Chapter 5
Boundedness of Solutions to a Certain System of Differential Equations with Multiple Delays

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Abstract In this chapter, we consider a system of differential equations of second order with multiple delays. Based on the Lyapunov–Krasovskii functional approach, we investigate the boundedness of solutions. The obtained results essentially complement and improve some known results in the literature.

5.1 Introduction

In recent years, the theory of delay differential equations (DDEs) with retarded arguments has provided a natural framework for the mathematical modeling of many real-world phenomena related to the engineering technique fields, mechanics, models of economic dynamics, optimal control problems, physics, chemistry, life sciences, medicine, atomic energy, information sciences, nerve conduction theory, the slowing down of neutrons in nuclear reactors, the description of traveling waves in a spatial lattice, among others. See, in particular, the books of Bellman and Cooke [1], Erneux [2], Kolmanovskii and Myshkis [3], Smith [4], and Wu et al. [5] for more applications of differential equations of retarded type. The concept of delay is related to the memory of systems, where past events influence the current behavior, and which could be useful for decision making. Further, finding solutions of DDEs is difficult and many times cannot be obtained in closed form. In the absence of a closed form, a viable alternative is studying the qualitative behavior of solutions. In this case, for nonlinear systems without and with delay, the Lyapunov function and Lyapunov–Krasovskii functional approach, respectively, provide a way to analyze the qualitative behavior of solutions (stability, instability, boundedness, asymptotic behaviors, global existence, etc.) of a system without explicitly solving the differential equations. However, since the method requires an auxiliary function or functional, which is not easy to find, it remains an open problem in the literature at this time. When we look at the related literature, the qualitative properties of solutions to second-order DDEs have been intensively discussed and are still being

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investigated. We refer the reader to the papers or books of Ahmad and Rama Mohana Rao [6], Anh et al. [7], Barnett [8], Burton [9], Burton and Zhang [10], Caldeira-Saraiva [11], Cantarelli [12], Èl'sgol'ts and Norkin [13], Gao and Zhao [14], Hale [15], Hara and Yoneyama [16, 17], Heidel [18], Huang and Yu [19], Jitsuro and Yusuke [20], Kato [21, 22], Krasovskiì [23], LaSalle and Lefschetz [24], Li [25], Liu and Huang [26, 27], Liu and Xu [28], Liu [29], Long and Zhang [30], Luk [31], Lyapunov [32], Malyseva [33], Muresan [34], Sugie [35], Sugie and Amano [36], Sugie et al. [37], Tunç [38–45], Tunç and Tunç [46], Yang [47], Ye et al. [48], Yu and Xiao [49], Yoshizawa [50], Zhang [51, 52], Zhang and Yan [53], Zhou and Jiang [54], Zhou and Liu [55], Zhou and Xiang [56], Wei and Huang [57], and Wiandt [58].

At the same time, very recently Omeike et al. [59] considered the following second-order nonlinear system of differential equations of the form

$$X'' + F\left(X, X'\right) X' + H(X) = P\left(t, X, X'\right). \tag{5.1}$$

Omeike et al. [59] proved two new results dealing with the boundedness of solutions of Eq. (5.1). In their work, the authors extended some known results, in the literature, on the boundedness of certain second-order nonlinear scalar differential equations to a system of second-order differential equations, Eq. (5.1). However, to the best of our knowledge of the literature, there is no work based on the results of Omeike et al. [59] to discuss the boundedness of solutions of certain systems of second-order DDEs.

In this chapter, we study the boundedness of solutions to the retarded system of differential equations with multiple constant delays,

$$X'' + F\left(X, X'\right) X' + \sum_{i=1}^{n} H_i\left(X\left(t - \tau_i\right)\right) = P\left(t, X, X'\right), \tag{5.2}$$

where $t \in \Re^+$, $\Re^+ = [0, \infty)$, $X \in \Re^n$, τ_i are positive constants with $t - \tau_i \geq 0$, F is a continuous $n \times n-$ symmetric matrix function, $H_i : \Re^n \to \Re^n$ and $P : \Re^+ \times \Re^n \times \Re^n \to \Re^n$ are continuous, and H_i are also differentiable with $H_i(0) = 0$. The existence and uniqueness of the solutions of Eq. (5.2) are assumed (see Èl'sgol'ts and Norkin [13]).We can write Eq. (5.2) in the differential system form

$$\begin{aligned} X' &= Y, \\ Y' &= -F(X, Y)\, Y - \sum_{i=1}^{n} H_i(X) \\ &\quad + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} J_{H_i}(X(s))\, Y(s) ds + P(t, X, Y) \end{aligned} \tag{5.3}$$

or

$$X' = Y,$$
$$Y' = -F(X,Y)\,Y - \sum_{i=1}^{n} H_i\Big(X(t-\tau_i)\Big) + P(t,X,Y)\,, \tag{5.4}$$

which were obtained by setting $X' = Y$ from Eq. (5.2), and $X(t)$ and $Y(t)$ are respectively abbreviated as X and Y throughout the chapter.

Throughout the chapter, the Jacobian matrices of $H_1(X), \ldots, H_n(X)$ will be given by

$$J_{H_1}(X) = \left(\frac{\partial h_{1i}}{\partial x_j}\right), \ldots,\; J_{H_n}(X) = \left(\frac{\partial h_{ni}}{\partial x_j}\right), \quad (i,\, j = 1,\, 2, \ldots, n)\,,$$

where $(x_1, \ldots, x_n)$ and $(h_{1i}), \ldots, (h_{ni})$ are the components of X and H_i, respectively. Moreover, we assume that the given Jacobian matrices exist and are continuous.

The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$, and $\lambda_i(A)$ are the eigenvalues of the real symmetric $n \times n$ matrix A. The matrix A is said to be negative definite when $\langle AX, X\rangle \le 0$ for all nonzero X in $\Re^n$. Finally, by $\operatorname{sgn} X$, we mean $(\operatorname{sgn} x_1, \operatorname{sgn} x_2, \ldots, \operatorname{sgn} x_n)$ and $\|\operatorname{sgn} X\| = \sqrt{n}$.

Motivated by the work in Omeike et al. [59], in this chapter we will improve and extend the results in [59] to DDE (5.2). This work is also a first attempt to obtain certain sufficient conditions on the ultimate boundedness of solutions of a vector Lienard equation with multiple delays; and it is a contribution to the subject in the literature and may be useful for researchers' work on the qualitative behaviors of solutions.

We need the following preliminary result.

Lemma 1. (Bellman [60]) Let A be a real symmetric $n \times n$ matrix and

$$\overline{a} \ge \lambda_i(A) \ge a > 0, \quad (i = 1, 2, \ldots, n)\,,$$

where $\bar{a}$ and a are constants.

Then

$$\overline{a}\|X\|^2 \ge \langle AX, X\rangle \ge a\|X\|^2$$

and

$$\bar{a}^2 \|X\|^2 \geq \langle AX, AX \rangle \geq a^2 \|X\|^2.$$

5.2 The Main Results

In this section, we introduce the main results.

Theorem 1. We assume that there exist some positive constants ρ, β, δ_f, $\varDelta_f$, α_i, β_i and τ such that the following conditions hold in Eq. (5.2):

(i) $\rho - \beta > 0$, the matrix F is symmetric,
$0 < \delta_f \leq \lambda_i (F(X, Y)) \leq \Delta_f$ for all X, Y,

(ii) $H_i(0) = 0,\ H_i(X) \neq 0,\ (X \neq 0),\ J_{H_i}(X)$ are symmetric, $\alpha_i \leq \lambda_i (J_{H_i}(X)) \leq \beta_i$ for all X, $\langle H_i(X), X\rangle \to +\infty$ as $\|X\| \to \infty$ or $\langle H_i(X), X\rangle \to -\infty$ as $\|X\| \to \infty$,

(iii)

$$\lim_{\|X\|\to\infty} \Big\{ \langle \alpha H_1 (X(t-\tau_1)), \operatorname{sgn} X \rangle + \cdots + \langle \alpha H_n (X(t-\tau_n)), \operatorname{sgn} X \rangle - 2\gamma\Delta_f \Big\} > 2\gamma\beta,$$

where

$$\alpha = \operatorname{sgn} \langle H_i (X(t-\tau_i)), \operatorname{sgn} X \rangle \quad \text{and} \quad \gamma = \sqrt{n},$$

(iv) $\|P(t, X, Y)\| \leq \beta$ for all t, X and Y.
If

$$\tau < \frac{\delta_f}{\sqrt{n}\sum_{i=1}^{n} \beta_i},$$

then there exists a positive constant D, whose magnitude depends only on the constants ρ, β, δ_f, $\varDelta_f$, α_i, β_i as well as on $F(X, Y)$, $J_{H_1}(X), \ldots, J_{H_n}(X)$ and $P(t, X, Y)$ such that every solution X of Eq. (5.2) ultimately satisfies

$$\|X\| \leq D, \ \left\|X'\right\| \leq D.$$

Proof We define a Lyapunov–Krasovskii functional $V = V_1 + V_2 = V_1(X_t, Y_t) + V_2(X, Y)$ by

$$V_1 = \sum_{i=1}^{n} \int_0^1 \langle H_i(\sigma X), X \rangle d\sigma + \frac{1}{2} \langle Y, Y \rangle + \sum_{i=1}^{n} \mu_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 d\theta ds \tag{5.5}$$

and

$$V_2 = \begin{cases} \alpha \langle Y, \operatorname{sgn} X \rangle & \text{if } \|Y\| \le \|X\|, \\ \langle X, \operatorname{sgn} Y \rangle & \text{if } \|X\| \le \|Y\|, \end{cases}$$

where s is a real variable such that the integrals $\int_{-\tau_i}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2 d\theta ds$ are nonnegative and μ_i are some positive constants to be determined later in the proof.

It is clear that $V_1(0,0) = 0$. Using the estimates $H_i(0) = 0$, $\frac{\partial}{\partial \sigma} H_i(\sigma X) = J_{H_i}(\sigma X) X$, $\lambda_i(J_{H_i}(X)) \ge \alpha_i$, $(i = 1, 2, \dots, n)$, we obtain

$$H_i(X) = \int_0^1 J_{H_i}(\sigma X) X d\sigma$$

so that

$$\begin{aligned} \int_0^1 \langle H_1(\sigma X), X \rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_{H_1}(\sigma_1 \sigma_2 X) X, X \rangle d\sigma_2 d\sigma_1 \\ &\ge \int_0^1 \int_0^1 \langle \sigma_1 \alpha_1 X, X \rangle d\sigma_2 d\sigma_1 \ge \tfrac{1}{2} \alpha_1 \|X\|^2, \end{aligned}$$

$$\begin{aligned} \int_0^1 \langle H_2(\sigma X), X \rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_{H_2}(\sigma_1 \sigma_2 X) X, X \rangle d\sigma_2 d\sigma_1 \\ &\ge \int_0^1 \int_0^1 \langle \sigma_1 \alpha_2 X, X \rangle d\sigma_2 d\sigma_1 \ge \tfrac{1}{2} \alpha_2 \|X\|^2, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} \int_0^1 \langle H_n(\sigma X), X \rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_{H_n}(\sigma_1 \sigma_2 X) X, X \rangle d\sigma_2 d\sigma_1 \\ &\ge \int_0^1 \int_0^1 \langle \sigma_1 \alpha_n X, X \rangle d\sigma_2 d\sigma_1 \ge \tfrac{1}{2} \alpha_n \|X\|^2. \end{aligned}$$

Then it follows from V_1 that

$$V_1 \geq \frac{1}{2}\left(\sum_{i=1}^{n}\alpha_i\right)\|X\|^2 + \frac{1}{2}\|Y\|^2 + \sum_{i=1}^{n}\mu_i \int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds$$

$$\geq D_1\left(\|X\|^2 + \|Y\|^2\right) + \sum_{i=1}^{n}\mu_i \int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds$$

$$\geq D_1\left(\|X\|^2 + \|Y\|^2\right),$$

where $D_1 = \min\left\{\frac{1}{2}\left(\sum_{i=1}^{n}\alpha_i\right), \frac{1}{2}\right\}$.

Further, it follows from the definition of V_2 that $|V_2| \leq \delta\|Y\|$, where δ is a positive constant. Hence, we can conclude

$$V \geq D_1\left(\|X\|^2 + \|Y\|^2\right) - \delta\|Y\|.$$

Then it is clear that the right-hand side of the last estimate tends to $+\infty$ when

$$\|X\|^2 + \|Y\|^2 \to +\infty.$$

Using a basic calculation, by the time derivatives of V_1 and V_2 along the solutions of (5.3), we have

$$\dot{V}_1 = -\langle F(X,Y)Y, Y\rangle + < \sum_{i=1}^{n}\int_{t-\tau_i}^{t} J_{H_i}(X(s))\,Y(s)ds, Y >$$
$$+ \langle Y, P(t,X,Y)\rangle + < \sum_{i=1}^{n}\left(\mu_i\tau_i\right)Y, Y > - \sum_{i=1}^{n}\mu_i\int_{t-\tau_i}^{t}\|Y(\theta)\|^2 d\theta$$

and

$$\dot{V}_2 = \begin{cases} -\alpha\Big\{\langle F(X,Y)Y, \operatorname{sgn} X\rangle + < \sum_{i=1}^{n} H_i(X(t-\tau_i)), \operatorname{sgn} X > \\ -\Big\langle P(t,X,Y), \operatorname{sgn} X\Big\}\Big\rangle \quad \text{if} \quad \|Y\| \leq \|X\|, \\ \langle Y, \operatorname{sgn} Y\rangle \quad \text{if} \quad \|X\| \leq \|Y\|. \end{cases}$$

In view of Lemma 1, the assumptions $\lambda_i(F(X,Y)) \geq \delta_f$, $\lambda_i(J_{H_i}(X)) \leq \beta_i$ and the estimate $2|a||b| \leq a^2 + b^2$ (with a and b real numbers) combined with the classical Cauchy–Schwartz inequality, it follows that

$$
\begin{aligned}
&-\langle F(X,Y)\,Y, Y\rangle \le -\delta_f \|Y\|^2, \\
&< \int_{t-\tau_i}^{t} J_{H_i}(X(s))\,Y(s)ds, Y > \le \|Y\| \left\| \int_{t-\tau_i}^{t} J_{H_i}(X(s))\,Y(s)ds \right\| \\
&\le \sqrt{n}\beta_i \|Y\| \left\| \int_{t-\tau_i}^{t} Y(s) \right\| ds \\
&\le \sqrt{n}\beta_i \|Y\| \int_{t-\tau_i}^{t} \|Y(s)\| ds \\
&\le \frac{1}{2}\sqrt{n}\beta_i \int_{t-\tau_i}^{t} \left(\|Y(t)\|^2 + \|Y(s)\|^2 \right) ds \\
&\le \frac{1}{2}\sqrt{n}\beta_i \tau_i \|Y\|^2 + \frac{1}{2}\sqrt{n}\beta_i \int_{t-\tau_i}^{t} \|Y(s)\|^2 ds
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{V}_1 \le{}& -\delta_f \|Y\|^2 + \left(\sum_{i=1}^{n} \mu_i \tau_i \right) \|Y\|^2 + \tfrac{1}{2}\left(\sqrt{n} \sum_{i=1}^{n} \beta_i \tau_i \right) \|Y\|^2 \\
&- \sum_{i=1}^{n} \left(\mu_i - \frac{1}{2}\sqrt{n}\beta_i \right) \int_{t-\tau_i}^{t} \|Y(s)\|^2 ds + \langle Y, P(t,X,Y) \rangle .
\end{aligned}
$$

Let $\mu_i = \frac{1}{2}\sqrt{n}\beta_i$, $\tau = \max \tau_i$, and $\overline{\delta} = \sqrt{n}\sum_{i=1}^{n} \beta_i$. Then it is clear that

$$
\begin{aligned}
\dot{V}_1 &\le - \left\{ \delta_f - \sqrt{n} \sum_{i=1}^{n} \left(\beta_i \tau_i \right) \right\} \|Y\|^2 + \langle Y, P(t,X,Y) \rangle \\
&\le - \left(\delta_f - \delta\tau \right) \|Y\|^2 + \langle Y, P(t,X,Y) \rangle .
\end{aligned}
$$

If $\tau < \frac{\delta_f}{\overline{\delta}}$, then we can obtain, for some positive constant ρ, that

$$
\dot{V}_1 \le -\rho \|Y\|^2 + \langle Y, P(t,X,Y) \rangle .
$$

In view of the last estimates for $\dot{V}_1$ and $\dot{V}_2$, if $\|Y\| \le \|X\|$, then

$$
\begin{aligned}
\dot{V} \le{}& -\rho \|Y\|^2 + \langle Y, P(t,X,Y) \rangle - \alpha \langle F(X,Y)\,Y, \operatorname{sgn} X \rangle \\
&- \alpha < \sum_{i=1}^{n} H_i\left(X(t-\tau_i) \right), \operatorname{sgn} X > + \alpha \langle P(t,X,Y), \operatorname{sgn} X \rangle
\end{aligned}
$$

and if $\|X\| \leq \|Y\|$, then

$$\dot{V} \leq -\rho\|Y\|^2 + \langle Y, P(t, X, Y)\rangle + \langle Y, \operatorname{sgn} Y\rangle .$$

Hence, it follows that

$$\begin{aligned} \dot{V} \leq -\rho\|Y\|^2 + |\langle F(X,Y)\,Y, \operatorname{sgn} X\rangle| - \alpha < \sum_{i=1}^{n} H_i\,(X\,(t-\tau_i))\,, \operatorname{sgn} X > \\ + (\|Y\| + \gamma)\ \|P(t,X,Y)\| \quad \text{if } \|Y\| \leq \|X\| \end{aligned} \tag{5.6}$$

and

$$\dot{V} \leq -\rho\|Y\|^2 + (\|P(t,X,Y)\| + \gamma)\,\|Y\| \quad \text{if } \|X\| \leq \|Y\| \ . \tag{5.7}$$

The assumption

$$\begin{aligned} \lim_{\|X\|\to\infty} \Big\{ & \alpha\,\langle H_1\,(X\,(t-\tau_1))\,, \operatorname{sgn} X\rangle + \cdots + \alpha\,\langle H_n\,(X\,(t-\tau_n))\,, \operatorname{sgn} X\rangle \\ & - 2\gamma\Delta_f \Big\} > 2\gamma\beta \end{aligned}$$

with $\alpha = (\operatorname{sgn}\langle H_i\,(X\,(t-\tau_i))\,, \operatorname{sgn} X\rangle)$, and $\gamma = \sqrt{n}$, β are positive constants, implies the existence of finite constants $\alpha_0 > 0$ and $D_2 > 0$ such that $\|X\| \geq \alpha_0$.

By assumption (iii) of Theorem 1, $\|X\| \geq \alpha_0$ implies that

$$\begin{aligned} & \alpha\,\langle H_1\,(X\,(t-\tau_1))\,, \operatorname{sgn} X\rangle + \cdots + \alpha\,\langle H_n\,(X\,(t-\tau_i))\,, \operatorname{sgn} X\rangle \\ & \quad - 2\gamma\Delta_f - 2\gamma\beta \geq D_2. \end{aligned} \tag{5.8}$$

Let

$$\alpha_1 = \max\{1, \alpha_0, \mu\}$$

with $\mu = \rho^{-1}(\beta + \gamma)$.

We claim, for some finite positive constant $D_3 > 0$, that

$$\dot{V} \leq -D_3 \quad \text{if } \|X\| \geq \alpha_1.$$

In fact, if $\|Y\| \leq \|X\|$, then it is clear that $\dot{V}$ satisfies (5.6), and if $\|Y\| \geq 1$, then, by the assumptions $0 < \delta_f \leq \lambda_i\,(F\,(X,Y)) \leq \Delta_f$, $\|P(t,X,Y)\| \leq \beta$, and $\rho - \beta > 0$, we have

$$
\begin{aligned}
\dot{V} &\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > -\left\|Y\right\|\left(\rho\left\|Y\right\| - \gamma\Delta_f\right) + \beta\left(\left\|Y\right\| + \gamma\right) \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > -\left(\rho\left\|Y\right\| - \gamma\Delta_f\right) + \beta\left(\left\|Y\right\| + \gamma\right) \\
&= -\left(\rho-\beta\right)\left\|Y\right\| - \alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > +\gamma\left(\beta + \Delta_f\right) \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > +2\gamma\left(\beta + \Delta_f\right).
\end{aligned}
\tag{5.9}
$$

By noting assumption (iii) of Theorem 1,

$$
\lim_{\|X\|\to\infty}\Big\{\alpha\left\langle H_1\left(X\left(t-\tau_1\right)\right), \operatorname{sgn} X\right\rangle + \cdots + \alpha\left\langle H_n\left(X\left(t-\tau_n\right)\right), \operatorname{sgn} X\right\rangle - 2\gamma\Delta_f\Big\} > 2\gamma\beta
$$

it can be followed from (5.9), for some positive constant D_2, that

$$
\dot{V} \le -D_2 \text{ if } \|X\| \ge \alpha_1. \tag{5.10}
$$

Next, we suppose that $\|Y\| \le 1$. Then

$$
\begin{aligned}
\dot{V} &\le -\rho\|Y\|^2 + \gamma\Delta_f\left\|Y\right\| - \alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > +\beta\left(\left\|Y\right\| + \gamma\right) \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > +\gamma\Delta_f + \beta\left(1+\gamma\right) \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > +2\gamma\left(\beta + \Delta_f\right).
\end{aligned}
$$

Hence, it can be concluded that estimates (5.8) and (5.10) still hold in this case.

We now consider estimate (5.7) when $\|X\| \le \|Y\|$. If $\|X\| \ge \alpha_1$ with $\alpha_1 = \max\{1, \alpha_0, \mu\}$, then $\|Y\| \ge \alpha_1$. Hence,

$$
\dot{V} \le -\rho\|Y\|^2 + \left(\beta+\gamma\right)\left\|Y\right\| = -\left\|Y\right\|\left\{\rho\left\|Y\right\| - \left(\beta+\gamma\right)\right\} \le -1
$$

if $\|Y\| \ge \mu = \rho^{-1}\left(\beta+\gamma\right).$

This means that if $\|X\| \geq \alpha_1$, then $\dot{V} \leq -1$. In view of the last estimate and (5.10), $\dot{V} \leq -D_2$, if $\|X\| \geq \alpha_1$, it follows that

$$\dot{V} \leq -D_3,$$

where $D_3 = \max\{1, D_2\}$.

To conclude the end of the proof, we suppose, on the contrary, that $\|X\| \leq \alpha_1$. Let $\|Y\| \geq \alpha_1$. Then $\|Y\| \geq \|X\|$. Hence,

$$\dot{V} \leq -\rho\|Y\|^2 + (\beta + \gamma)\|Y\| = -\|Y\|\{\rho\|Y\| - (\beta + \gamma)\} \leq -1$$

if $\|Y\| \geq \rho^{-1}(\beta + \gamma)$.

Then, in view of the last estimate and $\dot{V} \leq -D_2$ if $\|X\| \geq \alpha_1$, it follows that

$$\dot{V} \leq -D_3,$$

where $D_3 = \max\{1, D_2\}$. Therefore, we can conclude that

$$\dot{V} \leq -D_3 \quad \text{if } \|X\|^2 + \|Y\|^2 \geq 2\alpha_1.$$

The proof of Theorem 1 is complete. □

Our second main result is the following theorem.

Theorem 2. Let all assumptions of Theorem 1 hold, except (iii), and assume that

(i) $$\lim_{\|X\|\to\infty}\Big\{\alpha\langle H_1(X(t-\tau_1)), \operatorname{sgn} X\rangle + \cdots + \alpha\langle H_n(X(t-\tau_n)), \operatorname{sgn} X\rangle - 2\gamma\Delta_f\Big\} > 2\gamma\beta*,$$

where

$$\beta* = \max\left\{\frac{\gamma}{8}(\Delta_f + \beta)^2(\rho - \beta)^{-1}, \beta\right\},$$

and

(ii) $\|P(t, X, Y)\| \leq \beta\|Y\|$ for all $t, X, Y \in \Re^n$.

If

$$\tau < \frac{\rho}{\sqrt{n}\sum_{i=1}^{n}\beta_i},$$

then there exists a positive constant D, whose magnitude depends only on the constants ρ, β, δ_f, Δ_f, α_i, β_i as well as on $F(X, Y)$, $J_{H_1}(X), \ldots, J_{H_n}(X)$, and $P(t, X, Y)$ such that every solution X of Eq. (5.2) ultimately satisfies

$$\|X\| \leq D, \ \left\|X'\right\| \leq D.$$

Proof The main tool to prove Theorem 1 is the Lyapunov–Krasovskii functional $V = V_1 + V_2$ used in Theorem 1. We use the same procedure as that used in the proof of Theorem 1. The proof of Theorem 2 is immediate when we show the following estimates:

$$V \to \infty \quad \text{as } \|X\|^2 + \|Y\|^2 \to \infty \tag{5.11}$$

and

$$\dot{V} \leq -D_0 \quad \text{if } \|X\|^2 + \|Y\|^2 \geq D_1. \tag{5.12}$$

The verification of (5.11) can be easily checked from the discussion made in the proof of Theorem 1. Therefore, we omit the details to verify (5.11), $V \to \infty$ as $\|X\|^2 + \|Y\|^2 \to \infty$. To verify estimate (5.12), we benefit from estimates (5.6) and (5.7), which are still valid in this case. In fact, in view of assumption (i) of Theorem 2, that is, the definition of an infinite limit, we can say that there are positive constants α_0 and D_4 such that $\|X\| \geq \alpha_0$ implies the existence of the following estimate:

$$\begin{aligned} &\alpha \left\langle H_1\left(X\left(t-\tau_1\right)\right), \operatorname{sgn} X\right\rangle + \cdots + \alpha \left\langle H_n\left(X\left(t-\tau_n\right)\right), \operatorname{sgn} X\right\rangle \\ &\quad - 2\gamma\Delta_f - 2\gamma\beta* \geq D_4. \end{aligned} \tag{5.13}$$

We also assume that there exists a positive constant ξ_0 such that $\|Y\| \geq \xi_0$ implies

$$-(\rho - \beta)\|Y\|^2 + \|Y\| \leq -1.$$

Let

$$\alpha_1 = \max\{1, \alpha_0, \delta_0\}. \tag{5.14}$$

We claim, for some finite positive constant $D_5 > 0$, that

$$\dot{V} \leq -D_5 \quad \text{if } \|X\| \geq \alpha_1.$$

To conclude the preceding claim; we consider the following two cases:
$\|Y\| \leq \|X\|$ and $\|X\| \leq \|Y\|$, separately.

In fact, if $\|Y\| \leq \|X\|$ and $\|Y\| \geq 1$, then in view of (5.6) and the assumptions of Theorem 2, it follows that

$$\begin{aligned}
\dot{V} &\le -\rho\|Y\|^2 + \gamma\Delta_f\,\|Y\| - \alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > \\
&\quad + \beta\,\|Y\|\,(\|Y\| + \gamma) \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > -(\rho-\beta)\left[\|Y\| - \tfrac{\gamma\left(\Delta_f+\beta\right)}{2(\rho-\beta)}\right]^2 \\
&\quad + \tfrac{\gamma^2\left(\Delta_f+\beta\right)^2}{4(\rho-\beta)} \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > + \tfrac{\gamma^2\left(\Delta_f+\beta\right)^2}{4(\rho-\beta)} \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > + 2\gamma\Delta_f + \tfrac{\gamma^2\left(\Delta_f+\beta\right)^2}{4(\rho-\beta)}.
\end{aligned}$$

Next, we suppose that $\|Y\| \le 1$. Then

$$\begin{aligned}
\dot{V} &\le -\rho\|Y\|^2 + \gamma\Delta_f\,\|Y\| - \alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > \\
&\quad + \beta\,\|Y\|\,(\|Y\| + \gamma) \\
&= -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > \\
&\quad + \gamma\left(\Delta_f + \beta\right)\|Y\| \\
&\le -\alpha < \sum_{i=1}^{n} H_i\left(X\left(t-\tau_i\right)\right), \operatorname{sgn} X > + 2\gamma\left(\beta + \Delta_f\right).
\end{aligned}$$

Hence, in view of (5.13), assumption (i) of Theorem 2, and (5.14), it follows in either case that if $\|X\| \ge \alpha_1$, then

$$\dot{V} \le -D_5,\ D_5 > 0, \tag{5.15}$$

hold. Thus, we have

$$\dot{V} \le -(\rho-\beta)\,\|Y\|^2 + \gamma\,\|Y\| \quad \text{for } \|Y\| \ge \gamma(\rho\text{-}\beta)^{-1}.$$

On the contrary, we now suppose that $\|X\| \le \alpha_1$ and assume $\|Y\| \ge \alpha_1$. In this case, it is clear that $\|Y\| \ge \|X\|$. Then in view of $\|Y\| \ge \alpha_1$, we get

$$\dot{V} \le -\left\{(\rho-\beta)\,\|Y\| - \gamma\right)\right\}\|Y\| \le -1$$

for $\|Y\| \ge \gamma(\rho-\beta)^{-1}$.

The last estimate and (5.15) together imply that

$$\dot{V} \le -D_5 \quad \text{if } \|X\|^2 + \|Y\|^2 \ge 2\alpha_1,$$

which verifies

$$\dot{V} \leq -D_0 \quad \text{if } \|X\|^2 + \|Y\|^2 \geq D_1.$$

The proof of Theorem 2 is complete. □

5.3 Conclusion

We have considered a system of second-order differential equations with multiple delays. By using the Lyapunov–Krasovskii functional approach, we proved two new theorems on the boundedness of solutions to the considered system. The obtained results complement and improve the recent results obtained by Omeike et al. [59].

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