# Chapter 2
# On Periodic Motions in a Time-Delayed, Quadratic Nonlinear Oscillator with Excitation

**Albert C.J. Luo and Hanxiang Jin**

**Abstract** Analytical solutions of periodic motions in a time-delayed, quadratic nonlinear oscillator with periodic excitation are obtained through the finite Fourier series, and the stability and bifurcation analysis for periodic motions are discussed. The bifurcation trees of period-1 motion to chaos can be presented. Numerical illustration of periodic motion is given to verify the analytical solutions.

## 2.1 Introduction

The quadratic nonlinear oscillator is often used to describe boat motion under periodic ocean waves. To stabilize boat motions under waves, once the feedback is introduced, the boat motion equation will be a time-delayed dynamical system. In this chapter, the analytical solution of periodic motions in a time-delayed, quadratic nonlinear oscillator will be investigated for the stabilization of boat motion.

The study of periodic motions in dynamical systems dates back to the eighteenth century. In 1788, Lagrange [1] developed the standard Lagrange form to obtain the method of averaging and used this method for the periodic motions of three-body problems. In the nineteenth century, Poincaré [2] developed perturbation theory to determine the periodic motions of celestial bodies. In 1920, van der Pol [3] employed the method of averaging for the periodic solutions of oscillation systems in circuits. In 1928, Fatou [4] gave the first proof of the asymptotic validity of the method of averaging through the existing theorems of solutions of differential equations. In 1935, Krylov, Bogoliubov, and Mitropolsky [5] further developed the method of averaging and applied it to periodic motions in nonlinear oscillators. In 1961, Bogoliubov and Mitropolsky [6] summarized the asymptotic perturbation methods in nonlinear oscillations. In 1964, Hayashi [7] employed perturbation methods, the method of averaging, and the principle of harmonic balance for the approximate solutions of nonlinear oscillators, and the stability of approximate

A.C.J. Luo (✉) • H. Jin
Southern Illinois University Edwardsville, Edwardsville, IL 62026-1805, USA
e-mail: aluo@siue.edu; hjin@siue.edu

 
A.C.J. Luo, H. Merdan (eds.), *Mathematical Modeling and Applications in Nonlinear Dynamics*, Nonlinear Systems and Complexity 14,
DOI 10.1007/978-3-319-26630-5_2

periodic solutions in nonlinear oscillators was determined by the improved Mathieu equation. In 1973, Nayfeh [8] presented multiscale methods for approximate solutions of periodic motions in nonlinear structural dynamics (also see Nayfeh and Mook [9]). In 1990, Coppola and Rand [10] developed the method of averaging with elliptic functions for the approximate of limit cycle. In 2012, Luo [11] developed a methodology for analytical solutions of periodic motions in nonlinear dynamical systems. In 2012, Luo and Huang [12] applied such a generalized harmonic balance method to the Duffing oscillator for approximate solutions of periodic motions, and Luo and Huang [13] gave the analytical bifurcation trees of period-*m* motions to chaos in the Duffing oscillator. In 2013, Luo [14] systematically proposed a methodology for periodic motions in time-delayed, nonlinear dynamical systems. In 2014, Luo and Jin [15] used such a technique to investigate periodic motion in a quadratic nonlinear oscillator with time delay.

In this chapter, the analytical solutions of period-*m* motions for such a time-delayed, quadratic nonlinear oscillator will be presented and the stability and bifurcation of period-*m* motions in the time-delayed nonlinear oscillator will be discussed. From the bifurcation trees of period-1 motion to chaos, numerical simulations will be carried out for comparison of analytical and numerical solutions of periodic motions.

### 2.1.1 Analytical Solutions

As in Luo and Jin [15], consider a periodically forced, time-delayed, quadratic nonlinear oscillator as

$$\ddot{x} + \delta\dot{x} + \alpha_1 x - \alpha_2 x^{\tau} + \beta x^2 = Q_0 \cos\Omega t, \tag{2.1}$$

where $x^{\tau} = x(t-\tau)$ and $\dot{x}^{\tau} = \dot{x}(t-\tau)$. The coefficients in Eq. (2.1) are $\delta$ for linear damping, $\alpha_1$ and $\alpha_2$ for linear springs, $\beta$ for quadratic nonlinearity, and $Q_0$ and $\Omega$ for excitation amplitude and frequency, respectively. The standard form of Eq. (2.1) is written as

$$\ddot{x} + f(x, \dot{x}, x^{\tau}, \dot{x}^{\tau}, t) = 0, \tag{2.2}$$

where

$$f(x, \dot{x}, x^{\tau}, \dot{x}^{\tau}, t) = \delta\dot{x} + \alpha_1 x - \alpha_2 x^{\tau} + \beta x^2 - Q_0 \cos\Omega t. \tag{2.3}$$

The analytical solution of period-$m$ motion for the preceding equation is

$$\begin{aligned}
x^{(m)*} &= a_0^{(m)}(t) + \sum_{k=1}^{N} b_{k/m}(t)\cos\left(\frac{k}{m}\Omega t\right) + c_{k/m}(t)\sin\left(\frac{k}{m}\Omega t\right),\\
x^{\tau(m)*} &= a_0^{\tau(m)}(t) + \sum_{k=1}^{N}\left[b_{k/m}^{\tau}(t)\cos\left(\frac{k}{m}\Omega\tau\right) - c_{k/m}^{\tau}(t)\sin\left(\frac{k}{m}\Omega\tau\right)\right]\cos\left(\frac{k}{m}\Omega t\right)\\
&\quad + \left[b_{k/m}^{\tau}(t)\cos\left(\tfrac{k}{m}\Omega\tau\right) + c_{k/m}^{\tau}(t)\sin\left(\tfrac{k}{m}\Omega\tau\right)\right]\sin\left(\tfrac{k}{m}\Omega t\right).
\end{aligned} \tag{2.4}$$

where $a_0^{\tau(m)}(t) = a_0^{(m)}(t-\tau)$, $b_{k/m}^{\tau}(t) = b_{k/m}(t-\tau)$, $c_{k/m}^{\tau}(t) = c_{k/m}(t-\tau)$. The coefficients $a_0^{(m)}(t)$, $b_{k/m}(t)$, $c_{k/m}(t)$ vary with time. The first and second order of derivatives of $x^{(m)*}(t)$ and $x^{\tau(m)*}(t)$ are

$$\begin{aligned}
\dot{x}^{(m)*} &= \dot{a}_0^{(m)}(t) + \sum_{k=1}^{N}\left[\left(\dot{b}_{k/m}(t) + \frac{k}{m}\Omega c_{k/m}(t)\right)\cos\left(\frac{k}{m}\Omega t\right)\right.\\
&\quad \left. + \left(\dot{c}_{k/m}(t) - \tfrac{k}{m}\Omega b_{k/m}(t)\right)\sin\left(\tfrac{k}{m}\Omega t\right)\right],\\
\dot{x}^{\tau(m)*} &= \dot{a}_0^{\tau(m)}(t) + \sum_{k=1}^{N}\left\{\left[\left(\dot{b}_{k/m}^{\tau}(t) + \frac{k}{m}\Omega c_{k/m}^{\tau}(t)\right)\cos\left(\frac{k}{m}\Omega\tau\right)\right.\right.\\
&\quad \left. - \left(\dot{c}_{k/m}^{\tau}(t) - \tfrac{k}{m}\Omega b_{k/m}^{\tau}(t)\right)\sin\left(\tfrac{k}{m}\Omega\tau\right)\right]\cos\left(\tfrac{k}{m}\Omega t\right)\\
&\quad + \left[\left(\dot{b}_{k/m}^{\tau}(t) + \tfrac{k}{m}\Omega c_{k/m}^{\tau}(t)\right)\sin\left(\tfrac{k}{m}\Omega\tau\right)\right.\\
&\quad \left.\left. + \left(\dot{c}_{k/m}^{\tau}(t) - \tfrac{k}{m}\Omega b_{k/m}^{\tau}(t)\right)\cos\left(\tfrac{k}{m}\Omega\tau\right)\right]\sin\left(\tfrac{k}{m}\Omega t\right)\right\},
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\ddot{x}^{(m)}(t) &= \ddot{a}_0^{(m)}(t) + \sum_{k=1}^{N}\left[\ddot{b}_{k/m} + 2\left(\frac{k}{m}\Omega\right)\dot{c}_{k/m} - \left(\frac{k}{m}\Omega\right)^2 b_{k/m}\right]\cos\left(\tfrac{k}{m}\Omega t\right)\\
&\quad + \left[\ddot{c}_{k/m} - 2\left(\tfrac{k}{m}\Omega\right)\dot{b}_{k/m} - \left(\tfrac{k}{m}\Omega\right)^2 c_{k/m}\right]\sin\left(\tfrac{k}{m}\Omega t\right).
\end{aligned} \tag{2.6}$$

Substitution of Eqs. (2.4)–(2.6) into Eq. (2.1) and averaging for the harmonic terms of $\cos(k\Omega t/m)$ and $\sin(k\Omega t/m)$ $(k = 0, 1, 2, \ldots)$ gives

$$\begin{aligned}
&\ddot{a}_0^{(m)} + F_0^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}\dot{\mathbf{z}}^{\tau(m)}\right) = 0,\\
&\ddot{b}_{k/m} + 2\tfrac{k\Omega}{m}\dot{c}_{k/m} - \left(\tfrac{k\Omega}{m}\right)^2 b_{k/m} + F_{1k}^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}\dot{\mathbf{z}}^{\tau(m)}\right) = 0,\\
&\ddot{c}_{k/m} - 2\tfrac{k\Omega}{m}\dot{b}_{k/m} - \left(\tfrac{k\Omega}{m}\right)^2 c_{k/m} + F_{2k}^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}\dot{\mathbf{z}}^{\tau(m)}\right) = 0,\\
&k = 1, 2, \ldots, N,
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&\mathbf{z}^{(m)} = \left(a_0^{(m)}, \mathbf{b}^{(m)}, \mathbf{c}^{(m)}\right)^{\mathrm{T}} \text{ and } \dot{\mathbf{z}}^{(m)} = \left(\dot{a}_0^{(m)}, \dot{\mathbf{b}}^{(m)}, \dot{\mathbf{c}}^{(m)}\right)^{\mathrm{T}},\\
&\mathbf{z}^{\tau(m)} = \left(a_0^{\tau(m)}, \mathbf{b}^{\tau(m)}, \mathbf{c}^{\tau(m)}\right)^{\mathrm{T}} \text{ and } \dot{\mathbf{z}}^{\tau(m)} = \left(\dot{a}_0^{\tau(m)}, \dot{\mathbf{b}}^{\tau(m)}, \dot{\mathbf{c}}^{\tau(m)}\right)^{\mathrm{T}};\\
&\mathbf{b}^{(m)} = \left(b_1^{(m)}, b_2^{(m)}, \cdots, b_N^{(m)}\right)^{\mathrm{T}} \text{ and } \mathbf{b}^{\tau(m)} = \left(b_1^{\tau(m)}, b_2^{\tau(m)}, \cdots, b_N^{\tau(m)}\right)^{\mathrm{T}},\\
&\mathbf{c}^{(m)} = \left(c_1^{(m)}, c_2^{(m)}, \cdots, c_N^{(m)}\right)^{\mathrm{T}} \text{ and } \mathbf{c}^{\tau(m)} = \left(c_1^{\tau(m)}, c_2^{\tau(m)}, \cdots, c_N^{\tau(m)}\right)^{\mathrm{T}};
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
F_0^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}, \dot{\mathbf{z}}^{\tau(m)}\right) &= \delta \dot{a}_0^{(m)} + \alpha_1 a_0^{(m)} - \alpha_2 a_0^{\tau(m)} + \beta f_0^{(m)},\\
F_{1k}^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}, \dot{\mathbf{z}}^{\tau(m)}\right) &= \delta\left(\dot{b}_{k/m} + \tfrac{k\Omega}{m} c_{k/m}\right) + \alpha_1 b_{k/m} - \alpha_2\Big[b_{k/m}^{\tau}\cos\left(\tfrac{k}{m}\Omega\tau\right)\\
&\quad - c_{k/m}^{\tau}\sin\left(\tfrac{k}{m}\Omega\tau\right)\Big] + \beta f_{k/m}^{(c)} + Q_0\delta_k^m\\
F_{2k}^{(m)}\left(\mathbf{z}^{(m)}, \dot{\mathbf{z}}^{(m)}; \mathbf{z}^{\tau(m)}, \dot{\mathbf{z}}^{\tau(m)}\right) &= \delta\left(\dot{c}_{k/m} - \tfrac{k\Omega}{m} b_{k/m}\right) + \alpha_1 c_{k/m} - \alpha_2\Big[c_{k/m}^{\tau}\cos\left(\tfrac{k}{m}\Omega\tau\right)\\
&\quad + b_{k/m}^{\tau}\sin\left(\tfrac{k}{m}\Omega\tau\right)\Big] + \beta f_{k/m}^{(s)},
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
f_0^{(m)} &= \left(a_0^{(m)}\right)^2 + \tfrac{1}{2}\sum_{i=1}^{N}\left(b_{i/m}^2 + c_{i/m}^2\right),\\
f_{k/m}^{(c)} &= 2a_0^{(m)} b_{k/m} + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{i/m} b_{j/m}\left(\delta_{i+j}^k + \delta_{j-i}^k + \delta_{i-j}^k\right)\\
&\quad + c_{i/m} c_{j/m}\left(\delta_{j-i}^k - \delta_{i+j}^k + \delta_{i-j}^k\right),\\
f_{k/m}^{(s)} &= 2a_0^{(m)} c_{k/m} + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{i/m} c_{j/m}\left(\delta_{i+j}^k + \delta_{j-i}^k - \delta_{i-j}^k\right).
\end{aligned} \tag{2.10}$$

Equation (2.7) can be expressed in the form of a vector field as

$$\dot{\mathbf{z}}^{(m)} = \mathbf{z}_1^{(m)} \text{ and } \dot{\mathbf{z}}_1^{(m)} = \mathbf{g}^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}; \mathbf{z}^{\tau(m)}, \mathbf{z}_1^{\tau(m)}\right), \tag{2.11}$$

where

$$\begin{aligned}
&\mathbf{g}^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}; \mathbf{z}^{\tau(m)}, \mathbf{z}_1^{\tau(m)}\right)\\
&= \begin{pmatrix}
-F_0^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}; \mathbf{z}^{\tau(m)}, \mathbf{z}_1^{\tau(m)}\right)\\
-\mathbf{F}_1^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}; \mathbf{z}^{\tau(m)}, \mathbf{z}_1^{\tau(m)}\right) - 2\mathbf{k}_1 \tfrac{\Omega}{m}\dot{\mathbf{c}}^{(m)} + \mathbf{k}_2\left(\tfrac{\Omega}{m}\right)^2 \mathbf{b}^{(m)}\\
-\mathbf{F}_2^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}; \mathbf{z}^{\tau(m)}, \mathbf{z}_1^{\tau(m)}\right) + 2\mathbf{k}_1 \tfrac{\Omega}{m}\dot{\mathbf{b}}^{(m)} + \mathbf{k}_2\left(\tfrac{\Omega}{m}\right)^2 \mathbf{c}^{(m)}
\end{pmatrix}
\end{aligned} \tag{2.12}$$

and

$$\begin{array}{l}\mathbf{k}_1 = \mathrm{diag}\,(1, 2, \ldots, N) \text{ and } \mathbf{k}_2 = \mathrm{diag}\left(1, 2^2, \ldots, N^2\right)\\ \mathbf{F}_1^{(m)} = \left(F_{11}^{(m)}, F_{12}^{(m)}, \ldots, F_{1N}^{(m)}\right)^{\mathrm{T}} \text{ and } \mathbf{F}_2^{(m)} = \left(F_{21}^{(m)}, F_{22}^{(m)}, \ldots, F_{2N}^{(m)}\right)^{\mathrm{T}}\\ \text{for } N = 1, 2, \ldots, \infty.\end{array} \tag{2.13}$$

Setting

$$\mathbf{y}^{(m)} \equiv \left(\mathbf{z}^{(m)}, \mathbf{z}_1^{(m)}\right),\ \mathbf{y}^{\tau(m)} \equiv \left(\mathbf{z}^{\pi(m)}, \mathbf{z}_1^{\pi(m)}\right),\ \text{and } \mathsf{f}^{(m)} = \left(\mathbf{z}_1^{(m)}, \mathbf{g}^{(m)}\right)^{\mathrm{T}}, \tag{2.14}$$

equation (2.11) becomes

$$\dot{\mathbf{y}}^{(m)} = \mathsf{f}^{(m)}\left(\mathbf{y}^{(m)}, \mathbf{y}^{\tau(m)}\right). \tag{2.15}$$

The steady-state solutions for periodic motion in Eq. (2.1) can be obtained by setting

$$\begin{array}{l}F_0^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{0}; \mathbf{z}^{\tau(m)}, \mathbf{0}\right) = 0,\\ \mathbf{F}_1^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{0}; \mathbf{z}^{\tau(m)}, \mathbf{0}\right) - \mathbf{k}_2\left(\frac{\Omega}{m}\right)^2\mathbf{b}^{(m)} = \mathbf{0},\\ \mathbf{F}_2^{(m)}\left(\mathbf{z}^{(m)}, \mathbf{0}; \mathbf{z}^{\tau(m)}, \mathbf{0}\right) - \mathbf{k}_2\left(\frac{\Omega}{m}\right)^2\mathbf{c}^{(m)} = \mathbf{0}.\end{array} \tag{2.16}$$

The $(2N+1)$ nonlinear equations in Eq. (2.16) are solved by the Newton–Raphson method. In Luo [11, 14], the linearized equation at the equilibrium point is given by

$$\Delta\dot{\mathbf{y}}^{(m)} = D\mathsf{f}\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right)\Delta\mathbf{y}^{(m)} + D^{\tau}\mathsf{f}\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right)\Delta\mathbf{y}^{\tau(m)}. \tag{2.17}$$

The corresponding eigenvalues are determined by

$$\left|\mathbf{A} + \mathbf{B}e^{-\lambda\tau} - \lambda\mathbf{I}_{2(2N+1)\times 2(2N+1)}\right| = 0, \tag{2.18}$$

where

$$\begin{array}{l}\mathbf{A} = D\mathsf{f}\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right) = \partial\mathsf{f}\left(\mathbf{y}^{(m)}, \mathbf{y}^{\tau(m)}\right)/\partial\mathbf{y}^{(m)}\big|_{\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right)},\\ \mathbf{B} = D^{\tau}\mathsf{f}\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right) = \partial\mathsf{f}\left(\mathbf{y}^{(m)}, \mathbf{y}^{\tau(m)}\right)/\partial\mathbf{y}^{\tau(m)}\big|_{\left(\mathbf{y}^{(m)*}, \mathbf{y}^{\tau(m)*}\right)}.\end{array} \tag{2.19}$$

The corresponding submatrices are

$$\begin{array}{l}\mathbf{A} = \begin{bmatrix}\mathbf{0}_{(2N+1)\times(2N+1)} & \mathbf{I}_{(2N+1)\times(2N+1)}\\ \mathbf{G} & \mathbf{H}\end{bmatrix},\\ \mathbf{B} = \begin{bmatrix}\mathbf{0}_{(2N+1)\times(2N+1)} & \mathbf{I}_{(2N+1)\times(2N+1)}\\ \mathbf{G}^{\tau} & \mathbf{H}^{\tau}\end{bmatrix},\end{array} \tag{2.20}$$

where

$$\mathbf{G} = \frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}^{(m)}} = \left(\mathbf{G}^{(0)}, \mathbf{G}^{(c)}, \mathbf{G}^{(s)}\right)^{\mathrm{T}}, \mathbf{G}^{\tau} = \frac{\partial \mathbf{g}}{\partial \mathbf{z}^{\tau}} = \left(\mathbf{G}^{\tau(0)}, \mathbf{G}^{\tau(c)}, \mathbf{G}^{\tau(s)}\right)^{\mathrm{T}}, \tag{2.21}$$

$$\begin{aligned}
\mathbf{G}^{(0)} &= \left(G_0^{(0)}, G_1^{(0)}, \cdots, G_{2N}^{(0)}\right), \\
\mathbf{G}^{(c)} &= \left(\mathbf{G}_1^{(c)}, \mathbf{G}_2^{(c)}, \cdots, \mathbf{G}_N^{(c)}\right)^{\mathrm{T}}, \\
\mathbf{G}^{(s)} &= \left(\mathbf{G}_1^{(s)}, \mathbf{G}_2^{(s)}, \cdots, \mathbf{G}_N^{(s)}\right)^{\mathrm{T}}; \\
\mathbf{G}^{\tau(0)} &= \left(G_0^{\tau(0)}, G_1^{\tau(0)}, \cdots, G_{2N}^{\tau(0)}\right), \\
\mathbf{G}^{\tau(c)} &= \left(\mathbf{G}_1^{\tau(c)}, \mathbf{G}_2^{\tau(c)}, \cdots, \mathbf{G}_N^{\tau(c)}\right)^{\mathrm{T}}, \\
\mathbf{G}^{\tau(s)} &= \left(\mathbf{G}_1^{\tau(s)}, \mathbf{G}_2^{\tau(s)}, \cdots, \mathbf{G}_N^{\tau(s)}\right)^{\mathrm{T}}
\end{aligned} \tag{2.22}$$

for $N = 1, 2, \ldots, \infty$ with

$$\begin{aligned}
\mathbf{G}_k^{(c)} &= \left(G_{k0}^{(c)}, G_{k1}^{(c)}, \cdots, G_{k(2N)}^{(c)}\right), \ \mathbf{G}_k^{(s)} = \left(G_{k0}^{(s)}, G_{k1}^{(s)}, \cdots, G_{k(2N)}^{(s)}\right), \\
\mathbf{G}_k^{\tau(c)} &= \left(G_{k0}^{\tau(c)}, G_{k1}^{\tau(c)}, \cdots, G_{k(2N)}^{\tau(c)}\right), \ \mathbf{G}_k^{\tau(s)} = \left(G_{k0}^{\tau(s)}, G_{k1}^{\tau(s)}, \cdots, G_{k(2N)}^{\tau(s)}\right)
\end{aligned} \tag{2.23}$$

for $k = 1, 2, \ldots, N$. The corresponding components are

$$\begin{aligned}
G_r^{(0)} &= -\alpha_1 \delta_r^0 - \beta g_r^{(0)}, \\
G_{kr}^{(c)} &= \left(\tfrac{k\Omega}{m}\right)^2 \delta_k^r - \delta \tfrac{k\Omega}{m} \delta_{k+N}^r - \alpha_1 \delta_k^r - \beta g_{kr}^{(c)}, \\
G_{kr}^{(s)} &= \left(\tfrac{k\Omega}{m}\right)^2 \delta_{k+N}^r + \delta \tfrac{k\Omega}{m} \delta_k^r - \alpha_1 \delta_{k+N}^r - \beta g_{kr}^{(s)},
\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}
g_r^{(0)} &= 2a_0^{(m)} \delta_r^0 + \sum_{i=1}^{N} \left(b_{i/m} \delta_i^r + c_{i/m} \delta_{i+N}^r\right), \\
g_{kr}^{(c)} &= 2\left(b_{k/m} \delta_r^0 + a_0^{(m)} \delta_k^r\right) + \sum_{i=1}^{N} \sum_{j=1}^{N} b_{j/m} \delta_i^r \left(\delta_{i+j}^k + \delta_{j-i}^k + \delta_{i-j}^k\right) \\
&\quad + c_{i/m} \delta_{i+N}^r \left(\delta_{j-i}^k - \delta_{i+j}^k + \delta_{i-j}^k\right), \\
g_{kr}^{(s)} &= 2\left(c_{k/m} \delta_r^0 + a_0^{(m)} \delta_{k+N}^r\right) + \sum_{i=1}^{N} \sum_{j=1}^{N} \left(c_{j/m} \delta_i^r + b_{i/m} \delta_{j+N}^r\right) \left(\delta_{i+j}^k + \delta_{j-i}^k - \delta_{i-j}^k\right),
\end{aligned} \tag{2.25}$$

for $r = 0, 1, \ldots, 2N$. The components relative to the time delay for $r = 0, 1, \ldots, 2N$ are

$$\begin{aligned}
G_r^{\tau(0)} &= \alpha_2 \delta_r^0, \\
G_{kr}^{\tau(c)} &= \alpha_2 \left[\delta_k^r \cos\left(\tfrac{k}{m}\Omega\tau\right) - \delta_{k+N}^r \sin\left(\tfrac{k}{m}\Omega\tau\right)\right], \\
G_{kr}^{\tau(s)} &= \alpha_2 \left[\delta_{k+N}^r \cos\left(\tfrac{k}{m}\Omega\tau\right) + \delta_k^r \sin\left(\tfrac{k}{m}\Omega\tau\right)\right].
\end{aligned} \tag{2.26}$$

The matrices relative to the velocity are

$$\begin{aligned}
\mathbf{H} &= \frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}_1{}^{(m)}} = \left(\mathbf{H}^{(0)}, \mathbf{H}^{(c)}, \mathbf{H}^{(s)}\right)^{\mathrm{T}}, \\
\mathbf{H}^{\tau} &= \frac{\partial \mathbf{g}^{(m)}}{\partial \mathbf{z}_1^{\tau(m)}} = \left(\mathbf{H}^{\tau(0)}, \mathbf{H}^{\tau(c)}, \mathbf{H}^{\tau(s)}\right)^{\mathrm{T}},
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\mathbf{H}^{(0)} &= \left(H_0^{(0)}, H_1^{(0)}, \cdots, H_{2N}^{(0)}\right), \\
\mathbf{H}^{(c)} &= \left(\mathbf{H}_1^{(c)}, \mathbf{H}_2^{(c)}, \cdots, \mathbf{H}_N^{(c)}\right)^{\mathrm{T}}, \\
\mathbf{H}^{(s)} &= \left(\mathbf{H}_1^{(s)}, \mathbf{H}_2^{(s)}, \cdots, \mathbf{H}_N^{(s)}\right)^{\mathrm{T}}; \\
\mathbf{H}^{\tau(0)} &= \left(H_0^{\tau(0)}, H_1^{\tau(0)}, \cdots, H_{2N}^{\tau(0)}\right), \\
\mathbf{H}^{\tau(c)} &= \left(\mathbf{H}_1^{\tau(c)}, \mathbf{H}_2^{\tau(c)}, \cdots, \mathbf{H}_N^{\tau(c)}\right)^{\mathrm{T}}, \\
\mathbf{H}^{\tau(s)} &= \left(\mathbf{H}_1^{\tau(s)}, \mathbf{H}_2^{\tau(s)}, \cdots, \mathbf{H}_N^{\tau(s)}\right)^{\mathrm{T}}
\end{aligned} \tag{2.28}$$

for $N = 1, 2, \ldots, \infty$, with

$$\begin{aligned}
\mathbf{H}_k^{(c)} &= \left(H_{k0}^{(c)}, H_{k1}^{(c)}, \cdots, H_{k(2N)}^{(c)}\right), \\
\mathbf{H}_k^{(s)} &= \left(H_{k0}^{(s)}, H_{k1}^{(s)}, \cdots, H_{k(2N)}^{(s)}\right); \\
\mathbf{H}_k^{\tau(c)} &= \left(H_{k0}^{\tau(c)}, H_{k1}^{\tau(c)}, \cdots, H_{k(2N)}^{\tau(c)}\right), \\
\mathbf{H}_k^{\tau(s)} &= \left(H_{k0}^{\tau(s)}, H_{k1}^{\tau(s)}, \cdots, H_{k(2N)}^{\tau(s)}\right).
\end{aligned} \tag{2.29}$$

for $k = 1, 2, \ldots, N$. The corresponding components are

$$\begin{aligned}
&H_r^{(0)} = -\delta\delta_0^r, H_{kr}^{(c)} = -2k\Omega\delta_{k+N}^r - \delta\delta_k^r,\ H_{kr}^{(s)} = 2k\Omega\delta_k^r - \delta\delta_{k+N}^r; \\
&H_r^{\tau(0)} = 0,\ H_{kr}^{\tau(c)} = 0,\ H_{kr}^{\tau(s)} = 0
\end{aligned} \tag{2.30}$$

for $r = 0, 1, \ldots, 2N$.

From Luo [11, 14], the eigenvalues of Eq. (2.17) are classified as

$$\left(n_1, n_2, n_3 \middle| n_4, n_5, n_6\right), \tag{2.31}$$

where $n_1$ is the total number of negative real eigenvalues, $n_2$ is the total number of positive real eigenvalues, $n_3$ is the total number of negative zero eigenvalues;

$n_4$ is the total pair number of complex eigenvalues with negative real parts, $n_5$ is the total pair number of complex eigenvalues with positive real parts, $n_6$ is the total pair number of complex eigenvalues with zero real parts. If $\mathrm{Re}\,(\lambda_k) < 0$ $(k = 1, 2, \cdots, 2\,(2N+1))$, the approximate steady-state solution $\mathbf{y}^*$ with truncation of $\cos(N\Omega t)$ and $\sin(N\Omega t)$ is stable. If $\mathrm{Re}\,(\lambda_k) > 0$ $(k \in \{1, 2, \cdots, 2\,(2N+1)\})$, the truncated approximate steady-state solution is unstable. The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation.

The harmonic amplitude and phase are defined by

$$A_{k/m} \equiv \sqrt{b_{k/m}^2 + c_{k/m}^2} \text{ and } \varphi_{k/m} = \arctan \frac{c_{k/m}}{b_{k/m}}. \tag{2.32}$$

The corresponding solution in Eq. (2.4) becomes

$$\begin{aligned} x^*(t) &= a_0^{(m)} + \sum_{k=1}^{N} A_{k/m} \cos\left(\frac{k}{m}\Omega t - \varphi_{k/m}\right), \\ x^{\tau *}(t) &= a_0^{(m)} + \sum_{k=1}^{N} A_{k/m} \cos\left[\frac{k}{m}\Omega\,(t-\tau) - \varphi_{k/m}\right]. \end{aligned} \tag{2.33}$$

Consider system parameters as

$$\delta = 0.05, \alpha_1 = 15.0, \alpha_2 = 5.0, \beta = 5.0, Q_0 = 4.5, \tau = T/4. \tag{2.34}$$

## 2.2 Numerical Illustrations

To verify the approximate analytical solutions of periodic motion in the time-delayed, quadratic nonlinear oscillator, numerical simulations will be completed through the midpoint discrete scheme. The initial conditions and the initial time-delay values in the range of $t \in (-\tau, 0)$ for numerical simulation are computed from the approximate analytical solutions. The numerical results are depicted by solid curves, but the analytical solutions are given by red circular symbols. The big filled circular symbols are initial conditions and initial time-delay response values. The initial starting and final points of the time delay are represented by the acronyms D.I.S. and D.I.F., respectively.

The displacement, velocity, trajectory, and amplitude spectrum of stable period-1 motion for the time-delayed, quadratic nonlinear oscillator are presented in Fig. 2.1 for $\Omega = 7.767$ with initial condition $(x_0 \approx -0.100171, \dot{x}_0 \approx 0.089894)$ with initial time-delayed responses. This analytical solution is based on 20 harmonic terms (HB20) in the Fourier series solution of period-1 motion. In Fig. 2.1a, b, for over 100 periods, the analytical and numerical solutions of the period-1 motion in the time-delayed, quadratic nonlinear oscillator match very well. The

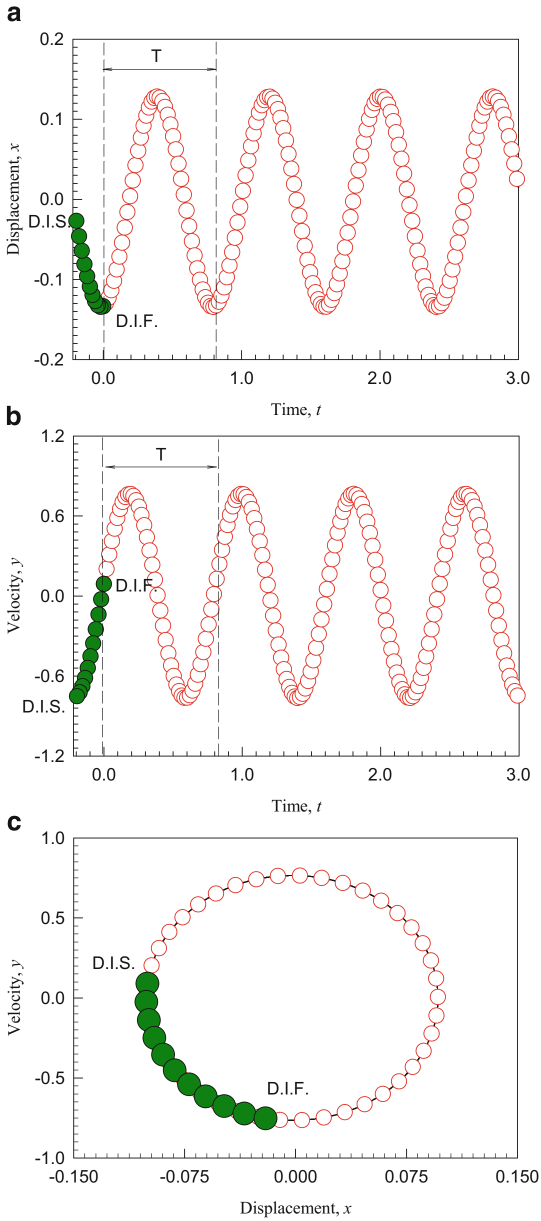
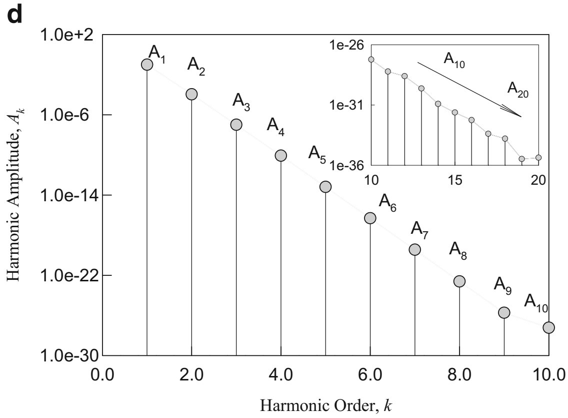



**Fig. 2.1** Analytical and numerical solutions of stable period-1 motion based on 20 harmonic terms (HB20) ($\Omega = 7.767$): (**a**) displacement, (**b**) velocity, (**c**) phase plane, and (**d**) amplitude spectrum. Initial condition ($x_0 \approx -0.100171, \dot{x}_0 \approx 0.089894$). Parameters: ($\delta = 0.05, \alpha_1 = 15.0, \alpha_2 = 5.0, \beta = 5.0, Q_0 = 4.5$, $\tau = T/4$)

**Fig. 2.1** (continued)




initial time-delayed displacement and velocity are represented by the large circular symbols for the initial delay period of $t \in (-\tau, 0)$. In Fig. 2.1c, analytical and numerical trajectories match very well, and the initial time-delay response in the phase plane is clearly depicted. In Fig. 2.1d, the amplitude spectrum versus the harmonic order is presented. The corresponding quantity levels of the harmonic amplitudes are given as follows: $a_0 \approx -2.4302\text{e-}3$, $A_1 \approx 0.0985$, and $A_k \in \left(10^{-36}, 10^{-4}\right)$ $(k = 2, 3, \ldots, 20)$. For the distribution of harmonic amplitudes, the harmonic amplitudes decrease with harmonic order nonuniformly. The main contribution for this periodic motion is from the primary harmonics. The truncated harmonic amplitude is $A_{20} \sim 10^{-36}$. For this periodic motion, one can use a harmonic term to get an accurate enough analytical solution.

From the bifurcation tree of period-1 motion to chaos in Luo and Jin [15], the stable period-1, period-2, period-4, and period-8 motions are presented in Fig. 2.2 at $\Omega = 1.897, 1.8965, 1.8920, 1.88906$ for illustrations of the complexity of periodic motions. The initial conditions for such stable periodic motions are listed in Table 2.1.

In Fig. 2.2a, the analytical and numerical trajectories of period-1 motion are presented. Such period-1 motion possesses two cycles and the initial time-delay conditions are presented. The harmonic amplitude distribution is presented in Fig. 2.2b. The main amplitudes of the period-1 motion in such a time-delayed, nonlinear system are $a_0 \approx -0.618722$, $A_1 \approx 0.309591$, $A_2 \approx 1.264949$, $A_3 \approx 0.086255$, $A_4 \approx 0.076064$, and $A_k \in \left(10^{-14}, 10^{-2}\right)$ for $k = 5, 6, \ldots, 20$. The second harmonic amplitude plays an important role in the period-1 motion.

In Fig. 2.2c, the analytical and numerical trajectories of period-1 motion are presented. Such period-1 motion possesses two cycles and the initial time-delay conditions are presented. The harmonic amplitude distribution is presented in Fig. 2.2d. The main amplitudes of the period-2 motion in such a time-delayed,

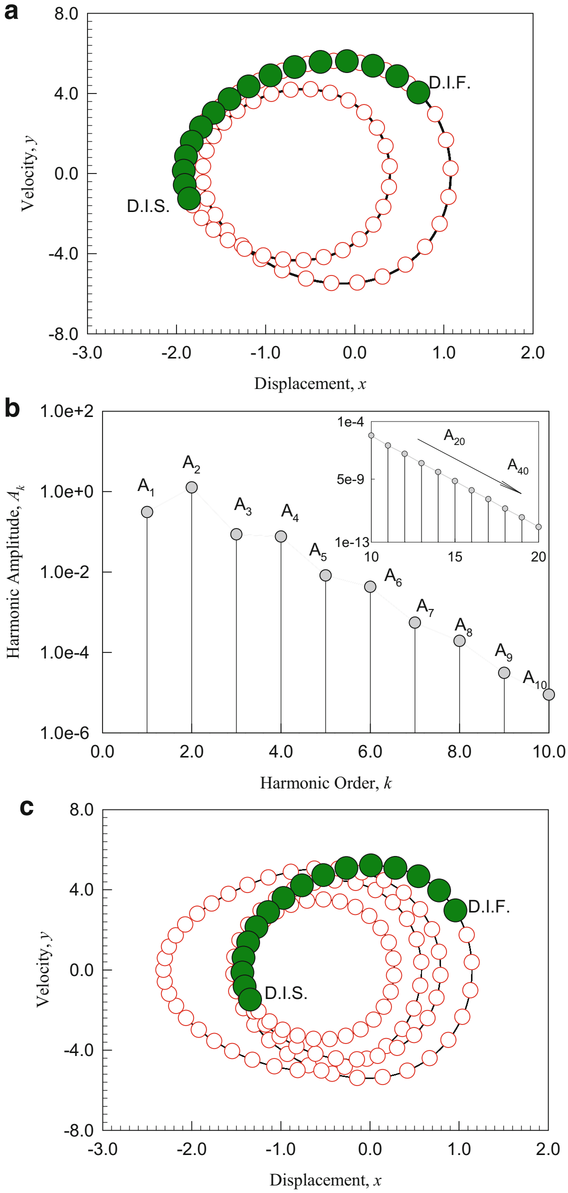



**Fig. 2.2** Phase plane and amplitude spectrum: (**a**) and (**b**) period-1 motion ($\Omega = 1.8970$, HB20); (**c**) and (**d**) period-2 motion ($\Omega = 1.8965$, HB40); (**e**) and (**f**) period-4 motion ($\Omega = 1.8920$, HB80); (**g**) and (**h**) period-4 motion ($\Omega = 1.88906$, HB80). Parameters: ($\delta = 0.05, \alpha_1 = 15.0, \alpha_2 = 5.0, \beta = 5.0, Q_0 = 4.5$, $\tau = T/4$)

**Fig. 2.2** (continued)

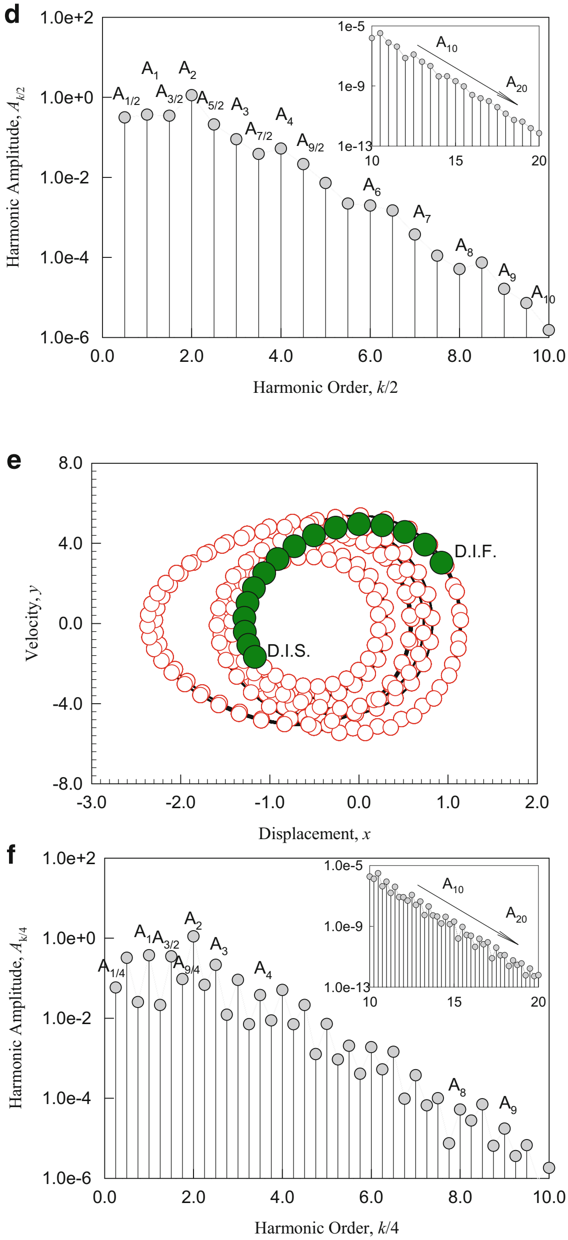
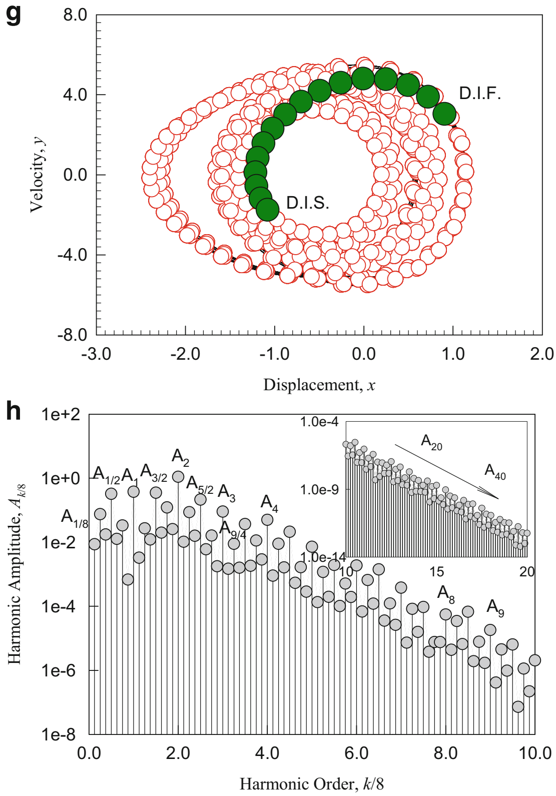


**Fig. 2.2** (continued)




nonlinear system are $a_0^{(2)} \approx -0.589080$, $A_{1/2} \approx 0.312662$, $A_1 \approx 0.366173$, $A_{3/2} \approx 0.345472$, $A_2 \approx 1.120050$, $A_{5/2} \approx 0.209455$, $A_3 \approx 0.089404$, $A_{7/2} \approx 0.038283$, $A_4 \approx 0.052349$, $A_{9/2} \approx 0.021267$, and $A_{k/2} \in (10^{-14}, 10^{-2})$ for $k = 10, 11, \ldots, 40$. The biggest contribution is from the harmonic term of $A_2 \approx 1.120050$.

In Fig. 2.2e, the analytical and numerical trajectories of period-4 motion are presented. Such period-4 motion possesses eight cycles and the initial time-delay conditions are presented. The harmonic amplitude distribution is presented in Fig. 2.2f. The main amplitudes of the period-4 motion are $a_0^{(4)} \approx -0.591813$, $A_{1/4} \approx 0.058286$, $A_{1/2} \approx 0.322076$, $A_{3/4} \approx 0.025289$, $A_1 \approx 0.373248$, $A_{5/4} \approx 0.021254$, $A_{3/2} \approx 0.351173$, $A_{7/4} \approx 0.094394$, $A_2 \approx 1.106125$, $A_{9/4} \approx 0.067732$, $A_{5/2} \approx 0.214359$, $A_{11/4} \approx 0.012157$, $A_3 \approx 0.090130$, $A_{13/4} \approx 7.042438\text{E-}3$, $A_{7/2} \approx 0.037581$, $A_{15/4} \approx 8.784526\text{E-}3$, $A_4 \approx 0.050681$, $A_{17/4} \approx 7.035358\text{E-}3$, $A_{9/2} \approx 0.021354$, $A_{19/4} \approx 1.263319\text{E-}3$, and $A_{k/4} \in (10^{-14}, 10^{-2})$ for $k = 20, 21, \ldots, 80$.

The analytical and numerical trajectories of period-8 motion are presented in Fig. 2.2g. Such period-8 motion possesses 16 cycles and the initial time-

**Table 2.1** Input data for numerical illustrations ($\delta = 0.05, \alpha_1 = 15.0, \alpha_2 = 5.0, \beta = 5.0, Q_0 = 4.5, \tau = T/4$)

| Figure no. | Ω | Initial condition $(x_0, \dot{x}_0)$ | Types | Harmonics terms |
|---|---|---|---|---|
| Figure 2.2a, b | 1.8970 | (0.713984, 4.045130) | P-1 | HB20 (stable) |
| Figure 2.2c, d | 1.8965 | (0.959465, 2.965047) | P-2 | HB40 (stable) |
| Figure 2.2e, f | 1.8920 | (0.926914, 3.026495) | P-4 | HB80 (stable) |
| Figure 2.2g, h | 1.88876 | (0.904503, 3.045649) | P-8 | HB160 (stable) |

delay conditions are presented clearly. As presented before, the harmonic amplitude spectrum is presented in Fig. 2.2h. The main amplitudes of the period-8 motion are $a_0^{(8)} \approx -0.594919$, $A_{1/8} \approx$ 8.668953e-3, $A_{1/4} \approx 0.075480$, $A_{3/8} \approx 0.017434$, $A_{1/2} \approx 0.324209$, $A_{5/8} \approx 0.012676$, $A_{3/4} \approx 0.033521$, $A_{7/8} \approx$ 6.822809e-4, $A_1 \approx 0.376686$, $A_{9/8} \approx$ 3.278184e-3, $A_{5/4} \approx 0.027110$, $A_{11/8} \approx 0.012213$, $A_{3/2} \approx 0.351086$, $A_{13/8} \approx 0.019842$, $A_{7/4} \approx 0.122173$, $A_{15/8} \approx 0.025622$, $A_2 \approx 1.099997$, $A_{17/8} \approx 0.010327$, $A_{9/4} \approx 0.087137$, $A_{19/8} \approx 0.015794$, $A_{5/2} \approx 0.214882$, $A_{21/8} \approx$ 5.998294e-3, $A_{11/4} \approx 0.016157$, $A_{23/8} \approx$ 1.775930e-3, $A_3 \approx 0.090622$, $A_{25/8} \approx$ 1.485620e-3, $A_{13/4} \approx$ 8.904199e-3, $A_{27/8} \approx$ 1.592552e-3, $A_{7/2} \approx 0.036887$, $A_{29/8} \approx$ 1.829681e-3, $A_{15/4} \approx 0.011286$, $A_{31/8} \approx$ 2.891636e-3, $A_4 \approx 0.050091$, $A_{33/8} \approx$ 9.021719e-4, $A_{17/4} \approx$ 8.953262e-3, $A_{35/8} \approx$ 1.640158e-3, $A_{9/2} \approx 0.021173$, and $A_{k/g} \in \left(10^{-14}, 10^{-2}\right)$ for $k = 37, 38, \ldots, 160$. The biggest contribution of the period-8 motion is still from the harmonic amplitude of $A_2 \approx 1.099997$.

## 2.3 Conclusion

In this chapter, the analytical solutions of period-$m$ motions in the time-delayed, quadratic nonlinear oscillator were obtained from the finite Fourier series expression. Based on such analytical solutions, the stability and bifurcation of period-$m$ motions of the time-delayed nonlinear oscillator were discussed. From the bifurcation trees of period-1 motion to chaos, numerical simulations were carried out to compare analytical and numerical solutions of periodic motions. The numerical and analytical solutions of periodic motions are well matched in such a time-delayed, quadratic nonlinear oscillator once enough harmonic terms are included in the finite Fourier series expression.

## References

1. Lagrange, J.L.: Mécanique Analytique, 2 vol. (édition Albert Blanchard Paris, 1965) (1788) [French]
2. Poincaré, H.: Méthodes Nouvelles de la Mécanique Celeste, vol. 3. Gauthier-Villars, Paris (1899) [French]
3. van der Pol, B.: A theory of the amplitude of free and forced triode vibrations. Radio Rev. **1**, 701–710 (1920). 754–762
4. Fatou, P.: Sur le mouvement d'un système soumis à des forces à courte periode. Bull. Soc. Math. **56**, 98–139 (1928) [French]
5. Krylov, N.M., Bogolyubov, N.N.: Methodes approchées de la mécanique non-linéaire dans leurs application à l'éetude de la perturbation des mouvements périodiques de divers phénomènes de résonance s'y rapportant. Academie des Sciences d'Ukraine, Kiev (1935) [French]
6. Bogoliubov, N.N., Mitropolsky, Y.A.: Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordon and Breach, New York (1961)
7. Hayashi, C.: Nonlinear Oscillations in Physical Systems. McGraw-Hill, New York (1964)
8. Nayfeh, A.H.: Perturbation Methods. Wiley, New York (1973)
9. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillation. Wiley, New York (1979)
10. Coppola, V.T., Rand, R.H.: Averaging using elliptic functions: approximation of limit cycle. Acta Mech. **81**, 125–142 (1990)
11. Luo, A.C.J.: Continuous Dynamical Systems. Higher Education Press/L&H Scientific, Beijing/Glen Carbon (2012)
12. Luo, A.C.J., Huang, J.Z.: Approximate solutions of periodic motions in nonlinear systems via a generalized harmonic balance. J. Vib. Control. **18**, 1661–1674 (2012)
13. Luo, A.C.J., Huang, J.Z.: Analytical dynamics of period-*m* flows and chaos in nonlinear systems. Int. J. Bifurcation Chaos **22**, 29 p (2012). Article No. 1250093
14. Luo, A.C.J.: Analytical solutions of periodic motions in dynamical systems with/without time-delay. Int. J. Dyn. Control **1**, 330–359 (2013)
15. Luo, A.C.J., Jin, H.X.: Bifurcation trees of period-*m* motions in a time-delayed, quadratic nonlinear oscillator under a periodic excitation. Discontin. Nonlinearity Complex. **3**, 87–107 (2014)