Topological Properties for Approximation Operators in Covering Based Rough Sets

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Abstract. We investigate properties of approximation operators being closure and topological closure in a framework of sixteen pairs of dual approximation operators, for the study of covering based rough sets. We extended previous results about approximation operators related with closure operators.

Keywords: Covering rough sets · Approximation operators · Topological closure

1 Introduction

The main concept of rough set theory is the indiscernibility between objects given by an equivalence relation in a non-empty universe set U. In this paper, we give necessary conditions for covering-based upper approximation operators to be closure operators. Three different definitions of approximation operators were presented in a general framework for the study of covering based rough sets by Yao and Yao in [20], element based definition, granule based definition and system based definition. For the element based definition, Yao and Yao consider four different neighborhood operators. In the granule based definition, they consider six new coverings defined from a covering $\mathbb{C}$. The covering $\mathbb{C}$ and the six new coverings define fourteen pairs of approximation operators. For the system based definition two new coverings are defined: $\cap$-closure($\mathbb{C}$) and $\cup$-closure($\mathbb{C}$). From these neighborhoods operators, new coverings and systems, it is possible to obtain twenty pair of dual approximation operators. But, as Yao and Yao noted [20], there are other approximations out of this framework. For example, Yang and Li present in [18] a summary of seven non dual pairs of approximation operators used by Żakowski [21], Pomykala [9], Tsang [13], Zhu [28], Zhu and Wang [30] and Xu and Whang [16]. Restrepo et al. present a framework of sixteen pair of dual approximations, unifying the two above frameworks, from duality, conjugacy and adjointness [11].

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Some topological connections with rough sets and generalized rough sets have been established. The relationships between topology and generalized rough sets induced by binary relations were studied in [1,5,8]. Q. Wu. proposes a study on rough sets which includes topological spaces, topological properties and homeomorphims [10]. L. Zhaowen investigates topological properties of compactness, separate and connectedness [23]. Finally W. Zhu [32] and G. Xun et al. [17] study topological characterization of some covering based approximation operators. Recently X. Bian et al. present a characterization of three types of approximation operators to be closure operators [2]. In this paper, we consider some previous characterizations of coverings and approximation operators to extend them to the framework proposed in [11].

The paper is organized as follows. Section 2 presents preliminary concepts about topology, rough sets, lower and upper approximations in covering based rough sets, the main neighborhood operators, and different coverings obtained from a covering $\mathbb{C}$. Section 3 presents topological characterization of coverings for some upper approximation operators. Section 4 presents necessary conditions for approximation operators to be closure operators. Finally, Sect. 5 presents some conclusions and future work.

2 Preliminaries

2.1 Pawlak's Rough Set Approximations

In Pawlak's rough set model an approximation space is an ordered pair $apr = (U, E)$, where E is an equivalence relation defined on a non-empty set U [7]. The equivalence relation E defines a partition of U, written as U/E. The set $[x]_E$ represents the equivalence class of x and $\mathscr{P}(U)$ represents the set of parts of U. According to Yao and Yao [19,20], there are three different, but equivalent ways to define lower and upper approximation operators: element based definition, granule based definition and subsystem based definition. According to the element based definition, for each $A \subseteq U$, the lower and upper approximations are defined by:

$$\underline{apr}(A) = \{x \in U : [x]_E \subseteq A\} = \bigcup\{[x]_E \in U/E : [x]_E \subseteq A\} \quad (1)$$

$$\overline{apr}(A) = \{x \in U : [x]_E \cap A \neq \emptyset\} = \bigcup\{[x]_E \in U/E : [x]_E \cap A \neq \emptyset\} \quad (2)$$

The first part of Eqs. (1) and (2) are called element based definition of approximation operators. The second part are called granule based definition.

Yao and Yao used the notion of a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection [20]. Given a closure system $\mathbb{S}$ over U, it is possible to construct its dual system $\mathbb{S}'$, containing the complements of each K in $\mathbb{S}$, as follows:

$$\mathbb{S}' = \{\sim K : K \in \mathbb{S}\} \quad (3)$$

The system $\mathbb{S}'$ contains $\emptyset$ and it is closed under set union. Given $S = (\mathbb{S}', \mathbb{S})$, a dual pair of approximation operators can be defined as follows:

$$\underline{apr}_S(A) = \bigcup\{K \in \mathbb{S}' : K \subseteq A\} \tag{4}$$

$$\overline{apr}_S(A) = \bigcap\{K \in \mathbb{S} : K \supseteq A\} \tag{5}$$

2.2 Closures

The notion of closure operator usually is used on ordered sets and topological spaces. We present some concepts about ordered structures, according to Blyth [3].

A family $\mathscr{C}$ of subsets of U is called a closure system if it is closed under intersections.

Closure Operators

Definition 1. *A map $c : \mathscr{P}(U) \to \mathscr{P}(U)$ is a* ***closure operator*** *on U if it is such that, for all $A, B \subseteq U$:*

1. *$c(A) = c[c(A)]$, (idempotent).*
2. *$A \subseteq B$ implies $c(A) \subseteq c(B)$, (order preserving).*
3. *$A \subseteq c(A)$, (extensive).*

Definition 2. *A map $c : \mathscr{P}(U) \to \mathscr{P}(U)$ is a* ***join morphims*** *if it is such that $c(A \cup B) = c(A) \cup c(B)$, for all $A, B \in \mathscr{P}(U)$.*

It is easy to see that a join morphism is an order preserving: $A \subseteq B \Leftrightarrow A \cup B = B$, so $c(A) \cup c(B) = c(B) \Leftrightarrow c(A) \subseteq c(B)$.

Topological Closure

Definition 3. *A topology for U is a collection τ of subsets of U satisfying the following conditions:*

1. *The empty set and U belong to τ.*
2. *The union of the members of each sub-collection of τ is a member of τ.*
3. *The intersection of the members of each finite sub collection of τ is a member of τ.*

The pair (U, τ) is called a topological space. The elements in τ are called open sets. The complement of an open set is called a closed set.

A family $\mathscr{B}$ is called a base for (U, τ) if for every non-empty open subset O of U and each $x \in O$, there exists a set $B \in \mathscr{B}$ such that $x \in B$. Equivalently, a family $\mathscr{B}$ is called a base for (U, τ) if every non-empty open subset O of U can be represented as union of a subfamily of $\mathscr{B}$.

The closure of a subset A of a topological space U is the intersection of the members of the family of all closed sets containing A.

Definition 4. *A **topological closure** operator on U assigns to each subset A of U a subset $c(A)$ such that (Kuratowski axioms):*

1. $c(\emptyset) = \emptyset$, *(minimal element)*
2. $c(A) = c[c(A)]$, *(idempotent).*
3. $A \subseteq c(A)$, *(extensive).*
4. $c(A \cup B) = c(A) \cup c(B)$, *(join morphism).*

The interior of a subset A of a topological space U is the union of the members of the family of all open sets contained in A. The interior operator on U is an operator which assigns to each subset A of U a subset A° such that the following statements are true.

1. $U^{\circ} = U$
2. $(A^{\circ})^{\circ} = A^{\circ}$
3. $A^{\circ} \subseteq A$
4. $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

Definition 5. *Let $f, g : \mathscr{P}(U) \rightarrow \mathscr{P}(U)$ be two self-maps. We say that g is the dual of f, if for all $A \in \mathscr{P}(U)$,*

$$g(\sim A) = \sim f(A),$$

where $\sim A$ represents the complement of $A \subseteq U$.

In a topological space an interior operator is the dual of a closure operator.

2.3 Covering Based Rough Sets

Covering based rough sets was proposed to extend the range of applications of rough set theory. In rough set theory the equivalence class of an element $x \in U$ can be considered as its neighborhood, but in covering based rough sets we need to consider the sets K in $\mathbb{C}$ such that $x \in K$.

Definition 6. [24] *Let $\mathbb{C} = \{K_i\}$ be a family of nonempty subsets of U. $\mathbb{C}$ is called a covering of U if $\bigcup K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.*

Definition 7. [4] *Let $(U, \mathbb{C})$ be a covering approximation space and $x \in U$. The set*

$$md(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K \wedge [\forall S \in \mathbb{C}(x \in S), (S \subseteq K \Rightarrow S = K)]\} \quad (6)$$

is called the minimal description of the object x.

Definition 8. [24] *A covering $\mathbb{C}$ is called unary if $|md(\mathbb{C}, x)| = 1$ for each $x \in U$.*

The notion of maximal description was introduced by W. Zhu and F. Wang in [31].

Definition 9. [31] *Let* $(U, \mathbb{C})$ *be a covering approximation space,* $K \in \mathbb{C}$*. If no other element of* $\mathbb{C}$ *contains* K*,* K *is called a maximal description in* $\mathbb{C}$*. All maximal descriptions for* $x \in U$ *in* $\mathbb{C}$ *are denoted as* $MD(\mathbb{C}, x)$*.*

Maximal description can be also be defined as:

$$MD(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K \wedge [\forall S \in \mathbb{C}(x \in S), (S \supseteq K \Rightarrow S = K)]\} \tag{7}$$

Definition 10. [20] *A mapping* $N : U \to \mathscr{P}(U)$*, such that* $x \in N(x)$ *is called a neighborhood operator.*

According to Eqs. (1) and (2), each neighborhood operator defines a pair of approximation operators, when we use the neighborhood $N(x)$ instead of the equivalence class $[x]_E$.

$$\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{8}$$

$$\overline{apr}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{9}$$

Element Based Definition. Equations (8) and (9) give the element based definition in covering based rough sets, analogous to Eqs. (1) and (2) in rough set theory.

From $md(\mathbb{C}, x)$ and $MD(\mathbb{C}, x)$, Yao and Yao define the following neighborhood operators:

1. $N_1^{\mathbb{C}}(x) = \bigcap\{K : K \in md(\mathbb{C}, x)\}$
2. $N_2^{\mathbb{C}}(x) = \bigcup\{K : K \in md(\mathbb{C}, x)\}$
3. $N_3^{\mathbb{C}}(x) = \bigcap\{K : K \in MD(\mathbb{C}, x)\}$
4. $N_4^{\mathbb{C}}(x) = \bigcup\{K : K \in MD(\mathbb{C}, x)\}$

The set $N_1^{\mathbb{C}}(x) = \bigcap md(\mathbb{C}, x)$ for each $x \in U$, is called the minimal neighborhood of x.

Granule Based Definition. Generalizing the granule based definitions given by the second parts of Eqs. (1) and (2), the following dual pairs of approximation operators based on a covering $\mathbb{C}$ were considered in [20]:

$$\underline{apr}'_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \subseteq A\} \tag{10}$$

$$\overline{apr}'_{\mathbb{C}}(A) = \sim \underline{apr}'_{\mathbb{C}}(\sim A) \tag{11}$$

$$\underline{apr}''_{\mathbb{C}}(A) = \sim \overline{apr}''_{\mathbb{C}}(\sim A) \tag{12}$$

$$\overline{apr}''_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\} \tag{13}$$

Generally a covering contains redundant information. For example, some definitions using all the sets of the covering and using the minimal sets only, are equivalent. Therefore, it is possible to consider only some particular elements of the covering.

From a covering $\mathbb{C}$ of U, we can define the coverings:

1. $\mathbb{C}_1 = \bigcup\{md(\mathbb{C}, x) : x \in U\}$
2. $\mathbb{C}_2 = \bigcup\{MD(\mathbb{C}, x) : x \in U\}$
3. $\mathbb{C}_3 = \{\bigcap(md(\mathbb{C}, x)) : x \in U\} = \{\bigcap(\mathscr{C}(\mathbb{C}, x)) : x \in U\}$
4. $\mathbb{C}_4 = \{\bigcup(MD(\mathbb{C}, x)) : x \in U\} = \{\bigcup(\mathscr{C}(\mathbb{C}, x)) : x \in U\}$
5. $\mathbb{C}_\cap = \mathbb{C} \setminus \{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\})\,(K = \bigcap \mathbb{K})\}$
6. $\mathbb{C}_\cup = \mathbb{C} \setminus \{K \in \mathbb{C} : (\exists \mathbb{K} \subseteq \mathbb{C} \setminus \{K\})\,(K = \bigcup \mathbb{K})\}$

Coverings $\mathbb{C}_\cap$ and $\mathbb{C}_\cup$ are called the $\cap$-reduction and the $\cup$-reduction of $\mathbb{C}$, respectively. The idea is eliminate the sets that can be expressed as intersection or union of other sets in the covering.

Using Eqs. 10 to 13, each covering defines two pairs of approximation operators, therefore for each covering we have fourteen pairs of dual approximation operators.

Closure System Based Definition. As a particular example of a closure system, [20] considered the so-called intersection closure $S_{\cap,\mathbb{C}}$ of a covering $\mathbb{C}$, i.e., the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set intersection. Similarly, the union closure of $\mathbb{C}$, denoted by $S_{\cup,\mathbb{C}}$, is the minimal subset of $\mathscr{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set union. It can be shown that the dual system $S'_{\cup,\mathbb{C}}$, defined by Eq. 3, forms a closure system. Both $S_\cap = ((S_{\cap,\mathbb{C}})', S_{\cap,\mathbb{C}})$ and $S_\cup = (S_{\cup,\mathbb{C}}, (S_{\cup,\mathbb{C}})')$ can be used to obtain two pairs of dual approximation operations, according to Eqs. (4) and (5).

According to the above three definitions Yao and Yao present twenty pairs of dual approximation operators, four from the element based definition, fourteen from granule based definition and two from the system based definition based on $\cap$-closure and $\cup$-closure.

The pairs of approximation operators $(\underline{apr}_N, \overline{apr}_N)$, $(\underline{apr}'_{\mathbb{C}}, \overline{apr}'_{\mathbb{C}})$ and $(\underline{apr}''_{\mathbb{C}}, \overline{apr}''_{\mathbb{C}})$ are dual pairs.

2.4 Other Framework of Lower and Upper Approximations

A summary of seven pairs of approximation operators for covering based rough sets was presented in [14,18]. In all cases only two lower approximations have been used. Żakowski first extended Pawlak's rough set theory from a partition to a covering in [21]. The second type of covering rough set model was presented by Pomykala in [9], Tsang [13] studied the third type. Zhu defined the fourth and the fifth types of covering-based approximation in [28,30]. Xu gave the definition of the sixth type in [16]. The seventh type of approximation operations can be found in [15].

Some of these approximation operators were already presented in Yao and Yao's framework, therefore we only present the different ones. The four upper approximations are listed as follows:

1. $H_1^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\bigcup\{md(\mathbb{C}, x) : x \in A - L_1^{\mathbb{C}}(A)\})$
2. $H_3^{\mathbb{C}}(A) = \bigcup\{N_2^{\mathbb{C}}(x) : x \in A\}$
3. $H_4^{\mathbb{C}}(A) = L_1^{\mathbb{C}}(A) \cup (\bigcup\{K : K \cap (A - L_1^{\mathbb{C}}(A)) \neq \emptyset\})$
4. $H_5^{\mathbb{C}}(A) = \bigcup\{N_1^{\mathbb{C}}(x) : x \in A\}$

where $L_1^{\mathbb{C}}(A)$ is the lower approximation defined as:

$$L_1^{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \subseteq A\} = \underline{apr}'_{\mathbb{C}}(A). \tag{14}$$

2.5 New Framework of Approximation Operators

Some equivalences and relationships among the operators in the two previous frameworks were studied and established by M. Restrepo et al. [11]. The Table 2 summarizes a framework of sixteen pairs of approximation operators, establishing equivalences among the two frameworks above, Yao and Yao's framework and Yang and Li's framework.

All the operators listed in Table 2 satisfy the relation $A \subseteq \overline{apr}(A)$ for all $A \subseteq U$, so they are upper approximations. Also, all they satisfy $\overline{apr}(\emptyset) = \emptyset$.

According to W. Zhu [24] operators 5, 6, 7 and 8 are not join morphisms and according to [25], operators 9, 10, 11 and 12 are not idempotent.

The following propositions show that $\underline{apr}_{N_3^{\mathbb{C}}}$ is an idempotent operator.

Proposition 1. *If $z \in N_3^{\mathbb{C}}(x)$ then $N_3^{\mathbb{C}}(z) \subseteq N_3^{\mathbb{C}}(x)$.*

Proof. If $z \in \underline{apr}_{N_3^{\mathbb{C}}}(x)$, we will show that $MD(\mathbb{C}, x) \subseteq MD(\mathbb{C}, z)$. In fact, if $K \in MD(\mathbb{C}, x)$, then $z \in K$, because $z \in N_3^{\mathbb{C}}(x)$. Now, if $S \supseteq K$ then $K = S$, because K is a maximal element. From $MD(\mathbb{C}, x) \subseteq MD(\mathbb{C}, z)$, we have that $\bigcap MD(\mathbb{C}, x) \supseteq \bigcap MD(\mathbb{C}, z)$ and therefore $N_3^{\mathbb{C}}(z) \subseteq N_3^{\mathbb{C}}(x)$.

Proposition 2. *The operator $\underline{apr}_{N_3^{\mathbb{C}}}$ is idempotent.*

Proof. Clearly $\underline{apr}_{N_3^{\mathbb{C}}}(\underline{apr}_{N_3^{\mathbb{C}}}(A)) \subseteq \underline{apr}_{N_3^{\mathbb{C}}}(A)$.

If $z \in \underline{apr}_{N_3^{\mathbb{C}}}(A)$, then $N_3^{\mathbb{C}}(z) \subseteq A$. We will show that $N_3^{\mathbb{C}}(z) \subseteq \underline{apr}_{N_3^{\mathbb{C}}}(A)$. In fact, if $w \in N_3^{\mathbb{C}}(z)$, by Proposition 1, we have that $N_3^{\mathbb{C}}(w) \subseteq N_3^{\mathbb{C}}(z) \subseteq A$, therefore $w \in \underline{apr}_{N_3^{\mathbb{C}}}(A)$. Since $N_3^{\mathbb{C}}(z) \subseteq \underline{apr}_{N_3^{\mathbb{C}}}(A)$ we have that $z \in \underline{apr}_{N_3^{\mathbb{C}}}(\underline{apr}_{N_3^{\mathbb{C}}}(A))$. So $\underline{apr}_{N_3^{\mathbb{C}}}(\underline{apr}_{N_3^{\mathbb{C}}}(A)) = \underline{apr}_{N_3^{\mathbb{C}}}(A)$ and $\underline{apr}_{N_3^{\mathbb{C}}}$ is an idempotent operator.

From duality, it is easy to establish the following corollary.

Corollary 1. *The operator $\overline{apr}_{N_3^{\mathbb{C}}}$ is idempotent.*

The following example shows that operators $\overline{apr}_{N_2^{\mathbb{C}}}$ and $\overline{apr}_{N_4^{\mathbb{C}}}$ are not idempotent operators.

Example 1. (Operators $\overline{apr}_{N_2^{\mathbb{C}}}$ and $\overline{apr}_{N_4^{\mathbb{C}}}$ are not idempotent).

Let us consider the covering $\mathbb{C} = \{\{1\}, \{1,2\}, \{2,3\}, \{4\}, \{1,2,3\}, \{1,4\}\}$ of $U = \{1,2,3,4\}$. The minimal description $md(\mathbb{C}, x)$, the maximal description $MD(\mathbb{C}, x)$ and the neighborhood operators are listed in Table 1.

$\overline{apr}_{N_2^{\mathbb{C}}}(\{1\}) = \{1,2\}$, while $\overline{apr}_{N_2^{\mathbb{C}}}(\{1,2\}) = \{1,2,3\}$, therefore $\overline{apr}_{N_2^{\mathbb{C}}}$ is not an idempotent operator. Similarly, $\overline{apr}_{N_4^{\mathbb{C}}}(\{2\}) = \{1,2,3\}$, while $\overline{apr}_{N_4^{\mathbb{C}}}(\{1,2,3\}) = \{1,2,3,4\}$, therefore $\overline{apr}_{N_4^{\mathbb{C}}}$ is not an idempotent operator.

Table 1. Illustration of neighborhood operator for the covering $\mathbb{C}$.

x	$md(\mathbb{C}, x)$	$\boldsymbol{MD}(\mathbb{C}, x)$	$N_1^{\mathbb{C}}(x)$	$N_2^{\mathbb{C}}(x)$	$N_3^{\mathbb{C}}(x)$	$N_4^{\mathbb{C}}(x)$
1	{{1}}	{{1,2,3},{1,4}}	{1}	{1}	{1}	{1,2,3,4}
2	{{1,2},{2,3}}	{{1,2,3}}	{2}	{1,2,3}	{1,2,3}	{1,2,3}
3	{{2,3}}	{{1,2,3}}	{2,3}	{2,3}	{1,2,3}	{1,2,3}
4	{{4}}	{{1,4}}	{4}	{4}	{1,4}	{1,4}

According to properties in Table 2, operators 2, 4, 9, 10, 11 do not satisfy idempotent property, operators from 5 to 8 are not join morphisms and operator 12 does not satisfy any property. Obviously each topological closure is a closure operator.

3 Topological Characterization of Upper Approximations

The following propositions are characterization of upper approximation operators presented in [17]. Similar results for lower approximations and their relation with interior operators, can be established from duality.

Table 2. Properties of upper approximations.

n	Upper approximation	Property		
		Idempotence	Order preserving	Join
1	$\overline{apr}_{N_1^{\mathbb{C}}} = \overline{apr}'_{\mathbb{C}_3} = H_6^{\mathbb{C}}$	Yes	Yes	Yes
2	$\overline{apr}_{N_2^{\mathbb{C}}}$	No	Yes	Yes
3	$\overline{apr}_{N_3^{\mathbb{C}}}$	Yes	Yes	Yes
4	$\overline{apr}_{N_4^{\mathbb{C}}} = \overline{apr}''_{\mathbb{C}} = \overline{apr}''_{\mathbb{C}_2} = \overline{apr}''_{\mathbb{C}_\cap} = H_2^{\mathbb{C}}$	No	Yes	Yes
5	$\overline{apr}'_{\mathbb{C}} = \overline{apr}'_{\mathbb{C}_1} = \overline{apr}'_{\mathbb{C}_\cup} = \overline{apr}_{S_\cup}$	Yes	Yes	No
6	$\overline{apr}'_{\mathbb{C}_2}$	Yes	Yes	No
7	$\overline{apr}'_{\mathbb{C}_4}$	Yes	Yes	No
8	$\overline{apr}'_{\mathbb{C}_\cap}$	Yes	Yes	No
9	$\overline{apr}''_{\mathbb{C}_1} = \overline{apr}''_{\mathbb{C}_\cup}$	No	Yes	Yes
10	$\overline{apr}''_{\mathbb{C}_3} = H_7^{\mathbb{C}}$	No	Yes	Yes
11	$\overline{apr}''_{\mathbb{C}_4}$	No	Yes	Yes
12	$\overline{apr}_{S_\cap}$	No	Yes	No
13	$H_1^{\mathbb{C}}$	Yes	No	No
14	$H_3^{\mathbb{C}}$	No	Yes	Yes
15	$H_4^{\mathbb{C}}$	Yes	No	No
16	$H_5^{\mathbb{C}}$	Yes	Yes	Yes

Proposition 3. [17] *$H_1^{\mathbb{C}}$ is a topological closure if and only if $\mathbb{C}$ is unary.*

Proposition 4. [17] *$H_1^{\mathbb{C}}$ is a topological closure if and only if there exists a topology τ, on U such that $\mathbb{C}$ is a base for (U, τ).*

Proposition 5. [17] *$H_2^{\mathbb{C}}$ is a topological closure if and only if $\{N_4^{\mathbb{C}}(x) : x \in U\}$ forms a partition of U.*

Proposition 6. [17] *$H_3^{\mathbb{C}}$ is a topological closure if and only if exists a topology τ, on U such that, $\{N_2^{\mathbb{C}}(x) : x \in U\}$ is a base for (U, τ) and for all $x \in U$, $\{N_2\mathbb{C}(x)\}$ is a local base at x for (U, τ).*

Proposition 7. [17] *$H_4^{\mathbb{C}}$ is a topological closure if and only if $\mathbb{C}$ is a base for some topology τ on U and (U, τ) consists of two disjoints subspaces U_1 and U_2, satisfying:*

1. *For $K, K' \in \mathbb{C}$ with $K \neq K'$, we have: $K \cap U_2 = K' \cap U_2 = \emptyset$ or $K \cap U_2 \neq K' \cap U_2$ and $\{K \cap U_2 : K \in \mathbb{C}\}$ is a partition of U_2.*
2. *(U_1, τ_1) is a discrete space and (U_2, τ_2) is a pseudo-discrete space.*

Propositions (3) to (7) are the characterization of operators 4, 13, 14 and 15 being topological closures, therefore closure operators. Also are closure operators 9, 10 and 11, because they are defined as $\overline{apr}''_{\mathbb{C}}$ for different coverings: $\mathbb{C}_1$, $\mathbb{C}_3$ and $\mathbb{C}_4$.

4 Algebraic and Topological Properties

From properties in Table 2, it is easy to establish the following propositions.

Proposition 8. *Operators 1, 3 and 16 are topological closure operators.*

Proposition 9. *Operators 1, 3, 5, 6, 7, 8 and 16 are closure operators.*

The following propositions show some properties of unary coverings.

Proposition 10. *A covering $\mathbb{C}$ is unary if and only if there exists a topology τ, on U such that $\mathbb{C}$ is a base for (U, τ).*

Proof. *It is a consequence of Propositions 1 and 2.*

Proposition 11. *For any covering $\mathbb{C}$, the covering $\mathbb{C}_3$ is unary.*

Proof. *The elements in $\mathbb{C}_3$ are the neighborhoods $N_1^{\mathbb{C}}(x)$ for $x \in U$, so $\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x)\}$ and it is unary, because the minimal description of each $x \in U$ has only an element, $|md(\mathbb{C}_3, x)| = 1$. Therefore $\mathbb{C}_3$ is unary.*

Proposition 12. *If $\mathbb{C}$ is unary, then $\mathbb{C}_1$ and $\mathbb{C}_\cup$ are unary.*

Proof. *The covering $\mathbb{C}_1$ is made of sets in minimal descriptions of $x \in U$. Since $\mathbb{C}$ is unary, clearly $\mathbb{C}_1$ is unary. $\mathbb{C}_\cup$ is unary, because $\mathbb{C}_1 = \mathbb{C}_\cup$, as was established in [12].*

Corollary 2. *If* $\mathbb{C}$ *is unary,* $\overline{apr}'_{\mathbb{C}_1}$, $\overline{apr}'_{\mathbb{C}_3}$ *and* $\overline{apr}'_{\mathbb{C}_\cup}$ *are topological closures.*

Proposition 13. *If* $\mathbb{C}$ *is unary, then* $\overline{apr}_{N_1} = \overline{apr}_{N_2}$.

Proof. Using the number of elements in $md(\mathbb{C}, x)$*, we can see that* $\bigcup md(\mathbb{C}, x) = \bigcap md(\mathbb{C}, x)$*, therefore,* $N_1^{\mathbb{C}}(x) = N_2^{\mathbb{C}}(x)$ *and* $\overline{apr}_{N_1^{\mathbb{C}}} = \overline{apr}_{N_2^{\mathbb{C}}}$.

Corollary 3. *If* $\mathbb{C}$ *is unary,* $\overline{apr}_{N_2^{\mathbb{C}}}$ *is a topological closure.*

Corollary 4. *If* $\mathbb{C}$ *is unary,* $\overline{apr}_{N_2^{\mathbb{C}}}$ *is a closure operator.*

The following example shows that unary covering is not a condition of closure operator for $\overline{apr}_{N_4^{\mathbb{C}}}$.

Example 2. (An unary covering such that $\overline{apr}_{N_4^{\mathbb{C}}}$ is not an idempotent operator).

Let us consider the covering $\mathbb{C} = \{\{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{1, 4\}\}$ of $U = \{1, 2, 3, 4\}$. The minimal description $md(\mathbb{C}, x)$, the maximal description $MD(\mathbb{C}, x)$ and the neighborhood operators are listed in Table 3.

Table 3. Illustration of minimal and maximal description.

x	$md(\mathbb{C}, x)$	$\boldsymbol{MD}(\mathbb{C}, x)$	$N_1^{\mathbb{C}}(x)$	$N_2^{\mathbb{C}}(x)$	$N_3^{\mathbb{C}}(x)$	$N_4^{\mathbb{C}}(x)$
1	{{1}}	{{1,2,3},{1,4}}	{1}	{1}	{1}	{1,2,3,4}
2	{{2,3}}	{{1,2,3}}	{2,3}	{2,3}	{1,2,3}	{1,2,3}
3	{{2,3}}	{{1,2,3}}	{2,3}	{2,3}	{1,2,3}	{1,2,3}
4	{{4}}	{{1,4}}	{4}	{4}	{1,4}	{1,4}

According to Table 3, $\mathbb{C}$ is unary.

$\overline{apr}_{N_4^{\mathbb{C}}}(\{2\}) = \{1, 2, 3\}$, while $\overline{apr}_{N_4^{\mathbb{C}}}(\{1, 2, 3\}) = \{1, 2, 3, 4\}$, therefore $\overline{apr}_{N_4^{\mathbb{C}}}$ is not a idempotent operator.

Proposition 14. *If* $\mathbb{C}$ *is a covering of* U *and* $\{N_4^{\mathbb{C}}(x) : x \in U\}$ *forms a partition of* U*, then* $\overline{apr}''_{\mathbb{C}}$ *is a topological closure.*

Proof. It is a consequence of Proposition 5.

5 Conclusions

This paper studies the properties of upper approximation operators for the study of coverings based rough sets, extending the results presented in [32] to the framework presented in [11]. We show that $\overline{apr}_{N_3^{\mathbb{C}}}$ is an idempotent operator. We give necessary conditions for topological closures and closure operators. For an unary covering $\mathbb{C}$, we have that operators 1, 2, 5, 8 and 16 are topological closures and therefore closure operators. Necessary condition for closure operators in the new framework are given. As future work we will establish sufficient conditions for topological closure and closure operators.

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