Bernard R. Hodgson · Alain Kuzniak
Jean-Baptiste Lagrange *Editors*

The Didactics of Mathematics: Approaches and Issues

A Homage to Michèle Artigue

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Preface

This book is based on a conference held in Paris on May 31–June 2, 2012, on the occasion of the retirement of Michèle Artigue from her position at Université Paris Diderot (Paris 7). Organised under the theme *The Didactics of Mathematics: Approaches and Issues. A Homage to Michèle Artigue*, the so-called Artigue Colloquium gathered more than three hundred colleagues from all over the world.

When Michèle Artigue announced that she was officially retiring from her position at Université Paris Diderot—but happily continuing her activities as an Emerita Professor, allowing her to be further involved in mathematics education research or other forms of involvement in support of the field— it became clear to us, as well as to numerous members of the international mathematics education community, that it was essential to mark in a very special way that moment in her truly remarkable career. The idea of hosting in Paris a colloquium in homage to her exceptional contribution to the development of didactics of mathematics over the last decades was very quickly and easily agreed upon.

This book, the outcome of the Artigue Colloquium, offers more than a mere reflection of its scientific content, as well-known researchers from the field have been invited to summarise the main topics where the importance of Michèle Artigue's contribution is widely recognised. Her multiple interest areas give to this volume its unique flavour of diversity.

We wish to use this opportunity in order to express our most sincere friendship to our colleague, Michèle Artigue, with whom each of us has intensively interacted over the past decades, in often quite different contexts.

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Alain Kuzniak
Jean-Baptiste Lagrange

Acknowledgements

We wish to express our deepest gratitude to the authorities of *Université Paris Diderot*, in particular, for providing us with the premises and facilities for hosting the colloquium in homage to Michèle Artigue held in Paris on May 31–June 2, 2012. We also thank the members of the *Laboratoire de didactique André Revuz*, for their enthusiastic reaction to the project and for their constant support.

Special thanks are also due to the members of the Scientific Committee[1] and Organising Committee[2] in charge of the Colloquium, for their collaboration in building a programme reflecting the vast scope of Michèle Artigue's contribution to the field, as well as for the actual handling of the various colloquium's sessions and events.

We also wish to thank more than three hundred colleagues from all over the world who were present at the colloquium, and contributed by their presence to the homage that we aimed at paying to our most esteemed colleague, thus making this scientific gathering a highly memorable and rewarding experience.

Special thanks are due to the whole editorial staff at Springer who supported us in the preparation of this book. But above all, we want to express our deepest gratitude to our editor Natalie Rieborn who, from the very outset of the project through its fruition, has brought us her enthusiastic support.

Starting from the many variants of the prose in English—today's *lingua franca*—written by some thirty different authors, the majority of whom (including the three editors of this volume) are non-native English speakers, and bringing this to a level of high-quality language, is a difficult but essential task. It was accomplished

[1]Pierre Arnoux (France), Ferdinando Arzarello (Italy), Bill Barton (New-Zealand), Carmen Batanero (Spain), Isabelle Bloch (France), Tânia Campos (Brasil), Corine Castela (France), Tommy Dreyfus (Israel), Bernard Hodgson (Canada) co-president, Alain Kuzniak (France), Jean-Baptiste Lagrange (France) co-president, David Rabouin (France), Kenneth Ruthven (UK), Kalifa Traoré (Burkina-Faso), Fabrice Vandebrouck (France), Laurent Vivier (France).

[2]Maha Abboud-Blanchard, Dominique Baroux, Christine Chambris, René Cori, Cécile de Hosson, Brigitte Grugeon, Christophe Hache, Alain Kuzniak president, Mariam Haspekian, Nathalie Sayac, Fabrice Vandebrouck, Laurent Vivier.

superbly by Kim Woodland, to whom we wish to express our thanks for her skilful and respectful suggestions that greatly contributed in making the authors' ideas be expressed in a nicer, and often clearer, language.

Bernard R. Hodgson
Alain Kuzniak
Jean-Baptiste Lagrange

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Chapter 1
Introduction: Perspectives on Didactics of Mathematics through Michèle Artigue's Contributions

Bernard R. Hodgson, Alain Kuzniak and Jean-Baptiste Lagrange

In 2012, a memorable conference on the didactics of mathematics was organised and held in Paris, a result of the determination of an entire community to honour one of its most distinguished members, Michèle Artigue. Given the exceptional character of their highly esteemed colleague, the organisers of the meeting expected the attendance of numerous researchers in the field, but the success largely exceeded their expectations: more than three hundred people from many parts of the world kindly participated so as to express both their admiration and their affection for Michèle. An especially emotional event in the symposium was undoubtedly its conclusion when, moving smoothly and graciously between French, Spanish and English, Michèle Artigue evoked her family origins in her native Pyrenees and her commitment to the development of education in all its forms. The importance of this commitment was recognised in 2013 by the International Commission on Mathematical Instruction (ICMI)—of which Michèle was herself Vice-President (1999–2006) and President (2007–2009)—when she was awarded the Felix Klein Medal, "*in recognition of her more than thirty years of sustained, consistent, and outstanding lifetime achievements in mathematics education research and development*". More recently, she was awarded the 2015 Luis Santaló Medal by the Latino-American educational community (*Comité Interamericano de Educación Matemática*—CIAEM), showing again her influence beyond borders.

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The book *The Didactics of Mathematics: Approaches and Issues. A Homage to Michèle Artigue* is the outcome of this conference, and it utilises the same general structure. However, it offers more than merely a reflection of the event, as various well-known researchers from the field have been invited to summarise the main topics where the importance of Artigue's contribution is widely recognised. Her multiple interest areas, as a researcher involved in a wider community, give to this volume its unique flavour of diversity. In the preparation of each chapter, authors were given the opportunity to use one important paper by Artigue to initiate their reflections about a given topic. It was not always easy to identify a clear or pre-established order among this abundant and diverse material, as the different themes discussed during the conference drew on the extremely rich scientific journey of our colleague. As a conclusion to the book, Michèle Artigue—who needless to say was given *carte blanche*—offers a few personal keys to understanding the main elements that guided her throughout her scientific choices.

Since the early 1970s and up to the present day, Michèle Artigue has been closely linked to the emergence and the development of the didactics of mathematics. By observing her exemplary professional history, one can witness a new and specific research domain taking form, as well as see the difficulties that accompanied its recognition by both the academic community and, more generally, the whole education community. Academic recognition relies on the elaboration of a research programme with a specific basis, both methodological and theoretical. Such recognition also passes through the definition of themes specific to the domain and the identification of its links and differences with related disciplines likewise interested both in an epistemological and cognitive perspective on mathematics. Finally, recognition from the education community implies that researchers abandon the comfort of their research labs to become involved in social and cultural debates.

Following this conception of recognition, we have organised this opening chapter around some of the major issues related to the past, the present, and the future of the didactics of mathematics, and more generally of mathematics education: didactics as a specific research domain, the role of theoretical frameworks, the relationship to connected fields of research, and finally, the way didactics considers its relationship with the outside world of mathematics teaching and learning.

1.1 Didactics of Mathematics as a Specific Research Domain

1.1.1 At the Very Beginning: The Year 1968

In order to honour Michèle Artigue, to delineate her scientific and academic path, and because of her close association with the genesis and development of didactics of mathematics, we have to review the evolution of this research domain, both in France and more generally around the world.

In 1993, a conference entitled *Twenty years of didactics of mathematics in France* was held in Paris. As pointed out in its title, the conference dated the birth of this research domain in France to 1973. In his address to that conference, Kilpatrick (1994) stressed the arbitrariness of such a date. Mentioning the outstanding figure of Sylvestre-François Lacroix in the XVIIIth century, he wondered whether we should not trace this birth to 200 years earlier.

Sylvestre-François Lacroix is the author of a remarkable publication about mathematics teaching, as well as of a mathematics textbook intended for high school teachers of that time. Inspired by Kilpatrick, we now consider this period of the history of teaching, making connections with today's concerns.

At the end of the eighteenth century, after the French Revolution, a first "massification" of education—actually rather relative—required enlarging the number of trained teachers. The solution to this problem was the establishment of the *École Normale de Paris* in the year III (Dhombres 1992). The implemented model was that of a pyramidal system disseminating knowledge from top to bottom, with students from districts all over France following the courses in the École Normale, and then redistributing knowledge to prospective teachers in their own districts.

In mathematics, the courses were entrusted to the best mathematicians, who in practice were not really concerned about the "art of teaching", but rather eager to communicate their own creations, often the most advanced of their time. For instance, Monge explained the theory and methods of descriptive geometry, of which he was the inventor. "Pedagogical" concerns, and more precisely concerns for a "method", are nevertheless present in the work of Lacroix, who was Lagrange's assistant during that period. Lacroix published two books on algebra, in which he adopted two radically different positions relative to student work. The first book, published in 1797, is an adaptation of a book by Clairaut chosen because its structure follows "the process of invention" (Ehrhardt 2009). Writing a second book two years later, Lacroix then adopted a synthetic structure, saying that he was now convinced "that it is necessary to reduce the part of invention, and that after the student overcomes the first difficulties […], he does not need a presentation of contents following the way they were invented" (Ehrhardt 2009, p. 15).

Two centuries later, with the New Math reform, the same tendency was observed to assign the development of curricula and the organisation of education to distinguished mathematicians. Again, the main orientation was provided by books for teachers produced by these mathematicians, such as Dieudonné rejecting the geometry of the triangle in favour of a geometry based on linear algebra—as is the case in advanced mathematics. The same problems and tensions were thus encountered when developing education to adapt to the rapidly changing world of the 1960s. During that period, France aimed at a deep transformation of school education, and substantial financial and institutional support was provided. Prominent mathematicians were given the task of educational design, thus triggering conflicts between school mathematics and research mathematics. The dilemma again arose of choosing between learning approaches based on invention, and more systematic methods that logically sequenced mathematical propositions.

Kilpatrick concludes by favouring the year 1968 as the birth of didactics of mathematics, when all these conditions helped the emergence of a research domain about mathematics teaching/learning: 1968 was the year of “tranquil revolutions” in a booming Western world, of the emergence of New Math in the curricula, as well as an important increase in the number of secondary school students. In France it is also the year that the *Instituts de Recherche sur l’Enseignement des Mathématiques* (IREMs) network was established and, most importantly with regard to this book, 1968 marks Michèle Artigue’s first year of teaching.

1.1.2 Reproducibility of Didactical Situations: Towards a Normal Science

At its inception, didactics benefited from solid institutional supports, among them the network of the above-mentioned IREMs, thus involving university teaching staff and facilitating the preparation and development of teachers in charge of implementing the new mathematics reform.

Nowadays (2015), didactics of mathematics—even though this precise term may still be a subject of debate within the international community—seems to be widely recognised as a research field. We can even speak of a “normal science”, or more precisely of a normal domain of research.

Indeed, a common assertion by the instigators of didactics of mathematics, in France if not elsewhere, pertains to the scientific nature of the research project. For instance, Brousseau declared that:

> The didactics of mathematics presents itself, a priori, quite naturally, as the science of specific conditions for the provoked acquisition of mathematical knowledge. (Brousseau 1994, p. 51).

Can we however speak of a “normal science” the way Kuhn defines it? Such a question leads us to consider the issue of “scientificity” and experimenting in didactics of mathematics. The notion of falsifiability introduced by the philosopher Popper (1935) allows to draw a line between science and non-science. Certain explicative or predictive statements of a scientific theory should be tested through experiments. Therefore, contrarily to the *Magna Didactica* written by Comenius in 1638, the new didactics is experimental and seeks to take into account contingency. This in turn raises the issue of what is an experiment in didactics.

Michèle Artigue’s contribution to this matter is crucial, questioning the reproducibility of didactical situations and developing the notion of *ingénierie didactique* in her “thèse d’état” and her frequently quoted 1990 paper (Artigue 1990a) as a method for validating hypotheses in didactics. Hence, a method of research and validation (and so of refutation) of didactical approaches was initiated.

> An experimental design based on the didactical achievements in class, i.e., the design, implementation and analysis of teaching sequences. (Artigue 1990a, p. 285).

The notion of *ingénierie didactique*, and more generally issues related to empirical studies in didactics, remain vibrant in the field and recently (in 2009) the French Didactics Summer School was entirely devoted to this topic. Specifically, questions of validity remain important regarding the number of individuals involved in an empirical study, the duration, the influence of contextual factors, the variability of curricula, and so on. In addition, the influence of settings in an empirical study, distinguishing between "laboratory experiment" and "ordinary classroom", should not be left aside.

1.1.3 The French School of Didactics of Mathematics Within an International Community

As already mentioned above, the mere use of the expression 'didactics of mathematics' may still be seen as a source of debate within the international community. In spite of such a lack of consensus and without entering into the linguistic or practical issues that may be attached to it, this term is most helpful in the present book in order to stress the specificities and the impact of the approach through which issues concerning the teaching and learning of mathematics are generally addressed in France. As is well known, one can even speak of a 'French School' of didactics of mathematics (or of mathematics education, in the usual English parlance). Most of Michèle Artigue's work can be seen as taking its source from this French tradition of *didactique des mathématiques*, and it was considered important in this book to propose some reflections on the connections with other approaches and contexts encountered in 'mathematics education'—connections in which Michèle has herself played a crucial role.

In a paper orchestrated by Tommy Dreyfus and Kenneth Ruthven, and inspired by the metaphor of travel—*Didactique goes travelling: its actual and potential articulations with other traditions of research on the learning and teaching of mathematics* (Chap. 2)—four colleagues, with interest and experience in research in the didactics of mathematics from both a French and an international perspective, were invited to reflect on the main issues underlying the actual and potential connections between the French approach and others. Arcavi first stresses the importance, in spite of the inherent difficulties, of establishing an extensive and intensive 'dialogue' between different research traditions. Kilpatrick then uses the notion of translation as a paradigm to address these difficulties, but emphasises that what is at stake is more than a mere translation of language, from French to English (or eventually other languages), but also, and more importantly, a translation between cultures. Boero takes these comments further, pointing to the fact that even a certain proximity of languages, such as may be the case with French and Italian, does not necessarily eliminate all potential obstacles for communication. Finally, Radford reminds us of key epistemological approaches on which didactique drew to develop its analysis of the genesis of new knowledge and draws our attention to two very different strands, one emphasising the internal logic of a discipline as the motor of its development, another urging attention to the broader sociocultural context.

1.2 The Multiplicity of Theoretical Frameworks: Networking Theories

In an article dealing with the notion of example spaces, Goldenberg and Mason (2008) ended their presentation with a fierce attack against the vagueness of definitions in mathematics education, due to the absence of an axiomatic framework. They reiterate that we frequently come across multiple usages of similar terms, and usages of multiples terms having similar meanings. This attack may come as a surprise to French researchers, as for a long time the French didactics has already been poking fun at its own propensity to generate theoretical frameworks—actually motivated by its scientific orientation which implies strong theoretical work.

The problems began with the proliferation of frameworks and paradigms. However this can be seen as the normal state for an emerging science, or as an intrinsic characteristic of a field which is so extended and complex that it needs different theoretical approaches to report properly. Conducting crucial experiments would then determine which, among all the theories, would lead to a "normal science" based on a paradigm (or paradigms) accepted by the majority.

Evidence of the field's scientificity would arise from its ability to avoid fragmenting into several, and sometimes ideological, chapels, and to construct the field upon a coherent networking of various approaches and methods. In order to make progress on the question of theorisation and the multiplicity of frameworks, it is useful to go back to simple questions often repeated by Michèle Artigue in order to communicate beyond unconnected frameworks:

What is the issue that the researcher wants to address?

What is the nature of the institutional context in which this issue arises?

Normally, the theoretical framework should adjust to the questions and explain certain essential differences. This constant return to the initial issue motivating research should help to stay clear of two frequent obstacles:

- On the one hand, accumulating research within theoretical frameworks, without referring to questions.
- On the other hand, omitting the context by applying exotic or exogenous theoretical frameworks.

In this book, Kidron and Bikner-Ahsbahs present, in their chapter entitled *Networking different theoretical perspectives* (Chap. 3), the efforts of mathematics education researchers in understanding how theories can be successfully connected while respecting their underlying conceptual and methodological assumptions, a process called "networking theories". Both authors had the privilege of collaborating with Michèle Artigue and other colleagues in exploring ways of handling the diversity of theories. They describe and explain the reasons for networking, as well as the expected difficulties of the networking process. They characterise different cases of networking and provide methodological reflections on the difficulties and benefits that accompany the networking.

Grugeon-Allys, Godino and Castela accept these differences between researchers, each engaged in a different theoretical reflection not so easy to bring closer one to the other. While acknowledging the fact that the creativity of researchers, which gave birth to a number of theories, may have created problems in the community, they offer a triple viewpoint on these matters in their chapter, *Three perspectives on the issue of theoretical diversity* (Chap. 4). The first perspective examines the richness of a multidimensional approach based on the mobilisation and networking of various well-identified theories, enabling a segmentation of reality that is well suited to the study of didactic phenomena. The second considers a possible methodology for reducing theoretical diversity based on an upward integration within an onto-didactical framework. Finally, the third perspective examines from a social viewpoint the multiplicity of theories in the didactics of mathematics and the search for connections.

1.3 Didactics of Mathematics and Related Research Fields

Didactics of mathematics covers a wide area related both to mathematics and to the conditions of its transmission and appropriation by various institutions. This naturally involves borrowing from other already highly structured fields—of course from mathematics itself, as well as from its epistemology and history—and from more distant fields such as psychology, semiotics and sociology.

1.3.1 Mathematics

The early development of didactics of mathematics was supported by leading mathematicians, such as Freudenthal in the Netherlands, Rouche in Belgium, and Revuz and Glaeser in France. These initial and very close links between mathematics didactics and mathematics itself were maintained throughout the years, evidenced by didactics research teams in France which are often still today part of university scientific departments. Such is the case, for instance, with the *Laboratoire de Didactique André Revuz*, of which Michèle Artigue is a member, which belongs both to the mathematics and the physics departments of *Université Paris Diderot*. This relationship with researchers in mathematics ensures the epistemological vigilance necessary to any didactic development.

These comments about the importance of mathematics per se clearly apply to Michèle Artigue herself. She started her own academic career as a mathematician with research projects in mathematical logic. But she very early became interested in the teaching of analysis, with a particular attention to the research conducted by Reeb and Lutz (see Lutz et al. 1996) concerning the use of non-standard analysis in teaching. Her interest in the teaching of mathematical analysis remained unfailing throughout the years and justifies the choice of this theme in the symposium.

Analysis is considered in this book from a didactical viewpoint in Oktaç and Vivier's *Conversion, change, transition… in research about analysis* (Chap. 5). In this chapter, the authors offer a personal and original synthesis of research in the didactics of mathematical analysis. The presentation stresses once more the necessity of cross-approaches to this specific mathematics domain, where students are led to embrace a number of complex and diverse notions—e.g., real numbers, functions, limits—that are both central to and emblematic of analysis. The authors show how theories such as socio-epistemology, APOS, or the use Duval's semiotic registers or Chevallard's praxeologies can enrich didactical as well as epistemological questioning.

Two further chapters of this book pertaining not only to analysis, but also to the field of mathematics as a whole, deal with Digital technology and mathematics education. The general background to this theme was clearly captured by the title of the very first so-called 'ICMI Study', *The influence of computers and informatics on mathematics and its teaching* (Howson and Kahane 1986). Artigue's contribution to this theme was of primary importance.

Issues concerning the role and impact of digital technologies in the teaching and learning of mathematics are addressed here under two headings. In their chapter, *Core ideas and key dimensions of Michèle Artigue's theoretical work on digital tools and its impact on mathematics education research* (Chap. 6), Kieran and Drijvers revisit Artigue's classical paper (2002) by drawing out what they consider to be the core theoretical ideas and key dimensions of the body of work on tools and tool use that Michèle not only elaborated, but also inspired others to develop further. They trace the evolutionary path of these core ideas, noting the ways in which they theorise the four general key dimensions: learner, teacher, tool, and mathematics. They focus on core theoretical ideas that have been central to Michèle's work and that have impacted in various ways on the research of others: the instrumental approach to tool use, instrumental genesis, the pragmatic-epistemic duality, the technical-conceptual connection, the paper-and-pencil versus digitally-instrumented-technique relationship, the institutional aspect, and the networking of theories.

In another chapter devoted to the theme of digital technologies, *The teacher perspective in mathematics education research—a long and slow journey still unfinished* (Chap. 7), Abboud-Blanchard, one of Michèle's first PhD students, draws our attention to the need for specific studies on teachers' use of digital tools for a better understanding of teaching practices in technological environments, of their determinants, and of their evolution dynamics.

1.3.2 Epistemology and History

The epistemological vigilance we just mentioned presupposes interactive exchanges with specialists in the history of mathematics and in epistemology. Michèle has clearly pointed this out in another of her fundamental papers, in which she discussed the possible connections between epistemology and didactics. Artigue (1990b) stressed the crucial need of epistemology for the researcher in didactics. She also pointed out that some knowledge of the history of mathematics is a key component of didactical research, in order either to understand the historical development of some mathematical concept, or to understand the shaping of mathematics as a cultural activity.

Historians do not provide direct answers to mathematics education research questions, for a number of structural reasons which Chorlay and de Hosson attempt to lay out in Chap. 8, *History of science, epistemology and mathematics education research*. Echoing Artigue's 1990 paper mentioned above, Chorlay and de Hosson discuss research practices at the intersection of two autonomous fields of knowledge: mathematics education research on the one hand, and history of mathematics on the other. The two main reasons they offer for the gap between history and didactics are the deep heterogeneity of the objects and contexts of study, and the epistemological differences between the two fields. Nonetheless, heterogeneity and autonomy do not imply incommensurability. The authors conclude by stressing that history does not teach, yet there is a lot to learn from it.

1.3.3 Psychology, Cognition, Semiotics and Sociology

An initial characteristic of the didactics of mathematics in France was the conspicuous presence of psychologists specialising in mathematical cognition.[1] In particular, Gérard Vergnaud, from the *Centre national de la recherche scientifique* (CNRS), participated in the first developments of didactics of mathematics. In his paper quoted above, Brousseau, in researching the link between epistemology and cognitive sciences, concluded by giving a new and enlarged definition of what didactics of mathematics was for him.

> The didactics of mathematics thus places itself within the framework of cognitive sciences as the science of the conditions specific to the diffusion of the mathematical knowledge that is useful to the functioning of human institutions. (Brousseau 1994, p. 52).

[1]This characteristic is of course not specific to France. One can also think of the *International Group for the Psychology of Mathematics Education* (PME), an ICMI-affiliated study group established in 1976, whose fifth conference was held in Grenoble in 1981. (Gérard Vergnaud was a member of the International Committee of PME at its beginning and also PME President in 1982.)

The bonds today seem to have weakened: mathematical cognition was not specifically mentioned at the conference in honour of Michèle Artigue, whereas it was very present in the 20-year of didactics conference. We assume that this gap results from the evolution of the two research fields. Today, cognitive sciences are a highly technical field of research, mainly aimed at understanding the functioning of the brain (e.g., the journal *Mind, Brain and Education*). Besides this aim, there is a societal demand for applications to education, and especially to mathematics. In recent years Michèle has stressed the need to consider this societal demand and, in concluding a booklet about gestures and embodied cognition with a special emphasis on cognitive sciences, she calls attention to “the need of working on the conceptual and methodological implications of the integration of these approaches of cognition into our research issues, reflecting on this occasion on the potentialities and limits of our theoretical frameworks to tackle these issues” (Lagrange et al. 2012, p. 36).

By contrast, the recent evolution of didactics of mathematics tends to favour research concerns centred on the forms of diffusion of specific knowledge within the institutions, with links to specific sociological approaches. Semiotics, in various forms, is a key component of most recent research studies.

1.4 Didactics of Mathematics and the Outside World of Mathematics Education

In his 1993 lecture, Kilpatrick (1994) also stressed the fact that researchers in North America tend to resist the expression *didactics of mathematics* and prefer to use instead *mathematics education* when referring either to the teaching activity or to the research field. ‘Didactics of mathematics’, de facto, allows a distinction between these two dimensions: research on the one hand; and its impact on teachers, on students through the education they receive, and also on curricula changes, on the other hand.

In all these dimensions, we can notice the specific input provided by Michèle Artigue and her ability to include all of them, without avoiding any. Her commitment to the implementation and assessment of digital tools in education has already been outlined above. What may be less known is that she was involved in the fashionable and institutional demand for Inquiry-Based Education (IBE), promoted today by most science and mathematics curricula on an international scale.

1.4.1 Inquiry-Based Education

In the chapter *Inquiry-based education (IBE): towards an analysing tool to characterise and analyse inquiry processes in mathematics and natural sciences* (Chap. 9), Ouvrier-Buffet, Bosdeveix and de Hosson (respectively from mathematics, biology and physics) offer a general overview of IBE and compare different experiences in various scientific domains. They stress that while many guidelines have been developed for helping both teachers and teacher educators to implement teaching-learning sequences involving inquiry processes, the specificities of the scientific knowledge involved is rarely taken into account. They propose a "checklist" as a tool for analysing inquiry-based sequences that are being implemented in mathematics and science classrooms.

1.4.2 The Researcher in the Wider Community

It is impossible not to mention Michèle Artigue's exceptional contribution in being, herself, a 'Researcher in the Wider Community'. Motivated by the numerous responsibilities that Michèle took on beyond her research work, a group of authors look in practical terms, in Chap. 10, at how a researcher may come to develop this kind of involvement, and at its goals and impacts.

In the first section of that chapter, Lagrange reflects on and draws lessons from responsibilities taken by Michèle in various institutions such as the IREMs (at the time of the colloquium in her honour, she was still President of the IREMs' scientific committee). Celia Hoyles then examines her personal involvement in the National Centre for Excellence in the Teaching of Mathematics (NCETM) in the UK, discussing how the Centre started and how it has evolved since 2006. Another influential and active researcher, Jill Adler, contributes to this theme by focusing on key developments in mathematics education in Africa that have emerged through the work of the International Commission on Mathematical Instruction (ICMI), in particular under the presidency of Michèle Artigue (2007–2009). Finally, French academician Jean-Pierre Kahane discusses more generally the role and position of researchers, and especially mathematicians, in contemporary society. He concludes with remarks on the place that mathematics has in civic life and on the eminent social role played in that connection by the teaching and learning of mathematics.

One aspect concerning the issue of the researcher as a member of a wider community is related to communication, and in particular to the choice of language(s) used in various public contexts, for instance, publications or conference presentations. We have already stressed above how Michèle Artigue, in her own oral presentation at the colloquium organised in her honour, made a point of using three languages which she speaks fluently, a testimony of her sensitivity to the diversity of the attendees' cultural environments. In a preliminary note about language in his contribution to this chapter, Kahane emphasises that this colloquium included three languages, which contributed to providing a successful setting, and he warns about the use of English as the preferred only possible language, especially in educational matters.

While acknowledging the convenience of having today English as a *lingua franca*, it could be argued that the use of only English in a field so diverse as didactics of mathematics could without doubt provoke an impoverishment—a risk for which the entire educational community must be fully aware. The Artigue symposium should be seen as a testimony to the importance of further encouraging multilingual conferences, even when editorial constraints compel using only one for the post-conference book.

1.4.3 "L'École Artigue"

In a final colloquium presentation reflected in this book, a different perspective is introduced concerning the relationship between didactics of mathematics and the global issue of the teaching and learning of mathematics: that of preparing the next generation of researchers in the field. Taking as a starting point the remarkable contribution of Michèle Artigue in this regard, Haspekian, Straesser and Arzarello propose, in their chapter, *Preparing young researchers in mathematics education: beyond simple supervising* (Chap. 11), not only a testimony to her personal accomplishments as a supervisor of numerous doctoral theses, but also a more global reflection on what it means to accompany and guide PhD candidates throughout their progression in their doctoral studies. They discuss the responsibilities of various parties involved in such an endeavour, and point to some of the main pitfalls that may occur.

1.4.4 Artigue's Didactic Aventure

In the concluding chapter written by Michèle Artigue, she underlines that the conference honouring her was a strong and emotional occasion for retrospective reflection and she uses this opportunity to convey to the new generations of didacticians some elements of a history which has shaped their field of study. She invites us to follow her on a didactic route which begins with the creation of the IREM of Paris 7 in January 1969 and ends at the conference in June 2012 in which she expresses her confidence in the future of this research domain.

Before leaving the reader to explore the chapters of this book, it is important to emphasise again the specific contribution of Michèle. She brings to us a precious existential theorem: There exists a personal and rich manner to reconcile all these points of view and develop research that interacts with social demand.

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Chapter 2
Didactique Goes Travelling: Its Actual and Potential Articulations with Other Traditions of Research on the Learning and Teaching of Mathematics

Abraham Arcavi, Paolo Boero, Jeremy Kilpatrick, Luis Radford, Tommy Dreyfus and Kenneth Ruthven

2.1 Introduction

The French tradition of research on the learning and teaching of mathematics, often referred to simply as 'didactique', has developed a range of theoretical tools. These tools share a common intellectual and professional hinterland, and although each is honed to analysis of particular types of didactical question, they have increasingly been used by French researchers in coordinated ways. As this movement towards a more systematically articulated didactique has developed within France, ideas from didactique have become sources of inspiration or points of reference for researchers outside France. Fresh questions have naturally arisen about the central concepts of didactique as they encounter new professional cultures and their associated intel-

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lectual traditions. At the Artigue colloquium, the last two authors of this chapter convened a round-table discussion to explore this issue. Each of the first four authors contributed by bringing a particular perspective inspired both by their professional contacts with Michèle Artigue and their own interests and experience.

The first two of these contributions looked, in rather different ways, at correspondences and contrasts between French didactique and some of the central theoretical frameworks influential in English-speaking research on the teaching of mathematics. Abraham Arcavi launches our discussion by comparing didactical engineering with design research and the didactical contract with sociomathematical norms.

2.2 Towards a Dialogue Between Traditions and Approaches in Mathematics Education

Abraham Arcavi

I am honored and pleased to be part of this homage to Michèle Artigue. It is a most appropriate opportunity to express my personal gratitude to her. I learned much from Michèle's writings and much more from the long and lively conversations I was very lucky to have with her on many occasions. The luck refers to the many opportunities we have had to encounter each other (including a mini-course in Sweden that we co-taught). However, the depth and the breadth of our conversations have less to do with luck and more to do with Michèle's friendly predisposition to engage in dialogue, with her special ways of communicating, and above all with her willingness to address my questions and comments, even if naïve—always with unusual patience, care and deep respect for her interlocutor. I was consistently impressed by her devotion as a teacher, as a "bridge builder" (between the knowledgeable and the less knowledgeable, between the French tradition and other schools of thought, between mathematicians and mathematics educators), and by the scope of her vast knowledge and wisdom. Inspired by Michèle's work and personality, I would like to argue for a comprehensive effort to enhance and extend the dialogue among traditions and communities in mathematics education, with depth, care, respect and patience. In particular, I think it is of special importance to continuously address, for example, the following question: Do the approaches of the French didactique overlap those of the other traditions, and if so, in what sense? Or maybe the issues are not the same after all?

I will briefly dwell on some aspects of this question by using examples and by reporting on a conversation with a colleague who works in different traditions of research and practice.

My departure point is the very title: “Didactique goes travelling”. The travel metaphor implies moving away from one’s comfortable and well-known territory to other, less known, places, to meet people who speak different languages and/or have different perspectives, and with whom we may talk about our places and ways of doing things. As a traveler for short, intermediate and long periods, including emigration and adaptation to a new place, I know that changing places often compels us not only to meet new perspectives, but also to profoundly revisit and make visible our own ideas and basic assumptions, which we tend to take for granted.

In this “travel”, I will rely on informal conversations with Michèle Artigue and on some of her lectures I attended.

2.2.1 Contrasting Traditions

In the 1980s, both the French didactique and the Anglo-Saxon traditions of mathematics education seemed to be at a similar juncture:

(a) Dissatisfaction with the methods of study (external methodologies such as tests and questionnaires, and overemphasis on statistical comparisons between experimental and control groups) as the main avenues to “scientificity”; and
(b) A need to move away from cognitive research mostly held in laboratories in order to capture “the complex intimacy of classroom functioning”. The Anglo-Saxon tradition started to focus on the socio-cultural aspects of teaching and learning, and the French didactique launched the Theory of Didactical Situations, the idea of Didactical Engineering and more (perhaps there is some chronological misrepresentation implicit here, since the roots of French didactique can be traced back to the 60s, therefore the ideas were not only a reaction to unsatisfactory extant theories and methods, however, their visibility and influence became apparent in the 80s). In a certain sense, the departure point and the ultimate goal of both traditions were similar: to capture the complexity of learning in classroom settings, and to understand the “ecology” beyond the “individual’s biology”.

In the following, I would like to consider two examples of ideas/constructs from both traditions, to juxtapose them in order to ponder points of tangency, the extent of the overlap and the character of the differences. Such juxtaposition is rather simplistic, and thus it may do some injustice to both the depth and the breadth of the ideas. Nevertheless, it is proposed as a starting point to launch a much needed dialogue, even if it only serves the purpose of a “balloon to shoot at”.

First example: Design research—Theory of Didactical Situations

Design research (Brown 1992)	Theory of didactical situations (Brousseau 1997; Artigue 2000)
• From work sites/assigned tasks towards communities of learners, where students take charge of their own learning • Engineer educational environments and simultaneously conduct experimental studies of those innovations • Study simultaneous changes in the system, concerning the role of students and teachers, the type of curriculum, the place of technology, and so forth This is intervention research to inform practice and which enables migration from experimental classrooms to average classrooms operated by and for average students • Autonomous learners • Work toward a theoretical model of learning and instruction rooted in a firm empirical base	• From cognising subjects to didactic situations: set of interactions between students, teachers and mathematics at play in classrooms • The didactic situation shapes and constrains the knowledge constructed • Without understanding the situation it is not possible to interpret the students' behaviors in cognitive terms • To understand teaching and learning processes and the ways they interact • To develop rational means for controlling and optimising didactical situations • Didactic, a-didactic situations and devolution processes • Confrontation between a priori analysis of engineering design and a posteriori analysis of data collected

Second example: Socio-mathematical norms—Didactical contract

Socio-mathematical norms (Yackel and Cobb 1996)	Didactical contract (Brousseau 1986; Artigue 2000; Sierpinska 1999)
• Normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient and mathematically elegant in a classroom • What counts as an acceptable mathematical explanation and justification • Influence the learning opportunities for both the students and the teachers • The students and the teachers may interactively elaborate the norms • Clarify how students develop beliefs and values	• Concept developed by Brousseau as a possible explanation for 'elective failure' in mathematics • Expectations of the teacher with respect to the students, and conversely, regarding the mathematical content • "Rules" pertinent to the knowledge taught • The rules of the didactic contract are implicit: the teacher and the students do not sign a chart of 'rights and obligations' but they are there and we know that they are there when they are broken

2.2.2 Reflections of a "Traveler"

In my pursuit of understanding and reflecting upon the differences between traditions, I thought that it may also be appropriate to bring vivid testimonies of a "traveler" among traditions. I was fortunate to meet Takeshi Miyakawa in Tsukuba University in Japan during 2005. Takeshi is currently a faculty member in the Department of Mathematics at the Joetsu University of Education. What is special about Takeshi's background is that he did his PhD in Grenoble, France, under the guidance of Nicolas Balacheff, and his post-doctoral fellowship with Patricio Herbst at the University of Michigan in the USA. So Takeshi has first-hand experience in three traditions of mathematics education. In preparation for the Artigue colloquium panel, I contacted Takeshi and he kindly agreed to reply to my questions. Here are some of them:

1. As a person who worked in both the French tradition of didactique and the Anglo-Saxon research tradition in mathematics education, what would you say are the main differences between them?
2. Did you find theoretical "points of tangency" between the two traditions?
3. Do you have any explanation as to why the French tradition and the Anglo-Saxon tradition do not engage in a deeper dialogue than seems to be the case?

Takeshi's answers are extensive and interesting. I include here some of the main points, taking into account what I think would best serve the promotion of the dialogue between traditions.

The French research tradition in mathematics education is relatively homogeneous. This is consonant with Laborde's (1989) description of research as based on the setting up of an original theoretical framework developing its own concepts and methods and satisfying three criteria: relevance in relation to observable phenomena, exhaustivity in relation to all relevant phenomena, and consistency of the concepts developed within the theoretical framework. It is described as a national program to be carried out neither by single researchers, nor by single teams, but by several institutional sites. The American tradition is much more heterogeneous: "Research in mathematics education is diverse. Very often, the theories from other fields of research, like linguistics, sociology, ethnomethodology, psychology, cognitive science, educational science, philosophy of mathematics, etc. are used. And the research work focuses on different aspects according to the adopted outside theory. It also seems, in my view, due to this diversity, that there is no sharp line between the research in math education and the educational research in general" (Miyakawa 2012).

Another difference in Takeshi's view is the centrality of the role and nature of mathematical content knowledge in the French tradition, whereas this may not always be the case in the USA. Even when there may be points of tangency regarding the objects of study, it is often quite difficult to "put them together" because the goals, the foci and the approaches to research are so different. Such differences should be pursued, captured and made explicit, as a pre-requisite for a fruitful dialogue.

I would like to conclude on the basis of a reflection by Takeshi, which may also contribute to pinpointing why the rapprochement between traditions is not happening as intensively as one might have expected: it is not easy to understand the French research tradition if you are in the American tradition, and vice versa. Understanding the other's perspective is not a simple endeavour (Arcavi 2007), and among its difficulties is undertaking a deep revision of one's own implicit assumptions, beliefs, biases and predispositions in one's own research tradition. It also implies the explication of the implicit—and this may require more than simply reviewing published research results.

However, says Takeshi, there have been several appeals followed up by actual research which seriously undertakes this rapprochement. The CERME working group on theory may be one of these examples.

I encourage the furthering of these attempts, and I also encourage the research community to be inspired by the intellectually sound and personally empathic approach so nicely pursued by Michèle.

2.3 First Entr'acte

As Miyakawa notes above, French didactique has traditionally been more strongly affiliated with mathematics in its disciplinary culture and social organisation than with mainstream educational studies and core social sciences, in contrast with English-speaking mathematics education. Jeremy Kilpatrick continues this discussion by examining the commensurability of these two traditions of study of the teaching of mathematics, and the conceptual frameworks that they have developed.

2.4 Lost in Translation

Jeremy Kilpatrick

> "La syntaxe française est incorruptible. C'est de là que résulte cette admirable clarté, base éternelle de notre langue. Ce qui n'est pas clair n'est pas français; ce qui n'est pas clair est encore anglais, italien, grec ou latin."[1]
>
> Antoine de Rivarol 1783

English-language speakers struggle to express what is meant by *didactique des mathématiques*. "Didactic of mathematics" sounds stilted, even when pluralised, and "mathematics education" is simply wrong. A common solution is to use the French words and let the Anglophone reader adopt the meaning he or she chooses. *L'ingénierie didactique* raises further hurdles. Connecting didactique to Anglophonic research in mathematics education guarantees that something will be lost—but what?

[1]"French syntax is incorruptible. It is from that that results this admirable clarity, the eternal foundation of our language. What is not clear is not French: What is not clear is still English, Italian, Greek, or Latin."

Almost two decades ago, in June 1993, a conference on 20 years of *didactique des mathématiques* was held in Paris. The report of that conference (Artigue et al. 1994), which focused on the contributions of Guy Brousseau and Gérard Vergnaud, marked a milestone in the development of didactique as a research enterprise. Brousseau's theory of didactical situations and Vergnaud's theory of conceptual fields, supplemented by the semiotic approach of Raymond Duval and the anthropological approach of Yves Chevallard, have in the ensuing years provided four important theoretical frameworks on which much of the field of French didactique has been built (Winsløw 2005).

During the past decades, a major role in bringing didactique onto the world stage has been played by Michèle Artigue, who through her many presentations and publications in French, English and Spanish has elucidated for numerous audiences the theory of didactical situations, didactical engineering, and the anthropological theory of didactics. Her interest in the integration of computer technologies into mathematics education, particularly at the university level, has led her to combine and elaborate the anthropological approach in didactics with the theory of instrumentation in cognitive ergonomics (Artigue 2002), a combination that has been especially productive in studying learning in computer algebra system environments. Over the past decade, in particular, Artigue has been a highly visible champion of all things didactique.

2.4.1 Didactique as Traveller

The theme *La didactique en voyage* (Didactique goes traveling) provides a challenging metaphor: Where does didactique choose to go? What does it do along the way? And what does it take along on its travels? A brief consideration of each of these questions serves to introduce my main concern: What is lost when didactique leaves its Francophonic nest?

2.4.1.1 Destinations: Places Travelled

The report of the conference (Artigue et al. 1994), held when didactique was just reaching adulthood, suggests that whereas during its youth it had reached many places within France, it had not managed to travel very far beyond the country's borders. Most of the affiliations of the contributors to the volume were with institutions in France, and only a handful were outside—a few European countries, the United States and Canada. Now that didactique is on the verge of middle age, that situation has greatly changed.

Didactique has ventured around the world, aided in large part by translations of important works into other languages. For example, the 1997 Spanish translation of

Chevallard's (1985) book on didactic transposition was enormously influential in bringing his ideas to Latin America. Publications in English (e.g., Artigue and Winsløw 2010; Barbin and Douady 1996; Herbst and Chazan 2009a; Laborde and Perrin-Glorian 2005; Winsløw 2005) brought work on didactique to a much wider audience, with research reports still coming primarily from France and its neighbors Spain and Italy, but increasingly from places such as Argentina, Denmark, Palestine, South Africa and Vietnam. The emphasis on studying teaching situations in mathematics classrooms using approaches anchored in the theoretical frameworks of didactique has given the work broad appeal.

2.4.1.2 Itineraries: Some Accomplishments

During its travels, didactique has expanded its reach (see Caillot 2007, for a brief history of that expansion across disciplines). At first, it seemed a simple traveler: Arising from the theory of didactical situations, it has been defined as "a research field whose central goal is the study of how to induce a student to acquire a piece of mathematical knowledge" (Warfield 2007, p. 86).

As a field of research, didactique seemed only part of the larger field of research, study, theory, and practice that is called, in English, *mathematics education*, and in fact, seemed only part of the research of other scholarly endeavors in that field. But over time, as didactical engineering and anthropological theory were added to its arsenal, didactique grew beyond the realm of research: "In French, the term '*didactique*' does not mean *the art or science of teaching*. Its purpose is far more comprehensive: it includes teaching AND learning AND school as a System, and so on" (Douady and Mercier 1992, p. 5).

Even a modest survey of the accomplishments of didactique over the past four decades would be beyond the scope of this chapter, so I mention only a few examples. Discussing the ways in which theories can provide a language to describe practice, Silver and Herbst (2007) note how Vergnaud's theory of conceptual fields has been used by US researchers: "Much of the empirical work [on the learning of additive and multiplicative structures] done by scholars like Carpenter, Fuson, Behr, and their associates concurs with Vergnaud's observations" (p. 52).

Silver and Herbst (2007) also give several examples of how research done in the US "can be seen as illuminating the origins of some [didactical] contractual difficulties and entanglements" (p. 55). They show how the construct of the didactical contract fits with the work of Mary Kay Stein and her colleagues (Stein et al. 2000) on the negotiation of task demands in the mathematics classroom. Silver and Herbst also connect the didactical contract to much other research on the work of the teacher. "The notion of didactical contract has turned out to be a particularly helpful notion to turn a descriptive theory of instruction based on [a] relational conception of teaching into an explanatory theory" (Silver and Herbst 2007, p. 63; see also Herbst 2003, 2006).

Sriraman and English (2010) term didactique "the French tradition" (p. 19), and they outline both the theory of didactical situations (which they term TDS) and the anthropological theory of didactics, noting that both are major theories of mathematics education. Linking TDS to other research on mathematics teaching, they assert:

> Taken in its entirety, TDS comprises all the elements of what is today called situated cognition. The only difference is that TDS is particularly aimed at the analysis of teaching and learning occurring within an institutional setting (p. 23).

2.4.1.3 Baggage: The Burden of Terminology

Despite the broad dispersion and wide-ranging accomplishments of didactique over the past decades, it has not had the influence outside the Francophone world that one might have expected, given the field's shift in focus to classroom situations:

> During the past 30 years, researchers in mathematics education from English-, French-, and Castilian-speaking regions have been giving increased attention to classroom instruction…. Communication among researchers across those language differences has however been scarce. Different theoretical traditions as well as cultural differences in how to write and edit scholarship have often contributed to exacerbate the obvious differences in language competence and thus discouraged mutual acknowledgment. (Herbst and Chazan 2009b, p. 13)

In my view, part of the communication problem is that didactique carries some heavy baggage stemming largely from the language it employs and its metaphors in particular. Pimm (1988) has noted what he calls "a fundamental metaphoric structure in English" (p. 31), which I would claim is present in French as well, in which one links together an adjective and a noun, with the adjective pointing "to a new context of application, and sometimes considerable effort is required to create a meaning for the whole expression" (p. 31). He gives some examples from science and mathematics, with the mathematical examples including *spherical triangle*, *complex plane*, and *differential geometry*. In a footnote, he notes two examples from didactique:

> Colette Laborde has pointed out to me that two terms central to the discipline of *didactique des mathématiques*, namely *contrat didactique* and *transposition didactique*…have had a difficult reception, in part precisely because of crossed interpretations. At one level, the last thing that a *contrat didactique* is is a contract (because it is completely tacit and unspoken); a *transposition didactique* usually involves a far greater alteration of material, structure and form than does a musical transposition. (p. 33)

In such metaphors, as Pimm (2010) later observed, "The semantic pressure is always piled onto the noun—it is never the adjective that has to shift or expand its meaning" (p. 613).

Another example is *l'ingénierie didactique*, which readily translates as didactical engineering. Although the adjective poses some problems that are discussed below, the more severe problem is posed by the noun. Didactical engineering is not engineering in the same sense as mechanical or electrical engineering. But, as Artigue (1994) observes, it is meant to "label a form of didactical work that is comparable to the work of an engineer" (p. 29). As she correctly notes, engineers "manage problems that science is unwilling or not yet able to tackle" (p. 30), and so didactical engineering is intended to handle such problems in didactics. *Didactical engineering* "has become polysemous, designating both productions for teaching derived from or based on research and a specific research methodology based on classroom experimentation" (p. 30). Whereas apparently, in French, *l'ingénierie didactique* has made the journey from metaphor to accepted, well-formulated usage, in English, *didactical engineering* still sounds awkward. Anglophones rarely use *engineering* when speaking of activities in education, and when they do, it is often used in a pejorative sense, suggesting the treatment of learners as material to be managed rather than educated.

Didactique, in creating a precise vocabulary for its work, has made extensive use of the fundamental metaphoric structure identified by Pimm, generating terms that need careful exegesis before they are used. Anglophones may find that English versions of those terms come laden with extra baggage that makes them difficult to interpret correctly.

2.4.2 Didactique in Translation

Like any metaphor, *la didactique en voyage* is limited in applicability. Didactique is not literally a traveler. So let me switch to another formulation and look at didactique as a body of work being translated from French to English. It was heartening to learn recently that, along with its neutral sense of "intended to instruct," *didactic* has much the same pejorative sense in French that it does in English and that neither sense was the one originally intended:

> About a quarter of a century ago, when some of us decided to use didactique as we do now, we were faced with two main obstacles. First, didactique was used in French essentially as an adjective, not as a noun. Second, the word had only two received meanings in common parlance, neither of which—understandably—tallied with what we had in mind.[2] (Chevallard 1999, p. 6)

Although the creators of didactique chose to give it a third—scientific—meaning, as discussed above, that did not solve the problem of the reader unfamiliar with that meaning.

[2]Chevallard argues in favor of using *didactics* in English rather than *didactique*. I agree with his position but have used *didactique* in this paper in the spirit of the panel to which it contributes.

For more than a decade, the editors of *Recherches en Didactique des Mathématiques* (RDM) have given me the opportunity, for many of the manuscripts accepted for publication, to edit the abstract in English that accompanies the resumen in Spanish and résumé in French. Occasionally, I am given the résumé and asked to produce an abstract, but usually my task is to polish the English of an abstract that an author has constructed. The unpolished abstracts run the gamut from elegant English prose to collections of English words that look as though they might have been produced by a computer-assisted translation program. With the latter kind of abstract, I struggle to find a good compromise between my understanding of the résumé and intelligible English. Perhaps if I were more fluent in French, I would be less sensitive to the difficulties that some authors apparently have in finding a reasonable English equivalent for the ideas they have expressed in French.

The challenge of polishing these RDM abstracts has made me acutely aware that translation from French to English, although at times fairly straightforward, is far from being a matter of simple equivalence. There is translation, and then there is interpretation, where it may be necessary to explain the intended meaning using words rather different from what a literal word-for-word translation would produce. French is known for its clarity, precision and rigor; English has other characteristics. A translation from French is bound to lose shades of meaning, particularly when one is dealing with the difficulties and complexities of didactique. Warfield (2007) put it well:

> Over the past decade I have made a sequence of translations [from French to English], each time feeling cleverer than the last, and each time discovering afterwards some nuance that has nonetheless disappeared. (p. 92)

Robert Frost once said, "Poetry is what gets lost in translation." When didactique is translated from the French milieu to that of English, it loses not only poetry but also clarity and nuance.

2.5 Second Entr'acte

One could imagine, then, that the nuance of didactique might be more easily grasped by speakers of other Romance languages. Certainly, didactique seems to have proved particularly influential in Spanish-speaking research on mathematics education. However, from his reflections on difficulties encountered in the dialogue between French didactique and an emerging Italian approach to research on mathematics education, Paolo Boero draws the lesson that research on mathematics education is strongly shaped by many (other) features of the local ecology.

2.6 Some Reflections on Ecology of Didactic Research and Theories: The Case of France and Italy

Paolo Boero

2.6.1 Introduction

Beginning with lively discussions at the French Summer Schools of Mathematics Didactics during the 80s, and her translation into French of my plenary lecture at PME-XIII (1989), Michèle Artigue has played an intensive and, in my opinion, crucial, role of 'mediator' in order to further the dialogue between the already well constituted tradition of French research in Didactics of Mathematics and the newly developing Italian research in the field.

While reflecting on the difficulties in establishing collaboration between the communities of French and Italian mathematics educators, I am now convinced that these difficulties do not derive only from researchers' characteristics and personal positions, but also (and perhaps mainly) from ecological conditions under which research in mathematics education develops. By comparing the cases of France and Italy, I will attempt to identify and describe some variables that are related to local conditions, and in my opinion, that influence research developments and their outcomes (cf Bartolini Bussi and Martignone 2013): the features of the school system (teacher's "mission" and degree of freedom); the economic constraints of research (conditions for funding), particularly in the initial stage; and especially the weight of the cultural environment (particularly, but not only, as concerns the field of mathematics). Discussing the influence of these variables may contribute to identifying potential meeting points and possibilities of productive collaboration between mathematics educators from different countries, and also to better understanding the roots of different theoretical elaborations and scientific productions developed in different countries.

2.6.2 Institutional and Economic Constraints

In the Italian reality, many of the studies performed concern educational (frequently, long term) innovation, which is conceived within theoretical frameworks that evolve according to the needs emerging from the analyses of what happens in the experimental classrooms. Needs may concern more advanced framings and/or the integration of new tools to improve teaching sequences and their analyses.

Such an evolutionary process, named 'research for innovation' by Arzarello and Bartolini Bussi (1998), also provides the researcher with an environment where basic research may take place. (An example of the relationships between the educational context shaped by research for innovation and didactical specific research is presented in Boero et al. 2009). The differences in the most typical research performed in France compared to Italy arise for various reasons. Later, I will consider some cultural factors; here I would like to deal with some important differences between France and Italy, which concern the institutional features of the school system, the teacher's role in the classroom, and the economic constraints that have influenced the development of Italian research in mathematics education (in particular, in the beginning, in the late70s and in the early80s).

Firstly, the Italian school system is much less rigidly organised than the French one; national programs and, more recently, the Indicazioni nazionali per il curriculum (National guidelines for curricula) are much less detailed and prescriptive than French programs and guidelines for teaching; and nothing exists in Italy comparable with the guidance and control functions exercised in France by inspectors (cf Bartolini Bussi and Martignone 2013). The Italian situation allows some teachers to violate not only official prescriptions, but also universally accepted didactical and pedagogical principles. However, the situation allows other teachers to engage in large scale innovative projects and to gradually develop their competence as true researcher-teachers who participate in research teams, while maintaining their teaching functions in their classrooms (Malara and Zan 2008).

Another aspect of the Italian situation concerns the role of the teacher: the teacher is conceived in official documents as well as in current mentality, within the school and outside the school, not only as a specialist in the teaching of one (or several) discipline(s), but also as an educator. This conception probably depends on the great interest of Catholic culture in educational issues, and on its strong influence, in the past, on teacher education for kindergarten and primary school. During past debates concerning how many years should be spent by a teacher in the same classroom (with the same students), the great importance of the educational side of the profession always resulted in the choice of long periods. Usually, a primary school teacher teaches the same students for 5 years, while in secondary school, a teacher teaches the same students for 3 years in lower secondary school and for 2–3 years in high school.

In such an institutional and educational context, broad long-term teaching projects may be developed and tested, frequently opened to extra-school reality, frequently extending over more than one year of teaching. These projects provide teachers and researchers with the opportunity to identify and appreciate the long term changes that educational choices produce, in comparison with other more or less traditional choices. As a consequence, when a teacher realises how the innovation resulting from collaborating with the researcher results in a different mastery of concepts and skills, in comparison with previous teaching cycles, then the teacher feels motivated to further engage in research work. Furthermore, both the teacher and the researcher are motivated to attempt to identify and analyse the mechanisms that have improved the teaching results, thus contributing to further improving the classroom work and develop interesting research work.

But such within-the-school reasons (which make the Italian situation different from the French) are not sufficient to explain: the importance of research for innovation in Italy; the strong presence of teachers in the Italian research teams; and the fact that the development of theoretical knowledge and toolkits is strictly connected with a perspective of didactical innovation (cf Bartolini Bussi and Martignone 2013), and not so much aimed at modeling the teaching process and what happens in ordinary classrooms (an aim so relevant in French didactics of mathematics).

Starting from the middle of the sixties, we find several mathematicians who engaged (from different positions—in favor of or against modern mathematics) as promoters of the reform of national programs, and then (since the beginning of the seventies) as organisers of projects for an alternative teaching of mathematics, in collaboration with teachers. Indeed, some Italian mathematics teachers—the best known was Emma Castelnuovo—were already engaged in the movement for the reform of mathematics teaching at the national and international level. Starting from the middle of the seventies, research grants by the National Research Council (including the first research fellowships to prepare researchers in the field of mathematics education) were specifically aimed at elaborating upon and experimenting with innovative educational projects, based on different methodological and epistemological assumptions. When some of the mathematicians and the teachers engaged in research in mathematics education served as members of ministry commissions for the development of new national programs (in the years 1977/79, for grades VI to VIII; and in the years 1983/85, for grades I to V), they were able to offer the principles and the results of their didactical innovations to many classrooms. Also in more recent years (starting from the year 2000) some mathematicians engaged in mathematics education research and some researcher-teachers from our research teams took part in the development of the national guidelines for curricula for all school levels. As a result, present official texts (particularly on the methodological and cultural side) and accompanying materials sponsored by the Italian Mathematical Union (UMI)—the Italian association of mathematicians—particularly the examples of good practice, largely rely upon the research for innovation work performed by the university-school research teams.

Such a history of the development of research for innovation in Italy, its funding (particularly in the first period), and its influence on the evolution of national programs and guidelines for curricula, may make it possible to better understand why Italian researchers in mathematics education prefer to engage in research with immediate or at least eventual innovations in the teaching of mathematics. It may also help to better understand why they prefer to engage in long-term teaching experiments and innovative projects, based on wide-ranging hypotheses—in spite of the dangers of such a choice for the scientific validity of results.

2.6.3 The Influence of the Cultural Context

While comparing research in mathematics education developed in France and in Italy, we may ask ourselves why the *problématique* of the mathematics-reality relationships and of the relationships between mathematics and other disciplines (like physics) is so relevant in Italian research, and why, on the contrary, research on the sociological and institutional sides is so little developed in Italy in comparison with France. My hypothesis is that those differences largely depend on the different cultural contexts in which research in mathematics education developed, and still develops, in France and in Italy.

As concerns mathematics, the lively debate in Italy among mathematicians about mathematics and its teaching during the sixties and the seventies resulted in a plurality of positions which corresponded rather well to the plurality of positions about *Modern Mathematics* among teachers engaged in teaching reform. During the same years, when many Modern Mathematics textbooks were distributed in Italian schools under the pressure of the in-service teacher preparation organised by the Italian Ministry of Education, according to OECD orientations, the Italian Mathematical Union promoted the translation into Italian of the School Mathematics Project (SMP). SMP was a project originating in the UK and linked to the tradition of mathematics teaching in that country. In the same period, Emma Castelnuovo's textbooks and volumes on mathematics teaching were popular among teachers engaged in elaborating new ways of teaching. Those positions, initiatives and materials opposed to Modern Mathematics contributed to developing experiences of lively relationships in the classroom between motivation of students, construction and development of mathematical knowledge, and knowledge of natural and social phenomena. Progressively those relationships were further developed and resulted in different educational orientations: those inspired by Freudenthal, based on the idea of vertical and horizontal mathematizing; more radical positions (mathematics develops both on the conceptual side and on the ways of thinking side according to the development of knowledge in suitable fields of experience) as in the projects of the Genoa research team led by me since the middle of the seventies; or positions resulting in the classroom construction of mathematical knowledge through problem solving activities within mathematics, but constantly open to applications to extra-mathematical problems.

By comparing what happened in France and in Italy we may identify deep differences which may be attributed to the influence of the Bourbaki team in the debate on mathematics and its teaching in France during the fifties and sixties. In fact, those differences originated far back in French history (since Descartes) and are still strong even today. In the French cultural tradition, mathematics is consistent, and its structural and formal features and its rigorous, coherent and unitary organisation are very relevant. Moreover, mathematical modeling (as well as statistics) belonged for long periods of the twentieth century to the field of applied mathematics, outside mathematics. While those positions have been dominant in France for long periods, in Italy some streams of research in geometry developed, based on intuition and spatial organisation, up to the middle of the twentieth century; and applied mathematics always belonged in Italy to the field of mathematics, in parallel with geometry, algebra and mathematical analysis.

Concerning sociological studies and the interest in the institutional aspects of schooling and their influences on teachers' choices, Italian research in mathematics education (and also in the sciences of education) is rather weak. When we consider phenomena related to sociological or institutional aspects, we usually refer to elaborations and tools taken from abroad (like Brousseau's construct of *Didactical Contract*, and Chevallard's *Theory of Didactical Transposition*). We may acknowledge the consequence of an important lack in Italian culture: nothing can be compared in Italy, as concerns the importance and the resonance of the cultural environment, with P. Bourdieu's or M. Foucault's constructions.

2.6.4 A Healthy (or Unhealthy?) Eclecticism

Should we borrow some tools or constructions from abroad? If we consider Italian scientific production in mathematics education, we may identify a strong tendency to use or adapt theoretical tools of different origins (borrowed from different disciplines, or derived from different theories in the field of mathematics education). The only constraints concern local coherency (when dealing with application to a given research problem). Some original contributions and constructs (like Mathematical Discussion by Bartolini Bussi 1996; or Semiotic Bundle by Arzarello et al. 2009; or Field of Experience Didactics by Boero and Douek 2008; or Semiotic Mediation by Bartolini Bussi and Mariotti 2008; or Cognitive Unity of Theorems by Boero et al. 2007) have a local scope and are usually integrated with other theoretical constructs borrowed from different theories. In the general framework of research for innovation (a kind of meta-theoretical framework), this kind of eclecticism is not only legitimate, but also favored according to the needs of didactic innovation and of an in-depth, comprehensive interpretation of what happens in the experimental classrooms. In Italy, we frequently compare mathematics education to the engineering sciences, and I find that this analogy may be rather well justified, in the Italian case. Here again we may identify the influences of an institutional context and a cultural context. Particularly at the beginning, and during the seventies and the eighties, the policy of funding oriented mathematics educators towards the improvement of mathematics teaching and the analysis of experimental situations. Moreover, in the Italian cultural context, the importance attributed to great coherent and autonomous theoretical constructions (in human sciences as well as in natural sciences) is not so strong as to orient research in that direction.

2.7 Third Entr'acte

Boero identifies an eclecticism in Italian research in mathematics education, in which ideas and tools are borrowed from disparate theories, even disciplines. Perhaps the greater maturity of *didactique* means that, as systematic relations between its constructs have come to the fore, they have taken on both an independent life and a new—almost objective—character, their roots in other disciplines transcended. In his contribution, Luis Radford reminds us of key epistemological approaches on which the youthful *didactique* drew to develop its analysis of the genesis of new knowledge. While both are, in some sense, historically informed, Radford draws our attention to two very different strands of historical epistemology, one emphasising the internal logic of a discipline as the motor of its development, another urging attention to the broader socio-cultural framing of the discipline.

2.8 Epistemology as a Research Category in Mathematics Teaching and Learning

Luis Radford

2.8.1 Introduction

In a seminal text, Artigue (1990) discusses the function of epistemological analysis in teaching. In 1995 she returns to this issue in her plenary conference delivered at the annual meeting of the Canadian Mathematics Education Study Group/Groupe canadien d'études en didactique des mathématiques. In my presentation, I draw on Artigue's ideas and inquire about the role of epistemology in mathematics teaching and learning. In particular, I ask the question about whether epistemology might be an element in understanding differences and similarities between current mathematics education theories.

As we know very well, mathematics came to occupy a predominant place in the new curriculums of the early 20th century in Europe. It is, indeed, at this moment that, in industrialised countries, the scientific training of the new generation became a social need. As Carlo Bourlet—a professor at the Conservatoire National des Arts et Métiers— noted in a conference published in 1910 in the journal *L'Enseignement Mathématique*:

> Notre rôle [celui des enseignants] est terriblement lourd, il est capital, puisqu'il s'agit de rendre possible et d'accélérer le progrès de l'Humanité toute entière. Ainsi conçu, de ce point de vue général, notre devoir nous apparaît sous un nouvel aspect. Il ne s'agit plus de l'individu, mais de la société.[3] (Bourlet 1910, p. 374)

However, if the general intention was to provide a human infrastructure with the ability to ensure the path towards progress (for it is in technological terms that the 20th century conceived of progress and development), it remains that, in practice,

[3]Our role [i.e., the teachers' role] is extremely serious, it is fundamental, because it is a matter of making possible and accelerating the progress of the whole of Humanity. Thus conceived of, from this general viewpoint, we see our duty in a new light. It is no longer a matter of the individual, but of society.

each country had to design and implement its curriculum in accordance with specific circumstances. Curriculum differences and implementation resulted, indeed, from internal tensions over political and economic issues, as well as national intellectual traditions and the way in which the school was gradually subjected to the needs of national capitalist production. These differences resulted also from different concepts of education. To give but one example, in North America, over the 20th century, the curriculum has evolved as it is pulled on one hand by a “progressive” idea of education—i.e., an education centered on the student and the discovery method—and, on the other, by ideas which organise the teaching of mathematics around mathematical content and the knowledge to be learned by the student. While proponents of the second paradigm criticise the first for the insufficiency of their discovery methods used to develop students’ basic skills in arithmetic and algebra, proponents of the first paradigm insist that, to foster real learning, children should be given the opportunity to create their own calculation strategies without instruction (Klein 2003). We see from this short example that the differences that underlie the establishment of a curriculum are far from circumstantial. They are, from the beginning, cultural. Here, they relate to how we understand the subject-object relation (the subject that learns, that is to say the student; and the object to learn, here the mathematical content) as mediated by the political, economical, and educational context. And it is within a “set of differences” in each country that the increasingly systematic reflection on the teaching and learning of mathematics resulted, in the second half of the 20th century, in the establishment of a disciplinary research field now called “mathematics education”, “didactique des mathématiques”, “matemática educativa”, “didattica della matematica”, etc.

As a result of its cultural determinations (which, of course, cannot be seen through deterministic lenses: they are determinations in a more holistic, dialectical, unpredictable sense), this disciplinary field of research cannot present itself as something homogeneous. It would be a mistake to think that the different names through which we call a discipline merely reflect a matter of language, a translation that would move smoothly from one language to another. Behind these names hide important differences, possibly irreducible, in the conception of the discipline, in the way it is practiced, in its principles, in its methods. They are, indeed, as the title of this panel indicates, research traditions.

The work of Michèle Artigue explores several dimensions of the problem posed by the teaching and learning of mathematics. In this context, I explore two of these dimensions.

The first dimension consists in going beyond the simple recognition of differences between the research traditions in mathematics teaching and learning. Artigue has played, and continues to play, a fundamental role in creating bridges between the traditions found in our discipline. She is a pioneer in the field of research that we now call connecting theories in mathematics education (e.g., Prediger et al. 2008). Artigue’s role in this field is so remarkable that there was, at the Artigue conference, a panel devoted to this field.

A second dimension that Artigue explores in her work is that of epistemology in teaching and more generally in education. She has also made a remarkable contribution to the point that there was also a panel on this topic at the conference. In what follows, I would like to briefly focus on the first dimension in light of the second. In other words, I would like to reflect on epistemology as a research category that provides insight in understanding differences and similarities in our research traditions.

2.8.2 Epistemology and Teaching

The recourse to epistemology is a central feature of the main theoretical frameworks of the French school of *didactique des mathématiques* (e.g., Brousseau 1983; Glaeser 1981). The recourse to epistemology, however, is not specific to mathematics. There is, I would say, in French culture in general, a deep interest in history. An inquiry into knowledge cannot be carried out without also raising questions about its genesis and development. In this context, one could hardly reflect on mathematical knowledge without taking into account its historical dimension. I can say that it is this passion for history that surprised me in the first place when I arrived in France in the early 1980s. In Guatemala, my native country, and perhaps in the other Latin American countries, as a result of the manner in which colonisation was conducted from the 16th century to the 19th century, history has a deeply ambiguous and disrupting meaning: it means a devastating rupture from which we will never recover and that continues to haunt the problem of the constitution of a cultural identity. In France, however, history is precisely that which gives continuity to being and knowledge—a continuity that defines what Castoriadis (1975) calls a collective imaginary. From this collective imaginary emanates, among other things, a sense of cultural belonging that not even the French revolution disrupted in France. Immediately after the French revolution, men and women certainly felt and lived differently from the pre-revolutionary period; however they continued to recognise themselves as French. With the disruption of aboriginal life in the 15th century (15th century as reckoned in accordance with the European chronology, of course, not to the aboriginal one), the aboriginal communities of the "New World" were subjected to new political, economical, and spiritual regimes that changed radically the way people recognised themselves. One may hence understand why the passion for history that I found in France was something new for me, as was also the idea of investigating knowledge through its own historical development.

The function of epistemology, however, is not as transparent and simple as it may first appear. And this function is even less transparent in the context of education. The use of epistemology in the context of education cannot be achieved without a theoretical reflection on the way in which epistemology can help educators in their research. It is precisely this reflection that Michèle Artigue undertakes in her 1990 paper in RDM and to which she returns in her plenary lecture delivered at the annual meeting of the Canadian Mathematics Education Study

Group/Groupe canadien d'études en didactique des mathématiques (Artigue 1995). Indeed, in these papers she discusses the function of epistemological analysis in teaching and identifies three aspects.

Firstly, epistemology allows one to reflect on the manner in which objects of knowledge appear in the school practice. Artigue speaks of a form of "vigilance" which means a distancing and a critical attitude towards the temptation to consider objects of knowledge in a naive, a naive non-historical way.

A second function, even more important than the first one, according to Artigue, consists of offering a means through which to understand the formation of knowledge. There is, of course, an important difference when we confront the historical production of knowledge and its social reproduction. In the case of educational institutions (e.g., schools, universities), the reproduction of knowledge is achieved within some constraints that we cannot find in the historical production of knowledge.

> Les contraintes qui gouvernent ces genèses [éducatives] ne sont pas identiques de celles qui ont gouverné la genèse historique, mais cette dernière reste néanmoins, pour le didacticien, un point d'ancrage de l'analyse didactique, sorte de promontoire d'observation, quand il s'agit d'analyser un processus d'enseignement donné, ou base de travail, s'il s'agit d'élaborer une telle genèse.[4] (Artigue 1990, p. 246).

The third function, which is not entirely independent of the first, and which is the one that gives it the most visibility to epistemology in teaching, is the one found under the idea of epistemological obstacle. Artigue wrote in 1990 that it is this notion that would come to an educator's mind if we unexpectedly asked the question of the relevance of epistemology to teaching.

Finally, the historical-epistemological analysis has undoubtedly refined itself in the last twenty years, both in its methods and in its educational applications (see, for example, Fauvel and van Maanen 2000; Barbin et al. 2008). We understand better the theoretical assumptions behind the notion of epistemological obstacle, its possibilities and its limitations.

My intention is not to enter into a detailed discussion of the notion of epistemological obstacle that educators borrow from Bachelard (1986) and that other traditions of research have integrated or adapted according to their needs (D'Amore 2004). I will limit myself to mentioning that this concept relies on a genetic conception of knowledge, that is to say a conception that explains knowledge as an entity whose nature is subject to change. Now, knowledge does not change randomly. Within the genetic conception that informs the notion of epistemological obstacle, knowledge obeys its own mechanisms. That is why, for Bachelard, the obstacle resides in the very act of knowing, it appears as a sort of "functional necessity". It is this need that Brousseau (1983, p. 178) puts forward when he says

[4]The constrains that govern these [educational] geneses are not identical to those that governed the historical genesis, but the latter remains nonetheless, for the didactician, an anchoring point, a kind of observational promontory when the question is to analyze a certain process of teaching, or a working base if the question is to elaborate such a genesis.

that the epistemological obstacles “sont ceux auxquels on ne peut, ni ne doit échapper, du fait même de leur rôle constitutif dans la connaissance visée”.[5]

This conception of knowledge as a genetic entity delimits the sense it takes in the different conceptual frameworks of the French school of didactique des mathématiques. More or less under the influence of Piaget, knowledge appears as an entity governed by adaptive mechanisms that subjects display in their inquisitive endeavours. These mechanisms are considered to be responsible for the production of operational invariants: this is the case in the theory of conceptual fields (Vergnaud 1990). As a result, this theory looks at these invariants from the learner’s perspective. But the adaptive mechanisms can also be understood differently: they can be considered as forms of action that show “satisfactory” results in front of some classes of problems. “Satisfactory” means here that they correspond to the logic of optimum or best solutions in the mathematician’s sense. This is the case in the theory of situations that looks at these forms of actions under the epistemological perspective. Beyond the boundary that defines the class of problem where knowledge shows itself to be satisfactory, these forms of action generate errors. That is to say, they behave in a way that is no longer suitable in the sense of optimal, mathematical adaptation. Knowledge encounters an obstacle. The crossing or overcoming of the obstacle ineluctably requires the appearance of new knowledge.

How far and to what extent do we find similar conceptions of knowledge in other educational research traditions? I would like to suggest that it is here where we can find a reference point that can allow us to find differences and similarities in our research traditions—sociocultural theories, critical mathematics education, socio-constructivist theories, and so on.

I mentioned above that in the genetic perspective on knowledge, the obstacle appears with a “functional necessity”. However, there are several ways to understand this need. In what follows I give two possible interpretations.

The first interpretation, and perhaps the most common, is to see this need as internal to mathematical knowledge. This would involve conceiving of mathematical knowledge as being provided, in a certain way, with its own “internal logic.” This interpretation justifies how, in the epistemological analysis, the centre of interest revolves around the content itself. Social and cultural dimensions are not excluded, but they are not really organically considered in the analysis (D’Amore et al. 2006). To use an analogy, these dimensions constitute a “peripheral axiom” which we can use or not, or use a bit if we will, without compromising the core theorems (or results) of the theory.

In the second interpretation, the development of knowledge appears intimately connected to its social, cultural and historical contexts. So we cannot conduct an epistemological analysis without attempting to show how knowledge is tied to culture, and without showing the conditions of possibility of knowledge in

[5]Epistemological obstacles “are those to which knowledge cannot and must not escape, because of their constitutive role in the target knowledge.”

historical-cultural layers that make this knowledge possible. It is here that we find Michel Foucault's conception of knowledge, whose influence in the French tradition of mathematics education has remained, surprisingly, relatively marginal.

What is important to note here is that behind these two interpretations of knowledge and its development are two different conceptions of the philosophy of history. In the first interpretation, history is intelligible in itself. In the second interpretation, history is not necessarily intelligible. To be more precise, in the first interpretation, in which the theoretical articulation goes back to Kant (1991), the conception of the history revolves around the idea of a reason that develops by self-regulation. History is reasonable in itself. There are aberrations and ruptures, of course, but if you look more closely, history appears intelligible to reason. Here, "history is a slow and painful process of improvement" (Kelly 1968, p. 362). In the second interpretation, in which theoretical articulation goes back to Marx (1998), history and reason are mutually constitutive. Their relation is dialectical. There is no regulatory, universal reason. The reason is historical and cultural. Their specific forms, what Foucault calls *epistemes*, are conditioned in a way that is not causal or mechanical, by its nesting in the social and political practices of the individuals. It is precisely the lack of such a nesting in the rationalist philosophies that Marx deplores in *The German Ideology*: "the real production of life appears as non-historical, while the historical appears as something separated from ordinary life, something extra-superterrestrial" (1998, pp. 62-63). He continues further: those theoreticians of history "merely give a history of ideas, separated from the facts and the practical development underlying them" (1998, pp. 64–65). In the Hegelian perspective (Hegel 2001) of history that Marx prolongs in his philosophical works, it is, indeed, in the socio-cultural practices that we must seek the conditions of possibility of knowledge, its viability and its limits. Reason is unpredictable and history, as such, is not intelligible in itself. It cannot be, because it depends on the reasons (always contextual and often incommensurable between each other) that generate it.

In this philosophical conception of history, what shape and role could the epistemological analysis have? And what could be its interest in different traditions of research on the teaching and learning of mathematics? Concerning the first question, one possibility is the use of a materialist hermeneutic (Bagni 2009; Jahnke 2012) that emphasises the cultural roots of knowledge (Lizcano 2009; Furinghetti and Radford 2008). Concerning the second question, the reasons already given by Artigue in the early 1990s seem to me to remain valid. These reasons can undoubtedly be refined. This refinement could be done through a reconceptualization of knowledge itself, reconceptualization that might consider the political, economical and educational elements that, as suggested previously, come to give their strength and shape to knowledge in general and to academic knowledge in particular. The topicalisation of epistemology in the different theoretical frameworks and the different traditions of research would be an anchor point to better understand their differences and similarities.

2.9 Concluding Comments

Abraham Arcavi, with the support of Takeshi Miyakawa, convincingly makes the point that establishing connections between theoretical frameworks is important for mathematics education as a scientific domain but is also very difficult, especially if these frameworks have arisen in different cultures and responded to different problématiques. A major reason for this difficulty lies in the implicit assumptions underlying the work of researchers and the questions they ask. Hence, extensive and intensive dialogues are needed to make progress. Abraham has shown a direction for such dialogue, and Takeshi has experienced it in the practice of his research in France, in the USA and in Japan. Nevertheless, such communication remains fraught with potential misunderstandings.

Jeremy Kilpatrick highlights these communicative difficulties from the point of view of "translation" in his contribution, but he shows how such translation must reach far deeper than language. A translation between cultures is involved cultures that incorporate different views of schooling and education, as well as different views about the role of theory in mathematics education research, as Paolo Boero has expounded eloquently and exemplified clearly in his contribution.

Radford takes a further step when he encourages us to follow Michèle Artigue's lead (of 25 years ago) in investigating the role of epistemology in mathematics teaching and learning. He explains how epistemology has the potential to lead beyond the mere recognition of the differences and difficulties of translation: refining the analysis of the epistemological foundations underlying different theories in different cultural contexts can lead to a deeper understanding of the differences and similarities and hence support building bridges.

The four contributors to this chapter point out that one needs a deep understanding of both cultures, the one translated from and the one translated into, in order to be able to build bridges, and they all point to Michèle as having developed such deep understanding in her own and foreign contexts of mathematics education research. In particular, Michèle's deep epistemological questioning has made an essential contribution to her being exemplary in connecting researchers from different cultures working in different paradigms.

The CERME working group on theory was mentioned repeatedly, and indeed a sustained effort at establishing deep bridges between theories has sprung from that working group and prompted a group of researchers to not only lead dialogues between theories but to look at different aspects of a classroom lesson by means of different theoretical frameworks, and to compare and connect these frameworks while trying to formulate and answer research questions. A comprehensive description of this effort has recently been published in book form (Bikner-Ahsbahs and Prediger 2014). Not surprisingly, one of the leaders in these efforts over the past decade has been Michèle.

All contributors have pointed to the central role Michèle has been playing and continues to play in many facets of mathematics education research (and practice—but that's for other chapters in this book). We cannot express it better than Abraham

Arcavi does in his piece, so we join him and, in the name of all authors of this chapter, repeat how impressed we are by her as a devoted teacher, as a "bridge builder" (between the knowledgeable and the less knowledgeable, between the French tradition and other schools of thought, between mathematicians and mathematics educators); and by the vast scope of her knowledge and wisdom.

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Chapter 3
Networking Different Theoretical Perspectives

Ivy Kidron and Angelika Bikner-Ahsbahs

In the introduction to the proceedings of the working group of CERME 4 on theories, Artigue et al. (2006) wrote

> …as a research community, we need to be aware that discussion between researchers from different research communities is insufficient to achieve networking. Collaboration between teams using different theories with different underlying assumptions is called for… (p. 1242).

As a consequence of this call for collaboration, a "networking group" coordinated by Angelika Bikner-Ahsbahs was created. Since 2006, we have collaborated in this group with Artigue and with a group of European researchers. This Networking Theories Group aims to advance the networking idea as a research practice. In our view, the networking of theories is not only another research approach. This chapter is inspired by our eight years of intense collaboration with Michèle Artigue and with the members of the Networking Theories Group. It is a result of our attempt to deal with the diversity of the theories involved in a fruitful way to advance our knowledge. This experience has been a joint experiment to empirically disclose and substantiate how theories can be brought together scientifically to solve problems with better results than if we had applied each theory on its own.

In the following sections, we describe our collaboration with Michèle Artigue and colleagues in the Networking Theories Group. The "problématique" that characterises the process of networking is described in Sect. 3.1, particularly the reasons for networking and the expected difficulties of the networking process.

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Methods and methodologies for networking are presented in Sect. 3.2. Then, the results of networking are discussed in Sect. 3.3. The final section is devoted to concluding remarks.

3.1 The "Problématique"

During the last decade researchers in mathematics education have devoted efforts to understanding how theories can be connected successfully while respecting their underlying conceptual and methodological assumptions, a process called 'networking theories'. In their introduction paper to the proceedings of the theory group at CERME 4, Artigue et al. (2006) wrote that "the central term that emerged from the working group was networking" (p. 1242):

> If we can develop and maintain a certain degree of networking between some of the advocates of the different theoretical stances that are currently evident within mathematics education, this will constitute an important step on the path towards establishing mathematics education as a scientific discipline. (Artigue et al. 2006, p. 1242)

Important questions arose from the theory group discussion:

Why networking? What are the aims of the effort of connecting theories?
What are the characteristics of the different cases of networking theories?
What are the problems which arise in the efforts of networking theories?

In the following, we attempt to answer these questions on the basis of the research undertaken by Michèle Artigue.

Why networking? What are the aims of the effort of connecting theories?

One important aim of connecting theories might be to capitalise on the benefits arising from drawing on the diversity of theories. Exploring ways of handling the diversity of theories may help researchers to better grasp the complexity of learning and teaching processes. Networking might also be a consequence of the rapid contemporary growth of different forms of communication and increasing international scientific cooperation (Radford 2008). These are without doubt all correct answers to the question 'why networking?'. Nevertheless, attempting to answer this question on the basis of the research undertaken by Artigue, we are especially interested in analysing how the networking of theories has emerged in the different paths of her research.

We first notice the importance of the epistemological dimension in Artigue's research from 1990 to 1995 (Artigue 1990, 1995). In 1990, Artigue refers to the important role of the epistemological analysis. In particular, she mentions that the epistemological analysis permits the formulation of important questions which enable mathematics educators to decide which elements of mathematical culture will be reproduced in the teaching of mathematics. At the same time, Artigue points out that it is not enough to ask questions of an epistemological nature, rather it is also important to build theoretical frameworks which will permit these questions to be adequately answered:

> Dans cette perspective, le travail du didacticien ne se limite d'ailleurs pas à intégrer ce questionnement de nature épistémologique à son activité. Il consiste aussi à construire les cadres théoriques permettant le travail sur de telles questions et la capitalisation des acquis didactiques. (Artigue 1990, p. 247)

As a consequence of Artigue's awareness of the crucial role of the theoretical dimension, beginning references to networking appear very early in her research, for example, in her research on the usage of CAS (see for example Artigue 2002). Kynigos (2012) wrote about the complexity of the instrumentation process and how Artigue found it necessary and operationally functional to draw some connections between cognition theory, cognitive ergonomics and the anthropological approach with its emphasis on institutions and praxeologies (Bosch and Chevallard 1999). We see here a beginning of networking theories long before the introduction of this term at CERME 4.

What are the characteristics of the different cases of networking theories?

As a consequence of the different aims of networking, the development of networking follows completely different profiles (Arzarello et al. 2007). We distinguish the top-down profile, in which researchers begin with different theoretical frameworks, and the bottom-up profile, in which researchers search for new theoretical tools only if the others turn out to be insufficient. As we already have described in Kidron and Bikner-Ahsbahs (2015, p. 226):

> We [thus] differentiate between different kinds of interest in the effort of networking theories. In some cases, the goal is to investigate the complementary insights that are offered when we analyze a given data with different theories (Kidron 2008; [Haspekian et al. 2013]). In some other cases, the researchers start with an empirical phenomenon with the aim of developing their understanding by means of connecting two or more different perspectives (Arzarello et al. 2009). In some further cases, the aim of networking is to satisfy the need for an enlarged framework in relation to some new domain of research (Lagrange and Monaghan 2010). In such cases, each theoretical tool turns out to be insufficient to properly analyse the data [bottom-up profile]. In other cases, the interest in the rich diversity of theories is to explore the insights offered by each theory to the others [top-down profile] and at the same time to explore the limits of such an effort. (Kidron et al. 2008; [Kidron et al. 2014; Haspekian et al. 2013])

What are the problems which arise in the efforts of networking theories?

First attempts to discuss the difficulties and benefits in networking are described in Bikner-Ahsbahs et al. (2010) and in Prediger et al. (2010). Conditions for a productive dialogue between theorists are discussed in Monaghan (2010). Kidron and Monaghan (2012) consider the complexity of dialogue between theorists. They refer to three points as the source of problems which arise in the efforts of networking theories: the relevance of data; the different priorities of each theory with regard to the focus of analysis; and the differences in the use of language in different theories.

A first source of difficulty in networking theories is that the same piece of data might be relevant for one theory but not relevant for another:

> It is through a methodological design that data is first produced; then the methodology helps the researcher to "select" some data among the data that was produced but also helps the researcher to "forget" or to leave some other data unattended. (Radford 2008, p. 321)

In our own 'networking experience' with Michele Artigue and other colleagues, the question of data relevance was crucial. This is described in Kidron et al. (2008) with the networking of three theories: Theory of Didactic situations (TDS), Abstraction in Context (AiC), and Interest Dense Situations (IDS). The data required to complete the appropriate analysis from the point of view of each framework was different. This is one example among others in which the notion of "minimal unit of analysis" (Artigue et al. 2012) is raised; especially the questions: What constitutes a significant unit for a didactic analysis? What are the minimal units of reality considered as pertinent in a given research paradigm in order to permit the analysis of the observed facts?

The different priorities of each theory with regard to the focus of analysis are another source of difficulty. There might be different questions of interest for different frameworks. This is illustrated in Kidron et al. (2014) in which three theories were involved: TDS and AiC (which we have already mentioned), and a third theory, the Anthropological Theory of the Didactic (ATD). The authors explain the meanings of the terms 'context' (for AiC), 'milieu' (for TDS), and 'media-milieus dialectic' (for ATD), each being a cornerstone for the theory while all attempting to theorise specific contextual elements.

> …in relation to the role of the teacher, TDS researchers might ask what milieu the teacher is making available to the students and how she is managing its evolution in order to establish a meaningful connection with the mathematical knowledge aimed at. AiC researchers might ask how the teacher's intervention influences the students' construction process as described by means of the RBC epistemic actions. ATD researchers in their turn might ask what responsibilities the teacher and the students are assuming in the media-milieus dialectics and what conditions enable them to manage it. (Kidron et al. 2014, p. 159)

Kidron et al. (2008) analyse how social interactions are viewed in three different theories. For the three theories (TDS, IDS and AiC), social interactions are important. Nevertheless, each theory has different priorities with regard to the focus of analysis. Therefore, the ways the theories view social interactions are different. For IDS researchers, it is the learning itself; for AiC researchers it is considered to be a part of the context; and for TDS researchers, contextual factors are part of the situation, the system of relationships between students, teacher and mathematics.

The differences in the use of language in different theories are also a source of difficulty. We note, especially, the plurality of meanings of a single word. Even the term 'epistemic actions' is used differently in AiC and IDS. Also, the term 'milieu' in ATD is not equivalent to the 'a didactic milieu' which is used in TDS.

3.2 Methods and Methodologies of Networking

Michèle Artigue has contributed in different ways to methodologies of the networking of theories. In her early work on didactical engineering (DE) (Artigue 1994), she combined two theories: The theory of didactical transposition (Chevallard 1991, 1992) and the theory of didactic situations (Brousseau 1986) which both "shape and determine, to a certain extent, the current approach [of didactical engineering as a framework for the conceptions of teaching products]" (Artigue 1994, p. 27). Artigue emphasised that both theories share a systemic approach to didactical engineering: "[…] despite their different focuses of interest, these two theories link up on one essential point related to our topic [didactical engineering]: They emphasize the need to envisage the study of didactical phenomena within a systemic approach" (p. 28). Due to their complementarity, the two approaches can fruitfully be networked in the methodology of didactical engineering. This was substantiated by Artigue in that she compared and contrasted both theories according to their strengths and weaknesses in contributing to the methodology of didactical engineering and finally proposed to link both, for example, by including constraints into the a priori analysis that stem from both theories: Constraints of an epistemological and of a cognitive nature referring to a student who interacts with the milieu were complemented by "constraints of a didactical nature" that refer to the institutional dimension of mathematical knowledge (Artigue 1994, p. 32.ff.). Note that this work on theories is not uncommon for French researchers, the field of didactics being much concerned by theoretical consistency.

This methodological direction has been further developed by Artigue to investigate the use of technology in building mathematical concepts (Artigue 2002). Although Artigue did not talk about the networking of theories at that time, the research she reported on and the development of the instrumental approach mirrors its essence: The necessity of looking to institutional conditions regarding the use of CAS calculators as well as to students' conceptualisation led to the coordination of the Anthropological Theory of the Didactic and the notions of instrument and instrumental genesis taken from cognitive ergonomics. This coordination being implemented into didactical engineering as a research methodology can be regarded as an early piece of the networking of two theories.

In a recent paper, Artigue (2015) traces the origin of DE back to the early eighties and describes its development in parallel with the development of the theory of didactical situations on the one hand and of the anthropological theory of the didactic on the other. She identifies two directions in which DE has expanded and changed. By referring to Barquero (2009), she indicates a change of the DE by its application to studying more open problems incorporated into the Anthropological Theory of the Didactic by a new paradigm. By referring to Perrin-Glorian (2011), Artigue proposes DE of second generation as a way of researching how DE can be used to implement a design into the practice of schooling. She finishes her reflection with the claim: "[…] that the transition from research to development needs specific forms of research, extending our view of the ways didactical engineering and educational research can be connected" (Artigue 2015, p. 493). This foreshadows a possible direction in which networking methodologies may develop to match theoretical and practical concerns. Interestingly, the idea of networking methodologies was raised at the last CERME conference in the working group on theoretical perspectives. In one of the sessions, methodology was an explicit focus (Hickman and Monaghan 2013). Questions about the place of methodology in theoretical frameworks and the relationship of this issue to the methodology for networking theories were discussed.

The investigation of the role of theories is explicitly addressed in the Technology Enhanced learning in Mathematics (TELMA) project by the method of cross-experimentation (Cerulli et al. 2008). The starting point was "a need to get a deeper insight on the role played by the theoretical frameworks each team uses in its own research" (p. 202). The researchers used the idea of experimenting "with a dynamic digital artifact [DDA] developed by another team, in another educational context, and under other theoretical perspectives" (Artigue and Mariotti 2014, p. 335). This method allowed the teams to explore how far the theories and constructs involved in the design process may influence its experimentation and what kind of constraints the alien user experienced. In their final reflection, the authors valued cross-experimentation as a "joint methodology to help different developing and experimenting teams to make explicit their assumptions and the set up of their experimental investigations" (Cerulli et al. 2008, p. 212).

This method of cross-experimentation has been further developed in two directions. In the ReMath project, it is used again as a method for educational ICT research but explicitly "developed and consolidated as a networking methodology" (Artigue and Mariotti 2014, p. 333) to enrich the process of theorising. This was in contrast to TELMA, that focused more on the a posteriori analysis of the experiments in the ReMath project, and also the "design of […] DDAs and the scenarios of use became an essential dimension" (p. 333). One interesting result of this enterprise was the deepened insight that instrumentation can be regarded as a boundary object which allows interpretation from different theoretical points of view.

Both the authors of this chapter worked with Tommy Dreyfus on a joint project titled "effective knowledge construction in interest-dense situations". In this project, cross-experimentation was also adopted as an idea to develop cross-methodologies that seemed typical of the methodologies normally used in networking projects (Kidron and Bikner-Ahsbahs 2015; Bikner-Ahsbahs and Kidron 2015). For example, in the above-mentioned project, a cross-task design began with separate designs of tasks, which were exchanged, piloted and analysed. The results were exchanged again and a revision of the tasks was initiated; finally, the teams decided upon common tasks which could be used by the different theory teams in the networking approach of research. Similarly, cross-analyses of a common dataset started with separate analyses, followed by exchange of the results, co-revisions of the analyses and by exchange again. The final step aimed to merge the analyses by the use of networking strategies (see for example, Prediger et al. 2008) as far as possible towards integration.

3.3 The Results of Networking

As we began our networking experience, we realised the difficulties that can accompany the networking process. Sometimes, it seemed that it might be impossible to use our own theory to analyse a transcript obtained from another research project that had been designed within another theoretical perspective. After our networking experience we discovered that it is possible and that this cross analysis of a transcript is a very rewarding and enriching effort. We have learnt that the difficulties point to the benefits. As an example of the results of networking, we realised that trying to define the data each theorist missed allowed us not only to better understand the other theory, but also to better understand our own theory with its basic assumptions. Assumptions are sometimes tacit: "even researchers who are quite explicit about the theoretical frameworks they use, are usually not explicit about, and can even be unaware of the assumptions underlying their theoretical approach" (Artigue et al. 2006, p. 1241). In our experience of the networking of theories, a specificity of this approach became apparent: its strengths in disclosing tacit assumptions.

Entering a dialogue between theorists is not an easy task. It requires what Artigue calls "an effort of decentration", that is, an ability to go beyond our own focus of analysis in trying to understand our respective didactical cultures, and to identify interesting similarities and complementarities between our perspectives as well as boundary objects that could support connections. As a consequence of this effort we may better see the limits of our respective theoretical lenses and also what could be gained from networking.

In the following section, we describe the results of networking from our collaboration with Artigue and colleagues within the Networking Theories Group (mentioned earlier in the Introduction section). Some of these results are described in the book Networking of theories as a research practice in mathematics education (Bikner-Ahsbahs, Prediger and The Networking Theories Group 2014). This book results from our fruitful collaboration during the last decade. Five theories were introduced in the networking process: APC; the theory of Action, Production and Communication (Arzarello and Sabena 2014); TDS, the Theory of Didactical Situations (Artigue et al. 2014); ATD, the Anthropological Theory of the Didactic (Bosch and Gascón 2014); AIC, the theory of Abstraction in Context (Dreyfus and Kidron 2014); and IDS, the theory of Interest-Dense Situations (Bikner-Ahsbahs and Halverscheid 2014). We also collaborated with Artigue in two cases studies.

In one case study, the networking was conducted between IDS and TDS (Bikner-Ahsbahs et al. 2014). Inspired by Ferdinando Arzarello, both groups of researchers immediately identified phenomena well known in the traditions the researchers normally work in from the same set of video data. Bikner-Ahsbahs, coming from social constructivism as established by Bauersfeld and his colleagues, observed a phenomenon called funnel (interaction) pattern. Artigue and Haspekian, as TDS-researchers, identified a Topaze effect. The identification of different phenomena in the same set of data was a surprise and initiated a networking process resulting in further conceptualising the phenomena. Based on the definitions of the two phenomena, separate analyses were carried out and resulted in the insight that the data neither showed a complete Topaze effect nor a complete funnel pattern. A subsequent literature review carried out by Artigue revealed that:

> The Topaze effect naturalized too quickly in the sense that it was so directly adopted and integrated in didactic analyses that the notion remained somehow not sufficiently worked out. (Bikner-Ahsbahs et al. 2014, p. 209)

Comparing and contrasting both phenomena and their theoretical backgrounds by analysing the same data allowed the researchers to discover a common underlying theme about teaching and learning in classrooms: Teachers and students sometimes attempt to hold on to the fiction that mathematics is learnt although understanding the mathematical content is missing or only takes place in a superficial way. The two phenomena capture this fiction but in different ways. The funnel pattern describes it as a pattern of social interaction based on routine actions, whereas the Topaze effect, coming from a strong epistemological background, focuses more on the mathematics to be learnt and explains this kind of fiction as a specific attempt to avoid breaking the didactic contract. A coordinated reflection on the networking process revealed that the theoretical understanding of the two theories involved was progressing with respect to being more explicit about their strengths and weaknesses and allowed re-conceptualizing the two phenomena as extreme types that only partly appeared in the data. The additional value of the new status of the two phenomena as extreme types mirrors the methodological strength of networking, in that analysing the difference between these extreme types and the empirical situations allowed clarification as to why the data conformed to neither a Topaze effect nor a funnel pattern.

In the second case study, the networking was conducted between AiC, TDS and ATD (Kidron et al. 2014). The three theoretical approaches are sensitive to issues of context but, due to their differences in focus, context is not theorised and treated in the same way in each of the three theories. In this networking case, the authors observed how the dialogue between the three theories appeared as a progressive enlargement of the focus, demonstrating the complementarity of the approaches and the reciprocal enrichment, without losing what is specific in each one. As a consequence of the networking, the authors observed how the three theories complement each other. AiC offers a fine-grained analysis of the students' epistemic processes and makes subtle evolutions visible in the process of construction of knowledge. TDS and ATD offer to AiC the benefits of a more systematic engagement in a priori analysis for anticipating the possible effect of contextual characteristics on epistemic actions.

The analyses illustrate the differences between the three theories as well as the shared epistemological sensitivity which can be noticed in the a priori analyses provided by each theory. This epistemological sensitivity was the point of contact which permitted the dialogue between the theories to begin. The three theories share the aim of understanding the epistemological nature of the episode, while, at the same time, each theory accesses data in its own ways. For AiC researchers, the focus is on the epistemic process itself whereas TDS and ATD researchers are interested in the question of how this process is made possible.

We have already mentioned another collaboration from both chapter authors with Artigue and colleagues on networking theoretical approaches (Kidron et al. 2008) in which the focus is on how each of the three frameworks—TDS, AiC and IDS—take into account social interactions in learning processes. Observing how each theoretical framework has its own way of considering the role of social interactions in the learning process, the authors point out that:

> […] the different views the three theories have in relation to social interactions force us to reconsider the theories in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks. (Kidron et al. 2008, p. 253)

Reconsidering the theories "in all their details", the authors ask how it is possible to establish links between the theoretical approaches without becoming embroiled in contradictions between the basic assumptions underlying each theory. The authors were aware of the substantial difficulties involved in the attempt to connect the theoretical approaches. Therefore, they raised the question: What can (and what cannot) be the possible aims of such an effort? The authors concluded that the aim was to develop meta-theoretical tools able to support the communication between the different theoretical languages, which enable researchers to benefit from their theories' complementarities.

In summarising the results of networking, we pay attention to contributions to the CERME groups on theory. We have already mentioned the CERME 4 theory group (Artigue et al. 2006) in which the crucial role of the theoretical dimension was analysed. The work was continued at CERME 5 in Artigue's plenary, where she used digital technologies as a window on raising theoretical issues in mathematics education (Artigue 2007). At CERME 7, Artigue et al. (2012) considered the potential offered by the anthropological theory of the didactic (ATD) for addressing the issue of networking between theories through the extension of the notion of praxeology—which is at the core of ATD—from mathematical and didactical praxeologies to research praxeologies. Extending the notion of praxeology results in "networking praxeologies, which can be situated at a meta-level with respect to ordinary research praxeologies" (Artigue and Mariotti 2014, p. 352). These considerations directed the participants at the CERME 7 group to the challenging task of networking 'approaches for networking theories' (Kidron et al. 2012).

Artigue's long journey with respect to networking includes important projects with other researchers, such as the TELMA and ReMath projects that we mentioned in the previous section on methods and methodologies in the networking process. The ReMath enterprise is analysed in a special issue of Educational Studies in Mathematics (Kynigos and Lagrange 2014). In an article of this issue, Artigue and Mariotti (2014) describe the methodological constructs that have been developed and used in ReMath and the results of the networking activity: The identification of possible connections and complementarities between frameworks, the identification and elaboration of boundary objects between cultures, and the progressive building of a shared theoretical framework regarding semiotic representations.

3.4 Concluding Remarks

Exploring issues in the networking of theories is not a purely academic exercise. As Michèle Artigue described in her comment during the colloquium in Paris, the main issue is to solve problems in mathematics education. Therefore, the networking of theories is explored in order to find out how far this approach assists in finding solutions or further clarifying problems. We share this point of view and aspire to better understand the construction of mathematical knowledge towards the end result of enriching learners' mathematical experiences. After networking and enriching our own theoretical frameworks, we need to consider how to apply our work to solve problems which occur in the classroom. This new approach will provide us with strategies and methodologies to take into account different kinds of milieus and to answer research questions focusing on the teaching and learning of mathematics.

Meanwhile, the enterprise of the networking of theories has attracted many more researchers. These researchers not only research collaboratively but also introduce a further diversity of theories. However, the novelty of this area of research includes the practical block of networking praxeologies, both tasks and techniques, which are not yet well established; they are rather art-craft objects whose potential needs to be tested in action, and progressively refined (Artigue and Mariotti 2014, p. 251).

In conclusion, research using the networking of theories approach seems to provide an interesting potential for advancing our field, but it is still at the early stages. Combining different perspectives such as cognitive, epistemological, socio-cultural, and including semiotic and institutional aspects is a real challenge. In Artigue's work, we observe how the way she integrates her experience in different domains of research facilitates the fruitful interaction between these perspectives in the networking process. As mentioned by Radford (in this book), "Artigue has played, and continues to play, a fundamental role in creating bridges between the traditions found in our discipline. She is a pioneer in the field of research that we now call connecting theories in mathematics education".

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Chapter 4
Three Perspectives on the Issue of Theoretical Diversity

Brigitte Grugeon-Allys, Juan D. Godino and Corine Castela

4.1 Introduction

Since the birth of the didactics of mathematics in the 1970s, the research community has aimed to build theories that may be used as models for studying phenomena in the teaching and learning of mathematics, within a *milieu* designed for their teaching. A survey of the literature reveals the development and multiplication, both in France and abroad, of a great number of theories for appreciating complex and multifarious phenomena in many different cultures, examined according to a variety of inputs and levels of analysis. These theories involve an interplay of different didactic concepts and tools. The creativity of researchers has created certain problems in the community, such as 'internal difficulties relating to the communication and capitalisation of knowledge, and external difficulties when holding discussions with other communities, explaining the state of the art on a specific topic to non-specialists, or guiding didactic efforts in a well-argued manner' (Artigue 2009, p. 307). Over the last decade, researchers have shared thoughts and views on the relationships between these theories at such events as the CERME congress[1] and in European projects such

[1] CERME: Congress of European Research in Mathematics Education.

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as Technology Enhanced Learning in Mathematics (TELMA) and ReMath. This is clearly reflected in the publication of special journal issues (Morgan and Kanes 2014; Prediger et al. 2008a; Skott et al. 2013) and books such as the Networking of Theories in Mathematics Education (Bikner-Ahsbahs and Prediger 2014). Michèle Artigue's active participation in all these endeavours warrants the devotion of an entire chapter to this theme.

This chapter proposes three different perspectives on this topic, expressed in the first person. The first of these examines the richness of a multidimensional approach based on the mobilisation and networking of various well-identified theories, enabling a segmentation of reality that is suited to the study of didactic phenomena. To do so, Grugeon-Allys highlights the identity and limitations of didactic theories developed in France during the early days of research in the didactics in mathematics, along with their functionality and complementarity. The second perspective defends a possible methodology for reducing theoretical diversity. Godino does this by drawing on the theory known as EOS (Entidades primarias de la ontología y epistemología). The third perspective is a contribution by Castela, who examines a social viewpoint of the multiplicity of theories in the didactics of mathematics and the search for connections. After considering networking based on the notion of praxeology, she proposes some new perspectives by borrowing from the anthropological theory of didactics and the field theory applied to science respectively.

4.2 The Richness of Didactic Theories and Their Networking for a Multidimensional Approach of Didactic Phenomena

I will begin with a point of view developed by Artigue (2009), whereby 'a given theory cannot claim to encompass everything and explain everything' and 'the coherence and strength of a theory lies primarily in what it relinquishes' (Ibidem, p. 309). Artigue singles out two guiding principles in her reflections:

- 'the consideration of the relations between theoretical frameworks cannot be achieved without identifying and respecting their respective coherencies and limitations',
- '[one should] examine theories and their development in terms of functionalities, tracing theoretical objects to the needs that these objects fulfill or at least attempt to fulfill' (Artigue 2009, pp. 309–310).

In order to determine the identity of theories and their boundaries, Radford (2008) compares the systems of principles, the methodologies and the types of research questions on which they are based.[2] This constitutes a means of deepening

[2]A theory can be seen as a way of producing understandings and ways of action based on: A system, *P*, of *basic principles,* which includes implicit views and explicit statements that delineate

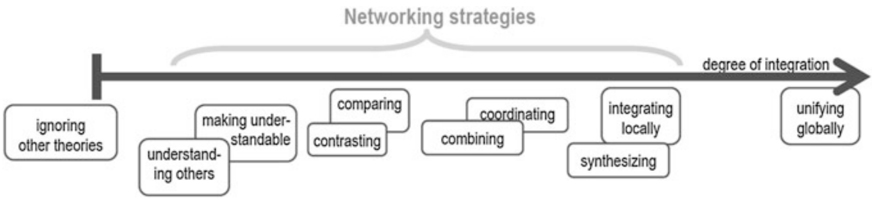

Fig. 4.1 Landscape of strategies for connecting theoretical approaches (Prediger et al. 2008b, p. 170)

the understanding of theories in relation to their research paradigms, and investigating their limitations and potential connections. It defines the boundary of a theory as 'the "edge" that a theory cannot cross without a substantial loss of its own identity. (…) behind such an edge, the theory conflicts with its own principles' (Radford 2008, p. 323). To determine the types of connections between theories, Radford studies the structure and aim of the connection. He considers the theory-networking scale (Prediger et al. 2008b) that distinguishes between several connection types (Ibidem, p. 318). 'Comparing' consists of searching for similarities and differences, whereas 'contrasting' has to do with emphasising differences. 'Coordinating theories' amounts to selecting coherent elements from different theories in order to investigate certain research problems. 'Combining theories' tends to involve juxtaposition.

To take this point of view further, I shall hypothesise that a multidimensional approach based on the mobilisation and networking of well-identified theories—each of which uses a particular conceptual filter to segment reality and mobilise its objects of study—can be conducive to the study of complex didactic phenomena. The connection between theories can take several forms and involve varying degrees of integration, culminating with the local level.

To argue in favour of the contributions of a multidimensional approach, I will examine the salient features of the identity and limitations of the founding theories that emerged in the field of research in the didactics of mathematics in France from the 1970s onwards.

(Footnote 2 continued)

the frontier of what will be the universe of discourse and the adopted research perspective; and a *methodology*, *M*, which includes techniques of data collection and data interpretation as supported by P. A set, *Q*, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified) (Radford 2008, p. 320).

4.2.1 *Main Theories: The Theory of Conceptual Fields, the Theory of Didactic Situations and the Anthropological Theory of Didactics*

These theories aim to study, describe and explain the processes of the teaching and learning of mathematics by assuming that the very nature of mathematical *savoir* and practices influences these processes. They share the following principles: the fundamental role of mathematical *savoir* and its epistemology, and the role of problem solving for the learning and teaching of mathematics. These theories developed according to the historical, scientific and cultural conditions of the 1970s as well as the individual background of the researchers, in response to research questions and methodological choices that led the researchers to distinguish their theory from other theories. I will not attempt to present in a few words the fundamental ideas or the concepts developed by each theory, as this would be impossible. I only wish to shed light on certain elements that lie at the core of each of the theories, which constitute their strength and delineate the boundaries between them, at a given stage in the development of research in the didactics of mathematics.

4.2.1.1 A Theory Centred on the Modeling of Knowledge Acquisition

The Theory of Conceptual Fields (TCF) developed by Vergnaud is centred on the modeling of knowledge acquisition. Vergnaud endeavours to bring research in developmental cognitive psychology closer to issues pertaining to teaching and academic learning. Two major concerns form the backbone of TCF.

First, Vergnaud makes a strong hypothesis: that of a link between the creation of *connaissance* and the structure of mathematical *savoir*. He defends the idea that it is essential to study mathematical concepts in relation to the situations that will enable their conceptualisation. To him, one of the major challenges in research in didactics in the 1980s was to characterise and classify problems, in the psychological sense of the term,[3] that give a concept its meaning and function. Thus, Vergnaud (1990) models a 'concept' by means of a 'triplet of three sets (S, I, L): the set S of situations that give meaning to the concept, the set I of invariants that form the basis of the operationality of schemes (the signified, or *signifié*), and the set L of linguistic and non-linguistic forms that enable the symbolic representation of the concept, its properties, the situations and the handling procedures (the signifier, or *signifiant*) (Ibidem, p. 61, personal translation). The conceptualisation of a given concept, built by a given student, at a given moment, corresponds to a set of situations in which

[3]'By problem, I mean (…) all situations in which one must discover the relations and develop exploration, hypothesis and verification activities in order to produce a solution' (Vergnaud 1986 p. 52). In this paper, Vergnaud uses the terms 'problem', 'situation' and 'situation-problem' interchangeably.

the concept is applied, the representations that enable it to be represented, the invariants, rules of action and acting attributes that appear in the course of the associated activity through procedures, as well as the perceived scope of validity; hence the disparities between students' conceptions and the targeted concepts that are taught.

This point of view leads Vergnaud to consider that one cannot comprehend the development of a concept without setting it in the context of a long time period in a much broader system that is directly linked to the mathematical content of the problems[4]—a conceptual field. A conceptual field encompasses a set of situations that is progressively mastered by reference to several concepts and the set of concepts that contribute to the mastery of the situations. To understand how a student develops and adapts a concept requires a segmentation of mathematical *savoir* into relatively large domains in order to study the long-term evolution of such processes through a set of relatively diverse situations.

In TCF, students are psychological subjects.[5] The questions that are studied are connected to the object of study: students' conceptualisation processes over the long term. The methodology is anchored in cognitive psychology approaches, centred on the study of schemas developed by students during problem solving, which calls for a microscopic scale of analysis. Vergnaud focuses—from the student's and teacher's perspective—on the role of the structure of problems and carries out classifications. But at the time of writing, he was not interested in the conditions for putting situations into practice in the classroom, the role of the teacher in the management of situations and interactions, or even the constraints to be taken into account in the educational system.

4.2.1.2 A Theory Centred on the Conditions of the Operation of Situations in an Educational System

In the theory of didactic situations (TDS), Brousseau (1986, 1998) develops conceptual tools for understanding what is at play in the classroom from a mathematical standpoint by focusing the study around the didactic situation, which is initially defined as a system of relationships between some students, a teacher and a mathematical *savoir*. The object of study in TDS corresponds to the conditions in which a teaching system may bring about optimal development of students' *connaissances* in relation to an existing *savoir* within an educational system. TDS studies didactic phenomena that involve interactions between the *savoir*, the students and the teacher as a whole. The didactic modelling of situations consists first and foremost in matching, to a given *savoir*, a minimal class of adidactic situations

[4]'The description of general stages of development such as those described by Piaget or other developmental psychologists does not enable us to understand the development of *compétences–connaissances* involved in problems' (Vergnaud 1986, p. 56).

[5]Vergnaud seems to use the terms 'student' and 'child' independently of each other.

that make this *connaissance* appear to be the optimal and independent means of solution in these situations with regard to the *milieu*. This class of situations can be created by a game involving the cognitive variables[6] and the didactic variables of a fundamental situation. The 'Race to 20' is a prototypical example of this for Euclidian division. Brousseau (1986) defines conceptual tools for understanding the adidactic situations: situations of action, formulation and validation, as well as devolution and institutionalisation. Of utmost importance to him is the belief that meaningful learning of mathematics cannot be achieved if there is too much reliance on the teacher for problem solving. This explains the importance placed on the concepts of devolution and *milieu*, and on the duality between didactic and adidactic situations.

4.2.1.3 A Theory Focused on Institutions

The Anthropological Theory of Didactics (ATD) shifts the focus on research of didactic transposition, the *savoir savant* or scholarly knowledge questions and situations towards the institutions in which the situations 'live'. It studies new and broader questions. Indeed, Chevallard adopts a perspective of epistemological and institutional emancipation in relation to the institutions where the objects of *savoir* studied the didactics of mathematics 'live'. A *savoir* has not always existed: it is the result of human activities and depends on their position and function, according to the place, society and time period. In the educational system, the *savoir* to be taught should not be considered to be transparent as it is related to the institution in which it is taught. Chevallard (1985) distinguishes, through the process produced by mathematicians from the *savoir à enseigner* or knowledge to be taught in an institution, the *savoir enseigné* or knowledge taught by the teacher, and the *savoir appris* or knowledge learnt by the student. A student's learning is also determined by the institution where he or she is learning. ATD is a tool for modelling and analysing students' and teachers' activities in teaching institutions that enables an appreciation of the implicit constraints (*assujettissements*) at work. Chevallard (1992, 1999) develops the notion of praxeology to describe the creation and evolution of objects of *savoir* within an institution, as well as the institutional and personal relationships to these objects. The notion of praxeology encompasses on the one hand the types of tasks and the techniques for accomplishing them, the praxis, and on the other hand the discourse known as technology that justifies a technique and renders it intelligible along with the theory that, in turn, justifies this technology and renders it comprehensible, the logos. This notion provides a tool for analysing the structure of teaching in different institutions, and is particularly useful for understanding the transitions between two institutions in the different stages of didactic transposition. One of the most crucial contributions of ATD is the

[6]For example, those related to the categories of problems in conceptual fields pertaining to the *savoir* in question.

definition, prior to any study, of the reference epistemological model relating to a given *savoir* that forms the basis for the analysis of transposition phenomena. As Brousseau points out, 'this "anthropological" approach is perfectly in line with the theory of situations and completes it. It allows more direct access to a certain number of problems, especially those pertaining to macrodidactics and the relation to *savoir*' (Brousseau 2003).

Having reached the end of this first section, I would like to emphasise, apart from the usual issues, the following points based on the fields of action prioritised in these theories: the modelling of students' acquisition of *connaissances* in TCF, the modelling of adidactic situations as an optimal means of access to *savoir* in TDS, the creation and evolution of a *savoir* in different institutions, and the relationships to this *savoir* in ATD. These choices give rise to distinct methodologies at different levels of analysis, each with its functionalities—but also its limitations—for addressing new research questions.

4.2.2 Functionalities of Theories

I will now examine the importance and richness of these theories for studying new research questions and take into account different aspects of the didactic phenomena involved by distinguishing between dimensions at different levels of analysis. For any research and segmentation, the consideration of different dimensions can lead to subsegments, each of which is associated with a theoretical framework and an appropriate methodology. I will base my investigation on research pertaining to different ways of connecting elements (Fig. 3.1) derived from TCF, TDS and ATD.

4.2.2.1 Dynamics of a Multidimensional Approach: Complementarity Between Theories and Evolution of Research Questions

Here, I will illustrate the dynamics of a multidimensional approach by contrasting the differences between two theories, then coordinating them, according to two main inputs: the student and the institution.

The investigation focuses on the transition problems and notably the institutional discontinuities that are often involved in these transitions. We consider a study that examines the difficulties faced by 16- and 17-year-old students from 'lycées professionnels (LP)', who are among the best academically and yet fail in specially tailored adaptation classes aimed at preparing them for further education in 'lycées technologiques (LT)' (Grugeon 1997). One *savoir* lies at the core of this failure: elementary algebra. The frequent explanations for this failure are of a cognitive kind: the students' difficulties stem from their standard in knowledge of mathematics. A cognitive approach serves this type of reasoning and its associated conclusions. Adopting an anthropological approach enables us to overcome such negativity, view the cognitive through the institutional filter and put the problem being studied in a wider perspective encompassing all the transition problems. Thus, the contrasting of these two approaches encourages us to explore a new hypothesis: as the two institutions developed different institutional relationships to algebraic objects, the difficulties observed could be explained by the inadequacy of the personal relationships developed under the influence of the first institution, with respect to the expectations of the second institution. In order to carry out the investigation, I considered it necessary to define a 'sort of reference, independent of the institutions involved, yet positioned within their field of action (…), then build a (multidimensional) analysis framework based on this definition for analysing the institutional and personal relations' (Grugeon 1997, p. 170). This reference forms the basis of an epistemological analysis of the various relationships to algebra, through the types of problems in the algebraic domain, the algebraic objects and their properties, as well as the modes of representation used to solve them: it has been developed from a summary of research work carried out in the didactics of algebra (Chevallard 1985; Kieran 2007).

This study identified discontinuities in the institutions' programmes, which a more superficial examination would not only have failed to find, but which could also have led to misapprehensions. The analysis reveals that the dominant institutional relationship to algebra in LP is mainly structured around the use of formulae (for calculating rates or loan repayments), for example for writing equations, whereas in LT, it is focused on equation writing, equations and functions.

The results of this research clearly demonstrate that 'the relevance and importance of a theoretical construction are closely tied to the manner in which it forces us to change our filters, rendering visible what was previously invisible, forcing us to question our spontaneous interpretations, making apparently erratic, incoherent or counterproductive behaviour rational and understandable, in a word, changing our vision of the world and pointing out where we should direct our energy' (Artigue 2009, p. 320).

4.2.2.2 Scope and Adaptability of Theoretical Frameworks

I will now deal with the scope of theories and their adaptability, especially with regard to the conditions in which TDS may be used in the study of regular sessions. This involves a 'coordinating'-type connection between two theories that will be illustrated based on research into the issues surrounding 'regular' classes (Perrin-Glorian 1999; Perrin-Glorian and Hersant 2003). I will refer to certain points that were featured in Perrin-Glorian's contribution at the Artigue Colloquium.[7] Perrin-Glorian formulated the general problem in these terms: 'how to define situation when observing a sequence of sessions on proportionality in 6e [i.e. the first year of French secondary school] that one has not prepared?', unlike

[7]For all references to contributions to the Artigue Colloquium, the reader may refer to https://sites.google.com/site/colloqueartigue/short-proceedings.

what occurs in didactic engineering. Perrin-Glorian and Hersant (2003) propose reconstructing such situations after observing the sessions and analysing them a priori. In order to access the dynamics of the teaching occurring in regular classes, the implemented methodology segments reality by conducting successive zoom-ins that allow an appreciation of different scales of analysis, and aims to link the various levels involved: the microdidactic level of the situation and interactions, the local level of the teacher's long-term plan, and the macro level of the *savoir* and the institutional constraints based on programmes and manuals. This segmentation has led Perrin-Glorian and Hersant to call upon TDS concepts and tools at the local or microdidactic level, and ATD concepts and tools at the macroscopic level. ATD is used to analyse *savoir* and institutional constraints, by making a distinction between the various institutions, based on an epistemological construction of the *savoir* in question. At the local level of the session, TDS enables us to study the situations that come into play in the sessions considered to be of significance with regard to the didactic aim of the series. The definition of a situation as a game is primarily dependent on three components and their function in the situation: the didactic aim (targeted new *connaissances*), the material *milieu* that is or is not set up, the problem and the rules of the game (especially on how to win). The didactic contract is what enables us to interpret the game, whether from the perspective of the teacher or the students, by distinguishing between the *milieu*-related conditions that can be changed and those that cannot be changed by virtue of being connected to institutional constraints. The teacher's game consists of organising and regulating the student's game, and in leading the student to identify and formulate the *connaissances* that are necessary to win. At the microdidactic level, the analysis of significant episodes relies on the examination of feedback from the *milieu* and the evolution of the didactic contract. Here, Perrin-Glorian points out the distinction between a psychological subject and a social subject. Indeed, the study of the adidactic situation and the anticipation of procedures for solving it take into account the cognitive dimension of the epistemic subject. The social dimension is brought in at the microscopic level to study the interactions within the classroom, and at the macroscopic level to place didactic issues within social issues. Perrin-Glorian and Hersant (2003) were led to define new dimensions for analysing didactic contract, notably the area of mathematics concerned, the status of the *savoir* (degree of institutionalisation and familiarity) to the students with regard to tool–object dialectic (Douady 1986), and the properties of the *milieu* (possibilities of feedback that are open to interpretation by the students).

From this we may observe yet another illustration of the importance, richness and scope of the theories that highlights, at different levels, the dynamics of the coordination between TDS and ATD for dealing with a new research question: 'TDS manages the local level, especially matters pertaining to didactic engineering, whereas ATD manages the global level and the study of institutional relations by studying official documents, manuals and various teaching resources' (Artigue 2009, p. 314). Researchers use the complementarities between these theories as levers for conducting research on appropriate segmentations.

4.2.2.3 Different Cultures and Common Sensitivities: Contributions of a Comparison Between Theories

In this section, I would like to demonstrate how the contrasting of theories can enrich the study of a given theme, by bringing in different approaches and varied perspectives. I will refer to the Theory of Semiotic Mediation (TSM) (Bartolini Bussi and Mariotti 2008), which was developed in Italy based on Activity theory. I will examine the respective sensitivities of TDS and TSM to the sociocultural dimension of learning and to its semiotic dimension. Are they different? If the answer is yes, how do the didactic analyses and choices differ and do they enhance the relationship with these dimensions?

To do so, I will draw on certain points discussed in Mariotti's talk at the Artigue Colloquium. Mariotti refers to crossover experiments conducted between French researchers and Italian researchers in the ReMath project. Didactic analyses and choices differ significantly depending on whether the sociocultural sensitivity leans towards TDS or TSM. TDS is based on the a priori analysis of situations, students' actions and feedback from the *milieu*, with students being considered as epistemic subjects (cf. Sect. 4.2.1.2). A posteriori analysis examines the realisation of situations. The focus is, on the one hand, on the relationships between the *connaissances* that are called upon in the situation as well as the interaction between the actions and feedback of the *milieu*, and on the other hand, on the institutional techniques and technologies brought into play. By adopting the TSM perspective (Bartolini Bussi and Mariotti 2008), the priority is not to analyse either the productivity of the scenarios built around tools perceived to be sources of feedback, or the institutional constraints. Instead, the focus is on the semiotic mediations that take place in the classroom and the teacher's role in making them effective; the analysis concerns the indicators given by students that point to the accomplishment of the task, and the way in which these indicators develop.

There is another difference, which has to do with the mediating role of the teacher. TDS mainly focuses on the teacher's role in engaging students in mathematical activity (devolution) and in formulating and decontextualising, in terms of *savoir*, the *connaissances* developed by students and targeted in the situation during institutionalisation. From a TSM perspective, the critical issue is to show how personal significations are connected to the mathematical significations of the targeted and culturally established mathematical *savoir*. The teacher's role appears to be essential for enabling students to link the *connaissances* called upon in the situations to the *savoir*; this manifests itself in complex interactions occurring over time spans that far exceed those of the institutionalisation phases. This analysis points out the difference in the importance that the TDS and TSM approaches bring to social activities.

In summary, the research work presented in this first perspective illustrates the functionality of theories in their networking for studying new research questions, according to various objectives and levels of integration (Fig. 4.1): a contrasting connection for developing a problem and the associated methodology (Sect. 4.2.2.1), and for revealing the different aspects of an object of study based on the different underlying principles of theories (Sect. 4.2.2.3); a coordination of theories at different levels of analysis (Sect. 4.2.2.2); and a locally integrated construction in the last example born of the necessity to define new concepts for studying new research questions. The next perspective deals with the topic of unification, which will be developed and argued by Juan Godino in contrast to the option of connecting theories.

4.3 Hybridisation of Theories: The Case of the onto-Semiotic Approach

As we indicated above, the articulation of theoretical frameworks (networking theories) is receiving special attention. Several authors (Prediger et al. 2008b; Radford 2008; Artigue et al. 2009) consider that the coexistence of the various theories explaining the phenomena of a discipline, such as mathematics education, is to some extent inevitable and enriching, but it can also be a hindrance to its consolidation as a scientific field. As already analysed in the previous section (see Fig. 4.1), Prediger et al. (2008b) describe different strategies and methods for articulating theories, which range from ignoring each other, to their global unification.

Personally I believe that progress in any discipline, particularly in mathematics education, requires that we consider the "Occam's razor" or parsimony, economy or succinctness principle, used in logic and problem solving. This principle states that among competing hypotheses, those with fewer assumptions are preferable; in other words, the simplest explanation is usually the best. Applying Occam's razor to mathematics education research justifies the efforts made in the field to compare, articulate and unify theories.

But it is also necessary to consider the phrase attributed to Einstein: "Everything should be kept as simple as possible, but no more," which can be regarded as a formulation of the "Chatton's anti-razor" principle: "If an explanation does not satisfactorily determine the truth of a proposition, and is sure that it is true, you should find another explanation". The multiplicity of theories in mathematics education is a consequence of the implicit application of Chatton's anti-razor, while efforts of comparison, coordination and unification of theories results from the also implicit application of Occam's razor. It is important to acknowledge that both principles do not conflict and that a rational position on the multiplicity of theories should be to explore the synergy that exists between them.

4.3.1 Issues in Theories Unification

In this section I argue the necessity and usefulness of articulating (internal and local) mathematics education theories, using the example of four well known theories in “French didactic”: TCF (Theory of Conceptual Fields), Theory of Semiotic Representation Registers (TSRR, Duval 1995, 1996), TDS (Theory of Didactical Situation), and ATD (Anthropological Theory of Didactic). The first two theories focus their attention on the cognitive dimension (individual or subjective knowledge) while the last two basically study the epistemic dimension (institutional or objective knowledge). I, however, believe that the consolidation of mathematics education as a techno-scientific discipline should tackle issues such as:

- What are the problems, principles and methodologies addressed and used in each framework?
- What redundancies exist in the tools used by these frameworks? Are they incompatible?
- Can the cognitive and epistemic tools of different frameworks synergistically coexist?
- Is it useful to construct a theoretical system that takes into account the various dimensions involved (epistemic, cognitive, instructional and ecological), and to avoid redundancies? What would the primitive notions and basic postulates of this new system be?

It is clear that we cannot address these issues here, but only show the relevance and potential utility of moving towards a theoretical system that coherently articulates the epistemic and cognitive approaches, in order to achieve effective instructional designs. To meet this goal I briefly describe some basic notions from these theoretical models whose clarification, confrontation and articulation could be productive. I will briefly mention how these theories conceive *knowledge*, from the epistemic point of view in TDS and ATD, and from the cognitive point of view in the other theories. This is not the place to make a comparison and possible articulation of these theories and their various components; instead I try to exemplify a networking strategy based on the *rational* analysis and possible hybridisation of conceptual tools used in each case. The system of results developed by each theoretical framework is not discussed or articulated.

This strategy has given rise to the Onto-semiotic Approach (OSA) in Mathematics Education, which has been developed by Godino et al. (Godino and Batanero 1994; Godino et al. 2007) in an attempt to articulate these and other related theories from an approach they describe as onto-semiotic. These authors conceive the theories under two perspectives:

1. In a narrow sense, as “system of tools” (concepts, principles and methodologies) used to answer a set of characteristic questions of an inquiry field; this interpretation may be similar to the triplet given in Radford (2008)—Principles, Methods and Questions.

2. In an expanded sense, and in addition to the above components, the "system of results" (knowledge) obtained as result of applying the tools to the questions.

In principle, any theory can produce valuable knowledge for understanding the field and rationally acting upon it. However, various theories may be redundant, inconsistent, insufficient, or more or less effective for the intended work. The clarification, comparison and possible articulation of theories intend to develop a system of optimal conceptual and methodological tools which enhance research in the field. Such an articulation can be performed by rational analysis of the constituent elements of these theories and by developing new conceptual tools when mere amalgam of existing ones is not possible or appropriate. As I aim to demonstrate, this strategy has led to a new theoretical notion, onto-semiotic configuration (Fig. 4.2), which incorporates, in a hybrid or blending way, constituent elements of *concept*, *conception*, *scheme*, *mathematical praxeology*, and *semiotic register of representation*.

4.3.2 The Notion of Knowledge in the Theories Analysed

The theoretical contribution of Duval (1995) falls within the line of inquiry, which posits a mental (internal representations) nature for mathematical knowledge, and attributes an essential role in the processes of formation and apprehension of mental representations (*noesis*) to language and its various manifestations. The availability and use of various semiotic representation systems and their transformations are considered essential in the generation and development of mathematical objects. However, *semiosis* (production and apprehension of material representations) is not spontaneous and its mastery should be a goal of teaching. Particular attention should be given to the conversion between non-congruent semiotic representation registers. Duval's cognitive semiotics provides other useful notions for studying mathematical learning, such as types of discourse, and meta-discursive functions of language, functional differentiation and coordination of registers (Duval 1996).

The theory of conceptual fields (Vergnaud 1990, 1994) has introduced a set of theoretical concepts for analysing the construction of knowledge by learners (see Sect. 4.2.2.1). This is why we consider this theoretical model in the cognitive programme, recognising, however, that some theoretical notions (conceptual field) have an epistemic nature. Vergnaud's basic cognitive notion is that of scheme. The scheme is described as "the invariant organization of behavior for a class of given situations" (Vergnaud 1990, p. 136). The author states that "it is in the schemes where the subject's knowledge acts; they are the cognitive elements that allow the subject's action to be operative, and should be investigated." Each scheme is relative to a class of situations whose characteristics are well defined.

Vergnaud also proposes a notion of concept (Sect. 4.2.1.1) to which he attributes a cognitive nature by incorporating the operative invariants "on which rests the operationality of the schemes." This notion is different from the concepts and theorems that are found in science; he does not propose an explicit conceptualisation for them. Vergnaud first describes the notion of conceptual field as “a set of situations”, and then clarifies that we should also consider the concepts and theorems involved in solving such situations.

In the TSD the *savoir* (knowledge to teach) has a separate, preexisting cultural existence and, in a way, is independent of the individuals and institutions interested in its construction and communication. The main objective of the didactic of mathematics is the analysis of communication and reconstruction processes of such cultural knowledge by the subject, in the form of knowing within the didactical systems. The didactic transposition, developed in the ATD framework, recognises the adaptations of this knowledge for its study in the school context, giving rise to different epistemic varieties of the same knowledge.[8]

As for the notions used in the TDS to refer to “subject’s knowledge”, we find ‘representation’ in the sense of internal representation; at other times Brousseau uses the expression “implicit models” for such knowledge and representations. He interprets implicit models as “ways of knowing”, which do not operate in a way completely independent nor totally integrated in controlling the subject’s interactions.

The Anthropological Theory of Didactic has so far focused almost exclusively on the institutional dimension of mathematical knowledge. The notions of mathematical organisation and institutional relationship to the object are proposed to describe mathematical activity and the emerging institutional objects from such activity. The cognitive dimension is described in terms of “personal relationship to the object”, which is proposed as a substitute for the related psychological concepts (such as conception, intuition, scheme, and internal representation).

4.3.3 Towards an Integrative Theoretical System

The brief summary of the concepts used by the four theories to describe mathematical knowledge from the institutional (epistemic) and personal (cognitive) points of view suggests that the simple superposition or the indiscriminate use of them to describe the phenomena of didactic transposition and mathematical learning can only create confusion.

This is one reason why Godino and Batanero (1994) began to lay the foundations of an ontological, epistemological and cognitive model of mathematical knowledge based on anthropological and semiotic bases. With a style reminiscent of axiomatic works in mathematics, these authors began by defining the primitive notions of mathematical practice, institution, institutional and personal practices,

[8]Developing the ecology of knowledge is a characteristic feature of ATD, whose connection with TDS, and other theories, is considered possible and necessary, as explained B. Grugeon (see Sect. 3.2.2).

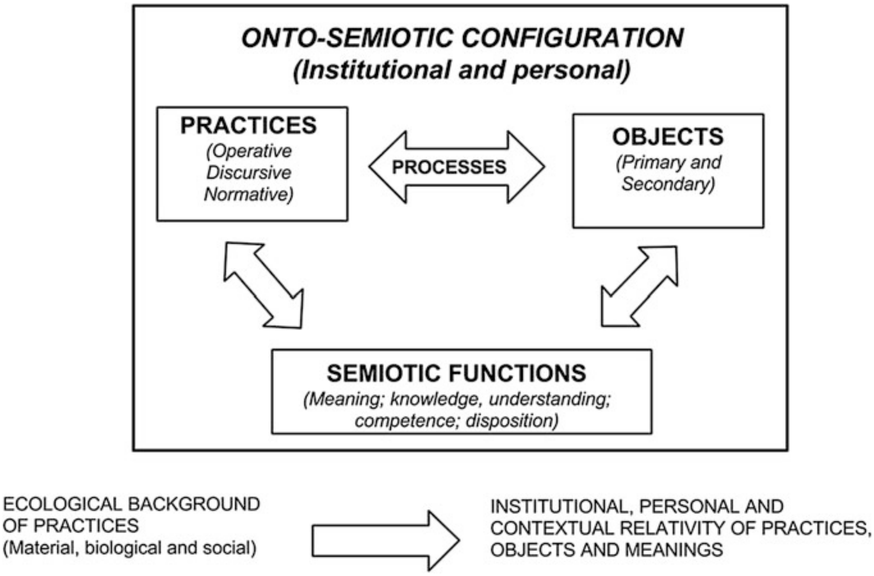

Fig. 4.2 Primary entities of the OSA ontology and epistemology

institutional and personal object, meaning of an institutional and personal object, and knowledge and understanding of the object. These notions were supplemented in later works with a typology of primary mathematical objects and processes as well as an interpretation of the notion of semiotic function. This notion is conceived as a triadic relationship between two objects, antecedent and consequent, according to a criterion or rule of correspondence, allowing the development of an operational notion of knowledge (meaning, understanding and competence) (Fig. 4.2). These notions may include those related to the epistemological and cognitive approaches used in mathematics education, as described in Godino et al. (2006).

In Fig. 4.2 the notions of practice, object, process (sequence of practices from which the object emerges) and semiotic function (tool which relates the various entities and that takes into account the object referential and operational meaning) are the key elements of the epistemological and cognitive modeling of mathematical knowledge proposed by the OSA. We might think that the onto-semiotic configuration is equivalent to the TCF conceptual triplet or the TAD praxeological quartet, however, the OSA has developed an explicit typology of objects (and processes) that enables more analytical and explanatory descriptions of mathematical activity than the other theoretical notions.

Specifically, the OSA proposes that in mathematical practices, the following six types of objects intervene: situations–problems, languages, concepts (in the sense of entities which are defined), procedures, propositions and arguments. These primary entities can also be seen from five dual points of view: personal–institutional;

ostensive–not ostensive; extensive–intensive; unitary–systemic; and expression–content (Godino et al. 2007).

4.3.4 Concordances and Complementarities

The theories mentioned (TSRR, TCF, TDS, ATD) put different weight on the personal and institutional dimension of mathematical knowledge and its contextual dependence. The OSA postulates that the systems of practices and the emerging objects are relative to the contexts and institutions in which the practices are carried out and the subjects involved in them (i.e., they depend on language games and forms of life, Wittgenstein 1973).

The description of an individual subject's knowledge about an object O can be undertaken in a comprehensive way with the notion of "systems of personal practices." Knowledge is also interpreted as the set of semiotic functions that the subject can establish where O is brought into play as an expression or content (signifier, signified). Within this system of practices, when asked to solve a type of problem–situation, we distinguish between those with operative or procedural character and those with discursive nature, and we obtain a construct closely related to the notion of praxeology (Chevallard, 1999), but only if we consider both a personal and an institutional dimension in the notion of praxeology.

We propose the different ways of "solving and communicating the solution" of certain types of problems related to a given object, for example, "function" as the answer to the question "what does the function object mean" for a person (or an institution)? This semiotic modeling of knowledge allows us to interpret the notion of schema as the cognitive configuration associated with a subsystem of practices relative to a class of situations or contexts of use, and the notions of concept-in-act, theorem-in-act and conception, as partial constituent components of such cognitive configurations.

The notion of conception (in its cognitive version) is interpreted in the OSA framework in terms of personal onto-semiotic configuration (systems of personal practices, objects, processes and relationships). In semiotic terms, when we ask for the meaning of a subject's "conception" about an object O the answer is "the system of operative and discursive practices that the subject is able to express and where the object is involved". This system is relative to some local and temporal circumstances and is described by the network of objects and processes involved.

Likewise, understanding and knowledge are conceived in their personal–institutional dual nature, involving therefore the system of operative, discursive and normative practices carried out to solve certain types of problem—situations. Subject's learning of an object O is interpreted as the subject's appropriation of the institutional meanings of O; it occurs through negotiation, dialogue and progressive linkage of meanings.

The notion of meaning is specified in the OSA framework. Meaning of a mathematical object is the content of any semiotic function and, therefore, according to the corresponding communicative act can be an ostensive or non-ostensive, extensive or intensive, personal or institutional object; it could refer to a system of practices, or a component (e.g., problem situation, notation, or concept). The notion of sense is interpreted as a partial meaning, that is, it refers to the subsystems of practice corresponding to certain frames or contexts of use.

In the OSA framework, Duval's notions of representation and semiotic register allude to a particular type of referential semiotic function between ostensive objects and (not ostensive) mental objects. The semiotic function generalises this correspondence to any type of objects and also includes other types of dependences between objects. For example, the ostensive expression $y = 2x$ refers to a particular mathematical function (conceptual entity, not ostensive). Between the two entities a representational semiotic function is established. In other situations the function $y = 2x$ can be on behalf of (represent) the class of first-degree polynomial functions, or the general function concept. Now the antecedent and the consequent of the semiotic function are conceptual entities. The mathematical function $y = 2x$ can be used to model certain practical situations, for example, to determine the cost of x kilograms of apples with a unit cost of 2 €. Here prevails the use or pragmatic meaning of the function concept: $y = 2x$ is defined by the system of practices the object participates.

The notion of sense in the TDS is restricted to the correspondence between a mathematical object and the various fundamental situations from which the object emerges and "gives its senses" (it can be described as "situational meaning"). This correspondence is undoubtedly crucial to provide the *raison d'être* for that object, its justification or phenomenological origin. But it is also necessary to take into account the semiotic functions or correspondences between the object and the remaining operative and discursive components of the system of practices from which the object comes from, understood either in cognitive or epistemic terms.

The TCF extends the notion of meaning as a "response to a given situation" introduced in TDS. This extension is meant to include, in addition to the situational component, the procedural (schemas) and discursive/normative (concepts and theorems) elements. The content considered as "meaning of a mathematical object for a subject" in TCF is nearly the wholeness described in the OSA as the "system of personal practices". However, the notion of semiotic function and the associated mathematical ontology provides a more general and flexible tool for didactic—mathematical analysis.

An essential aspect that allows us to distinguish between the theoretical models considered in this section is the dialectic between the institutional and personal duality, between epistemological and cognitive approaches, which often appear disjointed, resulting in extreme positions. In some cases the emphasis is on the personal dimension (TCF and TSRR), in others the institutional dimension (ATD and TDS), while in OSA a dialectical relationship between the two dimensions is postulated, so that it can help to coordinate the remaining theoretical models.

4.3.5 Hybridisation and Competition of Theoretical Frameworks

As we have explained, the OSA does not intend to build a "holistic theory that explains everything", but to advance the construction of a system of conceptual and methodological tools that allows researchers to conduct the macro and micro analysis of the epistemic, cognitive, instructional and ecological dimensions involved in mathematics teaching and learning processes. For the epistemic and cognitive notions analysed, the mere overlap or amalgamation of theoretical tools is not possible, given their heterogeneity and partiality, and the OSA has tried to develop a new framework with a clear hybrid character. The onto-semiotic configuration construct (Fig. 4.2) keeps a "family resemblance" with the notions of concept, conception, semiotic representation register, knowledge, and mathematical praxeology, but is not reducible to any of them, so it requires a specific designation. This notion can be more effective than the original notions, allowing researchers to analyse the micro and macro level of institutional and personal mathematical activity, and to better understand the relationships between both dimensions of mathematical knowledge. To prove this statement, however, would require a deeper analytical and experimental work than that produced in this brief presentation and that provided in Godino et al. (2006).

It is clear that this new entity competes with those already existing, and has to prove its relative effectiveness to solve the paradigmatic issues in the field. Progress is needed in comparing the results obtained from the emerging construct with other theoretical frameworks to test its possible survival.

The ecological analysis outlined of the emerging ideas should be complemented with the corresponding sociological analysis; it is not enough having generated a new potentially strong hybrid entity, it is necessary that social and material circumstances for its development are given. It is needed to attract new researchers involved in the study, to develop understanding and application of the new instruments, and to be able to achieve the necessary resources to conduct research, communicate, discuss and publish the results.

4.4 Social Perspective of the Multiplicity of Theories and the Search for Connections in the Didactics of Mathematics

The final section of this chapter is relatively adventurous in the sense that it offers—if not provides novel answers—to at least introduce new ways of examining the multiplicity of theoretical approaches by proposing to add a sociopolitical dimension to what has hitherto been an essentially epistemological approach. There are three parts to this text. The first part picks up on the key points of Bosch's contribution at the Artigue Colloquium and is centred on the notion of research praxeology, which is a significant contribution by Bosch, Artigue and Gascón to the reflections on Networking. The next two parts will, as mentioned earlier, attempt to open new doors by borrowing from the anthropological theory of didactics and the field theory applied to science respectively.

4.4.1 Theories as Components of the Praxeological Modelling of Scientific Research

4.4.1.1 Some Reference Points Relating to the European Dynamics of the Networking of Theories

As mentioned in the introduction to this chapter, schemes such as the Working Groups of the CERME congress and the two research projects, Telma and ReMath, have enabled the development of a community of European researchers who are able to understand each other and begin to build a common capital focused on interaction activities between theories.

Most notably, the ReMath project aims to develop certain interactive environments that are tested in conditions as close as possible to real teaching situations. The original version of this project also aimed to reduce the diversity of theoretical frameworks, as such diversity is viewed as one of the reasons why research in digital environments for human learning has a weak influence on practices. However, the initial work carried out quickly led to modifications in the formulation of this aspect of the project. The idea of creating a grand unifying theory that could incorporate contributions from the various views that exist in the field of research in the didactics of mathematics is considered to be a dead end.

ReMath has thus been directed towards the exploration of other ways of connecting theories. In their assessment of the work accomplished, Artigue and Mariotti (2014, p. 350) remark that:

> we have obtained practical results in terms of networking situated at different levels of the hierarchic scale presented in section I,[9] from comparison up to local integration in a few cases; we have also identified limitations to such networking especially when a design perspective is adopted, and coherent design choices must be made.

In several cases, research led to local integration between theory couples, to which Mariotti chose to devote her entire contribution at the 'Évolution des cadres théoriques' roundtable (see also Sect. 4.2.2.3). On the whole, however, the networking of theories was of limited scope. On the other hand, in the aforementioned text, Artigue and Mariotti emphasise the productivity of the work accomplished at a meta-level of reflection, a shared effort that could not have been achieved without the development of a specific language and set of tools. At the Artigue Colloquium,

[9]See Fig. 4.1.

Bosch centred her presentation on the contribution of research praxeology to the modelling of activities in the networking of theories and to the building of the necessary metalanguage. I will revisit this contribution in the following section.

4.4.1.2 The Notion of Research Paradigm, a Critical Tool for Thinking About the Complexity of the Networking of Theories

In the standard ATD model, research activities are described in terms of pointwise praxeologies [$T/\tau/\theta/\Theta$] where the notion of theory appears as the symbol Θ. This very model considers praxeological organisations that contain praxeologies with the same technology, followed by those of with the same theory; one refers to local and regional praxeological organisations. Artigue et al. (2011, 2012) only introduce the expression 'research praxeology', but it is clear that this refers to an amalgam of pointwise praxeologies located at the regional level. A praxeological organisation of research may thus be defined by:

- the types of problems that may or may not be addressed, which determines the acceptable research issues and the transformations that they will undergo to define objects of study;
- the corresponding methodologies that are deemed legitimate;
- the technologies of these research techniques, especially the rational arguments that produce them, and that justify and explain methodological choices; and
- the theory—I refer to the definition used in the ReMath group, which is not exactly that in ATD (see below)—of a structured system of knowledge that emerges from research practices and that, in return, conditions the three previous praxeological levels.

In a dynamic vision, such an amalgam can exist prior to the development of a theory. Artigue et al. (2011, 2012) stress the importance of the technological level of praxeologies: it is particularly pertinent when it comes to appreciating the nature and role of knowledge in the emergence phase of an initial amalgam, which will only become a praxeological organisation that is identifiable by a relatively substantial theory as research is carried out. But this lability at the theoretical level does not prevent the emerging organisation from attaining a certain social existence, which materialises at the very least through the development of a community of researchers who refer to it. At this stage of an iterative developmental process, some of the results thus obtained, especially didactic phenomena,[10] create new types of problems and new methodologies. The knowledge produced is integrated into the technology of new praxeologies that come to be included within the growing amalgam. Artigue et al. (2011, 2012) provide the example of didactic transposition:

[10]According to an expression by G. Brousseau, a didactic phenomenon is a regularity whose stability is established by research.

this is a phenomenon that, well before its inclusion in ATD, gave rise to the development of henceforth indispensable praxeologies in research work based on ATD and TDS.

In the above, we may discern the existence of an object that does not appear in the praxeological model: a social organisation, without which praxeologies would neither develop nor organise themselves into an amalgam that could produce a theory. Here, the expression '**research paradigm**' refers to this epistemic and social pair. A research paradigm is also a social construction, a product of a certain history, the fruit of a search for consensus within a community of researchers specialising in amalgamated praxeologies, which may be conceptualised in ATD by considering a paradigm to be an institution.

The notion of the praxeological organisation of research clearly brings to mind Radford's proposal (Radford 2008, see 4.2 note 2), which analyses the concept of theory by means of the (P, M, Q) triplet. To these three components the authors of the book *Networking of Theories in Mathematics Education* have added a fourth, *Key constructs*.[11] They use this new model to present five theoretical approaches that played a major role in the studies reported in their book, the expression 'theoretical approach' being preferred to 'theory'. The two resulting quadruplets are thus brought closer together. If one were to consider that 'savoirs', understood as explicit and socially legitimised knowledge, is but little taken into account in Radford's proposal, then the 'Key constructs' component changes everything. Explicit principles and key constructs are elements that the praxeological model considers at the level of the 'Theory' component. Is there, then, such a thing as identity? If one considers the theories of mathematical praxeologies, the answer is no, unless the idea of key constructs is very broad. Besides, other types of knowledge are revealed by the 'Technology' component of praxeologies. Indeed, this model makes a distinction between two components of methodology: research techniques and their technology. This enables a convenient description of the process that accompanies what is referred to in Radford (2008, p. 322), under the heading 'Connecting the principles P1 of a theory τ1 and the methodology M2 of a theory τ2': when one paradigm borrows a research technique from another, the technology is reworked in such a way as to establish technique compatibility with the principles and theory of the borrowing paradigm. If one considers a research paradigm to be an institution, this is a form of the transpositive phenomenon that comes with the interinstitutional circulation of praxeologies (Chevallard 1999, p. 231; Castela and Elguero 2013, p. 132). In summary, the praxeological model still offers greater possibilities for incorporating and analysing the knowledge produced and used by a research paradigm.

Both models take into account—in what appears to me to be the same way—the existence of a field of questions associated with a theory (Q) or a paradigm (set of type T problems). However, praxeological modelling does not allow one to incorporate all the basic principles considered by Radford, since technologies and

[11]For example, the concepts of adidactic and didactic situations, milieu and didactic contract are presented as *key constructs* of TDS.

theories are composed of explicit and legitimised knowledge. Yet a research paradigm is not wholly subsumed by the theoretical component that symbolises it: however developed the latter may be, it cannot reveal everything about the reasons for the choices that constitute the identity of a paradigm—its point of view, to borrow the expression used by Bosch in her talk, or its boundary in the words of Radford (2008, p. 323, see 4.2). Some principles remain implicit. One of the effects of networking activities is to force researchers who call upon the paradigms involved to update the founding principles.

As mentioned above, a research paradigm, and especially the social organisation that it encompasses and that produces it, can begin to exist prior to the existence of a theory in the sense that has been used thus far, i.e., a structured system of knowledge. But in ATD, theory is the technology of technology in the eyes of the community that acknowledges praxeology, and can be very embryonic. In this way, it is possible for the phenomenon of praxeological amalgamation in a scientific community to rapidly manifest itself as the development of a theoretical instance, without being identifiable as a theory in the scientific sense. Initially, the theoretical instance of a paradigm may be strongly characterised by the personality of the researcher or small group of researchers who introduced it: certain philosophical or ideological options serve as a prelude to the priorities given to certain topics, and therefore to the problems studied and the results obtained, which by default leaves its mark on the technologies and subsequently the theory subjected to the scientific process of the assessment. Let us conjecture that as a scientifically recognised theory is being built, there is a tendency to forget the primary reasons that constitute the very foundations of the paradigm and whose signature is borne by all the components of the latter.

Thus, based on this first part, I draw attention to the fact that the work required to appraise the commensurability of two paradigms is not limited to an examination of the conceptual apparatus of each theory. This echoes one of the points highlighted by Artigue and Mariotti in the conclusion of their assessment of the ReMath project:

> Despite our maturity as researchers, we all discovered up to what point our knowledge of many of the theoretical frameworks involved in the project was superficial. It had been gained through reading articles, listening to presentations, and discussing with colleagues. It lacked the first hand experience provided by the actual use in a research project. In such conditions, misunderstanding and distortions are frequent. (Artigue and Mariotti 2014, p. 350)

What constitutes the essence, pertinence and efficiency of a paradigm are collectively the objects of research that it favours for reasons that are sometimes implicit; its research practices and its theory. How then, in these conditions, do we connect two paradigms and above all attain a true understanding of a 'foreign' paradigm? This is precisely the question to which the ReMath project has brought preliminary answers, a summary of which has been provided by Artigue and Bosch (2014) in terms of networking praxeologies. One might say that the analysis of the amalgamation of research praxeologies developed here belongs to the technology of several of the networking methodologies tested.

At this point, let us turn our attention to one particular outcome of the assessment conducted by the researchers involved in the ReMath project, that is, the fact that they were led to deconstruct their original injunction of reducing theoretical diversity in the didactics of mathematics in order to reconstruct a version that would most certainly be very different. This was accomplished from within the field of research in the didactics of mathematics, alongside a group of didactical studies on the design and testing of sequences by means of software, and is a key component of the networking praxeologies that were developed. I propose to carry on this deconstruction of the postulate on the harmful nature of the multiplicity of research paradigms and the necessity for integrative development in the didactics of mathematics by adopting a complementary approach; indeed, one must, conversely, search for tools beyond didactics in order to consider how it works.

4.4.2 Considering Research in Didactics as Being Externally Determined: A Second Contribution of ATD

The Anthropological Theory of Didactics is not just a theory in didactics: many of its key concepts also apply to other realms of reality. This stems from a certain consistency in the approach by the person to whom its development may be chiefly credited, Yves Chevallard: deconstruct the self-evidences ('allant de soi') and to do so, put things into perspective by immersing the realm being studied into another broader realm. Wishing to justify the use of the adjective 'anthropological' by clarifying its meaning, Chevallard (1999, p. 223) writes:

> Le point crucial […] est que la TAD situe l'activité mathématique, et donc l'activité d'étude en mathématiques, dans l'ensemble des activités humaines et des institutions sociales. Or ce parti pris épistémologique conduit quiconque s'y assujettit à *traverser* en tout sens –ou même à ignorer- nombre de frontières institutionnelles à l'intérieur desquelles il est pourtant d'usage de se *tenir*, parce que, ordinairement, on respecte le découpage du monde social que les institutions établies, et la culture courante qui en diffuse les messages à satiété, nous présentent comme allant de soi, quasi naturel, et en fin de compte *obligé*.

In the rest of the cited paper, the author introduces the postulate that the same praxeological model may be used for all human activities, including mathematics. This stance has enabled didactics developed according to the ATD paradigm to refrain from having viewpoints from the world of research in mathematics imposed on it without question, and to distance itself from the presentations of mathematical knowledge proposed in academic texts.

Thus, when Artigue, Bosch and Gascón use the notion of praxeology to consider research in didactics, they already do so from outside this realm of human activities, which is precisely what I propose to pursue in the following section.

4.4.2.1 Research in Didactics as Being Externally Determined in Its Workings

I will once again adopt the ATD point of view and consider research in didactics as an institution among other elements, determined by other institutions that encompass it or are juxtaposed with it and that it also, in return, determines. Thus, civilisation, society, school and pedagogy appear in the range of institutions that are considered in ATD to influence the discipline of mathematics and its constitutive praxeologies (Chevallard 2007, p. 737). This provides an idea of the diversity of institutional levels involved. We are led to consider the influence exerted by the various societies and geopolitical organisations that are home to research in didactics on the latter, its structure, actors and praxeologies. To appreciate this complex reality requires the consideration of a network of institutions, each possessing a unique identity, at the local level of states, the regional level of geographical or linguistic groups, and finally the global level. Signs of such a structure are easily identified in the organisation of the ICMI as well as in the range of journals devoted to the didactics of mathematics.

On the epistemological front, the theory we have put forward has important consequences, as it amounts to considering that the paradigms of research in didactics, and therefore the results produced, bear the mark of the local sociocultural and institutional contexts in which they appear. Returning to the paradigms involved in the ReMath project, the influence of the French and Italian sociocultural contexts on TDS and TSM is, for example, likely to have affected the role given to the teacher under either theory (see Sect. 4.2.2.3).

4.4.2.2 Research in Didactics as Being Externally Determined by Its Objects

In the above section, we touched on the determinations that affect research institutions, especially the social components of paradigms and via these, therefore, praxeologies. However, we must also consider that these very institutions determine the realm of reality that constitutes the object of study in the didactics of mathematics, that is, all the phenomena of passing down and learning associated with mathematical praxeologies. Noone can dispute the vast distance that separates the following two objects of study: on the one hand, the passing down of arithmetic techniques in the Aymara villages of northern Chile, whose culture developed specific calculation praxeologies, and on the other hand, the use of software in the French (or Italian) education system to promote the learning of algebra. Is the epistemic aim of research in didactics to bring to light universal regularities when the reality that it aims to study is, unlike that of physics for example, so diversified? Assuming that such shared phenomena do exist (didactic contract is often cited as an example), how can they explain the complexity of the two local realities described above? More importantly still, given that research in didactics is as much engineering as it is science, and has both technical and epistemic objectives, what results can they bring about? Can we postulate that the same tools enable us to meet the requirements of the various didactical institutions in the world, to understand any dysfunctions identified, and to develop solutions deemed acceptable by these institutions and their subjects? I propose to consider it more epistemologically justified to adopt the reverse postulate until there is evidence of its erroneous nature. It seems to me that the multiplicity of paradigms is a consequence of the epistemology of a science that aims to take action in the reality being studied in order to improve it, which highlights the local dimension.

One example is the ethnomathematic perspective developed in both South America and Africa in response to what emerged in the period between 1985 and 1990 as a need to '"*multiculturaliser*" *les curricula de mathématiques pour pouvoir améliorer la qualité de l'enseignement et augmenter la confiance en soi sociale et culturelle de tous les élèves.*'[12] (Gerdes 2009, p. 21). This consisted in coming up with solutions to the widespread failure in the learning of mathematics within an educational system that had not truly done away with the colonial vision of teaching and '[presenting] *la mathématique comme quelque chose d'"occidental" ou d'"européen", comme une création exclusive de la race blanche*' [13](ibidem, p. 31).

4.4.2.3 How to Meet the Epistemological Need for Mutual Understanding?

Research in the didactics of mathematics is therefore regarded as being determined by both the social and cultural environment in which it is carried out, in terms of its questions and answers as well as its agents and organisations. Such determinations arising from the same source are reflections of each other, which may be considered to be a factor of coherence and efficiency. But this vision contradicts the conception of science as an approach to building objective facts possessing a universal value of truth and, according to Bourdieu, resulting as much from a confrontation between scientists as from a confrontation with reality:

> The fact is won, constructed, observed, in and through the dialectical communication among subjects, that is to say through the process of verification, collective production of truth, in and through negotiation, transaction, and also homologation, ratification by the explicit expressed consensus - *homologein*- (and not only in the dialectic between hypothesis and experiment). (Bourdieu 2001, p. 73)

The extension to higher institutional levels of work leading to homologation, or etymologically speaking to a rational agreement on the same discourse, appears to be an intrinsic component of the scientific approach: '*The process of depersonalization, universalization, departicularization of which the scientific fact is the*

[12]"Multiculturalise" mathematics curricula in order to improve the quality of education and increase confidence in social and cultural self of all students.

[13]Mathematics as something "Western" or "European", as an exclusive creation of the white race.

product is all the more likely actually to take place the more autonomous and international the field is.' (ibidem, pp. 75–76)

It is clear that the multiplicity of theories and paradigms of didactical research does not facilitate this change in level as it stands in the way of mutual understanding. This advocates, at the very least, for a verification of the emergence of new theories. Reducing theoretical diversity has been the solution adopted in a certain number of sciences, especially the exact sciences, which are also the oldest. To consider that didactics should follow the same path by replacing all existing theories with integrative wholes is to postulate that the workings of the sciences in question are the only ones possible, to accept the idea that given its youth, didactics should have no other way forward than to align itself with others. This view is illustrated by J. Godino in Sect. 4.3 of this chapter. Conversely, I consider that such a strategy is not relevant and would otherwise cause us to lose much of what has been produced by the various paradigms. The work carried out in the ReMath project tested the fact that the mutual criticism required to build scientific objectivity could be achieved by means that do not presuppose a common theoretical framework. The praxeologies of homologation in didactics could well be different from those of the exact sciences.

In summary, it is my opinion that social and epistemological reasons justify why research in the didactics of mathematics should not be subjected to, as a self-evidence ('allant de soi'), a principle of reducing theoretical diversity derived from other sciences: the multiplicity of theories and paradigms in this scientific field is seen as being inherent to the eminently social nature of its object and to the intended intervention in the society in which it is found. But it must come up with its own means of attaining scientific homologation.

4.4.3 Research in Didactics as a Power Game

To conclude the work of deconstruction undertaken thus far, I will refer to the field theory by Bourdieu, and more specifically to how he uses it to analyse the workings of science in the aforementioned book, *Science de la science et réflexivité*.

4.4.3.1 Elements of the Field Theory Applied to Science

The scope of the present text makes it impossible to truly appreciate the force of the work described above. I will, notwithstanding, present certain elements of the field theory, which may not be familiar to all readers.

A field is characterised by a game that is played only by its agents, according to specific rules. The agents are individuals and structured groups; in science they are isolated scientists, teams or laboratories. The conformity of agents' actions to the rules of the game is partly controlled by objective visible means, but the key point of the theory, through the concept of *habitus*, is the inculcation of social field rules into the agents' subjectivity. This individual system of dispositions, partly embodied as unconscious schemes, constitutes an individual's right of entry into the field.

The field game is twofold. Firstly, it is productive of something that is the field-legitimised goal in the social space. The rules, and therefore the individual dispositions, are established to achieve this goal that every agent considers desirable. In the case of science, the goal is epistemic: tacitly accepting the existence of an objective reality endowed with some meaning and logic, scientists have the common aim of understanding the world and producing true statements about it. As seen above, Bourdieu adds a social dimension to the Bachelardian conception of the construction of scientific fact. Despite this social nature, scientific homologation produces objective statements about the world thanks to specific rules of scientific critical scrutiny, '*the reference to the real, [being] constituted as the arbiter of research*' (ibidem, p. 69).

Secondly, the game is a competition between agents, which results in an unequal distribution of some specific form of capital—a source of advantage in the game itself and a source of power over the other agents. Thus, a field, including the scientific one, appears to be

> a structured field of forces, and also a field of struggles to conserve or to transform this field of forces. […] It is the agents, […] defined by the volume and structure of the specific capital they possess, that determine the structure of the field […This one] defined by the unequal distribution of capital, bears on all the agents within it, restricting more or less the space of possible that is open to them, depending on how well placed they are within the field… (ibidem, pp. 33–34)

Capital includes several species, for instance, in science: laboratory equipment, funding and journal edition. Here, we focus on symbolic capital, especially its scientific modality.

> Scientific capital is a particular kind of symbolic capital, a capital based on knowledge and recognition. (ibidem, p. 34)
> A scientist's symbolic weight tends to vary with the distinctive value of his contributions and the *originality* that the competitor-peers recognize in his distinctive contribution. The notion of *visibility*, used in the American universitary tradition, accurately evokes the differential value of this capital which, concentrated in a known and recognized name, distinguishes its bearer from the undifferentiated background into which the mass of anonymous researchers merges and blurs. (ibidem, pp. 55–56)

This theory of science as a field challenges an idyllic vision of the scientific community, disinterested and consensual; however through the hypothesis of embodied dispositions, it avoids considering the scientists' participation in the capital conquest in terms of personal ambition or cynicism.

In summary, we bear in mind that within the field theory, scientific strategies are twofold.

> They have a pure – purely scientific- function and a social function within the field, that is to say, in relation to other agents engaged in the field. (ibidem, p. 54) Every scientific choice […] is also a strategic strategy of investment oriented towards maximization of the specific, inseparably social and scientific profit offered by the field. (ibidem, p. 59)

4.4.3.2 The Injunction to Unify Theories Seen as Being Related to a Power Game

Let us return to the main topic. Bourdieu's work leads us to consider that the production of independent theories, just like the call for their incorporation into wholes where they are engulfed, relates to the game of conquest and contestation of positions of power in the field. To be recognised as the producer of a theory gives both material and symbolic credit to a researcher. This is true of the major didactic theories introduced by persons whose names are ever present. This phenomenon fosters multiplication: there is greater potential, from an individual point of view, in creating one's own theory than in seeking approval for one's contributions via an existing theory. Since it is difficult to blend two well-developed paradigms, regulating individual productions appears to be an epistemological necessity.

The same does not necessarily apply to the intermediate structures of research: above all, I believe that the development of a specific paradigm is an asset to an emerging research community, a means of avoiding the domination of older communities whose general tendency is to impose their own paradigms as the only pertinent ones. I have already postulated that the need to unify paradigms could be epistemologically challenged by virtue of the diversity of the didactic reality depending on the societies and countries involved. Now, I question it as an obstacle to an autonomous organisation of didactical research in countries where the latter is just emerging. We have already discussed ethnomathematics, and now I mention socioepistomology, which has been deliberately developed by a group of Mexican researchers with the twofold aim of developing tools adapted to the educational reality of Latin America, and breaking away from what could be felt as a prolongation of colonisation through the dominance of North American and European theoretical frameworks in the didactics of mathematics.

To end this paper, I propose to invert the problem: if developing a paradigm is an empowering factor, then the question lies in the significance—from the point of view of the positions of the various subinstitutions in the field of didactical research, notably regional institutions (characterised, for example, by their geographical location or by a common language)—of the necessity of reducing theoretical diversity, especially when this involves not the integration of several theories into a whole that would preserve the contributions of each but rather a process of selection for the purposes of simplification. Let us recognise that the significance afforded to the issue of the multiplicity of paradigms in didactics is not only related to epistemological reasons, but is also a facet of the social game of the field.

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Chapter 5
Conversion, Change, Transition… in Research About Analysis

Asuman Oktaç and Laurent Vivier

5.1 Introduction

Analysis, whether at the secondary school or university level teaching, has been a didactical field of study for several decades, notably through the research current known as *Advanced Mathematical Thinking* (Tall 1991). The understanding of mathematical analysis concepts can be approached from different points of view. The following citation from Duval (2001, pp. 83–84) helps us understand the complexity of this endeavour and establish the focus of this chapter.

> [O]bserving the subject "in mathematical activity" is far from being a simple or unequivocal process, because the functioning of the subject can be analysed in terms of the objects to be manipulated, used or transformed […]. However the functioning of the subject can also be analysed in terms of the internal systems (of representation or other) that should be called upon so that the subject has access to mathematical objects and can manage and control their transformations.

The perspective adopted in this chapter concerns the "changes in points of view and in tools" that is involved when tackling problem situations in the process of learning mathematical analysis concepts. Regarding this issue, Artigue points out the following:

> [F]or more than twenty years, research has brought out the fact that the learning of mathematics is not a continuous process, that it necessitates reconstructions, reorganizations, even sometimes veritable breaks with earlier knowledge and modes of thought. This fact has often nourished a vision of a hierarchy in learning, conceived as the progression through a succession of stages, as a progression toward increasing levels of abstraction.

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> More and more, research shows that learning rests, in a quite decisive way, on the flexibility of mathematical functioning via articulation of points of view, "registers of representation", and "settings of mathematical functioning". Conceptualization appears also more and more dependent on the concrete and symbolic tools of mathematical work. This dependence, which concerns at the same time what is learned and the methods of learning, is particularly important to take into account today, because of the rapid evolution of tools resulting from technological advances (Artigue 1999, p. 1379).

Artigue (2006) indicates that research in advanced mathematics education is mainly concerned with "*the progression between the increasing levels of abstraction*" (p. 276), but that since the year 2000 new research approaches have been developed focusing on connection and flexibility. Examples of this trend include connections linked to changes in setting (Douady 1986); changes in mathematical domains (Kuzniak 2014; Montoya Delgadillo and Vivier 2014); conversions of register (Duval 1995), as suggested strongly by the above citations; as well as changes in points of view, as in the three perspectives raised by Vandebrouck (2011b) for functions.

A transition between two *stages* of APOS (Action-Process-Object-Schema) Theory (Arnon et al. 2014), can also be interpreted in this sense since it involves a change of point of view about a mathematical object (more information about this theory will be provided in the section on functions). The same is true concerning the notions of *concept definition* and *concept image* (Tall and Vinner 1981).

These theoretical notions provide a way of looking at mathematical topics from different angles, sometimes even simultaneously, bringing out different qualities and characteristics of the concepts involved. Knowing about different facets of a mathematical concept, including the special difficulties that might be attached to them, has strong implications for developing teaching strategies and designing problem situations in order to optimise student learning.

Artigue (1998) classified the difficulties that students experience in relation to learning elementary analysis into three categories: (1) those tied to the complexity of the mathematical objects such as real numbers, functions and sequences, that are still in construction when students start learning analysis; (2) those related to the notion of limit, including its conceptualisation and technical aspects; and (3) those that have to do with the rupture from algebraic thinking.

Topics such as function, limit, derivative, continuity, real number system, integral, sequence, series and differential equation are among those that form the heart of elementary and advanced analysis courses. The notion of infinity often plays an important role in their conceptualisation. For the purposes of this chapter, a selection of studies that are more directly related to its aim is made, due to the large amount of research concerning analysis and related topics (in particular, the theme of differential equations is not addressed here because of space considerations. On this topic the special issue 26(3) of the *Journal of Mathematical Behavior* (2007) can be consulted).

In the following section, the subject of real numbers will be addressed, focusing on the notions of rational and irrational numbers, decimal and non-decimal numbers, the numerical line and the completeness of **R**. The case of the equality

0.999… = 1 will be the topic of a specific discussion. Next, the notion of limit will be examined, elaborating on the case of sequences more specifically. Finally, the topic of functions will be covered, including their derivation and integration.

The role that technological tools (calculators, software, movement detectors) play in the learning of mathematical analysis, including the advantages that they provide in terms of utilising different registers of representation, has been studied quite extensively (Berry and Nyman 2003; Kaput and Roschelle 1997). However, Artigue (2008) warns against the limitations of the type of visualisations that these tools offer. The use of such technologies and their particularities will only be touched in this chapter as they relate to the chapter's general objective of offering a perspective on the learning of mathematical analysis, seen from the angle of changes in points of view.

5.2 The Real Numbers

5.2.1 Registers of Representation

Research studies conducted on real numbers are few, with the exception of those exploring the distinction between rationality and irrationality, and with some including questions related to usual number sets (Fischbein et al. 1995; Ghedamsi 2008; Sirotic and Zazkis 2007a; Voskoglou 2013; Zachariades et al. 2013; Zazkis and Sirotic 2010). Notwithstanding, beyond the rationality/irrationality distinction, these studies highlight the importance of changes in registers of representation (Duval 1995, 2006) in the teaching and learning of real numbers.

From a numerical point of view and related to the notion of irrationality, two registers of representation are particularly at play in teaching: the fractional register favoured over the decimal register for representing rational numbers. However, concerning the notion of non-decimality, Bronner (1997, 2005) claims that real numbers can also be perceived by means of the distinction between the finite decimal expansion that characterises decimal numbers, and the infinite expansion, that characterises non-decimal numbers (Bronner introduced the term *idecimal*, emphasising therefore the decimal register). Two advantages of this change in viewpoint can be identified. Firstly, the numbers can be written in a unique register of representation, even if the treatment rules are to be defined, and for the same reason one can have an idea about their definition. Secondly, the perspective employed is analytic and no longer algebraic. Indeed, the rational/irrational distinction presented at secondary school and at the beginning of university studies is essentially algebraic—it suffices to look at the classical proof of the irrationality of $\sqrt{2}$ to be convinced—and this register contains in a natural manner the notions of approximation and neighbourhood (the balls of radius 10^{-k} are cylinders of lexicographical order) that are at the heart of real analysis.

Alongside these two numerical registers, there is another important one: the geometric register where **R** is represented by a line equipped with a point of reference. This representation seems to be especially used, with apparently positive effects, in works concerning the rational/irrational distinction (Sirotic and Zazkis 2007b; Voskoglou 2013). Núñez et al. (1999), in their study on the continuity of functions, distinguish two ways of conceiving a line: (1) the line is a totality which is not constituted by elements and where the points are dots on the line; and (2) the line is a set of points and therefore the points constitute the line.

Few studies focus specifically on the topological links between the geometrical line and **R**; on this issue we cite the work of Castela (1996), who reports on a study carried out with 58 Grade 10 students. In particular, to the questions: *between two points A and B*, *is there one closer to B than all the others?* and *between the numbers 0 and 8*, *is there one that is larger than all the others?*, students' responses were coherent with 17 negative and 30 affirmative answers obtained to both questions simultaneously. Castela concludes in particular that the passage to a continuous set of numbers constitutes a complex learning process where one cannot rely on the points-numbers correspondence. We also refer to the Ph.D. thesis of Bergé (2004) who made a summary on this subject.

Although studies about different semiotic registers are of utmost importance and have gained the support of researchers in the field, there are also other theoretical notions employed in research with regard to changes in viewpoints. One such example is the distinction between opaque and transparent, used by Zazkis and Sirotic (2010) in their study on irrationality (which can also be applied to other notions). Another approach consists of the *global*, *local* and *pointwise* perspectives on functions as investigated by Vandebrouck (2011a, b)—see also Maschietto (2008) and Rogalski (2008) where the expression *point of view* is used instead of *perspective*. Using this approach, the decimal expansion allows the pointwise and local perspectives. The value of a number such as 0.333… or a = 0.12112111211112… constitutes an instance of a pointwise perspective, whereas the mastering of neighbourhoods, as in affirming that '0.12112 is in a ball of radius 10^{-5} around a' or that 'it is an approximation of a with a precision of 10^{-5}', is a local one. However this approach does not permit a global perspective as allowed by a geometrical representation or as in the form of an interval.

5.2.2 From the Development of Repeating Decimal to the Object of Real Number

The passage from a finite to an infinite number of digits poses two related problems: (1) thinking about this new way of writing as a mathematical object, namely a number; and (2) identifying the two kinds of representations of decimal numbers, namely finite expansion and infinite repeating expansion, in the perspective of dealing with usual numbers (if not, one obtains other set numbers which are not

Q nor **R**, sets with a lot of holes). This identification is not obvious and this fact can explain why so many researchers are interested in the emblematic relation 0.999… = 1 (see the next section).

These two problems are not recent; their roots can be traced back to the Eleatics' paradoxes about the existence of movement (Aristote, Physique VI). In Zenon's first argument, the *dichotomy*, if we fix the departure at 0 and the arrival at 1 (for a detailed discussion of this paradox, see Fischbein 2001): before arriving at 1, one has to arrive at the middle of the path, namely at 1/2; afterwards one also needs to arrive at the middle of the middle, namely at 1/4, etc. In base two, the remaining distances to be travelled are, successively, 0.1, 0.11, 0.111 etc. and, symmetrically, the distances already travelled are 0.1, 0.01, 0.001 etc.

On this issue, APOS Theory (Arnon et al. 2014; Dubinsky et al. 2005) highlights the passage between the *stages* of Action, where one deals with a finite number of digits, and of Process, where one perceives that the digits go on without coming to an end. However, if one remains at the Process stage, one sees that the movement can never start since there is always a distance remaining from the departure—and that is the paradox. Therefore, one stays with a potential infinity. But if one allows an actual infinity, or an Object in the sense of APOS Theory, one obtains 0.111… and, of course, 0.00…01 (the "…" indicate an infinite repetition). Consequently, the question turns into making sense of 0.111… and 0.00…01 and then identifying them respectively, with 1 (identical to 0.999… and 1 in base ten) and with 0.

One can therefore recognise the complexity of encapsulation; on the one hand it is necessary to perceive the infinite series as a total (actual infinity), and on the other hand this total should be conceived as a mathematical object. To respond to this problem, Dubinsky et al. (2013) propose in APOS theory the introduction of a new intermediary *stage* between Process and Object that they name *totality*. It is a matter of conceiving 0.999… as a total, with infinitely many 9's, before being able to have access to the object, namely the number. This point of view is closely related to an idea proposed by Vivier (2011), which distinguishes between two objects: the number and the period. Indeed, totality can be thought of as the encapsulation of the Process of the repetition of 9's that gives the object period, usually denoted by $\overline{9}$ (this is the passage from potential to actual infinity). What remains to be done is constructing $0.\overline{9}$ as a mathematical Object, namely as a number.

The study of Weller et al. (2009, also see Arnon et al. 2014, Chap. 8) provides elements for understanding how repeating decimal expansions can become numbers. These authors propose a teaching method for rational numbers in two registers: fractional and decimal. Operations on decimal expansions are performed with the aid of a programming language, with the calculations performed in the fractional register, internally. They note a clear improvement in quantity and in quality of knowledge about rationals and decimal expansions, and in particular about the double representation of decimals (on an experimental group of 77 and control group of 127 pre-service teachers). The results of this study can be interpreted as the construction of repeating decimals as numbers since basic arithmetic operations can be applied to them (Vivier 2011; Yopp et al. 2011). The Object "number" can emerge from this work, and this result agrees with the point of view presented by Artigue (2007, p 47) who, following Bronner, affirms that "numbers and calculus live in symbiosis".

In their mathematical study about an exotic base, Rittaud and Vivier (2012) have identified the Object period as a fundamental object for the comprehension of repeating decimal expansions. They propose a construction of **Q** through repeating decimals by means of simple algorithms of four basic arithmetic operations (these algorithms were developed during the 18th century by accountants (Hatton 1728; Marsh 1742) but they seem to have been forgotten completely by the history of mathematics). It becomes possible to work directly on these repeating expansions, without a programming language, contrary to suggestions from Weller et al. (2009).

Once the case of the rationals is settled (that is, the repeating decimals, including the equality between 0.999… and 1), there is a second step that needs to be overcome before arriving at **R**. If one only considers the periodic sequences, everything can be algebraised, either by a conversion in the fractional register, or by working directly with algorithms in the decimal register. The passage to **R** by the unlimited repeating decimal expansions needs to take into account a *totality* (in the sense of APOS), which does not, however, correspond in general to the encapsulation of a Process. One can very well consider the particular processes such as the one in 0.1011011101111011111… or in the Champernowne number 0.123456789101112… or still in the numbers defined by a limit such as e or π. However, more generally, the Process relative to a real number x is defined by the sequence of truncations of the expansion of x. Here there is an interplay between the Process and Object that cannot be described easily: it seems that x already exists as an Object and that the Process is second. This idea should be investigated in order to clarify the construction of these concepts.

5.2.3 The Case of the Comparison Between 0.999… and 1

Numerous studies about the comparison between 0.999… and 1 have been conducted in the past (Sierpinska 1985; Tall 1980; Tall and Schwarzenberger 1978, and many others). It is considered a key issue for entry into the real numbers, at a crossroads of several notions: completeness (no gap between 1 and 0.999…); the notion of limit (sum of the terms of a geometric series or of a sequence of truncations); infinitesimals (between standard and non-standard analysis); the double representation of decimals in the decimal system (or the equivalent with a different base); rational (and irrational) numbers; the impact on Euclidean geometry (given that every number is the abscissa of a point on a line equipped with a reference point and a unit vector); potential and actual infinities; and so on. Yopp et al. (2011) state that it is important for those who teach students in the final years of primary school to have knowledge related to this equality.

Wilhelmi et al. (2007) propose several ways of verifying the equality of two real numbers. However, they rely explicitly on the field of real numbers which is supposed to be already constructed, together with its properties. In fact, this is the case for all *proofs* for the equality between 0.999… and 1 (see for example Tall and Schwarzenberger 1978) that are based implicitly on the properties of an already constructed set, essentially **Q** and **R**. Assuming a different point of view, Rittaud and Vivier (2014) propose the idea that the equality between 0.999… and 1 (and as a result, equalities concerning all the decimal numbers) is a technology (Chevallard 1999) since it produces and explains calculation techniques. This technology is hidden, made up of the construction of **Q** by means of repeating decimal expansions.

As noted, for example by Mena-Lorca et al. (2014) or Ngansop and Durand-Guerrier (2014), the *proofs* concerning "0.999… = 1" generally do not convince the subjects, even if they accept the validity of them. The semiotic opposition between 0.999… and 1 seems to be too strong. It is noteworthy that in the question of comparison between 0.999… and 1, the response rate suggesting the equality varies little across different studies: in Tall (1980), 14 out of 36 (39 %) university students; in Mena-Lorca et al. (2014), 14 out of 40 (35 %) and 7 out of 19 (37 %) for two populations of mathematics teachers; and in Rittaud and Vivier (2014), 17 out of 43 (39 %) first-year university students. For a non-mathematician public on the other hand and not surprisingly, Vivier (2011) found that at the start of high school, out of 113 students who were interviewed, all responded the inequality to the comparison question between 0.999… and 1; in Weller et al. (2009), 150 out of 204 (73.5 %) pre-service teachers responded inequality.

Non-standard analysis developed by Robinson in the 1960s proposes an alternative to the equality in question (see for example the *vignette* of Artigue, "the revenge of the infinitesimals", at http://blog.kleinproject.org/). One can very well build a non-standard theory of numbers in which 0.999… < 1 and that could give some meaning to objects such as 0.00…01. Nevertheless, the construction is complex since it is necessary to pursue and consider, after the comma, infinitely many infinite series of digits. Despite the promising attempts of its introduction in teaching (Artigue 1991; Hodgson 1994), non-standard analysis remains marginal. Still, some researchers continue developing this point of view (Ely 2010; Katz and Katz 2010a, b) and they consider it especially important that students can develop, against traditional teaching, non-standard conceptions of numbers (Ely 2010). For example, Manfreda Kolar and Hodnik Čadež (2012, pp. 404–405) report that, to the question *what is the largest number?*, a student responded 99… and to the question *what number is closest to the number 0.5?*, 67 students responded 0.4999… while three responded 0.500…1. These answers are classified within a conception associated to potential infinity, although one can also note here a non-standard conception of numbers.

5.2.4 The Completeness of R

Apart Bergé's work (see below), there are few research studies about the completeness of **R** and this is also true of topology (we can cite Bridoux 2011, who focuses on the first notions of the topology of R^N at the university which she identifies as Formalising, Unifying and Generalizing (FUG) notions after Robert 1998). Of course, one can also consider the studies about the notion of limit, but these do not focus directly on the questions of topology; in some of these studies, issues about topology are discussed without being a main concern.

Similarly the relationship between the completeness of **R** and the notion of limit is far from being obvious, as evidenced by confronting the articles of Burn (2005) and Mamona-Downs (2010). Burn (2005) exposes a point of view on limits of numerical sequences that allows to free oneself from the completeness of **R**. He proposes an elaboration of the limit notion, based on the historical development and initial conceptions of students (especially the monotone convergence, as in Bloch 2000; see the discussion that follows) without specifying the numerical domain: indeed it suffices to have a subset of **R**, if possible stable for the usual operations, and that contains sequences converging to 0, such as **D** (the ring of finite decimals), **Q** or the algebraic numbers—hence the specific topology of **R** is no longer required. However, as it can be seen in the method of exhaustion proposed by Grégoire de Saint Vincent, it is necessary to come up with a candidate for the limit since the point of view developed by Burn cannot give the existence of a limit. For the existence, it is essential to refer to theorems, such as the theorem on the convergence of a monotone bounded sequence, and the characteristics of **R** as a complete space (see also Nardi 2008, Chap. 6).

Conversely, Mamona-Downs (2010) proposes basing oneself on the set of the terms of a sequence rather than the sequence itself. This has the advantage of dismissing all the dynamic aspects in favor of a static point of view. The reformulation of the notion of convergence is made with the help of the points of accumulation; existence is ensured by the Bolzano-Weierstrass theorem. It is, however, necessary to distinguish some particular cases, mainly consisting of constant sequences, since in this case the set is finite. This innovative point of view, based strongly on the topology of **R**, needs to be tested experimentally.

Thus the connections between the completeness of **R** and the limits of real sequences seem to be particularly complex. As for the question of the completeness of **R** proper, it seems that only Bergé has tackled it. Bergé (2010) studied the responses of 145 students to a test on completeness. The students were distributed in courses II, III and IV, where the completeness of **R** is explicit, notably with lower and upper bounds (in course I, completeness is implicit). In her study, she analysed two tasks. The first task asked students how to explain to a young student the fact that an increasing sequence bounded from above converges. Only 12 students mentioned explicitly the completeness. The great majority of the students admitted as a fact, without discussion, that the limit existed, and almost half of them provided a graph (two- or three-dimensional). Some students used an extra-mathematical metaphor in their explanations while others presented major confusions. However, one can think that there was a bias in the question since one should address "a young student" while in course I itself, the mathematical work did not require the explicit utilisation of the completeness.

The second task concerned the meaning of the completeness of **R** for the 21 students in courses III and IV. Bergé noted that 8 students, of which 7 were from course IV, had an operational conception of the completeness that could be useful in proofs; the other 13 students only had a natural vision that could not be of help in proofs.

While completeness was a question that Bergé judged as crucial, she remarked that its comprehension did not come from solving tasks and that it was necessary to think about curricula which aimed at conceptualising. We agree with Artigue (2006) on the necessity of thinking about the teaching of analysis, here for real numbers, on a long-term basis.

5.2.5 Conclusion on Real Numbers

In numerous studies on real numbers it appears that the comprehension of the set **R** depends on the distinction between rational and irrational. Yet, the notion of irrational number only shows the insufficiency of **Q** for geometry and algebra. This point of view is too limited for analysis (let's not forget that **R** is really needed for analysis). In our opinion, the study of real numbers, from the perspective of analysis, requires other points of view than that of rational/irrational distinction with the purpose of being able to understand in an *elementary* manner the set **R**.

We can consider the notion of *density* which is often brought up in research studies, but it is not characteristic of real numbers since the rationals and decimals also possess this property which is more related to the order rather than the topology of **R** (even if these two concepts are close for the set **R**, the points of view are quite different). More generally, the properties of **R** treated in research are often not specific to **R**. For example, the study, otherwise interesting, by Zachariades et al. (2013) proposes 25 questions, none of which really involves properties distinctive of **R**. Of course they are concerned about essential questions such as the density, the recognition of the nature of a number, or the conversion between representations. However, it is the interviews that reveal some central ideas about **R**, especially concerning the equality between 0.999… and 1 (which is, by the way, related to the set **Q**).

Where is, then, the essence of **R**? When, then, do we need **R**? It seems that it happens only in analysis, otherwise, one can very well concentrate on a subset (such as **Q** or the algebraic numbers, even if it means including π and some other transcendent numbers). Bergé (2006, 2008), in a study on university-level analysis, identifies the explanation of the completeness of **R** as an essential element of the passage from *calculus* to analysis (*calculus* is a type of algebraised analysis, a calculation on the objects of analysis). It also seems that in order to achieve at least a part of the essential features of **R**, it is necessary to focus, more or less directly, on the completeness of **R**. One can consider several notions such as the Intermediate Value Theorem, the fixed point theorem, the theorem/axiom about the increasing sequences bounded from above, the expansion of unlimited decimals, upper and lower bounds, etc. without forgetting the topological relationships between the numerical and geometrical viewpoints, between **R** and the line.

There also remains a question which does not seem to have been the object of research: what is the role of knowledge about the real numbers for the learning of analysis? This lack of attention contrasts with affirmations about the supposed importance of this role (see for example Bloch 2000, p. 117).

5.3 Limits

The notion of limit has been the subject of research in mathematics education for nearly 40 years (Cornu 1991; Davis and Vinner 1986; Robert 1982; Tall and Schwarzenberger 1978; Tall and Vinner 1981), including both sequences and functions. Research about the limits of functions is abundant, however, specific studies about series are few (see for example González-Martín et al. 2011; Martínez-Planell et al. 2012, or see Bagni 2005 for a historical point of view on teaching).

We mention two reasons for considering this concept in mathematics education. On one hand, research into the learning of limits points out that this notion presents serious difficulties for students (Cornu 1991; Sierpinska 1987). On the other hand, the notion of limit, as a fundamental notion of mathematical analysis, appears in different contexts such as continuity, differentiability, integration and approximation, so the associated difficulties with limits can also have an implication on the learning of these concepts.

The interplay between the formal and the intuitive seems to be especially important in the case of limits. Attempts at starting instruction with a formal definition can bring about memorisation of rules and procedures, rather than building necessary structures for understanding (Oehrtman 2008); this may be partially due to the algebraic notation involved in the formal definition or difficulties with quantification (Swinyard and Larsen 2012). Some authors suggest that success is more likely to occur when understanding of the formal aspects is built upon the spontaneous conceptions of students (Fernández 2004). However, as Artigue (1998) notes, between the intuitive and formal conceptions, there is a considerable qualitative gap.

Many researchers consider that the passage from a dynamic to a static conception of limit is at the heart of the difficulty experienced by many students (Tall 1992). The first section on limits is about models that students have about the notion of limit. These models are often in contradiction with the formal definition, which is the topic of the second section. This definition is recognised as being particularly difficult to acquire and is sometimes judged as useless by students. The third section concerns the support that the graphic register can constitute in learning limits. The fourth section deals more specifically with functions. We have chosen to address the notion of limit more in detail for sequences, with most of the research results being transferable to functions as well.

5.3.1 Models About the Notion of Limit

Robert (1982) noted three main types of representations about limit in university students, which consequently became the subject of several research studies:

- Dynamic models where "to converge" is described as "to get closer to"; this kind of model can be observed since the first introduction of the notion of limit and is favoured by verbal expressions and gestures. These models are not a priori incorrect but they are not precise enough, for example, $4 - 1/n$ gets closer to 5 (example taken from Dawkins 2012). They can *glide* towards a monotone conception, including, for a sequence (s_n) and its limit s, "the distance between s_n and s decreases".
- Static models that are not formalised and are more or less precise expressions of the formal definition such as "every interval around l contains all the u_n except for a finite number of them" (also see Roh 2008, 2010 in what follows).
- Monotone models where convergence is perceived by means of the case "monotone bounded" with the interpretation of the term "limit" as "barrier" or "wall". These models are incorrect since they are too partial.

In her 1982 study, Robert investigated the performances of students on a test and remarked that the three types of representations she found were quite differentiating: those who had a static model were successful; those with a monotone model failed the test; and among those with a dynamic model, half succeeded and half failed.

Davis and Vinner (1986) found the same types of conceptions, in particular: a sequence does not reach its limit; implicit monotony; the confusion limit/bound; and the limit is the last term. Likewise, Cornu (1991) studied students' conceptions associated with the use of words from everyday language, such as "limit", "to converge", "to tend towards" or "to approach". These terms induce dynamic conceptions tied to speed; one gets closer to an object (with a possible monotone conception) without reaching it. In relation to this issue, Tall and Schwarzenberger (1978, p. 46) reported on the response of a particular student: "$s_n \rightarrow s$ *means* s_n*gets close to s as n gets large*, *but does not actually reach s until infinity*." This conception seems to be reinforced through a verbalisation of the definition such as the following: "one can make s_n as close to s as one wants provided that one chooses n sufficiently large".

5.3.2 The Formal Definition

The formal definition with quantifiers, order of terms to be complied with, and the variables, ε and N, is a definition acknowledged as being difficult to understand (Nardi 2008, Chap. 6). Mamona-Downs (2001) proposes a detailed interpretation of different elements of this formal definition, patiently exposing its characteristics and role. It can also be deduced that this definition is difficult because of a double local point of view associated with the expressions "for all ε" and "for n sufficiently large".

In a graphical work with strips of length 2ε, Roh (2010) identifies four categories related to the interpretation of the expression "∀ε > 0": in category 1, students remain out of the discussion because they consider strips of length zero although ε = 0 is not allowed; in category 2, the expression "for all ε" is forgotten and only some strips, and sometimes even only one strip, are used; in category 3, students take into consideration an infinity of possible choices for ε but do not think about *making it tend towards 0* (it should be noted that this is not mentioned in the definition); category 4 consists of those students who respond correctly. Roh proposes a hierarchy of four categories associated with each one of the characteristics of ε and N as well as the relationship between them.

Similarly, Durand-Guerrier and Arsac (2005) discuss the use of variables and quantifiers in analysis in the case of a classical task in topology: show that if a closed set A and a compact set B are disjoint then d(A, B) > 0. Durand-Guerrier and Arsac (2005, p. 159) show the following proof proposed by a student to 22 university mathematics instructors:

1. $\forall \varepsilon > 0, \exists x \in A, \exists y \in B, 0 \leq d(x, y) < \varepsilon/2$
2. As $x \in A$, and A closed $\Rightarrow x \in \bar{A} \Rightarrow \exists x_n \subset \mathrm{A}, x_n \rightarrow x$
3. but, $d(x_n, y) \leq d(x_n, x) + d(x, y)$
4. And, as $x_n \rightarrow x, \exists n_0, n \geq n_0\ d(x_n, x) < \varepsilon/2$
5. thus for $n > n_0, d(x_n, y) < \varepsilon$ so $x_n \rightarrow y$ and as $x_n \subset A$, then:
6. $y \in \bar{A} = A$, and $y \in B \Rightarrow y \in A \cap B \Rightarrow A \cap B \neq \emptyset$

All the instructors acknowledge the importance of mathematical knowledge in order to be able to carry out a proof and point out the fact that x and (x_n) depend on ε. However, only two of them mention a logical analysis of the status of variables and quantifiers, while changes in status and in rules of inference are used implicitly (as often occurs in mathematics), with some of them being prohibited (here at line 5). Durand-Guerrier and Arsac offer a framework that can be used for a logical analysis of proofs and foresee the relevance that this issue might have for teaching.

Many research studies try to create or reinforce the relationships between the formal definition and the concept images about the notion of limit or graphical representations (see the next section for the supporting role of the graphical setting). Although a verbalisation and a geometrisation can aid in comprehension, it is important to keep a mathematical vigilance, so that the expressions remain correct (Nardi 2008, Chap. 6). In a similar way, we can question the usage of metaphors. Dawkins (2012) studies the cognitive trajectory of three students and identifies in particular the references to a metaphor used in classes where the terms of sequences are considered as stragglers to a party, with the party taking place at the limit of the sequence. Dawkins observes that one of the students presents numerous reorganisations of his *personal concept images* and *concept definitions* (Tall and Vinner

1981), making regular references to the metaphor and noting that he has serious difficulties in keeping a distance from this metaphor. One can also wonder if the usage of this metaphor has really constituted an aid for this student.

On the other hand, Hitt (2006) underlies the importance of conflicts and collaborative work in order to make spontaneous, non-institutional and functional representations emerge from students. In Hitt, examples concerning the definition of limit of a sequence can be found.

5.3.3 The Supporting Role of Graphical Setting

When introducing the notion of limit at the end of secondary school in France, Bloch (2000, Chap. 6) proposes a situation, designed and analysed within the framework of the Theory of Didactical Situations (Brousseau 1997), and based on a dialectical relationship between a convergent sequence and a sequence that tends to $+\infty$. The two proposed sequences are those defined by the perimeters and areas obtained during the construction of the Koch snowflake. This reliance on geometry allows, in particular, a control that turns out to be important when the calculators' capacities are surpassed. Validation criteria for the limits are obtained which in turn allows an institutionalisation of correct definitions, without necessarily presenting the formal definition. That being said, as Bloch warns, the fact of choosing strictly increasing sequences can become an obstacle and induce a monotone model.

Roh (2008, 2010), asks 11 undergraduate students to work on the notion of convergence by using two-dimensional graphical representations of pairs (n, u_n) of sequences and translucent strips to be placed on the graphs. They are strips of length 2ε with a central line (ε-*strips*), with the aim of having the inequality $|u_n - L| < \varepsilon$ appear on the graph. The students must choose between two *definitions*: the first one asks that an infinity of points be covered by all strips and the second one asks that only a finite number of points not be covered by all strips (only the second definition is correct). The idea is for the important elements of the formal definition to stand out, especially the fact that all the terms starting from a certain natural number should fall into the strip and not only an infinity of them, as well as the fact that one has to consider several ε's (actually an infinity of ε as close to zero as one wishes). The responses of these students are analysed in relation to their initial conceptions, including the conflicts (Roh 2008). The author recognises that it is difficult to change a conception while it is coherent for the subject, but affirms that offering a graphical work, such as the one with strips, helps provoking a conflict that can aid in confronting the obstacles.

Cory and Garofalo (2011) make use of the same graphical ideas. Making the assumption that mathematics teachers do not see the point of the formal definition, they use dynamic geometry software that allows for control over the size of the strips of length 2ε. They conclude that the work with a graphical-dynamical strip of length improves the *concept images* (Tall and Vinner 1981) of the three teachers who participated in the study, including that of the "unreached limit". Furthermore, this allows connections with the formal definition that is difficult to learn, notably the comprehension that involves quantifiers.

Alcock and Simpson investigate the characteristics of students who, according to them, are *visual* (Alcock and Simpson 2004) or *non-visual* (Alcock and Simpson 2005). Their study takes place at the beginning of two analysis courses (a standard course and a collaborative one); it is based on interviews that are performed over 8 weeks. They analyse two tasks: the first is about the logical relationship between *the sequence converges* and *the sequence is bounded* (see also Nardi 2008, p. 196); the second concerns the convergence of the series $\Sigma(-x)^n/n$. In their analysis they make use of four main indicators: use of a diagram; use of a gesture; explicit preference for diagrams and drawings over algebraic representations; and the search for a meaning beyond algebraic representation. With these indicators, the authors identify 9 students as visual and 7 as non-visual (2 other students are not classified).

The visual students quickly draw, almost the same images, which allows them to offer responses. The issue is to find out whether the students draw on, and in what manner, the formal definition. While the formal definitions are perceived as conventions to be implemented by an external authority, this is often accompanied by a rejection towards using them, with the trust in drawings being perceived as sufficient. These students experience difficulties engaging in processes of proof. The authors conclude their 2005 article by claiming that beyond the utilisation of drawings by teachers, which are widely adopted by visual students and rarely by non-visual students, it is precisely this relationship between the visual and formal representations that should be worked on. We also find in this work discussion about the coordination between registers again (see also the discussion with respect to the students E and H in Nardi 2008, pp. 196–197).

5.3.4 Limits of Functions

In connection with functions, the limit notion is involved in many analysis concepts such as continuity, derivatives, integrals and differential equations, having therefore a direct influence in their understanding. According to Artigue (1998), its role is more of a unifying concept than a tool for solving problems. Oehrtman (2008) contends that instruction on limits should not be carried out in isolation, rather the connections of limits to other notions should be explored, thus motivating a deep understanding of both the limit concept and the others in question. He adds that the design of limit instruction will depend on the goals that are set for students' learning, such as exposure to formal definitions and proofs; intuitive understanding; and de-emphasis of limits and alternative foundations of calculus.

Przenioslo (2004) argues that many conceptions that students in her research study held after completing a university calculus course had probably already been there at secondary school, and that the misconceptions formed at that level were not corrected by taking a university course in calculus.

Cottrill et al. (1996), in trying to come up with a model for how students might develop a formal understanding of the limit concept, provide a preliminary genetic decomposition, and after carrying out interviews they suggest that the formation of a dynamic conception of limit might be more difficult than most think, since it cannot be expressed as the interiorisation of an Action into a single Process. The revised genetic decomposition based on data suggests that it is formed by the coordination of two Processes (one where the domain element approaches a value, and the other where the function value approaches the limit) as part of a Schema. Another source of difficulty is identified as the lack of a powerful quantification conception. They suggest that writing computer programs as a pedagogical strategy might be helpful towards the construction of this concept. As a conclusion, Cottrill et al. (1996) argue the following:

> As opposed to some researchers who believe that a dynamic conception may hinder progress toward the development of a formal understanding of the limit concept, we believe that the difficulty in moving to a more formal conception of limit is at least partially a result of insufficient development of a strong dynamic conception (p. 190).

Swinyard and Larsen (2012) interpret this quote as referring to a passage from an informal to a formal understanding of limits, however they consider that the approach of Cottrill et al. (1996) is geared towards finding a candidate for the limit of a function, rather than checking that a certain value is the limit of a function at a certain point, the latter being the process involved in the definition of limit. They also point out that students often do not realise the difference between these two processes. Following a developmental research design and adopting the theoretical analysis of Cottrill et al. (1996) as a starting point, Swinyard and Larsen (2012) go on to describe a refined genetic decomposition of the limit concept. In their study, special importance was put on students' reasoning and, based on that, strategies were developed to overcome difficulties that they experienced; students were motivated to formulate their own definitions through specific tasks designed for that purpose.

Another important interplay with regard to limits of functions occurs among different representations of this concept. Hitt and Páez (2001) identified calculus students' difficulties when working on conversion tasks between numeric, graphic and algebraic representations of limits. They found that when given a graph, students were concentrating on the curve instead of the domain and the range of the function and were unable to interpret the graph in terms of limits. Students also relied on substitution when calculating limits.

Karatas et al. (2011) applied a test to Grade 12 students as well as to pre-service teachers in their first, second and third years of university. The test used tasks about limits and continuity, focusing on verbal, graphical and algebraic representations. The authors found that "the comprehension of graphical representation decreases with respect to education level of students" (p. 257). For example, when a function was given in a graphical context, students thought that it should be defined at a point in order for the limit to exist at that point, although in an algebraic context they did not have difficulty in finding the limits of discontinuous functions. The authors underlined the importance of making connections between limits and continuity.

Other studies (Blaisdell 2012; Duru 2011) report that in general students have less difficulty dealing with graphical representations compared to questions that involve symbolic and definitional aspects.

In a study conducted with high-school students, Elia et al. (2009) found that students with a conceptual understanding of limits could perform conversion tasks between graphical and algebraic representations, both ways. They also report that students who made use of algorithmic processes as a result of the didactic contract were not very successful in solving this kind of problems.

Hähkiöniemi (2006) makes use of the notions of *associative* and *reflective* connections in relation to representations, when researching student understanding of the limiting processes involved in the difference quotient. Associative connection between representations is defined as changing from one representation to another, and reflective connection happens when one representation is used to explain another. When working on the interview tasks, students used different representations of the limiting process related to the difference quotient, including graphical representation of secants converging to the tangent line, difference quotients over diminishing intervals, and average rate of change over diminishing intervals. Hähkiöniemi found that "Difficulty seemed to be in the structure of the limiting representations and their connections to formal mathematics" (p. 182).

5.3.5 Conclusion About Limits

One cannot but notice the contrast between the studies by Roh (2008, 2010) and by Cory and Garofalo (2011), and those by Alcock and Simpson (2004, 2005). Can images, aided by a graphical work, be effective for all students? Wouldn't it be necessary to target the interventions according to the profiles of students as Alcock and Simpson suggest? It should be noted that the study by Alcock and Simpson focuses mainly on the external manifestations of students. In particular, it is not because the non-visual students do not propose graphical images that they do not have these images "in their head". Here the difficulty in assessing the mathematical activity of a subject is acknowledged. This may be studied by research using activity theory.

Similarly, it can be noted that the majority of current research focuses on the notion of limit in its dimension of object hence neglecting the dimension of tool (Douady 1986). Yet sequences and limits often intervene as tools in mathematics, as indicated by González-Martín et al. (2011) for the series. Here, one can identify a line of research that remains to be explored in-depth.

Explicit studies on representational aspects of limits taking into account the relationship between the nature of different registers and related students difficulties are not abundant. This area seems to be a promising context for future research. This might help clarify the contrasting results that have been found in different studies.

5.4 Functions

The notion of function is one of the most basic and complex entities with which students come into contact; there is abundant research concerning its learning and teaching. The aim of this section is to give an overview of some research related to functions that draws on aspects related to calculus and analysis.

In relation to the learning of this concept, Artigue (1998) identified four basic sources of difficulties: (a) ones that are tied to the identification of what a function is, (b) those that are related to the breach between a process and an object conception, (c) ones that have to do with representing functions in different semiotic registers, and (d) difficulties with overcoming numerical and algebraic modes of thinking in favour of functional thinking. Sierpinska (1992) also mentions the manipulation of symbols and the language that is used in connection with functions as sources of problems for students. A common mistake is to confuse the physical attributes of a situation (such as the trajectory of an object) with the visual attributes of a graph modeling the situation (Bell and Janvier 1981; Oehrtman et al. 2008).

In the context of functions, three types of representations are commonly used: graphical, algebraic, and numerical. From early on, research demonstrated the difficulties students have with transiting from one representation to another, especially from graphical to algebraic expressions (Duval 1999). Tall (1996) considers that with the use of computer simulations new kinds of representations are added to the study of calculus:

- *enactive* representations with human actions giving a sense of change, speed and acceleration,
- *numeric* and *symbolic* representations that can be manipulated by hand or by computer, including the possibility of programming by the student,
- *visual* representations that can be produced roughly by hand or more accurately and dynamically on computers, and
- *formal* representations in analysis that depend on formal definitions and proof (p. 291).

He goes on to add that “enactive experiences provide an intuitive basis for elementary calculus built with numeric, symbolic and visual representations, but that mathematical analysis requires a higher level of formal representation” (p. 293).

5.4.1 Research Within APOS Theory

Different theories explain the cognitive development required to understand functions in different ways. We now turn to APOS, a cognitively oriented theory that models the construction of knowledge as moving through the stages (also known as conceptions) of *Action*, *Process*, *Object* and *Schema*. The mechanism that allows the transition from Action to Process is known as *interiorisation* and from Process to Object is *encapsulation* (Asiala et al. 1996). The mental constructions together with the mental mechanisms that are used in constructing a mathematical concept form part of a *genetic decomposition* of that concept, a model that shows a possible way in which students may construct their knowledge (Arnon et al. 2014). In what follows, the stages of APOS theory are illustrated for the function concept, and examples are provided as to what students are expected to perform under each conception.

An individual at the Action stage acts on a previously constructed object, such as number, as a result of external stimuli such as a formula or an algorithm; in the case of functions he or she can calculate, for example, the value of a given polynomial function at a given point. Oehrtman et al. (2008) mention that students whose understanding of functions is restricted to an Action conception experience difficulties such as thinking that "a piecewise function is actually several different functions", "reasoning dynamically" since it requires visiting each ordered pair, or "conceiving of domain and range as entire sets of inputs and outputs" (p. 157).

When asked to state what a function is, students with an Action conception tend to give responses such as the following: "A function is something that evaluates an expression in terms of x" or "A function is an equation in which a variable is manipulated so that an answer is calculated using numbers in place of that variable" (Breidenbach et al. 1992, p. 252).

Oehrtman et al. (2008) consider that students with an Action conception cannot think about the inverse of a function, since they are limited to carrying out procedural actions such as switching the variables or reflecting the graph with respect to the line $y = x$; they cannot think about the properties of a function, either. According to these authors, designing situations that involve different representations is helpful in promoting a Process conception of function. They posit that by using a Process conception, a covariational viewpoint can be developed, which in turn proves to be useful in visualising multi-variable functions.

By repeating actions and reflecting on them, an individual interiorizes them into Processes, which are dynamic transformations (Arnon et al. 2014) that are carried out internally, as opposed to Actions that are static. With this conception one can "think of a function in terms of accepting inputs, manipulating them in some way, and producing outputs without the need to make explicit calculations" (Arnon et al. 2014, p. 30). For example, a student with a Process conception can explain what a function is in the following manner: "A function is a statement that when given values will operate with these values and return some result" (Breidenbach et al. 1992, p. 252).

A Process conception of function is often needed to develop an understanding of the concepts of limit, derivative and definite integral (Oehrtman et al. 2008). However, without an Object conception, an individual cannot have a complete understanding of operations on functions, such as compositions.

By encapsulating the Process of function, an individual constructs a static entity, that is, an object, which can be transformed by means of applying actions on it. In the case of functions, this might be in terms of performing operations on the functions, or forming sets or other structures of functions. If necessary, an individual with an Object conception can *de-encapsulate* a function, that is, he or she can go back to the Process that gave rise to it. An individual with an Object conception can define a function K on a set F of real valued functions, for example: $K(f)(x) = f(-2x)$ for every f in F (Oktaç and Çetin 2016).

Finally, a function Schema can be constructed as a collection of mental constructions related to the function concept whose coherence would be determined by the individual's ability to decide whether a given problem situation can be resolved using functions. An individual who is at the Schema stage "could construct various systems of transformations of functions such as rings of functions, infinite dimensional vector spaces of functions, together with the operations included in such mathematical structures" (Dubinsky and McDonald 2001, p. 280).

5.4.2 Multiple Representations and Points of View

For Duval (2006), there are two kinds of transformations of objects from one semiotic representation to a different one. If one stays within the same semiotic register, the transformation is known as *treatment*; if one changes registers, then it is given the name of *conversion*. Success with conversions is in part due to their congruence, or transparency, and this property does not have to be reversible (Duval 2006). According to Duval, this explains why students have considerably more difficulty when they are asked to find an algebraic representation of a given graph, than when they identify the graph of a given function. The results of the review presented by Gagatsis et al. (2006) are in line with this observation. The reasons for this might be related on the one hand to the fact that the algebraic register is favoured in instructional practices and, on the other, to the iconic characteristics of visual representations (Gagatsis et al. 2006).

According to Bloch (2005), it is when conversion is implied that a task becomes non-routine. She gives the following examples as suggestions for teaching the concept of function at the secondary level: Using a graph, find the properties of the corresponding function and write them symbolically; find the algebraic equation of a function given in another register (such as graphical or numerical); construct graphs of functions that comply with certain restrictions; interpret equations as composed functions; and compose functions given by their graphs and find inverse graphs.

Duval (2008) asks the question "what enables us to recognize that two semiotic representations have nothing in common, yet stand for the same knowledge object?" (p. 45). Another related question is "how can one distinguish the represented object from the content of the representation used, if there is no access to mathematical objects apart from semiotic representations?" (p. 46). According to Duval (2008), the answer lies in establishing and discriminating correspondences between the relevant units of different representations. In the case of a linear function, two kinds of connections between graphs and the corresponding equations should be established (which can also be generalised to other types of functions as well): (1) a local one by reading relevant points such as intersections or points on a graph for the purpose of plotting, and (2) a global one focusing on the relationship between the visual properties of a graph such as the inclination and the related features of the equation, such as the coefficients (the terms *local* and *global* bear different meanings here than *local* and *global perspectives* introduced above). The key issue in Duval's approach seems to be that of access to mathematical objects.

Thompson (1994a, p. 39) is of the opinion that the idea of multiple representations presents problems; he cautions about their use in teaching the function concept:

> Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students.
>
> […]
>
> Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.

Kaput (1993) advocates an approach to teaching functions where different representations appear as representations of situations for which students can establish a link to formal mathematical systems. In his opinion, it is desirable for these representations to be presented simultaneously and to be acted on and controlled by students in rich environments. In agreement with this stance, Thompson (1994a, p. 40) mentions the following:

> The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them.

Thompson's warning is in line with the approach adopted in APOS Theory with regard to the issue of representations. Arnon et al. (2014) suggest that in constructing a Process conception as predicted by a genetic decomposition, each representation of a function such as directed graph, arrow diagram, set of ordered pairs, graph of points, table and expressions should be directly linked to the concept of function. "In dealing with a problem situation, which may call for a particular representation of the concept, the learner thinks of the concept in terms of that representation" (p. 180). In this way, the learner identifies the process that is represented at the start, then using her or his Process conception, expresses the function process in terms of the new representation (Arnon et al. 2014).

> [T]he reason students have so much trouble making the transition from one representation to the next is that they (are taught to) go directly from one representation to another without passing through the cognitive meaning of the concept (given by the genetic decomposition). (Arnon et al. 2014, p. 181)

For Duval, representations are more intimately related to mathematical concepts as the following quote shows:

> All separation between concept and semiotic representation is rejected. Just as there is no "problem" without "formulation", the same way there is no concept without semiotic representation. Naturally, one can change the register of representation of a concept, but one can never separate the concept from a semiotic representation. (2001, pp. 93–94).

Drawing on the notions developed by Rogalski (2008), and with the intention of investigating students' concept images further, Vandebrouck (2011a) considers three perspectives on functions as a way to explain difficulties students face when transiting from secondary school to university. He also explores the relationship between these perspectives and the process/object duality. When a *point-wise* perspective is adopted, a function is thought about in terms of a correspondence between two sets of numbers that comply with certain criteria. Representations that are often used in relation to this aspect are numerical formulas and tables of values (Vandebrouck 2011a). The *global* perspective takes into account the overall properties of a function and is "necessary to understand the notion of variation" (p. 2095), with a table of variations being an appropriate representation for it. According to Vandebrouck, although graphs can be used in relation to the two perspectives already mentioned in this paragraph, algebraic formulas can only be considered by students in a global context if they belong to well-known functions, since they do not readily present the function object as an entity. In order to develop a global sense of functions, students need to master representations that favour such a perspective, as in graphical and symbolic representations (Vandebrouck 2011b).

The *local* perspective is not usually addressed until the university level (Vandebrouck 2011a); from this perspective, notions such as continuity, differentiability and limit can be introduced in terms of neighbourhoods (Vandebrouck 2011b). In particular, Vandebrouck stresses the importance of the mastery of a global perspective on functions as a pre-requisite for working with a local perspective. In this context, algebraic formulas as well as graphs would be adequate representations, and establishing a connection between them is especially important.

5.4.3 Multi-variable Functions

Although research on functions is abundant, the same thing cannot be said about research on multi-variable functions. One example in this category is the study carried out by Trigueros and Martínez-Planell (2010, 2012) that aims to explain how students build their understanding of two-variable functions. They use APOS Theory to model the construction of the concept, together with Duval's theory of semiotic representations, with the aim of analysing student productions to determine the relationship between the flexibility in using these representations and understanding of the concept. For these authors, the two theories complement each other in a coherent manner; one of the ways they could interact is the enrichment of a Schema in terms of mastering different representations of functions of two variables. Trigueros and Martínez-Planell (2010) also mention that the use of a theory of representations added a new angle to the analysis carried out with APOS Theory.

Trigueros and Martínez-Planell (2010) attribute students' difficulties in carrying out treatments in the graphical register to their lack of coordination between the three schemas: those of $\mathbf{R}^3$, one-variable function and sets. Students had considerable difficulty in intersecting a fundamental plane (of the form $x = c$, $y = c$ or $z = c$ where c is a constant) with a given surface, when they were both given algebraically, or when the plane was given algebraically and the surface, geometrically. Trigueros and Martínez-Planell (2010) state that "This ability requires, according to our analysis, the possibility to do or imagine doing a succession of treatments and conversions of representations of the function of interest" (p. 17). Another source of difficulty when working with representations in $\mathbf{R}^3$ was identified as the lack of experience with interpreting and drawing projections into $\mathbf{R}^2$.

Trigueros and Martínez-Planell (2012) provide a genetic decomposition for the concept of two-variable functions that includes representations. After students had taken a course in multi-variable calculus in two universities (one in Mexico and another in Puerto Rico), the researchers carried out interviews for the purpose of testing their initial genetic decomposition; the interview questions included specific treatment and conversion tasks related to representations of two-variable functions.

The researchers found that the Action conception was identified as having to work with specific pairs of numbers when looking for the domain or the range of a function. In general, the students with this conception were able to find the graphical representations of points given in algebraic form, but they could not carry out the reverse conversion. Process conception was referred to as being able to think in terms of all the points that made up the domain or the range of a function. These students, for example, had "interiorized the actions of finding elements in the domain into the process of finding all the elements in the domain" (Trigueros and Martínez-Planell 2012, p. 375). Students with Process conception could generally perform conversions, but they presented difficulties with the types of conversions needed for an Object conception.

In the same study, only 4 out of 13 students demonstrated an Object conception. These students were generally able to recognise properties of two-variable functions and work with them in different registers and they were able to apply actions of treatment and conversion on them. All these students had developed a well-structured $\mathbf{R}^3$ schema.

5.4.4 Derivative

In this section and in the following one we consider two central concepts of calculus and analysis: derivative and integral. Promoting a conceptual understanding of these notions requires special strategies concerning different aspects of the teaching and learning process. In particular, understanding the mental processes involved in the construction of these concepts and the associated difficulties becomes of special importance.

As a result of their review, Sánchez-Matamoros et al. (2008) determine three contexts in which the development of the concept of derivative takes place: (1) in the relationship between the concepts of *rate of change* and *difference quotient*, (2) in systems of representation whose integration in teaching fosters conceptual understanding, and (3) in the relationship between the derivative of a function at a point and the derivative function. In what follows, some of these contexts will be illustrated by drawing on examples from research.

Derivative is a multi-faceted concept with which students need to work in different contexts. Zandieh (2000) developed a framework to analyse the concept of derivative, which consists of two main components: multiple representations and layers of process-object pairs. In this framework the representation viewpoint is described as follows:

> The concept of derivative can be represented (a) graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification; (b) verbally as the instantaneous rate of change; (c) physically as speed or velocity; and (d) symbolically as the limit of the difference quotient. Many other physical examples are possible, and there are variations possible in the graphical, verbal, and symbolic descriptions. (p. 105)

It is not very clear from this description why only the aspect “rate of change” is chosen as a verbal representation, since almost any facet can be described verbally.

In Zandieh’s approach, the aspects *ratio*, *limit*, *function* that are implied in the concept of derivative form its layers; each one of these layers can be thought of as a *process* or as an *object*. This information can be arranged into a matrix, where possible representations (or contexts) form its columns and different process-object layers form the rows.

> Each empty box in the matrix represents an aspect of the concept of derivative. For example, the box in the ratio row and the graphical column represents the slope of a secant line on a graph of the function whose derivative we are concerned about. (p. 106)

One of the conclusions that Zandieh (2000) draws from interview data by using the above-mentioned framework is that students in the same class can develop very different understandings of the derivative. She also points out a problem that students can have which consists in staying with a pseudo-object conception (citing Sfard 1992), which results from not properly understanding the underlying process of an object; the fact that many derivative problems can be solved using pseudo-objects makes it a challenge for developing instructional strategies.

Asiala et al. (1997) give a genetic decomposition of the derivative concept in terms of graphical and analytic paths which do not have to be disjoint. In what follows, we present this genetic decomposition adapting it into a table format for comparison purposes. From the fourth step, the two paths converge (see Table 5.1).

Table 5.1 Genetic decomposition of the derivative concept, adapted from Asiala et al. (1997, p. 407) into table format

Graphical stage	Graphical description	Analytical description	Analytical stage
1a graphical	The action of connecting two points on a curve of a function to form a chord which is a portion of the secant line through the two points together with the action of computing the slope of the secant line through the two points	The action of computing the average rate of change by computing the difference quotient at a point	1b analytical
2a graphical	Interiorisation of the actions in point **1a** to a single process as the two points on the graph get "closer and closer" together	Interiorisation of the actions in point **1b** to a single process as the difference in the time intervals gets "smaller and smaller," i.e., as the length of the time intervals gets "closer and closer" to zero	2b analytical
3a graphical	Encapsulation of the process in point **2a** to produce the tangent line as the limiting position of the secant lines and also produce the slope of the tangent line at a point on the graph of a function	Encapsulation of the process in point **2b** to produce the instantaneous rate of change of one variable with respect to another	3b analytical
4	Encapsulation of the processes in points **2a** and **2b**, in general, to produce the definition of the derivative of a function at a point as a limit of a difference quotient at the point		4
5	Coordination of the processes in points **2a** and **2b** in various situations to relate the definition of the derivative to several other interpretations		5
6	Interiorisation of the action of producing the derivative at a point into the process of a function f which takes as input a point x and produces the output value $f'(x)$ for any x in the domain of f'		6
7	Encapsulation of the process in point **6** to produce the function f' as an object		7
8	Reconstruction of the schema for the graphical interpretation of a function using the relationship between properties of functions and derivatives		8

Asiala et al. (1997) found that writing computer programs helped university students to move further along the stages of the genetic decomposition, since special activities promoted the mechanisms of interiorisation and encapsulation. According to these authors, a major issue in learning the concept of derivative occurs in establishing a "relationship between the derivative of a function at a point and the slope of the line tangent to the graph of the function at that point. This forms a foundation for understanding the derivative as a function which, among other things, gives for each point in the domain of the derivative the corresponding value of the slope" (p. 414). In this study they focused especially on students' ability to work with the graph of a function in the absence of a symbolic expression.

Based on their empirical observations, Asiala et al. (1997) proposed a revised genetic decomposition. In this new model, the role that the function concept plays, the need for an explicit symbolic expression, and connections between the paths, are taken into account.

Hähkiöniemi (2004) suggests that visual representations might be suitable for students to start establishing a relationship between functions and their derivatives. He adds that the students interviewed in his study were able to construct the concept of derivative as object and did not have difficulty in relating the visual representation of a tangent to the symbolic process of finding the slope of the tangent. However, the limit of the difference quotient caused problems in representations other than the symbolic. The author suggests that visual representations could be used in connection with differentiation rules, since this kind of representations seems to facilitate the learning of derivatives.

In another study, it was found that for most students the existence of an algebraic expression was necessary to answer any question about a function; if they were given a graph, they wanted a formula to be associated with it before they were able to work on it (Habre and Abboud 2006). For example, faced with the graph of a function, a student said: "In order to see if the derivative is increasing or decreasing, we must derive the function first" (p. 62). This kind of answer might be influenced by the teaching that the students received and consequently the didactic contract in question.

Vivier (2010) proposes an alternative approach to the one presented in Table 5.1, first by working on the object of tangent, in the graphical and algebraic settings, based on an adaptation of the method of Descartes for the determination of tangents. Afterwards, by means of an interplay between approximate and exact calculation, a transition towards analysis where the notion of instantaneous rate of change, as a limit of the average rate of change is proposed. In this approach computer work with zooms on curves as proposed by Tall (1985) and Maschietto (2008) is included. This proposition, tested on teachers (Páez Murillo and Vivier 2013), has shown the extent of a global conception in mathematics teachers' work as well as some difficulties related to the use of a conception associated with the derivation in a graphical task about tangents.

5.4.5 *Integral*

Thompson and Silverman (2008) consider that in order for students to construct an understanding of an accumulation function (usually represented by $F(x) = \int_a^x f(t)dt$), they need to coordinate three processes: defining formula of f, covariational relationship between x and f, and accumulation together with its quantification. This means for them to understand a space curve that is formed by points of the form $(x, f(x), \int_a^x f(t)dt)$. On the other hand, a Riemann sum represents the total amount of the derived quantity, where each element of the sum is a multiplication of two quantities, namely, $f(c)$ and Δx where $c \in [x, x + \Delta x]$. For students to think of an area under a curve as something other than area, they have to think about the accumulation of these incremental quantities (Thompson and Silverman 2008). The authors report that understanding a Riemann sum as a function is quite difficult for students.

Camacho Machín and DePool Rivero (2003) designed a Utility File to be used with the software DERIVE, in order to treat definite integral as an approximation in terms of areas of plane figures, in numerical and graphical perspectives. This made it possible to introduce the concept of definite integral before the concept of indefinite integral in a context of applications. Later, Camacho Machín et al. (2008) found that the use of this pedagogical strategy proved to be effective in the construction of the definite integral concept in the context of calculation of an area. They report that students had no difficulty in solving problems with continuous functions, including treatments and conversions among numerical, algebraic and graphical registers. However, when the function involved was defined by parts, the situation changed. Furthermore, application problems that did not explicitly ask for the calculation of an area proved to be difficult for the students.

As a result of their analysis of calculus textbooks, McGee and Martínez-Planell (2014) observed that when introducing and developing the notion of definite integral, the following registers were utilised: geometric, numeric, symbolic, and verbal (this last one in the context of applications). Within the symbolic register, the following representations were identified: expanded sum, sum in sigma notation, and the definite integral. They further observed that in the same textbooks, the treatment of double and triple integrals did not include the numerical register nor the expanded sum representation. The authors underlined that for comprehension of a concept to occur, conversion between registers should not simply stay at the level of conversions implied in a semiotic chain; instead, there should be simultaneous awareness (synergy) of the representations involved. After carrying out instruction with experimental and control groups (where the experimental group used materials designed to close the gap in relation to the representations identified as being absent in the textbooks), interviews were conducted with students from both groups. Based on the analysis of classroom observations, interviews and exam questions, it was concluded that the missing representations were necessary for many students to be able to comprehend the related treatments and conversions. It was also found that the more students had experience with different coordinate systems, the more they had awareness of the registers involved.

From an APOS perspective, Czarnocha et al. (2001) report that for an understanding of the concept of definite integral, coordination is necessary between the "visual schema of the Riemann sum and the schema of the limit of the numerical sequence" (p. 304). They emphasise that an Object conception of sequences is a pre-requisite for understanding the definite integral as a limit of the Riemann sums. Based on the genetic decomposition presented in Czarnocha et al. (2001), Boigues

Planes (2010) proposes a more detailed one in terms of three nested Schemas: partition of an interval, Riemann sums for a continuous function in an interval, and definite integral as the limit of a sequence of Riemann sums.

González-Martín (2007) designed a teaching sequence for the concept of improper integral, where the graphical register played a special role "in interpreting certain results as well as in predicting and applying divergence criteria" (p. 162). Algebraic register was used together with the graphical one so that students could articulate the two. This sequence was applied by means of a didactical engineering at a university. The author reports that as a result students accepted and recognised the graphical register as having a valid mathematical status, although at the beginning they might have expressed rejection.

5.4.6 Linking the Derivative and the Integral: The Fundamental Theorem of Calculus

Artigue (1995) frames the problematique of the teaching of analysis concepts as follows:

> Numerous research studies show, and with surprising convergence, that even if students can be taught to carry out in a more or less mechanical manner some calculations related to derivatives and integrals and to solve some standard problems, great difficulties are encountered to make them actually enter the field of calculus and to make them reach a satisfactory understanding of the concepts and methods of thinking, that are the center of this field of mathematics. These studies also show in a clear manner that faced with these difficulties, traditional teaching, in particular at the university level, although has other ambitions, tends to focus on an algorithmic and algebraic practice of calculus and to evaluate in essence competencies acquired in this domain. (p. 97)

The issues mentioned in this quote might explain why students have serious difficulties in developing a conceptual understanding of the Fundamental Theorem of Calculus (FTC). In this theorem, differential and integral calculus are intimately related, where the rate of change is coordinated with the amount of change of accumulation, two notions that are essential in understanding the FTC (Kouropatov and Dreyfus 2014; Thompson 1994b). According to Thompson and Silverman (2008), if an accumulation function is given by $F(x) = \int_a^x f(t)dt$, its comprehension (without focusing on Riemann sums) passes through three stages. First, a process conception of $f(x)$ as well as understanding of the covariational relationship between the independent variable and f must be achieved. These two aspects will then need to be coordinated with the process of accumulation together with its quantification. Finally, the three values x, $f(x)$ and $\int_a^x f(t)dt$ must be coordinated (Thompson and Silverman 2008). Establishing relationships between these elements and the rate of change of the accumulation function can lead to a conceptual understanding of the connection between derivative and integral, and of the FTC (Kouropatov and Dreyfus 2014).

5.4.7 Conclusion on Functions

The function concept has been studied extensively from the point of view of cognitive constructions and the use of representations. In some research studies, the use of these two approaches was found to be complementary and helpful in terms of explaining student difficulties and construction of concepts. There seems to be a need for further research, both theoretical and empirical, in order to explain in more depth the different standpoints involved in these approaches, including how they explain educational phenomena and if and how they might enhance our understanding of student learning.

5.5 Conclusion

The research studies that have been presented in this chapter are concerned, beyond the technical and procedural aspects, with the conceptualisation of the notions of analysis. This seems to be a particularly important and crucial issue for mathematics education researchers, as Artigue (1995, see the citation at the end of the section *Derivative and integral*) has pointed out. Ten years later, Artigue (2006) highlights new research studies concerning flexibility, connections between settings, registers, etc.

What stands out from these studies, independently of the notion in question, is the importance of an interplay between different registers, as well as the importance of the graphical register that constitutes a visual aid that is likely to favour understanding. At the *EM-ICMI Symposium* in 2000, Artigue asked the following questions:

> Up to what point are graphic visualizations efficient tools for supporting the development of mathematical knowledge in analysis? What kind of tasks, of problematic situations can allow the students to maximize the expected benefits of these techniques? What are their real potential and limits? Are they adequately exploited in current teaching with graphic calculators? Even if research allows us to approach these questions better today, they remain widely open. (Artigue 2003, p. 216).

Research continues into these still valid questions. Nevertheless, there seems to be a less extensive use of the graphical setting regarding real numbers, including for the topological questions, perhaps due to the unidimensionality of $\mathbf{R}$. It seems that real numbers constitute a research field to be invested in by researchers in mathematics education: what kind of aid can geometry provide for the learning of topology, especially of the completeness? What impact does knowledge of real numbers have on the conceptualisation of concepts in analysis?

However, the issue of changes in settings and domains, as well as the consideration of different contexts, does not seem to be taken into consideration in a specific manner in research studies. There might be a place for developing these aspects in analysis education. Similarly, we have observed that research into the dimension of object concerning the notions of analysis is well advanced however the same cannot be said of research about the dimension of tool. One exception is the didactic situation designed by Legrand (1997) in order to introduce the Riemann integral, about whose efficiency Artigue (2001) comments that "it strongly depends on the kind of scenario developed in order to organize students' encounter with this new facet of the integral concept. In a crucial way, this scenario plays on the social character of the learning process" (p. 214). This could be another line of research to develop.

Finally, the breach with the algebraic thinking identified by Artigue (1998) as the third problem posed by the learning of analysis (see the introduction), seems to be worth reconsidering from the perspectives of flexibility and connection. It could be interesting to consider a productive interplay between algebra and analysis, taking into account exact value, approximation and the nature of objects. As Artigue (2003) commented, when referring to the initiation to analysis:

> In the previous algebraic work, all the different components of an algebraic expression were given the same weight; solving an equation meant finding all the numbers satisfying that equation. In analysis, the management of algebraic expressions has to take into account the different orders of magnitude of the terms and to look for what is predominant and what can be neglected. Working with inequations is no longer playing the same game; it means combining this differentiated treatment of expressions with the play on intervals or neighbourhoods induced by the local perspective. The technical work thus deeply changes and becomes more complex, as well as the heuristics and the control processes. (p. 219)

Research in this direction can help identify student difficulties, as well as shed light on the ways for the construction of related concepts.

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Chapter 6
Digital Technology and Mathematics Education: Core Ideas and Key Dimensions of Michèle Artigue's Theoretical Work on Digital Tools and Its Impact on Mathematics Education Research

Carolyn Kieran and Paul Drijvers

6.1 Introduction

In 2002 Michèle Artigue published an article entitled *Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work*. That paper reflects a fundamental contribution to theory on the teaching and learning of mathematics in technological environments, and to instrumentation theory in particular. Clearly, Michèle's work[1] did not end with her 2002 paper; rather, the article presents important threads that she has continued to develop, and that have inspired other researchers in the field. As such, the paper has had an important influence on the international research agenda in the domain of technology-enhanced learning, as well as a considerable impact on recent research. This chapter, therefore, has two goals. The first goal is to revisit the central themes elaborated in that paper. The second is to follow the evolutionary paths of the paper's main themes and to outline some new directions that have emerged from them.

To achieve these goals, we distinguish the threads that are *general key dimensions*, which run through the body of Michèle's work, from the threads that are *core*

[1]Michèle would be the first to insist that the contributions we describe in this chapter were not hers alone, nor just those of her DIDIREM team in Paris, but were also based on collaboration that included a team in Rennes piloted by Jean-Baptiste Lagrange, and another in Montpellier piloted by Dominique Guin and Luc Trouche.

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theoretical ideas, which are interwoven into and provide specific perspectives on the key dimensions. The key dimensions are generic in nature; they include the mathematics, the teacher, the learner, and the tool—dimensions that are in fact touched upon in much of the research on the use of digital technology in mathematics classes. The cross-wise threads of core theoretical ideas are those particular notions that underpin and elaborate the ways in which the general dimensions are considered and without which the dimensional terms would be devoid of specific interpretation. In collaboration with others, Michèle has contributed uniquely to the generation of core theoretical ideas that have profoundly impacted the way in which we think about some of mathematics education's basic dimensions. We also believe that the metaphor of interweaving, which permeates this chapter, fits well with the kind of 'tinkering' that we all try to do in our work, and at which Michèle excelled.

6.2 The Importance of Theoretical Foundations

6.2.1 Towards a New Theoretical Framework

The first theme we identify in Artigue's (2002) IJCML article concerns the importance of theoretical foundations. In one of the first sections, entitled *A theoretical framework for thinking about learning issues in CAS environments*, Artigue emphasises the need that had been felt by her research group for a framework other than the ones that were then in use, in particular a framework that would avoid the traditional "technical-conceptual cut":

> In the mid-nineties, we thus became increasingly aware of the fact that we needed other frameworks in order to overcome some research traps that we were more and more sensitive to, the first one being what we called the "technical-conceptual cut" (Artigue 2002, p. 247).

In the search for such frameworks, she and her collaborators turned toward the anthropological theory of the didactic (ATD, or TAD within the French community) with its socio-cultural and institutional basis (Chevallard 1999) and the cognitive ergonomic approach with its tools for thinking about instrumentation processes (Rabardel 1995; Vérillon and Rabardel 1995). Together, these two theories formed the foundational principles for a new theoretical framework, the *instrumental approach to tool use*—a framework that was supported by the earlier research carried out by Artigue and her collaborators (e.g., Artigue et al. 1998; Guin and Trouche 2002; Lagrange 1999, 2000; Trouche 1997). This theoretical work is testimony to the importance Artigue attributed to what we consider an overall characteristic of her research, that of *theoretical frameworks* in the area of technology-enhanced learning. An important feature of this framework is the underlying process of combining, integrating, and adapting the two theoretical orientations for the specific purpose of investigating the opportunities and constraints of the use of digital tools in mathematics education.

It is noted that the combining of Chevallard's anthropological theory of the didactic (with its institutional aspects that impact upon the generic dimensions of teacher, learner, and mathematics) with the cognitive ergonomic approach of Vérillon and Rabardel (with its tool and learner dimensions) into the instrumental approach could be viewed as an early attempt at networking two theories before the term came into vogue—a notion that Artigue addressed in her plenary talk at CERME-5 in 2007. She remarked that this combining had been productive, even if at times it had yielded tensions:

> The difference [between the two frames] reflects in the evident tension existing between on the one hand the language of praxeologies and techniques used in the TAD, and on the other hand the language of schemes used by Rabardel. This tension between schemes and techniques, … between the institutional and the individual, has been extensively discussed in recent years … but up to now has not been overcome. … For me, this is a good illustration of the difficulties that one necessarily meets when trying to integrate two different logics, to build something starting from two different coherences. It shows the difficulties raised by the connection of theoretical frames (Artigue 2007, p. 75).

6.2.2 Further Developments and Impact: Networking of Theories

In order to follow the evolutionary paths of the 2002 paper's main themes and to outline some new directions that have emerged from them, we now address some further developments concerning the combination and confrontation of different theoretical frameworks. While the instrumental approach to tool use continued to develop in France and elsewhere during the years following the turn of the millennium, researchers who were conducting research on the use of digital tools in mathematical learning and teaching were adapting frames involving several other constructs, such as activity theory and social semiotics. The field was becoming marked by fragmentation with respect to the theoretical frameworks used in designing technological tools and in conducting research with these tools (Lagrange et al. 2003). This was making difficult not only productive collaboration among researchers but also the transporting of tools to educational contexts different from those for which they had initially been designed.

To overcome this theoretical fragmentation, the European project Technology Enhanced Learning in Mathematics (TELMA) was created, with Artigue one of the main collaborators. Project participants explored possibilities for connecting and integrating theoretical frames. According to Artigue et al. (2009, p. 218), "very soon, we became convinced that integration could not mean for us the building of a unified theory that would encompass the main theories we were relying on; the number and diversity of theories at stake made such an effort totally unrealistic." Artigue and her collaborators realised that in order to develop an integrated approach to research they needed a shared research practice so as to look at theories in operational terms. Such a practice also needed an appropriate methodology and

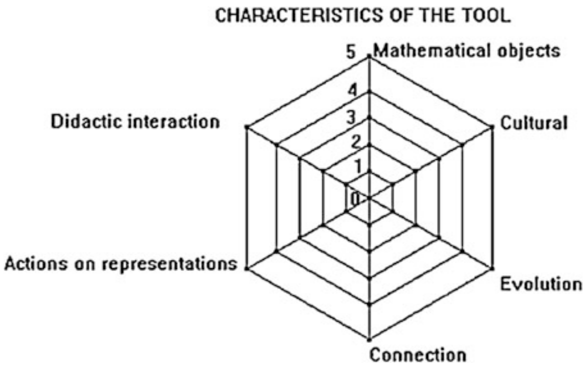

Fig. 6.1 Tool characteristic radar chart within the Integrative Theoretical Framework (ReMath Deliverable 1 2006)

instruments. Radar charts, for example, were used to help position the tools used in different studies (see Fig. 6.1).

Developing this shared research practice led to the constructs of didactical functionality and shared concerns, where tool characteristics, modalities of tool use, and educational goals were central. Tool characteristics included concerns related to ergonomics, semiotic representations, and institutional/cultural distances. Modalities of tool use included concerns related to the interaction with paper-and-pencil work, the social organisation and roles of the different actors, and the functions given over to the tool. Educational goals included concerns related to epistemological, cognitive, social, and institutional considerations. The several cross-experimentation[2] studies carried out by the various TELMA teams revealed that the concerns related to tool, tool use, and goals do indeed drive the entire experimentation process. The development of these concerns constitutes a major contribution by Artigue and her collaborators with respect to the theoretical elaboration of the *tool dimension* in research on technology-enhanced learning of mathematics. The work of the TELMA researchers in developing methodological and conceptual tools was to evolve further when the TELMA teams engaged in another project in continuity with their previous research: the ReMath project[3] (Representing Mathematics with Digital Media).

[2]The TELMA cross-experimentation studies involved pairs of teams coming from different theoretical cultures, but both using the same digital technology—a technology that was well known to one of the teams but alien to the other.

[3]The ReMath project relied on the TELMA meta-language of didactic functionalities and concerns, as well as the system of cross-experiments, but had somewhat different aims. It focused more specifically on representations and issues related to the design of digital artefacts and extended the TELMA methodology to include cross-case-study analyses. For further elaboration of the ways in which the ReMath project developed, modified, and extended the ideas initiated in the TELMA project, see the recently published Artigue and Mariotti (2014) paper, which appeared after this chapter was written.

One of Artigue's early initiatives within the ReMath project was the formulation of a first version of an integrative theoretical frame (ITF), a document that—we note with interest—began to use the language of *networking of theories*:

> The first version of the ITF is neither a theory, nor a meta-structure integrating the seven main theoretical frames used in ReMath into a unified whole. It is more a meta-language allowing the communication between these, a better understanding of the specific coherence underlying each theoretical framework, pointing out overlapping or complementary interests as well as possible conflicts, connecting constructs which, in different frameworks are asked to play similar or close roles or functions. … What has been achieved in TELMA … tends to show that the metaphor of *networking* is, as regards the idea of integrative perspective, better adapted than the metaphor of *unification*, but it only suggests some hints as regards the strategies we could engage for making this networking productive. (ReMath Deliverable 1 2006, p. 31, italics in the original document).

Artigue was not the only one to elaborate on this core idea of *networking of theories*; it received considerable attention at the 2005 Fourth Congress of the European Society for Research in Mathematics Education (CERME 4), as well as at successive ERME congresses (see also Bikner-Ahsbahs and Prediger 2006; Prediger et al. 2008). Some of the strategies proposed for networking theories included comparing, contrasting, coordinating or combining—in fact, strategies that bear a certain relationship to the approaches that were part of the ongoing discourse of researchers from the TELMA and ReMath projects. The interactions among the various researchers participating in the Theory Working Group at the ERME congresses, as well as the reflections of the networking group set up by Angelika Bikner-Ahsbahs and Susanne Prediger at CERME 5 to work between the ERME congresses, have not only advanced researchers' thinking about this emerging area (e.g., Artigue et al. 2005; Cerulli et al. 2005; Kidron et al. 2008; Artigue et al. 2010; Bikner-Ahsbahs et al. 2010) but have also served to stimulate an increase in the very activity of theorising within the field (e.g., Monaghan 2010, 2011; Drijvers et al. 2013a; Godino et al. 2013; Lagrange and Psycharis 2013).

More recently, Artigue et al. (2011) have proposed a broadening of the discussion on networking of theories to include the construct of research praxeologies. Artigue and her co-authors argue that talking about "theories," as in "networking theories," indicates only the theoretical part of research practice. They have therefore extended Chevallard's ATD notion of praxeology to elaborate the pivotal notion of *research praxeology:* It comprises the practice of research (with its task-technique block) along with its technological-theoretical discourse. Artigue et al. stress that research praxeologies are dynamic entities whereby changes in the practical block lead to evolution of the technological-theoretical block and vice versa (i.e., the technical-theoretical dialectic)—changes that involve considering the notion of research phenomena. They maintain that "networking between theoretical frameworks must be situated in a wider perspective than that consisting of the search for connections between the objects and relationships structuring these. … Our reflection tends to show that an approach in terms of research praxeologies can be productive for networking between theories, especially because it helps address the essential issue of the functionality of theoretical frameworks, by inserting these in systems of practices" (Artigue et al. 2011, p. 9).

In sum, our above review of recent literature shows that Artigue's (2002) article describing the interwoven roots of the instrumental approach to tool use was central to the later theoretical work of combining and integrating theoretical frameworks that grew into the networking of theories approach to research in mathematics education.

6.2.3 *Further Work and Impact: Ongoing Developments of ATD*

The above-mentioned Artigue et al. (2011) paper also reflects a second direction of follow-up work in the field of theoretical frameworks, in this case concerning ATD. In particular, researchers around the world have been inspired by Artigue's and her research group's insistence on avoiding the technical-conceptual cut. Her group's development of the idea that the technical has a strong conceptual element, especially during the period of the initial learning of a technique (Lagrange 2000), has been taken up not just in ensuing research involving digital tools (e.g., Nicaud et al. 2004; Boon and Drijvers 2005; Haspekian 2005; Martinez 2013) but also in the theorising of mathematical learning at large (Kieran 2013). As an example of the former, we refer to a research project on the interaction between the technical and the conceptual in the learning of algebra with CAS tools (Kieran et al. 2006), which was framed within the instrumental approach's task-technique-theory (TTT) adaptation of Chevallard's ATD. Within that project, Hitt and Kieran (2009) investigated in detail at close range the task-based activity of a pair of 10th grade students and documented, with the aid of a specially-developed notation (see Fig. 6.2), the ways in which students' emerging theories were systematically being revised as they engaged with CAS tools in concept-building actions within technique-oriented algebraic activity.

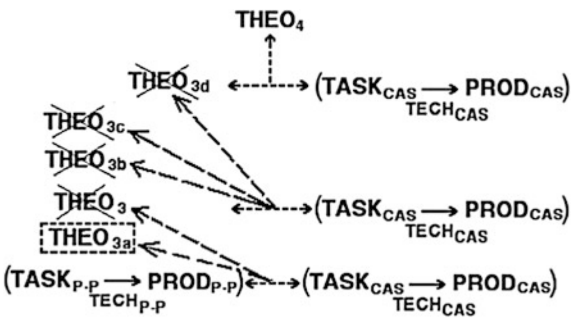

Fig. 6.2 Students' revisions of their theoretical explanations to account for task-based phenomena in a learning environment involving the use of CAS techniques (Hitt and Kieran 2009)

This core idea of the *technical-conceptual connection* (also referred to as the technical-theoretical connection), which was explored in the research of Artigue and her group (Artigue et al. 1998; Lagrange 1999, 2000) and further developed in the above more recent research, has provided a vital new theoretical tool for reflecting on the learning of mathematics. As such, it has led to a different way of thinking about the *learner dimension* within school mathematics, especially in the area of algebra. In this area, where the technical has for decades held sway and conceptual understanding considered all but an oxymoron, the work of Artigue and her colleagues in changing the relationship between technical skills and conceptual understanding has been truly ground breaking. We will come back to this technical-conceptual connection in Sect. 6.4.

6.2.4 Core Theoretical Ideas and Key Dimensions

To summarise Sect. 6.2, which has focused on Artigue's passion for theory, a main theme that has been highlighted is the importance of and need for theoretical foundations of research and development in the field of mathematics education. Two of the key dimensions that we have identified as being central to the theoretical advances that have been made are the *tool* and the *learner* dimensions. The theoretical threads that have been woven into, and have provided texture to, these dimensions include the core idea of the *instrumental approach to tool use* frame, with its concomitant core idea of the *technical-conceptual connection*—the latter yielding novel theoretical perspectives particularly with regard to the *learner dimension* in school mathematics. The *tool dimension* was significantly elaborated by the theorising initiated within the TELMA project and further developed within the ReMath project. Artigue's emphasis on theoretical foundations and the fact that these foundations can arise by a process of 'tinkering', integrating and adapting existing theoretical frameworks within the domain of study, or from outside, is another core idea of Artigue's work—a core idea that may be seen as *networking of theories* 'avant la lettre'.

6.3 Instrumental Approaches and Instrumental Genesis

6.3.1 The Complexity of Instrumental Genesis

In the previous section we drew attention to the emergence of the instrumental approach to tool use, based on principles from ATD and cognitive ergonomics. In our opinion, this instrumental approach was the first fundamental theoretical lens for studying the use of digital tools in mathematics education, and CAS in particular. It proved to be a major contribution to the field (Hoyles and Lagrange 2010) and underlines the importance of tools in use, which through their opportunities and constraints shape and are shaped by student knowledge. Instrumental approaches—we use the plural here because of the different variations that now exist for the theory—acknowledge the impact tools have on the ways in which students do and think about mathematics: “Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures” (Hoyles and Noss 2003, p. 341).

In line with Rabardel’s (1995) distinction between an artefact and an instrument, Artigue in her 2002 IJCML article points out that an instrument is a mixed entity that is part artefact and part cognitive schemes (see also Guin and Trouche 1999). We can summarise this in a ‘formula’: instrument = artefact + scheme. The process by which an artefact becomes an instrument is referred to as *instrumental genesis*—another core theoretical idea. This genetic process works in two ways: in one, the process is directed from the user toward the artefact in that the artefact becomes loaded with potentialities—the instrumentalisation of the artefact; in the other, the process is directed from the artefact toward the user in that the user develops schemes of instrumented action that permit an effective response to given tasks—the instrumentation of the user. An important contribution to our knowledge of using digital technology in mathematics education, now, is the notion that the use of cognitive tools such as advanced calculators or computers is neither self-evident nor trivial, and that the instrumental genesis needed is a complex and time-consuming process.

The research on instrumental genesis emanating from Artigue’s collaborative research group included doctoral theses that illustrated, for example, the diversity of the instrumental relationships that students studying the concept of limit develop with the digital technology of graphical and symbolic calculators (Trouche 1997, whose doctoral thesis was directed by Dominique Guin). Students’ conceptions and ways of interacting with the digital tools led Trouche to characterise five different student profiles: theorist, rationalist, scholastic, tinkerer, and experimentalist. Another thesis (Defouad 2000), which focused on the study of functional variation over the course of the school year and involved Grade 11 students equipped with the TI-92 CAS calculator, pointed to the complexity and fragility of the process of instrumental genesis. For Defouad’s students, instrumental genesis was found to progress slowly through various stages, beginning with the graphical application being used for exploration and solving, and evolving through to the symbolic application for the computation of exact values, at which point the graphical was being used primarily for anticipation and control. A key dimension of research with digital tools that is highlighted in both of these studies is that of the *learner* and the way in which his/her characteristics interact with those of the tool.

6.3.2 Further Developments and Impact: Instrumental Orchestration

The notion of instrumental genesis was followed up in several studies that identified instrumentation schemes and that documented the difficult process of building these up in students (e.g., see Fig. 6.3). However, it was not long before research related

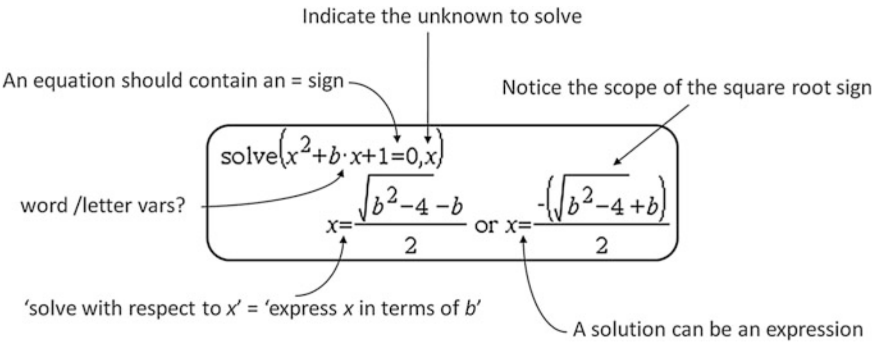

Fig. 6.3 Elements of an instrumentation scheme for solving equations in a CAS environment (Drijvers et al. 2013a)

to the core theoretical idea of instrumental genesis was to focus on the *teacher* dimension, both from the point of view of his/her role within the digitally enhanced learning environment and from the perspective of his/her own instrumental genesis.

The potential synergy between the instrumental approach and the role of the teacher led to Trouche's (2004) elaboration of the construct of instrumental orchestration: "the necessity (for a given institution—a teacher in her/his class, for example) of external steering of students' instrumental genesis" (p. 296). According to Trouche, an instrumental orchestration is defined by didactic configurations and their exploitation modes, the latter of which are aimed at providing students with the means to reflect on their own instrumented activity. In pointing to the instructional role involved in managing and fine-tuning an entire classroom of individualised instruments so as to bring out their collective aspects, Trouche integrates the individual concerns of the ergonomic frame with the institutional concerns of the ATD. Further research on teachers' instrumental orchestrations is reported in, for example, Drijvers and Trouche (2008) and Drijvers et al. (2010), and has resulted in some categorisations (see Fig. 6.4).

Teachers' instrumental genesis has also been an area of study that has evolved from the theoretical frame of the instrumental approach. Bueno-Ravel and Gueudet (2007), who participated in the GUPTEN (Genesis of Professional Uses of Technologies by Teachers) project spearheaded by Jean-Baptiste Lagrange, focused specifically on e-exercises and the way in which these artefactual resources become instruments for the teacher through a process of instrumental genesis. Artigue and Bardini (2010) studied teachers' instrumental geneses in a project involving the use of a new tool, the TI-Nspire CAS. In particular, they addressed the issue of the relationships between the development of mathematical knowledge and instrumental genesis and noted the impact of new kinds of instrumental distance (see Haspekian and Artigue 2007) and closeness that shape teachers' activities.

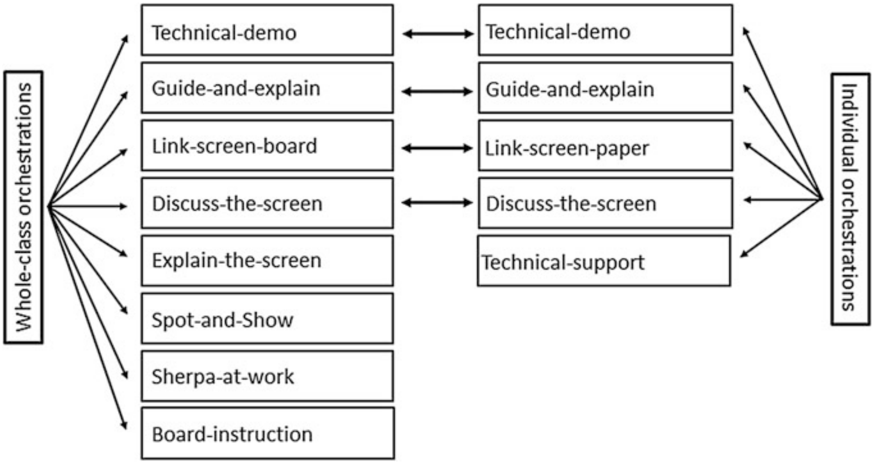

Fig. 6.4 A first inventory of teachers' orchestrations (Drijvers et al. 2013b)

6.3.3 Further Developments and Impact: The Documentational Approach

A further evolution of the research on teachers' instrumental geneses has been the theoretical transformation of this focus into a new frame that is referred to as the documentational approach of didactics (Gueudet and Trouche 2009). In this theoretical frame for studying teachers' documentation work, the artefact-instrument dialectic within instrumental genesis has been recrafted as the resource-document dialectic within the process of documentational genesis. The new 'formula' thus becomes: document = resource + scheme. This theoretical frame, which places documentation work at the core of teachers' professional growth, has been further developed in Gueudet and Trouche (2010) and Gueudet et al. (2012). As an elaboration, Sabra (2011) sketches the 'fabric' of a resource system for one particular teacher (see Fig. 6.5). Even more recently, this approach has evolved to take into account the way in which documentation work is also central to the professional activity of design researchers (Kieran et al. 2013).

6.3.4 Core Theoretical Ideas and Key Dimensions

To summarise, in Sect. 6.3 we have focused on the complexity of the use of digital tools and the corresponding instrumental genesis, and on the ways in which this construct had been applied and developed by Artigue's collaborators and by other researchers outside France. The dimensional threads that have been theoretically elaborated in that research include: the *tool* (and its use), the *learner,* and the

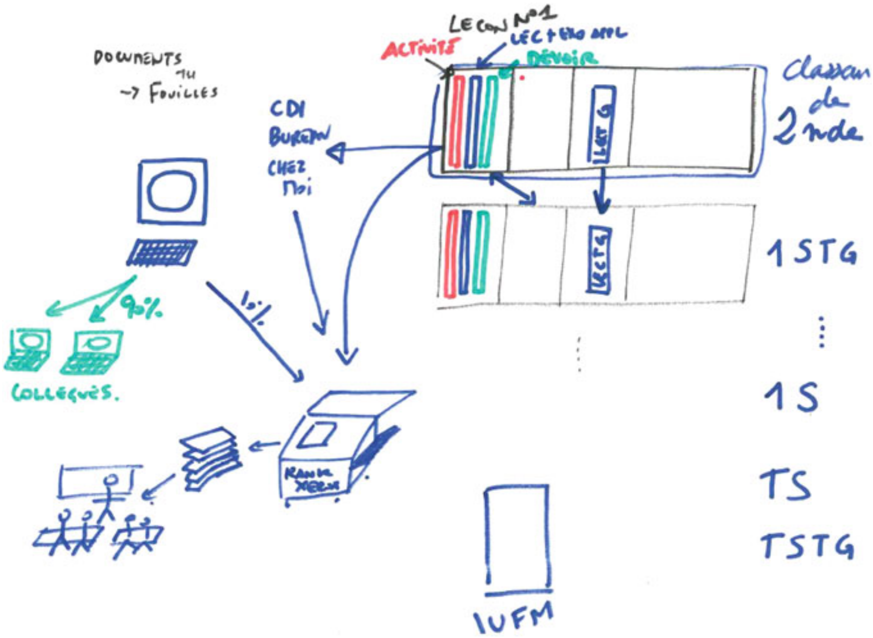

Fig. 6.5 An inventory of one teacher's resource system (Sabra 2011)

teacher. The core theoretical idea that has been interwoven through, and that has given a particular theoretical sense to, these dimensional threads has been the construct of *instrumental genesis*.

6.4 The Pragmatic-Epistemic Duality

6.4.1 The Pragmatic and Epistemic Value of Techniques

In Sect. 6.2, the avoidance of the technical-conceptual cut was mentioned as a hallmark of research on the use of digital tools in mathematics education—one that has been inherited from the instrumental approach to tool use. The technical aspects of using digital tools clearly incorporate a strong conceptual element and reconciling these two can be seen as an important component of instrumental genesis. Thus, while the conceptual is intricately interwoven with the technical within the core idea of the *technical-conceptual connection,* the role of technique in contributing to the development of the conceptual is central—and this brings us to the pragmatic-epistemic duality.

An important contribution of Artigue's (2002) article is the distinction she draws between the pragmatic and epistemic values of techniques. Within the instrumental approach, the *pragmatic* value of techniques refers to their “productive potential” (Artigue 2002, p. 248), while their *epistemic* value refers to “their contribution to the understanding of the objects they involve”, particularly during their period of learning when they constitute a source of questions about mathematical knowledge (see also Lagrange 2000). In her CERME-5 plenary lecture, Artigue (2007, p. 72) clarified an important point about this duality within the instrumental approach to tool use: “While technique is a fundamental object of the ATD, the ATD does not distinguish between the epistemic and pragmatic values of techniques; these terms come from cognitive ergonomy, but there they are linked to schemes and not to techniques.”

Taking the pragmatic-epistemic notion of the ergonomic approach and connecting it with the objects of the ATD was an astute move on Artigue’s part. Having already linked techniques to schemes by having the former designate the visible part of the latter, Artigue could then refer to the epistemic and pragmatic values of techniques. However, the appropriation of the pragmatic-epistemic duality within the instrumental approach allowed for much more than this. It provided for considering the ‘mathematical needs of instrumentation’ (a phrase that combined the mathematical underpinnings of the ATD with the instrumentational aspects of the ergonomic approach) and for these mathematical needs to be interpreted in terms of the epistemic value of instrumented techniques. In addition, it supported a pragmatic-epistemic perspective on the two ATD objects of technique and theory and highlighted the relationship between the two. As well, it opened up a discourse for comparing and contrasting the pragmatic and epistemic values of “official” mathematics with the pragmatic and epistemic values of instrumented mathematics. The multiple ways in which the notion of *pragmatic-epistemic duality* allowed for aligning the contributions of the ATD and of the ergonomic approach within the instrumental frame, as well as for operationalising their interactions, render it a truly core theoretical idea of Artigue’s work.

Three elements of Artigue’s research that are intertwined with the pragmatic-epistemic duality, but which can also be considered central notions in their own right, are the following: the institutional aspect, the task design component, and the mathematical dimension. The first element, the institutional aspect, refers to the educational, social and institutional contexts of techniques. In line with ATD, Artigue (2002) describes how teachers in French mathematics classes during the first year of a study were observed to have difficulty in giving adequate status to instrumented techniques. In contrast to the standard way in which paper-and-pencil techniques were explored, routinised, and institutionalised, the several digital techniques that were introduced suffered from ad hoc treatments that prevented them from becoming efficient and productive. The theoretical discourse accompanying the use of such techniques remained fragmentary and underdeveloped. Artigue points out that, while the “kinds of discourse which can be developed are well known for official paper and pencil techniques, … a discourse has to be constructed for instrumented techniques … a discourse that will call up knowledge which goes beyond the standard mathematics culture” (Artigue 2002, p. 261). The institutional roots of this difficulty are emphasised: “The institutional negotiation of the specific mathematical needs required by instrumentation [is] a negotiation which today is not an easy one" (p. 268). This *institutional aspect*, which was central to the ATD, remained a core theoretical idea that was threaded through all of Artigue's research (see Artigue 2012).

Second, in her discussion of the pragmatic-epistemic duality, Artigue relates the constructing of an adequate discourse for instrumented techniques to task design, that is, to the process of didactical engineering or ingénierie didactique. According to Artigue, developing appropriate situations and tasks for instrumental work was a challenge for the teachers involved in her research; they were unsure how to design tasks that make provision for developing the epistemic value of techniques. In this regard, Artigue (2002, p. 268) points out that "epistemic value is not something that can be defined in an absolute way; it depends on contexts, both cognitive and institutional; from the contextual [and mathematical] analysis of this potential to its effective realisation there is a long way, with situations to build, viability tests, and taking into account the connection and competition between paper and pencil and instrumented techniques." The latter remark highlights yet another core idea of her work: the *relationship between paper-and-pencil and digitally-instrumented techniques*. She notes that particular attention needs to be paid to the relationship between techniques for using digital tools and 'traditional' paper-and-pencil techniques: While both the pragmatic and the epistemic values are obvious for the case of "official" paper-and-pencil techniques in that "the epistemic value of a paper-and-pencil technique becomes evident through the details of its technical gestures" (Artigue 2002, p. 259), the epistemic value of instrumented techniques seems much less obvious.

Last but not least, a crucial step in the design of task sequences is a thorough analysis of the underlying mathematical domain. In commenting that more than the standard mathematics is called for when dealing with instrumented techniques, Artigue emphasises not only the mathematical needs of instrumentation but also the requirement for a deep a priori analysis of the mathematics embedded in the tool and its use. She thereby stresses the importance of elaborating the *mathematical dimension* within research studies—an emphasis that is shared by fellow researchers of the French *didactique* tradition (see also Brousseau 1997). In one of her examples, Artigue (2002) refers to the topic of equivalence of expressions and the problem of detecting equality for certain types of algebraic expressions in a CAS environment. She points out that the CAS tool can produce results—often quite surprising and unexpected—that go beyond what is usually faced in non-digital-technology-supported mathematics classrooms when algebraic expressions are to be simplified. In her ensuing discussion of the mathematical needs required for an efficient instrumentation, which she expresses in terms of the epistemic value of instrumented techniques, Artigue (2002, p. 260) suggests that the epistemic has to be provided for by constructing a mathematical discourse around it: "The epistemic value of instrumented gestures is something that must be thought about and reconstructed; in the teaching process, it has to be developed through an adequate set of situations and tasks".

6.4.2 Further Developments and Impact: The Institutional Aspect

The institutional/cultural aspect of the instrumental approach was highlighted in the work of the TELMA and ReMath projects, where institutional considerations figured into the three main theoretical developments of the two projects: tool characteristics, modalities of use, and educational goals. This aspect was also reflected in the practice of the participating research teams, as witnessed by their own institutional/cultural approaches to research. More recently, Artigue (2012) in her MERGA plenary presentation on multiculturalism in mathematics education research returned explicitly to the institutional aspect of Chevallard's ATD theory:

> Sensitivity to the cultural dependence of mathematics education must be supported by appropriate constructs and methodological tools for being productive. With the development of socio-cultural approaches, the field of mathematics education today offers a diversity of theoretical frameworks and constructs for such a purpose. As with many French colleagues, due to my cultural environment, I have found a support in the Anthropological Theory of Didactics (ATD). In this theory initiated by Chevallard, indeed, an initial postulate is that human knowledge emerges from practices which are institutionally situated thus a fortiori culturally situated (p. 6).

The attention paid to institutional conditions and constraints is also manifest in the documentational approach of didactics (Gueudet and Trouche 2009). As shown in Fig. 6.6, institutional influences may hinder or enhance teachers' documentational genesis to an important extent.

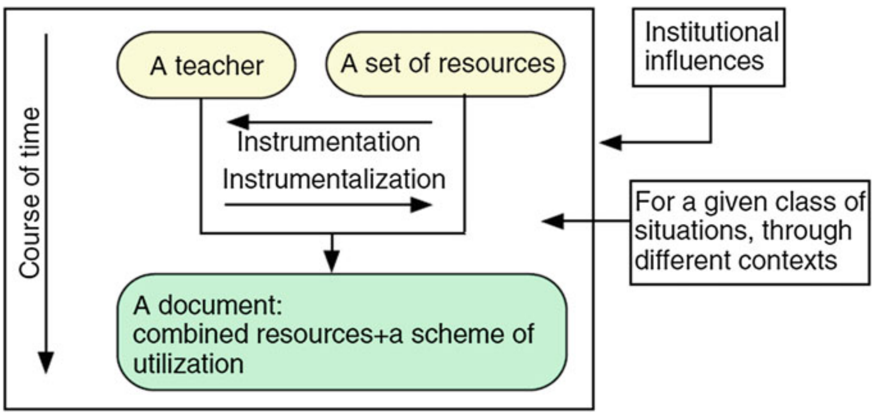

Fig. 6.6 The institutional aspect in documentational genesis (Gueudet and Trouche 2009)

6.4.3 Further Developments and Impact: Task Design and Mathematical Analysis

The potential interactions between, on the one hand, the pragmatic and epistemic values of techniques and, on the other hand, techniques instrumented digitally and paper-and-pencil techniques, served as a basis for designing tasks in a CAS study on equivalence reported by Kieran and Drijvers (2006). Task-sequences were designed that would invite both technical and theoretical development, as well as their co-emergence. One of the observations of the study was that most students wanted to be able to produce themselves, by means of paper and pencil, the results that were output by the CAS whenever the CAS results could not be explained by their existing technical and conceptual knowledge. That is, CAS and paper-and-pencil techniques were found to be interrelated epistemically and co-constitutive of students' theoretical development. However, it was also found that the a priori mathematical analysis of the notion of algebraic equivalence, which had guided the initial design of the study, did not go far enough. Data from student work indicated that the mathematical analysis by the task designers had to be further developed because it had not adequately taken into account the importance of domain considerations and transitivity in students' evolving conceptual understanding of equivalence (Kieran et al. 2013; also see Fig. 6.7). This led to a deeper theoretical elaboration of the dimensional thread related to the underlying mathematics and, at the same time, confirmed once again the importance of Artigue's insistence on the mathematical needs of instrumentation.

6.4.4 Core Theoretical Ideas and Key Dimensions

To summarise, Sect. 6.4 has highlighted the importance in Artigue's work of the dimensional thread related to the *mathematics*, that is, to the requirement for deep a priori mathematical analysis of the needs of instrumentation and for developing

We shall restrict ourselves to single-variable expressions. We just implicitly saw that there are two definitions for the equivalence of two expressions $f(x)$ and $g(x)$, equivalence for which we will use the usual notation $f(x) \equiv g(x)$:

- A *syntactic definition*: $f(x)$ and $g(x)$ are equivalent if and only if we can establish their equality by symbol manipulation, using rules recognized as true for the set $\mathcal{E}$.
- A *semantic definition*: $f(x)$ and $g(x)$ are equivalent if and only if for every element a in $\mathcal{E}$ we have an equality between $f(a)$ and $g(a)$ (we shall refer to this particular definition as Semantic Definition of Equivalence, Version 1).

Fig. 6.7 Extract from a mathematical analysis of the notion of algebraic equivalence in the Kieran et al. study (2013) adequate situations and tasks for instrumental work. Interwoven with the key mathematical dimension have been the three core theoretical ideas of the *pragmatic-epistemic duality*, the *relationship between paper-and-pencil and digitally-instrumented techniques*, and the *institutional* aspect.

6.5 Closing Remarks

In this chapter, we have revisited Michèle Artigue's classic 2002 IJCML article and have drawn out what we consider to be the core theoretical ideas and key dimensions of the body of work on tools and tool use that Michèle not only elaborated but also inspired others to further develop. We have traced the evolutionary path of these core ideas, noting the ways in which they theorised the four general key dimensions of learner, teacher, tool, and mathematics. Without claiming to be exhaustive in our selection, we have focused on seven core theoretical ideas that have been central to Michèle's work and that have impacted in various ways the research of others: the instrumental approach to tool use, instrumental genesis, the pragmatic-epistemic duality, the technical-conceptual connection, the paper-and-pencil versus digitally-instrumented-technique relationship, the institutional aspect, and the networking of theories.

We realise that we have discussed these core theoretical ideas as if they were separable, one from the other. Of course, they are all related, with each but the last being an intrinsic part of the frame of the instrumental approach to tool use. However, while the core idea that is the instrumental approach to tool use is an overarching one that subsumes most of the others, several of its component core ideas merited being singled out and discussed individually. Some have been further developed in various ways—sometimes without involving the use of digital tools—and have even taken on lives of their own. This was noted, for example, with the core theoretical idea of instrumental genesis, one strand of which has evolved into documentational genesis and the frame of the documentational approach. Another is the core theoretical idea of the technical-conceptual connection that has been applied more broadly in recent research on mathematical learning.

The dimension of tools and tool use has been at the heart of Michèle's work on instrumentation and thus has been central to her theoretical work. Nevertheless, her contributions extend beyond this dimension. Michèle's theoretical ideas have had a profound impact on the ways in which we think about some of the other basic dimensions of mathematics education, such as the learner, the teacher, and the mathematics. The further developments and impact of the core ideas and key dimensions that we have described in this chapter are clear testimony to the richness of Michèle Artigue's theoretical contributions, for which we have much to be thankful.

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Chapter 7
Digital Technology and Mathematics Education: The Teacher Perspective in Mathematics Education Research—A Long and Slow Journey Still Unfinished

Maha Abboud-Blanchard

7.1 Introduction

Roughly two decades have passed since a UNESCO report (1992) gave an overview of the impact of computers and calculators on mathematics education at the end of the eighties and in the early nineties. Following the first ICMI Study on technology (1985), the editors wished to update some outcomes of this study and to republish some others in order to make them available to mathematics educators throughout the world. In the chapter entitled “Teacher education and training”, Bernard Cornu declares:

> However, computers are now very common in society; they are used in many domains of daily life. In many countries national plans for computer equipment in schools have been achieved, and so a lot of computers are available in schools. Much educational software has been produced, and it is often of high quality. The use of computers does indeed become easier. […] Current and future teachers must be prepared for this evolution. It is not enough to master the knowledge and some pedagogical strategies and tools. Teachers must be able to deal with all the evolution which will happen, and to adapt to different kinds of pupils (pp. 87–88).

Despite significant advances in technological tools and environments for mathematics teaching, and in educational research related to this field, it is hard to claim nowadays that encouraging teachers to integrate technology into their practices is no longer necessary or a priority among institutional policymakers. In a recent UNESCO report (Artigue 2011), Michèle Artigue, referring to the 17th ICMI study (Hoyles and Lagrange 2010), states:

> Technologies have undeniably enriched the possibilities of experimentation, visualisation and simulation; they changed the relation to calculation, the relation to geometric figures. […] However, in spite of their undeniable potential for enhancing the teaching and learning

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> of mathematics and their many positive achievements, they have to date had little effect even in education systems that strongly encourage their use. Recent work on teachers' practices in computer environments is beginning to give insights into this situation, and forms of training properly adapted to teachers' needs are being considered. Nevertheless, the issue of widespread effective use of these technologies in basic mathematics education remains for the moment unresolved. (p. 35)

Indeed, research focusing on teachers' practices in mathematics classes is a relatively recent phenomenon within the history (even though short) of mathematics education and technology. During the first decade of use of technology the role of teacher and its eventual changes was not a central issue for several reasons, especially the opposition between technical and conceptual work, prevailing in the discourse of innovation and research and also the underestimation of instrumental issues. Technology by itself was considered to foster changes in the teacher's role. This first trend gradually gave way to research that recognised and attempted to better understand the challenges that teachers face in the presence of digital technology (DT) and the need to rethink new possibilities for doing mathematics and addressing classroom management issues (Healy and Lagrange 2010).

In this chapter I will review from this latter perspective the trajectory of Michèle Artigue, showing how by directing doctoral theses, by conducting national innovative projects and by participating in projects crossing cultural and educational contexts, she played a substantial role in the research in this area. The following sections are not conceived as a continuous chronological path but rather as milestones based on episodes from Michèle's "long story with technology".[1]

7.2 From Students' Tasks to Teachers' Practices

In the early nineties, the French Ministry of Education asked a group of researchers headed by Artigue to work together with a group of expert teachers to identify the potential offered by a computer algebra system (CAS), DERIVE, for the teaching and learning of mathematics at secondary level (Abboud et al. 1995; Artigue 1997). The observations made during this collaboration clearly showed that integrating CAS into mathematics teaching was absolutely not a matter of simply adding a new artefact into a classroom. Indeed, the researchers asked the expert teachers to design lessons that they considered to be evidencing the power of CAS for mathematics learning. The teachers then provided lessons plans and their rationale as well as their expectations on how the lesson plans would take place in their classrooms (computer rooms). By observing these lessons, the research team shed light on the fact that the mathematical dynamics in the classroom within the lesson in progress resulted from a balance between two opposite tendencies. The first one favoured reflexive work, as expected by teachers. The second focused on productive

[1]The title of the chapter is inspired by a wording used by Artigue (2012, p. 25).

strategies, reducing thus the global coherence of students' activity. This was especially the case when students used what were named "fishing strategies": multiplying trials without spending time understanding computer feedback up to the moment something easily interpretable (in terms of the task's performing) happened. The research also found that the role of the teachers and their management of the lesson were essential for maintaining this adequate, but complex, balance. It was the didactic expertise of the teacher that prevented the second tendency (productive strategies) gaining precedence over the first one (reflexive work).

Broadly, this study showed that in addition to thinking out the affordances of digital technologies (DT) and the ways it affects mathematical learning, it became essential to investigate the role of the teacher within technology-mediated lessons. Simultaneously other researchers, in different contexts, came to the same conclusion. This was confirmed by a meta-analysis of over 600 international publications (published before 2000) on DT in the teaching and learning of mathematics (Lagrange et al. 2003). Starting from this period, the emphasis in many research studies began to shift toward aspects of teachers' practices when DT was integrated into their teaching, and on the complexity of this integration.

For instance, in the French sphere, Laborde and colleagues (Laborde 2001) highlighted that appropriation of DT by teachers was a long process. Reporting on teaching scenarios using dynamic geometry and their evolution over 3 years, Laborde stressed:

> We assume that really integrating technology into teaching takes time for teachers because it takes time for them to accept that learning might occur in computer-based situations without reference to a paper-and-pencil environment and to be able to create appropriate learning situations. But it also takes time for them to accept that they might lose part of their control over what students do (p. 311).

This is consistent with Ruthven et al. (2005), whose research tackling the use of dynamic geometry shows that teachers may constrain the potential of technology in order to retain control of the classroom. Indeed, teachers tend to reduce the exploratory dimension of DT in order to control students' explorations and to avoid students encountering situations that could obscure the underlying rule or could require explanations that go beyond the narrow scope of the lesson.

Adopting a more holistic perspective in understanding the key factors of teachers' activities and roles, Monaghan (2004) used Saxe's cultural model centred on emergent goals under the influence of four parameters. Using this model enabled Monaghan to locate critical influences on teachers' practice which he then used to explore the complexities of integrating technology into teaching. A central feature of this model was the notion of emergent goal that proved adequate to spot and interpret phenomena that frequently occur in technology based lessons, resulting of a gap between what the teacher was expecting and what really happened. Lagrange and Ozdemir (2009) also used this model to analyse episodes encountered by experienced teachers, marked by improvisation and uncertainty. By contrasting, in similar lessons, the classroom activity of two teachers (one positively disposed towards the use of technology and the other not), they showed that suitable settings and favourable parameters did not guarantee less complexity in DT integration.

More generally, one can conclude from these examples and many others in education research that different approaches and frameworks were developed with the same purpose of better understanding the way teachers use technologies. Particular orientations can also be encountered that offer a wide range of theoretical and methodological constructs to examine teachers' use of technology in classrooms. A recent volume, "*The mathematics teacher in the digital era*", edited by Clark-Wilson et al. (2014), provides a more detailed and current overview of this domain.

Returning to the issue that opened this section, I point out that after the DERIVE project, a second project followed, also directed by Artigue, that studied the uses of symbolic calculators. The outcomes and theoretical work resulting from these two projects led Artigue and her colleagues to introduce a new and currently well-known framework, the Instrumental Approach (IA), which is presented and detailed in Chap. 6 (Kieran and Drijvers in this volume). It is worth noting that IA was first used as means of studying students' instrumental genesis; and a teacher perspective was afterwards introduced, that is, the notion of instrumental orchestration (Trouche 2004), also used by Drijvers et al. (2010).

7.3 Towards Geneses of Technology Uses

In 2003, Artigue and her colleagues was in charge of a regional French project involving more than 50 Grade 10 mathematics teachers (Artigue et al. 2007). The project aimed to study volunteer teachers' uses of web-based resources (Electronic-Exercise-Bases (EEB)) over a period of three years. The study was qualitative and the data was collected from lesson preparation plans, class observations, and responses to questionnaires and interviews. Most of these teachers were familiar with the use of technology in the classroom at the beginning of the project. The goal of the project was pragmatic in that it involved observing the potential of EEB in ordinary classes, with an emphasis on helping the weaker students (Abboud-Blanchard et al. 2007). The data analysis addressed the general questions: Why and how do teachers use EEB? What effect does this use have on their teaching activity? The outcomes emphasised the impact of using EEB in three phases of the teachers' activity: preparing lessons; interacting with students during a lesson; and reflecting after the lesson on a comparison between what was prepared and the effective activity of students. Outcomes mainly referred to teachers' wish to control the students' activity contrary to specificities of the EEB, and to teachers' focus on mathematical process in contrast with the EEB focus on answers only.

Data mining was also used in a subsequent national research project that explored the geneses of technology uses in different educational contexts: the

GUPTEN[2] project (Lagrange 2013). The aim of this project was to better understand how the practices of the teachers involved in the regional project evolved over time and the factors that shaped this evolution. Indeed, the GUPTEN project had a challenging perspective which was to identify the ordinary "uses" of DT that teachers developed, beyond the original perspective of innovation, and by searching for a state of equilibrium between paper-and-pencil traditional practices and the response to an external (institutional and social) demand to integrate DT into mathematics teaching and learning.

Multiple frameworks were used in this national project. The researchers attempted to combine the Instrumental Approach with a framework used in the French educational field to study teaching practices: the Double Approach (DA), with the Activity Theory as its frame of reference. The DA was introduced and developed by Robert and Rogalski (2005) to incorporate, on the one hand, a didactical perspective, which views the teachers' activities that involve task choices and classroom management as a key factor affecting students' activities, and on the other hand an ergonomic perspective, which considers teachers as professionals having craft knowledge, beliefs and previous experience whilst working in given institutional and social conditions. We also used the distinction between "productive" activity and "constructive" activity emphasized by cognitive ergonomists like Samurçay and Rabardel (2004). Indeed, by their actions, the subject (the teacher in this case) modifies the situation but also changes him or herself, i.e., develops his or her knowledge or builds new knowledge. The fact that the Instrumental approach and the DA could both be considered as expanding Activity Theory, by adding and articulating mathematics didactic perspective, ensured a certain a priori consistency and continuity on a meta-level within the national research study.

The main contribution of this work was to provide a theoretical construct which could be used to grapple with the complexity of the emergence and evolution of teachers' practices in technology-based lessons. This theoretical construct was a thoughtful way of modelling geneses of technology uses (for more details, see Abboud-Blanchard and Lagrange 2006, Abboud-Blanchard and Vandebrouck 2012). It focused on the development of technology uses by teachers. Considering the processes of instrumental genesis of specific artefacts by the teacher, the notion of geneses of uses transcends these processes by taking into account the globality and stability of the teacher's practices (with and without DT). The geneses of uses are considered as patterns of development in three levels of practice' organisation, related both to temporality and to goals in the teaching activity: the micro level of "automatisms" and elementary gestures; the local level related to management issues and to teacher-students interactions; and the global level referring to preparations and scenarios.

As explained at the end of the above section, the notion of instrumental genesis was introduced and developed in research into mathematics education by Michèle Artigue and her colleagues. They also drew from the Anthropological Theory of

[2]Genèses d'Usages Professionnels des Technologies Numériques chez les Enseignants.

Didactics (ATD), highlighting the role of instrumented techniques and of their interaction with paper-and-pencil techniques, and introducing the twofold characteristics of techniques-"pragmatic" and "epistemic"-that Rabardel already identified for instrumental schemes.

As to the recent development that the GUPTEN project proposes, it broadly draws on the development of the instrumental approach by Rabardel and colleagues. Indeed, artefacts are considered not as isolated but as inscribed into systems of artefacts; the subjects' activity often implies the use of multiple instruments (Rabardel and Bourmaud 2005).

Relative to the two main conceptual frameworks inspiring these developments, Lagrange (2012) recently attempted to evaluate the contribution of ATD and of activity theory (and so to the ensuing IA and DA) in view of overarching issues related to the use of DT by mathematics teachers. He emphasised that, although very different in their nature and roots, the two theories *start from a common vision of knowledge as the product of human activity in social and cultural contexts* (p. 33). He then demonstrated how, in a specific research study, the use of the two frameworks was possible and insightful.

These diverse efforts that aim to compare, connect and integrate theoretical frames are in accordance with Michèle Artigue's current preoccupations about the networking of theories (see for example Artigue and Mariotti 2014). This trend in Michèle's research is further developed in Chap. 6 (Kieran and Drijvers in this volume) and Chap. 3 (Kidron et al. in this volume).

7.4 Improving Teacher Education

In the proceedings from a Conference on ICT in school mathematics, Artigue (1998) entitled her paper: "*Teacher training as a key issue for the integration of computer technologies*". The paper was her first published overview of obstacles to the integration of ICT; she claimed that *the poor sensitivity* of teacher training to these obstacles partly explains *its poor efficiency*. Indeed, research projects in which she was involved and doctoral thesis that she was directing (see for example Abboud 1994) brought her to highlight the fact that a current tendency, at the time, in teacher training was to consider that disturbances due to the presence of technologies can be avoided by *careful preparation and proper choice* of a situation's variables relative to students' tasks. In the closing section of the text, she states:

> Teacher training based on innovative values and militancy has shown evident limits. For the reasons mentioned above, our personal conviction is that such resistant obstacles will not be overcome without giving didactic analysis a more important role in teacher training and without providing teachers with didactic tools allowing them to analyse transpositive processes, to identify the didactic variables of situations and pilot them, analyse their professional techniques and the way these are modified by the use of computer technologies (p. 127).

Several years later, in a text synthesising her current research, Artigue once more attempted to identify and analyse the difficulties and challenges in technology integration (Artigue 2004). She ended her discussion with the following statement:

> Training has not been able up to now, to go beyond the primitive phase of pioneer militancy that is associated to the entrance of any kind of newness in the educational system. This fact leads to a general underestimation of the necessary changes in the professional work of the teacher, of the mathematical, technological and didactic expertise required if one wants to have computer technologies really benefits mathematics teaching and learning (p. 221).

She also emphasised that educational policies continued to overlook how DT participates in *destabilising established routines* and increasing the complexity of teaching activity. She added that necessary changes must be supported by insightful knowledge and appropriate training.

This situation described by Michèle has gradually become a major concern of current research. Most teacher education researchers are themselves teacher educators studying the teachers with whom they work. Some others are focusing on the kinds of knowledge developed in teacher training courses and on training strategies used by teachers' educators (Abboud-Blanchard 2013; Emprin 2007). In the 17th ICMI Study (Hoyles and Lagrange 2010), several papers analysed views and options of teacher education courses in mathematics and technology. They offered features by which teacher education courses might be characterised, especially those of changes in teachers' role, activity and practices. In their text, Grugeon et al. (2010) describe a number of teacher development courses implanted in different cultural and educational contexts. Even if each course had its own consistency, the authors tried to determine the underlying options and hypotheses; a categorisation model according to the content and the teaching strategies followed. Six types of course content were identified: the potential of software for learning, the evolution of curriculum due to technology use, instrumental genesis, the reworking of old tasks and the creation of new tasks for use with technology, appropriate new teaching abilities, and working with technology in various professional contexts. Four main strategies were also identified: demonstration (showing how to), role playing (teacher as student), 'in practice' (teacher as reflective practitioner), and learning communities. It is interesting to note that the first strategy, demonstrating good practices, was the only one common to all five courses. The authors hoped through this classification to provide useful support for future work in analysing and describing teacher education projects. Reflecting on the latter, Artigue (2010) declared in her closing chapter:

> But we find also in the different contributions some evidence that we are now ready to enter a new phase, and that the Study can efficiently contribute to this new phase through the analysis it provides of current practices and of their resulting effects, through the methodological and conceptual tools it proposes, through the positive and substantial examples it presents of teacher preparation and professional development programs. These examples moreover show that the technology itself offers now new and powerful tools for supporting and accompanying the professional development of teachers in that area, seen as a collective and collaborative enterprise […] (p. 471)

Shifting into a 'higher level', new educational studies have emerged that tackle the topic of educating the educators.[3] New approaches focus on the need for a much closer coordination between research, development, design, and practice, and acknowledge the impact that educating teacher educators has on improving teacher education itself, which has, in turn, repercussions for a wider implantation of technology in everyday teaching practices. Still, to be efficient, this implies long-term projects and innovative design of teacher education courses (Abboud-Blanchard and Robert 2013). This cannot, though, be the task of individuals, neither left in the sole hands of researchers (Artigue 2009). Instead, it requires specific structures able to organise and evaluate the effects of such approaches on teacher educators' professional development.

All of the previous issues and results were discussed within the session dedicated to this topic[4] in the International Conference "Hommage à Michèle Artigue" (in Paris 2012). But at the same time, open questions emerged such as: What main criteria are needed, and must be displayed, in teacher education to help teachers efficiently organise technology-based sequences? How can questioning the balance between pragmatic and epistemic values of technology (Artigue 2002) be integrated in pre-service and in-service teacher education? How do we adapt current teacher education to new generations of "plugged in and connected" teachers? Following on from this last question, Artigue points out that we are living in a new digital era and are witnessing the arrival and spread of new artefacts which shape our personal and professional lives, including smartphones, touch-sensitive screens, mathematical applets, and diverse mobile technology devices. She also stresses the important role that social networks play in communication modes (Artigue 2013). She deduces that traditional paths in teacher education must be improved especially by further reflecting on the potentialities of e-learning and on the growth of new ways of teaching such as the Massive Online Open Courses (MOOC).

To summarise, I illustrated throughout this section how the discourse of Artigue on teacher education kept coherence and even a certain recurrence over the passage of time. By synthesising issues and outcomes from national and international research, I pointed out that teacher education remains a key factor for any possible evolution of DT integration in educational systems. It surely illustrates how difficult these issues are to deal with and that the research community has to develop new directions to challenge visions related to teachers' professional development and to expand this research area.

[3]See for example the Conference on international approaches to scaling up professional development in maths and science education, December 2014 in Essen, Germany.

[4]This session was coordinated together with Colette Laborde; special thanks to her.

7.5 Concluding Comments

Throughout this chapter I have attempted to revisit Michèle Artigue's work on the teacher's perspective in technology-based education. I used episodes from her research work to show how she contributed, often in a collaborative way, to introduce new guidelines and new perspectives for research in this field. Nevertheless it is also a fine-grain analysis based on her work within diverse research projects, which led Artigue to provide reliable insights into the complex issue of integrating technologies into educational systems.

I would like to add that Michèle had, and still has, a role of actor beyond her research activity, often positioned at an institutional level; intervening and working to improve mathematics teaching and learning in technology environments. One could say that she is playing a dual role, by creating efficient conceptual frames and methodological tools, and at the same time working in close collaboration with actors. For instance, she regularly designed and intervened in teacher education courses proposed by the Institute of Research in Mathematics Education (IREM) that she headed for several years and still is an active member of the board. She also continued to work with secondary teachers to experiment with innovation, to produce educational resources, and to participate in projects addressing professional development. Indeed, to accompany the slow evolution of incorporating technology into school mathematics, researchers have to take into account a vision close to what teachers experience in their everyday professional lives.

Finally, I would like to quote Michèle from a recent plenary conference entitled: *Teacher education and technology: a major challenge*[5]:

> […] to teach math in this digital era, it is not only to learn to integrate in teaching practices technologies such as calculators, dynamic geometry, spreadsheets or computer algebra systems (CAS) which have long been the emblem of this integration, and this even if the effective integration of these 'old' technology is still marginal. It is learning to take advantage of many resources provided by the digital world for teaching and learning, and new modes of social interaction and communication that it promotes. […] More than ever, the need to thorough studies of teachers' professional development related to digital technologies is topical; the need for a better understanding of teaching practices in technological environments, of their determinants, and of their evolution dynamics. (Artigue 2013, pp. 5–6, our translation)

Indeed, this is another challenge for all educational researchers to continue towards new directions and perspectives in a journey that is still unfinished…

[5]The original French title is: La formation technologique des enseignants: un défi majeur.

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Chapter 8
History of Science, Epistemology and Mathematics Education Research

Renaud Chorlay and Cécile de Hosson

8.1 Introduction

In her paper of 1990 entitled *Epistémologie et didactique*, Michèle Artigue reflected on 10 years of practice within the French mathematics education[1] community, while stressing the *need* for epistemology for the working researcher. First, she underlined the need for epistemological awareness as an *experience* for the researcher, enabling distance between the researcher and their personal mathematical culture; second, she pointed out that some *knowledge* of the history of mathematics was of a key component of didactical research, either to understand the historical development of a mathematical concept, or to understand the shaping of mathematics as a ruled cultural activity.

This chapter will, to a large extent, directly echo the original Artigue paper, starting with a common take on "epistemology". Of course, the word has many meanings: rather than the noun "epistemology", which seems to denote a well defined research field, we will use the adjective "epistemological" to denote the *endeavour* of deriving insight from knowledge/awareness of the history of mathematics that is

[1]We will use the following acronyms: MER = mathematics education research, HM = history of mathematics.

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relevant from a ME research perspective.[2] Also, in spite of the fact that it discussed several well-known and important papers from the 1980s, the Artigue paper was not a literature review. Rather, the literaturereferenced by Artigue was discussed in detail from a *methodological* perspective, focusing on the use of concepts from MER and science education research such as "epistemological obstacle" and "conception".

This chapter will also focus on issues of method, although with a shift of emphasis. Instead of focusing directly on didactical concepts, we will mainly discuss research practices at the intersection of two autonomous fields of knowledge: MER on the one hand, and the history of mathematics on the other; in this context, "history of mathematics" will denote the outcome of the work of historians of mathematic.

It seems to us that since Artigue's 1990 paper, the interactions between the two research communities have become less intense; certainly not because the need for epistemological inquiry which Artigue clearly spelled out has faded, but probably from the conjunction of two factors. First, a growing professionalisation of the two research communities occurred. Second, the theory of didactical situations assigned a role to the history of mathematics *within* a didactical theoretical framework, making it easier for the ME researcher to engage with history. The multiplication of theoretical frameworks, and the decline of interest in concepts such as "epistemological obstacle" probably made it less clear, in particular for early-career ME researchers, *why* and *how* interactions with the historical community could benefit them.

We will first endeavour to describe the structural differences between the two fields of research: MER and the history of mathematics (HM). We will then use a methodological viewpoint to analyse several classical and more recent works in MER, in order to document the range of possible interactions and stress fruitful leads. Our range of examples will cover recent works in physics education research to the extent that they bring to light complementary connections between the histories of science and didactics.

[2]This implies that we do not plan to cover another research topic which lies at the intersection of the didactical and historical domains, namely, the use of history in a teaching (or teacher-training) context. To learn more about this research-line, we refer the reader to Fauvel and van Maanen (2000) or to Jankvist (2009) for recent survey papers. For a thought-provoking reflection on the topic, we recommend Fried (2007). To learn more, the HPM group (International Study Group on the Relations between History and Pedagogy of Mathematics) has a website: http://www.clab.edc.uoc.gr/hpm/. We will not touch on the history of education either; for a recent and comprehensive survey, see Karp and Schubring (2014).

8.2 Two Autonomous Fields of Research

8.2.1 Structural Differences

MER and HM are two different and autonomous disciplines: each has its own empirical field of investigation, its own set of legitimate questions, its own way of validating claims, its own reference works, etc. That fact may be self-evident; however, we feel it should be taken into account in order to pave the way for fruitful collaborations. It is also a fact that ME researchers and historians of mathematics have often been speaking at cross purposes: when historians of mathematics read what ME researchers say about the history of mathematics, the typical reaction goes: "this is not history, but a sketchy reconstruction of history framed within a-historical categories; what really happened is really much more complicated than that, you know …"; which ME researchers are usually fully willing to acknowledge while wondering why historians would deny them the right to make *heuristic* use of the HM, usually in a preliminary phase to their main investigation. For them, learning *about* history (which is one of the things historians do) is a means to learn something *from* history (which is not what historians do). Reciprocally, ME researchers may sometimes be surprised by the lack of theoretical frameworks in the work of historians, since such frameworks provide the main tools for describing and analysing specific issues, and enable researchers to integrate their particular study within a growing and soundly-structured body of knowledge about the learning of mathematics in educational contexts. Even though some historians occasionally borrow concepts from some theoretical frameworks,[3] they usually feel they have no use for theoretical frameworks from MER, because they don't use theoretical frameworks at all!

The purpose of this chapter is not to claim that these common misunderstandings are only the result of the relative isolation of the two communities, and that they would soon fade if both groups of researchers decided to work together with an open mind. Quite the contrary, we think these misunderstandings point to differences which are *structural*, and our purpose is to sketch ways of living with this fact.

Since the intended audience of this chapter is that of ME researchers, we would like to briefly describe some elements of the work of historians. Of course, our approach is descriptive and not normative.

MER and HM have at least this in common: contrary to what research mathematicians do, the object of their investigation is not *mathematics*, and this object is not studied primarily *mathematically*.[4] Rather, both historians and ME researchers

[3]For instance, in the workshop on *Epistémologie et didactique* at the Artigue conference, historians Dominique Tournès and Renaud Chorlay mentioned their use (or their interest) in concepts such as "*changement de cadre, de registre, de point de vue*", "*niveau méta*", and "*dialectique outil/objet*".

[4]At least not *only*, or even *primarily*, when it comes to ME. This does not mean that knowing as much mathematics as possible, even very contemporary mathematics, is not very helpful for both RME and HM. We will touch on this below.

study how *agents engage* with mathematics, in a context which can be described; mathematics is necessary to make sense of this engagement and this context, but cannot possibly be the only background tool.

Beyond this common agent-based approach, dissimilarities become striking: ME researchers study learners, while historians of mathematics tend to focus on experts.[5] ME researchers have direct access to the living agents they study, which means empirical data can be gathered, hypotheses can be put to the test in finely-tuned conditions, and cognitive processes can be investigated. Historians of mathematics have indirect access to the agents they study, and it is part of their field to attempt to assess what biases exist (for example, critique of sources, and careful methodological reflection on corpus delineation). Historians of mathematics have to deal with events that happened once, but that can be understood, compared, and to some extent, fit into narratives[6]; MER has an experimental side to it, and can aim for invariants and reproducibility.[7]

The fact that historians depend heavily on the availability of sources and do not explicitly rely on theoretical frameworks does not imply that their work is purely descriptive and erudite. To use Kuhn's phrase, historians *solve puzzles*, just as any researcher does, whatever their field. We would like to illustrate this agent-based, puzzle-solving approach from three different angles.

8.2.2 Echoing Questionnaires

First, let us mention the kind of questions that historians aim to tackle. A very general and context-free list of questions can be found, for instance, in Catherine Goldstein's (1999, 187–188) methodological paper[8]:

> At a given period in time, what were the networks, the social groups, the institutions, the organizations where people practiced mathematics or engaged with mathematics? Who

[5]This is of course a huge oversimplification: practitioners of mathematics need not be "professional" or "research" mathematicians: see, for instance, Dominique Tournès' work on the mathematics of engineers in the 19th century; they need not even be experts: see, for instance, works on the mathematics of merchants, either in the paleo-Babylonian period (Cécile Michel), or at the turn of the 16th century in Occitan France (Maryvonne Spiesser). Again, the term "professional mathematician" would need to be historically situated, since the current meaning refers to something which stabilised in the second half of the 19th century. For instance, Stevin, Viète, Fermat and Leibniz were trained in law and worked as top civil servants.

[6]The *scale* of the narratives—from the very local to the global study of the development of something that matters to us today (be it negative numbers, the circle, or proofs by contradiction)—is maybe where the tension between learning *about* the history of mathematics and learning *from* the history of mathematics is the most perceptible.

[7]In her paper, Artigue pointed to another difference, that between historical genesis and artificial genesis. We do not wish to mention it here, being wary of terms such as "historical genesis".

[8]What follows is based on Goldstein's list, but is not a direct translation.

were mathematicians? In what conditions did they live; in what conditions did they carry out mathematical work? How were they educated and trained? What did they learn?

Why did they work in mathematics, in what preferred domain? What did this domain mean to them? (…) Where did mathematicians find problems to be solved? What were the form and origins of these problems? Why was some result considered as very important, or of lesser importance? According to which criteria? What was considered to be a solution to a problem? What had to be proven, and what did not require a proof (tacitly or explicitly)? Who decided so? When was a proof accepted or rejected? When was an explicit construction deemed indispensable, optional or altogether irrelevant?

When, where and how mathematics were written? Who wrote, and for whom? For instance, were new results taught, were they printed, were they applied? What got transmitted? To whom was it transmitted, in which material and intellectual conditions?

What changed and what remained fixed (and according to what scale, to which criteria)?

This list strikingly echoes the list of questions which Guy Brousseau considered to be meaningful for MER when he attempted to derive didactically-relevant insight from a study of the history of mathematics. When discussing Georges Glaeser's paper of 1981 on the epistemological obstacles relative to negative numbers, Brousseau summed up Glaeser's approach, and then pointed to what he would consider to be the more relevant questions:

This formulation shows what failed Diophantus or Stevin, seen from our time and our current system. We thus spot some knowledge or possibility which failed 16th century authors and prevented them from giving the "right" solution or the proper formulation. But this formulation[9] hides the necessity to understand by what means people tackled the problems which would have required the handling of isolated negative quantities. Were such problems investigated? How were they solved? (…) What we now see as a difficulty, how was it considered at the time? Why did this "state of knowledge" seem adequate; relative to what set of questions was it reasonably efficient? What were the advantages of this "refusal" to handle isolated negative quantities, or what drawbacks did it help avoid? Was this state stable? Why were the attempts at changing it doomed to fail, at that time? Maybe until some new conditions emerge and, some "side" work be done, but which one? These questions are necessary for an in-depth understanding of the construction of knowledge [*pour entrer dans l'intimité de la construction de la connaissance*] (…).[10]

In both lists, we can see that a focus on *agency* does not mean that the object of study is a freely creative cognitive agent. Quite the contrary: agents are born in a world which preexists and constrains[11] their actions. When it comes to mathematical activity, constraints come from a great variety of sources, ranging from the material environment (a Chinese abacus is not an electronic calculator) to epistemic values (such as rigour, generality, simplicity, accuracy, applicability) and epistemic categories (such as definition, justification, proof, example, algorithm, analysis/synthesis, principle). The historical contingency of these constraints does not imply that they have no a-historical components, be they mathematical properties (a rule such as "minus times minus equals minus" is not compatible with

[9]"formulation" denoting Glaeser's analysis.

[10]Quoted in Artigue (1990, p. 252) Trans RC. Our emphasis.

[11]The word "constraint" should not be taken negatively: depending on viewpoints, such an "element of context" can prove to be both a hindrance and a source of opportunities.

distributivity of × over +) or semiotic properties (an algebraic shorthand with no parentheses—such as Cardano's—has different properties from Bombelli's). Making historical sense of how actors engage with mathematics involves understanding how they act *within* a given set of constraints, what *meaning* they give to their actions, and in what respect these actions *alter* the system of constraints.

8.2.3 An Example

Let us now flesh out this notion of agent-based approach—this focus on mathematical agency—from another angle. We will use the diagram below (see Fig. 8.1) to illustrate several methodological points.

The very same diagram (Fig. 8.1) appears in two of the most influential works in the history of mathematics: Euclid's *Elements* and Descartes' *La Géométrie*. One could argue that not only the diagram is the same, but also the mathematical content is the same; however, the parts these diagrams play in both works are strikingly different.

In Euclid's *Elements*[12] (ca. 300 BCE), this diagram comes with proposition 14 of book II, a proposition which solves the following construction problem: to construct a square equal (in area) to a given rectangle. If the sides of the rectangle are equal (in length) to FG and GH, then the perpendicular IG is the side of the sought-after square, which Euclid proves using proposition 47 of book I (which we call Pythagoras' theorem[13]). At the end of book I, a series of propositions established that, for any given polygon, a rectangle with the same area could be constructed (with ruler and compass only), hence proposition 14 provides the final positive solution to the problem of quadrature of polygons (i.e., to transform any polygonal area into a square). In turn, this fact implies that—at least for polygons—area is a well behaved magnitude: areas can be compared (since square areas can) and added (since the Pythagorean construction provides a means to add square areas). On this basis, a modern reader would conclude that a theory of measure is possible for polygonal areas; the modern reader also knows that this requires the set of real numbers. Euclid was well aware of the fact that the theory of well-behaved geometrical magnitude (even line-segments, for which comparison and addition are straightforward) requires more than natural numbers and their ratios. The solution he presented in book V is a number-free solution, based on the notion of ratio of magnitudes and not of measure. The positive result of II.14 also points to open

[12]Heath (1908) provides a convenient edition which is available online. For a recent critical edition, see Euclide (1990–2001).

[13]Another proof is given by Euclid in proposition 8 of book VI. This proof rests on the notion of similar triangles: in modern notations, IGF, IGH and IFH being similar, $\frac{FG}{GI} = \frac{GI}{GH}$, hence $GI^2 = FG \times GH$. But the notion of similarity is introduced in book VI, since it depends on the theory of ratios from book V.

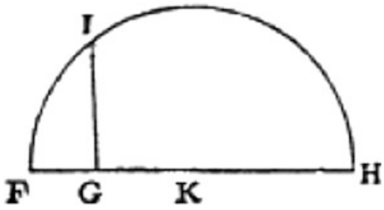

Fig. 8.1 First diagram from Descartes' Geometry

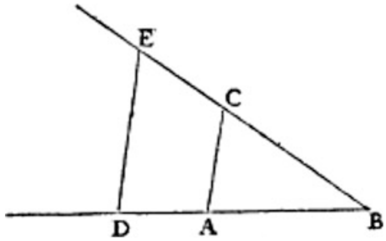

Fig. 8.2 Second diagram from Descartes' Geometry

questions in the theory of magnitudes, in particular the extension of the theory beyond the case of polygons (the case of the circle being of prime importance).

The same diagram (Fig. 8.1) appears on the second page of Descartes' *La Geométrie* (1637[14]). Along with another diagram (Fig. 8.2), he aims to define operations on segments[15]; operations for which he would use the same names as for arithmetical operations. In Fig. 8.2, if AB denotes a unit segment, then BE will be called the "product" of segments BC and BD.

In Fig. 8.1, if FG is the unit segment, then IG will be called the "square root" of GH. Descartes then adds that he would not only use the same names as those of arithmetical operations, but that he would also resort to the same signs as in algebra: letters for segments (known or unknown), and symbols such as $\times$ and $\sqrt{\ }$ for the above mentioned constructions. The project was to use the means of algebra (rewriting rules, elimination in simultaneous equations, identification in polynomial equalities, method of indeterminate coefficients) to capture and analyse geometrical relations between segments; among such relations, those expressed by one equation in two unknowns capture plane curves.[16]

This specific Cartesian project is quite different both from Euclid's, and from what we call either algebra or coordinate geometry. In the *Elements*, proposition II.14 solved an *area* problem; in terms of magnitudes, considering two line-segments could lead either to a new segment (by concatenation, which can be seen as a form of addition), or to an area (that of a rectangle, which can be seen as a form of multiplication), or to a ratio (which is not a geometrical entity, but not a number either). On the contrary, Descartes uses elementary constructions (with an

[14]See, for instance, Descartes (1954).

[15]"segments" as magnitudes; location is irrelevant.

[16]Not *all* plane curves, but this is another issue.

ad hoc unit segment) to define operations such as "times", "divide" or "root" as internal operations within the domain of segments; this enables him to make free use of algebraic symbolism while warranting geometrical interpretability.[17] This system, however, involves no global coordinate system; it does not even involve coordinates, if by coordinates we mean (real) numbers, since no such numbers play any part in the system.[18]

The fact that Descartes' system is an algebra of segments has other far-reaching consequences. Let us mention one of general epistemological importance. At first, when we read in *La Géometrie* (Descartes 1954, p. 303) that the solution of equation $z^2 = az + b^2$ (z being unknown, a and b known) can be expressed by

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb},$$

we feel we are on familiar ground. However, we need to recall that this formula is not a symbolic summary for a list of arithmetical operation on numbers, but is a symbolic summary for a geometrical construction program; a ruler-and-compass construction program which involves two concatenations, two multiplications (Fig. 8.2), the construction of a "square-root" segment (Fig. 8.1), and three mid-points. This, in turn, means that the algebraic manipulation of formulae and equations deals with the transformation and comparison of geometric construction programs. Here, the comparison with Al-Khwarizmi (ca 820 CE) is striking:

> Roots and numbers equal to squares; for instance, if you say: three roots and four in numbers are equal to one square.
>
> Procedure: halve the number of roots, you get one and one half; multiply it by itself, you get two and one quarter; add four, you get six and one quarter; work out the root, which is two and one half; add half the number of roots – that is one and one half – you get four, which is the root of the square; and the square is sixteen. (Rashed 2007, p. 106). (Trans. RC).

With its purely rhetorical algebra and its use of generic examples, Al-Khwarizmi's text may look less familiar than Descartes' formula. However, it presents a *bona fide* list of operations which enables one to solve an equation, in a numerical context. In the rhetorical context, algorithms are easy to express, but not so easy to compare, transform and calculate upon.[19] One of the properties of

[17] In a more traditional interpretation, if a and b denoted a segment, then a^4 had no geometrical interpretation (unless you are willing to work with hypervolumes of four dimensions!), nor had formula $aa + b$ (areas and segments cannot be added; a standard way out of this predicament is to introduce a unit segment, and replace $aa + b$ by the homogeneous $aa + b.1$).

[18] One could argue that whole numbers, and even positive rational numbers are implicitly present, since concatenation warrants the existence of n-uples of a segment, and divisibility by the n-uple of the unit segment warrants the existence of the n-th part of any segment.

[19] For an in-depth analysis of meta-level practices in a rhetorical/algorithmic context, we refer the reader to Karine Chemla's work on Ancient Chinese mathematics. For instance, Chemla and Shuchun (2004), in particular pp. 21–34, and pp. 36–39.

Descartes' system is that its symbolic algebra allows for calculation to operate on algorithms; the fact that the basic steps of the algorithms involved are ruler and compass constructions and not numerical operations is irrelevant, and testifies to the meta-level function of symbolic algebra.

This interplay between the familiar and the not-so-familiar (yet understandable) may feel disorienting at first, but this disorienting effect is positive, as Artigue stressed. It has a *critical* function, helping the researcher to distance him or herself from his/her own mathematical culture[20]; and a *heuristic* function, suggesting new viewpoints on seemingly familiar notions, for instance, the role of symbolism in algebra, or the role of real numbers in geometry (as measures and as coordinates). At least two other functions can be mentioned. First, it helps identify *problems* to which there are no straightforward answers, for instance, what should we consider to be the geometrical analogue of numerical multiplication, at least for one-dimensional objects? In particular, should the analogue of the product be one-dimensional or two-dimensional? A long series of different–yet mathematically sound—constructions provides different answers to this question, including dimension-changing solutions (going down with the dot product, or up with the exterior product). Secondly, it helps question the notion of identity. It could be argued that, from a purely mathematical point of view, Euclid and Descartes rely on the *same* content associated with Fig. 8.1; this probably makes sense, but is not necessarily very helpful, either to the historian or to the ME researcher. Indeed, researchers in both fields aim to analyse how *content* depends on, for instance, semiotic resources, or intended use.

To conclude this example from Euclid-Descartes, we would like to explain why we chose such an example. On the one hand, the example is relatively small scale; we did not need to include it in any large-scale narrative on the "stages" in the history of geometry for this sketchy comparison to serve the four functions listed above of epistemological inquiry. On an even smaller scale, the comparison with a short passage of Al-Khwarizmi could play a relevant part even with no background "big picture" on the history of algebra, or even on the *Kitāb al-Jabr wa-al-muqābala*. On the other hand, to compare the uses of the same diagram required that its role in the whole structure of the works (the *Elements* and *La Géométrie*) be analysed. It requires some knowledge of history to make sense of highly sophisticated but largely forgotten theoretical constructs such as the classical theory of ratios or the 17th century research program of construction of equations. This knowledge cannot derive from a quick look at short extracts from the original sources, and probably not even from a lengthy examination of the complete books; here, we depend on secondary sources and the work of professional historians such as Bernard Vitrac (Euclide 1990–2001) for Euclid and Bos (2001) for Descartes.

[20]"*A un premier niveau, l'analyse épistémologique est, me semble-t-il, nécessaire au didacticien pour l'aider à mettre à distance et sous contrôle les 'représentations épistémologiques' des mathématiques induites par l'enseignement*" (Artigue 1990, p. 243).

8.2.4 Solving Puzzles, Crafting Puzzles

We would now like to illustrate the puzzle-solving side of research in HM, by giving short descriptions of a selection of recent works in that field. We also wish to illustrate, with six examples, the difference between a naïve question and a research question. Indeed, it is natural to begin with a naïve question, and curiosity is the first driver in all fields of inquiry; however, delineating a specific question, on the basis of available documents and of the state of research (historiography) is a key stage when practicing research-level historical investigation. Since this sample is not a survey, we can also proceed chronologically.

Example 1 Contrasting rational reconstructions: In Proust (2012), the author studies the algorithm displayed in paleo-Babylonian tablets when working out the reciprocals of large numbers in the sexagesimal system. The clay tablets display instances of calculations, but no general descriptions of the method (much less any justifications), which is why historians endeavour to come up with reconstructions of the algorithm. A pioneer in the history of Babylonian mathematics, Otto Neugebauer (1899–1990) reconstructed an algorithm on the basis of a few tablets-an algorithm which required that additions be used along the way. However, in the *floating point* sexagesimal number system, and in the purely numerical context of these tablets, addition is not possible (whereas products and reciprocals make perfect sense). On the basis of a much larger sample of tablets, Proust reconstructed a different algorithm, one which is fully compatible with a floating point arithmetic.

Example 2 Re-problematising familiar practices: In his now classic work, *The Shaping of Deduction in Greek Mathematics*, Netz (1999) attempted to re-historicise the endeavours of the Greek mathematicians of the classical and Hellenistic periods, with the aim of helping us to question many things we take for granted. This is difficult for at least two reasons: first, some of the basic elements of practice displayed in these texts—in particular, the practice of discussing lettered diagrams using only explicit axioms and formerly established results—is so familiar to us that we cannot imagine how a few men strove to establish this specific cultural form based on the background of other cultural activities; and second, because we feel we know that mathematics was a central intellectual activity in these periods, as many texts from Plato and Aristotle seem to indicate. Netz established that this was *not* the case, and discussed why Plato and Aristotle distorted our perception of historical realities. In a stimulating review of this erudite book, Latour (2008) emphasised the extent to which it echoed central methodological trends in the social history of science.

Example 3 Describing a reception as a form of hybridation: The question of the circulation of mathematics between different cultural areas—and not only different periods—is also a central field of investigation. For instance, in a paper published in 1996, Karine Chemla discussed the introduction of “western” mathematics in 17th century China by Jesuit missionaries. It was usually thought that, in this period, the indigenous Chinese tradition of mathematics was to a large extent forgotten in

China, and that western mathematics had been adopted passively. Actually, studying the Chinese sources leads to a more nuanced picture. In particular, when Jesuit Matteo Ricci and Chinese scholar Li Zhizao collaborated to write a treatise of arithmetic based on Clavius' *Epitome arithmeticae*, they ended up with much more than a translation: Li added many elements from the indigenous tradition, in particular the *fangcheng* algorithm to solve simultaneous linear equations.[21] This work of synthesis did not stir interest in the West; in China, however, the introduction of western mathematics revived scholarly interest in classical Chinese mathematics and triggered comparative studies of both traditions.

Example 4 Renewing our understanding of a crucial step in the development of mathematics: Our image of the beginning of calculus was significantly altered in the 1990s by the publication of a hitherto little-studied Leibnizian manuscript (Knobloch 1993). Until then, our understanding of the calculus according to Leibniz was based on miscellaneous short texts (tracts, letters), in which few attempts at justification were given. It was generally thought that even though Leibniz claimed that his calculus could be justified by the rigorous methods of the Ancients, he actually relied on infinitesimals (which, he admitted, were only "useful fictions"). With the *De quadratura arithmetica*, which was written before the invention of his calculus, Leibniz wrote a long treatise, in a deductive style which both emulated and improved the exhaustion proof-scheme in a way that, were it to be reformulated in a symbolic and numerical context, would be closer to the current ε-δ proofs than those of the Ancients. Moreover, the term "fiction" was already used in this context, to denote abbreviations for calculations dealing only with finite quantities.

Example 5 Unravelling a forgotten branch of mathematical analysis: It is well-known that for the founders of calculus, the prime goal was the study of curves defined by ordinary differential equations, in a geometrical or physical context. Pen and paper, and formulaic solutions were not the whole story, as was demonstrated by the deep and original work of Dominique Tournès (following the work of Henk Bos). Tournès (2003) studied the intense work on graphical methods and graphing devices carried out from the very beginning (for example, Leibniz, Newton, Jean Bernoulli, Euler), up until the advent of digital instruments in the second half of the 20th century. This work brings to light a great wealth of largely forgotten mathematical ideas and techniques, shows the continuity between the algebraic research program on the "construction of equations" (as in Descartes) and the late-17th and 18th century researches on ODEs, and documents the deep connections between the most theoretical considerations on the one hand, and the demand for approximation methods (be they graphical, mechanical or numerical) in the engineering communities on the other hand. Since 2003, Tournès furthered his work on integration

[21]Similar to Gaussian elimination.

instruments,[22] and headed a collective research program on the mathematical and professional contexts of numerical calculation.[23]

Example 6 Studying the emergence of a *meta* level articulation: In the didactics of analysis, it is customary to distinguish between point-wise, local and global properties of functions. The distinction between "local" and "global" is now widely accepted in the scholarly mathematical world, but it was not always the case. Studying the emergence of an explicit local-global articulation is tricky for a number of reasons. It concerns more or less all mathematics[24]; the *meta* level terms "local" and "global" have definitions which differ in every specific mathematical context—actually they can be used with no definitions at all. Moreover, the question of the *explicit* is crucial. When, at the turn of the 20th century, some mathematicians began to explicitly express such a distinction, was the general context one in which it was clear to everyone that this mattered (though it went without saying), or one in which no clear distinction was made between local and global statements, resulting in a wealth of faulty proofs and ambiguously-worded theorems? These questions were addressed in Chorlay (2011), who provided answers based on a combination of quantitative and qualitative methods.

This short list of examples illustrates how historians endeavour to design non-trivial questions, the means they use to answer these questions, and the kind of answers they tend to consider relevant and innovative. Although historians provide a great wealth of material that is of prime interest in MER, they do not usually provide this material in a form which directly meets the needs of the MER community.

8.3 Methodological Discussion of Some Classical Works

In the second part of this chapter, we would like to review several papers from the didactics of mathematics and the sciences which explicitly carried out *epistemological* investigations in the sense of Artigue. Our primary goal is methodological: we will present only a brief selection of papers in order to discuss concepts such as obstacles, aspects of a concept, stages and didactical reconstruction.

[22]An exhibition of integration instruments was organised at the Paris Museum of Technology (CNAM): http://culturemath.ens.fr/histoire%20des%20maths/htm/expo_aire/expo_aire.htm.

[23]For more information, see http://www.sphere.univ-paris-diderot.fr/spip.php?rubrique64.

[24]Indeed, analysis and geometry are not the only fields concerned. In the 1920s, H. Hasse introduced the so-called "local-global" principle in number theory; "local algebra" is a branch of commutative algebra, etc.

8.3.1 Epistemological Obstacles

In a paper published in 1990, Artigue discussed in detail two studies which aimed to investigate epistemological obstacles in the sense of Brousseau[25]: Glaeser's (1981) paper on negative numbers, and Sierpinska's (1985) paper on limits. We will focus on the second paper, so as to summarise and further Artigue's methodological analysis.

8.3.1.1 One Example in Analysis

Sierpinska's paper is divided into two very different parts. The first part describes and analyses two classroom experiments, which included high school students who had received no prior teaching on limits or the derivative. In both situations, the students had to determine the tangent to a given curve at a given point, on the basis of a loose, intuitive and non-verbal description of the sought-after object. As Sierpinska points out, in contemporary mathematics, the notion of limit is the relevant tool for performing this task. Hence, by focusing on what students do and say (when trying to describe and justify what they do), she aims to capture what their conception of a limit is; to discover to what extent it differs from the current formal definition; and to reveal what obstacles stand in the students' way when trying to pass from naïve and context-dependent conceptions to efficient general procedures and proto-definitions (*définitions opératoires*).

In the second part of the paper, Sierpinska turns to the history of mathematics to establish the epistemological natures of the obstacles that were identified:

> If a given behaviour manifests itself in history just as it does with today's students, then we are justified in regarding it as a specific feature of the development of the given concept, as opposed to an effect of teaching conditions. (Sierpinska 1985 8), (Trans. RC)

On the basis of two surveys on the history of mathematics, Sierpinska (1985, p. 38) lists and classifies "obstacles" in the history of mathematics, as the Fig. 8.3 shows.

Artigue summarises Sierpinska's list of obstacles (adapted from Artigue 1990, p. 253):

- *Horror infinity*[26] brings together the obstacles which stem from the refusal to consider passing to the limit as an operation; those stemming from an automatic use of algebraic methods designed for the handling of finite quantities to the case of infinite quantities; those consisting in transferring all properties of the terms of

[25]Brousseau's concept was derived from Bachelard's historical epistemology.

[26]Sierpinska (1985, p. 39) uses Cantor's phrase to bring together what, to her, stems from the refusal to consider the *actual* infinite.

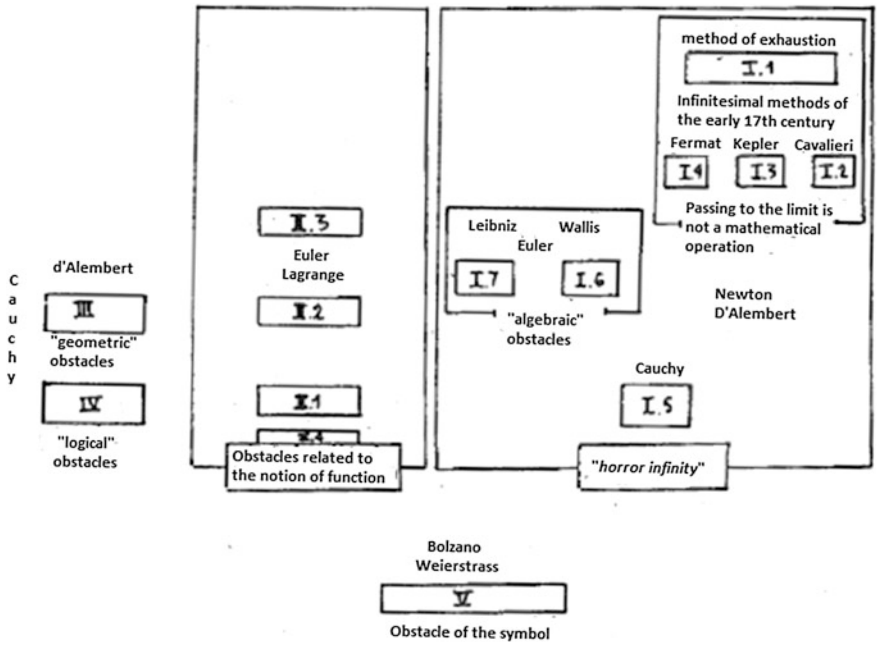

Fig. 8.3 Sierpinska's classification of epistemological obstacles regarding limits

a convergent sequence to its limit[27]; and eventually those obstacles which consist of regarding the passage to the limit as a physical movement, or a way of coming ever closer.

- *Obstacles related to the notion of function* include failure to identify the underlying functions; restriction to a sequence; monotonous reduction[28]; failure to distinguish between limits and upper/lower bounds.
- *Geometric obstacles*: geometric intuition generates serious obstacles which stand in the way of the formulation of a rigorous definition, both by preventing the determination of what is to be taken to be the difference/distance between two geometric magnitudes, and by conflating the notion of limit with that of point in the closure of a subset.
- *Logical obstacles* reflect the failure to use quantification, or to take into account the order of quantifiers.
- *The obstacle of the symbol* reflects the reluctance to coin a specific symbol (such as $\lim_{\dots}$ …) for the operation "passing to the limit".

[27]Including sequences of functions. For instance, when assuming that the limit of a sequence of continuous functions is continuous.

[28]This denotes the implicit assumption that a converging sequence is monotonous, at least from a certain rank onward.

Before we list what we take to be structural methodological shortcomings in this paper, it is only fair to underline several of its most positive features. The paper has a heuristic value, and indeed, several of the problems it points out have become central in the didactics of analysis[29]: such as the inadequacy of everyday language to capture the fundamental concepts of analysis; complicated and deceiving connections between limit-positions of geometric objects, limits of associated magnitudes, and limits of associated numerical functions; sequential reduction and monotonous reduction; implicit transfer of calculation rules and of properties from the finite to the infinite, or from a converging sequence to its limits; and the need, at some point in the curriculum, to regard a numerical sequence or a numerical function as *one* object and not only as a collection of numbers. Also, as is often noted, the didactics (and the teaching) of analysis is particularly thorny because of the entanglement of all the basic concepts (limit, derivative, integral, function, the real continuum); hence, the difficulty in designing experimental situations which are rich and open-ended enough to yield interesting data, yet not so rich and open-ended that all possible problems crop up and interfere. Finally, we are not aware of any well-established theoretical framework (or even reasonably sea-worthy set of conceptual tools) adapted to the description and analysis of epistemological obstacles. In this respect, Sierpinska's paper both points to a problem of general significance for MER, and offers a stimulating proposal.

8.3.1.2 Focusing on "Obstacles" in MER

As Artigue pointed out, the notion of "epistemological obstacle" may not be the best conceptual tool to describe the wealth of phenomena captured here, both in the classroom experiments and in the historical literature. Here, we first focus on the purely didactical aspects.

For Brousseau, an obstacle (be it didactic or epistemological[30]) is an element of knowledge (either explicit or "in act") which proves valid and efficient in some contexts, but becomes systematically error-generating in other contexts. A standard example is that of the comparison of decimal numbers: some rules which are valid and efficient for natural numbers (for instance, the more digits it takes to write a number, the larger the number) become invalid and error-generating when applied with decimals (it takes three digits to write 1,23 and only two to write 1,3; but $1{,}23 < 1{,}3$). An obstacle is epistemological if it depends on mathematical facts only, regardless of teaching paths. When confronted with an error-generating in-act-theorem, it is not always easy to tell whether it is epistemological or

[29]In particular, we refer the reader to the papers of Bernard Cornu, Aline Robert and Maggy Schneider.

[30]We will not discuss ontogenetic obstacles. For a further discussion of the concept of obstacle, we recommend (Duroux 1983, 52–54).

didactic; Brousseau suggested that a distinctive feature of epistemological obstacles was their presence in mathematics of the past.

In the classroom experiments, students face several challenges: changing registers or frames (from the graphical to the numerical); experiencing the inadequacy of the expressive resources available to them (everyday language, body motion, basic symbolic algebra); hitting upon a paradoxical core (division by zero yielding a finite result, straight line defined by two points which overlap); and regarding a process/procedure as an object/concept (the notion of *procept* would later be designed to describe this phenomenon). The fact that, when confronted with these challenges, students do not come anywhere close to an ε-δ definition of the derivative as limit of the differential quotient probably does not qualify as an *error*.

A difference between a statement and a reference statement (say, a given formal definition) may be telling without pointing to any difficulty, misconception or error. Some differences may indicate difficulties, but not all difficulties are errors. All errors do not derive from epistemological obstacles. The epistemological character of some obstacles is not always easy to ascertain; and, more often than not, the history of mathematics may not be useful or even necessary to ascertain it.

To substantiate the last statement, let us mention the case of the "natural numbers/decimals" obstacle, which is clearly (partly) epistemological but with no significant historical basis. The same holds for a well-documented error-generating belief, namely, all the magnitudes associated with a given class of objects vary in the same way: for instance, a polygon has a length and an area; but it is not true that when the length increases, the area always increases as well. This error-generating belief is probably epistemological insofar as reveals some aspect of the acts of knowing and forecasting, independently of teaching paths and curricula. As Artigue (1990, p. 261) stressed, it is reminiscent of Bachelard's original notion of epistemological obstacle in physics, and points to a larger class of obstacle-generating features of the understanding process. As such, instances of errors ascribable to this infelicitous thought-process may very well be found in historical texts. If some were, they would probably not be indicative of any global dynamic of the development of mathematical thought; and if none were found in history, it would not make this error-generating belief any less important in mathematics education.

8.3.1.3 Do "Obstacles" Help Us to Make Sense of History?

The notion of "obstacle" is probably not the best conceptual tool to make sense of the historical data collected either. Here we spell out several arguments, and suggest alternative investigative paths.

First of all, this notion depends on that of error. Writing, for instance, that Archimedes failed to consider "passing to the limit" as an operation, and avoided the use of the actual infinite which (Sierpinska claims) is a core element of the Weierstrassian ε-δ definition of a limit, seems slightly disrespectful and highly questionable (respectively). Archimedes' proofs by exhaustion may be written in a style differing from the current standards in the first year of tertiary education; it does not make them mathematically incorrect, and does not testify to any cognitive shortcoming. Whether or not passing to the limit is an "operation" (and has to be seen this way) seems debatable; so is the assertion to the effect that the ε-δ definition of limits depends on the actual infinite. As to the last point, one could argue that the ε-δ definition is actually very close to the Archimedean proof-scheme. For instance, to establish that two areas are equal, Archimedes proves that this difference is less than any given area. To establish this, he relies on a fact which is made explicit in Euclid's *Elements*, and warrants convergence to zero for a class of sequences of magnitudes.[31] As to Weierstrass, many of his contemporaries regarded his construction of the real numbers and his definition of the limit as a major step toward the reduction of analysis to arithmetic, and the subsequent elimination of infinites. When comparing proofs by exhaustion with the current Weierstrassian definition, their fundamental similarity raises a series of potentially fruitful questions: what is the difference between a uniform, formal[32] proof-scheme, and a definition? What is the role of symbolic notations (the question is all the more tricky since Weierstrass worked in a partly rhetorical context when discussing limits)? Why was the Archimedian proof-scheme criticized in the 17th century, and how are these criticisms related to the emergence of calculus?

As we mentioned earlier, all conceptual difficulties are not errors. The case of negative numbers could also illustrate this point: from the middle of the 17th century to the middle of the 19th century, some mathematicians, mathematics educators and philosophers debated the meaning and legitimacy of isolated negative numbers, and of the multiplication rule (Schubring 2005). This phenomenon was quite independent from the fact that negative coefficients and the multiplication of negatives had been used for centuries.[33] The same holds for imaginary numbers or calculus: at some points, their meaning and legitimacy were discussed in spite of the consensus on how to use them. Two fruitful questions would be: what were the arguments? Why were these issues controversial in some contexts and not in others?[34]

[31] Proposition 1 of book X reads: "Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out" (Heath 1908, vol. 3, p. 14). In the context of real numbers, it means that any positive sequence which is bounded above by a geometric sequence with common ratio ½ tends to zero.

[32] "Formal" does not necessarily mean "symbolic". Historians of ancient Greek mathematics convincingly argue that the rhetoric of mathematical texts is so constrained, and so distinct from an ordinary use of the Greek language, that it can indeed be called "formal". See, for instance, Netz (1999).

[33] On negative coefficients and the rule of sign in ancient Chinese mathematics, see Chemla and Schuchun (2004, pp. 606–609).

[34] For instance, D. Rabouin showed that the use of $\sqrt{-1}$ in algebra was *not* controversial in the late 16th and 17th centuries, but *became* controversial in the late 17th century. This is unpublished work, unfortunately.

Gathering historical data to document differences between various statements and a reference definition can also bring to light the multiplicity of aspects[35] of a given concept. For instance, in her paper, Artigue listed a number of aspects of the notion of a tangent to a curve: such as straight-line such that no other straight-line can be inserted between it and the curve (local convexity); straight-line defined by two infinitely close points; and straight-line defined by the direction of the velocity vector of any point gliding along the curve. These aspects are not mathematically equivalent and from an epistemological viewpoint their ecology is different; they target different classes of curves and of problems and the associated signifiers (be they graphical, symbolic or rhetoric) are different. Similarly, historical investigation (based on texts) and didactical investigations (based on live empirical data) could go hand in hand; which does not mean they would be carried out along the same line or with the same expected outcome. Firstly, because the didactical study attempts to capture the total and personal cognitive structure of students associated with a mathematical concept (such as Tall and Vinner's concept image); while historians study context-embedded rational actors, not cognitive subjects. Secondly, historical texts usually display the mature and genre-dependent productions of mathematical experts. It does not mean that all they say is correct, but it means they usually make meta-level choices which reflect both a large overview of mathematics, and the intended middle-scale structure of the text (such as letter, research paper, treatise, textbook). For instance, it is true that reading the first lesson of Lagrange's *Théorie des fonctions analytiques* (first published in 1797) shows a specific definition of the derivative: if $f(x)$ denotes of function of variable x, then substituting $x + i$ for x (i being an indeterminate quantity) gives rise to a development of the form

$$f(x+i) = f(x) + ip + i^2q + \cdots$$

where p is a function of x only; function p will be called the derivative of f, denoted by $f'(x)$ (we are paraphrasing) (Lagrange 1867, p. 21). However, the complete title of the book shows that this is a choice: *Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toutes considérations d'infiniment petits, d'évanouissants, <u>de limites</u>*[36] *et de fluxions, et réduits à l'analyse algébrique des quantités finies.* In the introduction, Lagrange (1867, trans. RC, p. 16) even discussed the definition of the derivative as the limit of the differential quotient: "One has to acknowledge that this idea, in spite of being right in itself, is not clear enough to serve as principle for a science whose certainty must be founded on evidence, and most of all, not clear enough to be presented to beginners" . To make sense of this statement would require that the meaning of "limit", "algebraic

[35]For a discussion of the possible meanings of the terms "conceptions", "aspects" or "viewpoints" (relative to a mathematical concept), see Artigue (1990, pp. 265–274). We will use the term "aspect" to avoid the more cognitive and personal connotation of "conception".

[36]The underlining is ours.

analysis", "principle" and "evidence" be investigated. Since his definition depends on a theorem on the existence of power-series expansions for functions, Lagrange established in a later chapter that this expansion holds "generally"; to make sense of the latter statement would require that the meaning of the terms "function" and "general" be investigated. Finally, Lagrange also engages in proofs in which the limit-definition of the derivative is used in a ε-δ fashion (Chorlay 2013), demonstrating his ability to switch between aspects when necessary. The fact that expert mathematical thinkers (not necessarily mathematicians and not necessarily of the past) exert meta-level control on a variety of representations is a well-identified key to advanced mathematical thinking, as Artigue stressed (Artigue et al. 2007).

Finally, gathering historical documents is only one of the ways to document the multiplicity of aspects making up a mathematical concept. In particular, studying the avatars of the concept in various fields of contemporary mathematics can, to some extent, serve a similar purpose. For instance, as far as the derivative is concerned, differential geometry and algebraic geometry offer new vistas.[37] The standard differential geometric approach provides an extension of calculus in which the primitive notion ramifies into several distinct notions (differentials, Lie derivative, covariant derivative with respect to a connection), with different invariance properties, each of them involving new spaces associated with the original domain. From a dynamic and epistemological viewpoint, this testifies to a process of conceptual differentiation rather than to a process of reduction to a unique correct definition on the basis of formerly loose and partly faulty proto-concepts. The algebraic geometric approach is further from the standard calculus approach, since it relies neither on real numbers nor on notions of limit, velocity or approximation. Extending the original multiplicity-of-intersection approach to the derivative, the scheme version of algebraic geometry provides rigorous notions of double point or infinitesimal thickening of a point (or even a subvariety).

On the one hand, this multiplicity of viewpoints/aspects within mathematics (be it contemporary or historical) for a given concept provides food for thought; on the other hand, it raises methodological questions. As to the food-for-thought part, let us mention two fields of investigation: first, the epistemological analysis of the aspects (mathematical properties, ergonomic and semiotic properties) and of their connections to other aspects of the same notion. Second, the study of the connections between scholarly knowledge and school knowledge (as documented in syllabi); the history of education, and the theory of didactic transposition provide tools for the latter. As to the methodological issues, we will mention only one. For a working mathematician endeavouring to give mathematical answers to mathematical questions, a given theoretical context provides a stable and unquestioned framework including definitions, fundamental theorems and standard proof techniques. For the researcher in mathematics education as well as for the historian of

[37] In her paper of 1990, Artigue mentioned non-standard analysis as an alternative rigorous theoretical context.

mathematics, the very notion of a background reference mathematical theory is problematic in a number of ways: which reference mathematical theory should be chosen for a specific investigation? What is the role of the reference theory in the didactical or historical investigation? Is a reference theory needed at all?

8.3.2 Large Scale Narratives: Episodes or Stages

We now proceed with our methodological discussion by presenting several papers written by researchers in mathematics education and mathematics educators with a sustained interest in the history of mathematics. This will enable us to discuss two issues: the relevance of large scale narratives dealing with "stages" or "genesis"; and the heuristic use of history in didactics.

In 2007, two papers published in the same issue of *Educational Studies in Mathematics* suggested different ways of using a historical perspective to question the teaching of algebra: Katz's (2007) *Stages in the history of algebra with implications for teaching*; and *Syntax and meaning as sensuous, visual, historical forms of algebraic thinking*, by Puig and Radford (2007).

The abstract of the Katz paper reads:

> In this article, we take a rapid journey through the history of algebra, noting the important developments and reflecting on the importance of this history in the teaching of algebra in secondary school or university. Frequently, algebra is considered to have three stages in its historical development: the rhetorical stage, the syncopated stage, and the symbolic stage. But besides these three stages of expressing algebraic ideas, there are four more conceptual stages which have happened along side of these changes in expressions. These stages are the geometric stage, where most of the concepts of algebra are geometric ones; the static equation-solving stage, where the goal is to find numbers satisfying certain relationships; the dynamic function stage, where motion seems to be an underlying idea, and finally, the abstract stage, where mathematical structure plays the central role. The stages of algebra are, of course not entirely disjoint from one another; there is always some overlap. We discuss here high points of the development of these stages and reflect on the use of these historical stages in the teaching of algebra. (Katz 2007, p. 185)

The distinction between a study of the semiotic aspects and the "conceptual" aspects provides depth to the analysis by spanning a two-dimensional grid, and suggests further investigations into the connections between both aspects. However, whether or not Katz identifies "stages" in the history of "algebra" is questionable; actually, the core of the paper is less schematic, displaying Katz's historical culture and sense of nuance. For instance, in regard to the "stage" aspect, Katz mentions Al-Tusi's (d. 1274) study of the number of solutions for a class of third-degree equations in terms of the maximal value reached by what we would call the left-hand side of the equation; this exemplifies a typically "functional" way of thinking in an algebraic context, long before the 17th century. As to "algebra": whether or not the algorithmic solving of riddles bearing on width and area in Babylonian clay tablets, or establishing a list of basic geometric identities as in book II of the *Elements*, exemplify "algebraic" practices is highly questionable, and has been (and sometimes still is) hotly discussed in the historical community. Katz is well aware of this fact and points out in the introduction of the paper that the term "algebra" is not well-defined. Babylonian methods and Greek "geometric algebra" may not be stages in the history of "algebra"; however it is necessary to study them since they provide the necessary context to understanding something which clearly belongs to the history of algebra, namely Al-Khwarizmi's *Kitab al-jabr*. Comparison is the key methodological tool, and what is specific in Al-Khwarizmi can only be spelled out by comparing it to other mathematical practices; deciding whether or not these practices are algebraic is a somewhat byzantine question.

More generally, setting out to identify "the stages" in the "development /historical genesis" of a theory or concept is a methodologically risky endeavour, even for careful thinkers who warn their readers against dubious analogies between ontogeny and phylogeny, and abstain from drawing direct implications for teaching. We've already mentioned two reasons: the use of a-historical categories (such as "algebra") to select and organise historical content on a very large scale raises intrinsic methodological difficulties, and leads to simplifications which leave out all that does not fit within the frame; but then, what are we to learn from a history which is not the *actual* history? This delineation of homogeneous blocks also wipes out the role of agents. For instance, in *La Géométrie*, Descartes presents a method to determine the tangent to algebraic curves; however, his correspondence shows that he also relies on ingenious kinematic arguments when studying non-algebraic curves such as the cycloid.

There are two further difficulties. First, thinking in terms of "stages of development" has a definite positivistic flavour: in this setting, the next stage *replaces* the previous one; the fact that the next one comes *after* the former one is implicitly taken to mean that it is *better* in some way, and this improvement is taken to be an adequate explanation or cause for the historical change. Instead of using loaded terms such as "stage", lighter terms could probably be used, such as "aspects", "viewpoints" or "conceptual polarities". What Katz convincingly points out is that algebraic equations can be (and have been) considered from a static-numerical viewpoint, or from a more dynamic-functional viewpoint. Viewpoints can coexist, and their relative virtues may depend on circumstances. Moreover, if one becomes obsolete at some point in time, its obsolescence can be studied as a historical phenomenon: for whom does it become obsolete? Was the drive for change conceptual, semiotic, instrumental or institutional? Was the change actively promoted by some, or did things just fade away and die out? We do not mean to say that loosely defined terms such as "aspect" or "polarities" are the only descriptive tools which didactical analysis should use when attempting to derive relevant information from the history of mathematics. Rather, we argue that starting the investigation with a priori conceptual tools such as "stages" (and "obstacles") not only leads to many difficulties, it also hides a wealth of interesting phenomena. However, we do not discuss here the conceptual tools relevant to *later* analysis of these phenomena in a way which makes full use of what history can offer *and* provides relevance from a mathematics education perspective.

Second, more often than not, large scale narratives are implicitly expected to cover *all* the stages. For instance, in his paper, Katz mentions a final structural stage, in spite of the fact that it is not central to his argument and relies on a rather dated view of the history of modern mathematics. Indeed, recent historical work has greatly improved our understanding both of the history of algebra in the 19th and early 20th centuries,[38] and of the history of structuralism, both in algebra (Corry 2004) and in other branches of mathematics (Chorlay 2010).

The transition from rhetorical algebra to symbolic algebra is also the focal point of Puig and Radford (2007). Their semiotic analysis focuses on the way meaning was/is attached either to words or to symbols; and they point out the fact that the connection between signs and what they represent is different when writing an isolated equation, and when the equation is algebraically transformed to yield solutions. On the basis of their knowledge of rhetorical and geometrically flavoured algebraic practices (such as that of Al-Khwarizmi), Katz, Puig and Radford question what they take to be the commonly held view that algebra grew from arithmetic, and that it is only natural that school curricula should follow the same path.

The latest point illustrates what Artigue mentioned as one of the main roles of historical knowledge for the researcher in mathematics education, namely to distance oneself from one's own training-induced mathematical knowledge and image of mathematics. The role of historical knowledge is used heuristically so as to suggest new vistas, or to identify elusive articulations. This heuristic use bears on epistemological moments rather than actual historical episodes, even if these moments are suggested by historical texts and can be illustrated with thought-provoking excerpts. As such, their heuristic value is not affected by the fact that, for instance, Al-Khwarizmi's practice is more multi-faceted when it comes to the relationship between algebra, geometry and arithmetic. Indeed, in the *Kitab al-jabr*, Al-Khwarizmi usually resorts to geometry to justify numerical algorithms. However, on several occasions, for lack of a justification "by the (geometric) cause" (*al-'illa*), he resorts to justification "by the expression" (*al-lafz*) (Rashed 2007, pp. 49–56). On several occasions, algebraic rewriting rules (in a rhetorical context) are derived from the numerical algorithms in the Indian numerical system (base 10 positional system) which Al-Khwarizmi had introduced in his book, *Hindu Art of Reckoning*.[39] Just as well, when analysing the emergence of symbolic algebra in the 16th and early 17th centuries, a historian would probably feel the need to bring into the picture elements which were central for mathematicians at that time, in particular, the distinction between analysis and synthesis, as well as the method of false position (*regula falsi*).

A third way of making heuristic use of the history of mathematics over a long time-scale is illustrated by Artigue and Deledicq's text of 1992 on *Quatre étapes dans l'histoire des nombres complexes: quelques commentaires épistémologiques*

[38]To learn more about recent works, a good starting point is Brechenmacher and Ehrhardt (2010).

[39]For instance, see Rashed (2007, pp. 122–124) for an example of what we would write $(10+x)^2 = 100+20x+x^2$.

et didactiques. The authors study four groups of texts related to four stages in the history of complex numbers: the use of new and uninterpreted operators in the symbolic algorithms for solving 3rd degree equations in the Italian Renaissance; the dispute over the logarithm of complex numbers, and the multivaluedness of the measurement of angles at the beginning of the 18th century; the first geometric representations of complex numbers at the turn of the 19th century; and the formal algebraic constructions of complex numbers by Cauchy (as residue classes of real polynomials modulo $X^2 + 1$) and Hamilton (C as $R \times R$). The method of identifying "stages", "episodes" or "key moments" is clearly epistemological. Artigue and Deledicq select four meaningful epistemological challenges (such as: to extend the symbolic system in order to provide more uniform algorithms to solve some equations), without aiming for a (questionable) comprehensive narrative in terms of successive homogeneous stages. It so happens that, in the case of complex numbers, these four epistemological challenges do correspond to well-delineated historical contexts.[40] Again, the fact that they include an episode such as the dispute over complex logarithms—which played a significant role in history but does not clearly correspond to a contemporary teaching issue—illustrates the "distancing ourselves from what we think complex numbers are about" function of history; a function which, as we mentioned earlier, knowledge of advanced contemporary mathematics can serve just as well.

For each group of texts, Artigue and Deledicq explicitly distinguish between three types of commentary: historical, epistemological, and didactical. They call "epistemological" the comments which relate to the nature of mathematics or the nature of the active engagement with mathematics, regardless of teaching contexts and learning issues. In this case, the "didactical" comments do not relate directly to teaching and learning issues, since the authors do not wish to draw quick conclusions from the study of the action of mathematical "experts" such as Cardano or Cauchy. Rather, the didactical comments relate to two different aspects: the comparison between the historical situations/challenges and teaching situations/challenges; and the conceptual tools which researchers in mathematics education are familiar with when describing and analysing engagement with mathematics in teaching contexts. For instance, after remarking that the formal constructions of Cauchy and Hamilton used real numbers but took place before the formal constructions of the set of real numbers, Artigue and Deledicq stress the similarity with teaching contexts in which the order of the introduction of notions differs from the purely deductive order. More often than not—and this is typical of

[40]The word "select", and the fact that the criteria for selection are quite explicit, is of great methodological importance. From a historical point of view, a study of the early 19th century context would probably require that geometric calculations (vector and barycentric calculus) and complex analysis be brought into the picture.

a heuristic approach—the final output of the reflection is a series of questions rather than answers, or the delineation of research projects.[41]

In this context, the distinction between an epistemological comment and a didactical cannot be absolute. However, even when similar terms can be used in both cases, a slight but definite difference in meaning remains. For instance, discussing (on an epistemological level) the tool/object polarity leads to questions such as: when did mathematicians consider that symbolic tools such as $\sqrt{-1}$ challenged the notions of number or magnitude? What were the relative roles of the geometric semantic and the formal constructions in the passage from complex-numbers-as-tools to a new and autonomous class of objects? In the didactical context, the *dialectique outil/objet* of Régine Douady provides tool to analyse the function of a given concept in a given teaching context, and to design teaching paths.

8.3.3 Epistemological Analysis of an Advanced Field: Linear Algebra

In the 1990s, Jean-Luc Dorier's dissertation and subsequent publications exemplify the rare case in which both historical and didactical issues are studied at research level, simultaneously. Dorier combined a clear methodological distinction between the two fields of inquiry, and a long-run practice of *co*-problematisation.[42]

The topic of study was linear algebra and its teaching in the transition from secondary to tertiary education. The starting point was a diagnosis of the didactical system around 1990 in France, and its structural shortcomings. In higher-secondary education, students dealt with free-moving vectors (defined either as translations, or as equivalence classes of couples of points) in the context of Euclidean geometry.

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[41]For instance: "*Est-il possible de faire vivre dans le système didactique le processus analogique de façon plus conforme au rôle qu'il joue dans l'activité scientifique, et si oui à quelles conditions ? Il y a là encore un champ de questions auxquelles la didactique à l'heure actuelle n'apporte pas de réponse satisfaisante*" (Artigue and Deledicq 1992, p. 49).

[42]Dorier summarized his approach in his habilitation dissertation: "*Notre position méthodologique est finalement assez simple à résumer. Il s'agit de disposer, dans un premier temps, d'une analyse historique de la genèse du savoir visé, établie de façon indépendante de l'analyse didactique, qui est cependant l'origine et le but de la recherche.* (…) *l'analyse historique constitue une banque de données que sous-tend déjà une réflexion épistémologique. Ce travail historique est en général conduit de façon parallèle avec les premières analyses didactiques et peut s'appuyer pour une part plus ou moins grande, sur des recherches déjà existantes. De cette confrontation initiale ressortent les premières hypothèses didactiques, qui vont permettre d'éclairer certaines difficultés d'enseignement ou d'apprentissage au niveau global. Il s'ensuit une deuxième analyse didactique de ces difficultés visant à préciser et valider (ou invalider) les hypothèses. Ces analyses sont alors confrontées à l'analyse historique dans une dialectique de nature épistémologique. Ce processus se poursuit ainsi, permettant de mettre en place les éléments du dispositif expérimental, visant à l'élaboration d'une genèse artificielle contrôlée par l'explicitation et la détermination des variables didactiques*" (Dorier 2000b, p. 29).

These vectors differed from those used in physics (which are usually fixed), and with their 3D-geometrical interpretation and their products (dot and cross), they also differed strikingly from the elements of an abstract, axiomatically-defined, vector space. As a result, the teaching of abstract linear algebra in the first year of higher education usually combined a poorly-motivated abstract approach as far as lessons were concerned (and those who tried to abstract the general structure from the geometrical case faced foreseeable difficulties) and repetitive algorithmic tasks as far as exercises were concerned; tasks for which the general theory was usually not necessary, since a reasonable command of linear systems could usually solve the problem.

Dorier endeavoured to both analyse the reasons for this state of affairs, and design alternative teaching paths. In both cases, the analysis and the proposals were based on his knowledge of history; however, the analysis and the proposals do not derive directly from history, for at least two reasons. First, there is no such thing as *the* history of linear algebra, and the problem of delineation of the object of study is a purely historical problem. Second, because of the heterogeneity of the two fields of research, Dorier selected what he considered to be key episodes in the history of mathematics, analysed them from both a historical and epistemological viewpoint, and used these studies as raw material for his didactical reflection. This didactical reflection combined other elements, both theoretical and empirical.

A key episode took place in the inter-war period, with the formulation and (partial) adoption in the mathematical community of axiomatic algebraic structures:

> This final step has its roots in the late nineteenth century, but only really started after 1920. It corresponds to the axiomatization of linear algebra, that is to say the reconstruction, with the concepts and tools of a new axiomatic central theory, of what used to be operative (but not explicitly theorized or unified) methods for solving linear problems. It is important to realize that axiomatization did not, in itself, allow mathematicians to solve new problems, but it gave them a more universal approach and language to be used in many varied contexts (functional analysis, quadratic forms, arithmetic, geometry …). In fact, the theory of determinants, which was very prosperous in the first half of the nineteenth century, is sufficient to solve all linear problems in finite dimension. (Dorier 1995, p. 176)

This epistemological reading of a historical episode paves the way for didactical analysis. Contrary to many concepts taught at primary and secondary levels, the concepts of abstract linear algebra do not primarily fulfil a problem solving function; rather, they serve formalising, unifying and generalising purposes (FUG concepts[43]). This specific connection between new concepts and more familiar elements of knowledge create specific teaching challenges: painstakingly designed problem-solving situations will probably not be adequate; and abstracting, unifying and formalising on the basis of analogies between many different specific fields is probably not a feasible teaching path. For FUG concepts, alternative teaching strategies can be based on using reflective analysis with students, carried out at a *meta* level:

[43]We refer the reader to the references in Dorier (1995) for the concepts developed by Robert, Robinet and Rogalski.

> My hypothesis, based on the epistemological analysis presented above, is:
> students have to anticipate the power of generalization due to the use of vector spaces.
> In this sense, I tried to build a teaching sequence which introduces the learner to a condensed form of reflective analysis, which has been proved to be one of the fundamental stages in the genesis of unifying and generalizing concepts. (…) The sequence, built on the basis of an epistemological analysis, creates an artificial context, which motivates the explicitation of the vector space axioms by the students themselves. (Dorier 1995, p. 186)

The adjective "artificial" testifies to the fact that this is not a rediscovery approach. Of course, both the challenge (teaching a FUG concept) and the design (enabling students to reflect *on* the properties of the mathematics they are dealing with) are pretty specific to late-secondary and tertiary education. However, they are not specific to abstract algebra, as Robert's work on the notion of limit shows.

A different historical investigation was motivated by didactical and epistemological reasons. Starting from the hypothesis that a key concept for learners was that of linear dependence (and its avatars: dimension, rank of a family of vectors, rank of a linear system of equations), Dorier investigated historical episodes of explicit formulations of similar concepts; as mentioned above, mathematicians were familiar with properties of linear dependence long before they were integrated into the abstract-algebraic theory of vector spaces, and had efficient tools to deal with linear problems. Without attempting to cover everything even barely related to linear-thinking, Dorier focused on the work of several authors, in particular Euler (1707–1783), Grassmann (1809–1877) and Frobenius (1849–1917). His in-context then comparative studies enabled him to make out several (mathematically equivalent) viewpoints on linear dependence, thus providing epistemological depth to the mathematical concept, and to identify conditions for the formulation of a more abstract and context-independent notion of linear dependence.

In his paper on Cramer's paradox in the theory of algebraic curves, Euler introduced a notion which Dorier christened "inclusive dependence": in a linear system,[44] the equations are not independent if at least one of them is "included" in the others, a phenomenon which manifests itself through an obstruction to the usual elimination procedure. In his paper of 1875 on the Pfaff problem, Frobenius also discussed linear systems, and the context is also elimination theory—although with the central tool of determinants, which was not the case for Euler. In spite of these similarities, the Frobenius paper explicates many fundamental notions which were only implicit in Euler: the fact that the set of solutions is stable under linear combinations; the fact that other systems of equations can be considered equivalent to the first, in two different (but equivalent) ways: they have the same linear space of solutions; and they consist of linear combinations of the first equations, satisfying certain conditions (we would now regard this as a change of basis for a

[44] Including systems with a non-null right-hand side.

subspace of the dual space). The duality between coefficients for the equations[45] and solution n-uples is brought to light, as is the relationship between the ranks m and n-m of the two systems of n-uples. Leaving technicalities aside, we can say that Frobenius formulated fairly general concepts by considering *sets* of solutions and sets of equations, the properties and equivalent representations of such sets, and a numerical characterisation of their "size" (the rank).

On the basis of the analysis we just outlined, Dorier designed a teaching module in which the notions of linear dependence and independence were gradually formulated in an ever more abstract way, starting from the context of linear equations. Instead of determinant theory (which was the historical context), Dorier relied on Gaussian elimination. At several points, *meta* level questions were discussed such as generality of the procedure, invariance of the rank, and the relative virtues of various proto-definitions of the concept of linear independence.

Of course, this summary does not do justice to a decade of investigations. In particular, Dorier also studied the transposition process through which the abstract algebraic structure of the 1930s was gradually divided and transformed into teaching objects. In particular, he showed how the geometric vectors—whose history is quite different from the one we just recapitulated—were brought into the picture in the hope of paving the way for the abstract version. For a comprehensive view of this work, see Dorier (2000a).

8.3.4 History of Physics and Physics Education Research

The following detour through physics may be surprising. Nevertheless it enables us to discuss a new set of examples based on recent research which documents a wealth of fruitful opportunities for interaction between the history of science and science education research. This section of the chapter will also reflect the fact that the two authors come from history of mathematics and physics education research respectively.

8.3.4.1 "Obstacle" in Early Physics Education Research

This detour through physics education research echoes Artigue's paper. She positioned her thinking on the concept of "epistemological obstacle" in the wake of

[45]For instance, with $n = 3$: consider the simultaneous equations $\begin{cases} 2x + 3y - z = 0 \\ x - z = 0 \end{cases}$, the triples (2,3,−1) and (1,0,−1) form a base of the space of equations with the same solutions (i.e., a subspace of the dual space), and (−3, 1, −3) is one element of the solution space. { (2,3,−1) , (1,0,−1) } being a family of rank $m = 2$, the space of solution is of dimension 1 (i.e., 3 − 2). In dimension 3, of course, a geometric interpretation is available: the space of solutions is the line along which two intersecting planes in general position meet.

results developed by early physics education researchers, particularly those of Viennot (1979, 2001). In doing so, Artigue connects the concept of obstacle to more generic reasoning that could explain recurrent mistakes or confusions produced by both novices and experts facing several physical situations or problems to be solved. As an example, Artigue refers to the *linear causal reasoning* where several variables changing simultaneously are considered one after another, interacting in a chronological way (i.e., multiple variable relationships interpreted as temporal relationships). This general trend of reasoning (generally incompatible with rationality in physics) has been identified as a powerful obstacle "generator" in several domains of physics (for example, electrokinectics, thermodynamics, waves). Focusing on trends of reasoning, physics education research reactivated to a certain extent the Bachelardian concept of "epistemological obstacle". In the late 70s and early 80s, several physics education researchers sought to identify similarities between historical ideas and students' trends of reasoning.[46] This orientation very likely opened the way for a long-standing interest from physics education researchers (and more generally from science education researchers) in the history of science. Nevertheless, this interest has changed substantially. Indeed, the large volume of research which has been carried out worldwide tends to promote the history of science as a powerful science teaching (or training) tool, especially in physics education.

Our intention is not to provide an exhaustive review of the aims and findings produced by physics education researchers working on or with historical materials. Instead, we address the issue of the choices that underlie the exploitation of the history of science when used for educational aims. Indeed, specific educational purposes may reflect specific and heuristic terms for using historical materials. In this regard, the history of science is mainly used in order to: (1) address the learning of a given concept (or law); and (2) improve students' and/or teachers' views on the nature of physics (from both epistemological and social viewpoints). Even if these two approaches are rarely exclusive from each other, researchers in physics education often fail to provide an explicit justification for the choices which govern the way they use, extract and organise the historical material in their work and studies. In this last part of our chapter, our aim is to provide some guidelines or benchmarks for a more explicit justification of the choices taken by physics education researchers when using historical materials. These guidelines can also benefit mathematics education researchers.

8.3.4.2 History of Science and the Learning of New Concepts

The above-mentioned approach promoted by Dorier can form a fruitful answer to questions which inevitably underlie the elaboration of teaching-learning sequences

[46]The relevancy of such similarities has been discussed at length in the physics education research community. See, for example, Saltiel and Viennot (1985).

in physics involving the history of science. In our perspective, associating the history of science and physics education consists in creating an "epistemological dialectic" (Dorier 2000b, p. 10) between two inquiries: the first focuses on students' reasoning or conceptions concerning a given physical phenomenon; the second concentrates on constructing knowledge in the historical context. This dialectic allows: (1) specifying the didactic constraints shaping the teaching process; (2) extracting the historical elements to be reorganised according to these constraints; and (3) ensuring that these elements take place in the didactic system with the aim of favouring the acquisition of a given knowledge by students. This last stage requires assuming a specific reorganisation of the extracted historical elements. Such reorganisation takes the form of a teaching-learning sequence designed on the basis of historical elements and named "didactical reconstruction". In a didactical reconstruction, historical and an-historical elements are included and mixed up in order to address specific didactic constraints (such as targeted knowledge, students' current conceptions or reasoning, visibility of the historical material, and the usual functioning of the class).[47] The idea is not to provide teachers or students directly with a history of science but to identify learning levers from a specific historical inquiry involving first-hand written sources.[48] These levers are articulated and completed with an-historical elements chosen and organised according to specific educational and conceptual purposes. Consequently, a didactical reconstruction is neither objective nor exhaustive but appears constrained, as with any reconstruction project. Indeed, the historical elements retained by a physics education researcher, as well as the way he or she chooses to organise them can lead to reconstructions that differ from those of historians of science. Because the motivations are specific on both sides, they produce particular readings. The legitimacy of these readings is guaranteed, not through a possible closeness with an ideal historical route but through the "fertility" of the program which underlies them:

> There is no neutral reading, no reading that does not engage a previous decision for defining the detained events or for defining relevant materials, entities, mechanisms. Every time, a selection principle is applied that depends on the adopted program. Each story, each reconstruction, each model corresponds to a determined principle of reading. This principle remains from a program, that is, from a generic manner of explaining or giving sense to an object. (Berthelot 2002, p. 242, trans. CdH)

In this perspective, we admit that a didactic "program" exists that supports the search for elements that could favour a better appropriation of physic concepts and laws. When conducting research, using the history of science for physics learning in a conceptual perspective should take into account the type of reasoning a student can use concerning a phenomena to be studied. Searching, within the history of

[47]Unlike Mäntylä's work which also involves didactical reconstructions (Mäntylä 2012), our approach is based exclusively on the exploration of first hand historical sources.

[48]The visibility of the historical material within the teaching-learning sequence could also be discussed. Indeed, the history of science can form a source for an educational pathway without being explicitly exposed to students as such (see, for instance, de Hosson and Kaminski 2007).

science, for situations that could be transformed into problems to be investigated by students is a way of challenging the didactic use of the history of science. From this standpoint, a search for certain proximity between students' reasoning and ideas from the past can form a fruitful fulcrum for the historical literature.

The model of didactical reconstruction as we define it has two functions. First, it allows us to make more explicit and also to better understand the (implicit) choices governing some teaching-learning sequences based on historical grounds. It is in this way that we understand the principles governing Merle's (2002) teaching sequences concerning the horizon line (skyline) in elementary astronomy. This sequence is based on an argument developed by Aristotle to address the spherical shape of Earth: the modification of the aspect of the night-sky for observers travelling south. It also takes into account the way students explain (with a drawing) why observers situated either in the north or in the south of the same meridian do not see the same stars in the sky. Generally, the drawing they provide presents a meridian sometimes flat and sometimes curved, whereas the field of vision of each of the observers is represented by a cone. If this way of geometrising the visible space of an observer fits perfectly with observations described by Aristotle, it does not allow discrimination between ideas of a flat Earth and those of a round Earth. In fact, the geometrical tool used by Aristotle to solve the puzzle of the visible stars is not a cone but a tangent line to the meridian, today known by the term "horizon", and which forms the knowledge developed as Merle's sequence. Here we face an asymmetry (not explicitly specified by the author) between historical hypotheses and stakes on the one hand and didactic ones on the other hand. From an historical point of view, the notion of horizon serves as a model for explaining the changes in the night-sky in order to justify the idea of a spherical Earth; from a didactic point of view, the changes in the night-sky connected to the spherical shape of the Earth allow the construction of the notion of horizon. Students face an incoherence which leads them to admit that their tool of "spontaneous" reasoning does not allow them to conclude that the Earth is spherical. This incompatibility between Aristotle's reasoning process and their own conclusions leads them to build a new geometrising tool for the field of vision. Here, the history of science is not actively involved in resolving the problem posed to the students but the approach proposed by Merle can be interpreted in terms of a didactic reconstruction: the situation she grasped from history of science has been chosen in order to form a fruitful problem-to-be-solved by students. Her choice was governed by what she knew about students' conception of "horizon".

The second function of our model is to provide science education researchers with a framework for the design of a teaching-learning sequence. We have created and implemented several sequences according to the guidelines presented above (de Hosson and Kaminski 2007; de Hosson and Décamp 2014). In these sequences, a dialectic was settled in order to create a problem directly inspired by an historical episode that could meet students' interest and thus, be accepted by them.

A common conception hold by students concerning the relationship between force and motion was addressed (de Hosson 2011). It is difficult for students to admit that an object dropped from the top of the mast of a ship moving at a constant velocity lands at the bottom of the mast because it retains the horizontal movement of the ship. This difficulty echoes the problem staged by Galileo in his *Dialogue concerning the two chief world systems*. This proximity led us to elaborate a learning pathway in which some elements of Galileo's dialogue were selected and reorganised according to specific educational constraints. The relevancy of the sequences have been asserted in the "didactic engineering" framework (Artigue 1994) and rely on the identification of students with the characters staged by Galileo. Here, the concept of "epistemologic obstacle" (Bachelard 2002) is considered as heuristic since it allows the search for anchoring problems, i.e., problems inspired by the history of science that could be appropriated easily by students, particularly as difficulties or ideas or reasoning on both (historical and cognitive) sides could be considered, to a certain extent, to be closed.[49]

Beyond the restricted and specific perspective detailed above, the selection process conducted by the science education researcher working with historical materials can also aim to challenge students' ideas of the nature of science.

8.3.4.3 Nature of Science (NoS)

History of science seems to play a significant role in helping teachers and/or students to develop more appropriate conceptions of the scientific enterprise. Nevertheless, the research carried out by Abd-el-Khalick and Lederman shows that the use of the HoS to enhance teachers' NoS views operates under certain conditions (Abd-el-Khalick and Lederman 2000). In particular, they claim that only an explicit instructional approach that targets certain NoS aspects can enhance teachers' NoS views:

> Science educators cannot simply assume that coursework in HoS by itself is sufficient to help prospective science teachers develop desired understandings of NoS (Abd-el-Khalick and Lederman 2000, p. 1088).

Considering NoS as an expression that refers to "the epistemology of science, science as a way of knowing, or the values and beliefs inherent to the development of scientific knowledge" (Lederman 1992), some authors have used the history of science to promote a renewed image of the nature of science. This also engages different choices and foci. Indeed, if we wish to use the history of science to

[49]An unexpected consequence of the dialectic process we address is the role that physics education research can play in historical inquiry. As Kuhn (1977, p. 23) says: "Part of what I know about how to ask questions to dead scientists has been learned by examining Piaget's interrogation of living children". Focusing on students' difficulties can lead to pinpoint historical episodes reduced or ignored by historians (de Hosson and Kaminski 2007). The proximity between students' ideas, and ideas from the past, may be justified from a Bachelardian point of view if one considers that some trends of reasoning contain universal elements. In other words, a *pre-physics* exists (as expired domain of knowledge) in both individual and historical development which may rely on a-contextual forms of thinking. This is quite different in mathematics since *pre-mathematics* does not really exist.

influence students' understanding of science, we must treat historical material in ways which illuminate particular characteristics of science. In the research conducted by Maurines and Beaufils (2012), history of science has not been considered per se but as a means to introduce students to 20th century philosophical ideas of science in order to help them to acquire scientific literacy. This was considered to be a richer understanding of how science works mainly today, rather than in the past. Consequently, the authors sought to identify, through analysing the history of physics (the creation of the laws of refraction), which aspects of physics could be considered as temporal invariables. According to the authors, the intersection of the various studies on science is where the most authentic view of science is revealed. Thus, they based their analysis on the philosophy, history, psychology and sociology of science. From this perspective, scientific knowledge was considered to be the result of activities—intellectual and practical—performed by individuals, working collectively, in the socio-cultural context of a given historical period. Maurines and Beaufils elaborated a set of documents (a dossier) in order to address and reveal some consensual views on NoS (e.g., "relationships between scientists"). These documents were made up of two types: some comprised historical scientific information (e.g., facts, hypothesis, knowledge, experiment); while others were chosen in order to be analysed on the basis of the different dimensions (spatiotemporal range and degree of externality) which can be associated with the different characteristics of NoS. As an example, the texts of the dossier related to the objective 'relationships between scientists' provide not only some scientific information, such as the type of explanation advanced by a scientist about the law of refraction, but also some information on the interaction between this scientist and the others.

8.4 Conclusion

We conclude this methodological tour by commenting on a deceptively simple motto: *history does not teach, yet there is a lot to learn from it*. Historians do not provide *direct* (even if partial) answers to MER questions, for a number of structural reasons which we attempted to describe: the two main reasons being the deep heterogeneity of the objects of study (mathematics written in contexts different from ours, usually by experts/teaching and learning of mathematics); and epistemological differences between the two fields of study (which reflect this heterogeneity). Yet heterogeneity and autonomy do not imply incommensurability. We presented several cases of fruitful interactions, from the purely heuristic—albeit of a well-controlled nature—to instances of the dialectic of co-problematisation.

While discussing these examples, we touched upon theoretical frameworks and concepts of MER, but we did not focus on them. These concepts were designed to study teaching and learning situations, hence are probably not suited for historical investigation. However, when the research question is a MER question, didactical concepts do not only help organise data into an intelligible structure, they also play a key role in the first phases of an investigation, when shaping the question and delineating the object of study. How, then, can a research question framed by didactical concepts make fruitful use of the history of science without mistaking these concepts as tools for historical investigation (Barbin 1997)? We feel this tension is structural and call for methodological vigilance. In this chapter, we adopted an approach which is very close to Dorier's, by using few and rather loose concepts—such as *aspect*—when discussing *direct* contact between historical and didactical investigations. This does not imply that, in another phase of research, theoretical frameworks cannot play their part. Indeed, we feel a natural continuation of the methodological discussion presented in this chapter would be the study of the extent to which different theoretical frameworks assign a role to epistemological investigation—and the role it might play.

As Artigue pointed out, discussing facts, knowledge and concepts cannot be all there is when discussing the role of epistemological awareness for the researcher in science education. Various forms of *experience* are also central, such as distancing oneself from one's own mathematical culture, or enabling one to see new dimensions in otherwise rather flat and unproblematic elements of mathematics. We would like to conclude with a quotation from Michel Foucault, who, in a different context, strikingly described a similar experience:

> As for what motivated me, it is quite simple; I would hope that in the eyes of some people it might be sufficient in itself. It was curiosity – the only kind of curiosity, in any case, that is worth acting upon with a degree of obstinacy: not the curiosity that seeks to assimilate what is proper for one to know, but that which enables one to get free of oneself [*se déprendre de soi-même*]. After all, what would be the value of the passion for knowledge if it resulted only in a certain amount of knowledgeableness and not, in one way or another and to the extent possible, in the knower's straying afield [*égarement*] of himself? There are times in life when the question of knowing if one can think differently than one thinks, and perceive differently than one sees, is absolutely necessary if one is to go on looking and reflecting at all. (Foucault 1990, p. 8)

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Chapter 9
Inquiry-Based Education (IBE): Towards an Analysing Tool to Characterise and Analyse Inquiry Processes in Mathematics and Natural Sciences

Cécile Ouvrier-Buffet, Robin Bosdeveix and Cécile de Hosson

9.1 Introduction

Inquiry-based Education (IBE) is being promoted by most science and mathematics curricula on an international level. This expansion is motivated by a political will to challenge students' interest and motivation by changing the way science and mathematics are taught. The main goal of such promotion is to prevent the decline in young people's interest in key science and mathematics studies in society. This goal explicitly governs the European Union's request for generalising IBE in the classroom. The Rocard report "for a renewed science education in Europe" (2007) clearly deplores the failure of the "traditional" way of teaching science and promotes IBE as a way of improving students' interest in science and mathematics: "the origins of this situation [i.e. the decline] can be found, among other causes, in the way science is taught" (The Rocard report, European Commission 2007, p. 8). In this context, many funded programs have been created to support institutions in implementing IBE in the classroom.

Over the last decade, an important part of Michèle Artigue's research activity has been devoted to the monitoring and expert evaluation of educational projects that focus on IBE in the specific fields of Science and Mathematics Education. Her commitment to the Fibonacci project[1] is likely one of the meaningful contributions

[1] http://www.fibonacci-project.eu/.

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of Michèle Artigue to IBE. Funded by the European Union under the 7th Framework Program (for research and technological development), the Fibonacci project aimed at a large dissemination of inquiry-based science and mathematics education in Europe, through institutions such as universities, teachers training centers and research institutions). The main aim of the Fibonacci project was to:

- assist students to develop concepts for understanding the scientific aspects of the world around them through their own thinking, using critical and logical reasoning about evidence they have gathered, and
- help teachers lead students to develop the skills necessary for inquiry and the understanding of science concepts through the students' own activity and reasoning.

Many IBE-promoting projects (like the Fibonacci project) relied on almost tacit but certainly questionable assumptions: first, skills developed through IBE echo those of science activity (in this regard, inquiry is supposed to be a method of helping students develop a sophisticated understanding of the nature of science); second, involving students in inquiry processes enables a more effective appropriation of scientific knowledge. In other words, through IBE, students would learn science concepts more deeply, as well as develop their skills in practicing science. Such assumptions have been widely discussed (see for example Lederman 2005; Sandoval 2005; Kirschner et al. 2006; Pélissier and Venturini 2012) since many political decisions for implementing IBE in science and mathematics curricula have occurred largely independently of the research on teaching and learning (Lederman 2005), and also independently of the research on students' beliefs about the nature of science. Moreover, most of the teachers were neither prepared nor trained to teach in accordance with IBE, and they have also faced (and still face) a shortage of established and consensual framework that defines what IBE is from an operational perspective.

Involving researchers in didactics (in mathematics and in sciences) in IBE projects can address a part of this issue. In this regard, this chapter proposes to highlight epistemological features of scientific processes aimed at researchers first and teachers ultimately. The underlying goal is to value a characterisation of such scientific processes that could fit with more open ways of learning, and then to put forward an analysing tool to characterise and analyse inquiry processes in mathematics and natural sciences. The desire to join science and mathematics in the same educational framework (as promoted by several IBE projects) will form an essential element of our purpose. In this chapter, a checklist is created as a characterising and analysing tool for inquiry-based sequences that are implemented in science classrooms. This tool is elaborated upon through the definition of features chosen to fit with both science and mathematics teaching sequences while respecting the specificities of each area of knowledge involved in our paper (mathematics, biology and physics).

9.2 Inquiry-Based Education: An Overview

Dorier and Maass (2014) point out that various definitions of IBE exist. During the 2000s, the North-American National Science Education Standards (NSES) were revised. Dorier and Mass attempted to summarise them in five points:

- “Students create their own scientifically oriented questions.
- Students give priority to evidence in responding to questions.
- Students formulate explanations from evidence.
- Students connect explanations to scientific knowledge.
- Students communicate and justify explanations” (Dorier and Maass 2014, p. 301).

In the French curriculum, inquiry-based teaching in mathematics and science is based on seven “key-moments”: the choice of a “situation-problème” (problem situation); the appropriation of the problem by students; the formulation of conjectures, explanatory hypotheses, possible experimental designs; the investigation or the solving of the problem led by students; the discussion argued around the elaborated designs; the acquisition and the structuration of the knowledge, and the mobilisation of the knowledge. Hypotheses and conjectures are emphasised.

In fact, IBE comes from science education: it is “not traditionally used in mathematics education and its recent appearance seems to have fostered by the proliferation of projects addressing both mathematics and science education” (Artigue and Blomhøj 2013, p. 802).

Several didactical theories in mathematics education can enrich the implementation and the analysis of IBE in mathematics, as shown by Artigue and Blomhøj (2013): for example, the problem solving tradition (arising from the work of Pólya 1957), the theory of didactical situations (Brousseau 1997), the realistic mathematics education (initiated by Freudenthal), and the anthropological theory of didactics (developed by Chevallard). The underlying epistemological backgrounds of some of these didactical theories have some commonalities with science education, such as the concept of “epistemological obstacle” (Bachelard 2002).

9.3 Inquiry Processes in Mathematics and Sciences: The Role of Problems

Artigue (2012) underscores the apparent similarity between the inquiry process in mathematics and in science, but warns against the features of these disciplines.

> As pointed out in the *Fibonacci Background Resource Booklet Learning Through Inquiry*, mathematical inquiry presents evident similarities with scientific inquiry as described above. Like scientific inquiry, mathematical inquiry starts from a question or a problem, and answers are sought through observation and exploration; mental, material or virtual experiments are conducted; connections are made to questions offering interesting

> similarities with the one in hand and already answered; known mathematical techniques are brought into play and adapted when necessary. This inquiry process is led by, or leads to, hypothetical answers – often called conjectures – that are subject to validation.
>
> (…)
>
> Nevertheless, despite the existence of similarities with scientific inquiry, mathematical inquiry has some distinct specificities, both regarding the type of questions it addresses and the processes it relies on to answer them. (Artigue 2012, p. 4).

In fact, from an epistemological point of view, we have to study the specificities of the problems studied by the disciplines themselves (mathematics and sciences), and also the key features of each inquiry process. That being said, in each discipline exist different branches and as many kinds of problems and inquiry processes.

Moreover, each didactic takes into account the epistemological features of the different sciences. Some underlying ways of thinking about the evolution of sciences (e.g., mathematics, physics, biology) used by mathematics and scientist educators are common (see the work of Bachelard and Popper for instance).

We propose a modeling which retains a strong epistemological background and which has a didactical impact. We first focus on both the definition of the problems involved and the characterisation of the inquiry process that can put mathematics and sciences closer for didactical purposes, and then, in the next part, we present the features we retain in order to analyse inquiry processes in a common way (mathematics and sciences).

In modern usage, the term “science” is often associated with the way scientific knowledge is developed. Thus, it now seems difficult to separate science as a knowledge about something from science as a process (sometimes called “scientific method”) in which the mechanism of creating problems-to-be-solved plays a leading role: “The formulation of the problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill” (Einstein and Infeld 1938, p. 29).

Placing the notion of problem as a key point for science teaching and learning refers to a double epistemological and cognitive posture which points to the construction of scientific knowledge as an act of creating and solving a problem (Dewey 1938; Bachelard 2002): “A question well put is half answered; i.e. a difficulty clearly apprehended is likely to suggest its own solution, -while a vague and miscellaneous perception of the problem leads to groping and fumbling” (Dewey 1938, p. 140). As a consequence, we will consider the construction of a problem by students as a key element for implementing IBE.

Indeed, referring to Dewey’s views on science, inquiry cannot be viewed in isolation from the process that governs the formulation of a problem: “Inquiry is a progressive determination of a problem and its possible solution” (ibid., p. 110). According to Dewey, inquiry is the controlled or directed transformation of an indeterminate situation into one that is thus determinate. This transformation goes through an intermediate step where the indeterminate situation becomes problematic. Actually, “to see that a situation requires inquiry is the initial step in inquiry” (ibid., p. 107).

Such an epistemological stand which gives a central role to the problems in the construction of scientific knowledge has strong implications in science teaching. Thus, the sense of school knowledge lies in the link with the problems with which they maintain a dynamic relationship (Bachelard 2002; Fabre 2009). To Lhoste (2008, p. 55), it is the problem, its solution, and the relationship which links problem and solution that constitute scientific knowledge. The problem does not disappear with its resolution. Placing more importance on the problem which is not given but built, some authors (such as Fabre and Orange 1997), propose to focus on problem building (or 'problematisation') rather than problem solving.

Mathematics, physics and biology are recognised today as forming specific areas of knowledge–as forming specific "disciplines". In that sense, they incorporate expertise, people, projects, communities, challenges, studies, inquiries, and research areas that are strongly associated with academic areas of study or areas of professional practice. It means that the three disciplines have a well-defined sociological existence which echoes the consistency unit formed by sets of specialised books, research articles and academic training and research programs, for example. The three disciplines may share common features and purposes, and may also address common problems or investigate common situations; however, their specificities rely, in a substantial way, on objects, concepts, and languages that carry normative relationships inside a consistent discursive space marked by relatively permeable boundaries. It is necessary to explore the relevancy, which consists in considering IBE as a cross-border epistemological and didactical concept. The following paragraphs aim to provide some benchmarks that allow us to specify the nature of the problems involved in the three disciplines we focus on.

9.3.1 Focusing on Problems in Physics

Physics can be defined as a domain of knowledge that explores "inanimate nature" (Wigner 1960, p. 3), from infinitely large to infinitely small. This exploration can deal with structure, organisation, and movement of matter; it can involve elementary objects or interactions between objects, for example. As a scientific activity, physics is a corpus of knowledge (e.g., laws of mechanics, standard model, Lorentz equations) and processes (or activities) leading to the discovery (or the creation) of such knowledge. The generic process of the discovery of these laws is to translate natural phenomena (observable or not) combining measurable quantity in order to establish laws expressed mathematically. Whether theoretical or experimental, digital and/or observational, physics is instituted by the connection it has with experiment (taken here in the sense of empirical referent), which remains the "judge" of any activity in physics. As a consequence, the validation of knowledge in physics is absolutely based on "reproducibility".

Indeed, in spite of the baffling complexity of the world, certain regularities have been discovered. These regularities (immediate consequence of the invariance principle) are independent of many conditions that could have an effect on them, and the exploration of the conditions, which do and which do not influence a phenomenon, is part of the experimental process carried out by the physicist:

> All the laws of nature are conditional statements, which permit a prediction of some future events on the basis of the knowledge of the present, except that some aspects of the present state of the world (…) are irrelevant from the point of view of the prediction. (ibid., p. 5).

The speech of the physicist applies to a system extracted from the real world. It is structured by figures, graphs, mathematical symbols, or propositions formed by words. It allows predictions and relies on causal relationships established through measurements. In this context, the problems in physics are diverse: for example, explanation, creation of phenomena, of objects, predictions of behaviour (see Hacking 1983). But globally, their solutions take the form of laws which are assumed to govern the reason for the inanimate nature (why nature—matter, radiation—is as it is?), the how of its past (how did it get there?) and its future (what would happen if?).

9.3.2 Focusing on Problems in Biology

Appearing at the beginning of the 19th century, the term "biology" designates the science of life's phenomena. Biology distances itself from the other natural sciences by its studied object: life. But behind a term showing a unified character, biology encompasses a set of various disciplines (such as anatomy, physiology, biochemistry, genetics, evolution, ecology) distinguishing themselves by the nature of problems and the type of research method used/employed. Biology addresses two types of explanatory questions: functional and historical problems. Mayr (1982) describes this duality of biology as follows:

> The two biologies that are concerned with the two kinds of causations are remarkably self-contained. Proximate causes relate to the functions of an organism and its parts as well as its development, from functional morphology down to biochemistry. Evolutionary, historical, or ultimate causes, on the other hand, attempt to explain why an organism is the way it is. (Mayr 1982, p. 68)

Functional biology shares a methodological similarity with physical sciences, as it implements an experimental method, and acts on the real world to understand it. However, historical biology distinguishes itself from physics, as E. Mayr and S.J. Gould explain:

> Evolutionary biology, in contrast with physics and chemistry, is a historical science—the evolutionist attempts to explain events and processes that have already taken place. Laws and experiments are inappropriate techniques for the explication of such events and processes. Instead one constructs a historical narrative, consisting of a tentative reconstruction of the particular scenario that led to the events one is trying to explain. (Mayr 1982, p. 80).
>
> The issue of verification by repetition does not arise because we are trying to account for uniqueness of detail that cannot, both by laws of probability and time's arrow of irreversibility, occur together again. We do not attempt to interpret the complex events of narrative by reducing them to simple consequences of natural law; historical events do not, of course, violate any general principles of matter and motion, but their occurrence lies in a realm of contingent detail. (Gould 1990, p. 275)

Biology differs physical sciences by the nature of biological generalisations, which do not have the status of authentic laws. J. Gayon describes this difference:

> Are there laws in biology? If we understand by "laws" more or less evident regularities, we cannot deny that life sciences discover laws. They are sometimes weak causal laws (because biological systems are open), sometimes laws of development, the latter being more descriptive than explanatory. More fundamentally, the strong category of law, as universal statement of unlimited reach, is philosophically problematic in biological sciences because of the primary facts of the variation and the evolution of species. The ideal physical model of the law is there ineffective in that we have to deal with systems rules which are the product of a history, along with it requires of locality and contingency. Biological laws, if they exist, are always partial and relative to a context. (Gayon 1993, pp. 56–57, our translation)
>
> On the one hand, biological generalizations almost always have exceptions, and are limited to groups of given organisms (for example the genetic code is not really universal). On the other hand, and more significantly, it is not possible to interpret them as meaning that things should be so, even if the past history of life had not been what it has been on our planet. (Gayon 2005, p. 5, our translation)

Besides the explanatory character of biological problems, some problems are technical or practical. They answer the question "how can it be done?" (such as how to treat a disease or how to improve agricultural returns). As with other scientific disciplines, biology maintains a dialogical relationship to applied sciences. Indeed, biological research has consequences for medical, agronomical, and biotechnological research and, mutually, these disciplines are the source of new biological problems.

9.3.3 Focusing on Problems in Mathematics

This broad area usually implies philosophical questionings and a kind of dominant view of mathematics, including the following elements: mathematics is formal; mathematics is theorem-proving, deductive logic, axiomatic method; mathematics is a body of truth; the ontological issue of the existence of entities is crucial, and so on. We agree with Cellucci (2006) who lists and demolishes 13 standard assumptions about mathematics (what he calls "the dominant view") and follow Hersh's vision of mathematics (1999). Hersh (1999) suggests that mathematics has a "front", which consists of polished results that we show to the world and a "back" which consists of what we do to obtain these results. He argues that mathematics is a collective human construction. The mathematicians choose the concepts that interest them, but they do not get to choose how the concepts behave. Since Aristotle has defined mathematics as "the science of quantity", the focus of mathematics has evolved through several rich and diverse branches. Each summons up specific kinds of problems, concepts and ways of reasoning, from purely abstract to utilitarian contents. The processes of exploring new problems, objects, structures, and new ways of modeling, reasoning and proving, are always fundamental and a challenge for mathematicians.

The characterisation of problems, which are at the core of mathematical questionings, is a challenge too: we avoid the classical dichotomy (often criticised and too general) between real-world problems and abstract problems in order to provide some emblematic kinds of problems in mathematics. In fact, mathematical problems are very different in nature (they depend on the considered branch of mathematics) and they often involve different ways of reasoning. The following kinds of problems demonstrate this wide variety: for example, existence problems, optimisation problems, classification problems, axiomatic problems, modeling problems, defining problems, decision problems, enumeration problems, inverse problems, approximation problems.

In this section we have attempted to provide some features of what a problem is in both mathematics and natural sciences considering that some of these features could take place in a science classroom context. Actually, if one considers IBE to echo with what scientists do in their laboratories, one would also expect to find some of these features embodied in the way IBE is implemented in the classroom. Specifically, IBE should favour the creation, by students themselves, of problems of different natures. Moreover, the specificities of the different scientific disciplines should also echo with specific features in the problems and solving processes at stake in the classroom. In this regard, we can thus question the relevancy of a unifying pedagogical framework for both mathematics and science education. Nevertheless, since standards tend to unify science teaching under a unique framework, we should be able to provide some guidelines that rest on some common epistemological features.

9.4 Our Tool to Analyse Inquiry Processes

9.4.1 How to Characterise and Analyse Inquiry Processes in Mathematics and Science Education?

Dorier and Maass (2014) propose the model shown in Fig. 9.1, which integrates what could be meant by an inquiry-based teaching practice in science and mathematics. These essential ingredients in IBE make up a working definition of inquiry-based education developed by the PRIMAS project.[2]

We can find other kinds of tables or figures in different projects and studies, each of them focusing on one or more of the ingredients proposed in Fig. 9.1. For this paper, we used this model from PRIMAS because it is a good synthesis of the existing research in the field of IBE, as well as the inquiry-based learning processes that PRIMAS lists in the document available on the project website. These processes include the following (they are presented in a circle with overlapping features), taking into account that "IBE [Inquiry Based Education] is a way of teaching

[2]http://www.primas-project.eu.

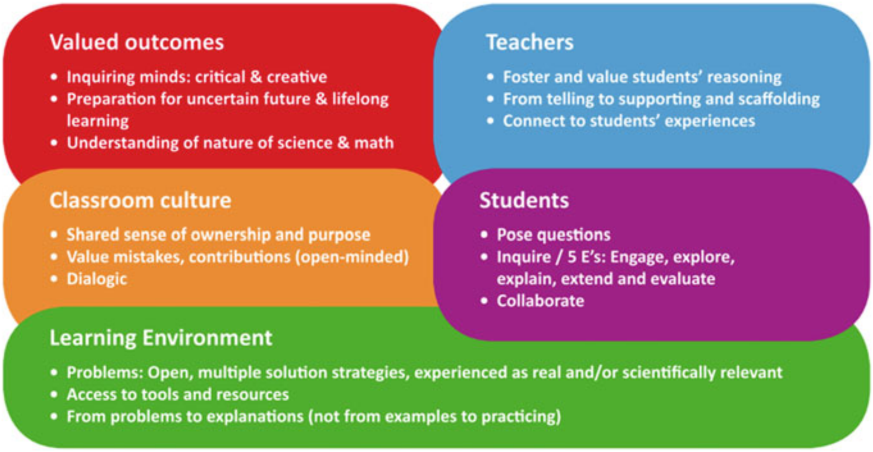

Fig. 9.1 Essential ingredients in inquiry-based education (Dorier and Mass 2014, p. 302)

and learning mathematics and science in which students are invited to work in the way mathematicians and scientists work" (The PRIMAS Project 2011, p. 10):

> (…) simplifying and structuring complex problems, observing systematically, measuring, classifying, creating definitions, quantifying, inferring, predicting, hypothesizing, controlling variables, experimenting, visualizing, discovering relationships and connections, and communicating. (The PRIMAS Project 2011, p. 10).

These actions belong to different levels, from the pragmatic to the theoretical.

We note that some features are transversal to the didactical analysis of inquiry processes (i.e., transversal to mathematics and science) in international research dealing with IBE: for example, the characterisation of the stages of the inquiry process (with some local differences between mathematics and science); the characterisation of the materials; the place of the written records; the composition of the students' groups; or the arrangement in the classroom. Furthermore, the Fibonacci project, for instance, proposes several ways to analyse the inquiry through the teacher's role and the students' activity. However, the epistemological characterisation of the problems and of their potential evolution, as well as the commitment of the teacher and the students in the evolution of both the problem and the questions, are not highlighted in the diagrams or forms resulting from projects in IBE. Yet a focus on the problems, on their evolution, and on the person who takes responsibility for the evolution of the problems at different stages of the process, would allow a better characterisation of the inquiry process itself.

In regard to the inquiry process, it is impossible to think about the method independently of the content. Scientific knowledge cannot be seen as the result of applying a stereotypical experimental method that addresses a scientific problem. Orange (2002, p. 84) explains that the problem commonly serves as a launch and as motivation, with an approach that aims to be general, and that the knowledge is a product which, once formulated, has a propositional existence. Orange (ibid., p. 88) questions the usual focus of science teaching on the experiment, which can lead to neglect of an essential point: the construction of problems. Such an approach, centered on the resolution of a problem, considerably limits the functions of the empirical data in the construction of scientific knowledge.

The inquiry process can thus appear in varied ways but it implies the construction and the putting in progressive tension between an empirical register, in which the relevant constraints are identified or developed, and a register involving explanatory models.

In order to develop a common point of view for mathematics and science to analyse inquiry processes, we then choose the following features: an accurate characterisation of the problems (in the discipline) and of their evolution during the inquiry process, the place and the role of both the students and the teacher (or the observer, or the researcher), the link between the stages of the inquiry process, and the evolution of the problems.

9.4.2 Our Features Used to Analyse Inquiry Processes

The following features (i.e., the characterisation of the notion of "problem", the definition of the moments of the inquiry, the role of both the teacher and the students, and the aims of the inquiry process) can be organised in a table. We present and define them first.

9.4.2.1 A Characterisation of the Notion of "Problem"

Our challenge is the following: we have to find a characterisation of the problems, which can lead to an inquiry process with a common viewpoint for both mathematics and science.

From a didactical point of view, the efficiency of the definition of "problems" arising from complexity theory has been proved through three research studies, which investigated epistemological and didactical aspects of particular mathematical concepts: Giroud (2011) studied the experimental dimension of mathematics; Modeste (2012) studied the concept of "algorithm"; and Ouvrier-Buffet (2013) studied the modeling of the defining processes in mathematics. This definition of "problem" is of course linked to the algorithmic notions of input and output, but it has the advantage of formalising a problem in an epistemological manner which is at first independent of didactical situations: this is a key point in our research on modeling problems from the perspective of studying inquiry processes. The didactical features can be integrated in this modeling of the concept of "problem" at a later stage, depending on didactical questionings and constraints.

In complexity theory, a "problem" is defined in the following way:

> For our purposes, a problem will be a general question to be answered, usually possessing several parameters, or free variables, whose values are left unspecified. A problem is

> described by giving: (1) a general description of all its parameters, and (2) a statement of what properties the answer, or solution, is required to satisfy. An instance of a problem is obtained by specifying particular values for all the problems parameters. (Garey and Johnson 1979, p. 4)

A problem is therefore a pair (I, Q) where I is the set of the instances of the problem (what is given at the beginning), and Q a question (or several questions) which integrate the instances (this question specifies the properties of the required solution).

This definition of "problem" is interesting in our study for three reasons:

- it appears usable in our three disciplines;
- it allows the characterisation (and then the analysis) of the evolution of the problem during an inquiry process, from an epistemological point of view: besides, the characterisation of the problem with the pair (I, Q) is made within the discipline and brings together the different stages of the problem;
- it connects the different stages of the problem with the different moments of the inquiry process (we define these moments in the next section).

Moreover, we propose to connect each problem (i.e., each problem that can be stated during an inquiry process) to the kinds of problems that exist in our disciplines. An overview of them is given above.

9.4.2.2 The Moments of the Inquiry Process and Their Aims

There is no unique characterisation of the inquiry processes in science and mathematics. We propose in this section a common vision of the inquiry process based upon our epistemological view of our disciplines (as presented above) and our crossed-view of the inquiry processes involved in our respective disciplines. In order to go further than existing descriptions, found in the literature, of several stages of an inquiry process, we define several moments: the aim(s) of each moment of the inquiry process should be described. We then propose nine non-linear and non-hierarchical moments of the inquiry process; it means that each moment can appear several times (or can also not appear) along a teaching sequence, in a spiral manner, following the chronological continuity of the inquiry.

The nine moments are:

- Exploration of the situation and construction of the problem
- Formulation of hypothesis/conjectures
- Test of hypothesis
- Modeling, for example, changing the model, the frame, or the scale
- Analysis, interpretation of the results, conclusion (first level of conceptualisation)
- Communication of the results and of their impact
- Generalisation of the results, the processes at stake, and the reuse of the process (second level of conceptualisation)

- Statement of new problems for the discipline
- Bibliographical research.

The aims of each moment depend on the situation and on the concepts involved. We can illustrate the aims of the moments of the inquiry process with the following examples: to state a temporary knowledge in order to experiment; to choose a relevant modeling; to valid or refute; to conceptualise; to generalise; or to prove.

To summarise, we have defined the moments of an inquiry process that are common to science and mathematics. Their aims specify them, and then the definition of these aims can distinguish the disciplines.

In order to keep the features of each discipline unspoiled (such as the counter-examples which appear mainly in mathematics), we decided to make these specificities (we call them "indicators", i.e., indicators of the process) appear in our characterisation of the inquiry process through clarifying the actions of both the teacher and the students.

9.4.2.3 Place and Role of the Teacher and of the Students—The Importance of Their Gestures During the Inquiry Process

The role of the teacher promoted by the implementation of IBE situations in classrooms is specific. In fact, several phases structure the activity of students when an IBE situation is implemented in the classroom. It often follows this pattern: individual work—cooperation with partners—presentation of ideas—discussion—summary and presentation of results. A variety of teaching methods for IBE exists and the goal of this paragraph is not to list these teaching methods. Instead, we focus on the role of the teacher and of the students from a different point of view.

We remind the reader here of an interesting table (Anderson 2002, p. 5—see Table 9.1), which compares a "traditional" transmissive teaching approach (called "old orientation") to an inquiry-oriented approach (called "new orientation"), taking into account the teacher role, the student role and the student work. We consider this table to be a good synthesis of the usual roles of the teacher and of the students, often quoted during an inquiry process.

In our epistemological perspective, we would like to emphasise the role of the teacher and the role of the students, regarding the previous moments of the inquiry process and to connect them to the evolution of the problem studied (in science or in mathematics). A central question for us is: "Who takes responsibility for the process and for the evolution of the problem?" In this connection, the gestures (in Gardes' sense, (2013)[3]), which consist of stating the problem, the questions, and

[3]Gardes (2013) models the concept of "gesture" in order to analyse the practices of mathematicians. This concept also appears relevant in considering the question of the transposition of the work of mathematicians to the classroom (several levels are considered: primary, secondary and university levels). Indeed, it can be used to analyse students' processes during research of a mathematical problem (the kind of problems used by Grades 2013 is mainly unsolved problems for

Table 9.1 A synthesis of the role of the teacher and of the student (Anderson 2002, p. 5) in two "opposite" teaching approaches

Predominance of old orientation	Predominance of new orientation
Teacher Role:	
As dispenser of knowledge	*As coach and facilitator*
Transmits information Communicates with individuals Directs student actions Explains conceptual relationships Teachers knowledge is static Directed use of textbook, etc.	Helps students process info Communicates with groups Coaches student actions Facilitates student thinking Models the learning process Flexible use of materials
Student Role:	
As passive receiver	*As self-directed learner*
Records teacher's information Memorizes information Follows teacher directions Defers to teacher as authority	Processes information Interprets, explains, hypoth Designs own activities Shares authority for answers
Student Work:	
Teacher-prescribed activities	*Student-directed learning*
Completes worksheets All students complete same tasks Teacher directs tasks Absence of items on right	Directs own learning Tasks vary among students Design and direct own tasks Emphasizes reasoning, reading and writing for meaning, solving problems, building from existing cognitive structures, and explaining complex problems

choosing the instances of the problem, for instance, appear crucial to us. The next paragraph synthesises our method for analysing an inquiry process.

9.4.2.4 Towards a Table to Characterise and Analyse Inquiry Processes

Bearing in mind the aforementioned epistemological and didactical reasoning, we structured our table for the characterisation and the analysis of inquiry processes with 4 main columns:

(Footnote 3 continued)

mathematicians, such as the Erdös-Straus conjecture). Gardes (2013) builds a new definition of "gesture", taking into account the research of Cavaillès (1994) and Châtelet and Longo (in Bailly and Longo 2003), with a theoretical background inscribed in the contemporary epistemology. She analyses the problem-solving process of mathematicians and students with several gestures, taking into account the syntaxic/semantic categorisation (Weber and Alcock 2004). Then, Gardes (2013) defines seven gestures during the exploration of the Erdös-Straus conjecture: to point out objects; to reduce the problem to a specific set of numbers (prime numbers); to introduce a parameter; to build examples and to explore them; to make local checks; to transform the starting/original equation; to implement an algorithm.

- the moments of the process;
- the aims of the moments;
- the characterisation of the problem from an epistemological point of view, anchored in the involved discipline, with 3 sub-columns (instances, questions, problem type);
- the indicators of the inquiry. These indicators follow the different moments of the process and enrich its characterisation with more contextualised elements.

The last criterion provides a representation of the distribution of the roles during the inquiry process ('T' for the teacher's indicators and 'S' the students'). It presents elements regarding the role of the teacher and the work of the students in the evolution of the inquiry, and more specifically, the entire process of the students and the complete guidance of the teacher appear clearly. In our table, we do not specifically develop the role of the teacher as coach and facilitator (the right-hand column of Anderson's table, see Table 9.1), which is clearly a necessary role of the teacher in situations involving inquiry processes. Table 9.2 synthesises our aforesaid criteria.

In Table 9.2, the first column provides a non-linear and a non-hierarchical list of the different potential moments of an inquiry process. Each one (of these moments) can appear several times or not appear at all, and the description of these moments is enriched by their aims. The list of these aims cannot be totally exhaustive but moves towards a global view of potential aims in our three disciplines (physics, biology, and mathematics). The column "Problems (in the discipline)—Type" mainly provides examples too.

In the next section, we add a further feature with a description of the aims of the inquiry process, in order to specify the place and the role of an inquiry process in a classroom (mainly a long-term process).

9.4.2.5 The Aims of the Inquiry Process

The implementation of situations implying an inquiry process in the classrooms can reveal several aims at different levels: from the level of the contents (concepts and skills) of the discipline to a meta-level regarding the discipline itself.

We synthesise these aims in the following way:

- the construction of knowledge in a discipline;
- the construction of theories (it can be a local theory);
- the construction of skills
  - specific to the inquiry process, or
  - linked to the use of an instrument/tool;
- the construction of abilities and skills for the statement of processes and results: this deals both with the scientific communication and written records;
- the learning and understanding of cultural and historical contents;

Table 9.2 Synthesis of our features to describe and analyse an inquiry process

Moments of the inquiry process (non-linear and non-hierarchical moments)	Aims of the moments	Problems (in the discipline)			Indicators of the inquiry process and sharing of roles
		Instances I	Questions Q	Type	
– Exploration of the situation and construction of the problem – Formulation of hypothesis /conjectures – Test of hypothesis – Exploration of conjectures, examples, counter-examples – Modeling, e.g., changing the model, the frame, the scale – Analysis, interpretation of the results, conclusion (first level of conceptualisation) – Search for a proof – Communication of the results and of their impact – Generalisation of the results, the processes at stake, the reuse of the process (second level of conceptualisation)	– Shifting from an indeterminate situation to a problematic one – Formulating a preliminary/temporary knowledge (for an experimental testing, for instance) – Choosing a relevant modeling (and changing the modeling…) – Validating or refuting the chosen modeling, the hypothesis, the conjecture – Conceptualising – Identifying an invariant (e.g., phenomenon, property) – Providing a proof – Generalising – Building a theory	Set of the instances of the problem (what is given at the beginning)	Question (or several questions) which integrates the instances (this question specifies the properties of the required solution)	– Study of a phenomenon – Seeking an order relation between quantities – Quantifying a phenomenon – Defining a new variable – Existence problems – Optimization problems – Classification problems – Axiomatic problems – Modeling problems	What students and teacher say and do

- the construction of interpersonal skills (speaking in public, respecting colleagues, and other social values);
- the understanding of what the discipline deals with, at a meta-level.

Clarifying the aims of an inquiry process is a necessary element which influences the way the analysis of this inquiry process should be performed.

9.5 Two Examples of the Use of Our Analysing Tool

The aim of this section is to provide examples of the use of our analysing tool in two disciplines: physics and mathematics. In order to present and analyse the situations, we first analyse the considered problem in the discipline, the potential moments of the inquiry process, and the use in a classroom. The complete tables appear in the appendices.

9.5.1 In Physics: Exploring the Bouncing Balls Phenomenon

Dewey's view on constructing problems in science has been turned into a didactic framework for creating a science teaching sequence based on inquiry. We sought a non-academic but familiar phenomenon—the bouncing balls—considered as a potential indeterminate situation that could be transformed into a problematic one. This bouncing balls phenomenon has been presented to prospective primary school teachers in the framework of a 3-h pre-service training session (Martinez et al. 2015). The students were asked to answer the following open-ended question: "You are scientists; you wish to study the bouncing balls. What are you interested in?" Starting from this question, the training session was divided into two different parts: in part 1, students had to formulate one problem that could be answered experimentally without support material (i.e., no balls, no measuring instruments); in part 2, after formulating a problem to be explored, students were provided with support material: balls of several kinds, of different volume and mass, measuring devices such as balances, rulers, chronometers) in order to solve the problem they constructed in part 1.

This activity has been carried out several times between 2011 and 2013 in France and in Colombia, involving a total of 13 groups of four people (8 French groups and 5 Colombian groups). The analysis allowed us to understand the elements that prevented or favoured the emergence of problems able to give birth: (1) to an investigation based on an effective experiment; and (2) to the construction of knowledge associated with the bounce phenomenon, such as the influence of physical parameters and the construction of characteristic quantities.

The French theoretical framework of learning-through-problematization (Fabre and Orange 1997) was applied to the analysis of the linguistic interactions between the preservice teachers involved in the problem construction activity. Table 9.2 (as used in Appendix 1) sums up and organises the different steps followed by one of the groups who succeeded in constructing one of the characteristic quantities of the physics of bounce: the coefficient of restitution. After wondering "why" balls bounce, students switched to a more investigable question (i.e., a question that can be answered with simple experiments and usual measuring devices), focusing on "how". They searched for a measurable quantity (the mass of the balls) that supposedly influences the bouncing balls phenomena (i.e., the heavier the ball is, the shorter the total length of time of the bounce). Noticing that the mass of the ball had no influence on the bounce phenomenon, students switched to another investigable problem: the ratio of two successive bounce heights.

The "bouncing balls" activity has been organised in order to favour possible investigable questions from a very open situation, which is not so usual in science teaching, which commonly provides problems to students. Here, the construction of the problem is challenged in the sense of Dewey: students are successfully involved in raising the indetermination of an indeterminate situation and the table we used allows us (as researchers) to identity different dimensions of the inquiry process.

9.5.2 *In Mathematics: An Example of a Defining Activity*

9.5.2.1 Defining Activities and Inquiry Processes in Mathematics

In fact, in mathematics education, defining activities are usually evoked during the study of proofs and of problem solving processes, rather than being studied on their own. To place the definitions in the core of the mathematical activity (i.e., an activity that builds new knowledge, brings new proofs and theories) actually reveals an epistemological interest and a didactical interest; besides, the construction of definitions is a component of the research process of mathematicians, as underscored by several authors (Lakatos 1976; Edwards and Ward 2008 for instance).

9.5.2.2 A Situation Involving a Defining Process

We chose to use the example of the construction of definitions of a particular mathematical object—the discrete straight lines—for several reasons. This object puts everybody (students, teachers, and also researchers) on the same level: discrete straight lines are accessible through their representations and are non-institutionalised in the classic curricula. Thus, students have neither preexistent definitions or properties or concept images of such objects nor expertise in problems involving discrete objects. Moreover, the questions of their definitions and of the equivalence of such definitions are crucial in ongoing mathematical research and can generate inquiry processes at different levels (such as problems of the construction of such discrete objects, recognising problems, axiomatic problems).

Studying defining situations of discrete objects is therefore interesting from an epistemological point of view and also from a didactical point of view (it allows a focus on the inquiry process). The epistemological and didactical study of the implementation of defining activities involving discrete straight lines with first-year university students is available in Ouvrier-Buffet (2006). Ouvrier-Buffet (2006) points out the different kinds of defining problems involving discrete straight lines and analyses the defining processes of students with epistemological tools involving the construction of definitions. She emphasises the in-action moments in particular (a kind of moment which comes before an explicit formulation of a property or a definition). The use of our table focusing on the inquiry process will bring a new perspective to the analysis of such situations.

9.5.2.3 The Use of Our Table

1. The problems in the discipline, described with Instances (I) and Questions (Q)

Following the mathematical elements presented in Ouvrier-Buffet (2006, p. 271) about the problématiques, we can identify different problems in mathematics involving discrete straight lines. The three types of problems described below provide general classes of problems. They can be refined as:

- Type 1: Classification problem. The starting point (I) is a set of objects built with coloured pixels on a regular squared table: some of them are examples of a discrete straight line and the others are non-examples of a discrete straight line, but they are non-identified as such. The question (Q) is linked to a classification activity (an explicit definition is not necessarily requested). Two problématiques can appear here: the first is to build discrete straight lines and the second is to recognise discrete straight lines. These problématiques lead to different properties of the mathematical object.
- Type 2: Axiomatic problem. The starting point (I) is a more complex set of objects than in Type 1 (e.g., discrete straight lines, discrete triangles and their non-examples). The question (Q) deals with the existence of a discrete geometry which could be in contradiction with the Euclidean axiomatics. The starting point can also be a discrete table with no coloured pixels on it.

- Type 3: Existence (and uniqueness) problem. The starting point (I) is the same as in Type 1 (with examples and non-examples of discrete straight lines). The question (Q) deals with the intersection of two discrete straight lines and of two discrete objects in general, and also with the number of discrete straight lines with two given pixels. It can lead to questions of Type 2.

2. The moments of the inquiry process

When one works on a defining activity, one can think that the only analysing tools are focused on the defining process. In fact, our table also provides an analysing tool for such situations in a wider perspective, with an explicit connection between the processes and the problems.

The table presented in Appendix 2 is based on the results of an experiment conducted with two groups of three first-year university students (the reader can refer to Ouvrier-Buffet 2006, Group A, and for more details, Ouvrier-Buffet 2013). This table underscores the evolution of the problems studied by the students and the nature of their questionings. The emphasis of the problems and of their evolution highlights the inquiry process and brings new elements to further expand the situation and to generate new inquiry processes through the definition of new problems in the discipline.

We do not focus on the scaffolding of the teacher, who clearly has the role of coach and facilitator (she/he recalls what the students did in particular)—that is the reason why the role of the teacher seems reduced.

9.6 Conclusion and New Perspectives

In this chapter, we have defined several features to study in both science and mathematics inquiry processes. These features clearly provide questionings regarding the epistemological backgrounds of our disciplines and the links between them.

We then emphasised the significance of the problems in our respective disciplines and proposed a tool to characterise these problems through three elements (Instances, Questions, Types). Focus then shifted to the formulation of problems and on the evolution of the studied problems, with regards to what the problem was in the considered discipline. Some emblematic problems in our three respective disciplines were given as examples (the list cannot be totally exhaustive without a complete epistemological analysis which depends on the considered concepts).

We also specified the moments of an inquiry process, in a common point of view for mathematics-science, and considering their respective aims. The indicators of the inquiry process allowed the characterisation of the role of both the teacher and the students, and then the identification of the inquiry process followed by the students, and the nature of the guidance of the teacher. These indicators are linked to the epistemological features which appear in the other columns of Table 9.2 (see above or in the Appendices). We then designed a tool (through a table) without taking into account the explanation of some pedagogical elements (e.g., how to constitute the groups of students, how to manage the moments of the inquiry process). Two main direct uses of such a tool can be:

- It allows a re-analysis of existing didactical materials: we have used the table to reanalyse existing excerpts involving an inquiry process in our respective disciplines; the relevance of the features was then proven. Furthermore, the epistemological content was linked to the process developed by the students and the problems they were able to generate. The process of the students was then characterised, as well as the guidance of the teacher.
- It brings an epistemological framework for designing didactical materials.

This tool can also be used to identify teacher practices, profiles and conceptions. Their view on the nature of mathematics and science can also be defined with additional interviews. In fact, several projects dealing with IBE have used Shulman's model of knowledge (1986) to identify teachers' content knowledge, pedagogical knowledge and pedagogical content knowledge. Several questions arise such as: What is the teachers' knowledge regarding the disciplines (mathematics/sciences)? What is their knowledge about students' mathematical/scientific conceptions? What is their knowledge about the ways of teaching mathematics/science? Even if previous research on IBE took these aspects into account (see for instance the PRIMAS project), the epistemology of the teachers is not identified. The tool we propose in this article in one of the frameworks that we can use to grasp a part of the epistemology of the teachers (of course, this epistemology is evolving and changing but a snapshot of it may be fruitful for didactical purposes).

In the same vein, we can also use the tool developed in this chapter to design pedagogical materials based on elements of the professional identity of teachers (teaching practices in the IBE framework among others).

In future, it should now be possible to conceive further research in order to preserve an epistemological background and to get to a finer analysis of inquiry processes. A new gate is opened with our tool for teaching and learning in mathematics and science in a transversal perspective and for teacher education.

Appendix 1

Moments of the inquiry process	Aims of the moments	Problems (in the discipline)			Indicators of the inquiry process and sharing roles (T teacher or researcher/S students)
		Instances	Questions	Type	
Exploring a situation and constructing a problem	Identifying an investigable problem	Balls, theoretical models	Different possible questions in physics, e.g., mechanics, dynamics, rheology, thermodynamics	Exploring a phenomenon	T: You are scientists; you wish to study the bouncing balls phenomenon. What are you interested in? S: Why do balls bounce? S: How do balls bounce? S: Does balls' mass affect the bounce?
Formulating a hypothesis	Formulating a temporary knowledge for experimental testing	Balls, theoretical model of the conservation of energy	Question referring to dynamics Question addressing the bounce modelling	Searching for an order of magnitudes for the involved quantities Quantifying a phenomenon	S: Balls' mass affects the bounce S: What are balls' measurable attributes of a bounce?
Modeling, change of models, scales, of frameworks	Choosing a compatible modeling with an experiment	Balls, theoretical models	Question addressing the bounce modeling	Quantifying a phenomenon	S: Comparing the height of the bounce before and after the impact. S: Computing the total numbers of bounces or the total length of time of the total bounce phenomenon

(continued)

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Moments of the inquiry process	Aims of the moments	Problems (in the discipline)			Indicators of the inquiry process and sharing roles (T teacher or researcher/S students)
		Instances	Questions	Type	
Testing the hypothesis	Validation or refutation of the modeling	Balls, theoretical models	Questions addressing the bounce	Quantifying a phenomenon	S: Observations: it is not possible to enumerate the last little bounces S: Experimentation: comparing the bounce heights before and after the impact with the ground
Analysing and interpretating the results (first level of conceptualisation)	Changing of interpretative framework	Balls, theoretical models	Questions addressing the bounce	Defining the parameters that do not impact a physical phenomenon	S: Conclusion: the quality of the bounce does not depend on the mass of the ball S: Formulating a new hypothesis
Formulating a hypothesis	Formulating a temporary knowledge for experimental testing	Balls, theoretical models of the coefficient of restitution	Questions referring to dynamic	Searching for a relationship between physical quantities	S: Successive heights ratio is constant
Testing the hypothesis	Validation of the hypothesis	Balls, theoretical models of the coefficient of restitution	Questions referring to dynamics	Searching for a relationship between physical quantities	S: Experimentation: measuring the three successive heights S: Observations: a constant ratio between the successive heights exists
Analysing and interpretating the results (first level of conceptualisation)	Generalisation of the conceptualisation	Balls, theoretical models of the coefficient of restitution	Questions referring to dynamics	Defining a new physical quantity	T: Institutionalising the "coefficient of restitution"

Appendix 2

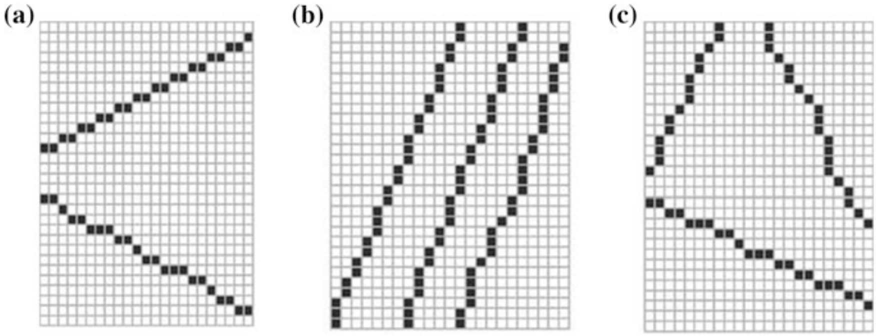
Moments of the inquiry process	Aims of the moments	Problems (in the discipline)			Indicators of the inquiry process and sharing roles (T teacher or researcher/S students)
		Instances	Questions	Type	
Exploration of the situation and construction of the problem	Identifying a treatable question	A regular discrete squared table is given	How to construct discrete triangles?	Axiomatic problem (with existence and uniqueness questions) Types 2 and 3	Session 1 S: Several ways to build a triangle are used (with three straight lines, with three angles, with three vertices)
Construction of a new problem	Simplifying the starting problem	Drawn triangles and non-triangles (non-identified as such)	How to build discrete straight lines?	Existence problem	S: The questions are: how to build straight lines, how to identify the existence of the intersection of two straight lines?
Formulation of conjectures and use of several models (e.g., analytic, algorithmic, geometric, arithmetic)	Formulating a preliminary knowledge (on discrete straight lines)	Drawn triangles and non-triangles, drawn straight lines and non-straight lines	How to characterise a discrete straight line/a discrete segment?	Defining problem	S: Several ways of building discrete segments come out Try with different mathematical models
Formulation of definitions	Formulating a preliminary knowledge (on discrete straight lines)	Previous drawings	Number of discrete straight lines by two given pixels?	Defining problem and uniqueness problem	S: How to define what is not a straight line? Formulation of in-actions definitions (the formulation of definitions is very

(continued)

(continued)

Moments of the inquiry process	Aims of the moments	Problems (in the discipline)			Indicators of the inquiry process and sharing roles (T teacher or researcher/S students)
		Instances	Questions	Type	
					close but not yet concretised by the students)
Formulation of conjectures (regarding definitions of discrete straight lines and of their equivalence)	Choosing a relevant modeling	About ten examples and counter-examples of discrete straight lines non-identified as such (see figures below)	Ask explicitly for delimitation between straight lines and non-straight lines.	Classification problem (to build discrete objects) Type 1	Session 2 S: The students explicit really different ways to build discrete segments T: The teacher/researcher asks about the limits of some characterisations
Test of conjectures/definitions And First level of conceptualisation	Choosing a relevant modeling	About ten examples and counter-examples of discrete straight lines non-identified as such (see figures below)	Ask explicitly for delimitation between straight lines and non-straight lines	Classification problem (to recognise discrete objects) Type 1	S: Tests of the students' characterisations on the given figures (see below).
Formulation of a new problem		Previous built discrete objects	How can one generalise a local definition?	Defining problem	S: How could we extend a discrete segment?

Examples and counter-examples of discrete straight lines



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Chapter 10
The Researcher in the Wider Community

Jill Adler, Celia Hoyles, Jean-Pierre Kahane and Jean-Baptiste Lagrange

10.1 A Quality Mathematics Education for All

Jean-Baptiste Lagrange

This section aims to draw some lessons from the way Michèle Artigue took on responsibilities in institutions at various levels in order to promote and enhance mathematics education. Drawing on Jean-Pierre Raoult's[1] friendly address and Bill Barton's[2] presentation during the International colloquium in honour of Michele Artigue, two institutions in France and one at an international level are considered.

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10.1.1 Positioning Our Research Field in the French Academic Setting

The first institution in France to be considered is the national peer-elected committee that discusses and makes decisions about the university teacher-researchers' academic careers. Although not well known abroad, this institution is very important for the development of a field of research in France, since a field cannot develop when researchers have no access to academic positions. As a new field, the "didactique des Mathématiques" needed to be recognised in a section existing in the institution and distinct from the "general education sciences" section. In the years since 1990, it was eventually decided that this section would be the "applied mathematics" section, given that "pure mathematicians" were particularly reluctant about acknowledging the field.

Statistician Jean-Pierre Raoult belonged to this section, as did Michèle Artigue, a recognised researcher in "didactique des mathématiques". Jean-Pierre Raoult recalls that only one another didactician worked alongside her, and that the two of them had difficulty processing all the didacticians' files submitted to the section. In addition, many mathematicians in the section were suspicious of mathematics education as a field of research. Thus, the task of the two didacticians could have been impossible. However, Michèle knew that a few members of the section, including Jean-Pierre Raoult, did not share the common condescending view on maths education. Research on mathematics teaching and learning appeared to Raoult to be both a necessity and a real scientific field, provided that studies in this domain relied on prior good quality field work, well-established frameworks, rigorous methods and finally on publications that would make sense outside the domain. Thus, on several occasions, his task dealt with a maths education file in collaboration with one of the two didacticians.

Most often, Jean-Pierre Raoult was interested in the maths education work he read for this task. However, he often relied upon Michèle for explanations. He most appreciated the way she answered him clearly, while leaving him free to make his own judgement about the quality of the file. Certainly, Michèle's participation in this committee and the type of relationship she maintained with the "applied" mathematicians on the committee was very important while the "didactique des mathématiques" was building its recognition inside the French academic milieu, allowing didacticians to obtain positions inside the newly created "Instituts Universitaires de Formation des maîtres".

10.1.2 Working in the IREMs Network

The second institution considered here is the network of "Instituts de Recherche sur l'enseignement des Mathématiques" (IREMs) and particularly the scientific committee of this network. The IREMs were created in a period of time marked both by the “mai 68” events and the emergence of “new maths” (“mathématiques modernes”) in the curriculum, as well as a large increase in the number of secondary students. Starting in September 1969, IREM were created in each region. The IREMs have been conceived as research institutes inside universities, bringing together university and secondary teachers in the assumption that everyone would bring specific knowledge to tackle the new maths “revolution”. Beyond teacher training (“recyclage”), a central aim has been using research methods to produce classroom resources. This is certainly “applied research”, but it also constitutes a terrain on which more fundamental research can develop. This was a great benefit for the French school of “didactique”.

The IREMS are federated in a network whose main structures are the Assemblée des Directeurs d’IREM (ADIREM), and a scientific committee whose role is to orient and evaluate research undertaken in the IREMs. Jean-Pierre Raoult was appointed as a member to this scientific committee in 2001. Probability and statistics teaching/learning was a growing topic of interest at secondary level, particularly because the new curriculum included tasks of simulation. The IREMs’ scientific committee did not previously include a specialist in probability and statistics, and Raoult was selected to address this gap.

While participating in committee meetings with Michèle, Raoult admired her wide knowledge, both of international research and of practical teaching in classrooms, as well as her ability to orient discussions at a high level. He noticed that Michèle rejected any dogmatism, instead favouring an inclusive approach to the various fields involved. Her understanding of difficulties experienced by IREM members, most often resulting from high teaching loads, did not prevent her from exercising the necessary critique of work done in the IREMs. Finally, in complex discussions, Michèle was able to propose synthesis that allowed the best conclusion to be reached. Thus, when Jean-Pierre Raoult became president of the committee, it was obvious to him that Michèle should be a member. She accepted in spite of her growing load of research project supervision and her increasing involvement in a number of international activities. Michèle had an eminent contribution to the meetings, including relevant advice and fruitful connections.

When it was time to step down as president of the committee, Jean-Pierre Raoult proposed Michèle as a successor and the committee approved. Consequently, at the time of the colloquium held in her honour, Michèle Artigue was president of the IREMs’ scientific committee. It was a difficult time for the IREMs network, in parallel with the current poor state of the French educational system, especially in regard to teacher preparation and in-service education. Difficulties existed not only from the political environment, but also from the sociological evolution that influences mathematics education: such as the growing influence of digital technologies and of distance education, the attraction of young people towards computer science and other fields, and decreasing confidence towards schools compared to other media. The way Michèle led the committee in this difficult context, in addition to her intensive research activity, is a brilliant and prominent example of how a researcher can take a role in the wider community. Jean-Pierre Raoult, along with other researchers, keeps trying to take part in this endeavour.

10.1.3 The International Committee for Mathematical Instruction

The institution considered in this section is the International Committee for Mathematical Instruction (ICMI). Michèle Artigue was president of ICMI from 2007 to 2009, followed by Bill Barton. In preparing to become president, Barton had a number of opportunities to collaborate with Michèle in ICMI activities. He reported on this collaboration during the colloquium in honour of Michèle Artigue.

The first example he gave occurred when the committee inspected Shanghai and Seoul as sites for ICME-12. In Shanghai, Michèle and Bill had been entertained on the Bund, taken to the Shanghai Circus, and had inspected the conference facilities. They were trying to establish the level of support for what they realised was a relatively trivial matter for Shanghai to host 4000 delegates for a week. Michèle was at her presidential best: with dignity equalling the epitome of Chinese diplomacy, she spoke in exactly the right tone to express ICMI's appreciation and respect while discreetly evaluating the actual level of support for an ICME gathering behind the politely expressed words.

A week later, in Seoul, the performance was repeated in what was, at the time, a scarily different context. The visit to the city hall coincided with a huge demonstration, and the delegation had to make its way into the building through a throng of chanting demonstrators, approach the four-deep cordon of riot police, thread through the small gap they conceded, walk through corridors of fully armed soldiers, and meet with the mayor in a small anteroom. Yet again Michèle found the right words and carefully observed protocol, while also eliciting the information needed. According to Bill, Michèle's confidence was epitomised as they sat in a black-windowed limousine, unable to move and hemmed in by riot police and protestors. With some experience from his student days in the 1960s, Barton realised that such a car would be the first target if things got out of control—and says he was tremblingly nervous. Not so Michèle: she understood enough of international protocol to realise that they would be, at that moment, the prime concern of the authorities. Sure enough, the Chief of Police soon smilingly rapped on the window and ushered the limousine through a gap that he had arranged in the protestors' cordon of minibuses.

Barton also related collaboration with Michèle during a two-day professional development event that took place in New Zealand. Like all teachers, participants were keen for the event to meet their expectations as they see their time as valuable, and are critical of it being used in impractical ways. Bill Barton had arranged the conference to coincide with the visit of the ICMI Executive and, of course, invited Michèle to deliver the keynote address. Much of the success of the whole event (and the $200,000 research project behind it) was in her hands. With the first hitch of her arm, the first endearing smile to those in the front row, the first words of engaging French English, and the first slide showing exactly how she would address matters of interest, Michèle captured her audience. Bill says that she touched each person in some way, showing she understood their trials, and speaking of their concerns with recognition—and, with her own particular stroke of brilliance, in a way that they received new insights about their classrooms. Here was a researcher who communicated directly, insightfully, and right on target. As a result, according to Bill, the two days were a “wild success”.

Finally, in his address, Bill Barton also evoked Michèle’s role in renewing a strong relationship between ICMI and the United Nations Educational, Scientific and Cultural Organization (UNESCO). In the 1970s, ICMI had a very strong relationship with UNESCO through Ed Jacobsen, who was appointed by UNESCO to focus on mathematics education. When Jacobsen left UNESCO he was not replaced, and for more than two decades the opportunities offered by a relationship languished unfulfilled. According to Bill, a major pillar of Michèle’s legacy to our international community is that the links in the chain have been repaired. Michèle, of course, does not do things by halves. An opportunity arose to assist UNESCO with a report. Despite her already full schedule, Michèle took this massive task on and completed it in such a way that it became a model for similar reports, and was elevated to the status of White Paper and UNESCO-guiding document.

However, that was not enough. With her toe in the door, Michèle levered it wide open, finding corridors to other sections of UNESCO so that the relationship was no longer dependent on one individual. According to Barton, Michèle chose not only to produce the UNESCO report, but to also form other relationships with UNESCO, fundamentally because she deeply and thoughtfully felt the need for “a mathematics education of quality for all”, a phrase she used to stress that it is not enough to get students into a mathematics classroom, but that all students, and especially the disadvantaged, need a *quality* mathematics education.

10.1.4 Two Lessons

Jean-Pierre and Bill’s talk that I drew on were full of genuine friendliness and admiration for Michèle. Many other colleagues, including myself, share these deep feelings and we are grateful to Jean-Pierre and Bill for finding the right words. I see also some useful lessons in these talks for making sense of a researcher’s activity beyond the strict production of knowledge in his or her field. The first lesson is that, for a field like mathematics education research (or “didactique des mathématiques”), there are crucial issues at stake and it would be a deep failure to ignore these: such as finding ways to establish the academic position of our field; maintaining relationships with school and university teachers in order to work together to find practical solutions to teaching problems; and developing relationships at an international level in order to promote mathematics education as a major partner in formulating educational policy.

The second lesson is that to take up the associated challenges, a researcher needs specific qualities and proficiencies, in addition to those that helped them to be recognised as a productive researcher. One has to understand the outside world, attending to the concerns of people interested in mathematics or education from a different point of view; and one has to be patient and collaborative, spending time and effort on tasks not directly linked to his or her research interests.

The next sections are authored by colleagues who are also widely recognised for the quality of their research and who also work beyond their field to the outside world. They provide additional visions on why, how, and for what outcomes they work.

10.2 Forging Bridges Between Research, Practice and Policy in Mathematics Education

Celia Hoyles

This section draws on the author's involvement as the first director of the National Centre for Excellence in the Teaching of Mathematics (NCETM) in the UK, which is in some aspects similar to the IREM network. It describes how the Centre started (and the role Michele Artigue played in this), how it has evolved since 2006, the invariant challenges and changes it has undergone, and what might be the Centre in the future.

10.2.1 Introduction

Mathematics education researchers pursue their own specific research questions without necessarily engaging with the practical and policy issues that have considerable bearing on mathematics teaching and learning. It might be considered important for the community to seek to enhance the relevance, utility, and accumulation of mathematics education research findings, and find ways to communicate messages from research to those that enact policy and practice (for a more elaborated argument, see Hoyles and Ferrini-Mundy 2012). In addition, there are benefits to exploiting the potential for engaging the mathematics education community in pursuing research questions that might have implications for policy. Michele Artigue has made invaluable contributions to all of these agendas: in France, but also more widely in international fora and the national debates that have taken place within different countries.

My goal here is to outline the challenges faced in making research professionally and publicly available in ways that might be used to inform the decisions and the practices of policy makers and teachers. Mathematics is 'problematic' for policy makers. The subject is highly regarded, tests tend to be high stakes and, in England at least, mathematics is widely conceived as difficult and procedural by those outside the mathematics community. An agenda for teaching and learning mathematics that is broader than calculation is often invisible to those outside the mathematics community, especially to policy makers, who most likely place high value (maybe their only value) on student and school test results and performance measures. Yet I would argue (with others of course) that progress in improving mathematics education can only be achieved when teachers resist narrowing the mathematical diet of their students to procedures to pass tests. Rather, for robust mathematical development, teachers must have the confidence to introduce a broader range of goals and activities.

One way to achieve this goal in England has been through setting up a national infrastructure for mathematics continuing professional development (CPD) in order to confer status, priority and obligation for evidence-based professional learning that is recognised and valued by all layers of the system and beyond. Thus the goal was, and is, that mathematics professional development will be an expectation and a responsibility for all those involved in teaching the subject, with politicians, local leaders and head teachers in schools all supporting this agenda. Teachers would then be supported in achieving a balance between helping their students perform better in tests, examinations, and international comparisons, but without sacrificing creativity and inquiry and without exerting so much pressure on students that they do not engage with the subject as soon as they are offered the choice. At the policy level, causes of a similar imbalance can be traced to the separation of the agenda for *teaching/learning* and the agenda for *standards*. By considering these issues from the perspective of the policy agenda in England, I will identify steps that have been taken to better align research, policy and practice.

10.2.2 A First Step: Giving Mathematics a Policy Voice Outside the Standards Agenda

The Advisory Committee on Mathematics Education (ACME) was established in 2007 by the Joint Mathematical Council of the United Kingdom and the Royal Society (RS), with the explicit support of all major mathematics organisations (www.acme-uk.org/). I was one of the founding members. ACME aims to act as a single voice for the mathematical community. ACME was formed after a period of many years during which there had been no conduit through which the mathematics community could have dialogue with government, despite the existence of a standards agenda that included mathematics.

Since its formation ACME has responded to all Government initiatives that touch upon mathematics and—crucially—written its own ‘proactive’ reports. The first of ACME’s reports was titled Continuing Professional Development for Teachers on Mathematics (ACME 2002). This report identified a ‘closed loop’ in mathematics teaching as a result of, what was then, diminishing numbers of students choosing to study mathematics post-16 (when mathematics is no longer compulsory) and following on to study undergraduate mathematics. It recommended the setting up of a sustained and developmental programme of CPD for all teachers of mathematics, across all phases and at all stages of their careers. This was to be provided through ‘a National Academy for Teachers of Mathematics and local mathematics centres’. The ACME report was launched at an event in London at which various influential members of the international mathematics community spoke, including Michele Artigue from France, Jeremy Kilpatrick from the United States of America, and Ruhama Even from Israel. The then Secretary of State, Charles Clarke, came to the launch in a dramatic fashion to announce to the large audience of teachers and policy makers that he supported the ACME proposal. This was the birth of what became the National Centre for Excellence in the Teaching Mathematics (NCTEM).

10.2.3 A Second Step: The National Centre for Excellence in the Teaching of Mathematics (NCETM)

The NCETM was eventually set up in 2006 by the UK Government and continues to operate at the time of writing (March 2012). I was the first Director. A new contract was agreed for the Centre to continue to March 2015. The Centre has a clear and ambitious vision. It aims to meet the professional aspirations and needs of all teachers of mathematics so that they can realise the potential of learners (for more detail, see Hoyles 2010). It is, however, a constant struggle to encourage teachers to see professional learning not as a threat or a punishment for being in some way deficient according to a standards agenda, but as an opportunity for new learning and inspiration.

The NCETM set out to meet its objectives through supporting a wide variety of mathematics education networks in the country, which include universities, subject associations and the whole range of CPD providers. At the same time, the National Centre encourages schools and colleges to learn from their own best practice through collaboration among staff and by sharing good practice locally, regionally and nationally. These collaborations take place face-to-face at national and regional events and in local network meetings across England, virtually, through interactions on the NCETM portal (www.ncetm.org.uk), and more recently through webinars. All methods of communication aim to support professional communities to discuss issues facing them (e.g., how to cope with new national initiatives such as a new curriculum, as well as perennial issues around teaching and learning: how to ask open questions in mathematics? how to design good formative assessments?). A portal is crucial but has of course to be regularly updated and improved to introduce new functionality (including Web 2.0), new design, and improved tools. It also implements “behind-the-scenes” improved search facilities. The statistics for the NCETM portal continue on an upward trend with over 80,800 registered users in March 2012, growing by approximately 1000 per week.

The NCETM signposts high quality CPD resources usually organised into microsites, which include departmental workshops that help teachers examine together a range of mathematical topics that “are hard to teach,” and sector-based magazines that offer monthly articles that are stimulating and timely. The site also points to useful CPD opportunities and courses offered by a range of providers in a constantly updated Professional Development Calendar, which identifies providers that hold a quality standard for CPD regularly monitored. The NCETM has developed self-evaluation tools (SETs) in mathematics content knowledge and mathematics-specific pedagogy. If teachers record limited confidence in any area, they are sign-posted to possible activities, on and off the portal, with which they might wish to engage to help them progress.

The NCETM has attempted to take forward into practice research indicating that involving teachers in collaborative reflection and enquiry pays dividends in producing real results in the classroom, and thus is an evidence-based initiative ripe for the policy arena (see for example, Krainer 2011; Roesken 2011). The Centre has organised a funded projects scheme, with over 300 projects funded and their reports posted at www.ncetm.org.uk/enquiry/funded-projects/view-all. The funded projects scheme provides resources to scaffold the research teachers may wish to carry out in collaborative groups within or across schools and colleges. Teachers bid for funds to pursue an enquiry and are provided with useful research “starting points” and references to promote building on previous work in the research community. The teachers write a report on their work, and reports and findings of the projects are posted on the portal and disseminated at NCETM events. Thus, learning is shared, and the impact maximised. Most, if not all, teacher groups find the experience of the research and the communication to others invaluable.

Many independent evaluation studies of the Centre have been conducted and I draw attention to just one (Gouseti et al. 2011). The authors noted that the modest amounts of funding provided by NCETM could have been provided using internal school funds. However, the researchers found clear benefits of having an *external* organisation provide the funding as a lever on school and district management and to confer status on the teachers’ work. Thus, funds and the recognition and validation of the process and outcomes through conferences, accreditation and award schemes, together proved a powerful incentive for professional learning.

10.2.4 Concluding Remarks

There are similar Centres in several other countries, a recent one being in the Federal Republic of Germany, where a national centre for mathematics teacher education has been established, funded by the Deutsche Telekom Foundation. An important research effort for the international mathematics education community might be to assess the impact of these centres and identify factors underpinning any successes that transcend national boundaries. Each country’s Centre has distinct goals, strategies, funding regimes and expected outcomes, but meta-analyses might usefully document the successes and challenges, and tease out some overarching principles and research findings that could have powerful implications for policy.

Many challenges remain: what type of evidence is needed to convince policy makers about needed resources or infrastructure in any one country, and can research form part of this evidence and, if it can, what form should it take? How can the findings be mediated so as to be meaningful for policy makers? In England, the picture of participation in mathematics post-16 and in university has shown quite dramatic improvement since around 2004. Which of the many initiatives initiated were critical for this upturn? Or, was it a matter of a cumulative effect? Those are important questions, worthy of investigation by future research.

10.3 Collaboration and Emergence: Reflections on (Some) Mathematics Education in Africa[3]

Jill Adler

This section focuses on key developments in mathematics education in Africa that have emerged through the work of the International Commission on Mathematical Instruction (ICMI), and been impacted on directly by the work of Michele Artigue. These developments, their strengths and weaknesses, are discussed and related to the influence of Michele.

10.3.1 Introduction

The past decade has seen increased mathematics education activity within and across a wide range of countries in Africa. The International Commission on Mathematics Instruction (ICMI), through its regional congresses and more recent outreach activities, has been instrumental in this development. The ICMI activities I focus on in this paper have all emerged during Michele Artigue's tenure first as Vice-President, then President, and currently Past President of ICMI. I joined Michele as co-Vice-President of ICMI in 2003–2006, and between 2007 and 2009, I continued as Vice-President under her presidency. During these years, and in many ways spearheaded by Michele, ICMI had a clear project: to expand activity in developing countries. "Espace Mathématique Francophone" (EMF) (discussed below) was underway, and there was keen support for another regional congress of ICMI, this time focused in and on Africa. AFRICME was launched with its first Congress in Johannesburg in 2005. In this section, I reflect on the *role of ICMI* in

[3] An earlier version of this paper was presented at the ICMI Centennial. See Adler (2008).

mathematics education activity in Africa, and conversely, on *the role of mathematics education in Africa* in ICMI, with focus on the AFRICME congresses.

This focus backgrounds ICMI's current development work. The Capacity and Networking Project (CANP) was launched with the first 'school' (a two-week workshop) held in Mali in September 2011. As elaborated on the ICMI website:

> CANP is a major international initiative in the mathematical sciences in the developing world to help exchange information, share the state of the art research, and identify strategies for addressing barriers to enhance mathematics education and to conduct to building a sustainable network for policy makers, scholars and practitioners across those targeted regions.

Both EMF and CANP have been initiated from 'outside', though in collaboration with, developing countries. Both contribute substantially to enabling networking across countries and continents. The inevitable tension is whether and how such initiatives come to be embedded in local activity, and extend beyond initial participants. Initiating, building and sustaining professional communities and collaborative activities can become even more pressing in contexts of poverty and constrained human and material resources. AFRICME is no exception. I focus here on mathematics education activity initiated and sustained within the continent. AFRICME is one such ICMI related activity. I discuss its emergence and development, and its strengths and weaknesses, as a fully regional activity. Of course, being internally driven does not remove, but rather repositions relations of power. For example, in the Southern African context in particular, the stronger South African economy, and instability in some neighbouring countries has drawn millions from other countries into South Africa's workforce, including many mathematics teachers. I return to these issues later.

10.3.2 Mathematics Education Across Africa

AFRICME emerged and functions alongside a considerable range of organised mathematical and mathematics education activity across Africa, some of which have been in effect for many years. For example, the African Mathematical Union (AMU) has a long history of activity on the continent, particularly through the Pan African Mathematics Olympiad (PAMO), and AMU-CHMA, the commission on the history of mathematics in Africa. The African Institute for Mathematical Sciences (AIMS), which focuses on postgraduate mathematics study, recently opened its second campus in Ghana. AIMS' first campus is in Cape Town, and these Institutes draw students from across the continent. More regionally, the Southern Africa Mathematical Sciences Association (SAMSA), and the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE) have regular conferences, promoting academic networking across countries. There are many other regional and local mathematics education activities and increasing collaboration and networking amongst academic colleagues. All the above initiatives are driven in and from Africa, and build collaborations and networks internally and then outwards towards and with the wider international community. It is beyond the scope of this chapter, and indeed my expertise, to do justice to the expansive and expanding work across Africa. I do not attempt such. I mention the above in order to provide some contextualisation for ICMI work in Africa.

As noted, the past decade has seen the emergence of two different kinds of regional ICMIs with conference-related activities involving participation in or from African countries. First was the CFEM (*Commission française pour l'enseignement des mathématiques*, the French Sub-Commission of ICMI)—an intervention driven by shared language. The CFEM has held conferences every three years since 2000, opening a space for communication on mathematics education across French-speaking countries with strong links into French-speaking African countries. Alternate conferences take place on the continent. Explicit attention to working in and across African countries is a central pillar of EFM, and one held up by Michele during her office in ICMI. Michele provided strong support for networking the Francophone community through EMF and with others drove the vision of paying critical attention to the African Francophone countries. The first EMF in 2000 saw a strong delegation from Africa, with the following EMF in 2003 being held in Tunisia. Over the years, both as VP and President of ICMI, Michele has been central in assuring this African presence at all conferences, as well as the alternation between North and South as sites for the EMF conferences.

Second to emerge was AFRICME. As an initiative in and from Africa, it was named the African regional congress of ICMI. In 2005, the first Africa regional ICMI congress took place in Johannesburg, with active support from the entire executive of ICMI, of which I was part. AFRICME1 marked the beginning of greater interaction, collaboration and exchanges focused on mathematics education among practitioners and researchers across the continent. AFRICME2, AFRICME3 and AFRICME4 have since followed in 2007 in Kenya, in 2010 in Botswana, and in 2013 in Lesotho. The organisation and academic aspects of this initiative are detailed below.

10.3.3 The Launch and Development of AFRICME

AFRICME1 was held at the University of the Witwatersrand, Johannesburg from 22 to 25 June 2005. One hundred and eighty delegates from sub-Saharan Africa, and particularly Southern, Central and Eastern Africa, attended the Congress. A considerable challenge in organising and hosting an Africa-focused Congress is finance. Across the continent, most practitioners in the field do not have direct access to travel funds. Fortunately, ICSU-South Africa contributed to congress costs, enabling us to part-subsidise registration for participants from Lesotho, Swaziland, Namibia, Botswana, Malawi, Mozambique, Zambia, Zimbabwe, Uganda and Kenya. Additional funding from the IMU-CDE enabled participation from African countries further afield, particularly Tunisia and Burkina Faso. South African participants had access to local funding. AFRICME1 set an agenda for this regional congress, with aims to stimulate regional collaboration and activity; promote regional and global contributions and interactions; highlight issues pertinent to mathematics education in developing countries; and share and showcase activities in and across countries in our region. These aims are aligned with overall ICMI goals and inform ongoing AFRICME activity.

The Congress theme of *Mathematics Teaching and Teacher Education in Changing Times* was supported by related national presentations. Participation was further invited through inputs into four focus strands, the first three of which were stimulated by an invited plenary panel presentation: Teacher Education (primary and secondary; initial and in-service); Information and Communication Technology (ICT) and Resources for teaching and learning; Indigenous Knowledge Systems (IKS), Ethnomathematics and the Curriculum; Teaching and Learning Mathematics (primary, secondary and tertiary). Members of the plenary panels were drawn from across the region, as well as from relevant international experts.

The Congress program included a dedicated 2 h slot for national presentations. Two problems surfaced as common across countries: the recruitment, retention and support of mathematics teachers across levels; and the mathematics content preparation of teachers. There are critical shortages of well-prepared mathematics teachers as well as recognition that the mathematics courses teachers are currently studying in their preparation years are not dealing with the kind of mathematical know-how needed for quality teaching. Many participants talked of the contradictory influence of the donor community in teacher education, particularly in professional development programmes. An outcome of deliberation during the Regional Meeting on the third day of the Congress was commitment to the development of a monograph, based on national presentations. The monograph was published and ready for distribution and sale at AFRICME2 (see Adler et al. 2007). The contributors are themselves mathematics teacher educators and active in the field of mathematics education across the continent and internationally. Their discussion of teacher education in their country is grounded in knowledge of practice, as well as knowledge of debates that frame the field more widely. None are nor claim to be systematically researched surveys. Each chapter offers the authors' perspective on mathematics teacher education in their country. The monograph is an important collective resource highlighting common challenges facing mathematics teacher education across twelve African countries, and a direct product and outcome of the emergence of AFRICME.

Reports on the congress identified key components of its resounding success. Firstly, there was extensive interaction and sharing of problems and ideas between *mathematicians, mathematics educators and teachers* across countries. This sharing across domains of expertise is a strength of ICMI. In many countries and contexts, mathematicians, tertiary level mathematics educators and government departments typically work in separation and all too often in conflict. Yet, all, though in different ways, are involved with the preparation of mathematics teachers, and all are concerned with the quality of mathematical learners leaving school.

Secondly, the Congress was Africa-focused i.e., the research, progress, problems and challenges shared during Congress sessions assumed a shared concern with conditions and challenges in and across African countries. This provided opportunity to grapple with real and large problems. For many of us who work extensively in mathematics education internationally, it is a common experience that the magnitude of the challenges we face in Africa is not easily appreciated. The scale and challenges of Africa are too often romanticised, exoticised or pathologised. We experienced a productive climate where we could share successes and challenges.

Third, this interaction across domains and across African countries enabled us to identify *key* shared problems, a central one of which is the recruitment and preparation, and then retention and professional support, for teachers. This is in turn exacerbated by the language diversity in and across various countries, and their particular colonial legacy. All are further affected by the prevalence and tragedy of HIV-AIDS, and these shaped the theme of AFRICME2.

The language of the conference was English—a constraint on full participation from, and discussion with, French and Portuguese speaking participants. Simultaneous translation would have greatly increased the costs of the Congress. This constraint is a weakness of AFRICME—to date all Congresses have been in English and only English, deterring participation more broadly across the continent.

Energy released in 2005 carried into the organisation of AFRICME2 in Kenya, Botwana in 2010, and Lesotho in 2013. At the same time, the relative economic strength of South Africa became visible as both costs for AFRICME and difficulties in raising funds in other countries increased. As is well known, travel across African countries is neither easy nor cheap, and while electronic communication improves daily, there were many more constraints in setting up AFRICME2, 3 and 4, and particularly in funding wide participation. Continuing with the aims and purposes that guide AFRICME activity, the AFRICME2 attracted 70 participants. Relatively large contingents attended from Kenya, South Africa, Nigeria and Botswana, a function of the host country and relative ease of access to funding in those countries. AFRICME3 and 4 drew greatest participation from Southern African states, a function of ease of access from neighbouring countries. ICMI provided start-up funding, and CDE funding enabled subsidisation of participants from countries further afield, and while small, the AFRICME conferences have included participation from Uganda, Burkina Faso, Malawi, Namibia, Zimbabwe, Mozambique and Rwanda.

The theme of AFRICME2 was *Embracing Innovative Responses to Challenges in Mathematics Instruction*. National presentations took this focus, and ranged from developments in Rwanda, to embracing technology in Botswana, and large national teacher development programmes across East Africa. As programme chair for AFRICME1, and in my role on the ICMI Executive, I was on the scientific committees for AFRICME2, and visited Kenya on two occasions as part of the congress and programme development. A principle we have tried to uphold is that there is some continuity across scientific committees from one congress to the next, so that experience and wisdom informs planning. The main organisers in Kenya however, were, for a range of complex reasons, not able to take on this role. Thus, I was also advisor to the scientific committee of AFRICME3, in a less direct role, also because my tenure in the executive ended at the end of 2009. AFRICME4 was supported by Prof Mellony Graven, the ICMI regional rep for South Africa. AFRICME4 participants from Kenya undertook to develop and maintain a newsletter and so facilitate ongoing collaboration and processes for the following conference.

In addition to this working committee, there are other visible spin-offs from AFRICME activity. Local communities are strengthened, and collaboration and networking across communities have started to develop. Kenya's mathematics education research community hosted a week's research workshop in July 2012, where scholars and postgraduate students interacted with invited scholars. A similar week's research workshop was held in Tanzania in November 2011, hosted and supported by the Aga Khan University. This workshop involved a range of participants in mathematics, science and language education research from predominantly East African countries, including Kenya, and it was encouraging to interact with emerging graduate students and young scholars.

In addition to the illumination of shared interests and concerns across countries, and the space and environment in which to discuss these with colleagues from other countries with similar or pertinent experience, open discussion in the Congresses has made visible some of the negative aspects of donor funding, particularly on social practices related to professional development. That donor led projects are largely not sustained once funding runs out is well known. Less evident are other impacts from 'aid'. One of the more contentious sessions at AFRICME2 was a presentation about the significance of professional associations—and a view presented in a plenary was that as professionals, teachers need to be encouraged to support their professional growth from their own resources, at least in part. Issues of dependency and entitlement versus independence and commitment were vigorously discussed, reflecting different orientations to, and effects of, 'aid' or 'support'.

AFRICME1 had wide participation, and while each of the following AFRICME congresses has drawn participants from a range of countries, numbers are still relatively small. I have already drawn attention to the two key constraints at work here. First are human resources, that is, sufficient numbers of local active scholars and practitioners who together can organise and support the logistics of an inter-national congress (albeit 'regional'), and be instrumental in its academic development. Second are financial resources. Each Congress has had start-up financial support from ICMI and some funds from the CDE for participation from African countries further afield. However, without additional local funding and support, wide participation is unlikely. AFRICME1 was able to draw substantial funding from research agencies in South Africa. Such resources are less accessible for countries that have since hosted the Congress.

10.3.4 Some Concluding Comments

In her role as both Vice and then President of ICMI, Michele Artigue brought a sensitivity and openness to working with African colleagues. She actively supported the development of AFRICME, and her collaboration with mathematics educationists in Francophone Africa enabled initial bridging across the language divides. She managed the demands of growing and shaping ICMI at the same time as ensuring ICMI's dominant work—its congresses and studies—broadened its gaze to include attention to the majority of the world's children who learn mathematics in conditions of poverty. Her detailed and insightful manuscript on basic mathematics education for UNESCO (Artigue 2011) has been influential in the setting up of CANP, and her two weeks of teaching and support in Mali are the clearest evidence that with respect to international collaboration and networking, and to both supporting and learning from those she works with, she walks her talk.

10.4 The Work of the Researcher, and Mathematics in CIVIC Life

Jean-Pierre Kahane

This last section discusses more generally the role and position of researchers, especially mathematicians, in contemporary society. After considering the place that mathematics has in civic life, the section concludes by emphasising the eminent social role of mathematics education.

10.4.1 A Preliminary Note About Language

Due to editorial constraints all chapters in this book are written in English. The colloquium in honour of Michele Artigue was delivered in three languages and it was appreciated as a success. There are many ways to speak of Michèle Artigue, and drawing on different languages, each with its own colour, could contribute to a satisfactory painting. My talk was in French, and Jean-Baptiste Lagrange was kind enough to translate into the present English version. If I had been asked to give my talk in English, the content would have been different. It was the case in another conference; I had already prepared a talk in French and was asked to give it in English just the night before. I had to entirely change the introduction and the conclusion, though it was essentially a mathematical talk. This is likely a common experience to many of us. In educational matters the use of English as the only possible language is a choice, not a necessity.

10.4.2 The Researcher is Primarily a Worker

Since there are millions of researchers and many research domains around the world, “the researcher” is an abstraction, like “the worker” is. Actually, the researcher is primarily a worker, and characterising the work of research is easier than characterising the work of the researcher. In any case, researchers are workers and their place in a community depends on the place of workers in this community. Recent experience in France is evidence that when the workers’ recognition declines, so does the researchers’ recognition.

Our hope is that workers will have a better position in the world, and so the researchers’ position will also improve. Reciprocally, is it true that when researchers are better off, the entire body of workers will also make progress? The answer is not straightforward. For centuries, research undertaken in prominent capitalist countries and recently, in European Community treaties, is seen as a weapon in the economic competition.

The link between research and innovation is not only a slogan; it actualises in the exploitation of researchers’ work for financial profit. As in sport, research is often dignified by emphasising individual performance, far from the actual research terrains where so many researchers work humbly. Society can recognise a small number of researchers as eminent, without recognising the role of the research work and more largely of the whole body of work.

Rather than highlight individuals’ achievements, I would like to support the idea of exemplarity of researchers’ work in general. This work combines personal initiative, collective work, cooperation and emulation, audacious projects, and constant world-wide peer evaluation. It involves debating, exchanging, communicating and publishing results and methods. In France for several decades, a stable environment allowed for efficient and flexible research. This, we should not only defend, but also promote. The French philosopher Montesquieu said that, in democracy, the citizens retain sovereignty and must do themselves what they know how to do. This is true also for workers and is of interest for all workers. Acting as an efficient professional is a civic obligation and this is what all researchers aim for. The researchers’ commitment to their work has value for all workers.

10.4.3 Mathematical Sciences

Mathematics is multifaceted and so is civic life. In this section I will explore some interconnections. Mathematics evolves as other sciences do, but with specific features. One feature is the intimate link with teaching; another is that ancient notions like numbers and figures retain a permanent and ever new interest; and another feature is the articulation between imagination and rigour.

In recent evolution, structures, models, and interactions inside mathematics and with other disciplines have been essential concerns. Attention is now placed on "mathematical sciences" as reflected by the work of the committee in which Michèle Artigue eminently contributed.[4] What is it about? In the committee, there were researchers in physical sciences and in computer science who did not have the same opinion on the role of informatics. Informatics plays an eminent role in modern life, as electronics does. Electronics is constitutive of physical sciences, so why separate computer science from mathematics? There was a danger of dwelling too long on this question. In contrast, the whole committee agreed that today mathematics is produced by physics and computer scientists as well as by mathematicians, even at the level of concepts and methods. Today, mathematical sciences come from everywhere, particularly from physics and computer sciences, and mathematics works like a sort of noria drawing from many sources: concentrating, distilling, simplifying, and making useable, ideas that often come even from outside the sources. For example, telephone engineers had no direct influence upon molecular biology; in contrast, the mathematical theory of information drew from telephone engineers' ideas that molecular biology found useful.

In mathematics, beauty is a guide. The beauty of a theorem has always been enjoyed. Etymologically, a theorem is something *before* what people stop and contemplate. In mathematics, beauty is mostly in simplicity. Henri Poincaré considered that definitions in science are chosen principally because they are easy to use. One can discuss this with regard to natural sciences. In contrast, it is a truth in mathematics. For example, defining a sphere by way of a centre and a radius provides a foundation, although it is counterintuitive with regard to ordinary experience: the centre of a spherical balloon cannot be reached. Children and adults, however, easily admit this abstraction because it is simple, beautiful and useful.

The cycle of conferences "one text, one mathematician"[5] organised by the French Mathematical Society is a true success. Themes play a role in this success, but another reason is that speakers succeed in being as direct and understandable as possible when addressing topics very new for the audience. They convey an excellent image of the mathematician in society.

10.4.4 Mathematics in Civic Life

In what dimensions does mathematical work concern civic life? What is its role now, and what can it be in the future?

Present-day mathematics' main contribution, if we want to be caustic, is to weapons and finance. All kind of mathematics is actually involved: financial mathematics as well as military mathematics are true and often bright mathematics, although directed by present dominant interests. My belief is that our grandchildren

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[4]CREM, commission de réflexion sur l'enseignement des mathématiques (see Kahane 2002).

[5]http://smf.emath.fr/cycle_texte_mathematiciens.

will have to work for a balanced world and for solidarity, and then we will start to see research in more varied directions. We already see new mathematics created and developed for mastering the data arising from other sciences. New mathematics will appear and old mathematics will reappear.

Mathematicians can valuably advise other citizens not to restrict research on matter immediately useful. For example, Euclide's prime numbers had no application for 20 centuries, and they continue to be the object of research with no visible application. Other advice is that nothing is garbage. We know how much past mathematics remains a source of inspiration and examples. We hope that the mathematics created today does not fall down and die because nobody takes over and continues the research.

Mathematics education is actually the main route by which collective values can expand to include present-day developments in mathematics. Teaching and learning mathematics, especially at university level, is crucial in order to ensure the transition between discovery and assimilation by society.

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Chapter 11
Preparing Young Researchers in Mathematics Education: Beyond Simple Supervising

Mariam Haspekian, Rudolf Straesser and Ferdinando Arzarello

11.1 Introduction

Preparing young people for research and supervising doctoral theses are two major ingredients of a researcher's activities. Whilst there are probably as many traditions as there are supervisors for carrying on this non-simple activity, there are certainly some essential processes and steps. What are these required keys to help a student become a scholar? Much more than concerning only two individuals—the student and the supervisor—the question is of community concern, as it is actually about how to make the doctoral student eventually gain the recognition of their peers.

The Artigue Colloquium offered an opportunity for discussing this researcher mission. This aspect, seldom considered in scientific debates, was taken up within a panel session, coined the *Artigue School*,[1] involving a few invited speakers and former students of Michèle—but the session also had as an aim to pay a tribute to Michèle Artigue's personal work as a supervisor of doctoral theses. Consequently, the expression 'Artigue School' intended not only to represent all those persons that Michèle Artigue has accompanied on the path of research, but also to emphasise ideas generated through such support and supervision. The session thus faced a

[1]The session (named *École Artigue* for the Colloquium) was organised by Ferdinando Arzarello, Mariam Haspekian, Bernard R. Hodgson and Avenilde Romo-Vázquez.

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delicate mission, as it aimed both at recognising a great professor, echoing the whole ‘Artigue School’, and at questioning in general terms the mentoring of students towards research in mathematics education. Moreover, the session was expected to be serious—but not boring, to offer some general reflection—but also remain personal, to be a true scientific activity—but also light and lively!

With these aims in mind, the session was composed of two parts. First, a presentation and homage to this specific activity, in which Michèle Artigue was herself strongly involved, offered by two of her former students, Mariam Haspekian and Avenilde Romo-Vázquez. Second, a more general reflection on preparing to research in mathematics education in the form of a roundtable, chaired by Ferdinando Arzarello, with two guests: Uffe Jankvist, a young researcher and representative of the YERME[2] group, and Rudolf Straesser, an experienced researcher and supervisor.

Building on what was presented at the Colloquium, this chapter aims to reflect these two strands, using the observations made about Michèle’s supervising activity as an occasion for raising the general issue of preparing young researchers to conduct research in mathematics education.

11.2 From Michèle Artigue’s Supervising Activity to More General Questions

11.2.1 The Diversity at the Heart of the Artigue School

The tribute given to Michèle Artigue was presented in the form of a two-voice dialogue echoing testimonies and reactions collected for the occasion from former doctoral students. A selection of these reactions was then used to sketch questions to be discussed in more general terms in the roundtable. We report here on these questions—but in a less personalised style than at the colloquium presentation.

We begin with a few interesting facts to describe the Artigue School (using data up to the year 2013):

- Michèle supervised some thirty theses (doctoral and *habilitation*) over 20 years, authored by 15 females and 12 males (Appendix 2 lists the doctoral theses)
- a total amount of about 11,500 pages, each one read several times by Michèle,
- Michèle supervised students from more than a dozen different countries.

But what is the actual activity hidden behind these numbers and facts? The various testimonies collected in Michèle’s case show that supervising doctoral studies can be seen as an activity carrying a threefold—at least—responsibility: professional, scientific and human.

[2] *Young European Researchers in Mathematics Education*, a strand within the ERME community.

First, supervision is a *professional* responsibility for all university professors in connection to their employment. Supervising young researchers is an activity quite demanding in terms of time and work, but is also one of the basic duties of an expert researcher. Depending on the country, this duty may be more or less officially inscribed in institutional texts.

However, these 'administrative' aspects are not always the ones that come to mind first: mentoring is, above all, a *scientific* responsibility assumed in reference to the community of researchers. This activity is essential for the very existence of research; it addresses the needs of new topics in the research field, and of developing future research strands.

The third window through which the preparation of young researchers can be analysed—and one strongly emphasised amongst the reactions to Michèle's work—is the *human* facet. This responsibility should not be underestimated. It is a difficult human activity to help a person become a researcher, facing all the various situations that a student may encounter throughout his/her personal life and position. In this context, there are two sets of constraints that are more often raised at the level of doctoral studies than at other levels: financial and family constraints. Such studies imply a minimum of financial 'security'. Moreover, in such a context, the student is typically not 'only' a student, living alone or with parents, but very often also with additional family constraints. Because of these fundamental constraints, the student's personal situations or difficulties frequently also become the supervisors'—at least to a certain extent.

These three crucial aspects relate to numerous and different qualities expected from the supervisor. Here are some (non-exhaustive) examples: tenacity, determination and engagement are needed for the first, professional, responsibility; open-mindedness, advisement, adaptability, diversity and expertise for the scientific responsibility; and humanity, respect for ideas and confidence, are needed for the final, human, responsibility. It is interesting that while all these words reflect Michèle's qualities as a supervisor, we can clearly use the same terms to qualify her own exceptional activity as a researcher per se.

Consequently, a central question is the interrelationship between these two activities of *supervising young researchers* and *carrying out high level research*: is one a condition for the other? This issue of interrelationship can also be seen through the diversity of research subjects in which Michèle was involved, either as a supervisor or as a researcher. The doctoral theses that she supervised concern the following areas: technology; university-level mathematics; transition from secondary school to university; teachers' practices and professional development; epistemology and didactics; transition to post-secondary mathematics; algebra; functions and analysis; geometry; modelisation; and connecting and integrating theoretical frameworks.

In conclusion, while many words could be used to describe the professional, scientific and personal activity of supervising, Michèle's involvement can also be characterised by another term: 'diversity'—diversity of ideas, diversity of places, and diversity of people. Michèle supervised many different types of students, including: a range of age groups; novice researchers as well as experienced teachers; people from different cultures; and those working across continents. Her students also prepared dissertations on a wide variety of topics, and utilised a range of methods, such as quantitative, qualitative and mixed methods.

11.2.2 Four Dimensions of Questions

One of the authors of this chapter (Haspekian) has previously been engaged in a reflection, with two other colleagues (Christine Chambris and Julie Horoks), on the mentoring of young researchers by experienced supervisors. While comparing perceptions and experiences of young researchers being mentored by different supervisors and looking for common, invariant questions, four dimensions were identified in order to evoke the task to be accomplished in such a context by the supervisor[3] (unpublished work). For the colloquium session, we borrowed these four dimensions as a good way to both pay tribute to an exceptional supervisor and to launch the discussion on learning to become a researcher in mathematics education:

1. *Degree of filiation*: this dimension refers to the interactions with the supervisor's own research;
2. *Productivity*: this dimension questions the way the supervisor fosters the progress of his or her student's work;
3. *Risk-taking and psychological stress* of the supervisor: this dimension indicates the fact that agreeing to mentor a doctoral student is also a kind of 'gambling'—on a person and on a topic;
4. *Handing over the reins*: this dimension points to a particular and crucial moment for a supervisor—when the student becomes a researcher on his/her own (and before in turn becoming a supervisor).

11.2.2.1 Degree of Filiation

The first dimension thus concerns the interactions with the supervisor's own research, passions, and the topics she is working on (or worked on some time ago, but reconsiders for the thesis). It can also concern the choice of a theoretical framework and/or methodology. Interactions with the supervisor's personal research are inevitable. The responsibility of supervising a thesis may lead the supervisor, more or less consciously, to push the student towards questions and methods with which she was or is struggling herself. This in turn raises two issues. In advance, the supervisor certainly perceives a range of possibilities for giving direction to the student—to what extent has she developed a sensible vision of the

[3]We wish to also thank Véronique Battie for the elaboration of these four dimensions.

future work? On the student's side, this also raises the more general question of the student's freedom, such as the freedom of choosing a research topic, e.g., a theoretical framework and/or a methodology. This question can vary greatly according to the supervisor/student relationship, but also depending on the institutional and cultural environment. Different countries may have different traditions about research, and may be more or less financed by private grants, eventually putting this element of freedom in danger.

11.2.2.2 Productivity

The second dimension is a more practical one: it concerns the stages and timeline of the supervising actions. How does the supervisor foster the progress of her student's work in the time available? What means does she use in order to help the student to write? What about the first production(s), the intermediate ones, and the read-through process (the comments on the written part)? What about the working meetings? Does a timeline exist?

The possible techniques are surely very different and to a large extent personal, but it should be possible to point out some invariants: designing a three-year plan, developing a good set of questions, getting the student to adopt a theoretical framework and a methodology, providing references, reading-through, receiving a well-structured text after the first drafts. However, these invariants must be more or less adapted to the student's personal constraints. Moreover, due to the specificity of the field of mathematics education, it is not unusual to meet doctoral students who are already teachers themselves, which often implies professional and family constraints on the student's side. Consequently, doctoral studies demand adaptation to a new environment, both in terms of culture and climate. Here again, the experience with Michèle raises two more general questions: the question of a *timeline of doctoral stages*, often organised differently according to the institutional culture, and the question of *adaptability*, which is not only a question of a subject's personality, but also one of institutional possibilities. How does the supervisor adapt to the different situations a doctoral student faces: for instance, a student or young teacher who wants to be better educated in the didactics of mathematics versus an experienced practitioner who wants to spend her 'leisure time' writing a doctoral thesis…? Is there a place for the latter type of doctoral student in every country?

11.2.2.3 Risk-Taking and Psychological Stress of the Supervisor

The level of a supervisor's stress is strongly connected to the risk-taking that thesis supervision represents. While the student, in the course of the doctoral study, is emotionally tested, the same is true of the supervisor, who in fact experiences a double insecurity, gambling both on a person and on a topic. This dimension is probably the most personal one, deeply dependent on the supervisor's character. It is certainly not a simple question of loving risk or loving gambling. It is more a question of trust in life, trust in every person. In Michèle Artigue's case, one can say that she acts as if nothing is impossible—and this, in return, creates conditions for wellbeing and makes the student feel more confident in the work to be accomplished. But Michèle's case does not seem to reflect the general case. According to Stubb (2012, p. 1):

> the number of doctoral students who never finish their degree is quite high in many countries. Depending partly on the discipline and the country in question, the numbers range between 30% and 50% (…), with some sources suggesting even higher attrition rates.

Thus, in the general case, three questions can be asked in order to limit risk and to deal with the time aspect when the doctoral process becomes problematic: How far ahead can the supervisor foresee the work to come when accepting a mentoring role? How does the supervisor manage when the research does not advance? How can the supervisor be of help to the student in overcoming personal difficulties?

11.2.2.4 Handing Over the Reins

The last dimension relates to the scientific future of the student, and her/his autonomy in research. How does the doctoral work prepare the student as a researcher? How does the experience as a Ph.D. student nurture the work of becoming both a future researcher and a future supervisor?

One of Michèle's techniques to address this issue is the immersion of the student in one of the supervisor's real-world research projects. Such an immersion is a good way of helping the student to grow on the research pathway. Most of Michèle's students heavily benefitted from the geographical and institutional proximity with the IREM[4] structure. This calls into question the influence of institutional (local and national) constraints on the supervisor's task, which may play out positively, but may also act as negative constraints in some cases. It is only within the complex webbing of these constraints that a supervisor can act (and be successful as Michèle was!). New questions are thus being addressed here: How important for the supervisor's job is the role played by local, institutional and national constraints?

[4]An IREM—*Institut de Recherche sur l'Enseignement des Mathématiques*—is a structure, partially financed by the Ministry of Education, where researchers in mathematics education, mathematics teachers and teacher educators can meet in order to carry out different activities together: research, teacher education sessions, or production and dissemination of teaching resources or innovations. There are 28 local IREMs throughout France.

11.3 Reflecting on the Education of Young Researchers in the Didactics of Mathematics

The preparation of young mathematics education researchers is an aspect seldom considered, although not totally inexistent, in scientific debates. Among the researchers who have already considered this issue are McAlpine and Amundsen (2011, 2012), Stubb (2012) and Pyhältö et al. (2009). One can also refer to the outcomes of the conference, *One Field, Many Paths: US Doctoral Programs in Mathematics Education* (Reys and Kilpatrick 2001). In this section, we aim to introduce some additional perspectives to complete the characteristics of this threefold activity, as we have presented it above.

We use as a starting point for our discussion the four dimensions structuring the supervisor's task, as introduced above, and then add perspectives arising from other studies.

11.3.1 The Questions Discussed in the Roundtable

The following four questions are related to the dimensions introduced in Sect. 11.2.2.

What is the interrelationship between the supervising activity and the supervisor's own research?

According to one of the panellists (Straesser), an obvious and necessary tension can be observed between the supervisor's research and a thesis: starting from the assumption that a doctoral student is an adult who has to somehow work on the 'same' question for a substantial number of years, it is absolutely necessary that he/she can personally identify with the topic and methods (the '*problématique*') of the thesis to be written. The chosen subject can be strongly oriented by the theoretical approach chosen. This can be in line with, but also in contrast to, the supervisor's ideas, experience and preferences, sometimes requiring a delicate balance. For instance, in France the two major theoretical approaches, namely the 'TAD' (sensu Chevallard) and the 'TDS' (sensu Brousseau), shed different lights on information and enable different analyses. The potential tensions, between the preferences of the supervisor and those of the student, can sometimes be reduced by bringing in a co-supervisor—quite common in some of the Nordic countries. However, a co-supervisor can also complicate the general balance by introducing an additional element into the communication/cooperation process between the doctoral student and the supervisor.

How does the supervisor foster the progress of her student's work?

Such a question about a possible 'timeline' or 'stages' on the road leading to a doctoral thesis, though not necessary in general for structuring the student's work, may presuppose a specific environment. In some places—and surely in the French '*écoles doctorales*'—there may be a set of (obligatory) doctoral courses and stages to follow on the way towards the defence of a doctoral thesis. This structure may be additionally strengthened by institutional constraints in certain local graduate schools and/or national regulations. Such structures and constraints may vary substantially from one country to another. For example, the structure of doctoral studies in Germany is quite different from, and less organised than, the way the process is handled in France. It is obvious that a delicate balance has to be found between wise institutional, maybe even national, regulations, and the possibilities and constraints of the (sometimes very) individual case of the doctoral student. A particular consideration in this respect is the role of courses for doctoral students offered on a national (or even international, as shown by courses in Denmark) basis. In a less developed environment (for a student, say, at a small university), such institutions can be crucial.

If there is a 'standard' timeline, which may even be strengthened by external, financial constraints (like stipends from different agents), questions about the inner organisation become crucial (such as the search for references; the choice of research methodology—maybe even at variance with the supervisor's habits, as mentioned above; learning and developing scientific writing; attending international conferences; and publishing in journals before graduation). The form of the dissertation may be an additional issue—awarding, for instance, a Ph.D. not for a 'book', but for a set of refereed journal publications collated together with a summary.

Another possible situation concerns the experienced 'practitioner', an able, scientifically well-educated teacher, who can afford, and wants, to spend her/his 'leisure time' writing a doctoral thesis (such cases really exist!). Is there a place in the 'system' for such doctoral 'students'? (Going back to the first dimension: Will she/he be accepted by a 'normal' supervisor closely linked to one of the dominant paradigms of the discipline?). This case often occurs in the research field of mathematics education. In France, however, there is no institutional help to facilitate doctoral studies by teachers. These doctoral studies then take 5 or 6 (or more) years, are self-financed, and carried out in conditions that are very different from 'direct' students, financed by doctoral funds, with no 'other' profession to fulfil than studying, and finishing in 3 years. Even if the quality, at the end, is the same, the financial impact is completely different between these two cases, the second student entering (and then progressing) a research career much more quickly than the first student.

There is also a gender issue, observable in Germany but which may be also valid in other countries: to put it in crude terms, is the 'normal' doctoral student male? The gender gap may be not too wide in mathematics education, but there are at least national differences in terms of family support, which often impact on the gender issue. To provide just one example from Germany—to a certain degree in contrast to France or Sweden—the societal support systems are not as helpful for female doctoral students (see the statistics in Mills et al. 2014, especially the low German provision of childcare for children up to three years of age—the graph on p. 6 of Mills et al. is reproduced in Appendix 1). Comparable constraints may be present to a certain degree in France, at least in the post-doctoral phase (see the fourth dimension, as discussed below).

What about the risk-taking and stress of the supervisor?

Both persons, the doctoral student and the supervisor, are caught in a linked gambling situation—and both should endeavour to make it a win-win situation. Apart from the nitty-gritty details (see some examples below), creating a win-win situation is most important, even if in some cases, the gambling develops into a lose-lose situation which is difficult to escape. If the deterioration is due to personal characteristics of the Ph.D. student and/or the supervisor, co-supervisors may again be a way out of this unwanted situation.

What about the student becoming a researcher on his/her own? How to accompany and guide this transition, from novice in research to an expert?

From the experience of one of the panellists (Straesser) at the Colloquium, there should be as many different ways as possible of achieving a Ph.D. thesis and becoming a recognised researcher. If writing a thesis can remain an 'affair' between the doctoral student and the supervisor—as is often the case in Germany—becoming a researcher needs more academic support and recognition. This can be helped by local or national doctoral programs—as was the case with the doctoral school sponsored by the Riksbankens Jubileumsfond in Sweden. In most cases, this implies for the doctoral student certain duties, if not compulsory courses, not directly linked to the topic of the dissertation, but helping with a certain standard for the dissertation and the overall research education of the doctoral student. Personal, local, institutional and national affordances and constraints are most important in the delicate balance inside a 'doctoral family'. It is interesting to note that in Germany, the expression 'Doktorvater' (doctoral father) is traditionally used to designate the supervisor—nowadays duly enlarged to 'Doktormutter' (doctoral mother) so to reflect both the appropriate 'role model' and the actual situation. As is the case in parenting, everything has to be done in order to help the doctoral student maintain this delicate balance, to further her/his success in academic life, and to empower her/him to educate in turn 'doctoral children' by her/himself.

Additional questions can also be raised, related to the issue of the education of researchers beyond the supervisor, and even beyond the university. Such questions can be connected to other studies.

11.3.2 Perspectives from Other Studies

A panel session at a recent PME conference (PME 37, Kiel 2013) was devoted to the discussion of issues concerning the education of young mathematics education researchers (Liljedahl et al. 2013). From a synthesis of different studies providing recommendations for the improvement of doctoral education, the contributors listed a number of different questions that can be explored (Liljedahl et al. 2013, pp. 1–71): the composition and housing of mathematics education Ph.D. programs; the research preparation of mathematics education Ph.D. students; their mathematical preparation; their mathematics education preparation; their teaching preparation; and the policy pertaining to the mathematics education of Ph.D. students.

Two main issues were then raised:

- the education of young mathematics education researchers goes beyond Ph.D. programs and beyond universities; and
- young researchers need mentorship from their supervisors and, more broadly, from the field of mathematics education.

These issues arising from the conference can relate to our questions, particularly to the question of freedom (what is the actual freedom in research for a Ph.D. student?—see the discussion under the first dimension in Sect. 11.2.2), the question of external/internal constraints (which can influence the productivity of a Ph.D. student), and the question of turning the student into a recognised researcher (entered into an international research community and able to suitably develop her/his own research project). But they also point, more strongly than we did, to the importance of the political dimension. According to Liljedahl et al. (2013, pp. I–71):

> young researchers need mentorship not only from their supervisors but also from, and within, the field of mathematics education. In this regard, consideration as to whether or not an organization actively partakes in the mentorship of young researchers becomes a political one.

In a similar spirit, Stubb (2012) stresses the importance of communities (scholarly communities or communities of practice, as defined by Lave and Wenger (1991) for helping undergraduates to move "from the periphery to the centre of research activity":

> This environment provides the grounds for learning competencies such as understanding one's own discipline, the ways to conduct research, and the ways to communicate the research to others, both inside and outside the scholarly community. (Stubb 2012, p. 18)

Thus, one key process in supervision is certainly to promote this dynamic integration of the student within various communities, including the central academic research community. But this needs to be put into balance with the fact that pertinent research results can also stem from dissensions between current practices and an individual's thinking.

These studies also suggest that the question of supervising is not merely a one-person question, but should be inscribed in the more general question of "how doctoral students are taken as members of academic communities" (Stubb 2012, p. 81).

While these issues underline the importance of local/national politics for education to research, the PME panel also stressed—of course—the importance of the supervisor's own work. It may be helpful to remember the (impressive) list of needed qualities that were evoked during that panel. These can be seen as complementing those discussed in the Artigue School roundtable, and fit well into our four-dimension scheme delineating the preparation of young researchers in mathematics education:

For the first dimension (*Degree of filiation*), one can quote the following needed qualities or actions: find areas of personal interest of the doctoral student, discuss anything the student is finding problematic, and learn about the student's own research (or communities).

For the second dimension (*Productivity*): become familiar with the scientific discourse in mathematics education, help with networking, invite the student into a research study and into other research projects, encourage the student to attend sessions organised by national grant bodies, provide significant feedback, provide support, do not leave the student on their own, aid cooperation with other doctoral students, provide resources, and encourage other researchers with expertise in the subjects to assist.

For the third dimension (*Risk-taking and psychological stress*): reward learning processes, be flexible without ready-set roles or traditions, do not be too rigid in providing advice, help in finding financial support, and consider mentorship not only about cognitive processes but also for wellbeing.

Finally, for the fourth dimension (*Handing over the reins*): develop professional identity for the student beyond being a school teacher or a teacher educator, treat the Ph.D. student as a collaborator rather than a student, introduce the students to other researchers during conferences and through international organisations, and provide opportunities for ongoing conversations at international conferences, by email, or face-to-face.

The panorama would not be complete, however, if we do not refer to another issue, one rarely mentioned in studies (for instance, in McAlpine and Amundsen 2011, p. 37): that of 'pleasure'. The feeling of fulfilment that goes with the accomplishment and defence of a thesis is undoubtedly a feeling shared by both the student and the supervisor.

Returning to Michèle Artigue, who so intensively put all of these points into practice in her contact with her students, we can say that far more than the simple title of 'supervisor' printed on the theses, in reality, Michèle embodies remarkably well the PME panel conclusion: "*Mentorship of young scholars goes beyond simple supervising.*"

How and when did Michèle learn how to endorse this role? Our chapter definitely demonstrates to what extent mentoring is a multifaceted task, requiring a triple responsibility and complexified by a particular teacher-student relationship—while, at the same time, hardly reducible to only this relationship—and subject to local, national, internal and external constraints. However, there seems to be no formal political regard to this complexity and no preparation for researchers to navigate this complex task—sometimes even its institutional recognition is weak. The entire academic world acts as if supervising comes naturally. In practice, without receiving any formal professional accompaniment, most supervisors 'learned' from peers, sometimes from observing experienced colleagues, or learned alone, from their own experience as a former doctoral student, in a learn-by-doing process. Consequently, our chapter concludes by emphasising the importance of professionally preparing academics to take up this triple responsibility of supervising doctoral students.

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Appendix 1

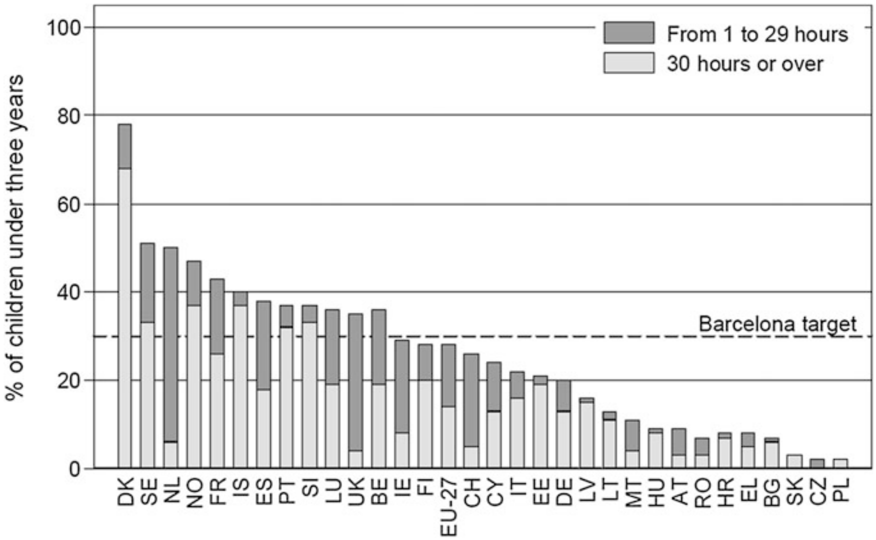

Fig. A.1 Percentage of children up to three years of age cared for by formal arrangements by weekly time spent in care, 2010. *Source* Eurostat, ilc_caindformal, extracted: 12 December 2013. *Note* Eurostat has flagged that for Finland (FI) there has been a break in the time series for both data points (From Mills et al. 2014, p. 6)

Appendix 2: 20 years of supervision

List of theses directed or codirected by Michèle Artigue [from 1993 to 2013]

Antoine Dagher	Environnement Informatique et apprentissage de l'articulation entre registres graphique et algébrique de représentation des fonctions	1993
Michelle Lauton	Enjeux et réalités de l'enseignement des mathématiques en IUT dans les départements de gestion : le cas des mathématiques financières	1994
Maha Abboud-Blanchard	L'intégration de l'outil informatique à l'enseignement secondaire des mathématiques : symptômes d'un malaise : Un exemple : l'enseignement de la symétrie orthogonale au collège	1994
Brigitte Grugeon	Etude des rapports institutionnels et des rapports personnels des élèves à l'algèbre élémentaire dans la transition entre deux cycles d'enseignement : BEP et Première G	1995
Georges Kargiotakis	Contribution à l'étude de processus de contrôle en environnement informatique : le cas des associations droites-équations	1997
Marléne Alves Dias	Les problèmes d'articulation entre points de vue cartésien et paramétrique dans l'enseignement de l'algèbre linéaire	1998
Badr Defouad	Etude de genèses instrumentales liées à l'utilisation de calculatrices symboliques en classe de Première S	2000
Fréderic Praslon	Continuités et ruptures dans la transition terminale S/deug sciences en analyse : le cas de la notion de dérivée et son environnement	2000
Agnès Lenfant	De la position d'étudiant à la position d'enseignant : l'évolution du rapport à l'algèbre de professeurs stagiaires	2002
Michela Maschietto (Ferdinando Arzarello)	L'enseignement de l'analyse au lycée : les débuts du jeu local-global dans l'environnement des calculatrices	2002
Caroline Bardini (Michel Serfati)	Le rapport au symbolisme algébrique : une approche didactique et épistémologique	2003
Véronique Battie (Michel Serfati)	Spécificités et potentialités de l'arithmétique élémentaire pour l'apprentissage du raisonnement mathématique	2003
Analia Bergé	Un estudio de la evolución del pensamiento matemático: el ejemplo de la conceptualización del conjunto de los números reales y de la noción de completitud en la enseñanza universitaria	2004
Eric Laguerre (François Colmez)	Une ingénierie didactique pour l'enseignement du théorème de Thalès au collège	2005
Mariam Haspekian	Intégration d'outils informatiques dans l'enseignement des mathématiques : étude du cas des tableurs	2005
Avenilde Romo-Vazquez (Corine Castela)	La formation mathématique des futurs ingénieurs	2009
Sonia Ben Nejma (Lalina Coulange and Faouzi Chaabane)	D'une réforme à ses effets sur les pratiques enseignantes - Une étude de cas : l'enseignement de l'algèbre dans le contexte scolaire tunisien	2009

(continued)

(continued)

Jean-Philippe Georget	Activités de recherche et de preuve entre pairs à l'école élémentaire : perspectives ouvertes par les communautés de pratique d'enseignants	2009
Laurent Souchard	Les logiciels tuteurs fermés: institutions d'apprentissage et d'enseignement ? : le cas du début du secondaire	2009
Ridha Najar (Houcine Chebbi)	Effets des choix institutionnels d'enseignement sur les possibilités d'apprentissage des étudiants. Cas des notions ensemblistes fonctionnelles dans la transition Secondaire/Supérieur	2010
Grégory Train (Maha Abboud-Blanchard)	Le tableau blanc interactif, un outil pour la classe de mathématiques ?	2013

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Chapter 12
Epilogue. A Didactic Adventure

Michèle Artigue

The conference from which this book originated was for me especially emotional, as the reader can easily understand. As I write these lines after reading the revised versions of the different chapters and also the text I had prepared for the closing lecture, the same emotion arises again. My first words will be to express my deepest gratitude to all those who make it possible for me to experience such an unforgettable moment in which I was surrounded by so much esteem and affection, and the associated feelings of scientific collaboration and shared values.

Beyond the homage, the festive moments and the pleasure of meeting so many colleagues and friends, this conference was remarkable for the quality of scientific exchanges between participants from very diverse horizons and fields of expertise, crossing world regions and generations. Many themes of major interest for me were addressed in the lectures, roundtables and workshops. Purposely selected by the scientific committee, the themes reviewed the main lines of my scientific engagement. During the conference, we could measure the important progression in thinking modes and didactic knowledge on each theme over the last few decades, and the resulting book reflects this progression beautifully. In this chapter, I come back to some of these themes as I did in the closing lecture, intertwining my comments with a more general reflection on the didactic adventure through my personal experience. Such a conference, indeed, is an occasion for retrospective reflection on the person who is honoured, and also an opportunity for conveying to the new generations of didacticians[1] some elements of a history which has shaped their field of study.

[1] In this chapter, I purposely use the word "didactician", a literal translation of the French word "didacticien" instead of the word "mathematics educator" usual in the Anglo-Saxon literature.

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12.1 First Steps in Didactics: Audacity and Ingenuity

In this section, I discuss elementary school mathematics—numbers, magnitudes, computation, geometry—that I began to seriously work on as educational issues in the mid seventies. I completed this work under the leadership of André Revuz, who was the first director of the IREM Paris 7 created in 1969. These first steps took place at the elementary school Almont 1, an experimental school attached to the IREM whose functioning was inspired by the school Michelet attached to the COREM,[2] recently created by Guy Brousseau in Bordeaux. Together with François Colmez and Jacqueline Robinet, we were responsible for the organisation of mathematics teaching in that school and we were given a lot of freedom. This was our experimental terrain. At that time, I was a young university lecturer with a recent Ph.D. in logic, full of certitude and thinking everything possible. I worked part-time in the IREM for a few years, acculturating in-service teachers to the modern mathematics introduced by the 1970 curricular reform.

Our didactic audacities were fed by the COREM realisations that we accessed through the "Colmez connection". François Colmez's father was the Director of the IREM of Bordeaux and François was an old friend of Guy Brousseau. Thanks to these privileged relationships, I had the chance to experiment with situations and learning progressions that would later become "classics": the extension of the number field from whole numbers to rational and decimal numbers through the measure of the thickness of sheets of paper (Brousseau et al. 2014); the teaching of multiplication from counting the number of cells in rectangular grids to the teaching of the algorithm per gelosia (Brousseau 1973); and the initiation of grade 3 pupils to random phenomena through predicting the composition of opaque bottles containing red and blue bowls (Brousseau 1972; Brousseau et al. 2002). Our audacities were also reinforced by the parallel work developed by Marie-Jeanne Perrin and Régine Douady in an elementary school in Montrouge with an exceptional teacher, Mrs. Latour. We introduced letters and formulas to express generality and dependences from grade 3 (Artigue et al. 1979). We used electric boxes built by my husband at the Technological Institute in Cachan to introduce the students to Boolean logic, Escher's tiling to approach geometrical transformations and the associated invariants, and so on. These didactic audacities, implemented by competent and passionate teachers with whom we spent at least one day per week, were almost systematically successful. At the time didactics was emerging as a scientific field, and experiencing the power of its constructions in the terrain of schools, as far as they were developed, was fascinating. This is, I think, difficult for those who enter the field today to understand what it meant for a young researcher (as I was at that time) to engage in the didactic adventure in such conditions.

[2]COREM: Centre d'Observation et de Recherche pour l'Enseignement des Mathématiques, created in 1973 (cf. http://guy-brousseau.com/le-corem/).

That didactic audacity was valuable and productive. I do not disavow it, but it went along with a lot of ingenuity. At that time, for instance, I did not suspect that generalisation could be a problematic process; I was unconscious of the networks of constraints conditioning the life of ordinary didactic systems. In the experimental school, we did not hesitate to free ourselves from these constraints, with the support of teachers and parents, thanks to the indisputable legitimacy given to our actions by the IREM institution. I was even more unconscious of the social and cultural forces of which I was, as were so many others, the instrument. Regarding generalisation, no researcher today would dare to show such ingenuity. Even at that time, many researchers, and especially our mentor Guy Brousseau, were not so naïve. This is fortunate because we have all learnt the high price to pay for naïve actions in educational systems. We have learnt, and we have built conceptual and methodological tools that help us approach the complexity of didactic action.

The up-scaling issue, while still open, is unanimously considered one of the major challenges we have to face as a community. And we are well aware that, to address it efficiently, we must be creative; in particular, we must distance ourselves from the forms of dissemination of didactic knowledge that have prevailed up to now. All around the world, innovation and research is developing in this area, often within projects transcending national frontiers as has been the case in Europe in recent years with projects aimed at the large scale dissemination of inquiry-based teaching practices such as those evoked in the chapter co-authored by Robin Bosdeveix, Cécile de Hosson and Cécile Ouvrier-Buffet [cf. also (Maaß and Artigue 2013)].

12.2 Generalisation and Reproducibility

It was only in the eighties that I realised the difficulty of generalisation in the educational field, first through the theoretical work I carried out on reproducibility for my doctorate, then through research and development at undergraduate level. Quite early, in fact, Guy Brousseau questioned the reproducibility of didactic situations, and exhibited the phenomenon of didactic obsolescence at play in the reproduction of the didactic constructions elaborated in the COREM by the same teachers, year after year. In my doctorate, I approached this issue from a mathematical perspective. I investigated the vision of reproducibility of didactic situations conveyed, more or less explicitly, by the didactic literature; I tried then to build a mathematical model of it to study (Artigue 1986). This was a stochastic model simple enough to make possible some direct computations that I complemented by using simulations. The results were clear. They invalidated the vision of reproducibility conveyed by the literature and showed that, if such reproducibility was observed, it could not generally result from the reasons and characteristics invoked. Other forces were at play whose action and mechanisms remained tacit. Obviously, the model allowed researchers to expect the appearance of some regularities but, as is the case in complex dynamic systems, situated at other structural levels than those usually expected. This led me to articulate a kind of principle of incertitude between internal reproducibility (a priori aimed at, conserving the meaning of actions and discourses despite possible variations in the trajectories) and external reproducibility (at the more superficial level of classroom trajectories). According to this principle, any effort made to ensure external reproducibility had a systematic cost in terms of internal reproducibility. This result showed that the phenomenon of obsolescence identified by Guy Brousseau was not at all due to a specific conjuncture.

However, even if the modeling process had been supported by experimental data collected in previous research, this was pure theoretical work. In order to progress, I had to come back to the terrain. At that time, I was no longer involved in the piloting of the elementary school Almont 1, and had engaged in research and development work at undergraduate level. The motive had been the creation of an experimental mathematics and physics course for first year university students, still under the auspices of the IREM (Artigue 1981). Once again, this was a very innovative design and during the first year of collaboration with physicists we met only one serious problem: the coordination of our respective perspectives regarding the teaching of differentials. This was the origin of my interest in the didactics of calculus and analysis, one of the conference themes with a devoted chapter in this book, co-authored by Asuman Oktaç and Laurent Vivier. While being oriented towards a specific perspective—that of interactions between semiotic systems of representation—the chapter overcomes the limitations potentially induced. In this area that has been extensively investigated over more than three decades, the authors beautifully show how the fundamental questions raised by the teaching and learning of key concepts (such as those of real number with the topological dimension so crucial in analysis, function of real variables, limit, derivative and integral), are regularly reworked, mobilising new approaches, exploiting technological advances. They also demonstrate how knowledge accumulates progressively. However, in reading this chapter, one can also measure up to what point knowledge is still fragmented and partial, and often in an insufficient state of consolidation, to inform curricular choices and teaching practices in a convincing way.

Returning to reproducibility issues, the connection between my research in the didactics of analysis and my theoretical work only occurred some years later. The context was that of the description, reproduction and dissemination of didactical engineering for the teaching of differential equations that I developed together with Marc Rogalski and his team for the experimental section they created at the University of Lille 1 (Artigue 1989; Artigue and Rogalski 1990). I made this connection by thinking in terms of *types* of situations rather than *particular* situations, and by questioning the conditions for robustness of these types of situations. I tried to overcome the trap of linear descriptions and to open the dynamics of these situations, envisaging, for instance, possible bifurcations. I also tried to approach more explicitly the key issue of the sharing of mathematical responsibility between teacher and students than was usual in classical engineering design.

In fact, the research carried out evidenced the existence of obstacles to generalisation in that precise case. These resulted from the following characteristic of the didactical engineering. In order to ensure its viability with first year students, we were obliged to legitimate theorems and proofs combining graphical and analytical formulations and arguments, which violated the didactic contract prevailing in analysis university courses at that time. At the University of Lille 1, in the experimental section, an important work was carried out at the beginning of the academic year on the graphical register of representation, resulting in a change of status of this register; the rules of the standard didactic contract were broken and some graphical arguments became legitimate.

However, we had to acknowledge that this situation was exceptional. This state of affairs in fact resulted in uses of this didactical engineering that, with the exception of the experimental section, were systematically reduced to the first situations of the qualitative approach, despite the acknowledged interest of the whole design. This clearly showed that the ecological viability of this particular didactical engineering depended on conditions regulating the teaching of analysis and, more globally, the status given to graphical representations in mathematics education. These conditions were situated at higher levels of the hierarchy of didactic codetermination than the didactical engineering itself, as I can express today by using a construct of the Anthropological Theory of Didactics (ATD) that did not exist at that time (Chevallard 2002). This example illustrates the fact that the extension of any didactical engineering, beyond the experimental and ecologically protected environment where it has generally been developed and tested, must seriously take into account these different levels. All those today engaged in design research in mathematics education are sensitive to this point, even if they do not use the same words to express this sensitiveness (Swan 2014).

Difficulties of generalisation in didactics are not limited to those evoked so far, in some sense internal to a given didactical system, and which can be grasped through a “vertical” analysis as the one we have sketched above. Their nature is also “horizontal” according to the distinction introduced in Artigue and Winslow (2010), because mathematics education is a field geographically and culturally situated. We all know today, even when we belong to dominant cultures—and mine is certainly one of them in the field of mathematics education—how our insufficient sensitivity to the diversity of social and cultural contexts has been the source of hegemonic visions, of abusive generalisations and exportations. We have built, and continue to build, the necessary tools to better include this cultural and social diversity. We are supported and even pushed in that direction by the changing balances in the world, by the multiplication of regional networks, and also by the vigilance exerted by researchers in mathematics education relying on critical and socio-political approaches (Skovsmose 2014).

Obviously a lot remains to be done but visions have changed and the conference evidenced this change. This book reflects it in diverse ways. I will mention here the chapter entitled “Didactics goes travelling”, with the interesting perspectives developed by Abraham Arcavi and Luis Radford, for whom, personally, didactic cross-breeding has been consubstantial to the didactic adventure, as well as the contribution by Jeremy Kilpatrick focusing on linguistic issues, and that by Paolo Boero. Boero, for instance, shows how the Italian didactic culture, a didactic culture geographically close to the French didactic culture and with which it has strong links over decades, is also very distant from it, and he identifies sources of this distance situated at different levels of the hierarchy of codetermination mentioned above. During the conference, I also found very insightful the first words by Luis Radford pointing out how the relationship with history could be different for researchers from Latin America whose cultures have been denied by colonisation and researchers from colonial countries, and how this shaped their historical and epistemological sensitiveness. The linguistic issue addressed by Jeremy Kilpatrick is also essential and, due to the domination of English as the language for scientific communication, it creates a form of cultural domination transcending the usual distribution between dominants and dominated. Voluntarily, the conference was thus trilingual, and voluntarily I used its three languages during my closing lecture.

Jeremy Kilpatrick is right when describing the loss which often accompanies translation, saying: “When didactique is translated from the French milieu to that of English, it loses not only poetry but also clarity and nuance”. This is the main rationale for the project Lexicon launched in 2014 by David Clarke from the University of Melbourne, in which my laboratory is engaged together with laboratories from eight other countries. The idea for this project emerged from observing the diversity of didactical and pedagogical terms existing in different languages and cultures to express what happens in a mathematics classroom, and the fact that many of these terms do not have an English equivalent. The project aims to create a multi-linguistic lexicon that will combine national lexicons and where each national term will be precisely defined and illustrated by insightful examples. An English international lexicon will then be established and coordinated with the multi-lingual lexicon. It is not by chance that the ICMI, about a decade ago, decided to establish a new regional network, EMF, the Francophone Mathematical Space, for the first time organised around a linguistic community.

12.3 Epistemology, History of Sciences and Didactics

One of the themes of the conference was the relationship between epistemology, history and didactics, and the article that I published on this theme in 1990 was used by several contributors as a starting point (Artigue 1990). Epistemological reflection, relying both on mathematical and historical inquiries, is indeed for me an essential component of didactical work. It is essential to the understanding of the mathematical or scientific concepts that mathematics education wants to make the students learn, and to the understanding of their rationale. It helps understand some of the difficulties raised by the learning of these. It provides ideas to design learning trajectories and teaching situations. Beyond that, as accurately stressed in the corresponding chapter of the book co-authored by Renaud Chorlay and Cécile de Hosson, it also has a function of vigilance regarding the educational world, and helps question its naturalisations and evidences. As made clear in this chapter, to carry such an epistemological reflection, the didactician does not need to become an historian or an epistemologist of mathematics. Even if this were her(his) desire, this would most often remain out of range. The two forms of scientific work, including their *problématiques*, their methodologies, their material and conceptual resources, are quite different, making the double acculturation demanding and costly. However, epistemological reflection requires that communication be possible, aware and respectful of differences and specificities.

I had the opportunity to evolve in environments that allowed and even favoured such communication, and was thus offered the possibility of cultivating epistemological sensitiveness. This happened first at the IREM of Paris where, quite early, Jean-Luc Verley created the group M.:A.T.H. (Mathematics: Approach through Historical Texts) with secondary teachers passionate about the history of mathematics, and where Michel Serfati also created a seminar of epistemology. From the year 2000, this potential was enriched by the creation of a new Doctorate School in my university, in charge of doctoral studies both in epistemology and history of sciences, and in didactics. Attached to this School are my didactic laboratory and one of the best laboratories in the history and philosophy of sciences, the laboratory SPHERE.[3]

With Maryvonne Hallez from M.:A.T.H., I carried out the historical inquiry which supported research into the notion of differential developed within the GRECO Didactique[4] (Groupe Maths et Physique – Enseignement supérieur 1989). The article published in 1990 (mentioned above) and the work on complex numbers referred to in the chapter by Renaud Chorlay and Cécile de Hosson were directly inspired by the work carried out in the master course entitled "History and didactics of mathematics" created by Régine Douady, Jean-Luc Verley and I at the mathematics department of the University Paris 7 in 1985.

I learnt a lot from Jean-Luc Verley and his prodigious culture but, unfortunately, he had no interest at all in didactics as a research field, which certainly limited the possible results from our collaboration. The situation became clearly different when, in early 2000, I collaborated with Michel Serfati in the supervision of Véronique Battie's Ph.D. on the teaching of arithmetic (Battie 2003) and Caroline Bardini's Ph.D. on the relationship to algebraic symbolism (Bardini 2003). The resulting theses, defended before juries including historians and epistemologists of mathematics, attest to the fecundity of these interactions between didactics and epistemology. Today in our laboratory, interactions are certainly favoured by the existence of the doctoral school and also by the importance of the history of science in the research of didacticians such as Cécile de Hosson. The contributions at the conference and the chapter she co-authored with Renaud Chorlay from the

[3] www.sphere.univ-paris-diderot.fr/.

[4] The GRECO Didactique was a temporary collaborative structure created by the National Center for Scientific Research (CNRS) to support research projects in the didactics of mathematics and physics.

laboratory SPHERE provide a good vision of the progression of reflection on the potential of interactions between history, epistemology and didactics, and also of the conditions to be satisfied in order to realise this potential.

In fact, during the conference, epistemological reflection was not confined to the activities of the thematic group devoted to these issues, which is described in the chapter mentioned above. It was more widely present in the work on the didactics of analysis already evoked, and also in the reflection developed around the idea of inquiry-based learning. The chapter co-authored by Cécile Ouvrier-Buffet, Cécile de Hosson and Robin Bosdeveix evidences the importance of such a reflection to understand what is likely to unify investigative approaches in the different scientific fields, but also to understand their specificities in individual fields. In recent years, we particularly developed this reflection in the modeling group of the IREM of Paris to which Robin Bosdeveix belongs, and I also tried to foster it in the different European projects on inquiry-based education in which I have participated as a scientific expert since 2010.[5]

12.4 Theories, Their Roles and Interactions

During the conference, a lot of attention was devoted to theoretical issues, in particular through references to the instrumental approaches of technology integration on the one hand, and through questions of interaction and networking between theoretical frameworks on the other hand. This attention to theoretical issues is like a filigree along the book chapters, but is especially central to three of them: those respectively co-written by Paul Drijvers and Carolyn Kieran; by Ivy Kidron and Angelika Bikner-Ahsbahs; and by Corine Castela, Juan Diaz Godino and Brigitte Grugeon-Allys.

The chapter co-written by Paul Drijvers and Carolyn Kieran focuses on the instrumental approaches of technology integration. This is supplemented by the chapter authored by Maha Abboud-Blanchard, who approaches the same integration issue from the perspective of teaching practices and teacher education, and who also ascribes a key role to instrumental approaches and provides an excellent overview of the topic. My name is usually attached to the development of such approaches from the mid-90s, particularly because of an article published in the *International Journal of Computers for Mathematical Learning* (Artigue 2002) which had a notable impact. However, as rightly stressed by Drijvers and Kieran, this development was a collective enterprise. In my opinion, it illustrates how research develops quite well:

[5]These are the projects Fibonacci (www.fibonacci-project.eu), Primas (www.primas-project.eu), Mascil (www.mascil-project.eu) and Assist-Me (www.assistme.ku.dk).

- the construction of a *problématique*, nurtured by knowledge and experience, but also by the identification of shortcomings and obstacles, in that case, for instance, the technical-conceptual opposition;
- decisive encounters, in that case, with cognitive ergonomy, through the insightful book published by Pierre Rabardel in 1995 (Rabardel);
- sudden intuitions (like that of imbalance between the pragmatic and epistemic valence of techniques induced by the use of digital technologies);
- the testing of ideas and conjectures through their organised confrontation with the empirical world; and
- the unexpected facts that this confrontation produces (for instance, the important time spent by students in solving technical problems generated by the interaction with CAS software, the productivity of what we called “fishing strategies”, the change in status and role given to the different CAS applications (symbolic, graphic) accompanying instrumental genesis, the underestimated importance of the mathematical needs of instrumental genesis, the poor institutional treatment of instrumented techniques), and the resulting progression of inquiry and knowledge.

The development of this approach shows the alternating moments of empirical research and structuration of knowledge, the rebounds of research, the progressive extension of *problématiques*, the dissemination of ideas and results, along with a proliferation of contributions, constructions, interpretations, and so on. It also illustrates the power gained by introducing words to express intuitions and ideas, for example, how simply assigning epistemic and pragmatic values to techniques has opened a multiplicity of new perspectives.

The instrumental approach is certainly a theoretical construction, but the most important, and the chapter by Drijvers and Kieran makes it clear, is that it made possible a change in vision that, for me, became unavoidable when I started working on the integration of CAS. From this point of view, this approach turned to be a fabulous tool; however, I did not expect that it would disseminate so easily. In retrospect, the reason is probably that this construction met a need more broadly shared in the research community, and also that it only relied on a few concepts and was thus relatively easy to appropriate. It is also worth noticing that although many researchers contributed to its development, so far the construction has kept its global consistency through successive extensions. In the literature, the focus is often put on the theoretical side of the construction. I will come back to this point in the next section, but would like to emphasise here once again that what really makes this construction valuable is its functionality and the results it has made accessible via the research praxeologies it contributed to, since its emergence twenty years ago.

Also stressed in these chapters the instrumental approach to which I contributed results from a theoretical combination between cognitive ergonomy and the Anthropological Theory of Didactics (ATD). This combination generated some tensions, especially between a cognitive entry through the idea of scheme and an institutional entry through the idea of technique. These two entries are in some sense incommensurable. The tensions generated were extensively discussed and differently managed by researchers, but they did not impact on the productivity of the instrumental approach. This kind of combination, in fact, appears quite frequently in didactics and does not make me a pioneer on networking issues. Since the eighties, within the French didactic community we have regularly combined theoretical approaches, for instance, the theory of didactical situations, the theory of conceptual fields, the tool-object dialectics, the theory of semiotic registers of representation, and ATD. The Summer Schools of Didactics of Mathematics gave us regular opportunities to collectively question some of these connections. In her contribution to the chapter cited above, Brigitte Grugeon describes several convincing examples, including from her own research. Reading this contribution reminded me of a memorable theoretical experience of my own. Supervising Brigitte's Ph.D. in the early nineties, I measured for the first time how a shift in perspective, in that case from a standard cognitive perspective on the teaching and learning of algebra to an institutional perspective supported by ATD, could radically change the perception of a problem and open the researcher to explanations and solutions that would have remained inaccessible otherwise. This crucial experience had a decisive influence on my vision of transition issues in mathematics education.

The combining of theoretical approaches is thus not a recent practice. What is certainly more recent is the international awareness of the difficulties created by the so-called fragmentation of the mathematics education field—something I never felt within my own didactic community—and the development of networking between theoretical frameworks as a proper *problématique* in the field. This *problématique* may appear to some, especially those who are at the periphery of the research world, as something non-essential, academic and "nombrilist", without potential impact on practice. And it is true that this networking can easily drift towards a purely intellectual game. As a community, we have to be vigilant to avoid such perversion. However, in my opinion, it answers a deep need for both fundamental research and didactic action.

Many times, in recent years, due to my ICMI responsibilities I have been confronted with questions about existing knowledge on particular educational issues that might inform teaching practices, curricular decisions or teacher education. Faced with such questions, most often I was unable to give a clear answer, and often even unable to orient my interlocutor towards a set of references that would help her(him) develop a coherent and synthetic vision. Of course, things are not so simple in education as in mathematics. We must accept that most of the certainties we acquire are, except for the most general ones, situated both in time and space, and that it is difficult to know their exact domain of validity. The question of how research knowledge may inform practice in particular contexts is a difficult question, still insufficiently addressed. Nevertheless, the theoretical explosion of the field, the diversity of approaches, constructions, discourses, and the lack of connection, substantially increases the difficulties of capitalisation and dissemination. For that reason, I am convinced that we must work to overcome the current state, and that this must be a collective enterprise, identified as such. As is clear from the chapters in this book mentioned above, and from the detailed publications they refer to, in recent years evident progress has been made, although the results remain local. The different projects developed at a European level in which I was involved have clearly helped to take the measure of the task, of its difficulty, to build methodologies, to establish categories and illustrate them, and also to build many connections and show their interest.

These studies have also revealed to what extent theoretical diversity deeply permeates our research practices, making connection efforts directly situated at the level of theoretical objects hopeless. We certainly underestimated this point until recently. Personally in the last decade, I have learnt the price to pay in order to overcome the current state, the necessary effort of decentration, and the uncompromising questioning required to understand the actual use we make of theoretical frameworks beyond their mere ritual invocation. I have also learnt the necessity of developing specific devices that can allow us to take our research practices as objects of study without distorting them, as well as the importance of developing metalanguages to support joint work and communication. One example is the metalanguage of key concerns, initially created in the TELMA European team, then refined in the project ReMath, which I also used as a guide when, together with Morten Blomøj, I investigated what the major didactic approaches have to offer to the conceptualisation of inquiry-based learning in mathematics education (Artigue and Blomøj 2013). As explained by Ivy Kidron and Angelika Bikner-Ahsbahs in their chapter, with Marianna Bosch and Josep Gascón, we made the conjecture that ATD, which was familiar to us, could support such awareness and work. In ATD, theories are indeed inserted in praxeologies, and as I wrote at the beginning of this paragraph, what needs to be connected are not just theoretical constructs, but the praxeologies these constructs contribute to and which, in return, contribute to their development. Once again, this may appear an insignificant change of perspective, but this reframing of networking issues proved useful for analysing the networking efforts undertaken so far, as well as their outcomes and potential, but also their limitations, as shown in the chapter co-written with Marianna Bosch (Artigue and Bosch 2014).

I also learned how this collaborative work, for it is necessarily collaborative work, can be exciting and enriching when engaged with seriously and with an open mind. When, at the end of the CERME5 Conference, Angelika Bikner-Ashbahs proposed the creation of a small research group to address networking issues, I could not imagine either that we would work so many years together on the same video, nor that this work would take us so far from where we began. As can be expressed using the language of the theory of didactical situations, for each of us the group acted as an antagonist (although empathic) *milieu*, where approximate statements were no longer possible, where the functionality of theoretical objects and their power, were systematically questioned. It was really productive as shown in (Bikner-Ahsbahs and Prediger 2014). Perhaps, in the future, this type of work, which is still in an emerging state, will help didactics to better travel and make exchanges more effective. Perhaps it will help us to better capitalise didactic knowledge, and to make it more usable by people other than those directly engaged in its production or close to the production sites. However, in this area too, much remains to be done.

Regarding this issue of theoretical frameworks and the importance attached to them, in closing this section I would like, however, to express some concerns. In my opinion, internationally, our community overvalues theoretical work. Using the language of ATD, I would say that, regarding research praxeologies, there is a dangerous overvaluation of the logos block with respect to the praxis block. Such a concern may surprise, after having read in the chapter co-written by Carolyn Kieran and Paul Drijvers that I am passionate about theory. However, this is not the image I have of myself and of my work as a researcher. Rather, I have the impression that, unlike other colleagues, I quickly get tired of theoretical work and that, with a certain pragmatism, I just do what seems to me to be necessary in order to think and act. Due to my culture, my needs in this area may be higher than the international average, which can lead to this impression. However, I could not, for instance, like Juan Diaz Godino, spend years building a construction like the EOS approach (Entidades primarias de la ontología y epistemología) that he presents in the book, in the quest for a theory integrating cognitive, semiotic and institutional perspectives. The contribution by Corine Castela seems to me especially insightful in deepening the reflection on such overvaluation of theoretical work. Relying on Bourdieu's work, Corine accurately and beneficently attracts our attention to the underground of the theoretical game, and the power relationships and identity issues it involves, that the reflection should not neglect. It is also interesting to read what Renaud Chorlay writes in the chapter already mentioned that he co-authored with Cécile de Hosson: "Even though some historians occasionally borrow concepts from some theoretical frameworks,[6] they usually feel they have no use for theoretical frameworks from MER, because they don't use theoretical frameworks at all!". The shift thus highlighted necessarily challenges us.

12.5 The Didactician in the City

The life of a didactician is, however, much more than the academic life of a researcher, and at seeing the conference program, I was happy to discover that one of its themes was "The researcher in the city". As was recalled at the beginning of the conference, my first area of research was mathematical logic. My Ph.D. dealt with issues of recursion, and then I worked on non-standard models of arithmetic and, perhaps a prescient choice, on bicommutability between theories. However, despite the pleasure I found in mathematical research, as I engaged in didactic activities, participated in the emergence of this new scientific field and was

[6]For instance, in the workshop on *Epistemology and didactics* at the conference, the historians Dominique Tournès and Renaud Chorlay mentioned their use (or their interest) in concepts such as '*change of setting*', '*register of representation*', '*viewpoint*', '*meta-level*', '*tool-object dialectics*'.

captivated by the enthusiasm and debates accompanying this emergence, and as I engaged in the innovative projects of the IREM Paris 7, I could not help but contrast these two types of activity from the point of view of their social engagement and role. I do not deny that the mathematician has a role to play in the city, as argued eloquently by Jean-Pierre Kahane at the conference, but s(he) can easily forget that role, relegating it to the background of her(his) activity. Some may disagree with me, but I claim here that this is not possible for a didactician, especially for a didactician of mathematics. Even if the constraints imposed today upon research, whatever its domain, create harmful tensions, there is no doubt that the quest for knowledge in this field remains primarily fed by external motivations and a profound desire to contribute to improving the teaching and learning of mathematics.

How could a didactician forget the problematic relationship to mathematics of a large proportion of the population? How could s(he) forget that this problematic relationship is not a law of nature, that school also has a huge responsibility for it? How could s(he) be blind to the fact that, in our educational systems, mathematics as a discipline is engaged in complex power games and that often, instead of being an emancipation tool, it is one of the levers of school exclusion with all the resulting consequences? How could a didactician not wish his(her) research would help improve the teachers' professional life? How could s(he) not be engaged in the city?

As individuals and as a community, we thus have specific responsibilities that we cannot escape. In the context in which I lived, such awareness came quickly, probably because of the existence of the original structure of the IREM and its values: the links it wove among communities, between didactic research and the school terrain, although a privileged terrain, that of the classes of the IREM educators. Also contributing to this awareness was the early establishment of an autonomous didactic community and its institutionalisation in the early 1980s, with the creation of the journal *Recherches en Didactique des Mathématiques*, of the National Seminar and of the Summer School of Didactics of Mathematics, and beyond that, the role I played for 12 years as a didactician member of the National Council of Universities in charge of the qualification, recruitment and promotion of university academics, in the section "Applied mathematics and mathematical applications" to which most French didacticians were attached.

However, I also believe that the strength of these structures and their associated culture, while promoting local awareness and commitment, has also generated a tendency to autarky and self-sufficiency within this community, which I think has been less positive. I became aware of it gradually, as I observed the gap between the work that many colleagues undertook to publicise their research nationally and their limited efforts to communicate more widely, particularly outside the spheres of the traditional influence of French didactics and when communication required the use of English. For a long time, I had the impression that many researchers actually did not feel such a need for several reasons: their *problématiques* were sufficiently nourished by a very dynamic environment; the conceptual resources that this environment offered for research grew steadily making unnecessary the need to look for external resources; effective structures existed to share, discuss, communicate and disseminate research; and the tyranny of publishing in high impact factor international journals was much lower than in many other countries. The situation, of course, has evolved and the creation of the European Society for Mathematics Education (ERME) played a decisive role in this change with its regular conferences every two years, and its summer schools for young researchers.

Personally, I dived early into the international pool, participating in the establishment of doctoral courses in Spain since the 1980s, and interacting with Latin American researchers in Brazil, Mexico and Colombia. In these tasks, my knowledge of Spanish helped me greatly. I was also elected in 1990 to the International Committee of the International Group PME (Psychology of Mathematics Education), and became involved in the organisation of topic study groups at ICME congresses from 1988. However, my election to the Executive Committee of ICMI in 1998, at a moment of tension between ICMI and its mother structure, the International Mathematical Union (IMU), was for me a radical turn. The world suddenly expanded; the cultural dimension of educational issues in mathematics and the associated games of power and domination, intruded on my consciousness. Didactics took a more political dimension, that was unthinkable to understand through the lenses of a particular educational culture, whatever its acknowledged qualities. These responsibilities within ICMI, and the actions and encounters they allowed in a multiplicity of countries, with a multiplicity of actors beyond the educational sphere alone, have reinforced my vision of the role of the didactician in the city. They also reinforced my conviction that the strength of the solidarities and synergies among cultures, among communities and generations that ICMI tries to promote, is the only one able to move us towards a better world, to make us resist headwinds. They confirmed my desire to strengthen these synergies and solidarities at all levels, at all scales, and of course in the first place within my own culture while contributing to its necessary openness.

I felt this strength and the hope it conveys during the conference. We shared the desire and pleasure for fostering communication between our respective *problématiques* and knowledge; we lived a form of solidarity and aspirations that transcended our differences; and we also showed the ability of the didactic community to dialogue and collaborate with neighboring scientific communities, those of mathematicians, epistemologists and historians of science, didacticians of other disciplines, researchers in educational sciences, and more broadly, with all those without whom research advances would only benefit a small minority of individuals, either students or teachers. All this is clearly visible in this book that results from the conference.

12.6 Conclusion

As I wrote at the beginning of this chapter, a conference such as the one from which this book originated is, for the person honoured, an occasion to reflect on her/his professional life. When I reflect on my own professional life, I realise how lucky I was to connect with this field of mathematics education in its infancy, and to participate in the adventure of its emergence within a community so particular, thanks to the existence of the IREMs, to the support of mathematicians such as André Revuz, Georges Glaeser and many others, and to the creative power and the passionate commitment of researchers such as Guy Brousseau and Gérard Vergnaud. It is, I think, impossible for young researchers entering the field today to imagine the first summer schools of didactics or the early years of the national seminar, when our community was building and forging its identity, or to imagine the enthusiasm, the warmth of the relationships and at the same time, the heated debates, Homeric and endless.

For a long time I tried to reconcile this burgeoning passion for didactic research with the pleasure I took in mathematical research, even if my research was modest. It was definitely easier at that time because the didactic field was less professional than it is now. Then, one could start investigating quickly without lengthy reading and training, and stay aware of the main advances in the field without too much difficulty; at least I had this impression. This contact maintained with mathematics, beyond the sole educational sphere, was precious for me in many respects. It supported my epistemological questioning and associated vigilance; it was also a source of inspiration. For instance, the work I undertook with Véronique Gautheron and Emmanuel Isambert on the bifurcations of differential systems prompted me to think of the class system as a dynamic system, and to wonder about the stability levels, structures and patterns which might emerge in its dynamics. This relationship to mathematics surely also reinforced my desire that mathematicians and didacticians communicate and work together to serve the cause of mathematics education, and succeed in countering the forces that tend to separate their respective communities. It supported my ongoing commitment in this area.

As also pointed out in this chapter, my didactic adventure has been profoundly marked by the didactic community in which I grew up, but also, and particularly in the last two decades, by the encounters and the experiences I have lived and continue to live beyond this community, especially in countries on the periphery, which were a major source of questioning and enrichment. They showed me at what point some of my concerns and perspectives were limited and European-centered concerns of the privileged. They opened my mind to issues insufficiently addressed in French didactic research, such as those regarding mathematics education in multicultural, multi-lingual contexts, and to research and development advances that no educational system can now overlook with the current diversification of the school population that is occurring worldwide. Often also, the energy, commitment and human warmth that I met in less privileged countries and contexts have allowed me to recharge my batteries, to regain confidence and hope. This was precious because confidence and hope are needed in a field like ours. There is no doubt that knowledge accumulates, that our understanding of the complex teaching and learning processes progresses, and that efforts are being made daily to put these advances at the service of teacher education and didactic action. However, one cannot escape the impression that the same problems constantly re-emerge. One cannot fail to observe how, despite laudable declarations, educational systems are often abused and forced by inconsistent policies. Progressions and regressions intertwine in an endless battle whose outcome often seems uncertain. We also need resilience!

I will end this text with a personal note. In my closing lecture at the conference, I evoked my parents, and the fact that during my childhood and youth I wore their hope, and especially my father's hope that, through education, his children would enter a world whose access had been forbidden to him. I evoked the confidence they have shown in me, the freedom they gave me. They helped me to grow and to consider life with confidence and optimism. I also mentioned my family so firmly united, especially when life becomes difficult. But I would also like to express here my deepest gratitude to all those who have accompanied my didactic adventure and made it so rewarding: my didactician friends and colleagues around the world, the many teachers I have worked with, and all those with whom I have worked within the ICMI and many other institutions, not to mention my students from whom I learned so much and whose voice was beautifully expressed with irresistible humor by Mariam Haspekian and Avenilde Romo Vasquez at the conference. My didactic adventure is not over, but while continuing to work and to contribute, I am also pleased to pass the baton to the new generations, to my didactic children and grandchildren. They inherit from our history but I am sure that they will also be able to get free of much heaviness and many ways of seeing which prevented us from going further.

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