Consistency and Consensus of Intuitionistic Fuzzy Preference Relations in Group Decision Making

Huchang Liao and Zeshui Xu

Abstract Intuitionistic fuzzy preference relations (IFPRs) have turned out to be a useful structure in expressing the experts' uncertain judgments. In this chapter, we consider a group decision making problem where all the members of the group use the IFPRs to express their preferences over the candidate alternatives. Firstly, we describe such a group decision making problem mathematically in details. Then, different types of definitions for the consistency of an IFPR are reviewed, which can be divided into two sorts, i.e., the additive consistency and the multiplicative consistency. Once all the IFPRs are of acceptable consistency, we then introduce a consensus measure to depict the consensus degree of the experts. A consensus reaching procedure is given to help the experts modify their assessments and then obtain an agreement between the experts as to the choice of a proper decision. A numerical example is given to show the validation and computational process of the consensus reaching procedure.

Keywords Intuitionistic fuzzy preference relation · Consistency · Consensus · Consensus reaching procedure · Group decision making

1 Introduction

Group decision making takes place commonly in many domains of our daily life, including such significant ones as the managerial, financial, engineering, and medical fields. It has gained prominence owing to the complexity of modern-life decision problems. For a group decision making problem, a group of experts are getting together to express their individual opinions over the problem and then yield

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a final decision which is mutually agreeable. Very often, such group decision making problem involves multiple feasible alternatives, and the objective of the group decision making problem is to select the best alternative(s) from these mutually exclusive alternatives based on the preferences provided by the experts. In many cases, the experts can not determine their preferences in accurate numerical numbers but fuzzy terms [1]. Fuzzy set (FS) was proposed to represent the relationship between a set and an element by membership degrees rather than by crisp membership of classical binary logic. When all the preferences of the experts are determined by fuzzy numbers which denote the relative intensities between each pair of alternatives, a set of fuzzy preference relations can be established [2]. Let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives under consideration, and $E = \{e_1, e_2, \ldots, e_s\}$ be the set of decision makers, who are invited to evaluate the alternatives. The fuzzy preference relations $B^{(l)} = (b_{ij}^{(l)})_{n \times n}$ $(l = 1, 2, \cdots s)$ can be generated, where $0 \leq b_{ij}^{(l)} \leq 1$ and $b_{ij}^{(l)} + b_{ji}^{(l)} = 1$. $b_{ij}^{(l)}$ indicates the degree that the alternative x_i is preferred to x_j. Concretely speaking, the case $b_{ij}^{(l)} = 0.5$ indicates that there is indifference between the alternatives x_i and x_j; $b_{ij}^{(l)} > 0.5$ indicates that the alternative x_i is preferred to x_j, especially, $b_{ij}^{(l)} = 1$ means that the alternative x_i is absolutely preferred to x_j; $b_{ij}^{(l)} < 0.5$ indicates that the alternative x_j is preferred to x_i, especially, $b_{ij}^{(l)} = 0$ means that the alternative x_j is absolutely preferred to x_i.

   Although fuzzy preference relations can be used to represent the fuzzy and uncertain preferences of the experts in the process of group decision making, they still have some flaws due to the limitation of the fuzzy set itself. Since the membership function of a fuzzy set is only single-valued function, it can't be used to express the support and objection evidences simultaneously in many practical situations [3]. If not possessing a precise or sufficient level of knowledge of the problem domain in cognition of things due to the complexity of the socio-economic environment, people usually have some uncertainty in assigning the preference evaluation values to the objects considered, which makes the judgments of cognitive performance exhibit the characteristics of affirmation, negation and hesitation. In 1983, Atanassov [4] proposed the concept of intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-membership function and a hesitancy function. Such type of fuzzy set extension is essential in representing the imprecision and hesitation of the experts' cognition [5]. Till now it has been applied to many different fields, such as decision making [3, 6], fuzzy logics [7], fuzzy cognitive maps [8], topological space [9], medical diagnosis [10] and pattern recognition [11]. Given the underlying set X of objects, an IFS $\tilde{A}$ is a set of ordered triples, $\tilde{A} = \{(x, \mu_A(x), v_A(x)) | x \in X\}$, where μ_A and v_A are the membership and non-membership functions mapping from X into [0, 1] with the condition $0 \leq \mu_A + v_A \leq 1$. For each $x \in X, \mu_A(x)$ represents the degree of membership of the element x in X to the set $A \subseteq X$, and $v_A(x)$ gives the non-membership degree. The number $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is called the hesitant degree or the intuitionistic index of x to A. The FS do not leave any room for indeterminacy between each

membership degree and its negation, but, in the realistic recognition of experts, such "disagreement" and indeterminacy are very common and useful in describing their opinions in decision making. The introduction of this ignorance statement, which is represented as the hesitancy function in an IFS, is the most characteristic of the IFS [12]. In many cases, when the experts are not able to express their preferences accurately or they are unable or unwilling to discriminate explicitly the degree to which alternative is better than others especially at the beginning of evaluation [13], it is suitable to express their preference information in IFS and thus we can get a set of intuitionistic fuzzy preference relations (IFPRs) [14].

As for group decision making with IFPRs, there are several problems raised, the first one of which is how to judge whether the IFPRs are consistent or not. Consistency of IFPRs requires that the preferences given by the experts yield no contradiction. The lack of consistency for IFPRs may lead to inconsistent or incorrect results for a group decision making problem. Thus it has turned out to be a very important research topic in decision making with IFPRs, and many scholars have paid attention to this topic [6, 12, 14–22]. In this chapter, we would give detail review for the different kinds of consistency of IFPRs. As for those IFPRs without consistency, how to repair them is also a problem which needs to be solved. Generally, this can be done by two different kinds of methodologies, which are the automatic methods and the interactive methods [21, 23].

In the next of this chapter, we would focus on another important issue, i.e., the consensus of group decision making with IFPRs. The consistency checking process of IFPRs can be seen as a collection of individual decision making problems and it is easy to be done by extending the methodology of single expert decision making problem. While the consensus of group decision making is much more complicated because of the complexity introduced by the conflicting views of experts and the varying significance of those views in the decision making process [24]. Sometimes, one expert may determine his/her preferences based on his/her perception, but the others may not agree with it unless they are confident about the perception of the former expert. The consensus is very important in group decision making. Although we can yield a decision for a group decision making problem by aggregating all individual IFPRs into an overall IFPR, the result derived by this type of methodologies may be not much reasonable because some experts may not agree with the final result derived by the weighted averaging methodologies. Consensus is viewed as a pathway to a true group decision because it considers concerns and conflicting ideas without hostility and fear [25]. Till now, there is litter research on the consensus of group decision making with IFPRs. In the following of this chapter, we would pay attention to this issue and give some basic definitions.

The rest of this chapter is organized as follows: Sect. 2 describes the group decision making problem mathematically within the context of intuitionistic fuzzy circumstance. Section 3 reviews the different types of consistency for IFPRs, including the additive consistency and multiplicative consistency for IFPRs. The definition of acceptable consistent IFPR is also given in this Section. In Sect. 4, we present the difficulties in reaching consensus in the process of group decision making with IFPRs. Furthermore, we introduce a hard consensus measure to depict the consensus degree

of the experts. The consensus reaching procedure is given for helping the experts to reach group agreement. A numerical example is given to validate the procedure in Sect. 5. Section 6 ends the chapter with some concluding remarks.

2 Group Decision Making with Intuitionistic Fuzzy Preference Relations

A group decision making problem with intuitionistic fuzzy preference information can be described as follows: Let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives under consideration, and $E = \{e_1, e_2, \ldots, e_s\}$ be a set of experts, who are invited to evaluate the alternatives and then provide their preferences through pairwise comparison. The weight vector of the experts $e_l(l = 1, 2, \ldots, s)$ is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)^T$, where $\lambda_l > 0$, $l = 1, 2, \ldots, s$, and $\sum_{l=1}^{s} \lambda_l = 1$, which can be determined subjectively or objectively according to the experts' experience, judgment quality and related knowledge. In general, they can be assigned equal importance if there is no evidence to show significant differences among the decision makers or specific preference on some decision makers [22]. In the existing literature, many techniques have been developed for determining the decision makers' weights (for more information, refer to Refs. [26–28]). In this chapter, we assume that the weights of experts can always be given.

In many cases, if the problem is very complicated or the experts can not be able to give explicit preferences over alternatives because of vague information and incomplete knowledge about the preference degrees between any pair of alternatives, it is suitable to use the IFSs, which express the preference information from three aspects: "preferred", "not preferred", and "indeterminate", to represent their opinions. Motivated by the idea of IFS, Szmit and Kacprzyk [29] firstly proposed the concept of intuitionistic fuzzy preference relation (IFPR). Later, Xu [14] gave the simple and straightforward notion and expression for it.

Definition 1 [14] An intuitionistic fuzzy preference relation (IFPR) on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, where $\tilde{r}_{ij} = <(x_i, x_j), \mu(x_i, x_j), v(x_i, x_j), \pi(x_i, x_j)>$ for all $i, j = 1, 2, \cdots, n$. For convenience, we let $\tilde{r}_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$ where μ_{ij} denotes the degree to which the object x_i is preferred to the object x_j, v_{ij} indicates the degree to which the object x_i is not preferred to the object x_j, and $\pi_{ij} = 1 - \mu_{ij} - v_{ij}$ is interpreted as an indeterminacy degree or a hesitancy degree, with the conditions:

$$\mu_{ij}, \ v_{ij} \in [0, 1], \mu_{ij} + v_{ij} \leq 1, \mu_{ij} = v_{ji}, \mu_{ii} = v_{ii} = 0.5, \pi_{ij} = 1 - \mu_{ij} - v_{ij}, \text{for all } i, j = 1, 2, \ldots, n$$

$$(1)$$

Xu [14] also proposed the concept of incomplete IFPR in which some of the preference values are unknown. There are some algorithms to estimate the missing values for the incomplete IFPR [30]. For convenience, in this paper we assume that the experts can provide complete IFPRs.

Suppose that the expert e_l provides his/her preference values for the alternative x_i against the alternative x_j as $\tilde{r}_{ij}^{(l)} = (\mu_{ij}^{(l)}, v_{ij}^{(l)}), (i, j = 1, 2, \ldots, n, l = 1, 2, \ldots, s)$ in which $\mu_{ij}^{(l)}$ denotes the degree to which the object x_i is preferred to the object x_j, $v_{ij}^{(l)}$ indicates the degree to which the object x_i is not preferred to the object x_j, and $\pi_{ij}^{(l)} = 1 - \mu_{ij}^{(l)} - v_{ij}^{(l)}$ is interpreted as an indeterminacy degree or a hesitancy degree, subject to $\mu_{ij}^{(l)}, v_{ij}^{(l)} \in [0, 1], \mu_{ij}^{(l)} + v_{ij}^{(l)} \leq 1, \mu_{ij}^{(l)} = v_{ij}^{(l)}, \mu_{ii}^{(l)} = v_{ii}^{(l)} = 0.5,$ for all $i, j = 1, 2, \ldots, n, l = 1, 2, \ldots, s$. The IFPR $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)} \right)_{n \times n}$ for the lth expert can be written as:

$$\tilde{R}^{(l)} = \begin{pmatrix} \tilde{r}_{11}^{(l)} & \tilde{r}_{12}^{(l)} & \cdots & \tilde{r}_{1n}^{(l)} \\ \tilde{r}_{21}^{(l)} & \tilde{r}_{22}^{(l)} & \cdots & \tilde{r}_{2n}^{(l)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{r}_{n1}^{(l)} & \tilde{r}_{n2}^{(l)} & \cdots & \tilde{r}_{nn}^{(l)} \end{pmatrix} \tag{2}$$

For any a group decision making problem with s decision makers, we can obtain s individual IFPRs $\tilde{R}^{(l)}(l = 1, 2, \cdots, s)$ with the form of (2).

3   Consistency of Intuitionistic Fuzzy Preference Relations

Consistency is a very important issue for any kinds of preference relations, and the lack of consistency in preference relations may lead to unreasonable conclusions. There are several different forms of definition for the consistency of IFPRs, which mainly involve two sorts: the additive consistency and the multiplicative consistency.

3.1   Additive Consistency

The concept of additive consistency of an IFPR was motivated by the additive transitivity property proposed by Tanino [1] in 1984. It was proposed to represent the relationship among different preferences. A preference relation $R = (r_{ij})_{n \times n}$ is with additive transitivity if it satisfies $(r_{ij} - 0.5) + (r_{jk} - 0.5) = (r_{ik} - 0.5)$ for all $i, j, k = 1, 2, \cdots, n$. This can be interpreted as the intensity of preference of the alternative x_i over x_k should be equal to the sum of the intensities of preference of x_i over x_j and that of x_j over x_k when $(r_{ij} - 0.5)$ is defined as the intensity of preference

of x_i over x_j. Let $\omega_i(i=1, 2, \cdots, n)$ be the underlying weights of the alternatives and satisfies $\sum_{i=1}^{n} \omega_i = 1, \omega_i \in [0, 1]$. Then, an additive consistent fuzzy preference relation can be given as [15]: $r_{ij} = 0.5(\omega_i + \omega_j - 1)$, for all $i, j = 1, 2, \cdots, n$.

Based on the additive transitivity of a preference relation, different forms of definitions for additive consistency of IFPRs have been proposed.

Xu's additive consistency

For each IFS $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$, the condition $\mu_{ij} \leq 1 - v_{ij}(i, j = 1, 2, \cdots, n)$ always holds. Thus, the IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ can be transformed into an interval-valued complementary judgment matrix $\hat{R} = (\hat{r}_{ij})_{n \times n}$ where $\hat{r}_{ij} = (\hat{r}_{ij}^-, \hat{r}_{ij}^+) = [\mu_{ij}, 1 - v_{ij}](i, j = 1, 2, \ldots, n)$, and $\hat{r}_{ij}^- + \hat{r}_{ij}^+ = \hat{r}_{ij}^+ + \hat{r}_{ji}^- = 1, \hat{r}_{ij}^+ \geq \hat{r}_{ji}^- \geq 0, \hat{r}_{ii}^+ \geq \hat{r}_{ii}^- \geq 0.5, i, j = 1, 2, \ldots, n$. Based on the above transformation, Xu [16] introduced the definition of additive consistent IFPR.

Definition 2 [16] Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IFPR with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$, if there exists a vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, such that

$$\mu_{ij} \leq 0.5(\omega_i - \omega_j + 1) \leq 1 - v_{ij}, \text{for all } i, j = 1, 2, \ldots, n \tag{3}$$

where $\omega_i \in [0, 1](i = 1, 2, \ldots, n)$, and $\sum_{i=1}^{n} \omega_i = 1$. Then, $\tilde{R}$ is called an additive consistent IFPR.

Gong et al.'s additive consistency

Gong et al. [17]'s definition is also based on the transformation between the IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ and its corresponding interval-valued complementary judgment matrix $\hat{R} = (\hat{r}_{ij})_{n \times n}$. As for an interval-valued complementary judgment matrix $\hat{R} = (\hat{r}_{ij})_{n \times n}$, Gong et al. claimed that it is additive consistent if there exists a priority vector $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_n)^T = ([\omega_1^l, \omega_1^u], [\omega_2^l, \omega_2^u], \cdots, [\omega_n^l, \omega_n^u])^T$, such that $\hat{r}_{ij} = 0.5 + 0.2 \log 3$ $\omega_i / -0pt\omega_j = [0.5 + 0.2 \log 3^{\omega_{iu}/-0pt\omega_{ju}}, 0.5 + 0.2 \log 3^{\omega_{iu}/-0pt\omega_{jl}}]$ $(i, j = 1, 2, \ldots, n)$, and the priorities $\hat{\omega}_i$ can be interpreted as the membership degree range of the importance of the alternative x_i. Hence, with the additive consistency condition of interval-valued complementary judgment matrix $\hat{R} = (\hat{r}_{ij})_{n \times n}$, a new form of definition for additive consistent IFPR can be given as follows.

Definition 3 [17] Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IFPR with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$, if there exists a vector $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_n)^T = ([\omega_1^l, \omega_1^u], [\omega_2^l, \omega_2^u], \cdots, [\omega_n^l, \omega_n^u])^T$, such that

$$\mu_{ij} = 0.5 + 0.2 \log 3^{\omega_{il}/-0pt\omega_{ju}}, v_{ij} = 0.5 + 0.2 \log 3^{\omega_{jl}/-0pt\omega_{iu}}, \text{for all } i, j = 1, 2, \ldots, n \tag{4}$$

Then, $\tilde{R}$ is called an additive consistent IFPR.

Wang's additive consistency

According to additive transitivity, Wang [18] introduced a definition of additive consistent IFPR by directly employing the membership and nonmembership degrees of IFSs.

Definition 4 [18] An IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$ is called additive consistent if it satisfies the following additive transitivity:

$$\mu_{ik} + \mu_{jk} + \mu_{ki} = \mu_{kj} + \mu_{ji} + \mu_{ik}, \text{for all } i, j, k = 1, 2, \cdots, n \tag{5}$$

Let $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T = ((\omega_1^\mu, \omega_1^v), (\omega_2^\mu, \omega_2^v), \cdots, (\omega_n^\mu, \omega_n^v))^T$ be an underlying intuitionistic fuzzy priority vector of an IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, where $\tilde{\omega}_i = (\tilde{\omega}_i^\mu, \tilde{\omega}_i^v)$ $(i = 1, 2, \cdots, n)$ is an intuitionistic fuzzy value, which satisfies $\tilde{\omega}_i^\mu, \tilde{\omega}_i^v \in [0, 1]$ and $\tilde{\omega}_i^\mu + \tilde{\omega}_i^v \leq 1$. $\tilde{\omega}_i^\mu$ and $\tilde{\omega}_i^v$ indicate the membership and non-membership degrees of the alternative x_i as per a fuzzy concept of "importance", respectively. The normalization of $\tilde{\omega}$ can be done via the following definition:

Definition 5 [18] An intuitionistic fuzzy weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T$ with $\tilde{\omega}_i = (\omega_i^\mu, \omega_i^v)$, $\omega_i^\mu, \omega_i^v \in [0, 1]$ and $\omega_i^\mu + \omega_i^v \leq 1$ for $i = 1, 2, \cdots, n$ is said to be normalized if it satisfies the following conditions:

$$\sum_{j=1, j \neq i}^{n} \omega_j^\mu \leq \omega_i^v, \omega_i^\mu + n - 2 \geq \sum_{j=1, j \neq i}^{n} \omega_j^v, \text{for all } i = 1, 2, \cdots, n \tag{6}$$

With the underlying normalized intuitionistic fuzzy priority vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T$, an additive consistent IFPR $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n \times n}$ can be established as:

$$\tilde{r}_{ij}^* = (\mu_{ij}, v_{ij}) = \begin{cases} (0.5, 0.5) & if \quad i = j \\ (0.5\omega_i^\mu + 0.5\omega_j^v, 0.5\omega_i^v + 0.5\omega_j^\mu) & if \quad i \neq j \end{cases} \tag{7}$$

where $\omega_i^\mu, \omega_i^v \in [0, 1], \omega_i^\mu + \omega_i^v \leq 1$, $\sum_{j=1, j \neq i}^{n} \omega_j^\mu \leq \omega_i^v$, and $\omega_i^\mu + n - 2 \geq \sum_{j=1, j \neq i}^{n} \omega_j^v$, for all $i = 1, 2, \cdots, n$.

3.2 Multiplicative Consistency

The additive consistency is, to some extent, inappropriate in modeling consistency due to that its consistency condition is sometimes in conflict with the $[0, 1]$ scale used for providing the preference values [31]. However, the multiplicative

consistency does not have this limitation [32]. The main idea of multiplicative consistency is based on the multiplicative transitivity of a preference relation $R = (r_{ij})_{n \times n}$, which is characterized as $r_{ij}/r_{ji} = (r_{ik}/r_{ki}) \cdot (r_{kj}/r_{jk})$ for all $i, j, k = 1, 2, \cdots, n$. This relationship can be interpreted as the ratio of the preference intensity for the alternative x_i to that of x_j should be equal to the multiplication of the ratios of preferences when using an intermediate alternative x_k, in the case where r_{ij}/r_{ji} indicates a ratio of the preference intensity for the alternative x_i to that of x_j. In other words, x_i is r_{ij}/r_{ji} times as good as x_j. Inspired by the multiplicative transitivity and the relationship between the IFPR and its corresponding preference relations, several distinct definitions of multiplicative consistency were proposed for IFPRs.

Xu's multiplicative consistency of IFPR

Based on the transformation relationship between the IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ and its corresponding interval complementary judgment matrix $\hat{R} = (\hat{r}_{ij})_{n \times n}$, Xu [16] proposed the definition of multiplicative consistent IFPR.

Definition 6 [24] Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$ be an IFPR, if there exists a vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, such that

$$\mu_{ij} \leq \frac{\omega_i}{\omega_i + \omega_j} \leq 1 - v_{ij}, \text{ for all } i = 1, 2, \ldots, n-1; \ j = i+1, \ldots, n \qquad (8)$$

where $\omega_i \geq 0, (i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} \omega_i = 1$. Then, we call $\tilde{R}$ a multiplicative consistent IFPR.

Gong et al.'s multiplicative consistency of IFPR

Based on the transformation between an IFPR and its corresponding interval-valued fuzzy preference relation, Gong et al. [19] introduced a definition of multiplicative consistent IFPR.

Definition 7 [19] Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IFPR with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$, if there exists a vector $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_n)^T = ([\omega_1^l, \omega_1^u], [\omega_2^l, \omega_2^u], \cdots, [\omega_n^l, \omega_n^u])^T$, such that

$$\mu_{ij} = \frac{\omega_{il}}{\omega_{il} + \omega_{ju}}, v_{ij} = \frac{\omega_{jl}}{\omega_{jl} + \omega_{iu}}, \text{ for all } i, j = 1, 2, \ldots, n \qquad (9)$$

Then, $\tilde{R}$ is called a multiplicative consistent IFPR.

Xu et al.'s multiplicative consistency of IFPR

Xu et al. [30] proposed another definition of multiplicative consistent IFPR, which was based on the membership and non-membership degrees of IFSs, shown as follows:

Definition 8 [30] An IFPR $\tilde{R} = (r_{ij})_{n \times n}$ with $r_{ij} = (\mu_{ij}, v_{ij})(i, j = 1, 2, \ldots, n)$ is multiplicative consistent if

$$\mu_{ij} = \begin{cases} 0, & (\mu_{ik}, \mu_{kj}) \in \{(0, 1), (1, 0)\} \\ \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj} + (1 - \mu_{ik})(1 - \mu_{kj})}, & otherwise \end{cases}, \text{for all } i \leq k \leq j, \quad (10)$$

$$v_{ij} = \begin{cases} 0, & (v_{ik}, v_{kj}) \in \{(0, 1), (1, 0)\} \\ \frac{v_{ik}v_{kj}}{v_{ik}v_{kj} + (1 - v_{ik})(1 - v_{kj})}, & otherwise \end{cases}, \text{for all } i \leq k \leq j. \quad (11)$$

Liao and Xu's multiplicative consistency of IFPR

Liao and Xu [20] pointed out that the definition of Xu et al. [30] was not reasonable in some cases because with the above consistency conditions, the relationship $\mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = \mu_{ik} \cdot \mu_{kj} \cdot \mu_{ji}$ (for all $i, j, k = 1, 2, \cdots, n$) can not be derived any more. Then, they introduced a general definition of multiplicative consistent IFPR, shown as follows:

Definition 9 [20] An IFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$ is called multiplicative consistent if the following multiplicative transitivity is satisfied:

$$\mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = v_{ij} \cdot v_{jk} \cdot v_{ki}, \text{for all } i, j, k = 1, 2, \cdots, n. \quad (12)$$

Liao and Xu [20] further clarified that the conditions in Definition 8 satisfy (12), which implies the consistency measured by the conditions given in Definition 8 is a special case of multiplicative consistency defined as Definition 9 for an IFPR. Hence, in general, Definition 8 is not sufficient and suitable to measure the multiplicative consistency of an IFPR.

With the underlying normalized intuitionistic fuzzy priority weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n)^T$, a multiplicative consistent IFPR $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n \times n}$ can be established as [20]:

$$\tilde{r}_{ij}^* = (\mu_{ij}, v_{ij}) = \begin{cases} (0.5, 0.5) & if \ i = j \\ (\frac{2\omega_i^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2}, \frac{2\omega_j^\mu}{\omega_i^\mu - \omega_i^v + \omega_j^\mu - \omega_j^v + 2}) & if \ i \neq j \end{cases} \quad (13)$$

where $\omega_i^\mu, \omega_i^v \in [0, 1], \omega_i^\mu + \omega_i^v \leq 1, \sum_{j=1, j \neq i}^{n} \omega_j^\mu \leq \omega_i^v$, and $\omega_i^\mu + n - 2 \geq \sum_{j=1, j \neq i}^{n} \omega_j^v$, for all $i = 1, 2, \cdots, n$.

3.3 Acceptable Consistency of IFPR

Due to the complexity of the problem and the limited knowledge of the experts, the experts often determine some inconsistent IFPR. Perfect consistent IFPR is somehow too strict for the experts to construct especially when the number of objects is very large. Since in practical cases, it is impossible to get the consistent IFPRs, Liao and Xu [22] introduced the concept of acceptable consistent IFPR.

Definition 10 [22] Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IFPR with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})(i, j = 1, 2, \cdots, n)$. We call R an acceptable consistent IFPR, if

$$d(\tilde{R}, \tilde{R}^*) \leq \xi, \tag{14}$$

where $d(\tilde{R}, \tilde{R}^*)$ is the distance measure between the given IFPR $\tilde{R}$ and its corresponding underlying consistent IFPR $\tilde{R}^*$, which can be calculated by

$$d(\tilde{R}, \tilde{R}^*) = \frac{1}{(n-1)(n-2)} \sum_{1 \leq i < j < n}^{n} \left(\left(\left| \mu_{ij} - \mu_{ij}^* \right| + \left| v_{ij} - v_{ij}^* \right| + \left| \pi_{ij} - \pi_{ij}^* \right| \right), \tag{15}$$

and ξ is the consistency threshold. The corresponding underlying consistent IFPR $\tilde{R}^*$ can be yielded by (7) or (13).

As for those IFPRs of inconsistency, there are many procedures to improve the inconsistent IFPRs into acceptable consistent IFPRs (For more details, please refer to [21, 23]).

If all the IFPRs are of acceptable consistency, we can aggregate these IFPRs into an overall IFPR and then derive the ranking of the alternatives. Liao and Xu [22] proposed a simple intuitionistic fuzzy weighted geometric (SIFWG) operator to fuse the IFPRs. For s IFPRs $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)} \right)_{n \times n} (l = 1, 2, \cdots, s)$, their fused IFPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ with $\bar{r}_{ij} = (\bar{\mu}_{ij}, \bar{v}_{ij}, \bar{\pi}_{ij})$ by the SIFWG operator is also an IFPR, where

$$\bar{\mu}_{ij} = \prod_{l=1}^{s} \left(\mu_{ij}^{(l)} \right)^{\lambda_l}, \bar{v}_{ij} = \prod_{l=1}^{s} \left(v_{ij}^{(l)} \right)^{\lambda_l}, \bar{\pi}_{ij} = 1 - \bar{\mu}_{ij} - \bar{v}_{ij}, i, j = 1, 2, \ldots, n. \tag{16}$$

Liao and Xu [22] further proved that if all the individual IFPRs are of acceptable multiplicative consistency, then their fused IFPR by the SIFWG operator is also of acceptable consistency. This is a good property in group decision making with IFPRs because with this property there is no need to check the consistency of the fused IFPR and we can use it to derive the decision making result directly.

4 Consensus of Group Decision Making with IFPRs

4.1 Difficulties in Reaching Consensus

With all the above mentioned different types of consistency and the corresponding inconsistency repairing methods, we can get a set of consistent or acceptable consistent IFPRs. This is the precondition of deriving a reasonable solution for a group decision making problem. The group decision making problem is very complicated owing to the complexity introduced by the conflicting opinions of the experts. As to a group decision making problem with IFPRs, how to find a final solution which is accepted by all the experts is a great challenge. The consensus is very important in any group decision making problems. It can be defined as "*a decision that has been reached when most members of the team agree on a clear option and the few who oppose it think they have had a reasonable opportunity to influence that choice. All team members agree to support the decision.*" [25] Consensus is a pathway to a true group decision and it can guarantee that the final result been supported by all the group members despite their different opinions.

However, to find such a consensus result is very difficult because of some inherent differences in value systems, flexibility of members, etc. Generally, if all experts are wise and rational, they should agree with each other. But, in reality, disagreement among the experts is inevitable. In fact, the disagreement is just the valuation of group decision making.

In the process of group decision making, the target is to find a solution which is accepted by all the experts. Initially, the experts should be with no consensus, and thus they need to communicate with each other and modify their judgments. That is to say, the consensus reaching process should be an iterative procedure and it should be converge finally. In addition, a group decision making problem with too many times of iteration does not make sense because it wastes too many resources and is not worthy to be investigated by the experts.

4.2 Consensus Measures for Group Decision Making with IFPRs

The consensus reaching process refers to how to obtain the maximum degree of consensus or agreement between the set of experts [33]. To do so, we should first know how to measure the consensus degree among the experts. Although there is litter research focused on the consensus of IFPRs [34], we still can found many approaches to model consensus process in group decision making with other preference relations, such as fuzzy preference relation [35], incomplete fuzzy preference relation [37], and linguistic preference relation [36]. These consensus measures involve two parts: hard consensus measure and soft consensus measure. The hard consensus measure uses a number in the interval [0, 1] to represent the

consensus degree of the experts, while the soft consensus measure employs a linguistic label such as "most" to describe the truth of a statement such as "most experts agree on almost all the alternatives." As to group decision making with IFPRs, Szmidt and Kacprzyk [34] used an interval-valued measure of distance to represent the consensus of experts. They took the membership degrees and the hesitance degrees as two separate matrices and then used these two matrices to derive the upper bound and lower bound of the interval-valued consensus measure. Their work can be seen as the first attempt to measure the consensus of group decision with IFPRs. However, they did not include any procedures to reaching consensus. In the following, we would define a hard consensus measure of experts whose opinions are represented by IFPRs.

For a set of IFPRs $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n \times n} (l = 1, 2, \cdots, s)$ given by s independent experts $e_l(l = 1, 2, \ldots, s)$, since it is known that if all individual IFPRs are of acceptable consistency, their fused IFPRs $\bar{R} = (\bar{r}_{ij})_{n \times n}$ with the SIFWG operator is also of acceptable consistency, then, motivated by the distance measure of two IFPRs given as (15), we can introduce a hard consensus measure of the experts with IFPRs.

Definition 11 For a set of IFPRs $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n \times n} (l = 1, 2, \cdots, s)$ with $r_{ij}^{(l)} = (\mu_{ij}^{(l)}, v_{ij}^{(l)}, \pi_{ij}^{(l)})(l = 1, 2, \cdots, s)$ given by s independent experts $e_l(l = 1, 2, \ldots, s)$, whose weight vector is $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)^T$ with $0 \leq \lambda_l \leq 1$, $\sum_{l=1}^{s} \lambda_l = 1$, then the consensus of the lth expert is defined as

$$C_l = 1 - \frac{1}{(n-1)(n-2)} \sum_{1 \leq i < j < n}^{n} \left(\left(\left| \mu_{ij}^{(l)} - \bar{\mu}_{ij} \right| + \left| v_{ij}^{(l)} - \bar{v}_{ij} \right| + \left| \pi_{ij}^{(l)} - \bar{\pi}_{ij} \right| \right), \quad (17)$$

where $\bar{R} = (\bar{r}_{ij})_{n \times n}$ with $\bar{r}_{ij} = (\bar{\mu}_{ij}, \bar{v}_{ij}, \bar{\pi}_{ij}), \bar{\mu}_{ij} = \prod_{l=1}^{s} \left(\mu_{ij}^{(l)}\right)^{\lambda_l}, \bar{v}_{ij} = \prod_{l=1}^{s} \left(v_{ij}^{(l)}\right)^{\lambda_l}, \bar{\pi}_{ij} = 1 - \bar{\mu}_{ij} - \bar{v}_{ij}$ is the overall IFPR derived by the SIFWG operator.

4.3 Consensus Reaching Procedure with IFPRs

With the above consensus measure, the consensus reaching procedure for helping the experts, whose preferences are given as IFPRs, to reach consensus can be given as follows:

- Establish s IFPRs $\tilde{R}^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{n \times n} (l = 1, 2, \cdots, s)$ for s independent experts $e_l(l = 1, 2, \ldots, s)$;
- Check the consistency of each IFPR: for those IFPRs of unacceptable consistency, repair them until acceptable;
- Compute the consensus degree of each experts;

- Determine the minimum consensus bound of the experts, τ. For $C_l < \tau$, ask the expert e_l to modify the IFPR. A suggestion for the expert e_l to modify the IFPR is to change the preferences by the following formulas:

$$\mu_{ij}^{(l)'} = \left(\mu_{ij}^{(l)}\right)^{\zeta} \cdot \left(\bar{\mu}_{ij}\right)^{1-\zeta}, \tag{18}$$

$$v_{ij}^{(l)'} = \left(v_{ij}^{(l)}\right)^{\zeta} \cdot \left(\bar{v}_{ij}\right)^{1-\zeta}, \tag{19}$$

$$\pi_{ij}^{(l)'} = 1 - \mu_{ij}^{(l)'} - v_{ij}^{(l)'}. \tag{20}$$

- Articulate the decision via aggregating all the IFPRs whose consensus degrees are greater than the threshold τ into an overall IFPR.

5 Numerical Example

The following example concerning the selection of the global suppliers (adapted from [22]) can be used to illustrate the consensus reaching procedure for group decision making with IFPRs.

Example The current globalized market trend identifies the necessity of the establishment of long term business relationship with competitive global suppliers spread around the world. This can lower the total cost of supply chain; lower the inventory of enterprises; enhance information sharing of enterprises; improve the interaction of enterprises and obtain more competitive advantages for enterprises. Thus, how to select different unfamiliar international suppliers according to the broad evaluation is very critical and has a direct impact on the performance of an organization. Suppose a company invites three experts e_1, e_2 and e_3 from different field to evaluate four candidate suppliers x_1, x_2, x_3 and x_4. The weights of the experts are 0.3, 0.4, 0.3, respectively, which is established by the decision making committee according to the experts' expertise and reputation. Global supplier development is a complex problem which includes much qualitative information. In such a case, it is straightforward for the experts to compare the different suppliers in pairs and then construct some preference relations to express their preferences. Since the experts do not have precise information of the global suppliers, it is reasonable for them to use the IFSs to describe their assessments, and then three IFPRs can be established:

$$\tilde{R}^{(1)} = \begin{pmatrix} (0.5, 0.5) & (0.5, 0.2) & (0.7, 0.1) & (0.5, 0.3) \\ (0.2, 0.5) & (0.5, 0.5) & (0.6, 0.2) & (0.3, 0.6) \\ (0.1, 0.7) & (0.2, 0.6) & (0.5, 0.5) & (0.3, 0.6) \\ (0.3, 0.5) & (0.6, 0.3) & (0.6, 0.3) & (0.5, 0.5) \end{pmatrix},$$

$$\tilde{R}^{(2)} = \begin{pmatrix} (0.5, 0.5) & (0.6, 0.1) & (0.8, 0.2) & (0.6, 0.3) \\ (0.1, 0.6) & (0.5, 0.5) & (0.5, 0.1) & (0.3, 0.7) \\ (0.2, 0.8) & (0.1, 0.5) & (0.5, 0.5) & (0.4, 0.6) \\ (0.3, 0.6) & (0.7, 0.3) & (0.6, 0.4) & (0.5, 0.5) \end{pmatrix},$$

$$\tilde{R}^{(3)} = \begin{pmatrix} (0.5, 0.5) & (0.6, 0.2) & (0.8, 0.1) & (0.7, 0.2) \\ (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.1) & (0.2, 0.7) \\ (0.1, 0.8) & (0.1, 0.6) & (0.5, 0.5) & (0.2, 0.3) \\ (0.2, 0.7) & (0.7, 0.2) & (0.3, 0.2) & (0.5, 0.5) \end{pmatrix}.$$

Using the fractional programming models constructed by Liao and Xu [20], the underlying intuitionistic fuzzy weights for these three individual IFPRs are

$$\tilde{\omega}^{(1)} = ((0.3951, 0.4221), (0.1354, 0.8397), (0.0451, 0.8894), (0.2370, 0.6298))^T,$$
$$\tilde{\omega}^{(2)} = ((0.4137, 0.5517), (0.1552, 0.7069), (0.0862, 0.9138), (0.2069, 0.6897))^T.$$
$$\tilde{\omega}^{(3)} = ((0.4686, 0.4143), (0.1406, 0.7891), (0.0586, 0.9414), (0.1538, 0.6700))^T.$$

According to (13), the corresponding multiplicative consistent IFPRs can be generated:

$$\tilde{R}^{(1)*} = \begin{pmatrix} (0.5000, 0.5000) & (0.6228, 0.2134) & (0.7001, 0.0799) & (0.5001, 0.3000) \\ (0.2134, 0.6228) & (0.5000, 0.5000) & (0.5999, 0.1998) & (0.2999, 0.5250) \\ (0.0799, 0.7001) & (0.1998, 0.5999) & (0.5000, 0.5000) & (0.1182, 0.6213) \\ (0.3000, 0.5001) & (0.5250, 0.2999) & (0.6213, 0.1182) & (0.5000, 0.5000) \end{pmatrix},$$

$$\tilde{R}^{(2)*} = \begin{pmatrix} (0.5000, 0.5000) & (0.6315, 0.2369) & (0.7999, 0.1667) & (0.5999, 0.3000) \\ (0.2369, 0.6315) & (0.5000, 0.5000) & (0.5001, 0.2778) & (0.3215, 0.4286) \\ (0.1667, 0.7999) & (0.2778, 0.5001) & (0.5000, 0.5000) & (0.2500, 0.6001) \\ (0.3000, 0.5999) & (0.4286, 0.3215) & (0.6001, 0.2500) & (0.5000, 0.5000) \end{pmatrix},$$

$$\tilde{R}^{(3)*} = \begin{pmatrix} (0.5000, 0.5000) & (0.6667, 0.2000) & (0.8000, 0.1000) & (0.6093, 0.2000) \\ (0.2000, 0.6667) & (0.5000, 0.5000) & (0.6000, 0.2501) & (0.3366, 0.3683) \\ (0.1000, 0.8000) & (0.2501, 0.6000) & (0.5000, 0.5000) & (0.1950, 0.5118) \\ (0.2000, 0.6093) & (0.3683, 0.3366) & (0.5118, 0.1950) & (0.5000, 0.5000) \end{pmatrix}.$$

Thus, via (15), we can obtain $d(\tilde{R}^{(1)}, \tilde{R}^{(1)^*}) = 0.1382$, $d(\tilde{R}^{(2)}, \tilde{R}^{(2)^*}) = 0.2569$, and $d(\tilde{R}^{(3)}, \tilde{R}^{(3)^*}) = 0.2837$. Suppose $\xi = 0.3$, then all these three individual IFPRs are of acceptable multiplicative consistency. Thus, with the SIFWG operator, the overall IFPR of the group can be aggregated as

$$\bar{R} = \begin{pmatrix} (0.5000, 0.5000) & (0.5681, 0.1516) & (0.7686, 0.1320) & (0.5950, 0.2656) \\ (0.1516, 0.5681) & (0.5000, 0.5000) & (0.5578, 0.1231) & (0.2656, 0.6684) \\ (0.1320, 0.7686) & (0.1231, 0.5578) & (0.5000, 0.5000) & (0.2980, 0.4874) \\ (0.2656, 0.5950) & (0.6684, 0.2656) & (0.4874, 0.2980) & (0.5000, 0.5000) \end{pmatrix}$$

By (16), we can calculate the consensus of each expert, which are $C_1 = 0.8114$, $C_2 = 0.8160$, and $C_3 = 0.8292$, respectively. That is to say, the experts have at least 80 % consensus. If the minimum consensus bound τ of the experts is 0.8, then, we can say that all the experts in this decision making problem reach the group consensus.

On the other hand, if the consensus threshold given by the decision maker is much higher, for instance, $\tau = 0.85$, then, the experts need to modify there assessments to reach a much higher consensus.

Let $\zeta = 0.6$, with (18)–(20), the suggested IFPRs are

$$\tilde{R}^{(1)'} = \begin{pmatrix} (0.5000, 0.5000) & (0.5262, 0.1790) & (0.7267, 0.1117) & (0.5360, 0.2857) \\ (0.1790, 0.5262) & (0.5000, 0.5000) & (0.5827, 0.1647) & (0.2857, 0.6265) \\ (0.117, 0.7267) & (0.1647, 0.5827) & (0.5000, 0.5000) & (0.2992, 0.5521) \\ (0.2857, 0.5360) & (0.6265, 0.2857) & (0.5521, 0.2992) & (0.5000, 0.5000) \end{pmatrix}$$

$$\tilde{R}^{(2)'} = \begin{pmatrix} (0.5000, 0.5000) & (0.5870, 0.1181) & (0.7873, 0.1694) & (0.5980, 0.2857) \\ (0.1181, 0.5870) & (0.5000, 0.5000) & (0.5224, 0.1087) & (0.2857, 0.6872) \\ (0.1694, 0.7873) & (0.1087, 0.5224) & (0.5000, 0.5000) & (0.3556, 0.5521) \\ (0.2857, 0.5980) & (0.6872, 0.2857) & (0.5521, 0.3556) & (0.5000, 0.5000) \end{pmatrix}$$

$$\tilde{R}^{(3)'} = \begin{pmatrix} (0.5000, 0.5000) & (0.5870, 0.1790) & (0.7873, 0.1117) & (0.6559, 0.2240) \\ (0.1790, 0.5870) & (0.5000, 0.5000) & (0.5827, 0.1087) & (0.2240, 0.6872) \\ (0.1117, 0.7873) & (0.1087, 0.5827) & (0.5000, 0.5000) & (0.2346, 0.3643) \\ (0.2240, 0.6559) & (0.6872, 0.2240) & (0.3643, 0.2346) & (0.5000, 0.5000) \end{pmatrix}$$

Since $\tilde{R}^{(l)'}(l = 1, 2, 3)$ is simple geometric aggregated by $\tilde{R}^{(l)^*}(l = 1, 2, 3)$ and $\bar{R}$ respectively, according to the theorem of Liao and Xu [17], the new IFPRs $\tilde{R}^{(l)'}$ $(l = 1, 2, 3)$ should be still of acceptable consistency. Then we aggregate them together with the SIFWG operator and yield an overall IFPR:

$$\bar{R}' = \begin{pmatrix} (0.5000, 0.5000) & (0.5681, 0.1516) & (0.7686, 0.1320) & (0.5950, 0.2656) \\ (0.1516, 0.5681) & (0.5000, 0.5000) & (0.5578, 0.1231) & (0.2656, 0.6684) \\ (0.1320, 0.7686) & (0.1231, 0.5578) & (0.5000, 0.5000) & (0.2980, 0.4874) \\ (0.2656, 0.5950) & (0.6684, 0.2656) & (0.4874, 0.2980) & (0.5000, 0.5000) \end{pmatrix},$$

which is the same as $\bar{R}$ (It can be proven that $\bar{R}'$ is always the same as $\bar{R}$ with our aggregation methodology). Then, by (16), the consensus of each expert are calculated as $C_1 = 0.8875$, $C_2 = 0.8936$, and $C_3 = 0.8732$, respectively. Since each consensus degree of the experts is greater than the threshold $\tau = 0.85$, the group reaches the consensus.

6 Conclusion

IFPR is a powerful tool to express the experts' opinions in the process of group decision making. In this chapter, we have discussed the consistency of the IFPRs and the consensus of the experts in group decision making with IFPRs. Firstly, we have described the group decision making problem with IFPRs in details. Then, we have reviewed all the different kinds of definitions for the consistency of IFPRs, which involves two sorts, i.e., the additive consistency and the multiplicative consistency. Based on the consistency, the definition of acceptable consistency can be given. For those IFPRs which are of unacceptable consistency, the consistency repairing process should be employed to modify them. Once all the IFPRs are of acceptable consistency, we have introduced a hard consensus measure to depict the consensus degree of the experts within the group decision making. Furthermore, we have given a simple procedure for aiding the experts to reach group consensus.

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