Chapter 4

Overview

4.1 Functional Itô Calculus

Many questions in stochastic analysis and its applications in statistics of processes, physics, or mathematical finance involve *path-dependent functionals* of stochastic processes and there has been a sustained interest in developing an analytical framework for the systematic study of such path-dependent functionals.

When the underlying stochastic process is the Wiener process, the Malliavin Calculus [4, 52, 55, 56, 67, 73] has proven to be a powerful tool for investigating various properties of Wiener functionals. The Malliavin Calculus, which is as a weak functional calculus on Wiener space, leads to differential representations of Wiener functionals in terms of *anticipative* processes [5, 37, 56]. However, the interpretation and computability of such anticipative quantities poses some challenges, especially in applications such as mathematical physics or optimal control, where causality, or non-anticipativeness, is a key constraint.

In a recent insightful work, motivated by applications in mathematical finance, Dupire [21] has proposed a method for defining a non-anticipative calculus which extends the Itô Calculus to path-dependent functionals of stochastic processes. The idea can be intuitively explained by first considering the variations of a functional along a *piecewise constant* path. Any (right continuous) piecewise constant path, represented as

$$\omega(t) = \sum_{k=1}^{n} x_k 1_{[t_k, t_{k+1}[}(t),$$

is simply a finite sequence of 'horizonal' and 'vertical' moves, so the variation of a (time dependent) functional $F(t, \omega)$ along such a path ω is composed of

(i) 'horizontal increments': variations of $F(t, \omega)$ between each time point t_i and the next, and

(ii) 'vertical increments': variations of $F(t, \omega)$ at each discontinuity point of ω.

If one can control the behavior of F under these two types of path perturbations, then one can reconstitute its variations along any piecewise constant path. Under additional continuity assumptions, this control can be extended to any càdlàg path using a density argument.

This intuition was formalized by Dupire [21] by introducing directional derivatives corresponding to infinitesimal versions of these variations: given a (time dependent) functional $F\colon [0, T] \times D([0, T], \mathbb{R}) \to \mathbb{R}$ defined on the space $D([0, T], \mathbb{R})$

V. Bally et al., *Stochastic Integration by Parts and Functional Itô Calculus*,
Advanced Courses in Mathematics - CRM Barcelona, DOI 10.1007/978-3-319-27128-6_4

of right continuous paths with left limits, Dupire introduced a directional derivative which quantifies the sensitivity of the functional to a shift in the future portion of the underlying path $\omega \in D([0,T],\mathbb{R})$:

$$\nabla_\omega F(t,\omega) = \lim_{\epsilon \to 0} \frac{F(t,\omega + \epsilon 1_{[t,T]}) - F(t,\omega)}{\epsilon},$$

as well as a time derivative corresponding to the sensitivity of F to a small 'horizontal extension' of the path. Since any càdlàg path may be approximated, in supremum norm, by piecewise constant paths, this suggests that one may control the functional F on the entire space $D([0,T],\mathbb{R})$ if F is twice differentiable in the above sense and $F, \nabla_\omega F, \nabla_\omega^2 F$ are continuous in supremum norm; under these assumptions, one can then obtain a change of variable formula for $F(X)$ for any Itô process X.

As this brief description already suggests, the essence of this approach is pathwise. While Dupire's original presentation [21] uses probabilistic arguments and Itô Calculus, one can in fact do it entirely without such arguments and derive these results in a purely analytical framework without any reference to probability. This task, undertaken in [8, 9] and continued in [6], leads to a pathwise functional calculus for *non-anticipative functionals* which clearly identifies the set of paths to which the calculus is applicable. The pathwise nature of all quantities involved makes this approach quite intuitive and appealing for applications, especially in finance; see Cont and Riga [13].

However, once a probability measure is introduced on the space of paths, under which the canonical process is a semimartingale, one can go much further: the introduction of a reference measure allows to consider quantities which are defined almost everywhere and construct a *weak functional calculus* for stochastic processes defined on the canonical filtration. Unlike the pathwise theory, this construction, developed in [7, 11], is applicable to all square-integrable functionals *without any regularity condition*. This calculus can be seen as a non-anticipative analog of the Malliavin Calculus.

The Functional Itô Calculus has led to various applications in the study of path dependent functionals of stochastic processes. Here, we focus on two particular directions: martingale representation formulas and functional ('path dependent') Kolmogorov equations [10].

4.2 Martingale representation formulas

The representation of martingales as stochastic integrals is an important result in stochastic analysis, with many applications in control theory and mathematical finance. One of the challenges in this regard has been to obtain explicit versions of such representations, which may then be used to compute or simulate such martingale representations. The well-known Clark–Haussmann–Ocone formula [55, 56], which expresses the martingale representation theorem in terms of the Malliavin derivative, is one such tool and has inspired various algorithms for the simulation of such representations, see [33].

One of the applications of the Functional Itô Calculus is to derive explicit, computable versions of such martingale representation formulas, without resorting to the anticipative quantities such as the Malliavin derivative. This approach, developed in [7, 11], leads to simple algorithms for computing martingale representations which have straightforward interpretations in terms of sensitivity analysis [12].

4.3 Functional Kolmogorov equations and path dependent PDEs

One of the important applications of the Itô Calculus has been to characterize the deep link between Markov processes and partial differential equations of parabolic type [2]. A pillar of this approach is the analytical characterization of a Markov process by Kolmogorov's backward and forward equations [46]. These equations have led to many developments in the theory of Markov processes and stochastic control theory, including the theory of controlled Markov processes and their links with viscosity solutions of PDEs [29].

The Functional Itô Calculus provides a natural setting for extending many of these results to more general, non-Markovian semimartingales, leading to a new class of partial differential equations on path space —*functional Kolmogorov equations*— which have only started to be explored [10, 14, 22]. This class of PDEs on the space of continuous functions is distinct from the infinite-dimensional Kolmogorov equations studied in the literature [15]. Functional Kolmogorov equations have many properties in common with their finite-dimensional counterparts and lead to new Feynman–Kac formulas for path dependent functionals of semimartingales [10]. We will explore this topic in Chapter 8. Extensions of these connections to the fully nonlinear case and their connections to non-Markovian stochastic control and forward-backward stochastic differential equations (FBSDEs) currently constitute an active research topic, see for exampe [10, 14, 22, 23, 60].

4.4 Outline

These notes, based on lectures given at the Barcelona Summer School on Stochastic Analysis (2012), constitute an introduction to the foundations and applications of the Functional Itô Calculus.

- We first develop a *pathwise calculus for non-anticipative functionals* possessing some directional derivatives, by combining Dupire's idea with insights from the early work of Hans Föllmer [31]. This construction is purely analytical (i.e., non-probabilistic) and applicable to functionals of paths with finite quadratic variation. Applied to functionals of a semimartingale, it yields a functional extension of the Itô formula applicable to functionals which are continuous in the supremum norm and admit certain directional derivatives. This construction and its various extensions, which are based on [8, 9, 21] are described in Chapters 5 and 6. As a by-product, we obtain a method for constructing pathwise integrals with respect to paths of infinite variation but finite quadratic variation, for a class of integrands which may be described as 'vertical 1-forms'; the connection between this pathwise integral and 'rough path' theory is described in Subsection 5.3.3.

- In Chapter 7 we extend this pathwise calculus to a *'weak' functional calculus* applicable to square-integrable adapted functionals with *no regularity condition* on the path dependence. This construction uses the probabilistic structure of the Itô integral to construct an extension of Dupire's derivative operator to all square-integrable semimartingales and introduce Sobolev spaces of non-anticipative functionals to which weak versions of the functional Itô formula applies. The resulting operator is a weak functional derivative which may be regarded as a *non-anticipative* counterpart of the Malliavin derivative (Section 7.4). This construction, which extends the applicability of the Functional Itô Calculus to a large class of functionals, is based on [11]. The relation with the Malliavin derivative is described in Section 7.4. One of the applications of this construction is to obtain explicit and computable integral representation formulas for martingales (Section 7.2 and Theorem 7.3.4).
- Chapter 8 uses the Functional Itô Calculus to introduce *Functional Kolmogorov equations*, a new class of partial differential equations on the space of continuous functions which extend the classical backward Kolmogorov PDE to processes with path dependent characteristics. We first present some key properties of classical solutions for this class of equations, and their relation with FBSDEs with path dependent coefficients (Section 8.2) and non-Markovian stochastic control problems (Section 8.3). Finally, in Section 8.4 we introduce a notion of weak solution for the functional Kolmogorov equation and characterize square-integrable martingales as weak solutions of the functional Kolmogorov equation.

Notations

For the rest of the manuscript, we denote by

- S_d^+, the set of symmetric positive $d \times d$ matrices with real entries;
- $D([0,T],\mathbb{R}^d)$, the space of functions on $[0,T]$ with values in $\mathbb{R}^d$ which are right continuous functions with left limits (càdlàg); and
- $C^0([0,T],\mathbb{R}^d)$, the space of continuous functions on $[0,T]$ with values in $\mathbb{R}^d$.

Both spaces above are equipped with the supremum norm, denoted $\|\cdot\|_\infty$. We further denote by

- $C^k(\mathbb{R}^d)$, the space of k-times continuously differentiable real-valued functions on $\mathbb{R}^d$;
- $H^1([0,T],\mathbb{R})$, the Sobolev space of real-valued absolutely continuous functions on $[0,T]$ whose Radon–Nikodým derivative with respect to the Lebesgue measure is square-integrable.

For a path $\omega \in D([0,T],\mathbb{R}^d)$, we denote by

- $\omega(t-) = \lim_{s\to t, s<t} \omega(s)$, its left limit at t;
- $\Delta\omega(t) = \omega(t) - \omega(t-)$, its discontinuity at t;
- $\|\omega\|_\infty = \sup\{|\omega(t)|, t \in [0,T]\}$;
- $\omega(t) \in \mathbb{R}^d$, the value of ω at t;
- $\omega_t = \omega(t \wedge \cdot)$, the path stopped at t; and
- $\omega_{t-} = \omega\, 1_{[0,t[} + \omega(t-)\, 1_{[t,T]}$.

Note that $\omega_{t-} \in D([0,T],\mathbb{R}^d)$ is càdlàg and should *not* be confused with the càglàd path $u \mapsto \omega(u-)$. For a càdlàg stochastic process X we similarly denote by

- $X(t)$, its value;
- $X_t = (X(u \wedge t), 0 \le u \le T)$, the process stopped at t; and
- $X_{t-}(u) = X(u)\, 1_{[0,t[}(u) + X(t-)\, 1_{[t,T]}(u)$.

For general definitions and concepts related to stochastic processes, we refer to the treatises by Dellacherie and Meyer [19] and by Protter [62].