Chapter 1
Introduction: How Does Nontrivial Network Connectivity Affect Dynamical Processes on Networks?

When studying a dynamical process, one is concerned with its behavior as a function of time, space, and its parameters. There are numerous studies that examine how many people are infected by a biological contagion and whether it persists from one season to another, whether and to what extent interacting oscillators synchronize, whether a meme on the internet becomes viral or not, and more. These studies all have something in common: the dynamics are occurring on a set of discrete entities (the *nodes* in a network) that are connected to each other via *edges* in some nontrivial way. This leads to the natural question of how such underlying nontrivial connectivity affects dynamical processes. This is one of the most important questions in network science [228], and it is the core question that we consider in our tutorial.

Traditional studies of continuous dynamical systems are concerned with qualitative methods to study coupled ordinary differential equations (ODEs) [127, 292] and/or partial differential equations (PDEs) [63, 65], and traditional studies of discrete dynamical systems take analogous approaches with maps [127, 292].[1] If the state of each node in a network is governed by its own ODE (or PDE or map), then studying a dynamical process on a network entails examining a (typically large) system of coupled ODEs (or PDEs or maps). The change in state of a node depends not only on its own current state but also on the current states of its neighboring nodes, and a network encodes which nodes interact with each other and how strongly they interact.[2]

[1]Of course, nothing is stopping us from placing more complicated dynamical processes—which can be governed by stochastic differential equations, delay differential equations, or something else—on a network.

[2]In addition to current states, one can also incorporate dependencies on some of the previous states or even on entire state histories. As suggested both in this footnote and in the previous one, it is possible to envision scenarios that are seemingly arbitrarily complicated.

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An area of particular interest (because of tractability and seeming simplicity) is binary-state dynamics on nodes, whose states depend on the states of their neighboring nodes and which often have stochastic update rules. (Dynamical processes with more than two states are obviously also interesting.) Examples include simple models of disease spread, where each node is considered to be in either a healthy (*susceptible*) state or an unhealthy (*infected*) state, and infections are transmitted probabilistically along the edges of a network. One can apply approximation methods, such as mean-field approaches, to obtain (relatively) low-dimensional descriptions of the global behavior of the system—e.g., to predict the expected number of infected people in a network at a given time or as a function of time—and these methods can yield ODE systems that are amenable to analysis via standard approaches from the theory of dynamical systems.

Importantly, it is true not only that network structure can affect dynamical processes on a network, but also that dynamical processes can affect the dynamics of the network itself. For example, when a child gets the flu, he/she might not go to school for a couple of days, and this temporary change in human activity affects which social contacts take place, which can in turn affect the dynamics of disease propagation. We will briefly discuss the interactions of dynamics on networks with dynamics of networks (these are sometimes called "adaptive networks" [124, 272]) in this monograph, but we will mostly assume time-independent network connectivity so that we can focus on the question of how network structure affects dynamical processes that occur on top of a network. Whether this is reasonable for a given situation depends on the relative time scales of the dynamics on the network and the dynamics of the network.

The remainder of our tutorial is organized as follows. Before delving into dynamics, we start by recalling a few basic concepts in Chapter 2. In Chapter 3, we discuss several examples of dynamical systems on networks. In Chapter 4, we give various theoretical considerations for general dynamical systems on networks as well as for several of the systems on which we focus. We overview software implementations in Chapter 5. In Chapter 6, we briefly examine dynamical systems on dynamical (i.e., time-dependent) networks, and we recommend several resources for further reading in Chapter 7. Finally, we conclude and discuss some open problems and current research efforts in Chapter 8.