Chapter 7
Almost Periodic Solutions of Evolution Differential Equations with Impulsive Action

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Abstract In an abstract Banach space we study conditions for the existence of piecewise continuous, almost periodic solutions for semilinear impulsive differential equations with fixed and nonfixed moments of impulsive action.

7.1 Introduction

We consider the problem of the existence of piecewise continuous, almost periodic solutions for the linear impulsive differential equation

$$\frac{du}{dt} + (A + A_1(t)u = f(t,u), \quad t \neq \tau_j(u), \tag{7.1}$$

$$u(\tau_j(u) + 0) - u(\tau_j(u)) = B_j u + g_j(u), \quad j \in Z, \tag{7.2}$$

where $u : R \to X$, X is a Banach space, A is a sectorial operator in X, $A_1(t)$ is some operator-valued function, $\{B_j\}$ is a sequence of some closed operators, and $\{\tau_j(u)\}$ is an unbounded and strictly increasing sequence of real numbers for all u from some domain of space X.

We use the concept of piecewise continuous, almost periodic functions proposed in [7]. Points of discontinuities of these functions coincide with points of impulsive actions $\{\tau_j\}$. We mention the remarkable paper [18], where a number of important statements about the almost periodic pulse system were proved. Then these results were included in the well-known monograph [19]. Today there are many articles related to the study of almost periodic impulsive systems (see, for example, [1, 3]). In the papers [8, 23, 27, 28] almost periodic solutions for abstract impulsive differential equations in the Banach space are investigated.

In this chapter we consider the semilinear abstract impulsive differential equation in a Banach space with sectorial operator in the linear part of the equation and

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some closed operators in linear parts of impulsive action. Using fractional powers of operator A and corresponding interpolation spaces allows us to consider strong or classical solutions. Note that such equations with periodic right-hand sides were first studied in [17]. In equations with nonfixed moments of impulsive action, points of discontinuity depend on solutions; that is, every solution has its own points of discontinuity. Moreover, a solution can intersect the surface of impulsive action several times or even an infinite number of times. This is the so-called pulsation or beating phenomenon. We will assume that solutions of (7.1) and (7.2) don't have beating at the surfaces $t = \tau_j(u)$; in other words, solutions intersect each surface no more than once. For impulsive systems in the finite-dimensional case, there are several sufficient conditions that allow us to exclude the phenomenon of pulsation (see, [19], [22]). Unfortunately, in a Banach space this conditions cannot easily be verified. In every concrete case one needs a separate investigation.

We assume that the corresponding linear homogeneous equation (if $f \equiv 0$, $g_j \equiv 0$) has an exponential dichotomy. The definition of exponential dichotomy for an impulsive evolution equation corresponds to the definition of exponential dichotomy for continuous evolution equations in an infinite-dimensional Banach space [5, 9, 16]. We require that only solutions of a linear system from an unstable manifold be unambiguously extended to the negative semiaxis.

Robustness is an impotent property of the exponential dichotomy [5, 10, 16]. We mention the papers [4, 14, 25, 26], where the robustness of the exponential dichotomy for impulsive systems by small perturbations of right-hand sides is proved. In this chapter we prove robustness of the exponential dichotomy also by the small perturbation of points of impulsive action. We use a change of time in the system. Then approximation of the impulsive system by difference systems (see [9]) can be used. If a linear homogeneous equation is exponentially stable, we prove stability of the almost periodic solution of nonlinear equations (7.1) and (7.2). Following [17], we use the generalized Gronwall inequality, taking into account singularities in integrals and impulsive influences.

This chapter is organized as follows. In Sect. 7.2 we present some preliminary definitions and results. In Sect. 7.3, we study an exponential dichotomy of impulsive linear equations. Section 7.4 is devoted to studying the existence and stability of almost periodic solutions in linear inhomogeneous equations with impulsive action and semilinear impulsive equations with fixed moments of impulsive action. In Sect. 7.5 we consider impulsive evolution equations with nonfixed moments of impulsive action. In Sect. 7.6 we discuss the case of unbounded operators B_j in linear parts of linear parts of impulsive action.

7.2 Preliminaries

Let $(X, \|.\|)$ be an abstract Banach space and R and Z be the sets of real and integer numbers, respectively.

We consider the space $\mathscr{PC}(J, X)$, $J \subset R$, of all piecewise continuous functions $x : J \to X$ such that

i) the set $\{\tau_j \in J : \tau_{j+1} > \tau_j, j \in Z\}$ of discontinuities of x has no finite limit points;
ii) $x(t)$ is left-continuous $x(\tau_j + 0) = x(\tau_j)$ and there exists $\lim_{t\to\tau_j-0} x(t) = x(\tau_j - 0) < \infty$.

We will use the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|$, in the space $\mathscr{PC}(J,X)$.

Definition 1. The integer p is called an ε-almost period of a sequence $\{x_k\}$ if $\|x_{k+p} - x_k\| < \varepsilon$ for any $k \in Z$. The sequence $\{x_k\}$ is almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε-almost periods.

Definition 2. The strictly increasing sequence $\{\tau_k\}$ of real numbers has uniformly almost periodic sequences of differences if for any $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods common for all sequences $\{\tau_k^j\}$, where $\tau_k^j = \tau_{k+j} - \tau_k, j \in Z$.

By Samoilenko and Trofimchuk [21], the sequence $\{\tau_k\}$ has uniformly almost periodic sequences of differences if and only if $\tau_k = ak + c_k$, where $\{c_k\}$ is an almost periodic sequence and a is a positive real number.

By Lemma 22 ([19], p. 192), for a sequence $\{\tau_j\}$ with uniformly almost periodic sequences of differences there exists the limit

$$\lim_{T\to\infty} \frac{i(t,t+T)}{T} = p \tag{7.3}$$

uniformly with respect to $t \in R$, where $i(s,t)$ is the number of the points τ_k lying in the interval (s,t). Then for each $q > 0$ there exists a positive integer N such that on each interval of length q there are no more than N elements of the sequence $\{\tau_j\}$; that is, $i(s,t) \leq N(t-s) + N$.

Also, for sequence $\{\tau_j\}$ with uniformly almost periodic sequences of differences there exists $\Theta > 0$ such that $\tau_{j+1} - \tau_j \leq \Theta, j \in Z$.

Definition 3. The function $\varphi \in \mathscr{PC}(R,X)$ is said to be W-almost periodic if

i) the strictly increasing sequence $\{\tau_k\}$ of discontinuities of $\varphi(t)$ has uniformly almost periodic sequences of differences;
ii) for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$;
iii) for any $\varepsilon > 0$ there exists a relatively dense set Γ of ε-almost periods such that if $\tau \in \Gamma$, then $\|\varphi(t+\tau) - \varphi(t)\| < \varepsilon$ for all $t \in R$ that satisfy the condition $|t - t_k| \geq \varepsilon, k \in Z$.

We consider the impulsive equations (7.1) and (7.2) with the following assumptions:

(H1) A is a sectorial operator acting in X and $\inf\{Re\mu : \mu \in \sigma(A)\} \geq \delta > 0$, where $\sigma(A)$ is the spectrum of A. Consequently, the fractional powers of A are well defined, and one can consider the spaces $X^\alpha = D(A^\alpha)$ for $\alpha \geq 0$ endowed with the norms $\|x\|_\alpha = \|A^\alpha x\|$.

(H2) The function $A_1(t) : R \to L(X^\alpha, X)$ is Bohr almost periodic and Hölder continuous, $\alpha \geq 0$, $L(X^\alpha, X)$ is the space of linear bounded operators $X^\alpha \to X$.

(H3) We shall use the notation $U_\varrho^\alpha = \{x \in X^\alpha : \|x\|_\alpha \leq \varrho\}$. Assume that the sequence $\{\tau_j(u)\}$ of functions $\tau_j : U_\varrho^\alpha \to R$ has uniformly almost periodic sequences of differences uniformly with respect to $u \in U_\varrho^\alpha$ and there exists $\theta > 0$ such that $\inf_u \tau_{j+1}(u) - \sup_u \tau_j(u) \geq \theta > 0$, for all $u \in U_\varrho^\alpha$ and $j \in Z$. Also, there exists $\Theta > 0$ such that $\sup_u \tau_{j+1}(u) - \inf_u \tau_j(u) \leq \Theta$ for all $j \in Z$ and $u \in U_\varrho^\alpha$.

(H4) The sequence $\{B_j\}$ of bounded operators is almost periodic and there exists $b > 0$ such that $\|B_j u\|_\alpha \leq b\|u\|_\alpha$ for $j \in Z, \alpha \geq 0$, and $u \in X^\alpha$.

(H5) The function $f(t, u) : R \times U_\rho^\alpha \to X$ is continuous in u and is Hölder continuous and W-almost periodic in t uniformly with respect to $x \in U_\rho^\alpha$ with some $\rho > 0$.

(H6) The sequence $\{g_j(u)\}$ of continuous functions $U_\rho^\alpha \to X^\alpha$ is almost periodic uniformly with respect to $x \in U_\rho^\alpha$.

Remark 1. We assume that operators B_j are bounded and satisfy assumption **(H4)**. Many of our results are valid if the B_j are unbounded closed operators $X^{\alpha+\gamma} \to X^\alpha$ for $\alpha \geq 0$ and some $\gamma \geq 0$. We discuss this case in the last section.

We use the following generalization of Lemma 7 from [7] (also, see [6] and [19]):

Lemma 1. *Assume that a sequence of real numbers $\{\tau_j\}$ has uniformly almost periodic sequences of differences, the sequence $\{B_j\}$ is almost periodic, and the function $f(t) : R \to X$ is W-almost periodic. Then for any $\varepsilon > 0$ there exist a such $l = l(\varepsilon) > 0$ that for any interval J of length l there are such $r \in J$ and an integer q that the following relations hold:*

$$\|f(t + r) - f(t)\| < \varepsilon, \ t \in R, |t - \tau_j| > \varepsilon, j \in Z,$$

$$\|B_{k+q} - B_k\| < \varepsilon, \ \|\tau_k^q - r\| < \nu, k \in Z.$$

If A is a sectorial operator, then $(-A)$ is an infinitesimal generator of the analytical semigroup e^{-At}. For every $x \in X^\alpha$ we get $e^{-At}A^\alpha x = A^\alpha e^{-At}x$. Further, we shall use the inequalities (see [9])

$$\|A^\alpha e^{-At}\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \ t > 0, \ \alpha > 0,$$

$$\|(e^{-At} - I)u\| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha \|A^\alpha u\|, \ t > 0, \alpha \in (0, 1], u \in X^\alpha,$$

where $C_\alpha \in R$ is nonnegative and bounded as $\alpha \to +0$.

Definition 4. The function $x(t) : [t_0, t_1] \to X^\alpha$ is said to be a solution of the initial-value problem $u(t_0) = u_0 \in X^\alpha$ for Eqs. (7.1) and (7.2) on $[t_0, t_1]$ if

(i) it is continuous in $[t_0, \tau_k], (\tau_k, \tau_{k+1}], \ldots, (t_{k+s}, t_1]$ with the discontinuities of the first kind at the moments $t = \tau_j(u)$ of intersections with impulsive surfaces;
(ii) $x(t)$ is continuously differentiable in each of the intervals (t_0, τ_k), (τ_k, τ_{k+1}), $\ldots, (t_{k+s}, t_1)$ and satisfies Eqs. (7.1) and (7.2) if $t \in (t_0, t_1), t \neq \tau_j$, and $t = \tau_j$, respectively;
(iii) the initial-value condition $u(t_0) = u_0$ is fulfilled.

We assume that solutions $u(t)$ of (7.1) and (7.2) are left-hand-side continuous; hence $u(\tau_j) = u(\tau_j - 0)$ at all points of impulsive action.

Also, we assume that in the domain U_ρ^α solutions of (7.1) and (7.2) don't have beating at the surfaces $t = \tau_j(u)$; in other words, solutions intersect each surface only once.

7.3 Exponential Dichotomy

Together with Eqs. (7.1) and (7.2) we consider the corresponding linear homogeneous equation

$$\frac{du}{dt} + (A + A_1(t))u = 0, \quad t \neq \tau_j, \tag{7.4}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j), \quad j \in Z, \tag{7.5}$$

where $\tau_j = \tau_j(0)$. Denote by $V(t, s)$ the evolution operator of the linear equation without impulses (7.4). It satisfies $V(\tau, \tau) = I,\ V(t, s)V(s, \tau) = V(t, \tau),\ t \geq s \geq \tau.$

By Theorem 7.1.3 [9, p.190], $V(t, \tau)$ is strongly continuous with values in $L(X^\beta)$ for any $0 \leq \beta < 1$ and

$$\|V(t, \tau)x\|_\beta \leq L_Q(t - \tau)^{(\gamma-\beta)_-} \|x\|_\gamma, \tag{7.6}$$

where $(\gamma - \beta)_- = \min(\gamma - \beta, 0),\ t - \tau \leq Q,\ L_Q = L_Q(Q)$. Moreover,

$$\|V(t, \tau)x - x\|_\beta \leq L_{\beta,\nu}(t - \tau)^\nu \|x\|_{\beta+\nu}, \quad \nu > 0, \beta + \nu \leq 1. \tag{7.7}$$

Using the proof of Lemma 7.1.1 from [9], p. 188, one can verify the following generalized Gronwall inequality:

Lemma 2. *$a_1 \geq 0, a_2 \geq 0$, and $y(t)$ is a nonnegative function locally integrable on $0 \leq t < Q$ with*

$$y(t) \leq a_1 + a_2 t^{-\alpha} + b \int_0^t (t - s)^{-\beta} u(s) ds$$

on this interval; then there is a constant $\tilde{C} = \tilde{C}(\beta, b, Q) < \infty$ *such that*

$$y(t) \le \left(a_1 + \frac{a_2}{(1-\alpha)t^\alpha}\right)\tilde{C}(\beta, b, Q).$$

We will use the following perturbation lemma.

Lemma 3. *Let us consider the perturbation equation*

$$\frac{du}{dt} + (\gamma A + A_2(t))u = 0, \tag{7.8}$$

where $\gamma = Const > 0,\ A_2(t) : R \to L(X^\alpha, X)$.

For $Q > 0$, *there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \le \varepsilon_0$ *and* $|\gamma - 1| \le \varepsilon,\ \sup_t \|A_1(t) - A_2(t)\|_{L(x^\alpha, X)} \le \varepsilon$ *the evolution operators* $V(t,s)$ *of (7.4) and* $V_1(t,s)$ *of (7.8) satisfy*

$$\|V(t,s) - V_1(t,s)\|_\alpha \le R_1(\varepsilon),\ t - s \le Q, \tag{7.9}$$

where $R_1(\varepsilon)$ *depends on* Q, α, *and* $R_1(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. For definiteness let $\gamma > 1$. Solutions $x(t)$ and $y(t)$ of Eqs. (7.4) and (7.8) satisfy the following integral equations:

$$x(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^t e^{-A(t-s)}A_1(s)x(s)ds$$

and

$$y(t) = e^{-A\gamma(t-t_0)}x_0 + \int_{t_0}^t e^{-A\gamma(t-s)}A_2(s)y(s)ds.$$

Then

$$\begin{aligned}
&\|x(t) - y(t)\|_\alpha \le \|(I - e^{-A(\gamma-1)(t-t_0)})A^\alpha e^{-A(t-t_0)}x_0\| + \\
&+ \int_{t_0}^t \|(I - e^{-A(\gamma-1)(t-s)})A^\alpha e^{-A(t-s)}A_1(s)x(s)\|ds + \\
&+ \int_{t_0}^t \|A^\alpha e^{-A\gamma(t-s)}(A_1(s) - A_2(s))x(s)\|ds + \\
&+ \int_{t_0}^t \|A^\alpha e^{-A\gamma(t-s)}A_2(s)(x(s) - y(s))\|ds \le \\
&\le a_1(\varepsilon)\|x_0\|_\alpha + a_2\int_{t_0}^t (t-s)^{-\alpha}\|x(s) - y(s)\|_\alpha ds,
\end{aligned}$$

where $a_2 = C_\alpha \sup_s \|A_1(s)\|_{L(X^\alpha,X)}$ and $a_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. By Lemma 2, there exists a positive constant K_1 depending on α and Q such that

$$\|x(t) - y(t)\|_\alpha \le K_1 a_1(\varepsilon)\|x_0\|_\alpha = R_2(\varepsilon)\|x_0\|_\alpha.$$

Lemma 4. *Let us consider Eq. (7.4) and*

$$\frac{dv}{dt} + (A + A_2(t))v = 0, \tag{7.10}$$

such that $A_2 : R \to L(X^\alpha, X)$ is a bounded and Hölder continuous function.

Then for $Q > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and

$$\sup_t \|A_1(t) - A_2(t)\|_{L(X^\alpha,X)} \le \varepsilon$$

the evolution operators $V(t,s)$ of (7.4) and $V_1(t,s)$ of (7.10) satisfy

$$\|(V(t,s) - V_1(t,s))u\|_\alpha \le R_3(\varepsilon)|t - t_0|^{1-2\alpha+\delta}\|u\|_\delta,\ t - s \le Q, \tag{7.11}$$

where $R_3(\varepsilon) = R_3(\varepsilon, Q, \alpha)$ and $R_3(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Denote by $u(t)$ and $v(t)$ solutions of (7.4) and (7.10) with initial value $u(t_0) = u(t_0) = u_0$. They satisfy the inequalities

$$\begin{aligned}
\|u(t) - v(t)\|_\alpha &\le \int_{t_0}^t \|A^\alpha e^{-A(t-s)}(A_1(s) - A_2(s))u(s)\|ds + \\
&+ \int_{t_0}^t \|A^\alpha e^{-A(t-s)} A_2(s)(u(s) - v(s))\|ds \le \\
&\le C_\alpha L_Q \varepsilon \|u_0\|_\delta \int_{t_0}^t \frac{ds}{(t-s)^\alpha (s-t_0)^{\alpha-\delta}} + C_\alpha \|A_1\|_L \int_{t_0}^t \frac{\|u(s) - v(s)\|_\alpha ds}{(t-s)^\alpha} \le \\
&\le \varepsilon \|u_0\|_\delta R_4 + C_\alpha \|A_1\|_L \int_{t_0}^t \frac{\|u(s) - v(s)\|_\alpha ds}{(t-s)^\alpha}.
\end{aligned} \tag{7.12}$$

Applying Lemma 2 to (7.12), we obtain (7.11).

We define the evolution operator for Eqs. (7.4) and (7.5) as

$$U(t,s) = V(t,s) \text{ if } \tau_k < s \le t \le \tau_{k+1}$$

and

$$U(t,s) = V(t,\tau_k)(I + B_k)V(\tau_k,\tau_{k-1})\ldots(I + B_m)V(\tau_m,s) \tag{7.13}$$

if $\tau_{m-1} < s < \tau_m < \tau_{m+1} < \ldots < \tau_k \le t \le \tau_{k+1}$.

It it easy to verify that for fixed $t > s$ the operator $U(t,s)$ is bounded in the space X^α.

Definition 5. We say that the system (7.4)–(7.5) has an exponential dichotomy on R with exponent $\beta > 0$ and bound $M \geq 1$ (with respect to X^α) if there exist projections $P(t), t \in R$, such that

(i) $U(t,s)P(s) = P(t)U(t,s),\ t \geq s$;
(ii) $U(t,s)|_{Im(P(s))}$ for $t \geq s$ is an isomorphism on $Im(P(s))$, and then $U(s,t)$ is defined as an inverse map from $Im(P(t))$ to $Im(P(s))$;
(iii) $\|U(t,s)(1-P(s))u\|_\alpha \leq Me^{-\beta(t-s)}\|u\|_\alpha,\ t \geq s,\ u \in X^\alpha$;
(iv) $\|U(t,s)P(s)\|_\alpha \leq Me^{\beta(t-s)}\|u\|_\alpha,\ t \leq s,\ u \in X^\alpha$.

If the system (7.4)–(7.5) has an exponential dichotomy on R, then the nonhomogeneous equation

$$\frac{du}{dt} + (A + A_1(t))u = f(t), \quad t \neq \tau_j, \tag{7.14}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j, \quad j \in Z, \tag{7.15}$$

has a unique solution bounded on R

$$u_0(t) = \int_{-\infty}^{\infty} G(t,s)f(s)(x)ds + \sum_{j \in Z} G(t,\tau_j)g_j, \tag{7.16}$$

where

$$G(t,s) = \begin{cases} U(t,s)(I - P(s)), & t \geq s, \\ -U(t,s)P(s), & t < s, \end{cases}$$

is the Green function such that

$$\|G(t,s)u\|_\alpha \leq Me^{-\beta|t-s|}\|u\|_\alpha,\ t,s \in R. \tag{7.17}$$

Analogous to [9], p. 250, it can be proven that a function $u(t)$ is a bounded solution on the semiaxis $[t_0, +\infty)$ if and only if

$$u(t) = U(t,t_0)(I - P(t_0))u(t_0) + \int_{t_0}^{+\infty} G(t,s)f(s)ds + \sum_{t_0 \leq \tau_j} G(t,\tau_j)g_j,\ t \geq t_0. \tag{7.18}$$

A function $u(t)$ is a bounded solution on the semiaxis $(-\infty, t_0]$ if and only if

$$u(t) = U(t,t_0)P(t_0)u(t_0) + \int_{-\infty}^{t_0} G(t,s)f(s)ds + \sum_{t_0 > \tau_j} G(t,\tau_j)g_j,\ t \leq t_0. \tag{7.19}$$

Now we estimate $\|G(t,s)u\|_\alpha$ for $u \in X$. Let $t > s$ and $\tau_{m-1} < s < \tau_m,\ \tau_k < t < \tau_{k+1}$. Then

$$\begin{aligned}
&\|G(t,s)u\|_\alpha = \|U(t,s)(I-P(s))u\|_\alpha \le \\
&\le \|U(t,\tau_m)(I-P(\tau_m))\|_\alpha \|U(\tau_m,s)u\|_\alpha \le \\
&\le Me^{-\beta(t-\tau_m)} L_\Theta(\tau_m - s)^{-\alpha}\|u\| \le \tilde{M}e^{-\beta(t-s)}|\tau_m - s|^{-\alpha}\|u\| \qquad (7.20)
\end{aligned}$$

and

$$\begin{aligned}
&\|G(s,t)u\|_\alpha = \|U(s,t)P(t)u\|_\alpha \le \\
&\le \|U(s,t+1)P(t+1)\|_\alpha \|A^\alpha U(t+1,t)u\| \le \tilde{M}e^{-\beta(t-s)}\|u\|. \qquad (7.21)
\end{aligned}$$

If t_1 and t_2 belong to the same interval of continuity, then

$$\|P(t_1)u - P(t_2)u\|_\gamma \le \tilde{M}_1\|t_1 - t_2|^\nu\|u\|_{\gamma+\nu} \qquad (7.22)$$

since as in [9], p. 247,

$$\begin{aligned}
&\|P(t+h)u - P(t)u\|_\gamma \le \|P(t)u - V(t+h,t)P(t)u\|_\gamma + \\
&+\|V(t+h,t)P(t)u - P(t+h)u\|_\gamma \le \\
&\le \|(I - V(t+h,t))P(t)u\|_\gamma + \|P(t+h)(V(t+h,t)u - u)\|_\gamma.
\end{aligned}$$

Lemma 5. *Let the impulsive system (7.4) and (7.5) be exponentially dichotomous with positive constants β and M. Then there exists $\varepsilon > 0$ such that the perturbed systems*

$$\frac{du}{dt} + (A + \tilde{A}(t))u = 0, \quad t \ne \tilde{\tau}_j, \qquad (7.23)$$

$$\Delta u|_{t=\tilde{\tau}_j} = u(\tilde{\tau}_j + 0) - u(\tilde{\tau}_j) = \tilde{B}_j u(\tilde{\tau}_j), \quad j \in Z, \qquad (7.24)$$

with $\sup_j|\tau_j - \tilde{\tau}_j| \le \varepsilon, \sup_j\|B_j - \tilde{B}_j\| \le \varepsilon,\ \sup_t\|A_1(t) - \tilde{A}(t)\|_{L((X^\alpha,X)} \le \varepsilon$, *are also exponentially dichotomous with some constants $\beta_1 \le \beta$ and $M_1 \ge M$.*

Proof. In system (7.4) and (7.5), we introduce the change of time $t = \vartheta(t')$ such that $\tau_j = \vartheta(\tilde{\tau}_j), j \in Z$, and the function ϑ is continuously differentiable and monotonic on each interval $(\tilde{\tau}_j, \tilde{\tau}_{j+1})$.

The function ϑ can be chosen in piecewise linear form:

$$t = a_j t' + b_j,\ a_j = \frac{\tau_{j+1} - \tau_j}{\tilde{\tau}_{j+1} - \tilde{\tau}_j},\ b_j = \frac{\tau_j \tilde{\tau}_{j+1} - \tau_{j+1}\tilde{\tau}_j}{\tilde{\tau}_{j+1} - \tilde{\tau}_j} \text{ if } t' \in (\tilde{\tau}_j, \tilde{\tau}_{j+1}). \quad (7.25)$$

The function $\vartheta(t')$ satisfies the conditions

$$|\vartheta(t') - t'| \le \varepsilon,\ |\frac{d\vartheta(t')}{dt'} - 1| \le 2\varepsilon/\theta.$$

The system (7.4) and (7.5) in the new coordinates $v(t') = u(\vartheta(t'))$ has the form

$$\frac{dv}{dt'} + \frac{d\vartheta(t')}{dt'}\left(A + A_1(\vartheta(t')\right) v = 0, \quad t \ne \tilde{\tau}_j, \quad (7.26)$$

$$\Delta v|_{t'=\tilde{\tau}_j} = v(\tilde{\tau}_j + 0) - v(\tilde{\tau}_j) = B_j v(\tilde{\tau}_j), \quad j \in Z. \quad (7.27)$$

The system (7.26) and (7.27) has the evolution operator $U_1(t', s') = U(\vartheta(t'), \vartheta(s'))$. If the system (7.4) and (7.5) has an exponential dichotomy with projector $P(t)$ at point t, then the system (7.26) and (7.27) has an exponential dichotomy with projector $P_1(t') = P(\vartheta(t'))$ at point t'. Really,

$$\|U_1(t', s')(1 - P_1(s'))\|_\alpha = \|U(\vartheta(t'), \vartheta(s'))(1 - P(\vartheta(s'))\|_\alpha \le$$
$$\le Me^{-\beta(\vartheta(t') - \vartheta(s'))} \le Me^{2\varepsilon} e^{-\beta(t'-s')},\ t \ge s.$$

The inequality for an unstable manifold is proved analogously.

The linear systems (7.26), (7.27) and (7.23), (7.24) have the same points of impulsive actions $\tilde{\tau}_j, j \in Z$, and

$$\|\frac{d\vartheta(t')}{dt'} A_1(\vartheta(t')) - \tilde{A}(t')\| \le \|\frac{d\vartheta(t')}{dt'} A_1(\vartheta(t')) - A_1(\vartheta(t'))\| +$$
$$+\|A_1(\vartheta(t')) - A_1(t')\| + \|A_1(t') - \tilde{A}(t')\| \le K_2(\varepsilon),$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let $\tilde{U}(t', s')$ be the evolution operator for the system (7.23) and (7.24). To show that for sufficiently small δ_0 the system (7.23) and (7.24) is exponentially dichotomous, we use the following variant of Theorem 7.6.10 [9]:

Assume that the evolution operator $U_1(t', s')$ has an exponential dichotomy on R and satisfies

$$\sup_{0 \le t' - s' \le d} \|U_1(t', s')\|_\alpha < \infty \quad (7.28)$$

for some positive d. Then there exists $\eta > 0$ such that

$$\|\tilde{U}(t',s') - U_1(t',s')\|_\alpha < \eta, \quad \text{whenever } \ t - s \le d;$$

the evolution operator $\tilde{U}(t',s')$ also has an exponential dichotomy on R with some constants $\beta_1 \le \beta, M_1 \ge M$.

To prove this statement, we set for $n \in Z$

$$t_n = s' + dn, \quad T_n = U_1(s' + d(n+1), s' + dn), \ \tilde{T}_n = \tilde{U}(s' + d(n+1), s' + dn).$$

If the evolution operator $U_1(t,s)$ has an exponential dichotomy, then $\{T_n\}$ has a discrete dichotomy in the sense of [9, Definition 7.6.4].

According to Henry [9], Theorem 7.6.7, there exists $\eta > 0$ such that $\{\tilde{T}_n\}$ with $\sup_n \|T_n - \tilde{T}_n\|_\alpha \le \eta$ has a discrete dichotomy.

Now we are in the conditions of [9], Exercise 10, pp. 229–230 (see also a more general statement [5, Theorem 4.1]), which finishes the proof.

Let us estimate the difference $\|\tilde{T}_k - T_k\|_\alpha$. There exists a positive integer N such that each interval of length d contains no more than N elements of sequence $\{\tau_j\}$. Let the interval $[\xi_n, \xi_{n+1}]$ contain points of impulses $\tilde{\tau}_m, \ldots, \tilde{\tau}_k$ where $k-m \le N$. Denote by $V_1(t,s)$ and $\tilde{V}(t,s)$ the evolution operators of equations without impulses (7.26) and (7.23), respectively. Then

$$\begin{aligned}
&\|T_n - \tilde{T}_n\|_\alpha = \|U_1(\xi_{n+1},\xi_n) - \tilde{U}(\xi_{n+1},\xi_n)\|_\alpha \\
&\le \|(V_1(\xi_{n+1},\tilde{\tau}_k) - \tilde{V}(\xi_{n+1},\tilde{\tau}_k))(I+B_k)V_1(\tilde{\tau}_k,\tilde{\tau}_{k-1})\ldots(I+B_m)V_1(\tilde{\tau}_m,\xi_n)\|_\alpha + \\
&+\|\tilde{V}(\xi_{n+1},\tilde{\tau}_k)(B_k - \tilde{B}_k)V_1(\tilde{\tau}_k,\tilde{\tau}_{k-1})\ldots(I+B_m)V_1(\tilde{\tau}_m,\tilde{\xi}_n)\|_\alpha + \ \ldots \ + \\
&+\|\tilde{V}(\xi_{n+1},\tilde{\tau}_k)(I+\tilde{B}_k)\tilde{V}(\tilde{\tau}_k,\tilde{\tau}_{k-1})\ldots(I+\tilde{B}_m)(V_1(\tilde{\tau}_m,\xi_n) - \tilde{V}(\tilde{\tau}_m,\xi_n))\|_\alpha.
\end{aligned} \quad (7.29)$$

Using (7.9), we get that

$$\sup_n \|T_n - \tilde{T}_n\|_\alpha \le K_3(\varepsilon)$$

with some $K_3(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The exponentially dichotomous system (7.23) and (7.24) has Green's function

$$\tilde{G}(t,s) = \begin{cases} \tilde{U}(t,s)(I - \tilde{P}(s)), \ t \ge s, \\ -\tilde{U}(t,s)\tilde{P}(s), \ t < s, \end{cases}$$

such that

$$\|\tilde{G}(t,s)u\|_\alpha \le M_1 e^{-\beta_1|t-s|}\|u\|_\alpha, \ t,s \in R, \ u \in X^\alpha.$$

The sequence of bounded operators $T_n : X^\alpha \to X^\alpha$ defines the difference equation

$$u_{n+1} = T_n u_n, \ n \in Z, \quad (7.30)$$

with evolution operator $T_{n,m} = T_{n-1} \ldots T_m,\ n \geq m,\ T_{m,m} = I$. It is exponentially dichotomous with Green's function

$$G_{n,m} = \begin{cases} T_{n,m}(I - P_m),\ n \geq m, \\ -T_{n,m}P_m,\ n < m, \end{cases}$$

where $P_m = P(\xi_m)$.

The second difference equation

$$u_{n+1} = \tilde{T}_n u_n,\ n \in Z, \tag{7.31}$$

has the evolution operator $\tilde{T}_{n,m} = \tilde{T}_{n-1} \ldots \tilde{T}_m,\ n \geq m,\ \tilde{T}_{m,m} = I$.

By sufficiently small $\sup_n \|T_n - \tilde{T}_n\|_\alpha$, Eq. (7.31) is exponentially dichotomous with Green's function

$$\tilde{G}_{n,m} = \begin{cases} \tilde{T}_{n,m}(I - \tilde{P}_m),\ n \geq m, \\ -\tilde{T}_{n,m}\tilde{P}_m,\ n < m. \end{cases}$$

According to Henry [9], p. 233, the difference between two Green's functions satisfies equality:

$$\tilde{G}_{n,m} - G_{n,m} = \sum_{k \in Z} G_{n,k+1}(\tilde{T}_k - T_k)\tilde{G}_{k,m} \tag{7.32}$$

and estimation

$$\|\tilde{G}_{n,m} - G_{n,m}\|_\alpha = M_2 e^{-\beta_2 d|n-m|} \sup_k \|\tilde{T}_k - T_k\|_\alpha,\ n, m \in Z \tag{7.33}$$

with some constants $\beta_2 \leq \beta_1, M_2 \geq M_1$.

Now we can consider the difference of two Green's functions $\tilde{G}(t,s) - G_1(t,s)$. Let $t = s + nd + t_1, t_1 \in [0, d)$. Then

$$\begin{aligned} &\|\tilde{G}(t,s) - G_1(t,s)\|_\alpha = \\ &= \|\tilde{U}(s+nd+t_1, s+nd)\tilde{G}(s+nd, s) - U(s+nd+t_1, s+nd)G(s+nd, s)\|_\alpha \leq \\ &\leq \|(\tilde{U}(s+nd+t_1, s+nd) - U(s+nd+t_1, s+nd))\tilde{G}(s+nd, s)\|_\alpha + \\ &+ \|U(s+nd+t_1, s+nd)(\tilde{G}(s+nd, s) - G(s+nd, s))\|_\alpha. \end{aligned}$$

Using (7.33) and an estimation of the difference $\tilde{U} - U_1$ at a bounded interval as is done in (7.29), we get

$$\|\tilde{G}(t,\tau) - G_1(t,\tau)\|_\alpha \leq \tilde{M}_2(\varepsilon) e^{-\beta_2|t-\tau|}, \quad t, \tau \in R, \tag{7.34}$$

with $\tilde{M}_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By the definition of Green's function, we have

$$\|\tilde{P}(\tau) - P_1(\tau)\|_\alpha \leq \tilde{M}_2(\varepsilon) \text{ for all } \tau \in R. \tag{7.35}$$

Corollary 1. *Let the conditions of Lemma 5 be satisfied. Then for* $t \in R, |t - \tau_j| \geq \varepsilon, j \in Z$, *we have*

$$\|(P(t) - \tilde{P}(t))u\|_\alpha \leq \tilde{M}_3(\varepsilon)\|u\|_{\alpha+\nu}, \tag{7.36}$$

where $\nu > 0, \alpha + \nu < 1$, *and* $\tilde{M}_3(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Using (7.22) and (7.35), we get

$$\|(P(t) - \tilde{P}(t))u\|_\alpha \leq \|(P(t) - P(\vartheta(t)))u\|_\alpha +$$
$$+\|(P(\vartheta(t)) - \tilde{P}(\vartheta(t)))u\|_\alpha + \|(\tilde{P}(\vartheta(t)) - \tilde{P}(t))u\|_\alpha \leq \tilde{M}_3(\varepsilon)\|u\|_{\alpha+\nu}.$$

7.4 Almost Periodic Solutions of Equations with Fixed Moments of Impulsive Action

Consider the linear inhomogeneous equation

$$\frac{du}{dt} + (A + A_1(t))u = f(t), \quad t \neq \tau_j, \tag{7.37}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j, \quad j \in Z. \tag{7.38}$$

We assume that

(H7) the function $f(t) : R \to X$ is W-almost periodic and locally Hölder continuous with points of discontinuity at moments $t = \tau_j, j \in Z$, at which it is continuous from the left;

(H8) the sequence $\{g_j\}$ of $g_j \in X^{\alpha_1}, \alpha_1 > \alpha > 0$, is almost periodic.

Theorem 1. *Assume that Eqs. (7.37) and (7.38) satisfy conditions* **(H1)**–**(H3)**, **(H7)**, *and* **(H8)** *and that the corresponding homogeneous equation is exponentially dichotomous.*

Then the equation has a unique W-almost periodic solution $u_0(t) \in \mathscr{PC}(R, X^\alpha)$.

Proof. We show that an almost periodic solution is given by the formula (7.16). For $t \in (\tau_i, \tau_{i+1}]$, it satisfies

$$\|u_0(t)\|_\alpha \leq \int_{-\infty}^{t} \|A^\alpha U(t,s)(I - P(s))f(s)\|ds +$$

$$+ \int_t^\infty \|A^\alpha U(t,s)P(s)f(s)\|ds + \sum_{j\in Z} \|G(t,\tau_j)g_j\|_\alpha \leq$$

$$
\begin{aligned}
&\le \sum_{j\in Z} \|G(t,\tau_j)g_j\|_\alpha + \int_{\tau_i}^{t} \|A^\alpha V(t,s)(I-P(s))f(s)\|ds + \\
&+ \sum_{k=0}^{\infty} \int_{\tau_{i-k-1}}^{\tau_{i-k}} \|U(t,\tau_{i-k})(I-P(\tau_{i-k}))\|_\alpha \|A^\alpha U(\tau_{i-k},s)f(s)\|ds + \\
&+ \sum_{k=1}^{\infty} \int_{\tau_{i+k}}^{\tau_{i+k+1}} \|U(t,\tau_{i+k+1})P(\tau_{i+k+1})\|_\alpha \|A^\alpha U(\tau_{i+k+1},s)f(s)\|ds + \\
&+ \int_{t}^{\tau_{i+1}} \|A^\alpha V(t,s)P(s)f(s)\|ds \le \frac{2M}{1-e^{-\theta\beta}} \frac{C_\alpha \Theta^{1-\alpha}}{1-\alpha} \|f\|_{PC} + \\
&+ \frac{2M}{1-e^{-\theta\beta}} \sup_j \|g_j\|_\alpha \le \tilde{M}_0 \max\{\|f(t)\|_{PC}, \|g_j\|_\alpha\} \qquad (7.39)
\end{aligned}
$$

with some constant $\tilde{M}_0 > 0$.

Take an ε-almost period h for the right-hand side of the equation, which satisfies the conditions of Lemma 1; that is, there exists a positive integer q such that $\tau_{j+q} \in (s+h, t+h)$ if $\tau_j \in (s,t)$ and $|\tau_j + h - \tau_{j+q}| < \varepsilon$, $\|B_{j+q} - B_j\| < \varepsilon$.

Let $t \in (\tau_i + \varepsilon, \tau_{i+1} - \varepsilon)$. We define points $\eta_k = (\tau_k + \tau_{k-1})/2, k \in Z$. Then

$$
\begin{aligned}
&\|u_0(t+h) - u_0(t)\|_\alpha \le \sum_{j\in Z} \|G(t+h,\tau_{j+q})g_{j+q} - G(t,\tau_j)g_j\|_\alpha + \\
&+ \int_{-\infty}^{\infty} \|G(t+h,s+h)f(s+h) - G(t,s)f(s)\|_\alpha ds \le \\
&\le \int_{-\infty}^{\infty} \|(G(t+h,s+h) - G(t,s))f(s+h)\|_\alpha ds + \\
&+ \int_{-\infty}^{\infty} \|G(t,s))(f(s+h) - f(s))\|_\alpha ds + \sum_{j\in Z} \|G(t,\tau_j))(g_{j+q} - g_j)\|_\alpha + \\
&+ \sum_{j\in Z} \|(G(t+h,\tau_{j+q}) - G(t,\tau_j))g_{j+q}\|_\alpha . \qquad (7.40)
\end{aligned}
$$

Denote $U_2(t,s) = U(t+h,s+h)$. If $u(t) = U(t,s)u_0, u(s) = u_0$, is a solution of the impulsive equations (7.4) and (7.5), then $u_2(t) = U(t+h,s+h)u_0, u_2(s) = u_0$, is a solution of the equation

$$
\frac{du}{dt} + (A + A_1(t+h))u = 0, \quad t \ne \tau_{j+q} - h, \qquad (7.41)
$$

$$
\Delta u|_{t+h=\tau_{j+q}} = u(\tau_{j+q}+0) - u(\tau_{j+q}) = B_{j+q}u(\tau_{j+q}), \quad j \in Z. \qquad (7.42)
$$

We will use the notation $V_2(t,s) = V(t+h, s+h)$ for the evolution operator of an equation without impulses (7.41). Denote also $\tilde{\tau}_n = \tau_{n+q} - h, \tilde{B}_n = B_{n+q}$. Since Eqs. (7.4) and (7.5) are exponentially dichotomous, Eqs. (7.41) and (7.42) are exponentially dichotomous also with projector $P_2(s) = P(s+h)$.

The first integral in (7.40) is the sum of two integrals:

$$\int_{-\infty}^{\infty} \|(G(t+r, s+r) - G(t,s))f(s+r)\|_\alpha ds =$$
$$= \int_{-\infty}^{t} \|(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s)))f(s+r)\|_\alpha ds +$$
$$+ \int_{t}^{\infty} \|(U_2(t,s)P_2(s) - U(t,s)P(s))f(s+r)\|_\alpha ds. \tag{7.43}$$

We estimate the first integral in (7.43); the second integral is considered analogously.

$$\int_{-\infty}^{t} \|(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s)))f(s+r)\|_\alpha ds \le$$
$$\le \int_{\tau_i+\varepsilon}^{t} \|A^\alpha(V_2(t,s)(I-P_2(s)) - V(t,s)(I-P(s)))f(s+r)\| ds +$$
$$+ \int_{\tau_i-\varepsilon}^{\tau_i+\varepsilon} \|A^\alpha(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s)))f(s+r)\| ds +$$
$$+ \int_{\eta_i}^{\tau_i-\varepsilon} \|A^\alpha(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s)))f(s+r)\| ds +$$
$$+ \sum_{k=1}^{\infty} \int_{\eta_{i-k}}^{\eta_{i-k+1}} \|A^\alpha(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s)))f(s+r)\| ds. \tag{7.44}$$

Let us consider all integrals in (7.44) separately. By (7.36) and (7.11) we have

$$I_{11} = \int_{\tau_i+\varepsilon}^{t} \|A^\alpha(V_2(t,s)(I-P_2(s)) - V(t,s)(I-P(s)))f(s+r)\| ds =$$
$$= \int_{\tau_i+\varepsilon}^{t} \|A^\alpha((I-P_2(t))V_2(t,s) - (I-P(t))V(t,s))f(s+r)\| ds \le$$
$$\le \int_{\tau_i+\varepsilon}^{t} \|A^\alpha(P_2(t) - P(t))V_2(t,s)f(s+r)\| ds +$$
$$+ \int_{\tau_i+\varepsilon}^{t} \|A^\alpha(I-P(t))(V_2(t,s) - V(t,s))f(s+r)\| ds \le$$

$$\leq \left(\int_{\tau_i+\varepsilon}^{t} \frac{\tilde{M}_3(\varepsilon) L_Q ds}{(t-s)^{\alpha}} + \int_{\tau_i+\varepsilon}^{t} \frac{R_3(\varepsilon) ds}{(t-s)^{2\alpha-1}} \right) \|f\|_{PC} \leq \Gamma_1(\varepsilon) \|f\|_{PC}.$$

$$I_{12} = \int_{\tau_i-\varepsilon}^{\tau_i+\varepsilon} \|A^{\alpha} U(t,s)(I-P(s)) f(s+h)\| ds \leq$$

$$\leq \int_{\tau_i}^{\tau_i+\varepsilon} \|A^{\alpha}(I-P(t)) V(t,s) f(s+h)\| ds +$$

$$+ \int_{\tau_i-\varepsilon}^{\tau_i} \| \|A^{\alpha}(I-P(t)) V(t,\tau_i)(I+B_i) U(\tau_i,s) f(s+h)\| ds \leq$$

$$\leq \Big(\int_{\tau_i}^{\tau_i+\varepsilon} \frac{C_{\alpha} ds}{(t-s)^{\alpha}} + M\|I+B_i\| \int_{\tau_i-\varepsilon}^{\tau_i} \frac{C_{\alpha} ds}{(s-\tau_i)^{\alpha}} \Big) \|f\|_{PC} \leq$$

$$\leq \Gamma_2(\varepsilon) \|f\|_{PC}.$$

Analogously,

$$I_{13} = \int_{\tau_i-\varepsilon}^{\tau_i+\varepsilon} \|A^{\alpha} U_2(t,s)(I-P_2(s)) f(s+h)\| ds \leq \Gamma_3(\varepsilon) \|f\|_{PC},$$

where $\Gamma_j(\varepsilon) \to 0$ as $\varepsilon \to 0,\ j = 1, 2, 3$.

Using (7.11) and (7.36), we get

$$I_{14} = \int_{\eta_i}^{\tau_i-\varepsilon} \|A^{\alpha}(U_2(t,s)(I-P_2(s)) - U(t,s)(I-P(s))) f(s+r)\| ds =$$

$$= \int_{\eta_i}^{\tau_i-\varepsilon} \Big\| \Big((I-P_2(t)) V_2(t,\tilde{\tau}_i)(I+\tilde{B}_i) V_1(\tilde{\tau}_i,s) -$$

$$-(I-P(t)) V(t,\tau_i)(I+B_i) V(\tau_i,s) \Big) f(s+h) \|_{\alpha} ds \leq$$

$$\leq \int_{\eta_i}^{\tau_i-\varepsilon} \|(P_2(t)-P(t)) V_2(t,\tilde{\tau}_i)(I+B_i) V_2(\tilde{\tau}_i,s) f(s+h)\|_{\alpha} ds +$$

$$+ \int_{\eta_i}^{\tau_i-\varepsilon} \|(I-P(t))(V_2(t,\tilde{\tau}_i) - V(t,\tau_i))(I+B_i) V_2(\tilde{\tau}_i,s) f(s+h)\|_{\alpha} ds +$$

$$+ \int_{\eta_i}^{\tau_i-\varepsilon} \|(I-P(t)) V(t,\tau_i)(\tilde{B}_i - B_i) V_2(\tilde{\tau}_i,s) f(s+h)\|_{\alpha} ds +$$

$$+ \int_{\eta_i}^{\tau_i-\varepsilon} \|(I-P(t)) V(t,\tau_i)(I-B_i)(V_2(\tilde{\tau}_i,s) - V(\tau_i,s)) f(s+h)\|_{\alpha} ds \leq$$

$$\leq \Gamma_4(\varepsilon) \|f\|_{PC},$$

where $\Gamma_4(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The last sum in (7.44) is transformed as follows:

$$I_{15} = \sum_{k=1}^{\infty} \int_{\eta_{i-k}}^{\eta_{i-k+1}} \|A^{\alpha}(U_2(t,s)(I - P_2(s)) - U(t,s)(I - P(s)))f(s+r)\|ds =$$

$$= \sum_{k=1}^{\infty} \int_{\eta_{i-k}}^{\eta_{i-k+1}} \|(U(t,\eta_i)(I - P(\eta_i))U(\eta_i, \eta_{i-k+1})U(\eta_{i-k+1}, s) -$$

$$-U_2(t,\eta_i)(I - P_2(\eta_i))U_2(\eta_i, \eta_{i-k+1})U_2(\xi_{i-k+1}, s))f(s+h)\|_{\alpha} ds \le$$

$$\le \sum_{k=1}^{\infty} \int_{\eta_{i-k}}^{\eta_{i-k+1}} \Big\| \Big(U(t,\eta_i) - U_2(t,\eta_i))(I - P(\eta_i))U(\eta_i, \eta_{i-k+1})U(\eta_{i-k+1}, s) +$$

$$+U_2(t,\eta_i)((I - P(\eta_i))U(\eta_i, \eta_{i-k+1}) - (I - P_2(\eta_i))U_2(\eta_i, \eta_{i-k+1}))U(\eta_{i-k+1}, s) +$$

$$+U_2(t,\eta_i)(I - P_2(\eta_i))U_2(\eta_i, \eta_{i-k+1})(U(\eta_{i-k+1}, s) - U_2(\eta_{i-k+1}, s))\Big) f(s+h) \Big\|_{\alpha} ds.$$

As in the proof of Lemma 5, we construct in space X^{α} two sequences of bounded operators

$$S_n = U(\eta_{n+1}, \eta_n), \quad \tilde{S}_n = U_2(\eta_{n+1}, \eta_n), \quad n \in Z,$$

and corresponding difference equations

$$u_{n+1} = S_n u_n, \quad v_{n+1} = \tilde{S}_n v_n, \quad n \in Z.$$

Per our assumption, these difference equations are exponentially dichotomous with corresponding evolution operators

$$S_{n,m} = S_{n-1} \ldots S_m, \quad \tilde{S}_{n,m} = \tilde{S}_{n-1} \ldots \tilde{S}_m, \quad n \ge m,$$

and Green's functions

$$G_{n,m} = \begin{cases} S_{n,m}(I - P_m), \ n \ge m, \\ -S_{n,m}P_m, \ n < m, \end{cases} \qquad \tilde{G}_{n,m} = \begin{cases} \tilde{S}_{n,m}(I - \tilde{P}_m), \ n \ge m, \\ -\tilde{S}_{n,m}\tilde{P}_m, \ n < m, \end{cases}$$

where $P_m = P(\eta_m), \tilde{P}_m = P_2(\eta_m)$.

Analogous to (7.32) and (7.33), we obtain

$$\tilde{G}_{n,m} - G_{n,m} = \sum_{k \in Z} G_{n,k+1}(\tilde{S}_k - S_k)\tilde{G}_{k,m}$$

and

$$\|\tilde{G}_{n,m} - G_{n,m}\|_\alpha = M_1 e^{-\beta_1 \theta |n-m|} \sup_k \|\tilde{S}_k - S_k\|_\alpha, \ n, m \in Z \tag{7.45}$$

with some constants $\beta_1 \le \beta, M_1 \ge M$.

$$\begin{aligned}
&\|S_n - \tilde{S}_n\|_\alpha = \|U(\eta_{n+1}, \eta_n) - U_2(\eta_{n+1}, \eta_n)\|_\alpha = \\
&= \|V(\eta_{n+1}, \tau_n)(I + B_n)V(\tau_n, \eta_n) - V_2(\eta_{n+1}, \tilde{\tau}_n)(I + \tilde{B}_n)V_2(\tilde{\tau}_n, \eta_n)\|_\alpha \le \\
&\le \|(V(\eta_{n+1}, \tau_n) - V_2(\eta_{n+1}, \tilde{\tau}_n))(I + B_n)V(\tau_n, \eta_n)\|_\alpha + \\
&+ \|V_2(\eta_{n+1}, \tilde{\tau}_n))(B_n - \tilde{B}_n)V(\tau_n, \eta_n)\|_\alpha + \\
&+ \|V_2(\eta_{n+1}, \tilde{\tau}_n))(I + \tilde{B}_n)(V(\tau_n, \eta_n) - V_2(\tilde{\tau}_n, \eta_n))\|_\alpha
\end{aligned}$$

Here we assume for definiteness that $\tilde{\tau}_n \ge \tau_n$. We have

$$\begin{aligned}
&\|(V(\eta_{n+1}, \tau_n) - V_2(\eta_{n+1}, \tilde{\tau}_n))y\|_\alpha \le \|V(\eta_{n+1}, \tilde{\tau}_n)(V(\tilde{\tau}_n, \tau_n) - I)y\|_\alpha + \\
&+ \|(V(\eta_{n+1}, \tilde{\tau}_n) - V_2(\eta_{n+1}, \tilde{\tau}_n))y\|_\alpha \le \\
&\le \Gamma_5(\varepsilon)\|y\|_\alpha
\end{aligned}$$

and

$$\begin{aligned}
&\|(V_2(\tilde{\tau}_n, \eta_n) - V(\tau_n, \eta_n))y\|_\alpha \le \|(V_2(\tilde{\tau}_n, \tau_n) - I)V_2(\tau_n, \eta_n)y\|_\alpha + \\
&+ \|V_2(\tau_n, \eta_n) - V(\tau_n, \eta_n)y\|_\alpha \le \Gamma_6(\varepsilon)\|y\|_\alpha,
\end{aligned}$$

where $\Gamma_5(\varepsilon) \to 0$ and $\Gamma_6(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we get

$$\begin{aligned}
&\|S_n - \tilde{S}_n\|_\alpha \le \Gamma_5(\varepsilon)\|I + B_n\|\|U(\tau_n, \eta_n)\|_\alpha + \\
&+ \varepsilon\|U_2(\eta_n, \tau_n)\|_\alpha\|U(\tau_n, \eta_n)\|_\alpha + \Gamma_6(\varepsilon)\|U_2(\eta_{n+1}, \tilde{\tau}_n)\|_\alpha\|I + \tilde{B}_n\| \le \Gamma_7(\varepsilon)
\end{aligned}$$

and by (7.45)

$$\|U(\eta_i, \eta_{i-k}) - U_2(\eta_i, \eta_{i-k})\|_\alpha \le M_1 e^{-\beta_1 \theta k}\Gamma_7(\varepsilon), \tag{7.46}$$

where $\Gamma_7(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Continuing to evaluate I_{15}, we can obtain the inequalities

$$\begin{aligned}
&\|U_2(t, \eta_i)g\|_\alpha \le M_2\|g\|_\alpha, \\
&\|(U(t, \eta_i) - U_2(t, \eta_i))g\|_\alpha \le \Gamma_8(\varepsilon)\|g\|_\alpha,
\end{aligned}$$

$$\int_{\xi_{i-k}}^{\eta_{i-k+1}} \|(U(\eta_{i-k+1}, s) - U_2(\eta_{i-k+1}, s))f(s+h)\|_\alpha ds \le \Gamma_9(\varepsilon)\|f\|_{PC},$$

where $\Gamma_8(\varepsilon) \to 0$ and $\Gamma_9(\varepsilon) \to 0$ as $\varepsilon \to 0$, M_2 is some positive constant. Note that as earlier, $t \in (\tau_i + \varepsilon, \tau_{i+1} - \varepsilon)$.

Taking into account the last inequalities, we conclude that series I_{15} is convergent and there exists $\Gamma_{10}(\varepsilon)$ such that $I_{15} \le \Gamma_{10}(\varepsilon)\|f\|_{PC}$ and $\Gamma_{10}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Using estimations for $I_{11}, \ldots, I_{15}$, we get that there exists $\Gamma_{11}(\varepsilon)$ such that

$$\int_{-\infty}^{\infty} \|(G(t+r, s+r) - G(t,s))f(s+r)\|_\alpha ds \le \Gamma_{11}(\varepsilon)\|f\|_{PC} \tag{7.47}$$

and $\Gamma_{11}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By Lemma 1, $|\tau_{j+q} - \tau_j - h| < \varepsilon$; therefore, $\tau_j + h + \varepsilon > \tau_{j+q}$ (we assume that $h > 0$ for definiteness). The difference $G(t, \tau_j) - G(t+h, \tau_{j+q})$ is estimated as follows. Let $t - \tau_j \ge \varepsilon$. Then

$$\begin{aligned}
&\|(G(t,\tau_j) - G(t+h, \tau_{j+q}))g_{j+q}\|_\alpha = \\
&= \|(U(t,\tau_j)(I - P(\tau_j)) - U(t+h, \tau_{j+q})(I - P(\tau_{j+q})))g_{j+q}\|_\alpha \le \\
&\le \|(U(t,\tau_j)(I - P(\tau_j)) - U(t, \tau_j + \varepsilon)(I - P(\tau_j + \varepsilon)))g_{j+q}\|_\alpha + \\
&+\|(U(t,\tau_j + \varepsilon)(I - P(\tau_j + \varepsilon)) - U(t+h, \tau_j + \varepsilon + h) \times \\
&\times(I - P(\tau_j + \varepsilon + h)))g_{j+q}\|_\alpha + \|U(t+h, \tau_{j+q})(I - P(\tau_{j+q})))g_{j+q} - \\
&-(U(t+h, \tau_j + \varepsilon + h)(I - P(\tau_j + \varepsilon + h))g_{j+q}\|_\alpha.
\end{aligned} \tag{7.48}$$

The first and third differences are small due to the continuity of function $U(t,s)$ at intervals between impulse points:

$$\begin{aligned}
&\|(U(t,\tau_j)(I - P(\tau_j)) - U(t, \tau_j + \varepsilon)(I - P(\tau_j + \varepsilon)))g_{j+q}\|_\alpha \le \\
&\le \|U(t, \tau_j + \varepsilon)(I - P(\tau_j + \varepsilon))(U(\tau_j + \varepsilon, \tau_j) - I)g_{j+q}\|_\alpha \le \\
&\le \|(I - P(t))U(t, \tau_j + \varepsilon)\|_\alpha \|(U(\tau_j + \varepsilon, \tau_j) - I)g_{j+q}\|_\alpha \le \\
&\le Me^{-\beta(t-\tau_j-\varepsilon)} C_{1-\alpha_1+\alpha}\varepsilon^{\alpha_1-\alpha}\|g_{j+q}\|_{\alpha_1}, \\
&\|(U(t+h, \tau_j + \varepsilon + h)(I - P(\tau_j + \varepsilon + h)) - U(t+h, \tau_{j+q})(I - P(\tau_{j+q})))g_{j+q}\|_\alpha = \\
&= \|\|U(t+h, \tau_j + \varepsilon + h)(I - P(\tau_j + \varepsilon + h))(U(\tau_j + \varepsilon + h, \tau_{j+q}) - I)g_{j+q}\|_\alpha \le \\
&\le Me^{-\beta(t-\tau_j-\varepsilon)} C_{1-\alpha_1+\alpha}\varepsilon^{\alpha_1-\alpha}\|g_{j+q}\|_{\alpha_1}.
\end{aligned}$$

The second difference in (7.48) is estimated using inequality (7.46) and the following transformation:

$$\|U(t, \tau_j + \varepsilon)(I - P(\tau_j + \varepsilon)) - U(t+h, \tau_j + \varepsilon + h)(I - P(\tau_j + \varepsilon + h))\|_\alpha =$$

$$= \|U(t,\tau_j+\varepsilon)(I-P(\tau_j+\varepsilon)) - U_2(t,\tau_j+\varepsilon)(I-P_2(\tau_j+\varepsilon))\|_\alpha =$$

$$= \|U(t,\eta_i)(I-P(\eta_i))U(\eta_i,\eta_{j+1})U(\eta_{j+1},\tau_j+\varepsilon) -$$

$$-U_2(t,\eta_i)(I-P(\eta_i)U_2(\eta_i,\eta_{j+1})U_2(\eta_{j+1},\tau_j+\varepsilon)\|_\alpha \le$$

$$\le \|(U(t,\eta_i)-U_2(t,\eta_i))(I-P(\eta_i))U(\eta_i,\eta_{j+1})U(\eta_{j+1},\tau_j+\varepsilon)\|_\alpha +$$

$$+\|U_1(t,\eta_i)(P(\eta_i)U(\eta_i,\eta_{j+1}) - P_2(\eta_i)U_2(\eta_i,\eta_{j+1})U(\eta_{j+1},\tau_j+\varepsilon)\|_\alpha +$$

$$+\|U_2(t,\eta_i)P_2(\eta_i)U_2(\eta_i,\eta_{j+1})(U(\eta_{j+1},\tau_j+\varepsilon) - U_2(\eta_{j+1},\tau_j+\varepsilon))\|_\alpha.$$

Therefore,

$$\sum_{j\in Z} \|(G(t+h,\tau_{j+q}) - G(t,\tau_j))g_{j+q}\|_\alpha \le \Gamma_{12}(\varepsilon)\sup_j \|g_j\|_{\alpha_1}, \tag{7.49}$$

where $\Gamma_{12}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The second integral and first sum in (7.40) are estimated as in (7.39):

$$\int_{-\infty}^{\infty} \|G(t,s))(f(s+h)-f(s))\|_\alpha ds + \sum_{j\in Z} \|U(t,\tau_j)(g_{j+q}-g_j)\|_\alpha \le M_3\varepsilon$$

since h is ε-almost periodic of the right-hand side of the equation.

As a result of these evaluations, we get

$$\|u_0(t+h) - u_0(t)\|_\alpha \le \Gamma(\varepsilon) \ \text{ for } \ t\in R,\ |t-\tau_j| > \varepsilon,\ j\in Z,$$

with $\Gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. The last inequality implies that the function $u_0(t)$ is W-almost periodic as function $R \to X^\alpha$.

Corollary 2. *Assume that Eqs. (7.37) and (7.38) satisfy the following:*

i) conditions **(H1)**–**(H3)**, **(H7)**;
ii) the sequence $\{g_j\}$ *of* $g_j \in X^\alpha$ *is almost periodic;*
iii) the corresponding homogeneous equation is exponentially dichotomous.

Then the equation has a unique W-almost periodic solution $u_0(t) \in \mathscr{PC}(R, X^\gamma)$ *with* $\gamma < \alpha$.

Now we consider a nonlinear equation with fixed moments of impulsive action:

$$\frac{du}{dt} + (A + A_1(t))u = f(t,u), \quad t \ne \tau_j, \tag{7.50}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j+0) - u(\tau_j) = B_j u(\tau_j) + g_j(u(\tau_j)), \quad j \in Z. \tag{7.51}$$

Theorem 2. *Let us consider Eqs. (7.50) and (7.51) in some domain* $U_\rho^\alpha = \{x \in X^\alpha : \|x\|_\alpha \le \rho\}$ *of space* X^α. *Assume that*

1) the equation satisfies assumptions **(H1)**–**(H4)**, $\tau_j = \tau_j(0)$;
2) the corresponding linear equation is exponentially dichotomous;
3) the function $f(t,u) : R \times U_\rho^\alpha \to X$ *is continuous in u, W-almost periodic, and Hölder continuous in t uniformly with respect to* $u \in U_\rho^\alpha$ *with some* $\rho > 0$, *and there exist constants* $N_1 > 0$ *and* $\nu > 0$ *such that*

$$\|f(t_1,u_1) - f(t_2,u_2)\| \le N_1(|t_1 - t_2|^\nu + \|u_1 - u_2\|_\alpha);$$

4) the sequence $\{g_j(u)\}$ *of continuous functions* $U_\rho^\alpha \to X^{\alpha_1}$ *is almost periodic uniformly with respect to* $u \in U_\rho^\alpha$ *and*

$$\|g_j(u_1) - g_j(u_2)\|_\alpha \le N_1\|u_1 - u_2\|_\alpha,\ j \in Z,$$

for $t_1, t_2 \in R,\ u_1, u_2 \in U_\rho^\alpha$ *and some* $\alpha_1 > \alpha$;
5) the functions $f(t,0)$ *and* $g_j(0)$ *are uniformly bounded for* $t \in R, j \in Z$.

Then in domain U_ρ^α *for sufficiently small* $N_1 > 0$ *there exists a unique W-almost periodic solution* $u_0(t)$ *of Eqs. (7.50) and (7.51).*

Proof. Denote by $\mathscr{M}_\varrho$ the set of all W-almost periodic functions $\varphi : R \to X^\alpha$ with discontinuity points $\tau_j, j \in Z$, satisfying the inequality $\|\varphi\|_{PC} \le \varrho$. In $\mathscr{M}_\varrho$, we define the operator

$$(\mathscr{F}\varphi)(t) = \int_{-\infty}^{\infty} G(t,s)f(s,\varphi(s))ds + \sum_{j\in Z} G(t,\tau_j)g_j(\varphi(\tau_j)).$$

Proceeding in the same way as in the proof of Theorem 1, we prove that $(\mathscr{F}\varphi)(t)$ is a W-almost periodic function and $\mathscr{F} : \mathscr{M}_\varrho \to \mathscr{M}_\varrho$ for some $\varrho > 0$.

Next, $\mathscr{F}$ is a contracting operator in $\mathscr{M}_\varrho$ by sufficiently small $N_1 > 0$.

Hence, there exists $\varphi_0 \in \mathscr{M}_\varrho$ such that

$$\varphi_0(t) = \int_{-\infty}^{\infty} G(t,s)f(s,\varphi_0(s))ds + \sum_{j\in Z} G(t,\tau_j)g_j(\varphi_0(\tau_j)).$$

The function $\varphi_0(t)$ is locally Hölder continuous on every interval $(\tau_j, \tau_{j+1}), j \in Z$. Actually,

$$\varphi_0(t+\delta) - \varphi_0(t) = \int_{-\infty}^{\infty} G(t+\delta,s)f(s,\varphi_0(s))ds - \int_{-\infty}^{\infty} G(t,s)f(s,\varphi_0(s))ds +$$

$$+ \sum_{j\in Z} G(t+\delta,\tau_j)g_j(\varphi_0(\tau_j)) - \sum_{j\in Z} G(t,\tau_j)g_j(\varphi_0(\tau_j)) =$$

$$= \int_{-\infty}^{t} (V(t+\delta,t) - I)U(t,s)(I - P(s))f(s,\varphi_0(s))ds -$$

$$- \int_{t+\delta}^{\infty} (V(t+\delta,t)-I)U(t,s)P(s)f(s,\varphi_0(s))ds +$$
$$+ \int_{t}^{t+\delta} V(t+\delta,s)(I-P(s))f(s,\varphi_0(s))ds + \int_{t}^{t+\delta} V(t,s)P(s)f(s,\varphi_0(s))ds$$
$$+ \sum_{\tau_j<t} (V(t+\delta,t)-I)U(t,\tau_j)(I-P(\tau_j))g_j(\varphi_0(\tau_j)) +$$
$$+ \sum_{\tau_j>t} (V(t+\delta,t)-I)U(t,\tau_j)P(\tau_j)g_j(\varphi_0(\tau_j)).$$

Applying (7.7), (7.20), (7.21), and (7.39), we conclude that for every interval $t \in (t', t'')$ not containing impulse points τ_j, there exists a positive constant C such that $\|\varphi_0(t+\delta) - \varphi_0(t)\|_\alpha \leq C\delta^{\alpha_1-\alpha}$.

The local Hölder continuity of $f(t, \varphi_0(t))$ follows from

$$\|f(t,\varphi_0(t)) - f(s,\varphi_0(s))\| \leq N_1 \left(|t-s|^\nu + \|\varphi_0(t)-\varphi_0(s)\|_\alpha\right) \leq$$
$$\leq C_1 \left(|t-s|^\nu + |t-s|^{\alpha_1-\alpha}\right).$$

By Lemma 37, [19], p. 214, if $\varphi_0(t)$ is W-almost periodic and $\inf_k(\tau_{k+1} - \tau_k) > 0$, then $\{\varphi_0(\tau_k)\}$ is an almost periodic sequence.

The linear inhomogeneous equation

$$\frac{du}{dt} + (A + A_1(t))u = f(t, \varphi_0(t)), \quad t \neq \tau_j, \tag{7.52}$$
$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j) + g_j(\varphi_0(\tau_j)), \quad j \in Z, \tag{7.53}$$

has a unique W-almost periodic solution in the sense of Definition 4. Due to the uniqueness, it coincides with $\varphi_0(t)$.

Hence, the W-almost periodic function $\varphi_0(t) : R \to X^\alpha$ satisfies Eq. (7.50) for $t \in (\tau_j, \tau_{j+1})$ and the difference equation (7.51) for $t = \tau_j$.

Now we study the stability of the almost periodic solution assuming exponential stability of the linear equation. First, using ideas in [17], we prove the following generalized Gronwall inequality for impulsive systems.

Lemma 6. *Assume that $\{t_j\}$ is an increasing sequence of real numbers such that $Q \geq t_{j+1} - t_j \geq \theta > 0$ for all j, M_1, M_2, and M_3 are positive constants, and $\alpha \in (0, 1)$. Then there exists a positive constant $\tilde{C}$ such that the positive piecewise continuous function $u : [t_0, t] \to R$ satisfying*

$$z(t) \leq M_1 z_0 + M_2 \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} (t_j - s)^{-\alpha} z(s)ds + M_2 \int_{t_m}^{t} (t-s)^{-\alpha} z(s)ds +$$

$$+M_3 \sum_{j=1}^{m} z(t_j) \quad \text{for} \quad t \in (t_m, t_{m+1}] \tag{7.54}$$

also satisfies

$$z(t) \le M_1 z_0 \tilde{C} \left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right)^m. \tag{7.55}$$

Proof. We apply the method of mathematical induction. At the interval $t \in [t_0, t_1]$ the inequality (7.54) has the form

$$z(t) \le M_1 z_0 + M_2 \int_{t_0}^{\tau_1} (\tau_1 - s)^{-\alpha} z(s) ds.$$

By Lemma 2 there exists $\tilde{C}$ such that

$$0 \le z(t) \le M_1 z_0 \tilde{C}, \quad t \in [t_0, t], \tilde{C} = \tilde{C}(M_1, M_2, Q). \tag{7.56}$$

Hence, (7.55) is true for $t \in [t_0, t_1]$. Assume (7.55) is true for $t \in [t_0, t_n]$ and prove it for $t \in (t_n, t_{n+1}]$. Hence, for $t \in (t_n, t_{n+1}]$ we have

$$z(t) \le M_1 z_0 + M_2 \int_{t_0}^{t_1} (t_1 - s)^{-\alpha} z(s) ds + M_3 z(t_1) +$$

$$+M_2 \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} (t_j - s)^{-\alpha} z(s) ds + M_3 \sum_{j=1}^{n} z(t_j) + M_2 \int_{t_n}^{t} (t-s)^{-\alpha} z(s) ds \le$$

$$\le M_1 z_0 + M_2 \frac{Q^{1-\alpha}}{1-\alpha} M_1 z_0 \tilde{C} + M_3 M_1 z_0 \tilde{C} + M_2 \int_{t_n}^{t} (t-s)^{-\alpha} z(s) ds +$$

$$+ \sum_{j=2}^{n} \left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right)^j \left(M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right) M_1 z_0 =$$

$$= M_1 z_0 + M_2 \frac{Q^{1-\alpha}}{1-\alpha} M_1 z_0 \tilde{C} + M_3 M_1 z_0 \tilde{C} + M_2 \int_{t_n}^{t} (t-s)^{-\alpha} z(s) ds +$$

$$+ \sum_{j=2}^{n} \left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right)^{j-1} \left[\left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right) - 1\right] M_1 z_0 =$$

$$\le M_1 z_0 \left(1 + M_2 \frac{Q^{1-\alpha}}{1-\alpha} \tilde{C} + M_3 \tilde{C}\right)^n + M_2 \int_{t_n}^{t} (t-s)^{-\alpha} z(s) ds.$$

Hence, for $t \in [t_n, t_{n+1})$, the function $z(t)$ satisfies the inequality

$$z(t) \le C_1 + M_2 \int_{t_n}^{t} (t-s)^{-\alpha} z(s)ds,$$

where $C_1 = M_1 z_0 \left(1 + M_2 \frac{Q^{1-\alpha}}{1-\alpha}\tilde{C} + M_3 \tilde{C}\right)^n$. Applying (7.56) at the interval $(t_n, t_{n+1}]$, we obtain (7.55). The lemma is proved.

Theorem 3. *Let Eqs. (7.50) and (7.51) satisfy assumptions of Theorem 2 and let the corresponding linear equation be exponentially stable.*

Then for sufficiently small $N_1 > 0$, the equation has a unique W-almost periodic solution $u_0(t)$, and this solution is exponentially stable.

Proof. The existence and uniqueness of the W-almost periodic solution $u_0(t)$ follows from Theorem 2. We prove its asymptotic stability. Let $u(t)$ be an arbitrary solution of the equation satisfying $\|u(t_0) - u_0(t_0)\|_\alpha \le \delta$, where δ is a small positive number.

Then by $t \ge t_0$ the difference of these solutions satisfies

$$u(t) - u_0(t) = U(t,t_0)(u(t_0) - u_0(t_0)) + \int_{t_0}^{t} U(t,s)\Big(f(s,u(s)) -$$
$$-f(s,u_0(s))\Big)ds + \sum_{t_0 \le \tau_k < t} U(t,\tau_k)\left(g_k(u(\tau_k)) - g_k(u_0(\tau_k))\right).$$

Then for $t_0 \in (\tau_0, \tau_1)$ and $t \in (\tau_j, \tau_{j+1}]$ we have

$$\|u(t) - u_0(t)\|_\alpha \le \|U(t,t_0)\|_\alpha \|u(t_0) - u_0(t_0)\|_\alpha +$$
$$+ \int_{t_0}^{\tau_1} \|U(t,\tau_1)\|_\alpha \|V(\tau_1,s)(f(s,u(s)) - f(s,u_0(s)))\|_\alpha ds + \cdots +$$
$$+ \int_{\tau_{j-1}}^{\tau_j} \|U(t,\tau_j)\|_\alpha \|V(\tau_j,s)(f(s,u(s)) - f(s,u_0(s)))\|_\alpha ds +$$
$$+ \int_{\tau_j}^{t} \|V(t,s)(f(s,u(s)) - f(s,u_0(s)))\|_\alpha ds +$$
$$+ \sum_{t_0 \le \tau_k < t} \|U(t,\tau_k)\left(g_k(u(\tau_k)) - g_k(u_0(\tau_k))\right)\|_\alpha \le$$
$$\le Me^{-\beta(t-t_0)}\|u(t_0) - u_0(t_0)\|_\alpha + Me^{-\beta(t-\tau_1)} \int_{t_0}^{\tau_1} \frac{L_Q N_1}{(\tau_1 - s)^\alpha} \|u(s) - u_0(s)\|_\alpha ds +$$
$$+\cdots + Me^{-\beta(t-\tau_j)} \int_{\tau_{j-1}}^{\tau_j} \frac{L_Q N_1}{(\tau_j - s)^\alpha} \|u(s) - u_0(s)\|_\alpha ds +$$
$$+ \int_{\tau_j}^{t} \frac{L_Q N_1}{(t-s)^\alpha} \|u(s) - u_0(s)\|_\alpha ds + \sum_{t_0 \le \tau_k < t} Me^{-\beta(t-\tau_k)} N_1 \|u(\tau_k) - u_0(\tau_k)\|_\alpha.$$

Denote $v(t) = e^{\beta t}\|u(t) - u_0(t)\|_\alpha, M_2 = e^{\beta Q} M L_Q N_1, M_3 = M N_1$. Then

$$v(t) \leq M v(t_0) + M_2 \int_{t_0}^{\tau_1} \frac{v(s)ds}{(\tau_1 - s)^\alpha} + \cdots + M_2 \int_{t_j}^{t} \frac{v(s)ds}{(\tau_j - s)^\alpha} + M_3 \sum_{k=1}^{j} v(\tau_k).$$

Then by Lemma 6 we get

$$\|u(t) - u_0(t)\|_\alpha \leq M\tilde{C} e^{-\beta(t-t_0)} \left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right)^{i(t,t_0)} \|u(t_0) - u_0(t_0)\|_\alpha.$$

Therefore, if

$$\beta > p \ln\left(1 + M_2 \tilde{C} \frac{Q^{1-\alpha}}{1-\alpha} + M_3 \tilde{C}\right),$$

where p is defined by (7.3), then the W-almost periodic solution $u_0(t)$ of Eqs. (7.50) and (7.51) is asymptotically stable. This can be achieved by sufficiently small N_1.

7.5 Almost Periodic Solutions of Equations with Nonfixed Moments of Impulsive Action

We consider the following equation with points of impulsive action depending on solutions

$$\frac{du}{dt} + Au = f(t, u), \quad t \neq \tau_j(u), \tag{7.57}$$

$$u(\tau_j(u) + 0) - u(\tau_j(u)) = B_j u + g_j(u), \quad j \in Z. \tag{7.58}$$

Definition 6 ([11]). A solution $u_0(t)$ of Eqs. (7.57) and (7.58) defined for all $t \geq t_0$, is called Lyapunov stable in space X^α if, for an arbitrary $\varepsilon > 0$ and $\eta > 0$, there exists such a number $\delta = \delta(\varepsilon, \eta)$ that, for any other solution $u(t)$ of the system, $\|u_0(t_0) - u(t_0)\|_\alpha < \delta$ implies that $\|u_0(t) - u(t)\|_\alpha < \varepsilon$ for all $t \geq t_0$ such that $|t - \tau_j^0| > \eta$, where τ_j^0 are the times during which the solution $u_0(t)$ intersects the surfaces $t = \tau_j(u), j \in Z$.

A solution $u_0(t)$ is said to be attractive if for each $\varepsilon > 0, \eta > 0$, and $t_0 \in R$, there exist $\delta_0 = \delta_0(t_0)$ and $T = T(\delta_0, \varepsilon, \eta) > 0$ such that for any other solution $u(t)$ of the system, $\|u_0(t_0) - u(t_0)\| < \delta_0$ implies $\|u_0(t) - u(t)\|_\alpha < \varepsilon$ for $t \geq t_0 + T$ and $|t - \tau_k^0| > \eta$.

A solution $u_0(t)$ is called asymptotically stable if it is stable and attractive.

Theorem 4. *Assume that in some domain $U_\rho^\alpha = \{u \in X^\alpha, \|u\|_\alpha \leq \rho\}$, Eqs. (7.57) and (7.58) satisfy conditions (H1), (H3)–(H6), and*

1) all solutions in domain U_ρ^α intersect each surface $t = \tau_j(u)$ no more than once;
2) $\|f(t_1,u) - f(t_2,u)\| \le H_1|t_1 - t_2|^\nu,\ \nu > 0,\ H_1 > 0;$
3) $\|f(t,u_1) - f(t,u_2)\| + \|g_j(u_1) - g_j(u_2)\|_\alpha + |\tau_j(u_1) - \tau_j(u_2)| \le N_1\|u_1 - u_2\|_\alpha,$ *uniformly to $t \in R, j \in Z$,*
4) $AB_j = B_jA,\ \|f(t,0)\| \le M_0,\ \|g_j(0)\|_1 \le M_0, j \in Z$
5) the linear homogeneous equation

$$M_* = \frac{M_1}{1 - e^{-\beta_1\theta}}\left(1 + \frac{C_\alpha Q^{1-\alpha}}{1-\alpha}\right).$$

$$\frac{du}{dt} + Au = 0, \quad t \ne \tau_j, \tag{7.59}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j + 0) - u(\tau_j) = B_j u(\tau_j), \quad j \in Z, \tag{7.60}$$

is exponentially stable in space X^α

$$\|U(t,s)u\|_\alpha \le Me^{-\beta(t-s)}\|u\|_\alpha,\ t \ge s, u \in X^\alpha \tag{7.61}$$

where $\tau_j = \tau_j(0),\ \beta > 0$ and $M \ge 1$.
6) $N_1M_* < 1$ and $\rho \ge \rho_0 = M_0M_*/(1 - N_1M_*)$, where

Then for sufficiently small values of the Lipschitz constant N_1, Eqs. (7.57) and (7.58) have in U_ρ^α a unique W-almost periodic solution and this solution is exponentially stable.

Proof. 1. First, using the method proposed in [6], we prove the existence of the W-almost periodic solution. Let $y = \{y_j\}$ be an almost periodic sequence of elements $y_j \in X^\alpha$, $\|y_j\|_\alpha \le \varrho$. We consider the equation with fixed moments of impulsive action

$$\frac{du}{dt} + Au = f(t,u), \quad t \ne \tau_j(y), \tag{7.62}$$

$$u(\tau_j(y_j) + 0) - u(\tau_j(y_j)) = B_j u(\tau_j(y_j)) + g_j(y_j), \quad j \in Z. \tag{7.63}$$

By Lemma 5, if a constant N_1 sufficiently small, then corresponding to (7.62) and (7.63) the linear impulsive equation [if $f \equiv 0, g_j(y_j) \equiv 0, j \in Z$,] is exponentially stable. Its evolution operator $U(t,\tau,y)$ satisfies estimate

$$\|U(t,\tau,y)u\|_\alpha \le M_1e^{-\beta_1(t-\tau)}\|u\|_\alpha,\ t \ge \tau, \tag{7.64}$$

with some positive constants $M_1 \ge M, \beta_1 \le \beta$.

Equations (7.62) and (7.63) have a unique solution bounded on the axis which satisfies the integral equation

$$\tilde{u}(t,y) = \int_{-\infty}^{t} U(t,\tau,y)f(\tau,\tilde{u}(\tau,y))d\tau + \sum_{\tau_j(y_j)<t} U(t,\tau_j(y_j),y)g_j(y_j). \quad (7.65)$$

We choose $u_0(t,y) \equiv 0$ and construct the sequence of W-almost periodic functions

$$u_{n+1}(t,y) = \int_{-\infty}^{t} U(t,\tau,y)f(\tau,u_n(\tau,y))d\tau + \sum_{\tau_j(y_j)<t} U(t,\tau_j(y_j),y)g_j(y_j),\ n=0,1,\ldots.$$

The proof of the W-almost periodicity of $u_{n+1}(t,y)$ in space X^{α} is similar to the proof of Theorem 1.

One can verify that for sufficiently small $N_1 > 0$ the sequence $\{u_n(t,y)\}$ converges to the W-almost periodic solution $u^*(t,y) : R \to X^{\alpha}$ of Eq. (7.65). As in the proof to Theorem 2, we prove that $u^*(t,y)$ is the W-almost periodic solution of impulsive equations (7.62) and (7.63).

Let $t \in (\tilde{\tau}_i, \tilde{\tau}_{i+1})$, where $\tilde{\tau}_i = \tau_i(y_i)$. As in (7.39), we obtain

$$\|u^*(t,y)\|_{\alpha} \le \int_{-\infty}^{t} \|A^{\alpha}U(t,s,y)(f(s,0) + f(s,u^*(s,y)) - f(s,0))\|ds +$$

$$+ \sum_{\tau_j(y_j)<t} \|U(t,\tilde{\tau}_j,y)(g_j(0) + g_j(y_j) - g_j(0))\|_{\alpha} \le$$

$$\le \frac{M_1}{1-e^{-\beta_1\theta}} \left(\frac{C_{\alpha}\Theta^{1-\alpha}}{1-\alpha} \left(M_0 + N_1 \sup_t \|u^*(t,y)\|_{\alpha} \right) + M_0 + N_1 \sup_j \|y_j\|_{\alpha} \right).$$

Hence, by sufficiently small $N_1 > 0$

$$\sup_t \|u^*(t,y)\| \le \rho_0. \quad (7.66)$$

If we choose the almost periodic sequence $y^* = \{y_j^*\}, y_j^* \in X^{\alpha}$, such that

$$u^*(\tau_j(y_j^*), y^*) = y_j^*$$

for all $j \in Z$, then the function $u^*(t,y^*)$ will be exactly the W-almost periodic solution of Eqs. (7.57) and (7.58).

We consider the space $\mathcal{N}$ of sequences $y = \{y_j\},\ y_j \in X^{\alpha}$, with norm $\|y\|_S = \sup_j \|y_j\|_{\alpha}$ and map $S : \mathcal{N} \to \mathcal{N}$,

$$S(y) = \{u^*(\tau_j(y_j), y)\}_{j \in Z}.$$

By (7.66), S maps the domain $U_{\varrho}^{\alpha} \subset \mathcal{N}$ onto itself for $\rho = \rho_0$.

Now we prove that S is a contraction:

$$\|S(y)_j - S(z)_j\|_{\alpha} = \|u^*(\tau_j(y_j), y) - u^*(\tau_j(z_j), z)\|_{\alpha} \le$$

$$\leq \|u^*(\tilde{\tau}_j^1, y) - u^*(\tilde{\tau}_j^1, z)\|_\alpha + \|u^*(\tilde{\tau}_j^1, z) - u^*(\tilde{\tau}_j^2, z)\|_\alpha, \tag{7.67}$$

where $\tilde{\tau}_j^1 = \tau_j(y_j),\ \tilde{\tau}_j^2 = \tau_j(z_j)$.

Denote $\mathscr{J} = \cup \mathscr{J}_j$,

$$\mathscr{J}_j = (\max\{\tilde{\tau}_{j-1}^1, \tilde{\tau}_{j-1}^2\}, \min\{\tilde{\tau}_j^1, \tilde{\tau}_j^2\}] = (\tau_{j-1}'', \tau_j'].$$

Denote also $\xi_i = (\tau_i' - \tau_{j-1}'')/2,\ i \in Z$.

To estimate the difference $\|u^*(\tilde{\tau}_j^1, y) - u^*(\tilde{\tau}_j^1, z)\|_\alpha$, we apply iteration on n. Put $u_0(t,y) = u_0(t,z) = 0$. Then for $t \in (\tilde{\tau}_i'', \tilde{\tau}_{i+1}']$ we get

$$\begin{aligned}
&\|u_1(t,y) - u_1(t,z)\|_\alpha = \\
&= \| \sum_{k\leq i} A^\alpha U(t, \tilde{\tau}_k^1, y) g_k(y_k) - \sum_{k\leq i} A^\alpha U(t, \tilde{\tau}_k^2, z) g_k(z_k)\| \leq \\
&\leq \sum_{k\leq i} \|A^\alpha U(t, \tilde{\tau}_k^1, y)\left(g_k(y_k) - g_k(z_k)\right)\| + \|A^\alpha \left(U(t, \tilde{\tau}_i^1, y) - U(t, \tilde{\tau}_i^2, z)\right) g_i(z_i)\| + \\
&+ \sum_{k<i} \|A^\alpha \left(U(t, \tilde{\tau}_k^1, y) - U(t, \tilde{\tau}_k^2, z)\right) g_k(z_k)\| \leq \\
&\leq \sum_{k<i} M_1 e^{-\beta_1|t-\tilde{\tau}_k^1|} N_1 \|y_k - z_k\|_\alpha + \|A^\alpha e^{-A(t-\tilde{\tau}_i'')}(e^{-A(\tilde{\tau}_i''-\tilde{\tau}_i')} - I) g_i(z_i)\| + \\
&+ \sum_{k<i} \|(U(t, \xi_i, y)(U(\xi_i, \xi_{k+1}, y) U(\xi_{k+1}, \tilde{\tau}_k^1, y) - \\
&- U(t, \xi_i, z)(U(\xi_i, \xi_{k+1}, z) U(\xi_{k+1}, \tilde{\tau}_k^2, z)) g_k(z_k)\|_\alpha \leq \\
&\leq \frac{M_1 N_1}{1 - e^{-\beta_1 \theta}} \|y - z\|_S + C_\alpha C_0 (t - \tilde{\tau}_i'')^{-\alpha} |\tilde{\tau}_k'' - \tilde{\tau}_k'| \|g_i(z_i)\|_1 + \\
&+ \sum_{k<i} \|A^\alpha (U(t, \xi_i, y) - U(t, \xi_i, z)) U(\xi_i, \xi_{k+1}, y) U(\xi_{k+1}, \tilde{\tau}_k^1, y) g_k(z_k)\| + \\
&+ \sum_{k<i} \|A^\alpha U(t, \xi_i, z)(U(\xi_i, \xi_{k+1}, y) - U(\xi_i, \xi_{k+1}, z)) U(\xi_k, \tilde{\tau}_k^1, y) g_k(z_k)\| + \\
&+ \sum_{k<i} \|A^\alpha U(t, \xi_i, z) U(\xi_i, \xi_{k+1}, z)(U(\xi_{k+1}, \tilde{\tau}_k^1, y) - U(\xi_{k+1}, \tilde{\tau}_k^2, z)) g_k(z_k)\|.
\end{aligned} \tag{7.68}$$

To evaluate the difference $U(\xi_i, \xi_{k+1}, y) - U(\xi_i, \xi_{k+1}, z)$ we construct two sequences of bounded operators $X^\alpha \to X^\alpha$ defined by

$$T_n = U(\xi_{n+1}, \xi_n, y),\ \tilde{T}_n = U(\xi_{n+1}, \xi_n, z),\ n \in Z.$$

The corresponding difference equations $u_{n+1} = T_n u_n$ and $u_{n+1} = \tilde{T}_n u_n$ are exponentially stable. Their evolution operators

$$T_{n,m} = T_{n-1} \dots T_m, \ n \geq m, \ T_{m,m} = I,$$

and

$$\tilde{T}_{n,m} = \tilde{T}_{n-1} \dots \tilde{T}_m, \ n \geq m, \ \tilde{T}_{m,m} = I,$$

satisfy equality

$$\tilde{T}_{n,m} - T_{n,m} = \sum_{k<n} T_{n,k+1}(\tilde{T}_k - T_k)\tilde{T}_{k,m}, \ n \geq m. \tag{7.69}$$

Analogous to (7.32) and (7.33), we obtain

$$\|\tilde{T}_{n,m} - T_{n,m}\|_\alpha \leq M_2 e^{-\beta_2 \theta(n-m)} \sup_k \|\tilde{T}_k - T_k\|_\alpha, \ n \geq m, \tag{7.70}$$

with some $\beta_2 \leq \beta_1, M_2 \geq M_1$.

Now we estimate the difference $\|\tilde{T}_n - T_n\|_\alpha$:

$$\begin{aligned}
&\|T_n - \tilde{T}_n\|_\alpha = \|U(\xi_{n+1}, \xi_n, y) - U(\xi_{n+1}, \xi_n, z)\|_\alpha = \\
&= \|e^{-A(\xi_{n+1} - \tilde{\tau}_n^1)}(I + B_n)e^{-A(\tilde{\tau}_n^1 - \xi_n)} - e^{-A(\xi_{n+1} - \tilde{\tau}_n^2)}(I + B_n)e^{-A(\tilde{\tau}_n^2 - \xi_n)}\|_\alpha \leq \\
&\leq \|(e^{-A(\xi_{n+1} - \tilde{\tau}_n^1)} - e^{-A(\xi_{n+1} - \tilde{\tau}_n^2)})(I + B_n)e^{-A(\tilde{\tau}_n^1 - \xi_n)}\|_\alpha + \\
&+ \|e^{-A(\xi_{n+1} - \tilde{\tau}_n^2)}(I + B_n)(e^{-A(\tilde{\tau}_n^1 - \xi_n)} - e^{-A(\tilde{\tau}_n^2 - \xi_n)})\|_\alpha \leq \\
&\leq 2C_\alpha C_1(\theta/2)^{-1-\alpha}|\tilde{\tau}_n^1 - \tilde{\tau}_n^2|.
\end{aligned} \tag{7.71}$$

Therefore,

$$\begin{aligned}
&\|(\tilde{T}_{n,m} - T_{n,m})u\|_\alpha = \|(U(\xi_n, \xi_m, y) - U(\xi_n, \xi_m, z))u\|_\alpha \leq \\
&\leq M_2 e^{-\beta_2 \theta(n-m)} 2C_\alpha C_1(\theta/2)^{-1-\alpha} \sup_j |\tilde{\tau}_i^1 - \tilde{\tau}_i^2| \|u\|_\alpha, \ n \geq m.
\end{aligned} \tag{7.72}$$

To finish the estimation of (7.68), we consider the following two differences:

$$\begin{aligned}
&\|(U(t, \xi_i, y) - U(t, \xi_i, z)))u\|_\alpha \leq \|A^\alpha(e^{-A(t-\tau_i')}(I + B_i)e^{-A(\tau_i' - \xi_i)} - \\
&-e^{-A(t-\tau_i'')}(I + B_i)e^{-A(\tau_i'' - \xi_i)})u\| \leq \frac{4C_0 C_{1-\alpha}}{\theta(t - \tau_i'')^\alpha}|\tau_i'' - \tau_i'| \|u\|_\alpha.
\end{aligned} \tag{7.73}$$

$$\begin{aligned}
&\|(U(\xi_k, \tilde{\tau}_k^1, y) - U(\xi_k, \tilde{\tau}_k^2, z))u\|_\alpha = \|A^\alpha(I - e^{-A(\tau_k'' - \tau_k')})e^{-A(\xi_{k+1} - \tau_k'')}u\| \leq \\
&\leq C_0 C_1(\theta/2)^{-\alpha}|\tau_i'' - \tau_i'| \|u\|_\alpha.
\end{aligned} \tag{7.74}$$

Taking into account (7.70), (7.73), and (7.74), by (7.68) we obtain for $t \in (\tau_i'', \tau_{i+1}']$

$$\|u_1(t,y) - u_1(t,z)\|_\alpha \le N_1\|y-z\|_S \left(K_1' + K_2''(t-\tau_i'')^{-\alpha}\right), \tag{7.75}$$

where the positive constants K_1' and K_2'' don't depend on i.

Now we consider the $(n+1)$st iteration

$$\begin{aligned}
&\|u_{n+1}(t,y) - u_{n+1}(t,z)\|_\alpha = \\
&= \| \int_{-\infty}^t A^\alpha U(t,\tau,y) f(\tau,u_n(\tau,y))d\tau + \sum_{k\le i} A^\alpha U(t,\tilde\tau_k^1,y) g_k(y_k) - \\
&- \int_{-\infty}^t A^\alpha U(t,\tau,z) f(\tau,u_n(\tau,z))d\tau - \sum_{k\le i} A^\alpha U(t,\tilde\tau_k^2,z) g_k(z_k)\| \le \\
&\le \int_{-\infty}^t \|A^\alpha U(t,\tau,y)\left(f(\tau,u_n(\tau,y)) - f(\tau,u_n(\tau,z))\right)\|d\tau + \\
&+ \int_{-\infty}^t \|A^\alpha (U(t,\tau,y) - U(t,\tau,z)) f(\tau,u_n(\tau,z))\|d\tau + \\
&+ \sum_{k\le i} \|A^\alpha U(t,\tilde\tau_k^1,y)\left(g_k(y_k) - g_k(z_k)\right)\| + \\
&+ \sum_{k\le i} \|A^\alpha \left(U(t,\tilde\tau_k^1,y) - U(t,\tilde\tau_k^2,z)\right) g_k(z_k)\|.
\end{aligned} \tag{7.76}$$

Similar to (7.39), we get

$$\begin{aligned}
&\int_{\tau_i''}^t \|A^\alpha e^{-A(t-s)}\left(f(\tau,u_n(\tau,y)) - f(\tau,u_n(\tau,z))\right)\|d\tau + \\
&\sum_{k<i} \int_{\tau_k''}^{\tau_{k+1}'} \|A^\alpha U(t,\tau,y)\left(f(\tau,u_n(\tau,y)) - f(\tau,u_n(\tau,z))\right)\|d\tau + \\
&\le \frac{M_1}{1-e^{-\theta\beta_1}} \frac{C_\alpha \Theta^{1-\alpha}}{1-\alpha} N_1 \sup_{\tau\in\mathscr{J}} \|u_n(\tau,y) - u_n(\tau,z)\|,
\end{aligned}$$

$$\sum_{k\le i} \|A^\alpha U(t,\tilde\tau_k^1,y)\left(g_k(y_k) - g_k(z_k)\right)\| \le \frac{M_1}{1-e^{-\theta\beta_1}} N_1\|y-z\|_\alpha. \tag{7.77}$$

If $\|u_n(\tau,y)\|_\alpha \le \rho$ and $\|u_n(\tau,z)\|_\alpha \le \rho$, then for $t \in (\tau_i'', \tau_{i+1}']$

$$\sum_{k\le i} \int_{\tau_k'}^{\tau_k''} \|A^\alpha U(t,s,y)\left(f(s,u_n(s,y)) - f(s,u_n(s,z))\right)\|ds \le$$

$$\leq \sum_{k\leq i}\int_{\tau_k'}^{\tau_k''}\|U(t,s,y)f(s,u_n(s,y))\|_\alpha ds + \sum_{k\leq i}\int_{\tau_k'}^{\tau_k''}\|U(t,s,y)f(s,u_n(s,z))\|_\alpha ds \leq$$

$$\leq 2\sum_{k<i} M_1 e^{-\beta_1|t-\tau_k''|}(M_0+N_1\rho) + 2\int_{\tau_i'}^{\tau_i''}\|A^\alpha U(t,s,y)\|(M_0+N_1\rho)ds \leq$$

$$\leq \left(\frac{2M_1}{1-e^{-\beta_1\theta}} + \frac{2M_1}{1-\alpha}(t-\tau_i'')^{-\alpha}\right)(M_0+N_1\rho)N_1\|y-z\|_S, \tag{7.78}$$

since for $t > \tau_2 > \tau_1$

$$\int_{\tau_1}^{\tau_2}\frac{ds}{(t-s)^\alpha} \leq \frac{\tau_2-\tau_1}{(1-\alpha)((t-\tau_2)^\alpha}.$$

The second integral in (7.76) satisfies the following inequality:

$$I_2 = \int_{-\infty}^{t}\|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds \leq$$

$$\leq \int_{\tau_i''}^{t}\|A^\alpha(e^{-A(t-s)}-e^{-A(t-s)})f(s,u_n(s,z))\|ds +$$

$$+\int_{\tau_i'}^{\tau_i''}\|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds +$$

$$+\int_{\xi_i}^{\tau_i'}\|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds +$$

$$+\sum_{k<i}\int_{\xi_k}^{\xi_{k+1}}\|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds. \tag{7.79}$$

We consider all integrals in (7.79) separately.

$$I_{21} = \int_{\tau_i'}^{\tau_i''}\|A^\alpha U(t,s,y)f(s,u_n(s,z))\|ds \leq \frac{C_\alpha\|I+B_i\|(M_0+N_1\rho)}{(1-\alpha)(t-\tau_i'')^\alpha}|\tau_i''-\tau_i'|,$$

$$I_{22} = \int_{\tau_i'}^{\tau_i''}\|A^\alpha U(t,s,z)f(s,u_n(s,z))\|ds \leq \frac{C_\alpha\|I+B_i\|(M_0+N_1\rho)}{(1-\alpha)(t-\tau_i'')^\alpha}|\tau_i''-\tau_i'|,$$

$$I_{23} = \int_{\xi_i}^{\tau_i'}\|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds =$$

$$= \int_{\xi_i}^{\tau_i'}\|A^\alpha(U(t,\tilde\tau_i^1,y)U(\tilde\tau_i^1,s,y)-U(t,\tilde\tau_i^2,z)U(\tilde\tau_i^2,s,z))f(s,u_n(s,z))\|ds \leq$$

$$\leq \int_{\xi_i}^{\tau_i'} \|A^\alpha\Big((e^{-A(t-\tilde{\tau}_i^1)} - e^{-A(t-\tilde{\tau}_i^2)})(I+B_i)e^{-A(\tilde{\tau}_i^1-s)} -$$

$$-A^\alpha e^{-A(t-\tilde{\tau}_i^2)})(I+B_i)(e^{-A(\tilde{\tau}_i^1-s)} - e^{-A(\tilde{\tau}_i^2-s)})\Big)f(s,u_n(s,z))\|ds \leq$$

$$\leq \frac{2C_0 C_{\alpha_1} C_{1+\alpha-\alpha_1}}{(t-\tau_i'')^{\alpha_1}}\|I+B_i\|\frac{(\tau_i'-\xi_i)^{\alpha_1-\alpha}}{\alpha_1-\alpha}|\tau_i''-\tau_i'|.$$

The last sum in (7.79) is transformed as follows:

$$I_{24} = \sum_{k<i}\int_{\xi_k}^{\xi_{k+1}} \|A^\alpha(U(t,s,y)-U(t,s,z))f(s,u_n(s,z))\|ds =$$

$$= \sum_{k<i}\int_{\xi_k}^{\xi_{k+1}} \|(U(t,\xi_i,y)U(\xi_i,\xi_{k+1},y)U(\xi_{k+1},s,y) -$$

$$-U(t,\xi_i,z)U(\xi_i,\xi_{k+1},z)U(\xi_{k+1},s,z))f(s,u_n(s,z))\|_\alpha ds \leq$$

$$\leq \sum_{k<i}\int_{\xi_k}^{\xi_{k+1}}\Big(\|(U(t,\xi_i,y)-U(t,\xi_i,z))U(\xi_i,\xi_{k+1},y)U(\xi_{k+1},s,y)f(s,u_n(s,z))\|_\alpha +$$

$$+\|U(t,\xi_i,z)(U(\xi_i,\xi_{k+1},y)-U(\xi_i,\xi_{k+1},z))U(\xi_{k+1},s,y)f(s,u_n(s,z))\|_\alpha +$$

$$+\|U(t,\xi_i,z)U(\xi_i,\xi_{k+1},z)(U(\xi_{k+1},s,y)-U(\xi_{k+1},s,z))f(s,u_n(s,z))\|_\alpha\Big)ds.$$

To finish the estimation of integral I_{24} we use (7.72), (7.73), and (7.74):

$$\int_{\xi_k}^{\xi_{k+1}} \|A^\alpha(U(\xi_{k+1},s,y)-U(\xi_{k+1},s,z))f\|_\alpha ds \leq$$

$$\leq \int_{\tau_k''}^{\xi_{k+1}} \|A^\alpha(e^{-A(\xi_{k+1}-s)} - e^{-A(\xi_{k+1}-s)})f\|ds +$$

$$+\int_{\tau_k'}^{\tau_k''} \|A^\alpha(e^{-A(\xi_{k+1}-\tau_k'')}(I+B_k)e^{-A(\tau_k''-s)} - e^{-A(\xi_{k+1}-s)})f\|ds +$$

$$+\int_{\xi_k}^{\tau_k'} \|(e^{-A(\xi_{k+1}-\tau_k')}(I+B_k)e^{-A(\tau_k'-s)} - e^{-A(\xi_{k+1}-\tau_k'')}(I+B_k)e^{-A(\tau_k''-s)})f\|_\alpha ds \leq$$

$$\leq \tilde{K}C_{\alpha_1}(\xi_{k+1}-\tau_k'')^{-\alpha_1}\|I+B_k\||\tau_k''-\tau_k'|\|f\|$$

with some positive constant $\tilde{K}$. Therefore,

$$I_2 \leq \left(K_2'N_1 + \frac{K_2''N_1}{(t-\tau_i'')^{\alpha_1}}\right)\|y-z\|_S \tag{7.80}$$

with $\alpha_1 > \alpha$ and positive constants K_1' and K_2'' independent of i, k.

By (7.75), (7.79), and (7.80) we obtain for $t \in (\tau_i'', \tau_{i+1}']$

$$\begin{aligned}
&\|u_{n+1}(t,y) - u_{n+1}(t,z)\|_\alpha \le \\
&\le \sum_{k<i} \int_{\tau_k''}^{\tau_{k+1}'} \|A^\alpha U(t,\tau,y)(f(\tau,u_n(\tau,y)) - f(\tau,u_n(\tau,z)))\| d\tau + \\
&+ \int_{\tau_i''}^{t} \|A^\alpha U(t,\tau,y)(f(\tau,u_n(\tau,y)) - f(\tau,u_n(\tau,z)))\| d\tau + \\
&+ \left(K_3' + \frac{K_3''}{(t-\tau_i'')^{\alpha_1}}\right) N_1 \|y-z\|_S,
\end{aligned} \tag{7.81}$$

where the constants K_3' and K_3'' don't depend on n.

Let the nth iteration satisfy the inequality

$$\|u_n(t,y) - u_n(t,z)\|_\alpha \le \left(L_n' + \frac{L_n''}{(t-\tau_i'')^{\alpha_1}}\right) N_1 \|y-z\|_S, \ t \in (\tau_i'', \tau_{i+1}'],$$

with positive constants L_n' and L_n''. We estimate the $(n+1)$st iteration.

$$\begin{aligned}
&\|u_{n+1}(t,y) - u_{n+1}(t,z)\|_\alpha \le \left(K_3' + \frac{K_3''}{(t-\tilde{\tau}_i'')^{\alpha_1}}\right) N_1 \|y-z\|_S + \\
&+ N_1^2 \|y-z\|_S \sum_{k<i} \int_{\tau_k''}^{\tau_{k+1}'} \|A^\alpha U(t,s)\| \left(L_n' + \frac{L_n''}{(s-\tau_k'')^{\alpha_1}}\right) ds + \\
&+ N_1^2 \|y-z\|_S \int_{\tau_i''}^{t} \|A^\alpha U(t,s)\| \left(L_n' + \frac{L_n''}{(s-\tau_i'')^{\alpha_1}}\right) ds \le \\
&\le N_1^2 \|y-z\|_S \left(\sum_{k\le i} \int_{\tau_k''}^{\tau_{k+1}'} M_1 e^{-\beta_1|t-s|} \left(L_n' + \frac{L_n''}{(s-\tau_k'')^{\alpha_1}}\right) ds + \right. \\
&\left. + \int_{\tau_i''}^{t} M_1 (t-s)^{-\alpha_1} \left(L_n' + \frac{L_n''}{(s-\tau_i'')^{\alpha_1}}\right) ds \right) + \\
&+ \left(K_3' + \frac{K_3''}{(t-\tau_i'')^{\alpha_1}}\right) N_1 \|y-z\|_S \le \\
&\le \left(\frac{M_1}{1-e^{-\beta_1\theta}} \left(L_n' Q + \frac{L_n'' Q^{1-\alpha_1}}{1-\alpha_1}\right) + \frac{L_n'' M_1 2^{2\alpha}}{1-\alpha_1} (t-\tau_i'')^{1-2\alpha_1} + \right. \\
&\left. + \frac{L_n' M_1}{1-\alpha_1} (t-\tau_i'')^{1-\alpha_1} \right) N_1^2 \|y-z\|_S + \left(K_3' + \frac{K_3''}{(t-\tau_i'')^{\alpha_1}}\right) N_1 \|y-z\|_S \le
\end{aligned}$$

$$= \left(L'_{n+1} + \frac{L''_{n+1}}{(t-\tau''_i)^{\alpha_1}} \right) N_1 \|y-z\|_S. \tag{7.82}$$

One can verify that for sufficiently small N_1 the sequences L'_n and L''_n are uniformly bounded by some constants L'_* and L''_*.

Since the sequences $u_n(t,y)$ and $u_n(t,z)$ tend to limit the functions $u_*(t,y)$ and $u_*(t,z)$, respectively, we conclude by (7.82) for $t \in (\tau''_i, \tau'_{i+1}]$ that

$$\|u_*(t,y) - u_*(t,z)\|_\alpha \le \left(L'_* + \frac{L''_*}{(t-\tau''_{i+1})^{\alpha_1}} \right) N_{i+1} \|y-z\|_S$$

and

$$\|u^*(\tau'_{i+1}, y) - u^*(\tau'_{i+1}, z)\|_\alpha \le \left(L'_* + \frac{L''_*}{\theta^{\alpha_1}} \right) N_1 \|y-z\|_S. \tag{7.83}$$

Now we estimate the second summand in (7.67). Note that by our assumption $\tilde{\tau}^1_j < \tilde{\tau}^2_j$.

$$\|u^*(\tilde{\tau}^1_j, z) - u^*(\tilde{\tau}^2_j, z)\|_\alpha = \left\| \int_{\tilde{\tau}^1_j}^{\tilde{\tau}^2_j} \frac{d}{ds} u^*(s,z) ds \right\|_\alpha .$$

By Theorem 3.5.2, [9], at the interval $(\tilde{\tau}^2_{j-1}, \tilde{\tau}^2_j)$ the derivative satisfies

$$\left\| \frac{d}{ds} u^*(s,z) \right\|_\gamma \le \tilde{K}_1 (s - \tilde{\tau}^2_{j-1})^{\alpha-\gamma-1}$$

with some positive constant $\tilde{K}_1$ independent of j and initial value from U^α_ρ.

Then for $t \in (\tilde{\tau}^1_j, \tilde{\tau}^2_j)$

$$\left\| \frac{d}{ds} u^*(s,z) \right\|_\gamma \le \tilde{K}_1 \left(\frac{\theta}{2} \right)^{\alpha-\gamma-1} = \tilde{K}_2$$

and

$$\|u^*(\tilde{\tau}^1_j, z) - u^*(\tilde{\tau}^2_j, z)\|_\alpha \le \tilde{K}_2 |\tilde{\tau}^1_j - \tilde{\tau}^2_j| \le \tilde{K}_2 N_1 \|y-z\|_S. \tag{7.84}$$

By (7.83) and (7.84) we have

$$\|u^*(\tilde{\tau}^1_j, z) - u^*(\tilde{\tau}^2_j, z)\|_\alpha = \Gamma_9 \|y-z\|_S, \tag{7.85}$$

where $\Gamma_9 < 1$ uniformly for j and $y, z \in \mathcal{N}_{\varrho_0}$.

By (7.67), (7.83), and (7.85) we conclude that the map $S : \mathcal{N}_{\varrho_0} \to \mathcal{N}_{\varrho_0}$ is a contraction. Therefore, there exists a unique almost periodic sequence $y^* = \{y_j^*\}$ such that $u^*(\tau_j(y_j^*), y^*) = y_j^*$ for all $j \in Z$. The function $u^*(t, y^*)$ is the W-almost periodic solution of Eqs. (7.57) and (7.58).

2. Now we prove the stability of the almost periodic solution. Fix arbitrary $\varepsilon > 0$ and $\eta > 0$. Let $t_0 \in [\tau_0(0) + \eta, \tau_1(0) - \eta]$.

   The W-almost periodic solution $u_0(t)$ satisfies the integral equation

$$u_0(t) = U_0(t, t_0)u_0 + \int_{t_0}^{t} U_0(t, s)f(s, u_0(s))ds + \sum_{t_0 < \tau_j^0 < t} U_0(t, \tau_j^0)g_j(\tau_j^0), \quad (7.86)$$

where $\tau_j^0 = \tau_j(u_0(\tau_j^0))$ and $U_0(t, s)$ is the evolution operator of the linear equation

$$\frac{du}{dt} + Au = 0, \ u(\tau_j^0 + 0) - u(\tau_j^0) = B_j u(\tau_j^0), \ j = 1, 2, \ldots.$$

Let $u_1 \in X^\alpha$ such that $\|u_0 - u_1\|_\alpha < \delta$. The solution $u_1(t)$ with initial value $u_1(t_0) = u_1$ satisfies equation

$$u_1(t) = U_1(t, t_0)u_1 + \int_{t_0}^{t} U_1(t, s)f(s, u_1(s))ds + \sum_{t_0 < \tau_j^1 < t} U_1(t, \tau_j^1)g_j(\tau_j^1), \quad (7.87)$$

where $\tau_j^1 = \tau_j(u_1(\tau_j^1))$ and $U_1(t, s)$ is the evolution operator of the linear equation

$$\frac{du}{dt} + Au = 0, \ u(\tau_j^1 + 0) - u(\tau_j^1) = B_j u(\tau_j^1), \ j = 1, 2, \ldots.$$

By Lemma 5, for a sufficiently small Lipschitz constant N_1 the evolution operator $U_0(t, s)$ satisfies the inequality

$$\|U_0(t, s)u\|_\alpha \leq M_1 e^{-\beta_1(t-s)}\|u\|_\alpha, \ t \geq s, \quad (7.88)$$

with some positive constants $\beta_1 \leq \beta, M_1 \geq M$. Moreover, one can verify that for some domain $U^\alpha_{\tilde{\rho}}, \tilde{\rho} \leq \rho$, and $N_1 \leq N_0$ the evolution operator satisfies

$$\|U_1(t, s)u\|_\alpha \leq M_1 e^{-\beta_1(t-s)}\|u\|_\alpha, \ t \geq s, \ t, s \in [t_0, t_0 + T], \quad (7.89)$$

if the values $u_1(t)$ belong to $U^\alpha_{\tilde{\rho}}$ for $\tau_j^1 \in [t_0, t_0 + T]$.

At the interval without impulses, the difference between solutions $u_0(t) - u_1(t)$ satisfies the inequality

$$\|u_1(t) - u_0(t)\|_\alpha \leq \|e^{-A(t-t_1)}(u_0(t_1) - u_1(t_1))\|_\alpha +$$

$$+\int_{t_1}^{t}\|A^{\alpha}e^{-A(t-t_1)}(f(s,u_1(s))-f(s,u_0(s)))\|ds\le$$

$$\le M_1e^{-\beta_1(t-t_1)}\|u_0(t_1)-u_1(t_1)\|_{\alpha}+\int_{t_1}^{t}\frac{M_1N_1e^{-\beta_1(t-s)}}{(t-s)^{\alpha}}\|u_1(s)-u_0(s)\|_{\alpha}ds.$$

Then by Lemma 2,

$$\|u_1(t)-u_0(t)\|_{\alpha}\le M_1\tilde{C}e^{-\beta_1(t-t_1)}\|u_1(t_1)-u_0(t_1)\|_{\alpha},\ t-t_1\le Q. \tag{7.90}$$

Hence, if initial values belong to the bounded domain from X^{α}, then the corresponding solutions are uniformly bounded for t from the bounded interval.

Assume for definiteness that $\tau_j^0\ge\tau_j^1$ and estimate $|\tau_j^1-\tau_j^0|$ by $(u_1(\tau_j^1)-u_0(\tau_j^1))$.

$$\|(u_1(\tau_j^1)-u_0(\tau_j^0)\|_{\alpha}\le\|(u_0(\tau_j^1)-u_0(\tau_j^0)\|_{\alpha}+\|u_0(\tau_j^1)-u_1(\tau_j^1)\|_{\alpha}\le$$

$$\le\left\|\int_{\tau_j^1}^{\tau_j^0}\frac{d}{d\xi}u_0(\xi)d\xi\right\|_{\alpha}+\|u_0(\tau_j^1)-u_1(\tau_j^1)\|_{\alpha}\le$$

$$\le\tilde{K}_2|\tau_j^1-\tau_j^0|+\|u_0(\tau_j^1)-u_1(\tau_j^1)\|_{\alpha}.$$

Hence,

$$|\tau_j^1-\tau_j^1|\le\|u_0(\tau_j^0)-u_1(\tau_j^1)\|_{\alpha}\le\frac{N_1}{1-\tilde{K}_2N_1}\|u_0(\tau_j^1)-u_1(\tau_j^1)\|_{\alpha}. \tag{7.91}$$

We assume that $t\in(\tau_i'',\tau_{i+1}']$ and estimate the difference

$$\|u_0(t)-u_1(t)\|_{\alpha}=\|U_0(t,t_0)(u_0-u_1)\|_{\alpha}+\|(U_0(t,t_0)-U_1(t,t_0))u_1\|_{\alpha}$$

$$+\int_{t_0}^{t}\|U_0(t,s)f(s,u_0(s))-U_1(t,s)f(s,u_1(s))\|_{\alpha}ds+$$

$$+\|\sum_{t_0<\tau_j^1<t}U(t,\tau_j^1)g_j(\tau_j^1)-\sum_{t_0<\tau_j^0<t}U(t,\tau_j^0)g_j(\tau_j^0)\|_{\alpha}\le$$

$$\le\|U_0(t,t_0)(u_0-u_1)\|_{\alpha}+\|(U_0(t,t_0)-U_1(t,t_0))u_1\|_{\alpha}+$$

$$+\int_{t_0}^{\tau_1'}\|U_0(t,s)f(s,u_0(s))-U_1(t,s)f(s,u_1(s))\|_{\alpha}ds+$$

$$+\sum_{j=1}^{i-1}\int_{\tau_j''}^{\tau_{j+1}'}\|U_0(t,s)(f(s,u_0(s))-f(s,u_1(s)))\|_{\alpha}ds+$$

$$+\sum_{j=1}^{i-1}\int_{\tau_j''}^{\tau_{j+1}'}\|(U_0(t,s)-U_1(t,s))f(s,u_1(s))\|_{\alpha}ds+$$

$$+\sum_{j=1}^{i}\int_{\tau_j'}^{\tau_j''}\|U_0(t,s)f(s,u_0(s))-U_1(t,s)f(s,u_1(s))\|_\alpha ds+$$

$$+\int_{\tau_i''}^{t}\|U_0(t,s)f(s,u_0(s))-U_1(t,s)f(s,u_1(s))\|_\alpha ds+$$

$$+\sum_{j=1}^{i}\|U_0(t,\tau_j^0)g_j(\tau_j^0)-U_1(t,\tau_j^1)g_j(\tau_j^1)\|_\alpha. \tag{7.92}$$

Denote $v(t)=\|u_0(t)-u_1(t)\|_\alpha$. Assume that for $t\in[t_0,\tau_i']$ the values $u(t)$ belong to $U_{\tilde\rho}^\alpha$; hence, the evolution operators $U_0(t,\tau)$ and $U_1(t,\tau)$ satisfy (7.88) and (7.89) at this interval. By (7.92), analogous to the proof of (7.75), (7.79), and (7.80), we conclude that there exist positive constants M_2 and P_1 independent of i such that for $t\in\mathscr{J}_{i+1}$

$$v(t)\le M_1e^{-\beta_1(t-t_0)}v(t_0)+\int_{t_0}^{\tau_1'}\frac{M_2N_1}{(\tau_1'-s)^\alpha}e^{-\beta_1(t-\tau_1'')}v(s)ds+$$

$$+\sum_{j=2}^{i-1}\int_{\tau_{j-1}''}^{\tau_j'}M_2N_1e^{-\beta_1(t-\tau_j'')}v(s)ds+\sum_{j=1}^{i-1}P_1N_1e^{-\beta_1(t-\tau_j'')}v(\tau_j')+$$

$$+\frac{1}{(t-\tau_i'')^{\alpha_1}}\left(\int_{\tau_{i-1}''}^{\tau_i'}M_2N_1e^{-\beta_1(t-\tau_i'')}v(s)ds+P_1N_1e^{-\beta_1(t-\tau_i'')}v(\tau_i')\right)+$$

$$+\int_{\tau_i''}^{t}M_2N_1e^{-\beta_1(t-s)}(t-s)^{-\alpha_1}v(s)ds \tag{7.93}$$

with $\alpha_1>\alpha$. By (7.90), at the interval $[t_0,\tau_1']$ $v(t)$ satisfies

$$v(t)\le M_1\tilde{C}e^{-\beta(t-t_0)}v(t_0),\ t\in[t_0,\tau_1']. \tag{7.94}$$

By (7.93) and (7.94), for $t\in(\tau_1'',\tau_2']$ we get

$$v(t)\le M_1e^{-\beta_1(t-t_0)}v(t_0)+\frac{1}{(t-\tau_1'')^{\alpha_1}}\int_{t_0}^{\tau_1'}M_2N_1e^{-\beta_1(t-\tau_1'')}v(s)ds+$$

$$+P_1N_1e^{-\beta_1(t-\tau_1'')}(t-\tau_1'')^{-\alpha_1}v(\tau_1')+\int_{\tau_1''}^{t}M_2N_1e^{-\beta_1(t-s)}(t-s)^{-\alpha_1}v(s)ds.$$

Hence, for $M_3=M_2e^{\beta_1 Q},\tilde{C}_1=\tilde{C}/(1-\alpha),\ v_1(t)=e^{\beta_1 t}v(t)$ and $P_2=P_1e^{\beta_1\sup_j|\tau_j''-\tau_j'|}$

$$v_1(t) \le M_1 v_1(t_0)\left(1+\frac{N_1\tilde{C}(M_3\tilde{Q}+P_2)}{(t-\tau_1'')^{\alpha_1}}\right)+\int_{\tau_1''}^{t} M_2N_1(t-s)^{-\alpha_1}v_1(s)ds.$$

By Lemma 2

$$v(t) \le M_1\tilde{C}_1 v(t_0)e^{-\beta_1(t-t_0)}\left(1+\frac{N_1\tilde{C}(M_3\tilde{Q}+P_2)}{(t-\tau_1'')^{\alpha_1}}\right),\ t\in(\tau_1'',\tau_2']. \quad (7.95)$$

Denote $\tilde{Q} = \max_j\{1,(\tau_{j+1}'-\tau_j'')\}$ and $\tilde{\theta} = \min_j\{1,(\tau_{j+1}'-\tau_j'')\}$. Let us prove that

$$v(t)\le M_1\tilde{C}_1 v(t_0)e^{-\beta_1(t-t_0)}\left(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(t-\tau_j'')^{\alpha_1}}\right)\left(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(1-\alpha_1)\tilde{\theta}^{\alpha_1}}\right)^{i-1} \quad (7.96)$$

for $t\in(\tau_i'',\tau_{i+1}']i\ge 2$. We apply the method of mathematical induction. Assume that (7.96) is true for $t\in[\tau_{i-1}'',\tau_i']$ and prove it for $t\in[\tau_i'',\tau_{i+1}']$. Really, by (7.93) for $t\in[\tau_i'',\tau_{i+1}']$ we have

$$\begin{aligned}
&v(t)\le M_1e^{-\beta_1(t-t_0)}v(t_0)\Bigg(\big(1+(M_3\tilde{Q}+P_2)N_1\tilde{C}\big)+\\
&+\sum_{j=2}^{i-1}\mathscr{A}^jM_3N_1\tilde{Q}\tilde{C}_1+\sum_{j=2}^{i-1}\mathscr{A}^{j-1}\left(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{\tilde{\theta}^{\alpha_1}}\right)N_1P_2\tilde{C}_1+\\
&+\mathscr{A}^{i-2}\Bigg(N_1M_3\tilde{C}_1\Big((\tau_i'-\tau_{i-1}'')+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)(\tau_i'-\tau_{i-1}'')}{(1-\alpha_1)(\tau_i'-\tau_{i-1}'')^{\alpha_1}}\Big)+\\
&+N_1P_2\tilde{C}_1\Big(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(\tau_i'-\tau_{i-1}'')^{\alpha_1}}\Big)\Bigg)\Bigg)+\mathscr{B}(t)\le\\
&\le\mathscr{A}+\sum_{j=2}^{i-1}\mathscr{A}^{j-1}(1+N_1\tilde{C}_1(M_3\tilde{Q}+P_2)-1)+\frac{\mathscr{A}^{i-1}N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(t-\tau_i'')^{\alpha_1}}+\\
&+\mathscr{B}(t)\le\mathscr{A}^{i-1}\left(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(t-\tau_i'')^{\alpha_1}}\right)+\mathscr{B}(t).
\end{aligned}$$

where

$$\mathscr{A}=\left(1+\frac{N_1\tilde{C}_1(M_3\tilde{Q}+P_2)}{(1-\alpha)\tilde{\theta}^{\alpha_1}}\right),\ \mathscr{B}(t)=\int_{\tau_i''}^{t}\frac{M_2N_1}{(t-s)^{\alpha_1}}e^{-\beta_1(t-s)}v(s)ds.$$

Hence, for $t \in (\tau_i'', \tau_{i+1}']$, the function $v_1(t) = e^{\beta_1 t} v(t)$ satisfies the inequality

$$v_1(t) \leq \mathscr{A}^{i-1} \left(1 + \frac{N_1 \tilde{C}_1 (M_3 \tilde{Q} + P_2)}{(t - \tau_i'')^{\alpha_1}} \right) + M_2 N_1 \int_{\tau_i''}^{t} (t-s)^{-\alpha_1} v_1(s) ds.$$

Applying Lemma 2, we obtain (7.96).

Let $N_1 > 0$ be such that $\mathscr{A}^{i(t_0,t)} e^{-\beta_1 (t-t_0)} < e^{-\delta_1 (t-t_0)}$ for some positive δ_1. For the given $\varepsilon > 0$ and $\eta > 0$ we choose $v(t_0) = v_0$ such that

$$M_1 \tilde{C}_1 v_0 \left(1 + \frac{N_1 \tilde{C}_1 (M_3 \tilde{Q} + P_2)}{\eta^{\alpha_1}} \right) < \varepsilon.$$

This proves the asymptotic stability of solution u_0.

Example 1. Let us consider the parabolic equation with impulses in variable moments of time:

$$u_t = u_{xx} + a(t) u_x + b(t,x), \tag{7.97}$$

$$\Delta u \Big|_{t=\tau_j(u)} = u(\tau_j(u) + 0, x) - u(\tau_j(u), x) = -a_j u(\tau_j(u), x), \tag{7.98}$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \tag{7.99}$$

where the sequence of hypersurfaces τ_j is defined by

$$\tau_j(u) = \theta_j + b_j \int_0^{\pi} u^2(\xi) d\xi, \tag{7.100}$$

where the sequence of real numbers $\{\theta_j\}$ has uniformly almost periodic sequences of differences and $\theta_{j+1} - \theta_j \geq \theta \geq 1/2$,

$\{a_j\}$ and $\{b_j\}$ are almost periodic sequences of positive numbers,

$a(t)$ is a Bohr almost periodic function,

$b(t,x)$ is a Bohr almost periodic function in t uniformly with respect to $x \in [0, \pi]$.

Denote

$$X = L_2(0,\pi),\ A = -\frac{\partial^2}{\partial x^2},\ X^1 = D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

The operator A is sectorial with simple eigenvalues $\lambda_k = k^2$ and corresponding eigenfunctions

$$\varphi_k(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin kx,\ k = 1, 2, \ldots.$$

Operator $-A$ generates an analytic semigroup e^{-At}.

Let $u = \sum_{k=1}^{\infty} a_k \sin kx,\ a_k = \frac{1}{\pi}\int_0^{\pi} u(x) \sin kx dx$. Then

$$Au = \sum_{k=1}^{\infty} k^2 a_k \sin kx,\ A^{\alpha}u = \sum_{k=1}^{\infty} k^{2\alpha} a_k \sin kx,\ e^{-At} = \sum_{k=1}^{\infty} e^{-k^2 t} a_k \sin kx.$$

Hence,

$$X^{1/2} = D(A^{1/2}) = H_0^1(0, \pi).$$

Let us consider Eqs. (7.97)–(7.99) in space $X^{1/2} = D(A^{1/2}) = H_0^1(0, \pi)$:

$$\frac{du}{dt} + Au = f(t, u),\ u(\tau_j(u) + 0) = (1 - a_j)u(\tau_j(u)),\ j = 0, \ldots,$$

where $f(t, u) : R \times X^{1/2} \to X,\ f(t, u)(x) = a(t)u_x + b(t, x)$.

We verify that in some domain $\mathscr{D} = \{u \geq 0, \|u\| \leq \rho\}$ solutions of (7.97)–(7.99) don't have beating at the surfaces $t = \tau_j(u)$. Assume to the contrary that solution $u(t)$ intersects the surface $t = \tau_j(u)$ at two points t_j^1 and t_j^2, $t_j^1 < t_j^2$.

Denote $u(t_j^1) = u_1, u(t_j^2) = u_2, \tilde{u} = e^{-A(t_j^2 - t_j^1)}u(t_j^1 + 0)$. Then $u(t_j^1 + 0) = (1 - a_j)u_1,\ \tau_j(u_1)) = t_j^1,\ \tau_j(u_2)) = t_j^2$, and

$$u_2 = e^{-A(t_j^2 - t_j^1)}u(t_j^1 + 0) + \int_{t_j^1}^{t_j^2} e^{-A(t_j^2 - s)} f(s, u(s))ds.$$

We have

$$|\tau_j(u_2) - \tau_j(\tilde{u})| \leq b_j |\int_0^t (u_2(t, x) - \tilde{u}(t, x))(u_2(t, x) + \tilde{u}(t, x))dx \leq$$
$$\leq b_j \|u_2(t, x) - \tilde{u}(t, x)\|_{L_2} \|u_2(t, x) + \tilde{u}(t, x)\|_{L_2} \leq$$
$$\leq b_j \| \int_{t_j^1}^{t_j^2} e^{-A(t_j^2 - s)} f(s, u(s))ds\|_{L_2} \|u_2(t, x) + \tilde{u}(t, x)\|_{L_2}.$$

The function $f(t, u)$ satisfies $\|f(t, u)\|_X \leq K(1 + \|u\|_{X^{1/2}})$; hence, solutions of the equation without impulses exist for all $t \geq t_0$ and there exist positive constants M_1 and M_2 such that $M_2 \geq \sup_{u \in \mathscr{D}} \|f(t, u)\|_{L_2},\ M_3 \geq \sup_{u \in \mathscr{D}} \|u_2(t, x) + \tilde{u}(t, x)\|_{L_2}$. Therefore, $\tau_j(u_2) - \tau_j(\tilde{u}) \leq b_j |t_j^2 - t_j^1| M_2 M_3$. By sufficiently small $b = \sup_j b_j$ we have $bM_2M_3 < 1$ and

$$0 < t_j^2 - t_j^1 = \tau_j(u_2) - \tau_j(u_1) \le \tau_j(u_2) - \tau_j(\tilde{u}) + \tau_j(\tilde{u}) - \tau_j(u_1),$$

$$t_j^2 - t_j^1 \le \frac{1}{1 - bM_2M_3}(\tau_j(\tilde{u}) - \tau_j(u_1)) \le \frac{b_j((1-a_j)^2 - 1)}{1 - bM_2M_3}\|u_1\|_{L_2}^2 < 0.$$

This contradicts our assumption.

Corresponding to (7.97)–(7.99), the linear impulsive equation is exponentially stable in space $X^{1/2}$. By Theorem 4, for sufficiently small $b = \sup_j b_j$ and $a = \sup_t |a(t)|$ the equation has an asymptotically stable W-almost periodic solution.

7.6 Equations with Unbounded Operators B_j

Many results in our chapter remain true if operators B_j in linear parts of impulsive action are unbounded. We refer to [27], where the following semilinear impulsive differential equation

$$\frac{du}{dt} = Au + f(t,u), \quad t \neq \tau_j, \tag{7.101}$$

$$\Delta u|_{t=\tau_j} = u(\tau_j) - u(\tau_j - 0) = B_j u(\tau_j - 0) + g_j(u(t_j - 0)), \quad j \in Z, \tag{7.102}$$

was studied. Here $u : R \to X$, X is a Banach space, A is a sectorial operator in X, $\{B_j\}$ is a sequence of some closed operators, and $\{\tau_j\}$ is an unbounded and strictly increasing sequence of real numbers. Assume that the equation satisfies conditions $(H1)$, $(H3)$, $(H5)$, $(H6)$, and

(H4u) the sequence $\{B_j\}$ of closed linear operators $B_j \in L(X^{\alpha+\gamma}, X^{\alpha})$ is almost periodic in the space $L(X^{\alpha+\gamma}, X^{\alpha})$, for $\alpha \ge 0$ and some $\gamma \ge 0$.

As in [17], we assume that solutions $u(t)$ of (7.1), (7.2) are right-hand-side continuous; hence, $u(\tau_j) = u(\tau_j + 0)$ at all points of impulsive action. Due to such a selection we avoid considering operators $e^{-A(t-\tau_j)}(I + B_j)$ with unbounded operator B_j and can work with the family of bounded operators $e^{-A(t-\tau_j)}$.

Since the operator A is sectorial and operators B_j are subordinate to A, an evolution operator of a corresponding linear impulsive equation is constructed correctly. Now analogs of the theorems 2 and 3 can be proven.

Example 2 ([27]). We consider the following parabolic equation with impulsive action:

$$u_t = u_{xx} + f(t,x), \tag{7.103}$$

$$\Delta u\Big|_{t=\tau_j} = u(\tau_j, x) - u(\tau_j - 0, x) = b_k(\sin x)u_x + c_k x(\pi - x), \tag{7.104}$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \tag{7.105}$$

where $\{\tau_j\}$ is a sequence of real numbers with uniformly almost periodic sequences of differences, $\tau_{j+1} - \tau_j \geq \theta \geq 1/2$,

$\{b_j\}$ and $\{c_j\}$ are almost periodic sequences of real numbers,

$f(t,x)$ is almost periodic and locally Hölder continuous with respect to t and for every fixed t belongs to $L_2(0,\pi)$.

As in Example 1, denote

$$X = L_2(0,\pi),\ A = -\frac{\partial^2}{\partial x^2},\ X^1 = D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

The operator A is sectorial with simple eigenvalues $\lambda_k = k^2$ and corresponding eigenfunctions $\varphi_k(x) = \sin kx, k = 1,2,\ldots$.

Operators B_j have form $B_j = b_j \sin x \frac{\partial}{\partial x}$.

If $u = \sum_{k=1}^{\infty} a_k \sin kx,\ a_k = \frac{1}{\pi}\int_0^{\pi} u(x)\sin kx dx$, then

$$B_j u = b_j \sin x u_x = b_j \sin x \sum_{k=1}^{\infty} a_k k \cos kx = \frac{b_j}{2}(R-L)A^{1/2}u = b_j T A^{1/2} u,$$

where $Ru = \sum_{k=1}^{\infty} a_k \sin(k-1)x$ and $Lu = \sum_{k=1}^{\infty} a_k \sin(k+1)x$ are bounded shift operators in X. Hence, operators $B_j : X^{\alpha+1/2} \to X^{\alpha}$ are linear continuous, $\alpha \geq 0$.

By (7.13), the evolution operator for homogeneous equations (7.103) and (7.104) is

$$U(t,s) = e^{-A(t-s)},\ \text{if}\ \tau_k \leq s \leq t < \tau_{k+1},$$

and

$$U(t,s) = e^{-A(t-\tau_k)}(I+B_k)e^{-A(\tau_k-\tau_{k-1})}\ldots(I+B_m)e^{-A(\tau_m-s)}$$

if $\tau_{m-1} \leq s < \tau_m < \tau_{m+1}\ldots\tau_k \leq t < \tau_{k+1},\ m < k,\ k,m \in Z$.

Theorem 5. *Let $p\ln(1+b) < 1$, where p is defined by (7.3) and $b = \sup_j |b_j|$. Then Eqs. (7.103) and (7.104) with boundary conditions (7.105) have a unique W-almost periodic solution which is asymptotically stable.*

Proof. We show that the unique almost periodic solution of (7.103) and (7.104) is given as function $R \to L_2(0,\pi)$ by formula

$$u_0(t) = \int_{-\infty}^{t} U(t,s)\tilde{f}(s)ds + \sum_{\tau_j \leq t} U(t,\tau_j)\tilde{g}_j, \tag{7.106}$$

where $\tilde{f}(t) \equiv f(t,.) : R \to L_2(0,\pi)$, $g_j(x) = c_j x(\pi - x)$, $\tilde{g}_j = g_j(.) : Z \to L_2(0,\pi)$.

First, $u_0(t)$ is bounded in space X^α :

$$\int_{-\infty}^{t} \|U(t,s)\tilde{f}(s)\|_\alpha ds \le \int_{\tau_i}^{t} \|A^\alpha e^{-A(t-s)}\tilde{f}(s)\|ds +$$
$$+ \int_{\tau_{i-1}}^{\tau_i} \|A^\alpha e^{-A(t-\tau_i)}(I+B_i)e^{-A(\tau_i-s)}\tilde{f}(s)\|ds +$$
$$+ \sum_{k=2}^{\infty} \int_{\tau_{i-k}}^{\tau_{i-k+1}} \|A^\alpha e^{-A(t-\tau_i)}(I+B_i)e^{-A(\tau_i-\tau_{i-1})}\| \times$$
$$\times \prod_{j=i-1}^{i-k+2} \|(I+B_j)e^{-A(\tau_j-\tau_{j-1})}\| \|(I+B_{i-k+1})e^{-A(\tau_{i-k+1}-s)}\tilde{f}(s)\|ds, \quad (7.107)$$

where $t \in [\tau_i, \tau_{i+1})$. The first integral in (7.107) has upper bound

$$\int_{\tau_i}^{t} \|A^\alpha e^{-A(t-s)}\tilde{f}(s)\|ds \le \frac{C_\alpha}{1-\alpha}\|\tilde{f}\|_{PC}.$$

Next, we need the following inequality (see [17], p. 35):

$$\|A^\alpha T A^\beta e^{-At}\| = \frac{1}{2}\|A^\alpha(R-L)A^\beta e^{-At}\| \le \frac{4^\alpha+1}{2}\|A^{\alpha+\beta}e^{-At}\|. \quad (7.108)$$

Then by (7.108),

$$\|A^\alpha e^{-A(t-\tau_i)}(I+B_i)e^{-A(\tau_i-s)}\| \le \|A^\alpha e^{-A(t-s)}\| + \frac{5}{2}\|A^{\alpha+1/2}e^{-A(t-s)}\| \le$$
$$\le \left(C_\alpha(t-s)^{-\alpha} + \frac{5}{2}C_{\alpha+1/2}(t-s)^{-(\alpha+1/2)}\right)e^{-\delta(t-s)} \quad (7.109)$$

From Henry [9], p. 25, we have

$$\|A^\alpha e^{-At}\psi\| < b_\alpha(t)\|\psi\|,$$

where $b_\alpha(t) = (te/\alpha)^{-\alpha}$ if $0 < t \le \alpha/\lambda_1$, and $b_\alpha(t) = \lambda_1^\alpha e^{-\lambda_1 t}$ if $t \ge \alpha/\lambda_1$. Since $\|T\| = 1$ and $\lambda_1 = 1$, we have

$$\|(I+B_j)e^{-A(\tau_j-\tau_{j-1})}\| \le \|e^{-A(\tau_j-\tau_{j-1})}\| + |b_j|\|A^{1/2}e^{-A(\tau_j-\tau_{j-1})}\| \le$$
$$\le (1+|b_j|)e^{-(\tau_j-\tau_{j-1})} \quad (7.110)$$

if $\theta \ge 1/2$.

Let $0 < \varepsilon_1 < 1 - p \ln(1+b)$. Then there exists a positive integer k_1 such that for $k \geq k_1$

$$\frac{i(\tau_{i-k}, \tau_i)}{\tau_i - \tau_{i-k}} \ln(1+b) - 1 < -\varepsilon_1.$$

Denote

$$N_1 = \max_{1 \leq k \leq k_1} \exp\left(i(\tau_{i-k}, \tau_i) \ln(1+b) - (\tau_i - \tau_{i-k})\right).$$

Then

$$\prod_{j=i-k+1}^{i} \|(I+B_j)e^{-A(\tau_j - \tau_{j-1})}\| \leq (1+b)^{i(\tau_{i-k}, \tau_i)} e^{-(\tau_i - \tau_{i-k})} \leq$$

$$\leq N_1 e^{-\varepsilon_1(\tau_i - \tau_{i-k})} \leq N_1 e^{-\varepsilon_1 \theta k}. \tag{7.111}$$

For $t \in (\tau_i, \tau_{i+1})$, by (7.109) and (7.111) we get

$$\|U(t, \tau_{i-k})\|_\alpha \leq \|A^\alpha e^{-A(t-\tau_i)}(I+B_i)e^{-A(\tau_i - \tau_{i-1})}\| \times$$
$$\times \prod_{j=i-k+1}^{i-1} \|(I+B_j)e^{-A(\tau_j - \tau_{j-1})}\| \leq K_1 e^{-\varepsilon_1(t - \tau_{i-k})}$$

with constant K_1 independent of t and τ_{i-k}.

Using the last inequality, we obtain the boundedness of $\|u_0(t)\|_\alpha$. We can now proceed analogously to the proof of Theorem 1 and show the almost periodicity of $u_0(t)$.

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