Integrability and Non Integrability of Some n Body Problems

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Abstract We prove the non integrability of the colinear 3 and 4 body problem, for any positive masses. To deal with resistant cases, we present strong integrability criterions for 3 dimensional homogeneous potentials of degree -1, and prove that such cases cannot appear in the 4 body problem. Following the same strategy, we present a simple proof of non integrability for the planar n body problem. Eventually, we present some integrable cases of the n body problem restricted to some invariant vector spaces.

Keywords Morales-Ramis theory · Homogeneous potential · Central configurations · Differential Galois theory · Integrable systems

1 Introduction

In this article, we will consider the n body problem whose Hamiltonian is given by

$$H_{n,d} = T_{n,d}(p) + V_{n,d}(q) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} + \sum_{1\leq i<j\leq n} \frac{m_i m_j}{\|q_i - q_j\|}$$

The quadratic form T correspond to kinetic energy, V is the potential, which is a homogeneous function of degree -1 in q. The coordinates $q_1, \dots, q_n$ correspond respectively to the coordinates of the bodies $m_1, \dots, m_n$.

Already since Poincare and Bruns [1, 2], it is known that the n-body problem is for $n \geq 3$ not integrable in general. Bruns in [1] proved the non-existence of additional algebraic first integrals, later generalized by Julliard-Tosel [3], and more recent work like [4–6] prove the meromorphic non-integrability or non existence of meromorphic first integrals in some cases. All these proofs strongly suggest that the n-body problem is never integrable for $n \geq 3$, even in particular cases (as proven for example for the

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isosceles 3-body problem in [7]). The colinear problem (in dimension 1) is a priori more difficult than the non-integrability proof of the n body problem in the plane and higher dimension, because it needs fewer additional first integrals to be integrable. Recall that as the energy and the impulsion of the center of mass are first integrals, in dimension 1 we only need $n-2$ additional first integrals for integrability. We will see that even if the problem is not so easy as the planar case, it can be completely studied in the case $n=3,4$ through the bounding of eigenvalues of the Hessian of V at central configurations (see Definition 1). A similar trick allows to obtain a simple proof of the non integrability of the planar case with positive masses. In the opposite direction, the n-body problem also possesses explicit algebraic orbits, linked to central configurations [8]. Restricting the n-body problem to a vector space associated to a central configuration leads in particular to an integrable problem, although very simple. Still, as we will see, there are also less trivial invariant vector spaces of the n-body problem on which the potential is integrable.

In the integrability analysis of the n body problem, and in the more general case of homogeneous potential, the notion of central configuration/Darboux point plays a key role.

Definition 1 We consider the potential $V_{n,d}$ of the n body problem. We will say that $c \in \mathbb{C}^{nd}$ is a central configuration if there exists $g \in \mathbb{C}^d$, $\alpha \in \mathbb{C}$ such that

$$\frac{\partial}{\partial q_i} V(c_1 - g, \dots, c_n - g) = \alpha (c_1 - g, \dots, c_n - g) \qquad i = 1 \dots n$$

The scalar α is called the multiplier. We say that the central configuration is proper if $\alpha \neq 0$ (the case $\alpha = 0$ is called an absolute equilibrium). In the more general setting of V a homogeneous potential of degree -1, we call c a Darboux point if moreover $g = 0$.

We add this constant g in our Definition for the n body problem as the potential is in this case invariant by translation, and thus we do not (always) want to require that the center of mass be at 0. Our non-integrability proofs will be based on variational equations of the corresponding differential system near these central configurations. The main theorem behind these non-integrability proofs is the following

Theorem 1 (Morales et al. [9]) *Let V be a meromorphic homogeneous potential of degree -1 and c a Darboux point. If V is meromorphically integrable, then the identity component of the Galois group of the variational equation near the homothetic orbit associated to c is abelian at any order. Moreover, the identity component of the Galois group of the first order variational equation is abelian if and only if*

$$Sp(\nabla^2 V(c)) \subset \left\{ \tfrac{1}{2}(k-1)(k+2),\ k \in \mathbb{N} \right\}$$

Note also that in dimension 1, $V_{n,1}$ is a rational potential (thus univaluated on $\mathbb{C}^n$), but is not in higher dimension. In the complex domain, the potential $V_{n,d},\ d \geq 2$ is properly defined on an algebraic variety $\mathcal{S}$. An extension of Theorem 1 has been done in [10], and proves that in the n body problem, the necessary condition for integrability on the Galois group of variational equations still holds.

Such a Theorem can be either used for each central configuration separately, or simultaneously using some algebraic properties. In the case of the n body problem, a direct computation of central configurations is often too difficult. The colinear case with $n = 3, 4$ is still tractable, and we prove moreover that a complete computation of central configurations is not necessary, only upper bounds on eigenvalues of the Hessian matrix of $V_{n,1}$ at Darboux points is necessary.

Using the real algebraic geometry software RAGlib [11], we prove such an upper bound for $n = 3, 4$ and we conjecture that a similar upper bound always hold for any n. The software RAGlib is a Maple package, and the command we will mostly use is **HasRealRoots**. This command take in input a system of polynomials with rational coefficients, and (possibly) a set of polynomial inequalities. The answer is true/false, saying if the system has (at least) one real solution. This also allows to prove upper bounds for a (multivariate) rational function f by just looking for solutions of the equation $f = B$ where B is a (numerically guessed) upper bound. Note that the real conditions on the masses will be heavily used: in particular some polynomial integrability conditions cannot be satisfied in the real but would be in the complex.

We then prove very strong non-integrability Theorem that rules out any potential which satisfies these bounds. In the planar case, we also prove a similar upper bound, which holds moreover for any n. This allows to prove the non-integrability of the planar n-body problem. The main theorems of this article are the following

Theorem 2 *For any* $(m_1, m_2, m_3) \in {\mathbb{R}_+^*}^3$, *the potential* $V_{3,1}$ *is not meromorphically integrable. Moreover, if* $m_1 + m_2 + m_3 = 1$, *the variational equations near the unique real central configuration have an Abelian Galois group (over the base field* $\mathbb{C}(t)$*) up to an order*

- *greater than* 1 *if and only if there exist* $\rho \in \mathbb{R}_+^*$ *and* $k \in \{5, 9, 14\}$ *such that*

$$\begin{aligned} m_1 &= \frac{(\rho+1)(-8\rho^5+k\rho^5-12\rho^4+3k\rho^4-8\rho^3+3k\rho^3+3k\rho^2+3k\rho+k)}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \\ m_2 &= -\frac{(-8\rho^4+k\rho^4-28\rho^3+2k\rho^3+k\rho^2-40\rho^2-28\rho+2k\rho-8+k)\rho^2}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \\ m_3 &= \frac{(\rho+1)(k\rho^5+3k\rho^4+3k\rho^3-8\rho^2+3k\rho^2-12\rho+3k\rho-8+k)\rho^2}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \end{aligned} \qquad (E_k)$$

- *equal to* 2 *if and only if moreover* $m_1 = m_3$ *or* $(m_1, m_2, m_3) \in E_9$.

Theorem 3 *For any* $m_1, m_2, m_3, m_4 > 0$, $m_1 + m_2 + m_3 + m_4 = 1$, *the potential* $V_{4,1}$ *is not integrable. Moreover, near the unique real central configuration, there are at most* 14 *one dimensional irreductible algebraic curves in the space of masses for which the variational equations have virtually Abelian Galois groups at least up to order* 1. *At least one of them, and at most* 10 *of them correspond to masses for* *which the second order variational equations have a virtually Abelian Galois group. None of them have a variational equation whose Galois group is virtually Abelian at order* 5.

Theorem 4 *For any n-tuplet of positive masses, the planar n body problem is not meromorphically integrable.*

Theorem 5 *The planar* 5 *body problem with masses* $m = (-1/4, 1, 1, 1, 1)$ *restricted to the vector space*

$$W = \{q \in \mathbb{R}^{10},\ q_{1,1} = q_{1,2} = q_{2,1} + q_{4,1} = q_{2,2} + q_{4,2} = q_{3,1} + q_{5,1} = q_{3,2} + q_{5,2} = 0\}$$

is integrable in the Liouville sense.

The spatial $n+3$ *body problem with masses* $m = (m_1, \dots, m_n, -\alpha, 4\alpha, 4\alpha)$ *restricted to the vector space*

$$W = \Big\{q \in \mathbb{R}^{3(n+3)},\ \ q_{n+1,1} = q_{n+1,2} = q_{n+1,3} = q_{n+2,1} = q_{n+2,2} = q_{n+3,1}$$
$$= q_{n+3,2} = q_{n+2,3} + q_{n+3,3} = 0,\ q_{i,3}{}_{|_{i=1\dots n}} = 0,\ q_i{}_{|_{i=1\dots n}} = \beta R_\theta c,\ \beta, \theta \in \mathbb{R}\Big\}$$

where c is a central configuration of n bodies with masses $(m_1, \dots, m_n)$ *in the plane on the unit circle with center of mass at* 0*,* R_θ *being a rotation in this plane and* α *chosen such that the configuration c with the central mass* $-\alpha$*, is an absolute equilibrium is integrable in the Liouville sense.*

The Theorem 2 implies the non integrability of the colinear 3 body problem, which was already done in [12] using the systematic approach using all central configurations and a relation between the eigenvalues of Hessian matrices. This approach is hard to apply to more complicated systems as its cost is exponential in the number of central configurations. This is due to the fact that all central configurations are analyzed, even if only a few of them would probably be enough to conclude to non integrability. Also, the physical assumption that the masses are real positive is not used. In the next section, we thus make a more precise analysis of variational equations near the unique real central configuration, whose existence and uniqueness is a result of Moulton [13]:

Theorem 6 (Moulton [13]) *For any fixed positive masses* $m_1, \dots, m_n$ *with a fixed order of the masses, the colinear n body problem admits exactly one real central configuration.*

Remark that also in the not trivially integrable example we found, central configurations seem to play a key role. In particular, they all contain continuums of central configurations (the first case contains the famous 5 body central configuration of Roberts [14]). According to a conjecture of Smale, proved for $n = 4, 5$ in [8, 15], such continuums are not possible with positive masses.

2 The Colinear 3 Body Problem

2.1 Central Configurations

Proposition 1 (Euler) *We pose $c = (-1, 0, \rho)$ with $\rho \in \mathbb{C}\setminus\{0, -1\}$. If c is a central configuration of the colinear 3 body problem (corresponding to the potential $V_{3,1}$), then the following equation is satisfied*

$$\begin{aligned}(m_2 + m_3) + (2m_2 + 3m_3)\rho + (3m_3 + m_2)\rho^2 \\ -(3m_1 + m_2)\rho^3 - (3m_1 + 2m_2)\rho^4 - (m_1 + m_2)\rho^5 = 0\end{aligned} \tag{1}$$

In the colinear 3 body problem, we can always translate a central configuration because the potential is invariant by translation. Moreover, due to this definition, the set of central configurations is also invariant by dilatation, so for any central configuration $q \in \mathbb{C}^3$, after translation and dilatation, we can always write it $q = (-1, 0, \rho)$ with $\rho \in \mathbb{C}\setminus\{0, -1\}$. The biggest problem that authors on the subject (see [4]) seem to have encountered is the fact that we have a polynomial of degree 5, which is not very easy to use. We will see that the complexity of central configuration equations is not a problem at all if we consider the problem differently.

The Theorem 6 of Moulton suggests that we should work in the opposite way. We fix $\rho > 0$ and we seek the masses such that $c = (-1, 0, \rho)$ is a central configuration. We are then sure that if we consider all possible ρ we will then consider all positive masses (because for each triplet of masses, there is at least one ρ that is convenient). More precisely, we have

Proposition 2 *The set of masses m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 1$ and $c = (-1, 0, \rho)$ with*

$$\rho \in \mathbb{C}\setminus\{\rho, \quad \rho(\rho + 1)(1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4) = 0\} \tag{2}$$

is a central configuration, is an affine subspace of dimension 1 parametrized by

$$\begin{aligned}m_1 &= s \\ m_2 &= -\frac{3s\rho^3 + 3s\rho^4 + s\rho^5 + s - 1 + 3\rho s - 3\rho + 3\rho^2 s - 3\rho^2}{\rho(1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4)} \\ m_3 &= \frac{2\rho s + \rho^2 s + 2s\rho^3 + s\rho^4 + s - 1 - 2\rho - \rho^2 + \rho^3 + 2\rho^4 + \rho^5}{\rho(1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4)}\end{aligned} \tag{3}$$

Conversely, for each triplet of masses $(m_1, m_2, m_3) \in \mathbb{R}_+^{*\,3}$, $m_1 + m_2 + m_3 = 1$, there exists a central configuration of the form $(-1, 0, \rho)$ with condition (2) and $\rho \in \mathbb{R}_+^*$. Eventually, for $\rho \in \mathbb{R}$, $\rho \geq 1$, the m_1, m_2, m_3 are positive if and only if

$$s \in \left]0, \frac{1 + 3\rho + 3\rho^2}{(1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4)(1 + \rho)}\right[$$

Proof Using equation of Proposition 1, we get the following equations

$$\begin{pmatrix} 3\rho^3 - 3\rho^4 - \rho^5 & 1 + 2\rho + \rho^2 - \rho^3 - 2\rho^4 - \rho^5 & 1 + 3\rho + 3\rho^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is an affine equation and so the space of solutions is an affine subspace. Taking $\rho \in \mathbb{C} \backslash \{\rho, \quad \rho(1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4)\}$, the matrix has always maximal rank, and so the space of solution is of dimension 1, which we parametrize by s. Conversely, the Euler equation (1), thanks to Moulton's result for $n = 3$, has always exactly one real positive solution.

Finally, let us look at the case $\rho \in \mathbb{R}, \ \rho \geq 1$. We want the masses to be positive, and according to our parametrization, the masses are affine functions in s. An affine function changes sign at most once. Solving $m_i = 0$, we get

$$m_1 = 0 \Rightarrow s = 0$$

$$m_2 = 0 \Rightarrow s = \frac{1 + 3\rho + 3\rho^2}{3\rho^3 + 3\rho^4 + \rho^5 + 1 + 3\rho + 3\rho^2}$$

$$m_3 = 0 \Rightarrow s = \frac{1 + 2\rho + \rho^2 - \rho^3 - 2\rho^4 - \rho^5}{2\rho + \rho^2 + 2\rho^3 + \rho^4 + 1}$$

The last equality gives us for $\rho \geq 1$ $s \leq 0$ which is impossible because $m_1 \geq 0$. So m_3 does not change sign for any $s > 0$ and is positive. The positivity of m_2 gives us the constraint. □

Let us remark that the constraint $\rho \geq 1$ is in fact not a constraint, because using dilatation and the symmetry which consists of reversing to reverse the order of **all the masses**, we exchange ρ by $1/\rho$. After this first proposition, we can study the integrability of the colinear 3 body problem for real positive masses.

In the following, we will note $W(c) \in M_3(\mathbb{C})$ the 3×3 matrix such that

$$W(c)_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V_3(c) \tag{4}$$

where V_3 is the potential of the colinear 3 body problem and $c \in \mathbb{C}^3$.

2.2 Non-integrability

In this subsection, we will prove Theorem 2.

Lemma 7 *For any $\rho \in \mathbb{R}, \ \rho \geq 1$, there exists, among the masses $(m_1, m_2, m_3) \in \mathbb{R}_+^{*\,3}$ such that $m_1 + m_2 + m_3 = 1$ and $c = (-1, 0, \rho)$ is a central configuration for* *the triplet of masses* (m_1, m_2, m_3)*, at most* 3 *triplets of masses for which the Galois group of first order variational equation has a Galois group whose identity component is Abelian.*

Proof The matrix W for the central configuration of the form $c = (-\gamma + g, g, \rho\gamma + g)$ is given by

$$\frac{2}{\gamma^3}\begin{pmatrix} \frac{m_2+3m_2\rho+3m_2\rho^2+m_2\rho^3+m_3}{(1+\rho)^3} & -m_2 & -\frac{m_3}{(1+\rho)^3} \\ -m_1 & \frac{m_1\rho^3+m_3}{\rho^3} & -\frac{m_3}{\rho^3} \\ -\frac{m_1}{(1+\rho)^3} & -\frac{m_2}{\rho^3} & \frac{m_1\rho^3+m_2+3m_2\rho+3m_2\rho^2+m_2\rho^3}{(1+\rho)^3\rho^3} \end{pmatrix}$$

We need to choose γ, g such that the multiplier of the central configuration is -1 and the center of mass is at 0 (because we want an orbit of the form $c.\phi(t)$). We first compute the spectrum of W which gives

$$\left[0, \frac{4(2\rho^2+3\rho+2)}{(3\rho^3+3\rho^4+\rho^5+1+3\rho+3\rho^2)\gamma^3}, -\frac{2(s\rho^4+2s\rho^3-\rho^2+\rho^2 s-2\rho+2\rho s-1+s)}{\rho^3(1+2\rho+\rho^2)\gamma^3}\right]$$

where the masses m_1, m_2, m_3 are parametrized by s according to the formula (3). The constraint that the multiplier of c should be equal to -1 gives

$$\gamma^3 = -\frac{(s\rho^4+2s\rho^3-\rho^2+\rho^2 s-2\rho+2\rho s-1+s)}{\rho^3(1+2\rho+\rho^2)}$$

and so we get

$$Sp(W(c)) = \left\{0, 2, -\frac{4(1+\rho)\rho^3(2\rho^2+3\rho+2)}{(s\rho^4+2s\rho^3-\rho^2+\rho^2 s-2\rho+2\rho s-1+s)(1+2\rho+\rho^2+2\rho^3+\rho^4)}\right\}$$

Let us note $G(s, \rho)$ this last eigenvalue, which is a fractional linear function in s. The singularity in s of G is at

$$s = \frac{1+2\rho+\rho^2}{1+2\rho+\rho^2+2\rho^3+\rho^4}$$

This value of s corresponds to the case where the central configuration is in fact an absolute equilibrium. Indeed, we then have the multiplier of the central configuration equal to zero. This special case produces the following set of masses

$$(m_1, m_2, m_3) = \left(\frac{(\rho+1)^2}{1+2\rho+\rho^2+2\rho^3+\rho^4}, \frac{-\rho^2}{1+2\rho+\rho^2+2\rho^3+\rho^4}, \frac{(\rho+1)^2\rho^2}{1+2\rho+\rho^2+2\rho^3+\rho^4}\right)$$

The mass m_2 is always non-positive, and so this case is impossible. Now in the general case, we solve the equation

$$G(s, \rho) \in \left\{\tfrac{1}{2}(i-1)(i+2)\ \ i \in \mathbb{N}\right\}$$

and we obtain the following solutions

$$\begin{aligned} m_1 &= \frac{(\rho+1)(-8\rho^5+k\rho^5-12\rho^4+3k\rho^4-8\rho^3+3k\rho^3+3k\rho^2+3k\rho+k)}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \\ m_2 &= -\frac{(-8\rho^4+k\rho^4-28\rho^3+2k\rho^3+k\rho^2-40\rho^2-28\rho+2k\rho-8+k)\rho^2}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \\ m_3 &= \frac{(\rho+1)(k\rho^5+3k\rho^4+3k\rho^3-8\rho^2+3k\rho^2-12\rho+3k\rho-8+k)\rho^2}{k(1+2\rho^3+\rho^4+2\rho+\rho^2)^2} \end{aligned} \qquad (E_k)$$

with $k \in \{\frac{1}{2}(i-1)(i+2) \;\; k \in \mathbb{N}\}$. These solutions are not valid for $k=0$, but we have that if $G(s,\rho)=0$ then

$$(1+\rho)(2\rho^2+3\rho+2)=0$$

which is excluded because $\rho \in \mathbb{R}_+^*$.

Let us look now what happen if we restrict ourselves to positive masses. We take $\rho \geq 1$ and we look at the sign of the masses given by the curves (E_k). We already know according to Proposition 2 that the interval $I(\rho)$ to consider for s is the following

$$I(\rho) = \left[0, \frac{1+3\rho+3\rho^2}{(1+2\rho+\rho^2+2\rho^3+\rho^4)(1+\rho)}\right]$$

and noting that $(1+2\rho+\rho^2) > (1+3\rho+3\rho^2)/(1+\rho)$ for $\rho \geq 1$, the singularity of $G(s,\rho)$ is never in $I(\rho)$, and so for $\rho \geq 1$, $G(.,\rho)$ increases on $I(\rho)$.

Then $G(.,\rho)$ is a bijection of $I(\rho)$ on

$$G(I(\rho),\rho) = \left]\frac{4(1+\rho)\rho^3(2\rho^2+3\rho+2)}{(1+\rho^2+2\rho)(1+2\rho+\rho^2+2\rho^3+\rho^4)}, \frac{4(2\rho^2+3\rho+2)(1+\rho)^2}{1+2\rho+\rho^2+2\rho^3+\rho^4}\right[$$

Studying these functions, we prove that the interval $G(I(\rho),\rho)$ is decreasing when $\rho \geq 1$ increases. Knowing that $G(I(1),1) =]2,16[$, the only possible eigenvalues corresponding to a Galois group with an Abelian identity component are $5, 9, 14$. □

Let us now remark that the potential $V_{3,1}$ of the colinear 3 body problem can be reduced. Indeed, this potential is invariant by translation, and by making the symplectic variable change $p_i \longrightarrow \sqrt{m_i}p_i,\ q_i \longrightarrow q_i/\sqrt{m_i}$, the kinetic part in the Hamiltonian becomes the standard kinetic energy $(p_1^2+p_2^2+p_3^2)/2$. So the set of potential $V_{3,1}$ with parameters $(m_1,m_2,m_3) \in \mathbb{R}_+^{*\,3}$, $m_1+m_2+m_3=1$ is a set of homogeneous potentials of degree -1 in the plane.

Corollary 1 *The colinear 3-body problem with positive masses is not meromorphically integrable.*

Proof We proved that only the eigenvalues 5, 9, 14 are possible for integrability of the colinear 3-body problem. In [16], all potentials having these eigenvalues have been classified and they are not meromorphically integrable. □

Remark that in the limit case when two masses tend to 0, the potential $V_{3,1}$ after reduction is not singular and converges to a potential of the form $\alpha/q_1 + \alpha/q_2$, which has the eigenvalue 2 and is integrable.

2.3 Higher Variational Equations

Let us now compute exactly at which order the variational equations near the unique real Darboux point have a Galois group whose identity component is not Abelian. Indeed, using [16], we note that on the curves E_5, E_{14}, the potentials are integrable at most up to order 4, and on E_9 at most to order 6 (which reduces to 4 in our case, because the potential V_3 is real and integrable cases to order 5, 6 are complex).

2.3.1 At Order 2

To study the Galois group of second order variational equations, we apply Theorem 2 of [17]. We have however to take into account that the kinetic energy is $p_1^2/(2m_1) + p_2^2/(2m_2) + p_3/(2m_3)$ instead of $(p_1^2 + p_2^2 + p_3^2)/2$. This standard form of kinetic energy can be obtained by a symplectic change of variable. The Hessian matrix $\nabla^2 V(c)$ after this variable change is simply the matrix W defined in (4) (Fig. 1).

Lemma 8 *Let* $\rho \in \mathbb{R},\ \rho \geq 1$ *be a real number,* $k \in \{5, 9, 14\}$ *and masses* $(m_1, m_2, m_3) \in E_k$*. The variational equations at order* 2 *near the homothetic orbit associated to* c *have a Galois group whose identity component is Abelian if and only if the masses belong to the set*

$$\left\{\left(\frac{12}{35}, \frac{11}{35}, \frac{12}{35}\right), \left(\frac{24}{49}, \frac{1}{49}, \frac{24}{49}\right)\right\} \cup E_9$$

Proof We compute the third order derivatives of V at c. Denoting by X_2 the eigenvector of eigenvalue 2 and X_3 the eigenvector of eigenvalue k, we have

$$D^3V(X_2, X_2, X_2) = D^3V(X_3, X_3, X_2) = -\frac{3\sqrt{2\rho^2 + 3\rho + 2}\sqrt{2k}}{(\rho + 1)^2 g^{\frac{4}{3}} \sqrt{k - 2}\rho^{\frac{3}{2}}} \quad D^3V(X_3, X_2, X_2) = 0$$

$$D^3V(X_3, X_3, X_3) = \frac{-3\sqrt{2}\sqrt{2\,\rho^2 + 3\,\rho + 2}(\rho - 1)P(\rho)}{(1 + 2\rho^3 + \rho^4 + 2\rho + \rho^2)^3 \rho^{\frac{3}{2}} g^{\frac{4}{3}} (\rho + 1)^2 \sqrt{k(k - 2)m_1 m_2 m_3}}$$

where

$$g = \frac{-4(2\rho^2 + 3\rho + 2)}{\left(\rho^5 + 3\,\rho^4 + 3\,\rho^3 + 3\,\rho^2 + 3\,\rho + 1\right) k}$$

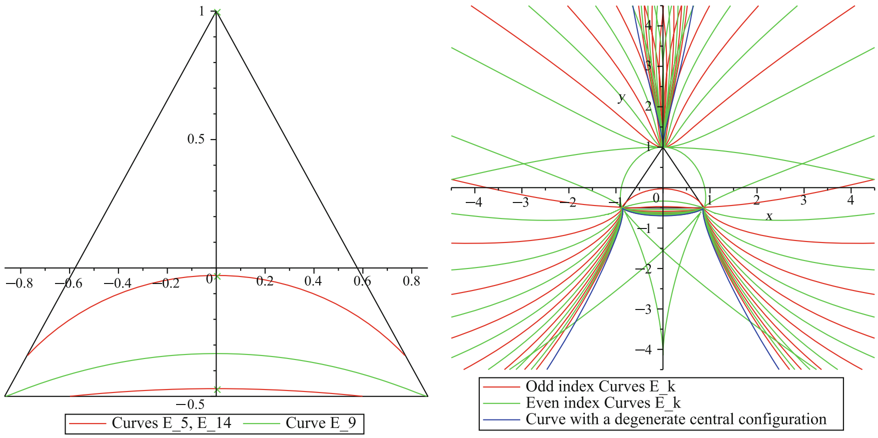

Fig. 1 Graph of the masses having a first order variational equation with a Galois group whose identity component is Abelian. The masses are represented in barycentric coordinates. The masses inside the *black triangle* are positive. Drawing the curves outside the positive masses reveals that the curves (E_k) accumulate on the curve (E_∞). They also intersect on the points $(m_1, m_2, m_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ which are the integrable cases (at the limit when the masses are going to zero through a limiting process)

$$P(\rho) = (k+10)\rho^6 + (5k+50)\rho^5 + (8k+120)\rho^4 + (7k+158)\rho^3 + (8k+120)\rho^2 + (5k+50)\rho + k + 10$$

According to [17], the condition for integrability of the second order variational equations are that some of these third order derivative should vanish. Using the table of [17], the three first third order derivatives never lead to an integrability condition, but the last one does. In particular, for $k = 5, 14$, the integrability condition is $D^3V(X_3, X_3, X_3) = 0$, and there is none for $k = 9$.

The only real positive solution of equation $(\rho - 1)P(\rho) = 0$ for $k = 5, 14$ is $\rho = 1$. Putting this in the parametrization of (E_5), (E_{14}), we obtain that the set of possible masses is given by

$$\left\{ \left(\frac{12}{35}, \frac{11}{35}, \frac{12}{35} \right), \left(\frac{24}{49}, \frac{1}{49}, \frac{24}{49} \right) \right\} \cup E_9$$

□

2.3.2 At Order 3

Let us now look at order 3. We will prove that V_3 is never integrable at order 3 near its unique real central configuration.

Lemma 9 *The potential V_3 is never integrable at order 3 at its unique real central configuration.*

Proof We will directly use the main Theorem of [18]. A convenient variable change sends the potential V_3 to a planar homogeneous potential with standard kinetic energy, and a rotation dilatation puts the central configuration at $c = (1, 0)$. We then find that the third order integrability condition can be written

$$-\frac{256}{715}a^2 + \frac{13824}{5005}c = 0, \quad b = 0, \quad k = 5$$
$$-\frac{475136}{57057}a^2 - \frac{753664}{101745}b^2 + \frac{19759104}{323323}c = 0, \quad k = 9$$
$$-\frac{2755788800}{7436429}a^2 + \frac{19729612800}{7436429}c = 0, \quad b = 0, \quad k = 14$$

where the constants a, b, c are

$$a = -\frac{3\sqrt{2\rho^2 + 3\rho + 2}\sqrt{2k}}{(\rho + 1)^2 g^{\frac{4}{3}}\sqrt{k - 2}\rho^{\frac{3}{2}}},$$
$$b = \frac{-3\sqrt{2}\sqrt{2\,\rho^2 + 3\,\rho + 2}(\rho - 1)P(\rho)}{(1 + 2\rho^3 + \rho^4 + 2\rho + \rho^2)^3\rho^{\frac{3}{2}}g^{\frac{4}{3}}(\rho + 1)^2\sqrt{k(k - 2)m_1m_2m_3}}$$
$$c = F(\rho, k)$$

where F is a rational fraction in ρ, k, and

$$g = \frac{-4(2\rho^2 + 3\rho + 2)}{\left(\rho^5 + 3\,\rho^4 + 3\,\rho^3 + 3\,\rho^2 + 3\,\rho + 1\right)k}$$

The constraint $b = 0$ for $k = 5, 14$ comes from order 2, and we already know the unique solution is $\rho = 1$. The other constraint gives

$$\frac{3024672}{1573}7^{\frac{2}{3}} \neq 0 \qquad \frac{2137106227200}{96577}7^{\frac{2}{3}} \neq 0$$

for $k = 5, 14$ respectively. For $k = 9$, the third order integrability constraint is

$$179523957 + 1436191656\,\rho + 5144769684\,\rho^2 + 11297844542\,\rho^3 + 17938383865\,\rho^4$$
$$+23104821764\,\rho^5 + 25814403801\,\rho^6 + 26361946842\,\rho^7 + 25814403801\,\rho^8 + 23104821764\,\rho^9$$
$$+17938383865\,\rho^{10} + 11297844542\,\rho^{11} + 5144769684\,\rho^{12} + 1436191656\,\rho^{13} + 179523957\,\rho^{14} = 0$$

This polynomial has no real positive root, and so the constraint is never satisfied. □

Remark 1 One could compute the third order integrability condition for any curve (E_k), and even test if this condition could be satisfied thanks to the holonomic approach of third order variational equations in [18]. Here the restriction $(m_1, m_2, m_3) \in {\mathbb{R}_+^*}^3$ is only for physical reasons, but a more complete study is possible. Still note that this constraint has allowed us to easily bound the eigenvalues, and then to study integrability near the unique real central configuration. If one would allow negative masses, or even complex masses, some results are no longer valid. Especially, there are complex masses which possess a non degenerate central configuration which is integrable at order 3.

3 The 4 Body Problem

The previous approach for non integrability proofs can be extended for more complicated systems, as the 4-body problem, for which a direct approach would be impossible due to the high number of central configurations. The difficulty of the problem of finding these central configurations is famous [8], thus we will try to require the least possible information on them. The most important quantity is the set of possible eigenvalues of Hessian matrices at the unique real central configuration. In particular, if this set is finite, then the classification approach of [16] is possible.

3.1 Eigenvalue Bounding

Following the method presented in [16], we will first try to prove a bound on eigenvalues of the Hessian matrices at Darboux points of $V_{4,1}$. In [16], the potential are planar, and so we need to operate a little differently. Instead of trying to bound directly these eigenvalues (whose expression could be complicated as they appear as roots of the characteristic polynomial), we simply bound the trace of the Hessian matrix. Indeed, the eigenvalues of the Hessian matrix are of the form $\{0, 2, \lambda_1, \lambda_2\}$, and so bounding the trace gives a bound on $\lambda_1 + \lambda_2$. Moreover, thanks to Theorem 1, we already know that for integrability we must have $\lambda_1, \lambda_2 \geq -1$, and thus we get also a bound on λ_1, λ_2 (Figs. 2 and 3).

Theorem 10 *We consider the colinear* 4 *body problem with positive masses, whose potential is given by* $V_{4,1}$. *Let* c *be the real central configuration (existence and uniqueness up to translation due to Theorem 6) with multiplier* -1. *Let* $W \in M_4(\mathbb{C})$ *be the matrix*

$$W_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V \quad i, j = 1 \ldots 4$$

Then $tr(W) < 70$.

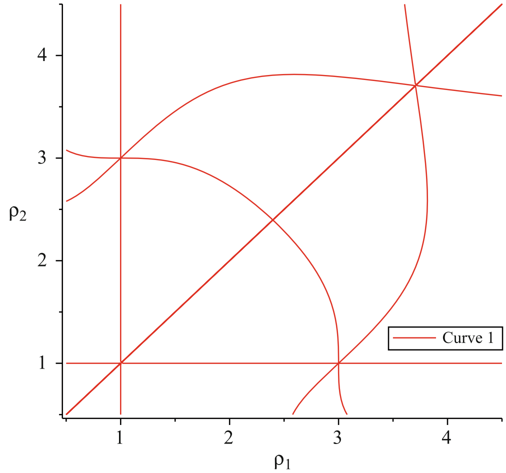

Fig. 2 Diagram of the bifurcations between the index i of the M_i that realize the maximum of $tr(W)$. The index $i(\rho_1, \rho_2)$ can only change on one of these curves. Moreover there exists a zone (near $\rho_1, \rho_2 = 1$) where the set S_{ρ_1,ρ_2} is empty. Numerical analysis gives a maximum around 69.74

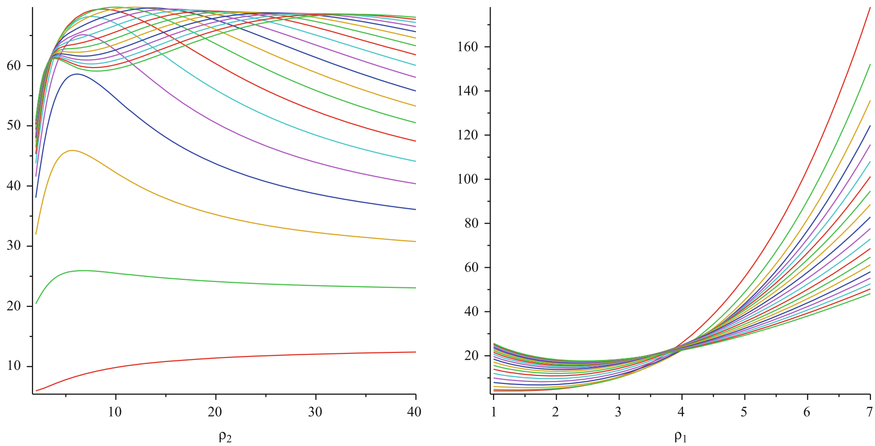

Fig. 3 Graph of the functions $M_i, i = 1 \ldots 4$. We see that M_2, M_3, M_4 are bounded but not M_1. This is why we proved that the curve M_1 has only to be considered for $\rho_1 \leq 5$, allowing us to bound the function

This value is not the optimal one which has a complicated algebraic expression. Still considering a better bound than this one is not useful as it will not allow us to reduce the number of exceptional cases we will have to deal with.

Proof We first remark that after translation, dilatation and changing the order of **all the masses**, a central configuration of V_4 can always be written in the form $c = (-\rho_1, -1, 1, \rho_2)$ with $\rho_1 \geq \rho_2 > 1$. Moreover, thanks to Moulton Theorem 6, we also know that for any fixed positive masses, there always exists a unique central configuration. So we will first fix our central configuration $c = (-\rho_1, -1, 1, \rho_2)$, and then compute the masses for which c is a central configuration. Moreover, we will assume that $m_1 + m_2 + m_3 + m_4 = 1$ because multiplying all the masses by a constant does not change the trace of the matrix W.

The equation of central configurations is a linear system in the masses, with 3 equations for 4 unknowns. The solution is of the form

$$(m_1, m_2, m_3, m_4) = (J_1(\rho_1, \rho_2, m_3), J_2(\rho_1, \rho_2, m_3), J_3(\rho_1, \rho_2, m_3), J_4(\rho_1, \rho_2, m_3))$$

where J_i are rational in ρ_1, ρ_2 and affine in m_3 (and $J_3(\rho_1, \rho_2, m_3) = m_3$). Now we compute the trace of matrix W, and we obtain that $tr(W)$ is also rational in ρ_1, ρ_2 and affine in m_3.

Lemma 11 *The functions J_i have no singularities for $\rho_1 \geq \rho_2 > 1$, and their coefficient in m_3 does not vanish for $\rho_1 \geq \rho_2 > 1$.*

Proof We simply build a polynomial whose factors are the denominators of the functions J_i and numerators of the coefficient in m_3 of the functions J_i. This polynomial has no real solutions for $\rho_1 \geq \rho_2 > 1$. □

So we can handle safely these J_i, and solve equations of the form $J_i = 0$ in m_3 without dealing with singular cases. We need to prove

$$\max_{J_i>0,\ i=1\ldots4,\ \ \rho_1\geq\rho_2>1} tr(W) < 70$$

Let us now remark that for fixed $\rho_1 \geq \rho_2 > 1$, the function $tr(W)$ in m_3 on the set

$$S_{\rho_1,\rho_2} = \{m_3 \in \mathbb{R}, J_i(\rho_1, \rho_2, m_3) > 0,\ i = 1\ldots4\}$$

has its maximum on the boundary of S_{ρ_1,ρ_2} (because $tr(W)$ is affine in m_3). So for fixed $\rho_1 \geq \rho_2 > 1$, the maximum on the possible m_3 has 4 possible values

$$M_i(\rho_1, \rho_2) = tr(W)(\rho_1, \rho_2, J_i(\rho_1, \rho_2, \cdot)^{-1}(0)) \quad i = 1\ldots4$$

Let us now prove the following Lemma

Lemma 12 *The following bounds hold*

$$M_2(\rho_1, \rho_2) \leq 69.9 \quad M_3(\rho_1, \rho_2) \leq 69.9 \quad M_4(\rho_1, \rho_2) \leq 69.9 \qquad \forall\ \rho_1 \geq \rho_2 > 1$$

$$M_1(\rho_1, \rho_2) \leq 69.9 \qquad \forall\ \rho_1 \geq \rho_2 > 1, \quad \rho_1 \leq 5$$

Proof These inequalities are automatically proved using RAGlib. □

Lemma 13 *If* $\rho_1 \geq 5$, $\rho_1 \geq \rho_2 > 1$, *then*

$$\max_{m_3 \in S_{\rho_1,\rho_2}} tr(W) \in \{M_2(\rho_1,\rho_2), M_3(\rho_1,\rho_2), M_4(\rho_1,\rho_2)\}$$

Proof We set $\rho_1 \geq 5$ with $\rho_1 \geq \rho_2 > 1$. Assume now that $S_{\rho_1,\rho_2} \neq \emptyset$ and $M_1(\rho_1,\rho_2)$ is the maximum of $tr(W)$ on S_{ρ_1,ρ_2}. Then the corresponding masses (m_1, m_2, m_3, m_4) should be all non-negative (recall that the maximum could be reached at the boundary of the domain of positive masses, so for non-negative masses). Solving equation $J_1(\rho_1, \rho_2, m_3) = 0$ in m_3, we get a rational fraction D in ρ_1, ρ_2. We now prove using RAGlib that

$$D(\rho_1,\rho_2) \leq 0 \qquad \forall \rho_1 \geq 5, \rho_1 \geq \rho_2 > 1$$

So the only possibility left for having all non-negative masses is that $m_3 = D = 0$. This implies that $M_1(\rho_1,\rho_2) = M_3(\rho_1,\rho_2)$ and so the Lemma follows. □

Using Lemma 13, we know that if $\rho_1 \geq 5$, $\rho_1 \geq \rho_2 > 1$, the maximum M_2, M_3 or M_4. These are bounded by 69.9 thanks to Lemma 12. For $5 \geq \rho_1 \geq \rho_2 > 1$, the maximum of $tr(W)$ can be any of the M_i, but due to Lemma 12, all of these are then bounded by 69.9. So

$$\max_{J_i > 0,\ i=1\ldots 4,\ \ \rho_1 \geq \rho_2 > 1} tr(W) < 70$$ □

3.2 Symmetric Central Configurations

For symmetric central configurations, several cases are possible which are not possible in the non-symmetric case. So we will analyze in this part the case where the real central configuration is of the form $(-\rho, -1, 1, \rho)$.

Lemma 14 *The function* $tr(W)$ *has no singularities for* $\rho_1 > \rho_2 > 1$, *and its coefficient in* m_3 *does not vanish for* $\rho_1 > \rho_2 > 1$.

This Lemma is immediately proved by RAGlib. For $\rho_1 = \rho_2$, the coefficient in m_3 of $tr(W)$ vanishes, making it a special case. On the other hand, this produces an additional symmetry that reduce the number of parameters by 1 and greatly simplify the formulas

Theorem 15 (Pacella [19]) *We consider the colinear* 4 *body problem potential* $V_{4,1}$ *with positive masses and the central configuration c with multiplier* -1 *(existence and unicity up to translation due to 6). Noting* $W \in M_4(\mathbb{C})$ *with*

$$W_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V$$

the spectrum of W is of the form $Sp(W) = \{0, 2, \lambda_1, \lambda_2\}$ *with* $\lambda_1, \lambda_2 > 2$.

This already allows to reduce somewhat the possible set of eigenvalues. We will now check if some curves (in the space of masses) corresponding to a pair of eigenvalues λ_1, λ_2 are non-empty for real positive masses.

Lemma 16 *If the potential $V_{4,1}$ with positive masses possesses a real central configuration of the form $(-\rho, -1, 1, \rho)$, then the spectrum of the Hessian matrix W at the real central configuration with multiplier -1 has the form $Sp(W) = \{0, 2, \lambda_1, \lambda_2\}$ with*

$$\{\lambda_1, \lambda_2\} \in \{\{5, 9\}, \{5, 14\}, \{9, 27\}, \{14, 44\}\}$$

Proof Using Pacella Theorem, we obtain a better minoration $\lambda_1, \lambda_2 > 2$. Knowing that $2 + \lambda_1 + \lambda_2 < 70$, we get the following possibilities

$$\begin{gathered}\{5,5\},\{5,9\},\{5,14\},\{5,20\},\{5,27\},\{5,35\},\{5,44\},\{5,54\},\{9,9\},\\ \{9,14\},\{9,20\},\{9,27\},\{9,35\},\{9,44\},\{9,54\},\{14,14\},\{14,20\},\{14,27\},\\ \{14,35\},\{14,44\},\{20,20\},\{20,27\},\{20,35\},\{20,44\},\{27,27\},\{27,35\}\end{gathered} \tag{5}$$

We first compute the characteristic polynomial of matrix W. Using the same notations as before, the characteristic polynomial has rational coefficients in ρ_1, ρ_2, m_3. Factoring it, we take out the $z(z-2)$ factor (corresponding to eigenvalues 0, 2) and we then get a degree 2 polynomial P in z. The coefficient in z corresponds to the trace of W, and is affine in m_3. We now put $\rho_1 = \rho_2$ in the expression of the characteristic polynomial. The coefficient corresponding to the trace only depends on ρ_2. The equation $P(z) = (z - \lambda_1)(z - \lambda_2)$ in ρ_2, m_3 gives rise to two equations in ρ_2, m_3, and we have moreover the constraint of positivity of the masses m_i which can be written as a function of ρ_2, m_3 with the functions J_i. This polynomial system of equations and inequalities has real solutions only for λ_1, λ_2 given by the Lemma. □

3.3 Reduction of Exceptional Curves

In this part, we will always assume that the real central configuration $(-\rho_1, -1, 1, \rho_2)$ is such that $\rho_1 > \rho_2$.

Lemma 17 *If the potential V_4 with positive masses is meromorphically integrable, then the real central configuration c with multiplier -1 has a Hessian matrix W with spectrum of the form $Sp(W) = \{0, 2, \lambda_1, \lambda_2\}$ with*

$$\begin{gathered}\{\lambda_1, \lambda_2\} \in \{\{5,5\},\{5,9\},\{5,14\},\{5,20\},\{5,27\},\{5,35\},\\ \{5,44\},\{5,54\},\{9,20\},\{9,27\},\{9,35\},\{9,44\},\{9,54\},\{14,44\}\}\end{gathered}$$

Proof Using Pacella Theorem, we obtain a better minoration $\lambda_1, \lambda_2 > 2$. Knowing that $2 + \lambda_1 + \lambda_2 < 70$, we get the possibilities (5). So we only need to eliminate the cases

$$\{9, 9\}, \{9, 14\}, \{14, 14\}, \{14, 20\}, \{14, 27\}, \{14, 35\},$$
$$\{20, 20\}, \{20, 27\}, \{20, 35\}, \{20, 44\}, \{27, 27\}, \{27, 35\}$$

We first compute the characteristic polynomial of matrix W. Using the same notations as before, the characteristic polynomial has rational coefficients in ρ_1, ρ_2, m_3. Factoring it, we take out the $z(z-2)$ factor (corresponding to eigenvalues 0, 2) and we then get a degree 2 polynomial P in z. The coefficient in z corresponds to the trace of W, and is affine in m_3. We then solve the equation $tr(W)(\rho_1, \rho_2, m_3) = 2 + \lambda_1 + \lambda_2$ in m_3 (using Lemma 14, this always produces exactly one solution) and put this solution in P. So the only equation we have to study is of the form

$$Z_0(\rho_1, \rho_2) = P_{\rho_1,\rho_2}(0) - \lambda_1\lambda_2 = 0 \qquad \rho_1 > \rho_2 > 1 \tag{6}$$

Using RAGlib, we prove that for λ_1, λ_2 in the upper 12 cases, this equation has no solutions. This proves the Lemma. □

Remark 2 Remark that all the remaining curves are non empty for $\rho_1 \geq \rho_2 > 1$, but this does not imply they are non empty for positive masses (contrary to the previous part where we have taken into account the positivity of the masses). Numerical evidence suggest that for positive masses, the only possible eigenvalues $\{\lambda_1, \lambda_2\}$ are

$$\{5, 9\}, \{5, 14\}, \{5, 20\}, \{5, 27\}, \{9, 20\}, \{9, 27\}, \{9, 35\}, \{9, 44\}, \{14, 44\}$$

but taking into account this additional constraint seems too complicated.

3.4 Second Order Variational Equations

Using the integrability table of [17], integrability at second order requires that some of the third order derivatives of the potential vanish. Considering only the eigenvalues λ_1, λ_2 (the other ones do not lead to any additional integrability condition) we obtain the following number of conditions (i.e. the number of third order derivatives that should vanish)

{5, 5}, {5, 14}, {5, 27}, {14, 44}	4 conditions
{5, 44}, {5, 20}, {5, 35}, {5, 54}	3 conditions
{5, 9}, {9, 27}, {9, 44}	2 conditions
{9, 35}, {9, 54}	1 condition
{9, 20}	0 condition

The main drawback is that we need a priori to compute the eigenvalues of the Hessian matrix, and due to the parameters, this is quite difficult in our problem. In particular, testing the constraint implies solving 2-variables polynomials of degree 172 and this seems too large to rule out real solutions (if there are none at all). Still in some cases, we can avoid this computation

Proposition 3 *Let V be a meromorphic homogeneous potential of degree -1 in dimension n, c a Darboux point of V with multiplier -1, and E a stable subspace of $\nabla^2 V(c)$. Assume that $\nabla^2 V(c)$ is diagonalizable and*

$$\exists B \subset \mathbb{N},\ \textit{with}\ \max(B) \leq 2\min(B) + 1,\ Sp\left(\nabla^2 V(c)\big|_E\right) \subset \{k(2k+3),\ k \in B\} \tag{7}$$

If the second order variational equation near the homothetic orbit associated to c has a Galois group whose identity component is Abelian then

$$D^3 V(c) \cdot (X, Y, Z) = 0 \qquad \forall X, Y, Z \in E$$

Proof Using the integrability table of [17], we see that the condition on eigenvalues (7) implies that the table A for such eigenvalues will only have zeros. So denoting $X_1, \ldots, X_p$ the eigenvectors associated to eigenvalues λ_i $i = 1 \ldots p$ of $\nabla^2 V(c)$, we obtain the integrability condition

$$D^3 V(c) \cdot (X_i, X_j, X_k) = 0 \ \ \forall i, j, k = 1 \ldots p$$

These p eigenvectors span the invariant subspace E, and so by multilinearity, this gives the Proposition. □

We try to avoid computing the eigenvectors associated to eigenvalues $\{\lambda_1, \lambda_2\}$ for the Hessian matrix of the real central configuration of V_4. In the cases $\{\lambda_1, \lambda_2\} \in \{\{5, 5\}, \{5, 14\}, \{5, 27\}, \{14, 44\}\}$, the hypotheses of Proposition 3 are satisfied using for E the stable subspace generated by the eigenvectors associated to λ_1, λ_2. And it appears that this subspace is much easier to compute. Remark also that when the two eigenvalues are equal, then finding the eigenvectors is not necessary as any vector in the corresponding eigenspace is an eigenvector.

Lemma 18 *We consider V_4 the potential of the colinear 4 body problem with positive masses, c the real central configuration with multiplier -1, and $W \in M_4(\mathbb{C})$ the matrix such that*

$$W_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V$$

If $Sp(W) = \{0, 2, 5, 5\}, \{0, 2, 5, 14\}, \{0, 2, 5, 27\}, \{0, 2, 14, 44\}$, then the potential V_4 is not meromorphically integrable.

Proof We want to consider the stable subspace E of W corresponding to eigenvalues λ_1, λ_2. We already know an eigenvector of eigenvalue 0, $v = (1, 1, 1, 1)$, and an eigenvector of eigenvalue 2, the vector $c = (-\rho_1, -1, 1, \rho_2)$. As the matrix is symmetric, the eigenspaces are orthogonal, and thus we have $E = \text{Span}(v, c)^{\perp}$. We obtain

$$E = \text{Span}((2, -1 - \rho_1, \rho_1 - 1, 0), (0, \rho_2 - 1, -1 - \rho_2, 2))$$

denoting w_1, w_2 these two basis vectors of E.

Let us first consider the non-symmetric case. As Lemma 14 applies, we can consider the polynomial $Z_0 \in \mathbb{R}[\rho_1, \rho_2]$ given by Eq. (6), and

$$Z_1 = D^3 V(c)(w_1, w_1, w_1), \quad Z_2 = D^3 V(c)(w_1, w_1, w_2)$$

$$Z_3 = D^3 V(c)(w_1, w_2, w_2), \quad Z_4 = D^3 V(c)(w_2, w_2, w_2)$$

We obtain a system of 5 equations in two variables (the polynomials Z_i being of degree 58), and we prove that this system has no solutions for $\rho_1 > \rho_2 > 1$. Thus the second order variational equation has not a Galois group with an Abelian identity component.

The symmetric case. Only the cases $Sp(W) = \{0, 2, 5, 14\}, \{0, 2, 14, 44\}$ are possible. We have $\rho_1 = \rho_2$, and then the condition to have these eigenvalues are of the form of two polynomials in ρ_2, m_3. The polynomials Z_i above are still defined, and are polynomials in ρ_2, m_3. This system of 6 equations has no real solutions for $\rho_2 > 1, m_3 > 0$, and thus the second order variational equation does not have a Galois group with an Abelian identity component. Thus the potential V_4 is not meromorphically integrable in these cases. □

4 Higher Variational Equations

Proof of Theorem 3 The still open cases are

$$\{5, 44\}, \{5, 20\}, \{5, 35\}, \{5, 54\}, \{5, 9\}, \{9, 27\}, \{9, 44\}, \{9, 35\}, \{9, 54\}, \{9, 20\} \tag{8}$$

The case $\{9, 20\}$ is particularly interesting (and difficult) as there are no integrability conditions at order 2, and numerical evidence suggest that this case is really possible for positive masses. So this curve gives masses for which all integrability conditions near the unique (up to translation) real Darboux point up to order 2 are satisfied.

In the same manner as in [16], we will compute for these remaining sets of eigenvalues higher variational equations. We only need to study real 3 dimensional homogeneous potentials of degree -1. Asuming there exists a real Darboux point c, after rotation we can assume that $c = (1, 0, 0)$ (and the potential is still real). Then the series expansion of V at c will be of the form

$$V(1+q_1, q_2, q_3) = q_1^{-1}\left(1 + \frac{1}{2}\left(\lambda_1 \frac{q_2^2}{q_1^2} + \lambda_2 \frac{q_3^2}{q_1^2}\right) + \sum_{i=3}^{\infty}\sum_{j=0}^{i} u_{i,j} \frac{q_2^{i-j} q_3^j}{q_1^i}\right) \quad (9)$$

As in [16], the main part of the algorithm consists of finding solutions in $\mathbb{C}(t)\left[\operatorname{arctanh}\left(\frac{1}{t}\right)\right]$ of a large system of linear differential equations, which are the k-th variational equations. These k-th variational equations are put in block triangular form to make computation faster. Only the last equation is solved through the variation of parameters technique and then its monodromy analyzed through commutativity condition of monodromy in [17].

Instead of computing a basis of solutions, we only compute several solutions, that through empirical evidence, will lead to the strongest integrability conditions. The output of the algorithm is a set of polynomial conditions on higher-order derivatives of the potential V at the Darboux point c, so here it will be polynomial conditions on the $u_{i,j}$. As presented in [16], if a non degeneracy type condition is satisfied (see [16] Definition 4.1), we will be able to express higher-order derivatives in function of lower order ones. In our cases, this will always be the case for variational equations of order ≥ 3 (but we are lucky, because it seems that if eigenvalues are spaced enough, degeneracy at any order is possible). This allows us in particular to express all derivatives of order ≥ 4 as functions of $u_{3,0}, u_{3,1}, u_{3,2}, u_{3,3}$. The possible series expansions are written in Appendix A. Variational equations up to order 4 have been analyzed. Still, at order 4, some combinations of eigenvalues are still possible, and thus looking at order 5 is necessary. However, a speed-up is possible in certain cases:

4.1 An Invariant Subspace of the 5-th Order Variational Equation

Lemma 19 *Let V be a real meromorphic homogeneous potential of degree -1 in dimension 3. Assume that V has a series expansion of the form*

$$V(1+q_1, q_2, q_3) = q_1^{-1}\left(1 + \frac{1}{2}\left(\lambda_1 \frac{q_2^2}{q_1^2} + \lambda_2 \frac{q_3^2}{q_1^2}\right) + \sum_{i=3}^{\infty}\sum_{j=0}^{i} u_{i,j} \frac{q_2^{i-j} q_3^j}{q_1^i}\right)$$

with $u_{3,1} = u_{4,1} = u_{5,1} = 0$ and $\lambda_1 \in \{5, 9, 14, 20\}$. Then V is not meromorphically integrable.

Proof The dynamical system associated to V is of the form $\ddot{q} = \nabla V(q)$. Let us compute $\partial_{q_3} V$

$$\partial_{q_3} V = q_1^{-1}\left(\lambda_2 \frac{q_3}{q_1^2} + \sum_{i=3}^{5}\sum_{j=2}^{i} j u_{i,j} \frac{q_2^{i-j} q_3^{j-1}}{q_1^i} + \sum_{i=6}^{\infty}\sum_{j=1}^{i} j u_{i,j} \frac{q_2^{i-j} q_3^{j-1}}{q_1^i}\right)$$

Thus we get that the series expansion of $\partial_{q_3} V$ at order 5 for $q_3 = 0$ is

$$\partial_{q_3} V = u_{6,1} \frac{q_2^5}{q_1^7} + O\left((q_2, q_3)^6 / q_1^8\right)$$

As we see, there is only one term left, and it is of order 5. Let us now look at the 5-th order variational equation.

This variational equation will have an invariant subspace $\mathcal{W}$ corresponding to the 5-th order variational equation of $\tilde{V}$, the restriction of V to the plane $q_3 = 0$. Let us now look the variational equation on $\mathcal{W}$. The potential $\tilde{V}$ is a 2-dimensional homogeneous potential of degree -1, and it has a Darboux point at $(1, 0)$. The eigenvalues of the Hessian matrix of $\tilde{V}$ at this point are $\{2, \lambda_1\}$. Now using [16], we know that for any choice of **real** $\tilde{V}$ with $\lambda_1 \in \{5, 9, 14, 20\}$, the 5-th order variational equation does not have a virtually Abelian Galois group. Thus the 5-th order variational equation of V does not have a virtually Abelian Galois group, and thus V is not meromorphically integrable. □

Remark 3 Note that physically, the condition $u_{3,1} = u_{4,1} = u_{5,1} = 0$ implies that the plane $q_3 = 0$ is invariant at order 4. At order 5, it is no longer invariant, however the derivatives in time of q_1, q_2 do not depend on q_3.

We now use Lemma 19. Looking at the series expansions in Appendix A we have computed, we see that for all of them except the last one, we have either $u_{3,1} = u_{4,1} = u_{5,1} = 0$ or $u_{3,2} = u_{4,3} = u_{5,4} = 0$ (or both). In the first case we can apply directly Lemma 19. In the second case, we just have to exchange q_2, q_3, and the hypotheses of Lemma 19 are satisfied. So except for the case $(\lambda_1, \lambda_2) = (9, 20)$, the hypotheses are satisfied and thus there is no real meromorphic homogeneous potential of degree -1 in dimension 3 with $c = (1, 0, 0)$ as a Darboux point of V with multiplier -1 and these pairs of eigenvalues are meromorphically integrable.

4.2 *The Case* {9, 20}

In the last subsection, we tried to avoid computing the Galois group of the 5-th order variational equation as it is computationally expensive. At order 4, the ideal $\mathcal{I}_4$ is zero-dimensional, but still has real solutions (given in Appendix A). As we can see in Appendix A, the previous Lemma does not apply for these eigenvalues. Thus it is necessary to compute completely the 5-th order variational equation. The coefficients of the series expansion at order 4 are polynomials in $u_{3,0}, u_{3,1}, u_{3,2}, u_{3,3}$ modulo the ideal $\mathcal{I}_4$. As the algorithm never needs to inverse an element of this ring (which contains zero divisors), it also works at order 5. The output (after one week of computation) is the ideal $\mathcal{I}_5$, which happens to be improper. Thus $\mathcal{I}_5 =< 1 >$, and so no solutions (even complex) at all. We deduce then

Lemma 20 *Let V be a real meromorphic homogeneous potential of degree -1 in dimension 3. Assume that $c = (1, 0, 0)$ is a Darboux point of V with multiplier -1 and $Sp(\nabla^2 V(c)) = \{2, 9, 20\}$. Then V is not meromorphically integrable.* □

5 The Planar *n*-Body Problem

Let us prove in this section Theorem 4. The main tool will be the following Theorem

Theorem 21 (Pacella [19] Theorem 3.1) *We consider the colinear n body problem with positive masses and c a configuration with multiplier -1, given by potential $V_{n,2}$. Noting $W \in M_n(\mathbb{C})$ with*

$$W_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V_{n,2}$$

the spectrum of W is of the form $Sp(W) = \{0, 2, \lambda_1, \dots, \lambda_{n-2}\}$ with $\lambda_i > 2,\ i = 1 \dots n-2$.

Proof For the n body problem in the plane, the Hessian matrix to compute is of the form $W \in M_{2n}(\mathbb{C})$

$$W_{i,j} = \frac{1}{m_i} \frac{\partial^2}{\partial q_i \partial q_j} V_{n,2}$$

with the notation $m_{i+n} = m_i$. Computing this matrix at a colinear central configuration, we obtain a matrix of the form

$$W = \begin{pmatrix} A & 0 \\ 0 & -\frac{1}{2}A \end{pmatrix}$$

Due to Pacella Theorem, we have moreover that $Sp(A) = \{0, 2, \lambda_1, \dots, \lambda_{n-2}\}$ with $\lambda_i > 2,\ i = 1 \dots n-2$. Then the spectrum of W is of the form

$$\{0, 2, \lambda_1, \dots, \lambda_{n-2}, 0, -1, -\tfrac{1}{2}\lambda_1, \dots, -\tfrac{1}{2}\lambda_{n-2}\}$$

According to the integrability condition of the Morales-Ramis Theorem 1, all allowed eigenvalues for integrability are greater or equal to -1. These conditions cannot be satisfied as $-\frac{1}{2}\lambda_i < -1,\ i = 1 \ldots n-2$. Thus the planar n body problem with positive masses is not meromorphically integrable. □

6 Integrable *n*-Body Problems

In this section, we will progress in the opposite way. Instead of trying to prove non integrability, we come from already known integrable cases, and we try to determine if after some transformations, they correspond to particular cases of the n body problem.

Proposition 4 *The potential V in n variables $q_1, \ldots, q_n$*

$$V(q) = \sum_{l=1}^{p} a_l \left(\sum_{j=j_l+1}^{j_{l+1}} q_j^2 \right)^{-1/2} \tag{10}$$

with $0 = j_1 < j_2 < \cdots < j_{p+1} \leq n,\ a_l \in \mathbb{C}$ is integrable in the Liouville sense. For any complex orthogonal matrix $R \in \mathbb{O}_n(\mathbb{C})$, the potential $V(Rq)$ is integrable in the Liouville sense.

Here the kinetic part is assumed to be $T(p) = \|p\|^2/2$ and so the potential V is associated to a Hamiltonian system with $H(p,q) = T(p) + V(q)$.

Proof The potential V of equation (10) is a decoupled linear combination

$$V(q) = \sum_{l=1}^{p} a_l V_l(q_{j_l+1}, \ldots, q_{j_{l+1}}), \qquad V_l(q_{j_l+1}, \ldots, q_{j_{l+1}}) = \left(\sum_{j=j_l+1}^{j_{l+1}} q_j^2 \right)^{-1/2}$$

These potentials are invariant by the rotation group $\mathbb{O}_{j_{l+1}-j_l}(\mathbb{C})$ and so are integrable. Thus the potential V is integrable. As integrability is preserved by any orthogonal transformation, the potential $V(Rq)$ will also be integrable in the Liouville sense. □

Although these potentials seem to have a quite simple expression, the orthogonal transformation R can mix the variables (the decomposition of V is not necessarily conserved). However, the potential can always be written

$$V(q) = \sum_{l=1}^{p} a_l Q_l(q)^{-1/2} \tag{11}$$

with Q_i quadratic forms. And as an orthogonal transformation conserves the rank of these quadratic forms, we have moreover $\sum_{l=1}^{p} \operatorname{rank} Q_l \leq n$.

The Hamiltonian of the n body problem in dimension d can be written

$$H(p,q)=\sum_{i=1}^{n}\frac{\|p_i\|^2}{2m_i}+\sum_{i<j}m_im_j\left(\sum_{k=1}^{d}(q_{i,k}-q_{j,k})^2\right)^{-1/2}$$

The kinetic part of $H_{n,d}$ is not $\|p\|^2/2$ (as in Proposition 4). To transform the kinetic part to the standard one, we only have to make the variable change $q_{i,k}\mapsto q_{i,k}/\sqrt{m_i}$. The potential now becomes

$$\tilde{V}_{n,d}=\sum_{i<j}m_im_j\left(\sum_{k=1}^{d}(q_{i,k}/\sqrt{m_i}-q_{j,k}/\sqrt{m_j})^2\right)^{-1/2}$$

This expression is similar to Eq. (11), but there could be too many quadratic forms. After reduction by translation, the potential becomes a $(n-1)d$-dimensional potential. There are $n(n-1)/2$ independent quadratic forms, and to be of the form (11), we need $n(n-1)d/2\le(n-1)d$, which implies $n\le 2$. So in general, the potential $\tilde{V}_{n,d}$ is not of the form (10), but it could be for some restricted cases.

Definition 2 We say that a vector space $W\subset\mathbb{R}^{nd}$ is an invariant vector space if

$$\forall q\in W,\quad \nabla V_{n,d}\in W$$

This definition generalizes central configurations, which correspond to the case $\dim W=1$. Needless to say, as it is more difficult to find the invariant vector spaces than to find central configurations, we will not try to be exhaustive in this search. Let us remark that we already know some invariant vector spaces such as the isosceles 3-body problem and the collinear 3-body problem (which is an invariant vector space of the planar 3 body problem). Several others can be found using symmetries.

Let us now establish some rules for finding vector spaces W and masses m such that $V|_W$ is of the form (10) up to an orthogonal transformation. A necessary condition is that it can be written under the form (11). So in the expression of $V|_W$ we should try to have the lowest possible number of independent quadratic forms (corresponding to mutual distances) with the lowest possible rank.

Remark that if we allow negative masses, some terms in the sum in $V_{n,d}$ could cancel each other, thus reducing greatly the number of quadratic forms. So it seems that finding examples will be easier when negative masses are allowed. And indeed, all interesting examples we will find require a negative mass. Let us now prove Theorem 5.

6.1 An Integrable 5 *Body Problem*

Proof The vector space W is of dimension $10 - 6 = 4$. The mass $-1/4$ is at the origin which is the center of mass of the system. On the vector space W, the 2-nd and the 4-th body are symmetric with respect to the origin, as well as the 3-th and 5-th bodies. Due to these symmetries, the vector space W is invariant. We can thus restrict our potential to W. Now computing the potential V on W we find $V_{5,2}\big|_W =$

$$\left((q_{2,1} - q_{3,1})^2 + (q_{2,2} - q_{3,2})^2\right)^{-1/2} + \left((q_{2,1} + q_{3,1})^2 + (q_{2,2} + q_{3,2})^2\right)^{-1/2} \tag{12}$$

There are only two quadratic forms each with rank two for each. As $2 + 2 = 4 = \dim W$, we are in the form (11). We can now try to put this potential in the form (10). This is done by the following orthogonal transformation

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

acting on $q_{2,1}, q_{3,1}, q_{2,2}, q_{3,2}$ in this order. Thus $V_{5,2}\big|_W$ is integrable. □

On W, the bodies are always on the edges of a parallelogram whose center is the origin (where lies the mass -1). Looking at the forces acting on the bodies, we see that they are not attracted by the center at all (because the repulsion of the central mass $-1/4$ exactly compensates the attraction of the opposite mass 1 at twice the distance). The masses are then only attracted by their neighbours. Looking at the expression of the potential (12), we see that the force acting on the center of vertices of the parallelogram (which are $(\pm q_{2,1} \pm q_{3,1}, \pm q_{2,2} \pm q_{3,2})$) is toward the center. Thus the motion of these centers are conics with focus at the origin.

Thus the motion of a body of mass 1 is the composition of two conic motions. The body has a conic motion whose focus is the center of mass of two bodies of mass 1, and this center of mass has a conic motion with focus at the origin. If the two conics are ellipses with rational period ratio, this leads to (algebraic) periodic orbits of the bodies (Fig. 4).

6.2 An Integrable $n + 3$ *Body Problem*

Proof The space W is of dimension 3. The forces between the n cocyclic masses and the central mass exactly compensate. The forces between the 3 last masses also compensate (as this is also an absolute equilibrium). So the only forces between the bodies are between the last two masses and the cocyclic masses. But due to symmetry

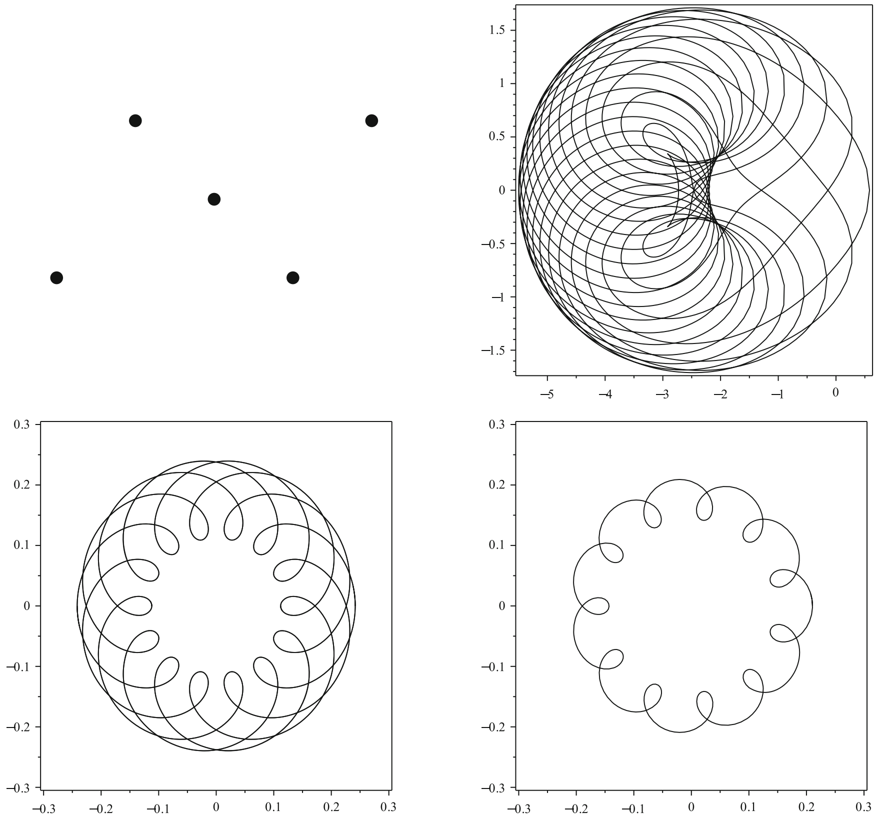

Fig. 4 The configuration of the 5 bodies and examples of motions of a body of mass 1 with ellipses with rational period ratio

and the fact that the masses are cocyclic, this force only involves one distance. Thus the potential is of the form

$$V = \gamma \left(q_{1,1}^2 + q_{1,2}^2 + q_{n+2,3}^2\right)^{-1/2}$$

This potential corresponds to a central force, and thus is integrable. □

Let us look at an example. The most known cyclic central configuration is the regular polygon. We have $m_1 = \cdots = m_n = 1$. The central mass (chosen to produce an absolute equilibrium) and the potential are then

$$-\alpha = -\frac{1}{2} \sum_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)^{-1} \qquad V = 4n\alpha \left(q_{1,1}^2 + q_{1,2}^2 + q_{n+2,3}^2\right)^{-1/2}$$

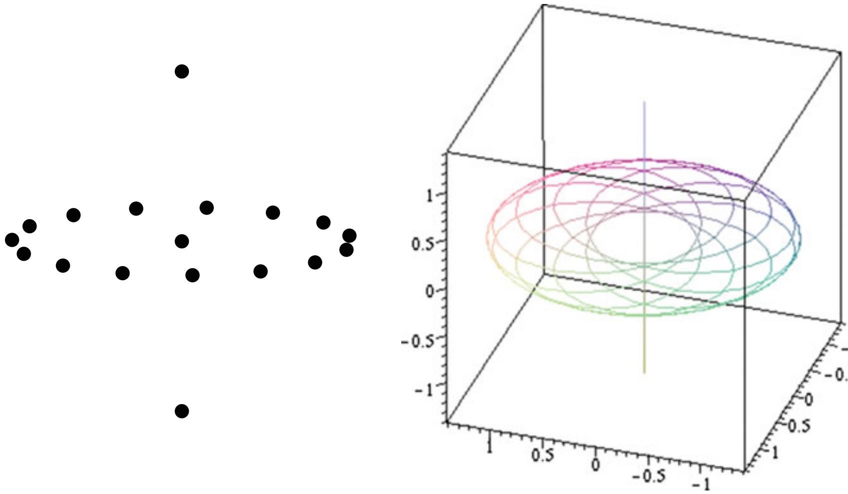

Fig. 5 A configuration of the $n+3$ bodies with a regular polygon and an example of motion of the bodies with an ellipse with non-zero inclination

respectively. The motion is the following: the n bodies describe conics in the plane, and the two symmetrical last bodies move along the vertical line. Note that the motion of the bodies on the vertical line is not determined by the motion of the bodies in the plane (this is not a rigid motion as in the case of central configurations). This vertical motion depends on the "inclination" of the conic orbit chosen for the above potential (Fig. 5).

Appendix A. Integrable Series Expansions at Order 4

The results are written in the following way. We give a series expansion of the form (9) of V, such that the k-th order variational equation of V near c has a virtually Abelian Galois group if and only if $(u_{3,0}, u_{3,1}, u_{3,2}, u_{3,3}) \in \mathcal{I}_k^{-1}(0)$. The sequence of ideals $\mathcal{I}_k$ is growing, and we compute these conditions up to order 4. For the eigenvalues in (8), they are given below. Remark that the Hilbert dimension of the ideals $\mathcal{I}_4$ greatly depend on eigenvalues, and that sometimes exceptional possible solutions appear in the 4-th order variational equation. In particular, the restriction of these series expansions to the planes in (q_1, q_2) and (q_1, q_3) does not always lead to integrable series expansion at order 4 on these planes.

$q_1^{-1}\left(1+\frac{5}{2}\frac{q_2^2}{q_1^2}+\frac{44}{2}\frac{q_3^2}{q_1^2}+u_{3,1}\frac{q_2^2q_3}{q_1^3}+\left(\frac{7}{459}u_{3,1}^2+\frac{175}{24}\right)\frac{q_2^4}{q_1^4}+\left(-\frac{734}{7775}u_{3,1}^2+\frac{130779}{622}\right)\frac{q_2^2q_3^2}{q_1^4}+\frac{682538736}{1082611}\frac{q_3^4}{q_1^4}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$ $\mathcal{I}_2=\mathcal{I}_3=<u_{3,0},u_{3,2},u_{3,3}>,\ \mathcal{I}_4=<u_{3,0},u_{3,1},u_{3,2},u_{3,3}>$

$q_1^{-1}\left(1+\frac{5}{2}\frac{q_2^2}{q_1^2}+\frac{20}{2}\frac{q_3^2}{q_1^2}+u_{3,3}\frac{q_3^3}{q_1^3}+\frac{175}{24}\frac{q_2^4}{q_1^4}+\frac{13575}{142}\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{105347}{840700}u_{3,3}^2+\frac{153200}{1201}\right)\frac{q_3^4}{q_1^4}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$ $\mathcal{I}_2=\mathcal{I}_3=<u_{3,0},u_{3,1},u_{3,2}>$ $\mathcal{I}_4=<u_{3,0},u_{3,1},u_{3,2},u_{3,3}>$

$q_1^{-1}\left(1+\frac{5}{2}\frac{q_2^2}{q_1^2}+\frac{35}{2}\frac{q_3^2}{q_1^2}+u_{3,3}\frac{q_3^3}{q_1^3}+\frac{175}{24}\frac{q_2^4}{q_1^4}+\frac{13575}{142}\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{108970043}{1516693800}u_{3,3}^2+\frac{8028474825}{20222584}\right)\frac{q_3^4}{q_1^4}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$ $\mathcal{I}_2=\mathcal{I}_3=<u_{3,0},u_{3,1},u_{3,2}>$ $\mathcal{I}_4=<u_{3,0},u_{3,1},u_{3,2},u_{3,3}>$

$q_1^{-1}\left(1+\frac{5}{2}\frac{q_2^2}{q_1^2}+\frac{54}{2}\frac{q_3^2}{q_1^2}+u_{3,3}\frac{q_3^3}{q_1^3}+\frac{175}{24}\frac{q_2^4}{q_1^4}+\frac{15646365}{60604}\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{663499046129}{14245925733270}u_{3,3}^2+\frac{93293112339}{97889959}\right)\frac{q_3^4}{q_1^4}+\frac{119177438935}{9191020828}\frac{q_2^2q_3^3}{q_1^5}+\right.$
$\left.\left(\frac{11164016229220693628221}{5008255522459502086348344}u_{3,3}^3+\frac{17921365151237750718}{239428834674960487}u_{3,3}\right)\frac{q_3^5}{q_1^5}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$ $\mathcal{I}_2=\mathcal{I}_3=\mathcal{I}_4=<u_{3,0},u_{3,1},u_{3,2}>$

$q_1^{-1}\left(1+\frac{5}{2}\frac{q_2^2}{q_1^2}+\frac{9}{2}\frac{q_3^2}{q_1^2}+u_{3,1}\frac{q_2^2q_3}{q_1^3}+u_{3,3}\frac{q_3^3}{q_1^3}+\left(\frac{7}{324}u_{3,1}^2+\frac{175}{24}\right)\frac{q_2^4}{q_1^4}+\left(\frac{19}{123}u_{3,1}^2+\frac{95}{246}u_{3,1}u_{3,3}+\frac{7965}{164}\right)\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{3289}{12060}u_{3,3}^2+\frac{13311}{536}\right)\frac{q_3^4}{q_1^4}+\left(\frac{3497470290125}{504750025968}u_{3,1}-\frac{664048660625}{504750025968}u_{3,3}\right)\frac{q_2^4q_3}{q_1^5}+\right.$
$\left.\left(\frac{2220239578205}{252375012984}u_{3,1}-\frac{921441864205}{252375012984}u_{3,3}\right)\frac{q_2^2q_3^3}{q_1^5}+\left(-\frac{187329008582632851}{386510093636424240}u_{3,1}+\frac{775252202045235367}{193270046818212120}u_{3,3}\right)\frac{q_3^5}{q_1^5}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$
$\mathcal{I}_2=\mathcal{I}_3=<u_{3,0},u_{3,2},\ \mathcal{I}_4=\left\langle u_{3,0},u_{3,2},u_{3,3}^5+\frac{189924461418873225}{1610583723485101}u_{3,3}^3-\frac{312185150911564368 75}{6442334893940404}u_{3,3},u_{3,3}^3+\frac{322885767353079225}{3221167446970202}u_{3,3}+\frac{103594358565795300}{1610583723485101}u_{3,1}\right\rangle$

$q_1^{-1}\left(1+\frac{9}{2}\frac{q_2^2}{q_1^2}+\frac{27}{2}\frac{q_3^2}{q_1^2}+u_{3,0}\frac{q_2^3}{q_1^3}+u_{3,2}\frac{q_2q_3^2}{q_1^3}+\left(\frac{3289}{12060}u_{3,0}^2+\frac{13311}{536}\right)\frac{q_2^4}{q_1^4}+\left(\frac{42887}{2155950}u_{3,2}^2+\frac{47791}{143730}u_{3,0}u_{3,2}+\frac{376407}{1597}\right)\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{11446993}{324669060}u_{3,2}^2+\frac{376298379}{1603304}\right)\frac{q_3^4}{q_1^4}+\right.$
$\left(\frac{207570596097655732674166455281418210645217993262286489869}{840193865795267874819893782623565474088703500063704000}u_{3,0}-\frac{16787633864550203502743667842716175416793657678815071}{33607754631810714992795727304942618963548140002524160}u_{3,2}\right)\frac{q_2^5}{q_1^5}+\left(\frac{41420797622285419653283151280696451 41}{5646615134224665663290954120421260}u_{3,0}-\right.$
$\left.\frac{1686976903148618227721001604738486 3}{5646615134224665663290954120421260}u_{3,2}\right)\frac{q_2^3q_3^2}{q_1^5}+\left(\frac{20533796486740053950290290586390781 97}{1389936033039917701733157937334464}u_{3,0}+\frac{121382613254803160782574481161900493}{6949680165199588508665789686672320}u_{3,2}\right)\frac{q_2q_3^4}{q_1^5}\left.+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$
$\mathcal{I}_2=\mathcal{I}_3=<u_{3,1},u_{3,3}>$ $\mathcal{I}_4=\left\langle u_{3,1},u_{3,3},u_{3,0}^5-\frac{183692999458343787808210893960821881229602107524169}{20377836297273288008691439130795431294539450561232}u_{3,0}^3+.\frac{5871817960967665190055792996619628724237316321335937 5}{40755672594546596017382878261590862549078901122464}u_{3,0},\right.$
$\left.u_{3,0}^3-\frac{3554174443372635952956958960222053446124173871757419}{20377836297273288008691439130795431294539450561232}u_{3,0}+.\frac{1350984520019956288893495927826867723730667397880 25}{20377836297273288008691439130795431294539450561232}u_{3,2}\right\rangle$

$q_1^{-1}\left(1+\frac{9}{2}\frac{q_2^2}{q_1^2}+\frac{35}{2}\frac{q_3^2}{q_1^2}+u_{3,0}\frac{q_2^3}{q_1^3}+u_{3,2}\frac{q_2q_3^2}{q_1^3}+u_{3,3}\frac{q_3^3}{q_1^3}+\frac{3289}{12060}u_{3,0}^2\frac{q_2^4}{q_1^4}+\left(\frac{299}{19730}u_{3,2}^2+\frac{6539}{19730}u_{3,0}u_{3,2}+\frac{1202985}{3946}\right)\frac{q_2^2q_3^2}{q_1^4}+\frac{2171}{27090}u_{3,2}u_{3,3}\frac{q_2q_3^3}{q_1^4}+\left(\frac{1084733}{30333876}u_{3,2}^2+\frac{108970043}{1516693800}u_{3,3}^2+\right.\right.$
$\left.\frac{8028474825}{20222584}\right)\frac{q_3^4}{q_1^4}+\left(\frac{99171}{8576}u_{3,0}^3+\frac{363467}{4824000}u_{3,0}\right)\frac{q_2^5}{q_1^5}+\left(\frac{12500754448143833461}{16013235835701805919400}u_{3,2}^3-\frac{106381137094520737}{9000423588396965634000}u_{3,3}^2u_{3,2}+\frac{236392873}{2319952050}u_{3,0}^2u_{3,2}+\frac{1575063218080739091500 63}{177111471909793942295 84}u_{3,2}+\right.$
$\left.\frac{203530243}{2667496}u_{3,0}\right)\frac{q_2^3q_3^2}{q_1^5}+\left(\frac{1108175418219424366845664 67}{1588606758916943651880510530000}u_{3,3}^3-\frac{3852292503751784909660 79}{12309652055185871405368}u_{3,3}\right)\frac{q_2^2q_3^3}{q_1^5}+\left(\frac{2133042536605191925}{57080871628797312}u_{3,2}+\frac{6522159327691787}{17980474563071153 28}u_{3,2}^3+\right.$
$\left.\frac{90416581343807626661}{1246212496854964472400 0}u_{3,2}u_{3,3}^2\right)\frac{q_2q_3^4}{q_1^5}+\left(\frac{1791170957429046156550020 21}{1758126054781922083471684 6}u_{3,3}+\frac{45081551340734816448303196649189}{8334836094207397117218388767450000}u_{3,3}^3\right)\frac{q_3^5}{q_1^5}\left.+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$
$\mathcal{I}_2=\mathcal{I}_3=<u_{3,1}>$ $\mathcal{I}_4=\left\langle u_{3,3}^3-\frac{26927363870719093699242 5}{6971035841270139539814}u_{3,2}^2u_{3,3}-\frac{19517430127889746138453959375}{60415643957674542678388}u_{3,3},\ \frac{296036655392605259124101 2}{296036655392605259124101}u_{3,0}u_{3,3}^3-\frac{200153459825026311801301}{592073310785210518248202}u_{3,2}u_{3,3}^3-\right.$
$\frac{95635407626659756078424400937 5}{296036655392605259124101}u_{3,0}u_{3,3}+\frac{16345103797398054141092774812 5}{296036655392605259124101}u_{3,2}u_{3,3},-\frac{52777633186609423650515300}{4618925995962145656953 1}u_{3,0}u_{3,2}u_{3,3}+u_{3,3}^3-\frac{3269020759479610828218554962 50}{200153459825026311801301}u_{3,3},$
$\left.u_{3,2}u_{3,3}^2+\frac{7692669733672620 20}{815441353231641}u_{3,0}u_{3,2}^2-\frac{36419152540918889 0}{5164461903800393}u_{3,2}^3+\frac{5093536748382230147212 5}{671380047494051 09}u_{3,2}\right\rangle$

$q_1^{-1}\left(1+\frac{9}{2}\frac{q_2^2}{q_1^2}+\frac{54}{2}\frac{q_3^2}{q_1^2}+u_{3,0}\frac{q_2^3}{q_1^3}+u_{3,2}\frac{q_2q_3^2}{q_1^3}+u_{3,3}\frac{q_3^3}{q_1^3}+\left(\frac{13311}{536}+\frac{3289}{12060}u_{3,0}^2\right)\frac{q_2^4}{q_1^4}+\left(\frac{8131447}{843807240}u_{3,2}^2+\frac{1469946609}{3125212}+\frac{46560929}{140634540}u_{3,0}u_{3,2}\right)\frac{q_2^2q_3^2}{q_1^4}+\frac{26821}{525690}u_{3,2}u_{3,3}\frac{q_2q_3^3}{q_1^4}+\left(\frac{3843346549}{105721155720}u_{3,2}^2+\right.\right.$
$\left.\frac{93293112339}{97889959}+\frac{663499046129}{14245925733270}u_{3,3}^2\right)\frac{q_3^4}{q_1^4}+\left(\frac{99171}{8576}u_{3,0}+\frac{363467}{4824000}u_{3,0}^3\right)\frac{q_2^5}{q_1^5}+\left(\frac{118579147678287}{965637379396}u_{3,0}+\frac{548369856819135481128554482500530673}{593939272542615654574305060222890 96}u3,2+\frac{2145942247517189587110709512396521 3}{7060785531047662221916925081604964920 0}u_{3,2}^3+\right.$
$\left.\frac{1543078654120439}{14556983494394700}u_{3,0}^2u_{3,2}-\frac{6532333876427547851207247 8}{18007002218330647802772268712859}u_{3,2}u_{3,3}^2\right)\frac{q_2^3q_3^2}{q_1^5}+$
$\left(-\frac{56399453236830854062047492240965325}{50148944984734256424203316413649 2}u_{3,3}+\frac{16450587697848383603452787103914 84}{49929543350426044052347426904339484 75}u_{3,3}^3\right)\frac{q_2^2q_3^3}{q_1^5}+\left(\frac{63108417379965952810933 53}{10832661651418700491360 4}u_{3,2}+\frac{10277293173559961713898 51}{43872279688245736990009620 0}u3,2^3+\right.$
$\left.\frac{30151163445704395316027384 98}{10096714493437453619876483672 25}u_{3,2}u_{3,3}^2\right)\frac{q_2q_3^4}{q_1^5}+\left(-\frac{40084954600004013999792520280720661661131 21}{29916876551082911336452430097420793501516}u_{3,3}+\frac{537412961592405139051640095845978898256260852721}{188802669023836001803818434124847208914422412210200}u_{3,3}^3\right)\frac{q_3^5}{q_1^5}\left.+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$
$\mathcal{I}_2=\mathcal{I}_3=<u_{3,1}>$ $\mathcal{I}_4=\left\langle u_{3,3}^3-\frac{2140503749348413383959897651802 1}{4495112199619735814187970929088}u_{3,2}^2u_{3,3}-\frac{38018197051893130425988535699449612 5}{11237780499049339535469927322 72}u_{3,3},\ \frac{403509244945150281569954332716559507 5}{18429960018440916838170680809260 8}u_{3,2}u_{3,3}-\right.$
$\frac{38018197051893130425988535699449612 5}{11237780499049339535469927322 72}u_{3,0}u_{3,3}-\frac{51588237214057}{1157111566196588}u_{3,2}u_{3,3}^3+u3,0u_{3,3}^3,-\frac{87760653723284948742355803723886 1}{82167457058793670335631729657 12}u_{3,0}u_{3,2}u_{3,3}+u_{3,3}^3-$
$\left.\frac{403509244945150281569954332716559507 5}{82167457058793670335631729657 12}u_{3,3},u_{3,2}u_{3,3}^2+\frac{58278294334415316709243773}{32748020504236080458960}u_{3,0}u_{3,2}^2-\frac{1046595675806171231682151 21}{12444247791609710574404 80}u_{3,2}^3+\frac{2746684168714569366809058005 55}{1244424779160971057440 48}u_{3,2}\right\rangle$

$$q_1^{-1}\left(1+\frac{9}{2}\frac{q_2^2}{q_1^2}+\frac{44}{2}\frac{q_3^2}{q_1^2}+u_{3,0}\frac{q_2^3}{q_1^3}+u_{3,2}\frac{q_2q_3^2}{q_1^3}+\left(\frac{3289}{12060}u_{3,0}^2+\frac{13311}{536}\right)\frac{q_2^4}{q_1^4}+\left(\frac{48787}{4090760}u_{3,2}^2+\frac{2031899}{6136140}u_{3,0}u_{3,2}+\frac{78370281}{204538}\right)\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{4690489}{129913320}u_{3,2}^2+\frac{682538736}{1082611}\right)\frac{q_3^4}{q_1^4}+\right.$$
$$\left(\frac{363467}{4824000}u_{3,0}^3+\frac{99171}{8576}u_{3,0}\right)\frac{q_2^5}{q_1^5}+\left(\frac{44243462785788303173707 5841}{938912143429199537485438760400}u_{3,2}^3+\frac{8173166259659}{78016448675700}u_{3,0}^2u_{3,2}+\frac{4779618603758363251509386 52440}{524225946747969741762703 30789}u_{3,2}+\frac{85215635549}{862536746}u_{3,0}\right)\frac{q_2^3q_3^2}{q_1^5}+$$
$$\left.\left(\frac{28213191044570641817}{979692671651440092 6600}u_{3,2}^3+\frac{15427883670599970273 11}{326564223883813364 22}u_{3,2}\right)\frac{q_2q_3^4}{q_1^5}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$$

$$\mathcal{I}_2=\mathcal{I}_3=<u_{3,1},u_{3,3}>\quad \mathcal{I}_4=\left\langle u_{3,1},u_{3,3},u_{3,2}^3-\frac{1222814898902417148900}{70922785617530321}-\frac{3628491376486815158}{212768356852590963}u_{3,0}u_{3,2}^2\right\rangle$$

$$q_1^{-1}\left(1+\frac{9}{2}\frac{q_2^2}{q_1^2}+\frac{20}{2}\frac{q_3^2}{q_1^2}+u_{3,0}\frac{q_2^3}{q_1^3}+u_{3,1}\frac{q_2^2q_3}{q_1^3}+u_{3,2}\frac{q_2q_3^2}{q_1^3}+u_{3,3}\frac{q_3^3}{q_1^3}+\left(\frac{13311}{536}+\frac{3289}{12060}u_{3,0}^2+\frac{11}{1809}u_{3,1}^2\right)\frac{q_2^4}{q_1^4}+\left(\frac{44}{135}u_{3,1}u_{3,0}+\frac{11}{270}u_{3,1}u_{3,2}\right)\frac{q_2^3q_3}{q_1^4}+\left(\frac{2959}{8772}u_{3,0}u_{3,2}+\frac{451}{2924}u_{3,1}u_{3,3}+\frac{11}{408}u_{3,2}^2+\right.\right.$$
$$\left.\frac{814}{6579}u_{3,1}^2+\frac{258255}{1462}\right)\frac{q_2^2q_3^2}{q_1^4}+\left(\frac{253}{2241}u_{3,1}u_{3,2}+\frac{121}{830}u_{3,2}u_{3,3}\right)\frac{q_2q_3^3}{q_1^4}+\left(\frac{2981}{86472}u_{3,2}^2+\frac{153200}{1201}+\frac{105347}{840700}u_{3,3}^2\right)\frac{q_3^4}{q_1^4}+\left(\frac{24121663415883}{456685207040}u_{3,0}-\frac{3676530375246235296032 7967}{2196201151998323963013120}u_{3,2}+\right.$$
$$\frac{89405326300298824860496291}{177892293311864241004062 72000}u_{3,2}^3-\frac{41358147528355848356789}{8222889690590546031104000}u_{3,0}u_{3,2}^2+\frac{134685326381 2693}{13769058992256000}u_{3,0}^2u_{3,2}-\frac{922257609158044398539993}{455711739039652222325222 40000}u_{3,2}u_{3,3}^2+\frac{363467}{4824000}u_{3,0}^3+\frac{230463041841}{4566852070400}u_{3,0}u_{3,1}u_{3,3}-$$
$$\left.\frac{32105330815398760859953}{151777431820034045736 9600}u_{3,1}u_{3,3}u_{3,2}\right)\frac{q_2^5}{q_1^5}+\left(\frac{1006362154607592365}{76272045092872776}u_{3,1}+\frac{1516843182851 94825}{5403817053617488}u_{3,3}-\frac{119025726972819079}{11265073504331898240}u_{3,2}^2u_{3,3}+\frac{23649118837130954071}{363298620514703718240}u_{3,0}u_{3,2}u_{3,3}+\right.$$
$$\left.\frac{5798437746842933}{10723874942903904936}u_{3,1}^2u_{3,3}+\frac{386205492458418970315 9}{85862492042850598854240}u_{3,1}u_{3,3}^2+\frac{24082362391042 82863}{13674052244250231 28128}u_{3,1}u_{3,2}^2+\frac{2556040771182469}{284638889347423632}u_{3,1}u_{3,0}u_{3,2}\right)\frac{q_2^4q_3}{q_1^5}+\left(-\frac{7295524609850}{130123321413}u_{3,1}+\frac{7533233096661}{105601052404}u_{3,3}-\right.$$
$$\left.\frac{114781940299}{12230075371440}u_{3,2}^2u_{3,3}+\frac{204029796932797}{1314733102429800}u_{3,0}u_{3,2}u_{3,3}-\frac{833363511404}{8316082876815}u_{3,1}^2u_{3,3}+\frac{2463793308929051}{26633641426812840}u_{3,1}u_{3,3}^2+\frac{46351753531771}{1166425453146132}u_{3,1}u_{3,2}^2-\frac{76656425825}{485605933866}u_{3,1}u_{3,0}u_{3,2}\right)\frac{q_2^3q_3^2}{q_1^5}+$$
$$\left(-\frac{7295524609850}{130123321413}u_{3,1}+\frac{7533233096661}{105601052404}u_{3,3}-\frac{114781940299}{12230075371440}u_{3,2}^2u_{3,3}+\frac{204029796932797}{1314733102429800}u_{3,0}u_{3,2}u_{3,3}-\frac{833363511404}{8316082876815}u_{3,1}^2u_{3,3}+\frac{2463793308929051}{26633641426812840}u_{3,1}u_{3,3}^2+\frac{46351753531771}{1166425453146132}u_{3,1}u_{3,2}^2-\right.$$
$$\left.\frac{76656425825}{485605933866}u_{3,1}u_{3,0}u_{3,2}\right)\frac{q_2^2q_3^3}{q_1^5}+\left(\frac{50525703076300}{161496415089}u_{3,2}-\frac{4235472447613}{52324838488836}u_{3,2}^3+\frac{3093278443}{4840858404}u_{3,0}u_{3,2}^2+\frac{4492719610495891}{187658834333418000}u_{3,2}u_{3,3}^2+\frac{13050298589}{34722700404}u_{3,1}u_{3,3}u_{3,2}\right)\frac{q_2q_3^4}{q_1^5}+$$
$$\left.\left(\frac{46809030}{1737847}u_{3,3}+\frac{1362858557}{173089561200}u_{3,2}^2u_{3,3}+\frac{68250852673}{4257725150000}u_{3,3}^3+\frac{133423}{51883632}u_{3,1}u_{3,2}^2\right)\frac{q_3^5}{q_1^5}+O\left(\frac{(q_1,q_2)^6}{q_1^6}\right)\right)$$

$$\mathcal{I}_2=\mathcal{I}_3=<0>\quad \mathcal{I}_4=\left\langle\frac{21492255493447480629464816241}{96826350676115640555159100}u_{3,1}-\frac{161978738053543177347}{58408235543590696}u_{3,0}+\frac{3682769787616585454443049170788929}{4017640277688287994053545828000}u_{3,2}-\frac{153908497065685013278741413}{5067633436123573432172 00}u_{3,3}+\frac{83773728539}{2056568535750}u_{3,1}^3+\right.$$
$$\frac{3602135236781333}{117779680042402500}u_{3,1}^2u_{3,2}-\frac{208416169135127627446573912789223}{756811308122677505856598167600000}u_{3,2}^3+\frac{7730882033}{41131370715}u_{3,0}u_{3,1}^2+\frac{4397878855122589301767627048843}{1505360636935661613092502600000}u_{3,0}u_{3,2}^2+\frac{111825024093141519961813257}{978171105112224639233200000}u_{3,2}^2u_{3,3}-$$
$$\frac{19202818906143244099}{2920411777179534800}u_{3,0}^2u_{3,2}+\frac{431284635777701402102430913273}{5702191228593158404692960050 00000}u_{3,2}u_{3,3}^2-\frac{7406139134798571607952467731}{10515339379956414871756900000}u_{3,0}u_{3,2}u_{3,3}-\frac{1981359607679726589}{584082355435906960}u_{3,0}u_{3,1}u_{3,3}+$$
$$\frac{16159160375636350412725999632347}{13882770318406657098519746200000}u_{3,1}u_{3,3}u_{3,2}+\frac{11541178916148758193157317}{36516770348537514437710000}u_{3,1}^2u_{3,3}-\frac{3628412075057407138506005535897}{7455629002060904322747250700000}u_{3,1}u_{3,3}^2-\frac{16997850549154609014088319603}{224276849473049251146368520000}u_{3,1}u_{3,2}^2+$$
$$\frac{209831610644740256848667017}{334578980271340473192265000}u_{3,1}u_{3,0}u_{3,2}+u_{3,1}u_{3,0}^2,\ \frac{11023564040554427666539761429}{619688644327140099553018 24}u_{3,1}-\frac{809893690267715886735}{467265884348725568}u_{3,0}+\frac{9500305040852292249125132917333559}{16456254577411227623643323711488}u_{3,2}-$$
$$\frac{910882769846955309793046 97}{32432853991190869965900 8}u_{3,3}+\frac{12565459497}{438734620960}u_{3,1}^3+\frac{18240141713679571}{964851138907361280}u_{3,1}^2u_{3,2}-\frac{537628219940107140701558350542803}{3099899118070487063988626094489600}u_{3,2}^3+\frac{7730882033}{65810193144}u_{3,0}u_{3,1}^2+\frac{11317999279608608041128669156553}{6165957168888469967226890649600}u_{3,0}u_{3,2}^2+$$
$$\frac{66181848063068086388215533}{626029507271823769109248000}u_{3,2}^2u_{3,3}-\frac{19202818906143244099}{4672658843487255680}u_{3,0}^2u_{3,2}+\frac{1133842119622475310043992533923}{2335617527231757682562236436480000}u_{3,2}u_{3,3}^2-\frac{4383204733717948930008199839}{6729817203172105517924416000}u_{3,0}u_{3,2}u_{3,3}-$$
$$\frac{1981359607679726589}{9345317686974511 36}u_{3,0}u_{3,1}u_{3,3}+\frac{41676784231893787208363119215647}{5686382722419366747553688043520 0}u_{3,1}u_{3,3}u_{3,2}+\frac{10045252530421753483940744 1}{39730246139208815708228480 0}u_{3,1}^2u_{3,3}-\frac{2147417526703479037810755093693}{4771602561318978766558240448000}u_{3,1}u_{3,3}^2-$$
$$\frac{8840853261257999339095182607}{143537183662751520733675852800}u_{3,1}u_{3,2}^2+\frac{327022808578066208908275469}{642391642120973708529148800}u_{3,1}u_{3,0}u_{3,2}+u_{3,1}u_{3,0}^2,\ \frac{8119705174366086529811522577}{48413175338057820277579550}u_{3,1}-\frac{161978738053543177347}{116816471087181392}u_{3,0}+$$
$$\frac{14870337582300636982677825606451679}{32141122221506303952428366624000}u_{3,2}-\frac{69826208977705124409476661}{253381671806178671608600}u_{3,3}+\frac{8860106111}{342761422625}u_{3,1}^3+\frac{28427099781479791}{1884474880678440000}u_{3,1}^2u_{3,2}-\frac{841516266837011306783082446930723}{605449046498142004685278534080000 0}u_{3,2}^3+\frac{7730882033}{82262741430}u_{3,0}u_{3,1}^2+$$
$$\frac{17705802748364933031308092640593}{12042885095485292904740020800000}u_{3,0}u_{3,2}^2+\frac{50733504972972215377690329}{489085552556112319616600000}u_{3,2}^2u_{3,3}-\frac{19202818906143244099}{5840823554359069600}u_{3,0}^2u_{3,2}+\frac{178235672009457430199004914523}{4561752982874526723754368040000000}u_{3,2}u_{3,3}^2-$$
$$\frac{3360065420713577247167215107}{5257669689978207435878450000}u_{3,0}u_{3,2}u_{3,3}-\frac{1981359607679726589}{1168164710871813920}u_{3,0}u_{3,1}u_{3,3}+\frac{65231513945362190404335844984847}{11106216254725325678815796960000}u_{3,1}u_{3,3}u_{3,2}+\frac{73950500303148142428237033}{310392547962568872720535000}u_{3,1}^2u_{3,3}-$$
$$\frac{1646161613169798251144263874409}{3727814501030452161373625350000}u_{3,1}u_{3,3}^2-\frac{6539529995537929093658254091}{112138424736524625573184260000}u_{3,1}u_{3,2}^2+\frac{120905494589403709828247761}{250934235203505354894198750}u_{3,1}u_{3,0}u_{3,2}+u_{3,1}u_{3,0}^2,\ \frac{73712383302787538188228465761}{387305402704462562220636400}u_{3,1}-$$
$$\frac{485936214160629532041}{233632942174362784}u_{3,0}+\frac{17774775882843616796760576626609403 33}{2571289777720504316194269329920 00}u_{3,2}-\frac{582369803672098936042126773}{202705337444942937286880 0}u_{3,3}+\frac{790993099357}{24678822429000}u_{3,1}^3+\frac{38119456437956873}{1675088782825280000}u_{3,1}^2u_{3,2}-$$
$$\frac{1117659694506847966268117264922169}{538176930220570670831358696960000 00}u_{3,2}^3+\frac{7730882033}{54841827620}u_{3,0}u_{3,1}^2+\frac{23544244037889590806056576606779}{10704786751542482581991129600000}u_{3,0}u_{3,2}^2+\frac{423131396695772871698910297}{3912684420448898556932800000}u_{3,2}^2u_{3,3}-\frac{57608456718429732297}{11681647108718139200}u_{3,0}^2u_{3,2}+$$
$$\frac{21102189756216048155978964167121}{36494023862996213790034944320000000}u_{3,2}u_{3,3}^2-\frac{28023870521327646500034450051}{42061357519825659487027600000}u_{3,0}u_{3,2}u_{3,3}-\frac{5944078823039179767}{2336329421743627840}u_{3,0}u_{3,1}u_{3,3}+\frac{779803805130262341084870907540869}{888497300378026054305263756800000}u_{3,1}u_{3,3}u_{3,2}+$$
$$\left.\frac{672102404038906146804563469}{2483140383700550981764280000}u_{3,1}^2u_{3,3}-\frac{1372944098655473706951657397373 7}{29822516008243617290989002800000}u_{3,1}u_{3,3}^2-\frac{176542672683304052381657155289}{2691322193676591013756422240000}u_{3,1}u_{3,2}^2+\frac{6532828008174407727920942563}{12044843289768257034921540000}u_{3,1}u_{3,0}u_{3,2}+u_{3,1}u_{3,0}^2\right\rangle$$

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