# Recent Developments in Deformation Quantization

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**Abstract** In this review an overview on some recent developments in deformation quantization is given. After a general historical overview we motivate the basic definitions of star products and their equivalences both from a mathematical and a physical point of view. Then we focus on two topics: the Morita classification of star product algebras and convergence issues which lead to the nuclear Weyl algebra.

**Keywords** Star products • Deformation quantization • Morita classification • Weyl algebra

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## 1 Introduction: A Historical Tour d'Horizon

In the last decades, deformation quantization evolved into a widely accepted quantization scheme which, on one hand, provides deep conceptual insights into the question of quantization and, on the other hand, proved to be a reliably technique leading to explicit understanding of many examples. It will be the aim of this review to give some overview on the developments of deformation quantization starting from the beginnings but also including some more recent ideas.

The original formulations of deformation quantization by Bayen et. al. aimed mainly at finite-dimensional classical mechanical systems described by symplectic or Poisson manifolds [5] and axiomatized the heuristic quantization formulas found earlier by Weyl, Groenewold and Moyal [56, 73, 89]. Berezin considered the more particular case of bounded domains and Kähler manifolds [7–9]. Shortly after it proved to be a valuable tool to approach also problems in quantum field theories, see e.g. the early works of Dito [41–43].

Meanwhile, the question of existence and classification of deformation quantizations, i.e. of star products, on symplectic manifolds was settled: first DeWilde and Lecomte showed the existence of star products on symplectic manifolds [39] in 1983 after more particular classes [38, 40] had been considered. Remarkably, also in 1983 the first genuine class of Poisson structures was shown to admit star products, the linear Poisson structures on the dual of a Lie algebra, by Gutt [57] and Drinfel'd [47]. In 1986 Fedosov gave a very explicit and constructive way to obtain star products on a symplectic manifold by means of a symplectic connection [53], see also [54, 55] for a more detailed version. His construction is still one of the cornerstones in deformation quantization as it provides not only a particularly nice construction allowing to adjust many special features of star products depending on the underlying manifold like e.g. separation of variables (Wick type) on Kähler manifolds [14, 61, 62, 79] or star products on cotangent bundles [19–21]. Even beyond the symplectic world, Fedosov's construction was used to globalize the existence proofs of star products on Poisson manifolds [36, 44].

Even though the symplectic case was understood well, the question of existence on Poisson manifolds kept its secrets till the advent of Kontsevich's formality theorem, solving his formality conjecture [63, 64, 67]. To give an adequate overview on Kontsevich's formality theorem would clearly go beyond the scope of this short review. Here one can rely on various other publications like e.g. [35, 52]. In a nutshell, the formality theorem proves a very general fact about smooth functions on a manifold from which it follows that every (formal series of) Poisson structures can be quantized into a star product, including a classification of star products. Parallel to Kontsevich's groundbreaking result, the classification of star products on symplectic manifolds was achieved and compared by several groups [10, 37, 58, 77, 78]. Shortly after Kontsevich, Tamarkin gave yet another approach to the quantization problem on Poisson manifolds [84], see also [65, 68], based on the language of operads and the usage of Drinfel'd associators. Starting with these formulations, formality theory has evolved and entered large areas of contemporary mathematics, see e.g. [1–3, 45, 46, 65, 66] to name just a few.

While deformation quantization undoubtedly gave many important contribution to pure mathematics over the last decades, it is now increasingly used in contemporary quantum physics as well: perhaps starting with the works of Dütsch and Fredenhagen on the perturbative formulations of algebraic quantum field theory [49–51] it became clear that star products provide the right tool to formulate quantum field theories in a semiclassical way, i.e. as formal power series in $\hbar$. Now this has been done in increasing generalities for various scenarios including field theories on general globally hyperbolic spacetimes, see e.g. [4, 22, 23, 59].

Of course, from a physical point of view, deformation quantization can not yet be the final answer as one always deals with formal power series in the deformation parameter $\hbar$. A physically reasonable quantum theory, however, requires of course convergence. Again, in the very early works [5] some special cases were treated, namely the Weyl-Moyal product for which an integral formula exists which allows for a reasonable analysis based on the Schwartz space. The aims here are at least two-fold. On one hand one wants to establish a reasonable spectral calculus for particular elements in the star product algebra which allows to compute spectra in a physically sensitive way. This can be done with the star exponential formalism, which works in particular examples but lacks a general framework. On the other hand, one can try to establish form the formal star product a convergent version such that in the end one obtains a $C^*$-algebra of quantum observables being a deformation, now in a continuous way, of the classical functions on the phase space. This is the point of view taken by strict deformation quantization, most notably advocated by Rieffel [81, 82] and Landsman [70], see also [16, 30–33] for the particular case of quantizable Kähler manifolds and [74–76] for more general symplectic manifolds. Bieliavsky and coworkers found a generalization of Rieffel's approach by passing from actions of the abelian group $\mathbb{R}^d$ to more general Lie group actions [11–13]. Having a $C^*$-algebra one has then the full power of $C^*$-algebra techniques at hands which easily allows to get a reasonable spectral calculus. However, constructing $C^*$-algebraic quantizations is still very much in development: here one has not yet a clear picture on the existence and classification of the quantizations. In fact, one even has several competing definitions of what one is looking for. It is one of the ongoing research projects by several groups to understand the transition between formal and strict quantizations in more detail.

Needless to say, in the above historical survey we can barely scratch on the surface of this vast topic: many aspects have not been mentioned like the role played of symmetries and reduction, the applications to concrete physical systems, various generalizations of deformation quantization to other geometric brackets, relations to noncommutative geometry, and many more. In the remaining part of this review we will focus on two aspects of the theory: first, we discuss the role of classification results beyond the notion of equivalence, i.e. isomorphism. Here we are particularly interested in the classification of star products up to Morita equivalence. Second, we give a short outlook on star products in infinite dimensions and problems arising there by investigating one particular example: the Weyl algebra of a vector space with a (quite arbitrary) bilinear form. Beside the purely algebraic construction we obtain a locally convex algebraic deformation once we start in this category.

## 2 From Poisson Manifolds to Star Products

In this section we give a more detailed but still non-technical motivation of the definition of star products and list some first examples.

The set-up will be a finite-dimensional phase space which we model by a symplectic or, more generally, a Poisson manifold $(M, \pi)$ where $\pi \in \Gamma^\infty(\Lambda^2 TM)$ is a bivector field satisfying

$$[\![\pi, \pi]\!] = 0. \tag{1}$$

Here $[\![\cdot\,,\cdot]\!]$ is the Schouten bracket and the condition is equivalent to the Jacobi identity for the Poisson bracket

$$\{f, g\} = -[\![[\![f, \pi]\!], g]\!] = \pi(\mathrm{d}f, \mathrm{d}g) \tag{2}$$

determined by $\pi$ for functions $f, g \in \mathscr{C}^\infty(M)$. One can then formulate classical Hamiltonian mechanics using $\pi$ and $\{\cdot\,,\cdot\}$. For a gentle introduction to Poisson geometry see [87] as well as [34, 48, 71, 85]. There are several important examples of Poisson manifolds:

- Every symplectic manifold $(M, \omega)$, where $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$ is a closed non-degenerate two-form, is a Poisson manifold with $\pi = \omega^{-1}$. The Jacobi identity (1) corresponds then directly to $\mathrm{d}\omega = 0$.
- Every cotangent bundle $T^*Q$ is a symplectic manifold in a canonical way with an exact symplectic form $\omega = \mathrm{d}\theta$ where $\theta \in \Gamma^\infty(T^*(T^*Q))$ is the canonical (or tautological) one-form on $T^*Q$.
- Kähler manifolds are particularly nice examples of symplectic manifolds as they possess a compatible Riemannian metric and a compatible complex structure.
- The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is always a Poisson manifold with a linear Poisson structure: the coefficient functions of the tensor field $\pi$ are linear functions on $\mathfrak{g}^*$, explicitly given by

$$\{f, g\}(x) = x_i c^i_{k\ell} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^\ell}, \tag{3}$$

  where $x_1, \ldots, x_n$ are the linear coordinates on $\mathfrak{g}^*$ and $c^i_{k\ell}$ are the corresponding structure constants of $\mathfrak{g}$. Since (3) vanishes at the origin, this is never symplectic.
- Remarkably and slightly less trivial is the observation that on every manifold $M$, for every $p \in M$ there is a Poisson structure $\pi$ with compact support where $\pi\big|_p$ has maximal rank.

To motivate the definition of a star product we consider the most easy example of the classical phase space $\mathbb{R}^2$ with canonical coordinates $(q, p)$. Canonical quantization says that we have to map the spacial coordinate $q$ to the position operator $Q$ acting on a suitable domain in $L^2(\mathbb{R}, \mathrm{d}x)$ as multiplication operator. Moreover, we have to assign the momentum coordinate $p$ to the momentum operator $P = -\mathrm{i}\hbar \frac{\partial}{\partial q}$, again defined on a suitable domain. Since we want to ignore functional-analytic questions at the moment, we simply chose $\mathscr{C}_0^\infty(\mathbb{R})$ as common domain for both operators. In a next step we want to quantize polynomials in $q$ and $p$ as well. Here we face the ordering problem as $pq = qp$ but $PQ \neq QP$. One simple choice is the *standard ordering*

$$q^n p^m \ \mapsto \ \varrho_{\mathrm{Std}}(q^n p^m) = Q^n P^m = (-\mathrm{i}\hbar)^m q^n \frac{\partial^m}{\partial q^m} \tag{4}$$

for monomials and its linear extension to all polynomials. More explicitly, this gives

$$\varrho_{\mathrm{Std}}(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p^r}\Big|_{p=0} \frac{\partial^r}{\partial q^r}. \tag{5}$$

Now this formula still makes sense for smooth functions $f$ which are polynomial only in $p$, i.e. for $f \in \mathscr{C}^\infty(\mathbb{R})[p]$. The main idea of deformation quantization is now to pull-back the operator product: this is possible since the image of $\varrho_{\mathrm{Std}}$ is the space of all differential operators with smooth coefficients which therefore is a (noncommutative) algebra. We define the *standard-ordered star product* by

$$f \star_{\mathrm{Std}} g = \varrho_{\mathrm{Std}}{}^{-1}(\varrho_{\mathrm{Std}}(f)\varrho_{\mathrm{Std}}(g)) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r} \tag{6}$$

for $f, g \in \mathscr{C}^\infty(\mathbb{R})[p]$. While it is clear that $\star_{\mathrm{Std}}$ is an associative product the behaviour with respect to the complex conjugation is bad: we do not get a $^*$-involution $\overline{f \star_{\mathrm{Std}} g} \neq \overline{g} \star_{\mathrm{Std}} \overline{f}$ since

$$\varrho_{\mathrm{Std}}(f)^\dagger = \varrho_{\mathrm{Std}}(N^2 f) \quad \text{with} \quad N = \exp\left(\frac{\hbar}{2\pi} \frac{\partial^2}{\partial q \partial p}\right), \tag{7}$$

as a simple integration by parts shows. We can repair this unpleasant feature by defining the Weyl ordering and the Weyl product by

$$\varrho_{\mathrm{Weyl}}(f) = \varrho_{\mathrm{Std}}(Nf) \quad \text{and} \quad f \star_{\mathrm{Weyl}} g = N^{-1}(Nf \star_{\mathrm{Std}} Ng). \tag{8}$$

Note that $N$ is indeed an invertible operator on $\mathscr{C}^\infty(\mathbb{R})[p]$. Again, $\star_{\mathrm{Weyl}}$ is associative. Then we get

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f} \quad \text{and} \quad \varrho_{\mathrm{Weyl}}(f \star_{\mathrm{Weyl}} g) = \varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g). \tag{9}$$

For both products we can collect the terms of order $\hbar^r$ which gives

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g) \tag{10}$$

with bidifferential operators $C_r$ of order $r$ in each argument. The explicit formula for $\star_{\mathrm{Weyl}}$ is slightly more complicated than the one for $\star_{\mathrm{Std}}$ in (6) and easily computed. We have

$$f \star g = fg + \cdots \quad \text{and} \quad f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + \cdots, \tag{11}$$

where $+\cdots$ means higher orders in $\hbar$. Also $f \star 1 = f = 1 \star f$. Note also that the seemingly infinite series in (10) is always finite as long as we take functions in $\mathscr{C}^\infty(\mathbb{R})[p]$.

The idea is now to axiomatize these features for $\star$ in such a way that it makes sense to speak of a star product on a general Poisson manifold. The first obstacle is that on a generic manifold $M$ there is nothing like functions which are polynomial in certain coordinates. This is a chart-dependent characterization which one does not want to use. But then already for $\star_{\mathrm{Weyl}}$ and $\star_{\mathrm{Std}}$ one encounters the problem that for general $f, g \in \mathscr{C}^\infty(\mathbb{R}^2)$ the formulas (6) and (9) will not make any sense: the series are indeed infinite and since we can adjust the Taylor coefficients of a smooth function in a rather nasty way, there is no hope for convergence. The way out is to consider *formal* star product in a first step, i.e. formal power series in $\hbar$. This yields the definition of star products [5]:

**Definition 2.1** A formal star product $\star$ on a Poisson manifold $(M, \pi)$ is an associative $\mathbb{C}[[\hbar]]$-bilinear associative product for $\mathscr{C}^\infty(M)[[\hbar]]$ such that

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g) \tag{12}$$

with

1. $C_0(f, g) = fg$,
2. $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$,
3. $C_r(1, f) = 0 = C_r(f, 1)$ for $r \geq 1$,
4. $C_r$ is a bidifferential operator.

Already in the trivial example above we have seen that there might be more than one star product. The operator $N$ interpolates between them and is invisible in classical physics: for $\hbar = 0$ the operator $N$ becomes the identity. As a formal series of differential operators it is invertible and implements an algebra isomorphism. This is now taken as definition for equivalence of star products: given two star products $\star$ and $\star'$ on a manifold, a formal power series $T = \sum_{r=0}^{\infty} \hbar^r T_r$ of differential operators $T_r$ with $T1 = 1$ is called an equivalence between $\star$ and $\star'$

if

$$f \star' g = T^{-1}(Tf \star Tg). \tag{13}$$

Note that $T$ is indeed invertible as a formal power series. Hence this is an equivalence relation. Conversely, given such a $T$ and $\star$ we get a new star product $\star'$ by (13).

We list now some basic examples of star products:

- The explicit formulas for $\star_{\mathrm{Std}}$ and $\star_{\mathrm{Weyl}}$ immediately generalize to higher dimensions yielding equivalent star products on $\mathbb{R}^{2n}$ and hence also on every open subset of $\mathbb{R}^{2n}$. Since by the Darboux Theorem every symplectic manifold looks like an open subset of $\mathbb{R}^{2n}$ *locally*, the question of existence of star products on symplectic manifolds is a global problem.
- For the linear Poisson structure (3) on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ one gets a star product as follows [57]: First, we note that the symmetric algebra $\mathrm{S}^\bullet(\mathfrak{g})$ over $\mathfrak{g}$ can be canonically identified with the polynomials $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ on the dual $\mathfrak{g}^*$. Then the PBW isomorphism

$$\mathrm{S}^\bullet(\mathfrak{g}) \ni \xi_1 \vee \cdots \vee \xi_k \;\mapsto\; \frac{(\mathrm{i}\hbar)^k}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} \in \mathcal{U}(\mathfrak{g}) \tag{14}$$

  from the symmetric algebra over $\mathfrak{g}$ into the universal enveloping algebra allows to pull the product of $\mathcal{U}(\mathfrak{g})$ back to $\mathrm{S}^\bullet(\mathfrak{g})$ and hence to polynomials on $\mathfrak{g}^*$. One can now show that after interpreting $\hbar$ as a formal parameter one indeed obtains a star product quantizing the linear Poisson bracket. This star product is completely characterized by the feature that

$$\exp(\hbar\xi) \star \exp(\hbar\eta) = \exp(\mathrm{BCH}(\hbar\xi, \hbar\eta)) \tag{15}$$

  for $\xi, \eta \in \mathfrak{g}$ with the Baker-Campbell-Hausdorff series BCH, see [19, 57].
- The next interesting example is perhaps the complex projective space $\mathbb{CP}^n$ and its non-compact dual, the Poincaré disc $\mathbb{D}_n$ with their canonical Kähler structures of constant holomorphic sectional curvature. For these, star products were considered by Moreno and Ortega-Navarro [72] who gave recursive formulas using local coordinates. Cahen, Gutt, and Rawnsley [30–33] discussed this in their series of papers of quantization of Kähler manifolds as one of the examples. The first explicit (non-recursive) formula was found in [17, 18] by a quantization of phase space reduction and extended to complex Grassmannians in [83]. Ever since these star products have been re-discovered by various authors.

We briefly comment on the general existence results: as already mentioned, the symplectic case was settled in the early 1980s. The Poisson case follows from Kontsevich's formality theorem.

**Theorem 2.2 (Kontsevich)** *On every Poisson manifold there exist star products.*

The classification is slightly more difficult to describe: we consider *formal Poisson structures*

$$\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]] \quad \text{with} \quad [\![\pi, \pi]\!] = 0. \tag{16}$$

Moreover, let $X = \hbar X_1 + \hbar^2 X_2 + \cdots \in \hbar\Gamma^\infty(TM)[[\hbar]]$ be a formal vector field, starting in first order of $\hbar$. Then one calls $\exp(\mathscr{L}_X)$ a formal diffeomorphism which defines an action

$$\exp(\mathscr{L}_X)\colon \Gamma^\infty(\Lambda^2 TM)[[\hbar]] \ni \nu \mapsto \nu + \mathscr{L}_X\nu + \frac{1}{2}\mathscr{L}_X^2\nu + \cdots \in \Gamma^\infty(\Lambda^2 TM)[[\hbar]]. \tag{17}$$

Via the Baker-Campbell-Hausdorff series, the set of formal diffeomorphisms becomes a group and (17) is a group action. Since $\mathscr{L}_X$ is a derivation of the Schouten bracket, it follows that the action of $\exp(\mathscr{L}_X)$ preserves formal Poisson structures. The space of orbits of formal Poisson structures modulo this group action gives now the classification:

**Theorem 2.3 (Kontsevich)** *The set of equivalence classes of formal star products is in bijection to the set of equivalence classes of formal Poisson structures modulo formal diffeomorphisms.*

In general, both moduli spaces are extremely difficult to describe. However, if the first order term $\pi_1$ in $\pi$ is symplectic, then we have a much easier description which is in fact entirely topological:

**Theorem 2.4 (Bertelson, Cahen, Gutt, Nest, Tsygan, Deligne, ...)** *On a symplectic manifold $(M, \omega)$ the equivalence classes of star products are in bijection to the formal series in the second deRham cohomology. In fact, one has a canonical surjective map*

$$c\colon \star \mapsto c(\star) \in \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]] \tag{18}$$

*such that $\star$ and $\star'$ are equivalent iff $c(\star) = c(\star')$.*

This map is now called the characteristic class of the symplectic star product. In a sense which can be made very precise [29], the inverse of $c(\star)$ corresponds to Kontsevich's classification by formal Poisson tensors.

## 3 Morita Classification

We come now to some more particular topics in deformation quantization. In this section we discuss a coarser classification result than the above classification up to equivalence.

The physical motivation to look for Morita theory is rather simple and obvious: in quantum theory we can not solely rely on the observable algebra as the only object of interest. Instead we also need to have a reasonable notion of states. While for $C^*$-algebras there is a simple definition of a state as a normalized positive functional, in deformation quantization we do not have $C^*$-algebras in a first step. Surprisingly, the notion of positive functionals still makes sense if interpreted in the sense of the ring-ordering of $\mathbb{R}[[\hbar]]$ and produces a physically reasonable definition of states, see [15]. However, the requirements from quantum theory do not stop here: we also need a super-position principle for states. Since positive functionals can only be added convexly, we need to realize the positive functionals as expectation value functionals for a $^*$-representation of the observable algebra on some (pre-) Hilbert space. Then we can take complex linear combination of the corresponding vectors to implement the super-position principle. This leads to the need to understand the representation theory of the star product algebras, a program which was investigated in great detail [24, 25, 27–29, 60], see also [86] for a review. The main point is that replacing the ring of scalars from $\mathbb{R}$ to $\mathbb{R}[[\hbar]]$ and thus from $\mathbb{C}$ to $\mathbb{C}[[\hbar]]$ works surprisingly well as long as we do not try to implement analytic concepts: the non-archimedean order of $\mathbb{R}[[\hbar]]$ forbids a reasonable analysis. However, the concept of positivity is entirely algebraic and hence can be used and employed in this framework as well.

In fact, one does not need to stop here: *any* ordered ring $\mathsf{R}$ instead of $\mathbb{R}$ will do the job and one can study $^*$-algebras over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ and their $^*$-representation theory on pre Hilbert modules over $\mathsf{C}$. For many reasons it will also be advantageous to consider representation spaces where the inner product is not taking values in the scalars but in some *auxiliary* $^*$-algebra $\mathcal{D}$.

*Example* Let $E \longrightarrow M$ be a complex vector bundle over a smooth manifold $M$. Then $\Gamma^\infty(E)$ is a $\mathscr{C}^\infty(M)$-module in the usual way. A Hermitian fiber metric $h$ now gives a sesquilinear map

$$\langle \cdot\,, \cdot \rangle : \Gamma^\infty(E) \times \Gamma^\infty(E) \longrightarrow \mathscr{C}^\infty(M) \tag{19}$$

which is also $\mathscr{C}^\infty(M)$-linear in the second argument, i.e. we have $\langle s, tf \rangle = \langle s, t \rangle f$ for all $s, t \in \Gamma^\infty(E)$ and $f \in \mathscr{C}^\infty(M)$. Moreover, the pointwise positivity of $h_p$ on $E_p$ implies that the map

$$\langle \cdot\,, \cdot \rangle^{(n)} : \Gamma^\infty(E)^n \times \Gamma^\infty(E)^n \longrightarrow M_n(\mathscr{C}^\infty(M)) = \mathscr{C}^\infty(M, M_n(\mathbb{C})) \tag{20}$$

is positive for all $n$ in the sense that the matrix-valued function $\langle S, S \rangle^{(n)} \in \mathscr{C}^\infty(M, M_n(\mathbb{C}))$ yields a positive matrix at all points of $M$ for all $S = (s_1, \ldots, s_n) \in \Gamma^\infty(E)^n$.

Using this kind of *complete positivity* for an inner product yields the definition of a pre Hilbert right module over a $^*$-algebra $\mathcal{D}$, where the inner product takes values in $\mathcal{D}$. Then again, we can formulate what are $^*$-representations of a $^*$-algebra $\mathcal{A}$ on such a pre Hilbert right module over $\mathcal{D}$. Without further difficulties this gives various categories of $^*$-representations of $^*$-algebras on inner product modules or pre Hilbert modules over auxiliary $^*$-algebras.

Having a good notion of $^*$-representations of $^*$-algebras it is a major talks to understand the resulting categories for those $^*$-algebras occurring in deformation quantization. From $C^*$-algebra theory we anticipate that already with the full power of functional-analytic techniques it will in general be impossible to "understand" the category of $^*$-representations completely, beside rather trivial examples. The reason is that there will simply be too many inequivalent such $^*$-representations and a decomposition theory into irreducible ones is typically an extremely hard problem. In a purely algebraic situation like for formal star product algebras, things are even worse: here we expect even more inequivalent ones which are just artifacts of the algebraic formulation. There are many examples of inequivalent $^*$-representations which, after one implements mild notions of convergence and hence of analytic aspects, become equivalent. From a physical point of view such inequivalences would then be negligible. However, it seems to be quite difficult to decide this *before* convergence is implemented, i.e. on the algebraic side.

Is the whole program now useless, hopeless? The surprising news is that one can indeed say something non-trivial about the $^*$-representation theories of the star product algebras from deformation quantization, and for $^*$-algebras in general. The idea is that even if the $^*$-representation of a given $^*$-algebra is horribly complicated and contains maybe unwanted $^*$-representations, we can still *compare* the whole $^*$-representation theory of one $^*$-algebra to another $^*$-algebra and ask whether they are equivalent as categories.

This is now the basic task of Morita theory. To get a first impression we neglect the additional structure of ordered rings, $^*$-involutions, and positivity and consider just associative algebras over a common ring of scalars. For two such algebras $\mathcal{A}$ and $\mathcal{B}$ we want to know whether their categories of left modules are equivalent categories. Now there might be many very strange functors implementing an equivalence and hence one requires them to be compatible with direct sums of modules, which is clearly a reasonable assumption. The prototype of such a functor is then given by the tensor product with a $(\mathcal{B}, \mathcal{A})$-bimodule. Since the tensor product with $\mathcal{A}$ itself is (for unital algebras) naturally isomorphic to the identity functor and since the tensor product of bimodules is associative up to a natural isomorphism, the question of equivalence of categories via such tensor product functors becomes equivalent to the question of *invertible bimodules*: Here a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is called invertible if there is an $(\mathcal{A}, \mathcal{B})$-bimodule ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ such that the tensor product ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ is isomorphic to $\mathcal{B}$ and ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is isomorphic to $\mathcal{A}$, always as bimodules.

The classical theorem of Morita now gives a complete and fairly easy description of the possible bimodules with this property: ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ has to be a finitely generated projective and full right $\mathcal{A}$-module and $\mathcal{B}$ is isomorphic to $\mathsf{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ via the left module structure, see e.g. [69].

Now the question is how such bimodules look like for star product algebras. Classically, the finitely generated projective modules over $\mathscr{C}^\infty(M)$ are, up to isomorphism, just sections $\Gamma^\infty(E)$ of a vector bundle $E \longrightarrow M$. This is the famous Serre-Swan theorem in its incarnation for differential geometry. As soon as the fiber dimension is non-zero, the fullness condition is trivially satisfied. Hence the only Morita equivalent algebras to $\mathscr{C}^\infty(M)$ are, again up to isomorphism, the sections $\Gamma^\infty(\mathsf{End}(E))$ of endomorphism bundles. The corresponding bimodule is then $\Gamma^\infty(E)$ on which both algebras act in the usual way. It now requires a little argument to see that for star products, an equivalence bimodule gives an equivalence bimodule in the classical limit $\hbar = 0$, i.e. a vector bundle. Conversely, the sections of every vector bundle can be deformed into a right module over the star product algebra in a unique way up to isomorphism. Thus for star products, we have to look for the corresponding module endomorphisms of such deformed sections of vector bundles. Finally, in order to get again a star product algebra, the endomorphisms of the deformed sections have to be, in the classical limit, isomorphic to the functions on a manifold again. This can only happen if the vector bundle was actually a line bundle over the same manifold. Hence the remaining task is to actually compute the star product of the algebra acting from the left side when the star product for the algebra on the right side is known. Here one has the following results:

**Theorem 3.1 (Bursztyn, W. [26])** *Let $(M, \omega)$ and $(M', \omega')$ be a symplectic manifolds and let $\star$, $\star'$ be two star products on $M$ and $M'$, respectively. Then $\star$ and $\star'$ are Morita equivalent iff there exists a symplectomorphism $\psi: M \longrightarrow M'$*

$$\psi^* c(\star') - c(\star) \in 2\pi \mathrm{i} \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{Z}). \tag{21}$$

*The difference of the above classes defines then a line bundle which implements the Morita equivalence bimodule by deforming its sections.*

This theorem already has an important physical interpretation: for cotangent bundles $T^*Q$ the characteristic classes $c(\star)$ can be interpreted as the classes of magnetic fields $B$ on the configuration space $Q$. Then a quantization of a charged particle in the background field of such a $B$ requires a star product with characteristic class $c(\star)$. Compared to the trivial characteristic class, $c(\star) = 0$, the above theorem then tells that quantization with magnetic field has the same representation theory iff the magnetic field satisfies the integrality condition for a Dirac monopole. Thus we get a Morita theoretic interpretation of the charge quantization for magnetic monopoles which is now extremely robust against details of the quantization procedure: the statement holds for all cotangent bundles and for all equivalent star products with the given characteristic class.

Also in the more general Poisson case the full classification is known. Here the actual statement is slightly more technical as it requires the Kontsevich class of the star products and a canonically given action of the deRham cohomology on equivalence classes of formal Poisson structures by gauge transformations. Then one obtains the following statement:

**Theorem 3.2 (Bursztyn, Dolgushev, W. [29])** *Star products on Poisson manifolds are Morita equivalent iff their Kontsevich classes of formal Poisson tensors are gauge equivalent by a $2\pi\mathrm{i}$-integral deRham class.*

## 4 Beyond Formal Star Products

Since formal star products are clearly not sufficient for physical purposes, one has to go beyond formal power series. Here several options are available: on one hand one can replace the formal series in the star products by integral formulas. The formal series can then be seen as the asymptotic expansions of the integral formulas in the sense of Taylor series of smooth functions of $\hbar$, which are typically not analytic: hence we cannot expect convergence. Nevertheless, the integral formulas allow for a good analytic framework.

However, if one moves to field theories and hence to infinite-dimensional systems, quantization becomes much more complicated. Surprisingly, series formulas for star products can still make sense in certain examples, quite unlike the integral formulas: such integrals would consist of integrations over a infinite-dimensional phase space. Hence we know that such things can hardly exist in a mathematically sound way.

This motivates the second alternative, namely to investigate the formal series in the star products directly without integral formulas in the back. This might also be possible in infinite dimensions and yield reasonable quantizations there. While this is a program far from being understood, we now present a class of examples with a particular physical relevance: the Weyl algebra.

Here we consider a real vector space $V$ with a bilinear map $\Lambda\colon V \times V \longrightarrow \mathbb{C}$. Then we consider the complexified symmetric algebra $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ of $V$ and interpret this as the polynomials on the dual $V^*$. In finite dimensions this is correct, in infinite dimensions the symmetric algebra is better to be interpreted as the polynomials on the (not necessarily existing) pre-dual. On $V^*$, there are simply much more polynomials than the ones arising from $\mathrm{S}^\bullet_{\mathbb{C}}(V)$. Now we can extend $\Lambda$ to a biderivation

$$P_\Lambda\colon \mathrm{S}^\bullet_{\mathbb{C}}(V) \otimes \mathrm{S}^\bullet_{\mathbb{C}}(V) \longrightarrow \mathrm{S}^\bullet_{\mathbb{C}}(V) \otimes \mathrm{S}^\bullet_{\mathbb{C}}(V) \tag{22}$$

in a unique way by enforcing the Leibniz rule in both tensor factors. If we denote by $\mu\colon \mathrm{S}^\bullet_{\mathbb{C}}(V) \otimes \mathrm{S}^\bullet_{\mathbb{C}}(V) \longrightarrow \mathrm{S}^\bullet_{\mathbb{C}}(V)$ the symmetric tensor product, then

$$\{a, b\}_\Lambda = \mu \circ (P_\Lambda(a \otimes b) - P_\Lambda(b \otimes a)) \tag{23}$$

is a Poisson bracket. In fact, this is the unique constant Poisson bracket with the property that for linear elements $v, w \in V$ we have $\{v, w\} = \Lambda(v, w) - \Lambda(w, v)$. Hence the antisymmetric part of $\Lambda$ determines the bracket. However, we will use the symmetric part for defining the star product. This will allow to include also standard-orderings or other orderings like Wick ordering from the beginning.

A star product quantizing this constant Poisson structure can then be found easily. We set

$$a \star b = \mu \circ \exp(zP_\Lambda)(a \otimes b) \tag{24}$$

where $z \in \mathbb{C}$ is the deformation parameter. For physical applications we will have to set $z = \frac{i\hbar}{2}$ later on. Note that $\star$ is indeed well-defined since on elements in the symmetric algebra, the operator $P_\Lambda$ lowers the degree by one in each tensor factor.

In a next step we want to extend this product to more interesting functions than the polynomial-like ones. The strategy is to look for a topology which makes the product continuous and which allows for a large completion of $\mathrm{S}^\bullet_{\mathbb{C}}(V)$. To start with, one has to assume that $V$ is endowed with a topology itself. Hence let $V$ be a locally convex Hausdorff space. In typical examples from quantum mechanics, $V$ is the (dual of the) phase space and hence finite dimensional, which makes the topology unique. In quantum field theory, $V$ would be something like test function spaces, i.e. either the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ or $\mathscr{C}^\infty_0(M)$ for a manifold $M$, etc. In this case $V$ would be a Fréchet or LF space.

We use now the continuous seminorms of $V$ to extend them to tensor powers $V^{\otimes k}$ for all $k \in \mathbb{N}$ by taking their tensor powers: we equip $V^{\otimes k}$ with the $\pi$-topology inherited from $V$. This means that for a continuous seminorm $p$ on $V$ we consider $p^{\otimes k}$ on $V^{\otimes k}$ and take all such seminorms to define a locally convex topology on $V^{\otimes k}$. Viewing the symmetric tensor powers as a subspace, this induces the $\pi$-topology also for $\mathrm{S}^\bullet_{\mathbb{C}}(V)$, simply by restricting the seminorms $p^{\otimes k}$. For the whole symmetric algebra we need to extend the seminorms we have on each symmetric degree. This can be done in many inequivalent ways. Useful for our purposes is the following construction. We fix a parameter $R \geq \frac{1}{2}$ and define

$$p_R(a) = \sum_{k=0}^{\infty} k!^R p^{\otimes k}(a_k) \tag{25}$$

for every $a = \sum_{k=0}^{\infty} a_k$ with $a_k \in \mathrm{S}^k_{\mathbb{C}}(V)$. Note that the sum is finite as long as we take $a$ in the symmetric algebra. Now taking all those seminorms $p_R$ for all continuous seminorms $p$ of $V$ induces a locally convex topology on $V$. Clearly, this is again Hausdorff. Moreover, all $\mathrm{S}^k_{\mathbb{C}}(V)$ are closed embedded subspaces in $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ with respect to this topology.

The remarkable property of this topology is now that a continuous $\Lambda$ will induce a continuous star product [88]:

**Theorem 4.1** *Let* $\Lambda: V \times V \longrightarrow \mathbb{C}$ *be a continuous bilinear form on* $V$*. Then* $\star$ *is a continuous associative product on* $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ *with respect to the locally convex topology induced by all the seminorms* $p_R$ *with* $p$ *being a continuous seminorm on* $V$*, as long as* $R \geq \frac{1}{2}$.

The proof consists in an explicit estimate for $a \star b$. Note that the topology can *not* be locally multiplicatively convex since in the Weyl algebra we have elements satisfying canonical commutation relations, thereby forbidding a submultiplicative seminorm.

**Definition 4.2 (Locally convex Weyl algebra)** Let $\Lambda: V \times V \longrightarrow \mathbb{C}$ be a continuous bilinear form on $V$. Then the completion of $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ with respect to the above locally convex topology and with the canonical extension of $\star$ is called the locally convex Weyl algebra $\mathcal{W}_R(V, \star)$.

Thus we have found a framework where the Weyl star product actually converges. Without proofs we list a few properties of this Weyl algebra:

- The locally convex Weyl algebra $\mathcal{W}_R(V, \star)$ is a locally convex unital associative algebra. The product $a \star b$ can be written as the absolutely convergent series

$$a \star b = \mu \circ \exp(zP_\Lambda)(a \otimes b). \tag{26}$$

- The product $\star$ depends holomorphically on $z \in \mathbb{C}$.
- For $\frac{1}{2} \leq R < 1$ the locally convex Weyl algebra $\mathcal{W}_R(V, \star)$ contains the exponential functions $\mathrm{e}^{\alpha v}$ for all $v \in V$ and all $\alpha \in \mathbb{C}$. They satisfy the usual Weyl relations. Note that not only the unitary ones, i.e. for $\alpha$ imaginary, are contained in the Weyl algebra, but all exponentials.
- The locally convex Weyl algebra is nuclear iff $V$ is nuclear. In all relevant examples in quantum theory this will be the case. In this case we refer to the *nuclear Weyl algebra*.
- If $V$ admits an absolute Schauder basis, then the symmetrized tensor products of the basis vectors constitute an absolute Schauder basis for the Weyl algebra, too. Again, in many situations $V$ has such a basis.
- The Weyl algebras for different $\Lambda$ on $V$ are isomorphic if the antisymmetric parts of the bilinear forms coincide.
- Evaluations at points in the topological dual $V'$ are continuous linear functionals on $\mathcal{W}_R(V, \star)$. Hence we still can view the elements of the completion as particular functions on $V'$.
- The translations by elements in $V'$ still act on $\mathcal{W}_R(V, \star)$ by continuous automorphisms. If $R < 1$ these translations are inner automorphism as soon as the element $\varphi \in V'$ is in the image of the musical map induced by $\Lambda$.

We now conclude this section with a few comments on examples. First it is clear that in finite dimensions we can take $V = \mathbb{R}^{2n}$ with the canonical Poisson bracket on the symmetric algebra. Then many types of orderings can be incorporated in fixing the symmetric part of $\Lambda$, while the antisymmetric part is given by the Poisson bracket. Thus all the resulting star products allow for this analytic framework. This includes examples known earlier in the literature, see e.g. [6, 80]. In this case we get a nuclear Weyl algebra with an absolute Schauder basis.

More interesting is of course the infinite dimensional case. Here we have to specify the space $V$ and the bilinear form $\Lambda$ more carefully. In fact, the *continuity* of $\Lambda$ becomes now a strong conditions since bilinear maps in locally convex analysis tend to be only separately continuous without being continuous. However, there are several situations where we can either conclude the continuity of a bilinear separately continuous map by abstract arguments, like for Fréchet spaces. Or one can show directly that the particular bilinear form one is interested in is continuous. We give one of the most relevant examples for (quantum) field theory:

*Example* Let $M$ be a globally hyperbolic spacetime and let $D$ be a normally hyperbolic differential operator acting on a real vector bundle $E$ with fiber metric $h$. Moreover, we assume that $D$ is a connection Laplacian for a metric connection with respect to $h$ plus some symmetric operator $B$ of order zero. In all relevant examples this is easy to obtain. Then one has advanced and retarded Green operators leading to the propagator $F_M$ acting on test sections $\Gamma_0^\infty(E^*)$. We take $V = \Gamma_0^\infty(E^*)$ with its usual LF topology. Then

$$\Lambda(\varphi, \psi) = \int_M h^{-1}(F_M(\varphi), \psi)\mu_g \tag{27}$$

is the bilinear form leading to the Peierls bracket on the symmetric algebra $S^\bullet(V)$. Here $\mu_g$ is the metric density as usual. The kernel theorem then guarantees that $\Lambda$ is continuous as needed. Thus we obtain a locally convex and in fact nuclear Weyl algebra from this. Now $\Lambda$ is highly degenerated. It follows that in the Poisson algebra there are many Casimir elements. The kernel of $F_M$ generates a Poisson ideal and also an ideal in the Weyl algebra, which coincides with the vanishing ideal of the solution space. Hence dividing by this (Poisson) ideal gives a Poisson algebra or Weyl algebra which can be interpreted as the observables of the (quantum) field theory determined by the wave equation $Du = 0$. It can then be shown that for every Cauchy surface $\Sigma$ in $M$ there is a canonical algebra isomorphism to the Weyl algebra build from the symplectic Poisson algebra on the initial conditions on $\Sigma$. Details of this construction can be found in [88], see also [4] for the background information on the wave equation.

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