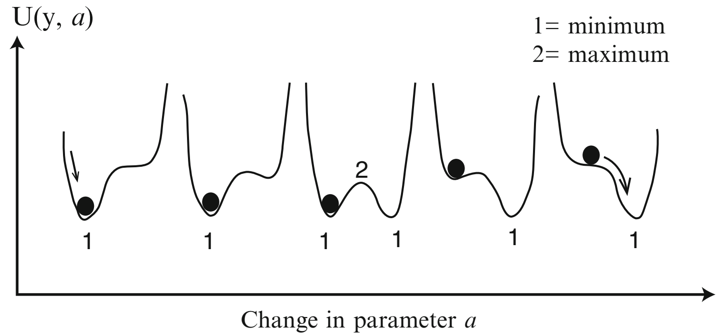

Fig. 9.1 The *sphere* represents the state of the system and can be at local minima or maxima—the equilibrium states. It is demonstrated how the sphere "jumps" from one position to another as the configuration of surface changes gradually

maxima or minima, with first and/or second derivative equal to zero. Next, observing across the five representations (Fig. 9.1), it is demonstrated how the sphere "jumps" from one position to another (local minimum) as the configuration of surface changes gradually (Castrigiano & Hayes, 2004; Gilmore, 1981).

This behavior may be described mathematically by postulating that the state of the system, y, will change over time t according to the equation

$$dy/dt = -\partial U(y/a)/\partial y \tag{9.1}$$

where $U(y/a)$ is the potential function and a is a vector of the control variables that affects the state of the system. The above equation characterizes a *gradient dynamical system*, which is at an equilibrium state, if the Eq. (9.1) equals zero (Feraro, 1978).

The equilibrium behavior of singular systems leads to multiple equilibria (multimodal distributions); thus abrupt changes in behavior might be expected as the system shifts from one equilibrium state to another. This "strange" behavior reflects *discontinuity* in mathematical sense. The concept of *discontinuity* is a fundamental issue in catastrophe theory and from the beginning it was a source of misconceptions that induced the criticism mentioned in the first section.

Catastrophe models become extremely complex, and less applicable, when number of the state and control parameters increase. However, the simplest and the most eminent one, the cusp catastrophe has numerous applications, and it is the best representation of the catastrophe theory models to be used also for didactic purposes. The cusp model describes the discontinuous behavior of a state variable as a function of just two independent variables. Considering that in traditional approaches a large number of independent variables are usually implemented when attempting to model changes, the choice of the cusp with merely two candidates has certainly an advantage; this justifies the widespread use and the applicability of the cusp catastrophe.

C*usp* describes the behavior as a function of the two control variables: asymmetry (a) and bifurcation (b). The potential function of the cusp catastrophe is expressed by the deterministic equation

$$U(y, a, b) = \frac{1}{4}y^4 - \frac{1}{2}by^2 - ay \quad (9.2)$$

The first derivative with respect to y is given by the equation

$$\frac{\partial U(y, a, b)}{\partial y} = y^3 - by - a \quad (9.3)$$

Setting $\partial U(y, a, b)/\partial y = 0$ gives rise to equilibrium function, which is geometrically represented by the three-dimensional surface. Note that Eq. (9.3) is cubic, a point with important consequences: small variation of the control variables can lead to abrupt shifts or jumps in the behavior y. This is an exclusive characteristic of the above function and the changes in the dependent variable are qualitatively different from other cases such as those in models with quadratic terms, where small continuous variation of independent variables is just accelerating y. Moreover, the changes in the cusp function are different from any sudden changes implied in a model, for instance, with a threshold function and also they are distinct from the shifts in logistic type functions, such as Rasch models, and from Markov models as well. Discontinuous changes in the cusp are sudden jumps occurring between regions of a smooth surface. This is a very important mathematical feature linked with primary epistemological issues related to nonlinearity. In real-world research, these discontinuous changes might imply a *qualitative* change within the system under investigation.

Further examination of the cusp model via its response surface reveals certain unique qualitative features, known as the *catastrophe flags*, which could be used to identify the presence of cusp catastrophe (Gilmore, 1981):

Bimodality: Refers to the probability distribution of the dependent variable, where two distinctly different modes exist or two simultaneously present sates.

Hysteresis: Is the effect, where cases with the same values of the two controls, asymmetry (a) and bifurcation (b), can be found in both distributional modes; that is, they can exhibit two types of behavior corresponding to both behavioral attractors; for a dynamical system hysteresis effect denotes memory for the path through the phase space of the system, in the sense that some point or areas of the system keep values from the preceded states.

Inaccessibility: The region on the response surface existing in between the two behavioral modes. This area is inaccessible in the sense that the corresponding behavior is unlikely to occur. The points within this area are pulled towards either attractor.

Divergence: Deviation from a linear relationship between the response and predictors demonstrated by two diverging response gradients—deviating paths towards the upper or the lower part of the surface.

Bifurcation point: The two divergent paths are joined at the bifurcation point at which the behavior is ambiguous, and beyond this point the system enters the *bifurcation set*, the area where discontinuous changes take place.

Sudden jumps: Abrupt changes between attractors, representing distinct behavioral modes, occurred even with slight changes in the control variables.

Among the above, sudden jumps in the value of the state variable, hysteresis, and bimodality are the most common flags constituting indicators for the presence of a cusp catastrophe in empirical data (Figs. 9.2 and 9.3). The identification of such flags encompasses a *qualitative* approach in evaluating the cusp as a model for data (see also Gilmore, 1981; Stewart & Peregoy, 1983; van der Maas & Molenaar, 1992; van der Maas, Kolstein, & van der Pligt, 2003).

Stochastic Catastrophe Theory

Catastrophe theory was developed initially for deterministic dynamical systems, whose basic processes entail change towards states of extrema (maximum or minimum), and it is perfectly applied to physical systems, e.g., to a pendulum

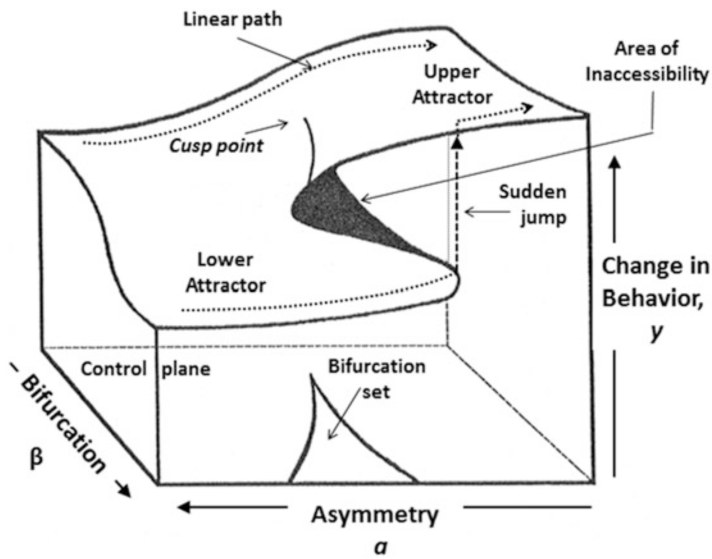

Fig. 9.2 Response surface of the cusp catastrophe model

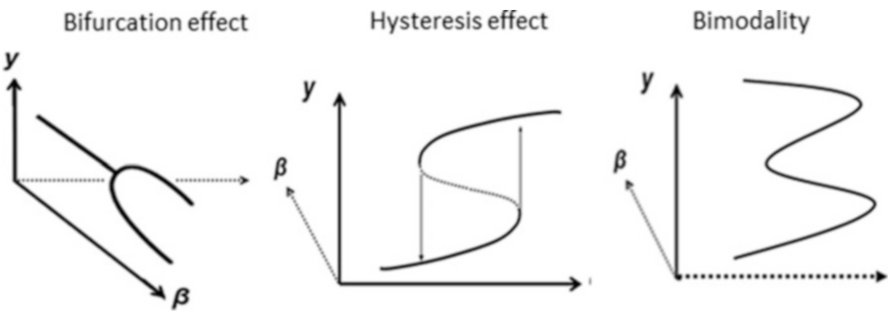

Fig. 9.3 Schematic representations of *bifurcation*, *hysteresis* effects, and *bimodality*

moving towards states of minimum potential energy. Theoretically, at least, the central idea seems to be applicable to human systems as well, by considering that the underlying processes of any social system always attempt to optimize some kind of "function," e.g., to maximize support or to minimize conflict.

Thinking stochastically, and focusing on the differential equation (9.1) holding for gradient dynamics, if the change in the dependent variable y is probabilistic rather than deterministic, then there is a probability density function over the rate of changes in y. On this idea Cobb (1978) set the basis for the development of *statistical catastrophe theory*. He restated catastrophe models using stochastic differential equations, where the assumed stochastic processes have stationary probability density functions of topological interest, which are receptive to statistical analysis.

The construction of stochastic catastrophe models starts by considering a deterministic system controlled by smooth potential function $U(y)$ and the relation (9.1)

$$dy/dt = -\partial U(y)/\partial y$$

The *singularities* of $U(y)$ are the points for which $\partial U/\partial y = 0$, while if they are *degenerate* ones the relation $\partial^2 U/\partial y^2 = 0$ also holds. In order to get a stochastic equation, a white noise term $dw(t)$ is added, so the differential equation becomes

$$dy = (-\partial U/\partial y)dt + \omega(y)dw(t) \tag{9.4}$$

The function $w(t)$ corresponds to standard Wiener process (Brownian motion), while the $\omega(y)$ modulates the intensity of the random input $dw(t)$ (Cobb, 1978). The increments of a Wiener process, $w(t + \Delta t) - w(t)$, are normally distributed with variance Δt. The function $\omega(y)$ determines the size of the variance of the noise and is called the *diffusion function*, which could be set to be constant. It was shown that the probability density function of the state variable y ultimately converges to a stationary one. Placing an error term in equation, the model becomes stochastic and the concept of persistence replaces the concept of stability in the deterministic one. Moreover, distinction between and within subject variability is allowed; thus stochastic catastrophe models can provide the means for investigating systems driven by underlying nonlinear processes (Cobb, 1978; Stewart & Peregoy, 1983).

Applying stochastic calculus and using Ito-Wright formulation finally a general equation is derived, which expresses that any differential equation can be presented as a probability density function *pdf*:

$$pdf(y) = \xi \cdot \exp\left[2\int^{x}(-\partial U/\partial y)ds/\varepsilon\right] \qquad pdf(y) = \xi \cdot \exp[2U(y)/\varepsilon] \tag{9.5}$$

where ε is the value of the variance function assuming to be constant and ξ is a constant introduced to ensure unity density.

In Cobb's stochastic catastrophe theory the derived stochastic differential equation is associated with a probability density that describes the distribution of the system's states in time. Thus, there is a unique relation between the potential function and the *pdf*. The stable and unstable equilibria of the potential function correspond to modes and antimodes of the *pdf*, respectively. A stochastic bifurcation occurs when the number of modes and antimodes changes as the control variables vary. By choosing a potential function one formulates the corresponding model. For instance using the canonical potential function for cusp catastrophe (Table 9.1) the corresponding probability density function is

$$pdf(y) = \xi \exp\left[-\frac{1}{4}y^4 + \frac{1}{2}by^2 + ay\right] \tag{9.6}$$

For empirical research, the next step was the development of statistical procedures to make estimates for the parameters for a specified hypothetical model, given a random sample of observations. Over the last decades various methods were developed based on maximum likelihood or least square optimization methods, so that given a set of empirical data, it becomes possible to test statistically hypotheses concerning the existence of degenerate singularities within the data.

Statistical and Methodological Issues

In this section, some crucial issues that appeared during the development of the stochastic catastrophe theory are highlighted, along with comments on the various methodological approaches and solutions. It is important to realize that catastrophe theory models, compared to the linear ones, are not easily workable and there are difficulties in developing evaluation procedures due mainly to the probability density functions, that is, the idiosyncrasy of the bimodality (or multimodality) and the non-triviality of the error variance. In addition, there are some strictly mathematical impediments concerning the nonlinear diffeomorphic transformation of the measurement, which however are not addressed here.[3]

From mathematical point of view the development of catastrophe theory involved primarily understanding of the critical points, that is, to determine how critical points behave via an, e.g., "equation of motion," which actually does not exist. Thus, the state of the system can be determined by fostering certain assumptions about the dynamics of the system (Gilmore, 1981, p 143). There are two *conventions* associated with these underlying assumptions, the *Maxwell convention* and the *delay convention*. The *Maxwell convention* considers that the system immediately jumps to a new equilibrium area. The state of the system is determined by the global minimum of the potential function. As the control parameters change, the state remains at the minimum as long as the current minimum remains the global minimum of the potential. When this minimum stops to be the global minimum, then the system state jumps to a new global minimum. The *delay convention* assumes that the system remains in the old equilibrium zone until the last possible point before it passes to the new equilibrium area. The state of the system is determined by the local minima of potentials. As the controls change, the state remains at the local minimum as long as the minimum exists. When the current minimum disappears, then the system's state jumps to a new local one. For the stochastic catastrophe theory, the above have crucial impact on the way the expected value of the bimodal distribution is estimated and affect the computation of the error variance and *scale*. It is recommended and worth trying to proceed with both conventions.

The various modeling techniques developed for testing catastrophe theory in empirical data are based on different assumptions and statistical approaches. A difference could be based on aforementioned *conventions*. Another difference lies in the presumed nature of variables; that is, they could be considered as univariate or multivariate. The univariates are measured directly as observable, while the multivariates are treated as *latent* variables with multiple indicators. Differences could also be based on the modeling formula, which could be the system's potential function or the derivative of the potential function.[4] Different optimization methods, such as the least squares or the maximum likelihood method, could also be implemented. Accordingly, different statistical tests and indexes are used for model evaluation; for example in the maximum likelihood method, BIC and AIC

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[3] Another mathematical issue of concern is that the classification scheme of the systems developed by Thom presupposes that the systems under consideration must be transformed to its canonical form using diffeomorphism transformations. Thus, the invariance under diffeomorphic transformation should hold. For the deterministic case it does. The stochastic version as developed by Cobb based on pdf is *not* invariant under nonlinear diffeomorphic transformation of the measurement. Statistical problems related to diffeomorphic transformation have not been addressed, while solution has been proposed for some cases, e.g., time series data (Wagenmakers, Molenaar et al., 2005).

[4] There are pros and cons to that choice, since as it has been pointed out that methods based on the derivative of potential function might reward the presence of unstable equilibrium states, while those based on the *pdf* might punish their presence, as these correspond to points in an area of the density function of low probability that lies in between two high-probability states.

criteria are implemented, while in least squares method the percent variance explained (R^2) is used as the effect size criterion for comparing a catastrophe model with the linear competitors. Besides the above criteria, a nonlinear model has to have all parameters statistically significant, while special attention should be given to certain parameters, e.g., the bifurcation factor in the cusp model, which plays a crucial role in the model specification and its interpretation. Practically wise, the different methods and the corresponding calculations could be performed either with popular software (e.g., IBM-SPSS, Statistica, Stata, SAS, Minitab) or with more specialized ones (e.g., GEMCAT, cuspfit in R). Specific concerns about the methodological choices, pros and cons, critiques, and debates could be found elsewhere (e.g., Alexander, Herbert, DeShon, & Hanges, 1992; Guastello, 1992; Guastello, 2011a, p. 275; van der Maas et al., 2003).

Finally, it is imperative to single out that the researcher be aware of the fact that in catastrophe theory analyses, like in any other methodological approach and stochastic procedure, assumptions and conventions always are made, which might inevitably limit the anticipated results and conclusions. Ergo, it is suggested that analyses might be strengthened by a combination of methods. Encouraging, however, is that the methodological assets of catastrophe theory nowadays support high-quality research, and thus are promising for the advancement of theory and practice in educational research, as it has been realized in other social sciences. Methodologically, when new research endeavors are initiated, it is important that statistical procedures are not merely applied to available data with a curve fitting philosophy, but rather, a research design is followed in model specification, which is sourcing out from a deeper understanding of the underlying mechanism and the dynamics of the system.

Sample Size and Research Design Philosophy

The sample size issue is in general an unexplored territory for nonlinear regression modeling. It is related to *statistical power*, which is the *odds* of rejecting the null hypothesis (*Ho*) given that it is actually false. Note that the issue arises from cases with very large samples that result in statistical significance, while the effects are very small. In the linear regime and for bivariate tests the statistical power analysis is rather a straightforward procedure, whereas for multivariate analysis, e.g., multiple regressions, the determination of sample size for a given power is a more complicated matter, since it depends on a number of factors, such as the intended effect size, overall R^2, the number of independent variables, the degree of correlation among them, and assumptions on their equal or unequal weights. Therefore, a lot of different procedures have been developed for determining the proper sample size.

For catastrophe theory models a concerned researcher has to rely on rubrics that developed for linear models with the same number of variables. For example, for a linear regression with three independent variables, medium effect, and intended power of 0.80, 55 cases might be the sample size (Maxwell, 2000). Recently, a Monte Carlo simulation-based method was reported, which was used to calculate statistical power and sample size for Guastello's polynomial regression cusp catastrophe model. A power curve is produced under different model specifications (e.g., different error term) and then it was used to determine sample size required for specified statistical power (Chen. Chen, Lin, Tang, Lio & Guo, 2014). Interestingly, sample size varies with measurement error. For power 0.85 and $\sigma = 1$ the sample size is 36 and becomes 100 for $\sigma = 2$. Thus, for this statistical approach, a moderate sample size is adequate for cusp analysis. Moreover, as far as the statistical significance is concerned for small samples, the results can be strengthened by implementing bootstrapping techniques (Stamovlasis, 2014a).

The sample size and the sampling adequacy in nonlinear analysis and modeling have been an issue of debate for some time where the "myth of million data points" has been untangled (Gregson & Guastello, 2005). A fundamental notion related to the issue in question is the *restriction of topological range*, which concerns the full ranges of data which the hypothesized dynamics are unfolding in. It is of paramount importance that the available data should cover the proper spectrum of values in order to capture the nonlinear effect associated with hypothesized model (Guastello, 1995). Given that nonlinear phenomena are manifested along with linear dependences, it is the researcher's responsibility not to just seek for merely a good curve fitting, but to also build first a theory-laden model, which satisfies aspects of the anticipated behavior in the context of system's dynamics.

Statistical Methods in Cusp Catastrophe Analysis

The contemporary stochastic catastrophe theory permits testing related hypotheses and examining the type of catastrophe structure that a set of observational data might possess. In this section, the cusp model analysis will be examined as the most eminent and applicable to behavioral sciences. In practice, when analyzing data one may start with the *qualitative* approach, seeking for catastrophe "flags," such as sudden jumps, hysteresis effects, and bimodality. For example, bimodality increases at higher values of bifurcation variable and it can be observed using the graphical representation showing the frequency distributions of the state variable at different levels of the bifurcation. However, the *quantitative* approach, which includes statistical procedures, merely, provides the sound evidence that the model fits the observational data. A number of methods and techniques have appeared in the literature based on different assumptions and statistical modeling. Some of them are more established, popular, or applicable; it is worth presenting, however, all the most contributing to development of the stochastic catastrophe theory and its application to behavioral sciences.

Model with Probability Density Function

First, Cobb (1978, 1981) starting from stochastic differential equations demonstrated that the cusp catastrophe can be represented by the cusp family of probability density function, such as the *pdf* in equation (9.6). The state variable is corrected for location and scale, $z = (y - \lambda)/\sigma$, while it is assumed to be *univariate*, but the control variables a and b, the asymmetry and the bifurcation factors, respectively, are assumed to be *multivariate* (latent). The canonical parameters a and b in the model depend on the two observed and measured control variables, i.e., c_1 and c_2, and they are expressed with the equations

$$a = a_0 + a_1 c_1 + a_2 c_2 \qquad b = \beta_0 + \beta_1 c_1 + \beta_2 c_2$$

The cusp catastrophe fitting procedure then involves the estimation of the parameters α_0, α_1, α_2, β_0, β_1, β_2, λ, and σ using maximum likelihood method (Cobb & Watson, 1980). A reliable computer program was developed for data analysis, which later had undergone some computational improvements (see Hartelman, van der Maas, & Molenaar, 1998) and it is free on the Web.

Based on the probability function a direct method has also been proposed for fitting the cusp model using nonlinear regression with least square procedures (Guastello, 2011b). The cusp model is compared with its linear alternatives and it has to be superior in terms of R^2. The method is easy to perform and the related statistics can be carried out with a usual software.

The GEMCAT Methodology

Oliva and his coworkers (1987) developed the GEMCAT methodology, primarily for cusp, but also for swallowtail and butterfly catastrophe. The mathematical formalization for the cusp assumes that the response Z and the two controls, asymmetry and bifurcation X and Y, are defined as *latent* variables, each measured by a number of observables:

$$Z = \sum_{k=1}^{k} \gamma_k Z_k \qquad X = \sum_{i=1}^{i} a_i X_i \qquad Y = \sum_{j=1}^{j} \beta_j Y_j$$

The equation $f(Z, X, Y) = \frac{1}{4}Z^4 - \frac{1}{2}YZ^2 - XZ$ defines the cusp function and its first derivative set equals to zero: $Z^3 - YZ - X = 0$. The estimation problem then is stated as

$$\min(a_i, \beta_j, \gamma_k) = \Phi = \|\varepsilon^2\| = \sum_{1}^{N} \left[Z^3 - YZ - X\right]^2 \tag{9.7}$$

where $\varepsilon =$ error and the summation is over the N observations. Given a set of empirical data for the response Z and the two controls, asymmetry X and bifurcation Y, one may estimate the impact of coefficients (α_i, β_j, γ_k) that define the corresponding latent variable, which minimize the function Φ. A modified control random search (CRS) algorithm was developed to estimate the desired parameters. The procedure, which is an *MLE* method, is equivalent to finding the best cusp catastrophe surface fitting to the empirical data. Analogous methodology and similar optimizing algorithms are followed for the other of catastrophe models. The GEMCAT program which is free on the Web provides a series of options, such as constraints on the coefficients (α_i, β_j, γ_k), standard errors for the parameters, a utility for testing competed nested models, chi-square statistics, standard likelihood ratio tests, and AIC statistic for fitting indices. In the latest version of the method (GEMCAT II, Lange, Oliva, & McDade, 2000), the technique was improved and inference is based on resampling techniques (jackknife and nonparametric bootstrap). The present program has been popular mainly among economic researchers.

Method of Difference Equations and Polynomial Regression Techniques

This model was developed by Guastello (1982, 1987, 2002, 2011), who followed a different approach. Starting from the deterministic equation $dz = (z^3 - yz - x)\, dt = 0$ by setting $dt = 1$ and inserting beta coefficients one gets the statistical formula:

$$\Delta z = z_2 - z_1 = \beta_1 {z_1}^3 + \beta_2 y z_1 + \beta_3 x + \beta_0 + \varepsilon \tag{9.8}$$

where ε is the error term. The polynomial regression technique approximates Cobb's stochastic form of Eq. (9.4) by a difference equation, which essentially results in a polynomial regression equation. The above equation is used to model the behavioral change $z_2 - z_1$ between two points in time, *Time 1* and *Time 2*, with behavioral outcomes z_1 and z_2, respectively. The difference equation in this formalism is assumed to imply a differential equation. Practically the equation implemented in data analysis contains often a quadratic term $\beta_4\, {z_1}^2$, which serves as a correction term associated with location, and it could be dropped if it is not significant or if it does not improve the model (Guastello, 2002). Data analysis with model includes testing the following alternative linear models:

$$\text{Linear 1} \quad \Delta z = \beta_1 x + \beta_2 y + \beta_0 \tag{9.9}$$

$$\text{Linear 2} \quad \Delta z = \beta_1 x + \beta_2 y + \beta_3 xy + \beta_0 \tag{9.10}$$

$$\text{Linear 3} \quad z_2 = \beta_1 x + \beta_2 y + \beta_3 z_1 + \beta_0 \tag{9.11}$$

z is the normalized behavioral variable, while x and y are the normalized asymmetry and bifurcation, respectively. The normalization procedure involves transformation of raw scores λ to z scores corrected for location and scale σ_s:

$$z = (\lambda - \lambda_{\min})/\sigma_s \tag{9.12}$$

Location correction is made by setting the zero point at $\lambda_{\min}$, the minimum value of λ, and the scale σ_s is the ordinary standard deviation of λ. The normalization is applied to the control variables as well. In some cases, the scale could represent the variability around the modes rather than around the mean (Guastello, 2002). The most competitive model is usually the *pre-post* linear model in Eq. (9.11).

In the above model the least square (OLS) method is used as optimization procedure. The distribution of the dependent measure at *Time 2* is expected to possess larger variance and it might exhibit bimodality. In order to demonstrate that a cusp catastrophe is the appropriate model to describe the outcome, its regression equation should account for a larger percent of the variance (R^2) in the dependent variable than the linear models. In addition, both the cubic and the product terms in Eq. (9.8) must have significant weights and/or the confidence intervals (95 % CI) should not span the zero point. The regression slopes, standard errors, *t*-tests, confidence intervals, and model fit for the cusp and the control linear models should be reported.

When modeling nonlinear phenomena, the inclusion of a nonlinear function in the model affects basic assumptions of standard measurement theory. In classical psychometric theory a measurement Y consists of a true score, T, and error term e. The percent unexplained variance is considered as error, while errors are assumed to be normally distributed and uncorrelated to each other and to true scores (*iid*). However, when a nonlinear function is included *dependent errors* (*de*) are expected to appear in the residuals. It has been shown that such non-*iid* errors (residuals) are indicative of nonlinear processes (Brock, Hseih, & Lebaron, 1990). The residual analysis could suggest that this might be the case. In nonlinear dynamical processes the score variance has four components:

$$\sigma^2(z) = \sigma^2(\text{linear}) + \sigma^2(\text{nonlinear}) + \sigma^2(\text{de}) + \sigma^2(\text{iid}) \tag{9.13}$$

The four components are the linear, the nonlinear, the dependent errors, and the *iid*. A linear model treats the last three components as errors [($\sigma^2(e)$], while the dependent errors are captured only by the proper and well-defined nonlinear model and could increase the variance explained (Guastello, 2002).

The difference equation model is affected by the restrictions and disadvantages of the OLS, e.g., under suboptimal condition the empirical coefficient may not be significant, while the bivariate correlations are. In those cases a cross-validation strategy is suggested by investigating collinearity effects among the control variables or other components of the model. Also, the order that the variables are entered in the OLS procedure could make a difference. It is recommended that all variables are entered simultaneously. In principle, the method considers the asymmetry and bifurcation as observables; however combination of candidate variables could be tested (e.g., Stamovlasis & Tsaparlis, 2012).

For enhanced generalization, *bootstrap* estimates have been recommended to cross validate the significance of the beta coefficients and the overall fitness of the model (Stamovlasis, 2014a). Note also that a large explained variance that might appear in some cases due to high linear correlations is not adequate to ensure a cusp structure. The fundamental components, such as the cubic term and especially the bifurcation term, have to be statistically significant (Guastello, 2011a, pp. 276).

The Cuspfit in R

Latest advances in catastrophe theory literature have presented methodological improvement and sophisticated software supported the analyses. The cusp package in R (Grasman, van der Maas, & Wagenmakers, 2009) combines the maximum likelihood approach of Cobb and Watson (1980) and the subspace fitting method proposed by Oliva et al. (1987).

The state-dependent variable y and the control variables of the cusp are considered as canonical variables, that is, they are smooth transformation of the actual state and control variables of the system. If there are n measured dependent variables $Y_1, Y_2, \ldots, Y_n$, then y is a linear weighted sum of them:

$$y = c_0 + c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n$$

Similarly the latent controls a and b are linear functions of the k measured independent control variables $X_1, X_2, \ldots, X_k$:

$$a = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_k X_k$$
$$b = b_0 + b_1 X_1 + b_2 X_2 + \cdots + b_k X_k$$

The fitting routine in R package performs maximum likelihood estimation of all the parameters in the above equations. The cusp program using one built-in optimization routine minimizes the negative log-likelihood L for a given set of experimental data, with respect to parameters, $\alpha_0, \alpha_1, \ldots, \alpha_k, b_0, b_1, \ldots, b_k, c_0, c_1, \ldots, c_n$:

$$L = \sum_{i=1}^{n} \log \Psi_i - \sum_{i=1}^{n} \left[-\frac{1}{4} y_i^4 + \frac{1}{2} b_i y_i^2 + a_i y_i \right] \tag{9.14}$$

In order to preserve stability and to control collinearity among predictors, standardized data are used.[5] A problem might arise from non-convergence of the optimization algorithm, which is overcome by providing alternative starting values (Grasman et al., 2009).

For statistical model fit evaluation, a number of diagnostic tools are provided. One is the pseudo-R^2 which is defined by the equation

$$\text{pseudo } R^2 = 1 - \frac{\text{ErrorVariance}}{\text{Var}(y)} \tag{9.15}$$

[5] The standardization is performed with QR decomposition, which is a mathematical procedure for obtaining accurate matrix decomposition using the modified Gram Schmidt re-orthogonalization method. It accounts for collinearity in the design matrix and the stability of the estimation algorithm (Press, Teukolsky, Vetterling, & Flannery, 2007).

The concept is analogous to the squared multiple correlation coefficient; however in the cusp catastrophe model the pseudo-R^2 is not the same as the measure of explained variance. It can take negative value if the error variance exceeds the variance of y, and for this reason it is not a reliable fit index. This is because the error variance is nontrivial and it is calculated based on predictions of *delay* or *Maxwell* rules. Recall that these estimation rules (conventions) are relevant to the concept of discontinuity as it was discussed earlier. Thom had from the beginning fostered the *Maxwell convention*, while in Cobb's method the *delay convention* was suggested. The *cuspfit* in R offers both conventions with the delay convention as the default.

Additional criteria typically used for evaluating the model fit of the cusp catastrophe are the following:

- The coefficients in the model should be statistically significant.
- Cusp model compared to the linear counterparts should be significantly better in terms of its likelihood.
- Cusp model could also be compared to the logistic function below:

$$y_i = \frac{1}{1 + \mathrm{e}^{-a_i/b_i^2}} + e_i \quad i = 1, \ldots, n$$

which does not possess degenerate critical points, but it can model steep changes mimicking abrupt transitions similar to the cusp (Hartelman, 1997). Besides the statistical part, however, it is important to note here that even though the logistic function is co-examined as an alternative model, it is not associated with rigorous theoretical interpretations as the cusp catastrophe (see the epistemology section).

- The use of AIC, AICc, and BIC for all alternative models should be in favor to cusp model. Especially the BIC can be used to compute approximation of the posterior odds for the cusp relative to the logistic curve, assuming equal prior probabilities (Wagenmakers, van der Maas, & Molenaar, 2005).

When analyzing with the *cuspfit* in R a difficulty arises if there are two or more dependent variables because in these cases the counterpart antagonistic linear regression model is not uniquely defined. Additional limitations are the absence of the alternative linear model with the interaction term and the lack of an effect size index, such as the R^2 in least square approach that could serve as a basis for comparison.

The advantage of the cuspfit method is that it can implement control variables as multivariate latent constructs and can be used in confirmatory analysis. When it is used in an exploratory approach the independent variables should not be assigned arbitrarily to the controls because the results might be very peculiar and uninterpretable. In order to improve estimations and get better results, it is recommended that before using the *cuspfit*, a factor analysis (e.g., PCA) should be applied, in order to identify the sets of potential candidates for control parameters.

Theoretical and Epistemological Issues

The following epistemological discussion focuses on the cusp catastrophe and its main features, which are fundamentally related to nonlinear dynamics. The cusp model reveals the pattern of behavior as a function of the two control variables, the asymmetry a and the bifurcation β, and it states that both linear and nonlinear changes in behavioral variable are expected depending on the values of the two controls.

The model interpretation via Fig. 9.2 suggests that at low values of β changes are smooth and a linear relation can better describe the relationship between the asymmetry and the response. At low values of a, changes occur over the lower mode and are relatively small. At high values of a, changes occur around the upper mode and are again small. At high values of β, however, changes are discontinuous and abrupt shift can be observed between the two modes or behavioral attractors. At the control surface we can observe the bifurcation set mapping in the unfolding of the surface in two dimensions. The cusp bifurcation set induces two diverging response gradients, which are joined at the *cusp point*. At the cusp point the behavior is ambiguous, while the two diverging gradients represent varying degrees of probability that a point be in the one or in the other behavioral mode (Guastello, 2002).

The three-dimensional response surface entails the *geometry of behavior*, which explicates that for certain values of the asymmetry a and the bifurcation β, a point, the *bifurcation point*, exists, beyond which the system enters the *bifurcation set,* the area where discontinuous changes occur. Points within the area of *inaccessibility* are unlikely to be observed, since they are pulled towards either behavioral attractor, and this is what introduces nonlinearity and uncertainty in the system, which, it is said, enters the chaotic regime. This behavior is also depicted on the other fundamental feature disclosed in the cusp structure, the *hysteresis effect*; that is, cases with the same values on control variables could be found either in the upper or the lower mode of the response surface.

The above geometry of behavior, which seems quite complicated to ordinary linear thought, is obviously *phenomenological*; that is, it apparently does not explain, but merely it describes the behavior. Thus, the crucial question, which entails explanation, is what kind of mechanism might force the state of the system to follow the response surface. This is a fundamental epistemological question to be answered (Zeeman, 1977).

Catastrophe theory models in science involve dissipating systems or potential-minimizing systems. The mathematical formalism using a potential function for a mechanical system, e.g., Zeeman's catastrophe machine, seems appropriate since by nature it is expected to obey some sort of deterministic type natural law. Epistemological questions arise, however, when attempting the application of catastrophe theory to "soft" science dealing with human behavior and related systems. Recall here that one of the points of criticism of catastrophe theory was the existence of potential function, which seems to arbitrarily appear in order to describe the sudden shifts in the system. The issue is related to argument originated from the confusion about the conception of discontinuous jumps (Zahler & Sussman, 1977), which ignores the *attractor notion*, a fundamental concept in nonlinear dynamics. The cusp model describes the shifts between stable states or distinct modes of behavior (behavioral attractors). This behavioral change might imply or be a *qualitative change*. This description is founded on the operation of a potential function, which mathematically is the proper tool to model shifts between attractors. The mathematical formalism of the cusp model assumes that the system is controlled by a "potential" function with two stable equilibria (Poston & Stewart, 1978). This assumption, for behavioral sciences, is not as arbitrary as it seems to be. Note that the assumption of linearity is also arbitrary, to the extent that there are no reasons for the behavior to follow straight lines; however the assumption of linearity being seemingly the simpler one is easier to accept.

Within complexity and nonlinear dynamics, epistemological arguments concerning human systems are advocates to the existence of attractors and dissipating mechanisms. One is that for human systems' behavior, an optimization process, analogous to energy dissipation or potential minimization process, can be reasonably assumed. A psychological system for instance could be sought as seeking to minimize cognitive dissonance, or to maximize the degree of adaptation (Saari, 1977). The concept of energy minimum is closely related to and it is a special case of the *attractor* concept, which by definition represents the stable state of a system operating in a dynamical equilibrium. Moreover, attractors at the psychological level can be assumed that originate from the brain functioning, which operates as nonlinear dynamical system possessing multiple coexisting attractors (Kelso, 1995; Freeman, 2000a, b; Freeman & Barrie, 2001). In addition, theoretical models on brain functioning based on neuropsychological evidences have provided mathematical description of its dynamics in perception and action, using the language of nonlinear dynamics. According to Nicolis and Tsuda (1999), brain functions as dissipative dynamical system, which is characterized by sensitive dependence on the initial conditions and the control parameters. These are manifested as chaotic behavior including bifurcations, braking symmetry, and multiplicity of behaviors beyond an instability point. In compensation to unpredictability due to the nonlinear character of the underlying process, the following hold for the system: (1) the existence of multiple attractors possessing invariant measures in the dynamical system governed by the interplay among the order parameters and (2) drastic reduction of degrees of freedom in the vicinity of a bifurcation and the emergence of essentially only a few dominant order parameters. These parameters may subsequently interact in a nonlinear fashion, giving rise to low-dimensional dissipative chaos. (3) Within such systems *information* is produced (Nicolis & Tsuda, 1999). The latter, the potential to produced information, is a property of nonlinear dynamical processes and it will be seen again in a later discussion on learning and creativity.

Answers to epistemological questions on phenomena, such as a bifurcation and hysteresis effects (Fig. 9.3), the interpretation of which seems too complicated for linear and reductionist ways of thought, are given by *self-organization theory*. The important feature of complex dynamical systems is the *emergent* properties that appear through *self-organization* processes. A cusp catastrophe for instance, when detected, is by virtue a state transition, and it is an emergent discontinuity. This finding at the behavior level has important philosophical implications targeting to ontological questions, since a bifurcation is the phenomenology of complex adaptive systems; it is in fact the signature of complexity and indicative of *self-organization* mechanisms (Nicolis & Nicolis, 2007). The notion *self-organization* has supported the development of the major scientific theories of nonlinear dynamics: Prigogine's non-equilibrium thermodynamics (Nicolis & Prigogine, 1977; Prigogine, 1961), Haken's synergetics (1983, 1990) and Thom's catastrophe theory (1975), even though they were grown with different rationales.

Self-organization can provide a *causal interpretation* of the bifurcations and state transitions within a nonlinear dynamical system and it is the process that occurs when a system is at a state of high *entropy* and far-from-equilibrium condition (Prigogine & Stengers, 1984). The structure that is taken on, which is an ordered state, allows the system to operate more efficiently and interestingly it does not require any outside intervention; this is the notion of "*order for free*" pointing out by Kauffman (1995, p. 17). *Self-organization* has been implemented for physical and biological systems as an explanatory theory; however it could be transferred to human system as well, for explaining emergent patterns observed in psychological processes (Grigsby & Osuch, 2007; Hollis, Kloos, & van Orden, 2008). It has been fostered for a causal interpretation of Piaget's theory of stepwise cognitive development (Molenaar & Raijmakers, 2000) and for interpreting the emergence of creativity (Stamovlasis, 2011).

A final point to be singled out is that the phenomenology of nonlinear systems is due to *self-organization* mechanism and on the other hand to the operation of coexisting attractors and the dynamics of the system. Bifurcation mechanism in a physical system such as Zeeman's catastrophe machine is nested in the operation of a potential function and the dynamics of the system (Zeeman, 1976). Similarly, when examining a cognitive or human system, its dynamic behavior is the formative cause of the ensuing bifurcation and the emergence of the new topological pattern in the state space of the system.

Note also that in psychological and educational sciences, the *processes* under examination regarding cognitive and human systems are more likely non-*ergodic*, and the hypothesized underlying *evolution equation* that describes the system over time is unknown. These two points are where catastrophe theory is filling the gap: it concerns sudden changes and it exemplifies that for studying these state transitions in a system, the evolution equation does not have to be known in advance; the description and the explanation of local observed behaviors can be attained with a small number of control parameters (Castrigiano & Hayes, 2004; Gilmore, 1981; Poston & Stewart, 1978; Thom, 1972, 1975, 1983). The above are also in accordance with primary postulates of nonlinear dynamical systems, where the principle of *dynamical minimalism* is assumed; that is, complex behaviors can be produced by simple rules and/or a few interacting variables. Thus, in constructing nonlinear models it is always sought to identify the simplest realistic set of assumptions and variables that finally produce theories that provide the simplest explanation of phenomena (Nowak, 2004; Vallacher & Nowak, 2009).

The above epistemological discussion concerns and applies to any process in educational research. The application of catastrophe theory, as a part of the meta-theoretical framework of nonlinear dynamics to a specific domain and discipline, does not ignore, but it essentially requires a local theory, which can provide the variables to implement as state and control factors.

Catastrophe Theory and Neo-Piagetian Premises in Learning Sciences

The Piagetian and Neo-Piagetian Theories

A requisite local theory that could serve as the bridge between science education research and nonlinear dynamics is the Piagetian and neo-Piagetian premises (Case, 1985; Pascual-Leone, 1970; Piaget, 1967; Piaget & Inhelder, 1969). They have been exceptionally appealing to educational sciences and they are the first on which catastrophe theory and nonlinear dynamics have been applied in a remarkable way. At earlier times, catastrophe theory has been connected to Piagetian stagewise development (Molenaar & Oppenheimer, 1985). A few interesting models had been proposed with the implementation of some core Piagetian concepts, such as the *assimilation* and *accommodation* processes, which were considered as controls determining the abrupt shifts between developmental stages, while discontinuities in the children responses in the vicinity of a transition from preoperational to concrete operational thought have been shown (Preece, 1980; Saari, 1977). A few decades ago, it has been pointed out that catastrophe theory analysis could embrace the traditional methodological approaches concerning stagewise cognitive development, and later the dynamic systems theory has been proposed as the unified framework of development (van Geert, 1991; van der Maas & Molenaar, 1992; van der Maas & Raijmakers, 2009).

The fundamental connection points between Piagetian and catastrophe theory are the notion of *equilibration* as applied to the former and the concept of *equilibrium* to the latter. Both are expressed mathematically by setting the first derivative of the dynamic system equation to zero. As it was pointed out in the epistemological section of this chapter, the *equilibrium* is implied by an *optimization process*, which is taking place within a dissipating system. This process allows the cognitive system to choose its internal states so that it maximizes the degree of adaptation, given the environmental inputs. Thus, from the beginning it was recognized that the inherent compatibility with catastrophe theory holds also for the neo-Piagetian theories, which can make available all the prerequisite psychological constructs for a catastrophe model specification.

The most representative within the neo-Piagetian premises is the *theory of constructive operators* (TCO), founded by Pascual-Leone (1970, 1987) as an account of individual differences in performance on mental tasks. According to TCO, cognitive processes involve a variety of *constructive operators*, each of which performs a specific function: the *M-operator* deals primarily with mental capacity, the *C-operator* with content knowledge, the *L-operator* with logical operations such as conservation and formal logic, the *F-operator* with field dependence/independence, and so on. The development of psychometric tests operationalizing the above mental resources allowed an array of applications in learning and educational sciences.

The merit of the neo-Piagetian framework as a scientific program with the Lakatotian sense (Lakatos, 1974) has demonstrated by its continuing evolution through the last decades (Pascual-Leone, 1970, 2000, 2013). Furthermore, it has supported a considerable amount of research at the behavioral level which has become the basis for further progress on theories of cognitive organization and growth that has determinedly added to our understanding about the architecture and function of mind (e.g., Demetriou, Efklides, & Platsidou, 1993; Demetriou & Efklides, 1994).

The neo-Piagetian theories emphasize the importance of a match between subject's mental operators and certain characteristics of mental tasks, for instance, the relation between *M*-operator and the mental demand of a task or between *F*-operator and the existing misleading information or "noise" in the data. Based on the above and given that numerous types of mental tasks or problems could be designed, a considerable amount of research has been carried out in the area of learning sciences, where individual differences associated with neo-Piagetian constructs have been shown to play a decisive role. The most known are the information processing capacity (*M*-capacity), the field dependence/independence or disembedding ability, the logical thinking (developmental level), and the convergent and divergent thinking. Note also that the information processing models (Baddeley, 1986) offer an analogue to *M*-capacity construct, the working-memory capacity, which has been linked to the well-known working memory *overload hypothesis* (Johnstone & El-Banna, 1986; Stamovlasis & Tsaparlis, 2001, 2005; Tsaparlis & Angelopoulos, 2000). It has been shown that the effect of the above variables is apparent in different types of mental tasks, such as algorithmic problems (Johnstone & Al-Naeme, 1991; Johnstone & El-Banna, 1986; Niaz, 1989), non-algorithmic problem solving (Lawson, 1983; Niaz, de Nunez, & de Pineda, 2000; Tsaparlis, 2005; Tsaparlis & Angelopoulos, 2000), and conceptual understanding (Danili & Reid, 2006; Kypraios, Stamovlasis, & Papageorgiou, 2014; Tsitsipis, Stamovlasis, & Papageorgiou, 2010, 2012; Stamovlasis, Tsistipis & Papageorgiou, 2010). Moreover, it has been shown that the effect of these individual differences is present at different ages from elementary school to the upper secondary education (Stamovlasis & Papageorgiou, 2012). Thus, the relationships between these individual differences and performance in learning sciences are well established, at least, in the linear regime.

Nonlinear Dynamics and Learning Science

In this section, the development of the framework for the application of catastrophe theory and nonlinear dynamics in science education is presented. It includes findings of inductive and deductive endeavors and implications for theory and the practice.

The first attention of nonlinear dynamical thinking to issues in science education was on problem solving, the most intriguing areas where neo-Piagetian constructs have been proved predictive variables. However, these effects have not been consistently observed across topics and ad hoc explanations were given to various contradictions, such as the unexpected failure of highly skilled students. On the other hand, it was clear that success could not be attributed to merely one variable and that some other individual differences interfere and cover up the effect of the hypothetical main predictor (Johnstone & Al-Naeme, 1991; Tsaparlis & Angelopoulos, 2000). The moderator role of some variables, e.g., field dependence/independence on information processing capacity, was evident, but there was lack of a comprehensible model that joins the synergetic role of these two mental resources. A response to this inquiry was the proposition of a cusp catastrophe model with the two above variables as controls. The effect of the two independent variables operationalizing two opponent processes is visualized as *force field dynamics*, where the outcome cannot be merely estimated as their weighted linear sum. Analysis of empirical data showed that for some cases the cusp catastrophe model was superior to its linear alternatives explaining a large portion of the variance of students' performance in chemistry problem solving (Stamovlasis, 2006). The above cusp structure, however, was not identified in every type of problem-solving data. Nonlinear models are not always better; that is, nonlinearity is not manifested everywhere.

The explanation to this was sought in the nature of mental processes and the differences that might exist among various tasks. There was need for reasonable justification, rooted, however, to fundamental theoretical premises. In science teaching, there are two types of cognitive tasks: The first are known as *exercises*. The students by applying a well-known solution path reach the answer successfully. The algorithm has usually been practiced, while the subjects are not necessarily aware about the strategy followed. On the other hand, there are "*real*" *problems*, where students cannot apply a learned procedure and the challenge is to find the solution path. Often it is said that those non-algorithmic problems require conceptual understanding and *high-order cognitive skills* (Tsaparlis & Zoller, 2003), implying an effective synergy of mental resources (e.g., neo-Piagetian constructs). Of course, in the school context all the above depend on what has been taught.

The answer to the question regarding nonlinearity manifested at the behavioral level is hidden in the differences between the two above categories of cognitive tasks; they correspond to two different processes, with distinct qualitative characteristics that determine the observed behavioral outcomes. A note of statistical interest is that the differences between the two types of problem solving are reflected in the empirical data and might become apparent in the briefing descriptive statistics. In easy and algorithmic problem solving, students' achievement scores are more likely distributed normally around the mean and most of the basic assumptions for linear modeling hold. A second statistical remark is that practically, in everyday school evaluation, students' scores conform to Gaussian distribution because they have to; that is, following *procrustean rationality*, which all the traditional evaluation theories propose, teachers and/or researchers tailor the assessment tests, so that they purposely produce bell-shaped curves in order to proceed with linear statistical analyses.

However, contrary to the ordinary thought, when exploring really challenging tasks, achievement scores are not recorded as normally distributed around the expected value, but deviation from normality, strong skewness, or even bimodality is often observed. Frequently, observations in these asymmetric distributions conform to the *inverse power law* or the *fractal distribution*. These are indicative for underlying dynamic processes where multiplicative rather than additive effects are taking place (West & Deering, 1995). Bimodality quite often appears also in very demanding tasks denoting bifurcation in a nonlinear process rather than the existence of two district subpopulations. In these cases, the implementation of conventional linear approach is proved inadequate and a nonlinear model, e.g., the cusp, arises then as a potential candidate. The reasons, however, for applying a nonlinear model to these empirical data are not merely statistical, but primarily are relevant to theoretical and philosophical issues. Algorithmic and non-algorithmic mental tasks belong to different categories as far as the nature of the underlying process is concerned. Algorithmic problem solving is a linear process, where predetermined and learned steps are followed. Non-algorithmic problem solving is a process with no predetermined scenario; each step is determined by the previous steps and there isn't a unique path to follow. The solution (if any) *emerges* from an iterative and recursive process, which is nonlinear and dynamic in nature. In this type of problems, nonlinearity at the behavioral level is more likely to be observed. Methodologically wise, yet, exploring empirical data obtained from such processes with linear models is an *epistemological fallacy* because the method is incompatible with the nature of the phenomenon being investigated (Stamovlasis, 2010, 2014b).

Based on the above theoretical premise, deductive endeavors have further supported the nonlinear hypothesis. A series of investigations have provided evidences for nonlinearity by the application of catastrophe theory in empirical data taken from science education research. Cusp catastrophe models explained students' achievement scores in chemistry and physics problem solving as a function of neo-Piagetian constructs that operationalize mental resources associated with the task execution. Those constructs were the information processing capacity (*M*-capacity or working memory capacity), logical thinking, disembedding ability, and divergent and/or convergent thinking. The dependent measure was the difference between the achievement scores in the prerequisite theoretical knowledge (z_1) and the problem-solving performance (z_2), while the least square technique (Guastello, 2002) was implemented. R^2 values were higher in the cusp compared to the linear alternatives. The nonlinear models were also supported by maximum likelihood estimates using cuspfit in R (Grasman et al., 2009), with fit criteria, such as AIC, AICc, and BIC (Stamovlasis, 2014a). The cusp structures do not appear in data originated from algorithmic problem solving and simple mental tasks. They are learned predetermined procedures, where the solution is actually known and nested in the algorithm. These are linear processes.

The crucial question for learning sciences is what the implications are. What have these nonlinear endeavors offered to science education, in theory and practice? Have they just provided an additional support to neo-Piagetian theories with new methodological tools? This is obviously true, but the main message is the crucial epistemological issues that challenge the dominant paradigm in educational research and practice.

In the epistemological section it was discussed that bifurcation and hysteresis effects are the signature of complex adaptive systems (CAS) and *self-organization* mechanisms. The findings via catastrophe theory models provide direct links to *self-organization* theory, and connect the behavioral level in education sciences with psychology and neuroscience, where the paradigm shift has already been attained. Thus, the above empirical research signified the departure from the mechanistic view of educational settings and set the framework for reconsidering, under the new perspective, the epistemological assumptions and the methodological issues in the existing local theories. It should be emphasized that the cusp models cited above are not advocates to the *reductionist view* for the role of individual differences and in general for any independent variables selected for describing and predicting phenomena in education. On the contrary, what the cusp models explicate is that given the protagonist role of decisive components in a nonlinear process the outcome might be ambiguous, due to the dynamics of the system and the sensitivity of the parameters.

Moreover, nonlinear dynamics and complexity challenge the conventional notion of *causality*, emphasizing the *emergent* nature of the outcomes through self-organization mechanism. The above concern the existing theories in educational sciences and in science education particularly, e.g., *constructivism* or *conceptual change* theories, which totally ignore, at least at the methodological level, the actual phenomena under investigation. Crucial debates and unanswered questions, such as those concerning the nature of conceptual change, could be resolved. For instance, the question, whether *conceptual change* is an outcome of a linear additive process modeled on the "architecture metaphor" or it is the outcome that *emerges* from a nonlinear dynamical process, could be addressed by implementing catastrophe theory. It is obvious that a new area of investigation opens that could elucidate crucial disputes and incoherent theoretical perspectives.

Coming to practical implications, based on rational explanations of students' failure, teaching strategies could be developed with the aid of the cusp response surface as a qualitative/metaphorical guide for manipulation of variables; for instance by reducing the "noise"-to-"signal" ratio one might induce "catastrophic success" (avoiding failure) for field-dependent students (see Stamovlasis, 2006). In addition, the identification of potential bifurcation variables in different cognitive tasks is crucial in learning sciences because these variables are more sensitive to the parameters and induce nonlinearity, turbulence, and uncertainty in the outcomes. For instance the moderating role of disembedding ability or logical thinking deficiencies beyond a threshold value might have a severe impact, leading abruptly to the overload phenomena (Stamovlasis & Tsaparlis, 2012). Catastrophe theory models could be applicable also to other educational processes at different level of complexity, e.g., at classroom or school level, where other variables, such as *motivation* or *performance climate*, under certain conditions, could operate as bifurcation variables for students' academic behavior (e.g., Sideridis & Stamovlasis, 2014; Sideridis, Stamovlasis, & Antoniou, 2015; Stamovlasis & Sideridis, 2014). It fact, a plethora of variables, individual, collective or environmental ones (Vygotsky, 1978), associated with educational process are potential candidates to be tested in a nonlinear context.

The new paradigm of nonlinear dynamics and complexity encourages further deductive endeavors and it signifies the departure from mechanistic views of the cognitive and educational processes and specifically of learning. Returning to the distinction between the two types of mental tasks, it was pointed out that the execution of algorithms and memorizing procedures are linear processes, which do not actually produce *information* (Nicolis, 1986, 1991). On the contrary, in *real* problem solving, where the system proceeds step by step in an iterative and recursive process without predetermined scenario, the solution *emerges* from the course of a nonlinear dynamical process driven by self-organization mechanisms. This theoretical remark affects obviously the definition of *learning*; algorithmic problem solving, like raw or parrot learning, is not "learning" per se (Stamovlasis, 2011). Novices attain learning outcomes if they involved in cognitive tasks mimicking processes that are nonlinear and dynamical in nature: the processes that produce *information*. Thus, educators and the scholars who develop curricula should be aware about this significant knowledge and should act accordingly. In science education the dominant and traditional instructing methodology stands on the opposite thesis, and persists in teaching algorithms, contributing essentially nothing to the issue of learning.[6]

It is noteworthy that most teaching practices have been developed on the computer view for mind, and it is rather amazing that they are still active, even though the theory has been proved flawed. The nonlinear dynamical nature of brain functioning as complex adaptive system operating far from equilibrium is the inherent property of mind that permits development that is not restricted by the

[6] A characteristic example is the plethora of problem-solving techniques taught in the Greek education system (and perhaps elsewhere) focusing on how to succeed in examinations in chemistry and physics, while students remain ignorant about the strategy followed or how to turn the implicit into explicitly. Behind this educational policy are wrong theoretical premises, that of computer metaphor for mind, and the hope that teaching problem solutions will enhance students' repertoire. This actually does not happen and rather it leads to *functional fitness*. The computer metaphor as theory of mind, applied to education, has been catastrophic for a novice's mind.

repertoire of the contributed components. This feature is a core element for nonlinear theories in psychology and behavioral sciences addressing human development, learning, and motor skill acquisition (e.g., Corrêa, Alegre, Freudenheim, Santos, & Tani, 2012; Molenaar & Oppenheimer, 1985; van Geert, 1991). It is relevant here to recall a debate and the criticism on Piaget's constructivist theory of stagewise cognitive development, around 1980s. Reductionist views (e.g., Fodor, 1980), refuting the nonlinear dynamical nature of human development, stated the alternative with the notion of "nativism", that is, certain features are "native" in the brain at birth, thus setting strictly programmed limitations to learning and development. The response to the criticism was decisive at that time, showing the possibility of acquiring more powerful structures, by fostering the nonlinear dynamical view of human development (Molenaar, 1986). That was merely a theoretical conjecture, and at that period along with the "adventures" of catastrophe theory and due to deficits in research methodology, the advancement of the new ideas delayed for two decades. Today the nonlinear dynamics and complexity framework returned in the scene with vigorous epistemological and methodological assets, as the new paradigm, alternative to linear and reductionist view of cosmos. Regarding educational issues in science teaching and in general the social and academic behavior, nonlinear dynamics is also filling the gap between genetic and environment dilemmas and offering a holistic view of reality that could amalgamate Piagetian and Vygotskian interpretations to a unified theory.

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