Preconditioners for Mixed FEM Solution of Stationary and Nonstationary Porous Media Flow Problems

Owe Axelsson, Radim Blaheta(✉), and Tomáš Luber

Institute of Geonics AS CR,
Studentska 1768, 70800 Poruba, Ostrava, Czech Republic
blaheta@ugn.cas.cz

Abstract. The paper concerns porous media flow in rigid or deformable matrix. It starts with stationary Darcy flow, but the main interest is in extending Darcy problem to involve time dependent behaviour and deformation of the matrix. The considered problems are discretized by mixed FEM in space and stable time discretization methods as backward Euler and second order Radau methods. The discretization leads to time stepping methods which involve solution of a linear system within each time step. The main focus of the paper is then devoted to the construction of suitable preconditioners for these Euler and Radau systems. The paper presents also numerical experiments for illustration of efficiency of the suggested numerical algorithms.

Keywords: Darcy flow · Poroelasticity · Saddle point systems · Preconditioners

1 Introduction

The porous media flow in rigid or deformable matrix are basically described by Darcy flow and Biot poroelasticity models, respectively. The stationary Darcy problem can be written in the form

$$\begin{aligned} K^{-1}v + \nabla p &= 0, \\ \operatorname{div}(v) \qquad &= Q \end{aligned} \tag{1}$$

with two physical fields, the Darcy velocity v and the fluid (pore) pressure p, which have to be determined in a domain Ω. Here $v = \phi v_f$ where ϕ is the porosity and v_f is the fluid velocity. The parameter K is the matrix of permeabilities divided by fluid viscosity (effective permeability, $K_{ij} = \kappa_{ij}/\nu_f$) and Q stands for the fluid source/sink term. The introduced Darcy flow model can be formulated variationally and discretized by a mixed finite element method, which leads to saddle point systems. The solution of these systems can be done by use of iterative methods with preconditioners based on the natural block structure. Efficient preconditioners can be based on regularization of the zero block and

I. Lirkov et al. (Eds.): LSSC 2015, LNCS 9374, pp. 3–14, 2015.
DOI: 10.1007/978-3-319-26520-9_1

subsequent formulation with augmented blocks. For stationary problems, the regularization implies necessity of strong augmentation and possible difficulties in solving the augmented block system.

The source term and accordingly also the velocity and pressure can be time dependent. In this case the Darcy model usually involves a flow retardation mechanism, which is provided by a small compressibility of fluid and/or deformation of the porous matrix. The time dependent Darcy model then has the following form

$$\begin{aligned} K^{-1}v \quad &+\nabla p &= 0, \\ \mathrm{div}(v) &+c_{pp}\tfrac{\partial}{\partial t}p &= Q. \end{aligned} \tag{2}$$

The time dependent Darcy model can be also discretized by the mixed finite elements in space and a suitable method in time. For the time discretization, we shall use stable methods such as the first order backward Euler or higher order Radau methods. After discretization, we solve the evolution problems by a time stepping procedure with solving saddle point systems within each time step. Compared with the stationary Darcy systems, the backward Euler systems are naturally regularized by the time derivative term, which influence the block preconditioners. For higher order Radau methods, we introduce additional preconditioners, which involve the solution of backward Euler type systems.

The porous media flow can be coupled with deformation of the porous matrix. The basic model in this respect is the Biot poroelasticity, which can be described by the equations

$$\begin{array}{lllll} -\mathrm{div}(C_{el} : \varepsilon(u)) & & +c_{up}\nabla p & = f, \\ & K^{-1}v & +\nabla p & = 0, \\ c_{pu}\frac{\partial}{\partial t}\mathrm{div}(u) & +\mathrm{div}(v) & +c_{pp}\frac{\partial}{\partial t}p & = Q. \end{array} \tag{3}$$

There are three physical fields in the domain Ω entering the above model, besides the velocity v and the fluid pressure p, it is the displacement u, which defines the small strain tensor $\varepsilon(u)$. Further, C_{el} is the elasticity tensor and $c_{up} = c_{pu} = \alpha$ are Biot-Willis coefficients. For simplicity, we assume $c_{up} = c_{pu} = 1$.

The organization of the paper is as follows. The next Section concerns discretization of the described porous media flow problems. The space discretization uses the lowest order Raviart-Thomas elements. The time dependent problems are then solved by time stepping methods with the solution of Euler and Radau systems in each step. The preconditioners for Euler and Radau systems are investigated in Sects. 3 and 4. Section 5 introduces a model problem and describes numerical experiments which illustrate the efficiency of the preconditioners.

2 Space and Time Discretization

The introduced problems can be formulated variationally and discretized by the Galerkin technique using proper function spaces. Namely, for $\Omega \subset R^d$ and decomposition of the boundary $\partial\Omega$ corresponding to different boundary conditions for flow $\partial\Omega = \Gamma_{v,p} \cup \Gamma_{v,v}$ and for mechanical response $\partial\Omega = \Gamma_{u,u} \cup \Gamma_{u,\sigma}$, we take

$$u_h \in U_h \subset U = \left\{u \in [H^1(\Omega)]^d,\ u = u_D \text{ on } \Gamma_{u,u}\right\},$$

where U_h corresponds to a finite element mesh division $\mathcal{T}_h$ of Ω into system of triangles for $d = 2$ or tetrahedra for $d = 3$. The functions from U_h are continuous on Ω and piecewise linear on elements of $\mathcal{T}_h$. Further,

$$v_h \in V_h \subset V = \{w \in H(\mathrm{div},\, \Omega),\ w \cdot \nu = q_n \text{ on } \Gamma_{p,v}\},$$

where V_h contains the lowest order Raviart-Thomas finite elements on the same division $\mathcal{T}_h$ as used for elasticity. Finally,

$$p_h \in P_h \subset P = L_2(\Omega),$$

where P_h contains functions piecewise constant on the same mesh $\mathcal{T}_h$.

After taking proper bases in U_h, V_h, P_h and establishing isomorphism between finite element functions and algebraic vectors $u_h \leftrightarrow u$, $v_h \leftrightarrow v$ and $p_h \leftrightarrow p$, we can introduce the finite element matrices,

$$\langle Au,\, w\rangle = \int_\Omega C\varepsilon(u_h) : \varepsilon(w_h)\,\mathrm{d}x \ \ \forall u_h,\, w_h \in U_h,$$

$$\langle Mv,\, z\rangle = \int_\Omega K^{-1} v_h \cdot z_h\,\mathrm{d}x \ \ \forall v_h,\, z_h \in V_h,$$

$$\langle M_p p,\, q\rangle = \int_\Omega p_h q_h\,\mathrm{d}x \ \ \forall p_h,\, q_h \in P_h,$$

$$\langle B_u u,\, q\rangle = \int_\Omega \mathrm{div}(u_h) q_h\,\mathrm{d}x \ \ \forall u_h \in U_h,\ q_h \in P_h,$$

$$\langle B_v v,\, q\rangle = \int_\Omega \mathrm{div}(v_h) q_h\,\mathrm{d}x \ \ \forall v_h \in V_h,\ q_h \in P_h.$$

Note that A, M, M_p are symmetric, $C = c_{pp} M_p$, A is positive definite if the displacement is prescribed on a part $\Gamma_{u,u} \subset \partial\Omega$ with a positive measure and M, M_p are always positive definite.

The discretization of time dependent Darcy and poroelasticity problems by mixed finite element methods results in a differential-algebraic (DAE) system of a general form

$$\mathcal{A}_1 \frac{\partial}{\partial t}\mathcal{U} + \mathcal{A}_0 \mathcal{U} = \mathcal{F},$$

where

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -C \end{bmatrix},\ \ \mathcal{A}_0 = \begin{bmatrix} M & B^T \\ B & 0 \end{bmatrix},\ \ \mathcal{U} = \begin{bmatrix} v \\ p \end{bmatrix}$$

for the time dependent Darcy problem and

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_u & 0 & -C \end{bmatrix},\ \ \mathcal{A}_0 = \begin{bmatrix} A & 0 & B_u^T \\ 0 & M & B^T \\ 0 & B & 0 \end{bmatrix},\ \ \mathcal{U} = \begin{bmatrix} u \\ v \\ p \end{bmatrix}$$

for poroelasticity.

The time discretization is performed by a sequence of time steps

$$0 = t_0 < t_1 < \ldots < t_N,$$

where $\tau_i = t_{i+1} - t_i$ are provided apriori or computed adaptively. To simplify the presentation, we shall assume constant time steps, $\tau_i = \tau$. For each time interval $\langle t_i,\, t_{i+1}\rangle$, we have

$$\int_{t_i}^{t_i+\tau} \mathcal{A}_1 \frac{\partial}{\partial t}\mathcal{U}\,\mathrm{d}t + \int_{t_i}^{t_i+\tau} (\mathcal{A}_0\mathcal{U} - \mathcal{F})\,\mathrm{d}t = \mathcal{A}_1\left(\mathcal{U}^{i+1} - \mathcal{U}^i\right) + \int_{t_i}^{t_i+\tau} (\mathcal{A}_0\mathcal{U} - \mathcal{F})\,\mathrm{d}t = 0.$$

The integration $\int_{t_i}^{t_i+\tau} (\mathcal{A}_0\mathcal{U} - \mathcal{F})\,dt$ has to be performed by a suitable approximate integration scheme. The use of the simple right-hand rectangle approximate integration provides the backward Euler method

$$\mathcal{A}_E\mathcal{U}^{i+1} = (\mathcal{A}_1 + \tau\mathcal{A}_0)\,\mathcal{U}^{i+1} = \mathcal{A}_1\mathcal{U}^i + \tau\mathcal{F}^{i+1} \;\; \forall i = 0, \ldots, N-1. \tag{4}$$

The backward Euler method is stable and suitable for the solution of stiff and DAE problems, but has only a first order time discretization error. As an example of higher order discretization methods, we use the third order L-stable method based on the second order Radau integration (RADAU IIA), see e.g. [1]. It uses two integration points $t_{i+1/3}$ and t_{i+1} in the interval $\langle t_i,\, t_{i+1}\rangle$. The position of $t_{i+1/3}$ and the weights are determined from the condition that the integration scheme should be exact for polynomials up to second order. It provides

$$\int_{t_i}^{t_i+\tau} \phi\,\mathrm{d}x = \frac{3}{4}\tau\phi(t_i + \tau/3) + \frac{1}{4}\tau\phi(t_{i+1}) \;\; \forall\phi(t) = \sum_{i=0}^{2}\alpha_i t^i.$$

The Radau method leads to the system

$$\mathcal{A}_R \begin{bmatrix} \mathcal{U}^{i+1/3} \\ \mathcal{U}^{i+1} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 + \frac{5\tau}{12}\mathcal{A}_0 & -\frac{\tau}{12}\mathcal{A}_0 \\ \frac{3\tau}{4}\mathcal{A}_0 & \mathcal{A}_1 + \frac{\tau}{4}\mathcal{A}_0 \end{bmatrix} \begin{bmatrix} \mathcal{U}^{i+1/3} \\ \mathcal{U}^{i+1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{A}_1\mathcal{U}^i + \frac{\tau}{12}(5\mathcal{F}^{i+1/3} - \mathcal{F}^{i+1}) \\ \mathcal{A}_1\mathcal{U}^i + \frac{\tau}{4}(3\mathcal{F}^{i+1/3} + \mathcal{F}^{i+1}) \end{bmatrix}. \tag{5}$$

Elimination of $\mathcal{U}^{i+1/3}$ provides a reduced system

$$\mathcal{A}_{RR}\mathcal{U}^{i+1} = \left(\mathcal{A}_1\mathcal{A}_0^{-1} - \frac{1}{3}\tau\right)\mathcal{A}_1\mathcal{U}^i + \frac{\tau}{4}\mathcal{A}_1\mathcal{A}_0^{-1}\left(3\mathcal{F}^{i+1/3} + \mathcal{F}^{i+1}\right) + \frac{1}{6}\tau^2\mathcal{F}^{i+1}$$

with the matrix

$$\mathcal{A}_{RR} = \frac{1}{6}\tau^2\mathcal{A}_0 + \frac{2}{3}\tau\mathcal{A}_1 + \mathcal{A}_1\mathcal{A}_0^{-1}\mathcal{A}_1\,. \tag{6}$$

The space and time discretization provides possibility to solve the porous media flow by time stepping algorithms which compute the vector of all unknowns by solving the corresponding system in each time step. The systems to be solved are (4), (5) and (6), depending on the chosen time discretization technique.

3 Preconditioners for the Euler Systems

The Euler system for the time dependent Darcy model has the form

$$\mathcal{A}_E = \mathcal{A}_1 + \tau \mathcal{A}_0 = \tau \begin{bmatrix} M & B^T \\ B & -\frac{1}{\tau} C \end{bmatrix},$$

which contains a regularization term in the (2,2) block. The regularization is based on pressure mass matrix multiplied by compressibility parameter which is typically constant in the whole domain. For the lowest order Raviart-Thomas elements the pressure mass matrix is diagonal, which allows an easy inverse of the matrix C. All these facts indicate that suitable preconditioners can be found of the augmented type, i.e.

$$\mathcal{P}_{ET} = \begin{bmatrix} M_C & B^T \\ & -\frac{1}{\tau} C \end{bmatrix}, \; \mathcal{P}_{ED} = \begin{bmatrix} M_C & \\ & -\frac{1}{\tau} C \end{bmatrix}, \; \mathcal{P}_{EDP} = \begin{bmatrix} M_C & \\ & \frac{1}{\tau} C \end{bmatrix}.$$

In all cases $M_C = M + \tau B^T C^{-1} B$ is the augmented matrix. For the Raviart-Thomas finite elements, M_C can be assembled as a sparse matrix, which allows to solve the inner system with M_C by various direct or iterative solvers. Note that small time steps improve conditioning, which is favourable.

As concerns the preconditioned systems, an analysis for both exact (the ideal case) and inexact solution of subproblems can be found in the literature, see e.g. [5–8]. Other applicable preconditioning techniques can be found e.g. in [4,8].

The Euler system for the poroelasticity problems has the form

$$\mathcal{A}_E = \mathcal{A}_1 + \tau \mathcal{A}_0 = \tau \begin{bmatrix} A & 0 & B_u^T \\ 0 & M & B^T \\ \frac{1}{\tau} B_u & B & -\frac{1}{\tau} C \end{bmatrix}.$$

$\mathcal{A}_E$ is not symmetric but the corresponding system can be easily symmetrized by row scaling which provides new system with the matrix $\tilde{\mathcal{A}}_E$,

$$\tilde{\mathcal{A}}_E = \begin{bmatrix} \frac{1}{\tau} & & \\ & 1 & \\ & & 1 \end{bmatrix} \mathcal{A}_E = \begin{bmatrix} A & 0 & B_u^T \\ 0 & \tau M & \tau B^T \\ B_u & \tau B & -C \end{bmatrix}$$

and suitable preconditioners for $\tilde{\mathcal{A}}_E$ can be again found of the augmented type form, i.e.

$$\mathcal{P}_{ET} = \begin{bmatrix} S_C & \bar{B}^T \\ & -C \end{bmatrix}, \; \mathcal{P}_{ED} = \begin{bmatrix} S_C & \\ & -C \end{bmatrix}, \; \mathcal{P}_{EDP} = \begin{bmatrix} S_C & \\ & C \end{bmatrix},$$

where $\bar{B} = \begin{bmatrix} B_u & B \end{bmatrix}$ and the pivot block has now a 2×2 structure

$$S_C = \begin{bmatrix} A + B_u^T C^{-1} B_u & \tau B_u^T C^{-1} B \\ \tau B^T C^{-1} B_u & \tau M + \tau^2 B^T C^{-1} B \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

As S_C is now more complicated, a question arises if it can be simplified by considering its block diagonal or block triangular part.

Proposition 1. *There is a constant* $0 \leq \gamma < 1$ *such that*

$$|\langle S_{12}v,\, u\rangle| \leq \gamma\sqrt{\langle S_{11}u,\, u\rangle}\sqrt{\langle S_{22}v,\, v\rangle}\ \ \forall u, v.$$

If c_{el} *is a positive constant such that*

$$c_{el}\,\|\mathrm{div}(u_h)\|^2_{L_2} \leq \langle Au,\, u\rangle$$

then $\gamma^2 \leq (1 + c_{pp}c_{el})^{-1}$. *For isotropic elasticity with Lamè constants* λ *and* μ, $c_{el} = \lambda$. *With the constant* γ, *we have the spectral equivalence*

$$(1-\gamma)\begin{bmatrix} S_{11} & \\ & S_{22}\end{bmatrix} \leq \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{bmatrix} \leq (1+\gamma)\begin{bmatrix} S_{11} & \\ & S_{22}\end{bmatrix}$$

Proof. The proof is based on the strengthened Cauchy-Schwarz-Bunyakowski constant γ. To show the estimate, we apply the CBS inequality to get

$$\begin{aligned} |\langle S_{12}v,\, u\rangle| &= \left|\left\langle \tau B_u^T C^{-1} Bv,\, u\right\rangle\right| = \left|\left\langle \tau C^{-1/2}Bv,\, C^{-1/2}B_u u\right\rangle\right| \\ &\leq \sqrt{\langle \tau^2 B^T C^{-1} Bv,\, v\rangle}\sqrt{\langle B_u^T C^{-1} B_u u,\, u\rangle}\,. \end{aligned}$$

Then

$$\begin{aligned} |\langle S_{12}v,\, u\rangle| &\leq \gamma\,\sqrt{\langle(\tau M + \tau^2 B^T C^{-1} B)v,\, v\rangle}\sqrt{\langle(A + B_u^T C^{-1} B_u)u,\, u\rangle} \\ &= \gamma\,\sqrt{\langle S_{11}u,\, u\rangle}\sqrt{\langle S_{22}v,\, v\rangle}\,. \end{aligned}$$

The estimate for γ comes from

$$\langle B_u u,\, C^{-1}B_u u\rangle \leq \frac{1}{c_{el}c_{pp}}\langle Au,\, u\rangle. \tag{7}$$

Details can be found in [10]. □

Note that the constant γ and the spectral equivalence are independent on discretization (represented by h and τ) and also does not depend on oscillations of permeability. On the other hand, γ depends on compressibility c_{pp} and mechanical stiffness of the porous matrix. A stiffer porous matrix will decrease the coupling between flow and deformation and provide a smaller value of γ.

Note that the estimate of γ could be improved by taking into account the contribution of M similarly as the contribution of A in (7). The contribution of M can be significant if the permeability is small. On the other hand, the qualitative result can be obtained without considering the contribution of M, which has two benefits - avoiding assumptions on oscillatory character of the permeability coefficients and avoiding the fact, that we should use h dependent inverse inequality to bound L_2-norm of $\mathrm{div}(v_h)$ by L_2-norm of v_h.

In the case of having a solver for the system $M + \tau B^T C^{-1} B$, which is robust with respect to coefficient oscillations (see e.g. [9]), the spectral equivalence above provide also a possibility to construct a robust solver for the poroelasticity problem.

4 Preconditioners for the Radau Systems

The more complex systems (5) and (6) arising from the Radau discretization of both time dependent Darcy and poroelasticity problems can be solved iteratively with very efficient preconditioners based on the solution of simpler Euler type systems with matrices

$$\mathcal{A}_1 + \tau \mathcal{A}_0.$$

Proposition 2. *Let $\mathcal{A}_{RR}$ be the matrix of the reduced Radau system (6). Then a suitable preconditioner is found in the form*

$$\mathcal{P}_{RR} = \left(\mathcal{A}_1 + \frac{1}{\sqrt{6}} \tau \mathcal{A}_0 \right) \mathcal{A}_0^{-1} \left(\mathcal{A}_1 + \frac{1}{\sqrt{6}} \tau \mathcal{A}_0 \right) . \tag{8}$$

The spectrum of the preconditioned matrix $\mathcal{P}_{RR}^{-1} \mathcal{A}_{RR}$ is real and lies in the interval $\left\langle 1 - \frac{1}{6+\sqrt{24}},\, 1 \right\rangle$.

Proposition 3. *Let $\mathcal{A}_R$ be the matrix of the full Radau system (5). Then a preconditioner can be taken in the triangular form*

$$\mathcal{P}_{R-T} = \begin{bmatrix} \mathcal{A}_1 + \frac{5\tau}{12} \mathcal{A}_0 & 0 \\ \frac{3\tau}{4} \mathcal{A}_0 & \mathcal{A}_1 + \frac{\tau}{4} \mathcal{A}_0 \end{bmatrix} . \tag{9}$$

The spectrum of the preconditioned matrix $\mathcal{P}_{R-T}^{-1} \mathcal{A}_R$ is real and lies in the interval $\langle 1,\, \frac{8}{5} \rangle$.

Proposition 4. *Let $\mathcal{A}_R$ be the matrix of the full Radau system (5). Then a preconditioner can be taken in the diagonal form*

$$\mathcal{P}_{R-D} = \begin{bmatrix} \mathcal{A}_1 + \frac{5\tau}{12} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_1 + \frac{\tau}{4} \mathcal{A}_0 \end{bmatrix} . \tag{10}$$

The spectrum of the preconditioned matrix $\mathcal{P}_{R-D}^{-1} \mathcal{A}_R$ is complex and lies in the interval $\left\{ z \in \mathbb{C} :\ Re(z) = 1 \ \&\ |Im(z)| \leq \sqrt{3/5} \right\}$.

Proof. The proof of Proposition 2 can be found in [2]. We shall show the proof of Proposition 3, the proof of Proposition 4 is similar.

A simple manipulation provides

$$\mathcal{P}_{R-T}^{-1} \mathcal{A}_R = I + \begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix},$$

where

$$E_{12} = \left(\mathcal{A}_1 + \frac{5\tau}{12} \mathcal{A}_0 \right)^{-1} \left(-\frac{\tau}{12} \right) \mathcal{A}_0 = -\frac{1}{5} \left(\frac{12}{5\tau} \tilde{\mathcal{A}}_1 + I \right)^{-1}$$

$$E_{22} = -\left(\mathcal{A}_1 + \frac{\tau}{4} \mathcal{A}_0 \right)^{-1} \left(\frac{3\tau}{4} \right) \mathcal{A}_0 E_{12} = \frac{3}{5} \left(\frac{4}{\tau} \tilde{\mathcal{A}}_1 + I \right)^{-1} \left(\frac{12}{5\tau} \tilde{\mathcal{A}}_1 + I \right)^{-1} .$$

Above $\tilde{\mathcal{A}}_1 = \mathcal{A}_0^{-1} \mathcal{A}_1$.

If $\mu \in \sigma(\tilde{\mathcal{A}}_1)$ then for the time dependent Darcy model there exists $z = (z_1, z_2)^T$ such that

$$\begin{aligned} \mu M z_1 + \mu B^T z_2 &= 0 \\ \mu B z_1 &= -C z_2 \end{aligned}$$

Thus $\mu = 0$ is the eigenvalue with the eigenvector $z = (z_1, 0)^T$, $z_1 \neq 0$. Moreover, if $\mu \in \sigma(\tilde{\mathcal{A}}_1)$ and $\mu \neq 0$ then $z_1 = -M^{-1}B^T z_2$ and μ fulfils $\mu B M^{-1} B^T z_2 = C z_2$. Therefore $\mu > 0$. Similarly, it is possible to show that $\sigma(\tilde{\mathcal{A}}_1) \subset \langle 0, \infty)$ for poroelasticity problem, see [2].

Consequently, $\sigma(E_{22}) \subset \left(0, \frac{3}{5}\right\rangle$ as $\lambda \in \sigma(E_{22})$ if $\lambda = \frac{3}{5}(\frac{4}{\tau}\mu + 1)^{-1}(\frac{12}{5\tau}\mu + 1)^{-1}$ for $\mu \in \sigma(\tilde{\mathcal{A}}_1)$, and $\mu \geq 0$. Finally,

$$\sigma\left(\mathcal{P}_{R-T}^{-1}\mathcal{A}_R\right) = 1 + (\{0\} \cup \sigma(E_{22})) \subset \left\langle 1, \frac{8}{5}\right\rangle. \qquad \square$$

5 Numerical Tests

The described preconditioners are tested on a poroelasticity model problem defined in the square domain $\Omega = \langle 0, 1\rangle \times \langle 0, 1\rangle$, see Fig. 1. We shall test the following material properties

(a) effective permeability $\log k(x) \in N(0, \tilde{\sigma})$, storativity $c_{pp} = 1$ (fast flow regime),
(b) effective permeability $\log 10^5 k(x) \in N(0, \tilde{\sigma})$, storativity $c_{pp} = 0.000165$ (modest flow regime like in sand).

Here $\xi \in N(\mu, \tilde{\sigma})$ denotes that the quantity ξ has a normal distribution with mean μ and variance $\tilde{\sigma}$. For $\tilde{\sigma} \neq 0$, we can therefore model the flow problem in heterogeneous porous medium, increasing $\tilde{\sigma}$ increases the contrast in coefficients. In practice there is a high range of possible effective permeability values. In geo-applications, $k = \kappa/\nu_f$ (permeability divided by viscosity) usually lies between 10^{-4} (highly fractured rock) and 10^{-16} (intact granite).

Other parameters are not oscillatory and we shall assume that they are constant in the whole domain Ω. The tests are performed for very soft elastic material Lamè constants $\lambda = \mu = 4$ ($E = 10$ and Poisson's ratio $\nu = 0.25$) as well as for stiffer material with $\lambda = 10^3$ and $\lambda = 10^6$ to confirm the behaviour described in Proposition 1. The Biot-Willis coefficient is also held constant $c_{up} = c_{pu} = 1$ in Ω.

For the elastic and flow part we use zero volume force $F_s = 0$ and zero volume source $Q = 0$. The boundary conditions are specified in Fig. 1. The problem uses zero initial conditions

$$u(x_1, x_2, 0) = 0, \;\; p(x_1, x_2, 0) = 0 \text{ for } (x_1, x_2) \in \Omega.$$

The elastic part is discretized by a finite element method on a regular grid Ω_h created by a division of Ω into $1/h^2$ small congruent squares and subsequent

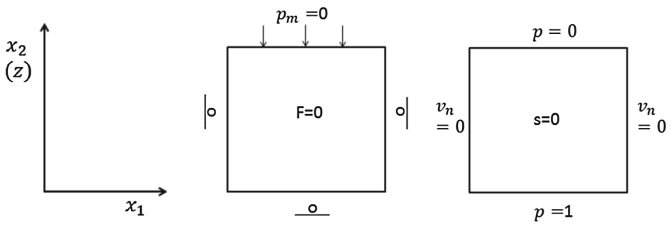

Fig. 1. The model problem in square Ω. Boundary conditions for elasticity are shown in the middle, boundary conditions for flow in the right square.

division of the squares into triangles. Then, the linear Courant elements are used. The flow part is discretized by a mixed finite element method on the same triangular grid with the lowest order Raviart-Thomas finite elements for the velocity and piecewise constant finite elements for the pressure. In this paper, we consider the time discretizations with fixed time steps, $\tau = 0.01$. The value of h is taken $h = 1/50$ in the reported experiments.

First, we test the efficiency of the iterative solution of the Euler system with the preconditioner

$$\mathcal{P}_{ED} = \begin{bmatrix} S_C & \\ & -C \end{bmatrix},$$

where

$$S_C = \begin{bmatrix} A + B_u^T C^{-1} B_u & \tau B_u^T C^{-1} B \\ \tau B^T C^{-1} B_u & \tau M + \tau^2 B^T C^{-1} B \end{bmatrix}$$

or S_C is replaced by its diagonal part $\tilde{S}_C$,

$$\tilde{S}_C = \begin{bmatrix} A + B_u^T C^{-1} B_u & \\ & \tau M + \tau^2 B^T C^{-1} B \end{bmatrix}.$$

Table 1. Numbers of GMRES iterations for solving Euler system with $\mathcal{A}_E$ using zero initial guess and relative residual accuracy $\varepsilon = 10^{-6}$. The numbers of iterations are averaged from the first ten time steps. The Euler system is preconditioned by $\mathcal{P}_{ED}$ with full and reduced Schur complements S_C and $\tilde{S}_C$, respectively. The systems with matrices S_C and $\tilde{S}_C$ are solved exactly. The model poroelasticity problem uses discretization $h = 1/50$, $\tau = 0.01$. The oscillations of effective permeability provide coefficient contrast $8.2 \cdot 10^5$ for $\tilde{\sigma} = 2$ and $6.8 \cdot 10^{11}$ for $\tilde{\sigma} = 4$.

	$\tilde{\sigma} = 0$				$\tilde{\sigma} = 2$				$\tilde{\sigma} = 4$			
λ	4	4	10^3	10^6	4	4	10^3	10^6	4	4	10^3	10^6
c_{pp}	1	10^{-3}	10^{-3}	10^{-3}	1	10^{-3}	10^{-3}	10^{-3}	1	10^{-3}	10^{-3}	10^{-3}
S_C	17	5	5	5	18	5	5	5	19	7	9	9
$\tilde{S}_C$	17	29	14	5	18	37	14	5	19	58	18	9

Table 2. The average numbers of GMRES iterations per one time step for Radau problem, block diagonal preconditioner $\mathcal{P}_{R-D}$, relative residual accuracy $\varepsilon = 10^{-6}$. The block systems are solved exactly. Column **Z** is for zero initial guess, **P** for initial guess taken from the previous time step. The model poroelasticity problem uses discretization $h = 1/50$, $\tau = 0.01$ and oscillations of effective permeability due to $\tilde{\sigma} = 2$.

$\log k \in N(0,\tilde{\sigma})$						$\log 10^5 k \in N(0,\tilde{\sigma})$					
$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$		$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$	
Z	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**
14	2.3	14	2.5	14	1.7	12	2.6	13	2.6	13	2.5

Table 3. The average numbers of GMRES iterations per one time step for Radau problem, block triangular preconditioner $\mathcal{P}_{R-T}$, relative residual accuracy $\varepsilon = 10^{-6}$. The block systems are solved exactly. Column **Z** is for zero initial guess, **P** for initial guess taken from the previous time step. The model problem and its discretization is the same as in Table 2.

$\log k \in N(0,\tilde{\sigma})$						$\log 10^5 k \in N(0,\tilde{\sigma})$					
$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$		$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$	
Z	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**
8	2.0	8	2.0	8	2.3	7	2.3	7.3	2.2	7	2.2

Table 4. The average numbers of GMRES iterations per one time step for the reduced Radau problem, preconditioner $\mathcal{P}_{RR}$, relative residual accuracy $\varepsilon = 10^{-6}$. The block systems are solved exactly. Column **Z** is for zero initial guess, **P** for initial guess taken from the previous time step. The model problem and its discretization is the same as in Table 2.

$\log k \in N(0,\tilde{\sigma})$						$\log 10^5 k \in N(0,\tilde{\sigma})$					
$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$		$\tilde{\sigma}=0$		$\tilde{\sigma}=2$		$\tilde{\sigma}=4$	
Z	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**	**Z**	**P**
5	1.4	5	1.5	5	1.7	4	2.2	4	2.1	4	2.1

We shall also examine the efficiency of preconditioners $\mathcal{P}_{R-D}$, $\mathcal{P}_{R-T}$, $\mathcal{P}_{RR}$ within Radau time steps, when the arising systems with Euler type matrices $\mathcal{A}_1 + c\tau\mathcal{A}_0$ are solved by a direct solution method (MATLAB backslash solver). The iterations are tested in two variants - with zero initial guess and with initial guess provided by the solution of system in the previous time step. As can be seen, the latter provides a significant reduction of the number of iterations. The case with zero initial guess somehow model the situation with adaptive time stepping, when the number of iterations are less reduced when the solution approaches the steady state. The results are summarized in Tables 2, 3 and 4.

6 Conclusions

The porous media flow problems are important in many applications. The numerical solution of these problems is not easy due to possible instabilities in the case of improper discretization and due to high heterogeneity and high contrast (oscillations) in the coefficients representing permeabilities, see also [3] and the references therein.

In this paper, we address both of the above mentioned aspects. A construction of preconditioner for Euler type systems in poroelasticity was shown, which is robust with respect to permeability oscillations and which can provide a fully robust and efficient solver if the subblock system corresponding to flow is solved by an inner robust solver, like solvers considered in [9].

The second focus is on solving still more complex systems arising in application of higher order Radau time integration method. It is shown that efficient preconditioning procedures to these systems can be created if solvers for the Euler systems are available. Preconditioning procedures can be applied to full or reduced Radau systems. The application to reduced system provides better convergence, the application to full systems brings more space for parallelization, which can be used for the whole matrix-vector multiplication as well as for the whole preconditioning in the case of diagonal $\mathcal{P}_{R-D}$ preconditioner. In this respect, we obtain efficiently preconditioned stable and accurate time discretization method.

Acknowledgement. This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (identification number CZ.1.05/1.1.00/02.0070).

References

1. Butcher, J.C.: Numerical Methods for Ordinary Differential Equations, 2nd edn. Wiley, Chichester (2008)
2. Axelsson, O., Blaheta, R., Kohut, R.: Preconditioning methods for high order strongly stable time integration methods with an application for a DAE problem. Numer. Linear Algebra Appl. (accepted)
3. Axelsson, O., Blaheta, R., Byczanski, P.: Stable discretization of poroelasticity problems and efficient preconditioners for arising saddle point type matrices. Comput. Vis. Sci. **15**(4), 191–207 (2012)
4. Axelsson, O., Blaheta, R., Hasal, M.: A comparison of preconditioning methods for saddle point problems with an application for porous media flow problems. Proceedings HPCSE 2015 (2015, submitted)
5. Axelsson, O., Blaheta, R.: Analysis of preconditioning methods based on augmentation. Math. Comput. Simul., special issue MODELLING 2014 **67**, 106–121 (2014). Kindly check and confirm the edit made in Ref.[5]
6. Axelsson, O.: Unified analysis of preconditioning methods for saddle point matrices. Numer. Linear Algebra Appl. **22**(2), 233–253 (2015)

7. Axelsson, O., Blaheta, R., Byczanski, P., Karatson, J., Ahmad, B.: Preconditioners for regularized saddle point problems with an application for heterogeneous Darcy flow problems. J. Comput. Appl. Math. **280**, 141–157 (2015)
8. Benzi, M., Golub, G.H., Liesen, J.: Numerical solution of saddle point problems. Acta Numerica **14**, 1–137 (2005)
9. Kraus, J., Lazarov, R., Lymbery, M., Margenov, S., Zikatanov, L.: Preconditioning of Weighted H(div)-Norm and Applications to Numerical Simulation of Highly Heterogeneous Media. Submitted, http://arxiv.org/pdf/1406.4455.pdf
10. Axelsson, O., Blaheta, Luber, T.: Robust preconditioners for flow problems in rigid and deformable porous media. In preparation