A Local Projection Stabilized Lagrange-Galerkin Method for Convection-Diffusion Equations

Rodolfo Bermejo, Rafael Cantón, and Laura Saavedra

Abstract We present and analyze a Lagrange-Galerkin (LG) method combined with a local projection stabilization (LPS) technique for convection dominated convection-diffusion-reaction equations. This type of stabilization improves the accuracy and performance of conventional LG methods when the diffusion coefficient is very small. Numerical tests support the results of the numerical error analysis.

1 Introduction

LG methods discretize the total derivative (the convective part of the equations) backward in time along the characteristic curves of the transport operator, this is a natural way of introducing upwinding in the discretization of the equations, but such an upwinding may not be strong enough to suppress the spurious oscillations that may appear when the solution is not smooth and the mesh is not fine enough. Good properties of LG methods are the following: (1) assuming that the integrals that appear in the formulation of LG methods are calculated exactly, it is easy to show that LG methods are unconditionally stable in the L^2-norm, therefore, they allow the use of a large time step without damaging the accuracy of the solution; (2) unlike the pure Lagrangian methods, LG methods do not suffer from mesh deformation; (3) they yield algebraic symmetric systems of equations; (4) the constant C in the error estimate is much smaller than the constant of the conventional Galerkin methods. However, an important drawback of LG methods is the calculation of some integrals whose integrands are functions defined in different meshes, because, in general, such integrals can not be calculated analytically, i.e., exactly, so one

R. Bermejo (✉) • R. Cantón
Departamento de Matemática Aplicada ETSII, Universidad Politécnica de Madrid, Madrid, Spain
e-mail: rbermejo@etsii.upm.es; r.canton@alumnos.upm.es

L. Saavedra
Departamento Funadamentos Matemáticos ETSIA,Universidad Politécnica de Madrid, Madrid, Spain
e-mail: laura.saavedra@upm.es

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has to use quadrature rules; this handicap is particularly serious when the diffusion coefficient is small, for in this case the calculation of such integrals has to be done with quadrature rules of high order to keep the method stable, see [2] and [10], and, therefore, it may become computationally expensive. We propose in this note a remedy to correct this drawback and to partially suppress the spurious oscillations that consists of combining LG methods with the LPS technique introduced and analyzed in many papers, for instance, [1, 4, 9] and [7] just to cite a few. LPS technique is a symmetric stabilizer that fits very well in LG methods because the combination of both yields algebraic symmetric systems of equations.

2 The Formulation of the Local Projection Stabilized Lagrange-Galerkin Method

Let $X := H_0^1(D)$, where $D \subset \mathbf{R}^d$ is a bounded domain with Lipschitz boundary ∂D, and ($d = 1, 2$, or 3). We consider the problem: find a function $c : [0, T] \to X$, $c(0) = u \in X$, such that for all $v \in X$

$$\left(\frac{Dc}{Dt}, v\right) + \varepsilon(\nabla c, \nabla v) + (\alpha c, v) = (f, v), \tag{1}$$

where $\frac{Dc}{Dt} := \frac{\partial c}{\partial t} + \mathbf{b} \cdot \nabla c$, $\mathbf{b} \in L^\infty(0, T; W^{1,\infty}(D)^d)$, $f \in L^2(0, T; L^2(D))$, $\alpha \in C([0, T]; C(\overline{D}))$, and $0 < \varepsilon \ll \|\mathbf{b}\|_{L^\infty(D\times(0,T))^d}$. To guarantee the existence and uniqueness of (1) we also assume that there is a real number $\beta \geq 0$ such that

$$\alpha - \frac{1}{2}\text{div}\, \mathbf{b} \geq \beta \quad \text{a.e. in } D \times (0, T). \tag{2}$$

Next, we consider a regular quasi-uniform partition D_h of D formed by simplices K, and the finite element space X_h associated with D_h. The space X_h has the following approximation property.

For $v \in H^{r+1}(D) \cap H_0^1(D),\ 1 \leq r \leq m$,

$$\inf_{v_h \in X_h} \left(\|v - v_h\|_{L^2(D)} + h\,\|\nabla(v - v_h)\|_{L^2(D)}\right) \leq Ch^{r+1}\,\|v\|_{H^{r+1}(D)}\,, \tag{3}$$

where m denotes the degree of the polynomials of X_h and $h = \max_K (h_K)$, h_K being the diameter of the element K. To apply the stabilization technique we also consider the discontinuous finite element space G_h defined on D_h such that we set $G_h(K) := \{q_h\,|_K \colon q_h \in G_h\}$. Then for each K we use the local L^2-projector $\pi_K : L^2(K) \to G_h(K)$ to define the fluctuation operator $\kappa_K := id - \pi_K$, where $id := L^2(K) \to L^2(K)$ is the identity operator. We shall make the following assumptions.

Assumption A1 *Let $s \in (0, \ldots, m)$ be the degree of the polynomials of the space G_h, the fluctuation operator κ_K satisfies the approximation property*

$$\|\kappa_K w\|_{L^2(K)} \le Ch^l \|w\|_{H^l(K)}, \ \forall w \in H^l(K), \ 0 \le l \le s+1. \tag{4}$$

A sufficient condition for Assumption **A1** is $P_s(K) \subset G_h(K)$, $P_s(K)$ being the set of polynomials of degree at most s defined in K

Assumption A2 *There is an interpolation operator $j_h : H^2 \cap X_h(D) \to X_h$ such that for all $w \in H^1(D)$, and for all $K \in D_h$*

$$\|w - j_h w\|_{L^2(K)} + h_K \|\nabla(w - j_h w)\|_{L^2(K)} \le Ch_K^l \|w\|_{H^l(K)} \ (1 \le l \le m+1). \tag{5}$$

We define in $[0, T]$ a uniform partition $\mathscr{P}_{\Delta t} := 0 = t_0 < t_1 < \ldots < t_N = T$ of uniform step Δt such that the numerical solution to problem (1) is a mapping, $c_h : \mathscr{P}_{\Delta t} \to X_h$, satisfying for all n, $0 \le n \le N-1$, the equations

$$\begin{cases} \dfrac{(c_h^{n+1} - c_h^n \circ X^{n,n+1}, v_h)}{\Delta t} + \varepsilon(\nabla c_h^{n+1}, \nabla v_h) + (\alpha^{n+1} c_h^{n+1}, v_h) \\ \\ +S_h(c_h^{n+1}, v_h) = (f^{n+1}, v_h) \quad \forall v_h \in X_h, \end{cases} \tag{6}$$

where $S_h(c_h^{n+1}, v_h)$ is the stabilization term given by

$$S_h(c_h^{n+1}, v_h) = \sum_K \tau_K (\kappa_K \nabla c_h^{n+1}, \kappa_K \nabla v_h)_K, \tag{7}$$

$\boldsymbol{\tau}_K$ being element-wise constant coefficients that depend on the mesh size, their optimal values are determined by the error analysis. In (6), f^{n+1} denotes the function $f(\cdot, t_{n+1})$ and $X^{n,n+1}$, which is a shorthand notation for $X(x, t_{n+1}; t_n)$ unless otherwise stated, denotes the position at time t_n of a particle that at time t_{n+1} will reach the point x; specifically, for $s, t \in [t_n, t_{n+1})$ the mappings $X(\cdot, s; t) : D \to D$ can be defined by solving the system of ordinary differential equations

$$\begin{cases} \dfrac{dX(x, s; t)}{dt} = \mathbf{b}(X(x, s; t), t), \\ X(x, s; s) = x \quad \forall x \in D. \end{cases} \tag{8}$$

3 Error Analysis

Our concern in this paper is to estimate the error of LG methods when they are stabilized by a local projection stabilization method, therefore to make clearer and shorter the analysis we shall consider the exact solution of (8); nevertheless, the calculation of a solution of (8) by a numerical method will contribute to the error of the local stabilized LG method, but such a contribution can be estimated using the methodology of [3].

For $u, v \in H_0^1(D)$ and a.e. $0 \le t \le T$, let us now define the time-dependent bilinear form

$$a(u, v; t) = \varepsilon\,(\nabla u, \nabla v) + (\alpha(\cdot, t)u, v)\,. \tag{9}$$

It is easy to see that $a(u, v; t)$ is symmetric, continuous and coercive so that for functions $u : [0, T] \to H_0^1(D)$

$$(a(u, u; t))^{1/2} = \left(\left\|\varepsilon^{1/2}\nabla u(t)\right\|_{L^2(D)}^2 + \left\|\alpha^{1/2}u(t)\right\|_{L^2(D)}^2\right)^{\frac{1}{2}}. \tag{10}$$

is an equivalent $H_0^1(D)$-norm, i.e.,

$$c_2\,\|u(t)\|_{H^1(D)} \le (a(u, u; t))^{1/2} \le c_1\,\|u(t)\|_{H^1(D)}\,, \tag{11}$$

where the constants $c_1 = \max(\varepsilon^{1/2}, \overline{\alpha}^{1/2})$ and $c_2 = \min(\varepsilon^{1/2}, \underline{\alpha}^{1/2})$, and $(\overline{\alpha}^{1/2}, \underline{\alpha}^{1/2}) = (\max_{(x,t)}\alpha(x, t), \min_{(x,t)}\alpha(x, t))$. Moreover, we define the mesh dependent norm

$$|||u(t)|||^2 := a(u, u; t) + S_h(u, u). \tag{12}$$

We will use the following continuous and discrete time dependent norms, noting that in the expressions that follow, when $r = 0$, $H^0(D) = L^2(D)$.

Continuous norms:

$$\begin{aligned}
&\|u\|_{L^\infty(L^\infty(D))} \equiv \|u\|_{L^\infty(0,T;L^\infty(D))} = \text{ess}\ \sup_{0\le t\le T}\|u(t)\|_{L^\infty(D)}\,,\\
&\|u\|_{L^\infty(H^r(D))} \equiv \|u\|_{L^\infty(0,T;H^r(D))} = \text{ess}\ \sup_{0\le t\le T}\|u(t)\|_{H^r(D)}\,,\ r \ge 0,\\
&\|u_t\|_{L^2(L^2(D))} \equiv \|u_t\|_{L^2(0,T;L^2(D))} = \left(\int_0^T \left\|\frac{\partial u}{\partial t}\right\|^2\right)^{1/2}.
\end{aligned} \tag{13}$$

Discrete norms:

$$\begin{aligned}
&\|u\|_{l^\infty(H^r(D))} \equiv \|u\|_{l^\infty(0,N;H^r(D))} = \max_{0\le n\le N}\|u^n\|_{H^r(D)}\,,\ r \ge 0,\\
&\|u\|_{l_2(H^r(D))} \equiv \|u\|_{l_2(0,N;H^r(D))} = \left(\Delta t \textstyle\sum_{n=0}^N \|u^n\|_{H^r(D)}^2\right)^{1/2},\\
&|||u|||_{l_2(0,N)} \equiv \left(\Delta t \textstyle\sum_{n=0}^N |||u^n|||^2\right)^{1/2}.
\end{aligned} \tag{14}$$

Next, we establish an estimate for the error $e^n = c^n - c_h^n$.

Theorem 1 *Let* $c \in L^{\infty}(0,T;H_0^1(D)\cap H^{m+1}(D))$, $c_t \in L^2(0,T;H_0^1(D)\cap H^{m+1}(D))$, $\dfrac{D^2c}{Dt^2} \in L^2(0,T;L^2(D))$, $0 < \Delta t < \Delta t_0 < 1$, *and* $0 < h < h_0 < 1$. *There exists a constant G independent of* Δt *and h such that*

$$\begin{aligned} &\|e\|_{l^{\infty}(L^2(D))} + |||e|||_{l_2(0,N)} \leq G\left(h^{m+1} + \sqrt{\tau+\varepsilon}h^m + \tau^{1/2}h^{s+1}\right. \\ &+T^{1/2}\min\left(\tfrac{K_4\Delta t}{\sqrt{\varepsilon}}, \tfrac{\|\mathbf{b}\|_{L^{\infty}(\mathbf{L}^{\infty}(D))}\Delta t}{h}, \sqrt{2}\right)\tfrac{h^{m+1}}{\Delta t} + \Delta t\ ) + \|u - j_h u\|_{L^2(D)}\,, \end{aligned} \tag{15}$$

where $\tau = \max_K(\tau_K)$ *with* $\tau_K = O(h^{\gamma})$ *and* $\gamma \geq 1$, $u = c(0)$, $K_4 = \|\mathbf{b}\|_{L^{\infty}(\mathbf{L}^{\infty}(D))} + K_5$, *and* K_5 *being another constant that depends on div* $\mathbf{b}$.

Proof A sketch of the proof goes as follows. We decompose the error at time instant t_{n+1} as

$$e^{n+1} = (c^{n+1} - j_h c^{n+1}) + (j_h c^{n+1} - c_h^{n+1}) \equiv \rho^{n+1} + \theta_h^{n+1}, \tag{16}$$

then the errors $\|e\|_{l^{\infty}(L^2(D))}$, and $|||e|||_{l_2(0,N)}$ are estimated by applying the triangle inequality and (5) to estimate ρ, so we need to estimate θ_h. To this end, we notice that for all n, $c_h^n = c^n - \rho^n - \theta_h^n$, so subtracting (6) from (1), and using the notation $a^{n+1}(\cdot,\cdot)$ to denote $a(\cdot,\cdot;t_{n+1})$, some simple operational work yields

$$\begin{aligned} &\left(\theta_h^{n+1} - \overline{\theta}_h^n, v_h\right) + \Delta t\varepsilon\left(\nabla\theta_h^{n+1}, \nabla v_h\right) + \Delta t\left(\alpha^{n+1}\theta_h^{n+1}, v_h\right) + \Delta t S_h\left(\theta_h^{n+1}, v_h\right) \\ &= \quad -\Delta t a^{n+1}(\rho^{n+1}, v_h) - \Delta t S_h(\rho^{n+1}, v_h) - \left(\rho^{n+1} - \overline{\rho}^n, v_h\right) \\ &\quad + \Delta t\left(\frac{c^{n+1} - \overline{c}^n}{\Delta t} - \frac{Dc}{Dt}\ |_{t=t_{n+1}}, v_h\right) + \Delta t S_h(c^{n+1}, v_h), \end{aligned} \tag{17}$$

where $\overline{g}^n := g(X(x,t_{n+1};t_n),t_n)$, $g(\cdot,t_n)$ being a generic function defined in D at time instant t_n. Letting $v_h = \theta_h^{n+1}$, see [2], we find that $(\theta_h^{n+1} - \overline{\theta}_h^n, \theta_h^{n+1}) \geq \frac{1}{2}(\|\theta_h^{n+1}\|^2_{L^2(D)} - \|\theta_h^n\|^2_{L^2(D)}) - \frac{\Delta t C}{2}\|\theta_h^n\|^2_{L^2(D)}$, where C is a positive constant independent of h and Δt, but dependent on div $\mathbf{b}$; then splitting $\rho^{n+1} - \overline{\rho}^n$ as $(\rho^{n+1} - \rho^n) + (\rho^n - \overline{\rho}^n)$ yields

$$\begin{aligned} &\tfrac{1}{2}\left(\left\|\theta_h^{n+1}\right\|^2_{L^2(D)} - \left\|\theta_h^n\right\|^2_{L^2(D)}\right) + \Delta t a^{n+1}(\theta_h^{n+1}, \theta_h^{n+1}) + \Delta t S_h(\theta_h^{n+1}, \theta_h^{n+1}) \\ &\leq -\Delta t a^{n+1}(\rho^{n+1}, \theta_h^{n+1}) - \Delta t S_h(\rho^{n+1}, \theta_h^{n+1}) - \Delta t S_h(c^{n+1}, \theta_h^{n+1}) \\ &\quad + \sum_{i=1}^{3}\left(z_i^{n+1}, \theta_h^{n+1}\right) + \frac{C}{2}\Delta t\left\|\theta_h^n\right\|^2_{L^2(D)} \end{aligned} \tag{18}$$

where

$$\begin{cases} z_1^{n+1} = -(\rho^{n+1} - \rho^n), \quad z_2^{n+1} = -(\rho^n - \overline{\rho}^n), \\ z_3^{n+1} = \Delta t \left(\dfrac{c^{n+1} - \overline{c}^n}{\Delta t} - \dfrac{Dc}{Dt} \mid_{t=t_{n+1}} \right). \end{cases} \tag{19}$$

Now, we estimate the terms on the right side. By Cauchy-Schwarz inequality and Young's inequality, $ab \leq \frac{\zeta}{2}a^2 + \frac{1}{2\zeta}b^2$, $a,\ b$ and $\zeta > 0$ real numbers, it follows that

$$\begin{aligned} \Delta t a^{n+1}(\rho^{n+1}, \theta_h^{n+1}) &\leq \Delta t \left(a^{n+1}(\rho^{n+1}, \rho^{n+1})\right)^{1/2} \left(a^{n+1}(\theta_h^{n+1}, \theta_h^{n+1})\right)^{1/2} \\ &\leq \frac{\Delta t}{2} a^{n+1}(\rho^{n+1}, \rho^{n+1}) + \frac{\Delta t}{2} a^{n+1}(\theta_h^{n+1}, \theta_h^{n+1}). \end{aligned} \tag{20}$$

Similarly,

$$\Delta t S_h(\rho^{n+1}, \theta_h^{n+1}) \leq \Delta t \left(S_h(\rho^{n+1}, \rho^{n+1}) + \frac{1}{4} S_h(\theta_h^{n+1}, \theta_h^{n+1}) \right). \tag{21}$$

Noting that $S_h(\rho^{n+1}, \rho^{n+1}) \leq \sum_K \tau_K \left\| \nabla \rho^{n+1} \right\|_{L^2(K)}^2$ and using Assumption **A2** it follows that

$$\Delta t S_h(\rho^{n+1}, \theta_h^{n+1}) \leq C \Delta t \sum_K \tau_K h_K^{2m} \left\| c^{n+1} \right\|_{H^{m+1}(K)}^2 + \frac{\Delta t}{4} S_h(\theta_h^{n+1}, \theta_h^{n+1}). \tag{22}$$

Similarly,

$$\Delta t S_h(c^{n+1}, \theta_h^{n+1}) \leq \Delta t \left(S_h(c^{n+1}, c^{n+1}) + \frac{1}{4} S_h(\theta_h^{n+1}, \theta_h^{n+1}) \right), \tag{23}$$

using Assumption **A1** with $l = s + 1$ it follows that

$$\Delta t S_h(c^{n+1}, \theta_h^{n+1}) \leq C \Delta t \sum_K \tau_K h_K^{2(s+1)} \left\| c^{n+1} \right\|_{H^{m+1}(K)}^2 + \frac{\Delta t}{4} S_h(\theta_h^{n+1}, \theta_h^{n+1}) \tag{24}$$

To estimate $\left(z_1, \theta_h^{n+1}\right)$, we note that by virtue of the Cauchy-Schwarz inequality

$$\left| \int_D \left(\int_{t_n}^{t_{n+1}} \rho_t dt \right) \theta_h^{n+1} dx \right| \leq \left\| \int_{t_n}^{t_{n+1}} \rho_t dt \right\|_{L^2(D)} \left\| \theta_h^{n+1} \right\|_{L^2(D)}, \tag{25}$$

hence, using Young's inequality yields

$$
\begin{aligned}
\left(z_1, \theta_h^{n+1}\right) &\leq \tfrac{3}{2} \left\| \rho_t \right\|^2_{L^2(t_n, t_{n+1}, L^2(D))} + \tfrac{\Delta t}{6} \left\| \theta_h^{n+1} \right\|^2_{L^2(D)} \\
&\leq C h^{2(m+1)} \left\| c_t \right\|^2_{L^2(t_n, t_{n+1}, H^{m+1}(D))} + \tfrac{\Delta t}{6} \left\| \theta_h^{n+1} \right\|^2_{L^2(D)} .
\end{aligned}
\tag{26}
$$

Next, by a Taylor expansion along the curves $X(x, t_{n+1}, t)$ it follows that

$$
\begin{aligned}
\left\| z_3^{n+1} \right\| &= \Delta t \left(\int_D \left| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \frac{D^2 c}{Dt^2} dt \right|^2 dx \right)^{1/2} \\
&\leq \frac{\Delta t}{\sqrt{3}}^{3/2} \left\| \frac{D^2 c}{Dt^2} \right\|_{L^2(t_n, t_{n+1}; L^2(D))},
\end{aligned}
\tag{27}
$$

then by using both the Cauchy-Schwarz and Young's inequalities yields

$$
\left| (z_3^{n+1}, \theta_h^{n+1}) \right| \leq \frac{1}{2} \Delta t^2 \left\| \frac{D^2 c}{Dt^2} \right\|^2_{L^2(t_n, t_{n+1}; L^2(D))} + \frac{\Delta t}{6} \left\| \theta_h^{n+1} \right\|^2_{L^2(D)} . \tag{28}
$$

To bound the term $(z_2^{n+1}, \theta_h^{n+1})$ we use Lemma 7 of [2] and obtain the following estimates:

Estimate 1:

$$
\begin{aligned}
(z_2^{n+1}, \theta_h^{n+1}) &\leq \left\| \rho^n - \rho^n \circ X^{n,n+1} \right\|_{L^2(D)} \left\| \theta_h^{n+1} \right\|_{L^2(D)} \\
&\leq \Delta t \min \left(K_1 \left\| \nabla \rho^n \right\|_{L^2(D)}, K_2 \left\| \tfrac{\rho^n}{\Delta t} \right\|_{L^2(D)} \right) \left\| \theta_h^{n+1} \right\|_{L^2(D)} \\
&\leq \tfrac{3}{2} \Delta t \min \left(K_1^2 \left\| \nabla \rho^n \right\|^2_{L^2(D)}, K_2 \left\| \tfrac{\rho^n}{\Delta t} \right\|^2_{L^2(D)} \right) + \tfrac{\Delta t}{6} \left\| \theta_h^{n+1} \right\|^2_{L^2(D)},
\end{aligned}
\tag{29}
$$

where $K_1 = K_3 \left\| \mathbf{b} \right\|_{L^\infty(\mathbf{L}^\infty(D))}$, and K_2 and K_3 being constants depending on div $\mathbf{b}$. Noticing that by virtue of Assumption **A2** we can set

$$
\begin{aligned}
&\min \left(K_1^2 \left\| \nabla \rho^n \right\|^2_{L^2(D)}, K_2 \left\| \tfrac{\rho^n}{\Delta t} \right\|^2_{L^2(D)} \right) \\
&\qquad \leq C \min \left(\frac{\left\| \mathbf{b} \right\|^2_{L^\infty(\mathbf{L}^\infty(D))} \Delta t^2}{h^2}, 2 \right) \frac{h^{2(m+1)}}{\Delta t^2} \left\| c \right\|^2_{l^\infty(0,N; H^{m+1}(D))},
\end{aligned}
\tag{30}
$$

then one gets the estimate

$$(z_2^{n+1}, \theta_h^{n+1}) \leq C\Delta t \min\left(\frac{\|\mathbf{b}\|^2_{L^\infty(\mathbf{L}^\infty(D))}\Delta t^2}{h^2}, 2\right) \frac{h^{2(m+1)}}{\Delta t^2} \|c\|^2_{l^\infty(0,N;H^{m+1}(D))} + \frac{\Delta t}{6} \left\|\theta_h^{n+1}\right\|^2_{L^2(D)}. \tag{31}$$

Estimate 2
A second estimate, see [6], is the following

$$\begin{aligned}(z_2^{n+1}, \theta_h^{n+1}) &\leq \left\|\rho^n - \rho^n \circ X^{n,n+1}\right\|_{H^{-1}} \left\|\nabla\theta_h^{n+1}\right\|_{L^2(D)} \\ &\leq \Delta t K_4 \|\rho^n\|_{L^2(D)} \left\|\nabla\theta_h^{n+1}\right\|_{L^2(D)},\end{aligned} \tag{32}$$

where H^{-1} is the dual of $H_0^1(D)$, $K_4 = \|\mathbf{b}\|_{L^\infty(\mathbf{L}^\infty(D))} + K_5$, and K_5 being another constant that depends on div $\mathbf{b}$. By using again **A2** we obtain that

$$(z_2^{n+1}, \theta_h^{n+1}) \leq C\Delta t \left(\frac{K_4^2 \Delta t^2}{\varepsilon}\right) \frac{h^{2(m+1)}}{\Delta t^2} \|c\|^2_{l^\infty(0,N;H^{m+1}(D))} + \frac{\Delta t \varepsilon}{4} \left\|\nabla\theta_h^{n+1}\right\|^2_{L^2(D)}. \tag{33}$$

Next, substituting the estimates calculated above into (18), adding from $n = 0$ to $N-1$ and arguing as in [2] we find out that the estimates of $(z_2^{n+1}, \theta_h^{n+1})$ give the term

$$\min\left(\frac{K_4^2 \Delta t^2}{\varepsilon}, \frac{\|\mathbf{b}\|^2_{L^\infty(\mathbf{L}^\infty(D))} \Delta t^2}{h^2}, 2\right) \frac{h^{2(m+1)}}{\Delta t^2} \|c\|^2_{l^\infty(0,N;H^{m+1}(D))}. \tag{34}$$

Then the application of Gronwall inequality and the triangle inequality, as we say at the beginning of the proof, yields the estimate (15).

4 Numerical Examples

Example 1 In this example, borrowed from [8], we consider the domain $D = (0,1)^2$ and the partition D_h generated from a uniform square mesh of size h by dividing the squares using the diagonals from the left lower corner to the right upper corner. The prescribed solution is $c(x,t) = t\cos(xy^2)$ for the parameters $\varepsilon = 10^{-8}$, $\mathbf{b} = (2,-1)$, $\alpha = 1$ and $T = 1$. The non-homogenous Dirichlet boundary conditions and the forcing term f are chosen such that the prescribed solution satisfies (1). The finite element spaces used in this example are: $X_h = \{v_h \in C^0(\overline{D}) : v_h|_K \in P_1^{\text{bubble}}(K)\}$ and $G_h = \{q_h \in L^2(D) : q_h|_K \in P_0(K)\}$.

Table 1 Error for different meshes with $\Delta t = 0.0001$

h	$Err_1, \tau_K = 100h$	$Err_1, \tau_K = 10h$	$Err_1, \tau_K = h$
1/8	3.07E−06	3.03E−06	2.79E−06
1/16	1.53E−06	1.50E−06	1.37E−06
1/32	7.60E−07	7.32E−07	6.76E−07
1/64	3.67E−07	3.51E−07	3.36E−07
1/128	1.74E−07	1.69E−07	1.67E−07

We show in Table 1 the error $Err_1 := \left(\Delta t \sum_{n=0}^{N} \epsilon \|c^n - c_h^n\|^2_{H^1(D)}\right)^{1/2}$ for different values of h and τ_K. Since the time step is so small, then the errors represented in the table can be considered spatial errors. By simple inspection we notice that the numerical solution is not sensitive to the value of τ_K, and $Err_1 = O(h)$ according to Theorem 1 because, in this case with $m = 1$, the term that controls the error estimate is $\min(\frac{\Delta t}{\sqrt{\varepsilon}}, \frac{\|\mathbf{b}\|_{l^\infty(\mathbf{L}^\infty(D))}\Delta t}{h}, 1)\frac{h^{m+1}}{\Delta t} = \frac{\|\mathbf{b}\|_{l^\infty(\mathbf{L}^\infty(D))}\Delta t}{h}\frac{h^{m+1}}{\Delta t} = O(h^m)$.

Example 2 In this example, taken from [5], $D := (0,1)^2$ and the partition D_h is formed by triangles obtained by dividing uniform squares of size h by diagonals that go from the left upper corner to the right lower corner. The velocity field $\mathbf{b}(x,y) = \nabla\phi$, where $\phi(x,y) = (1-\cos 2\pi x)(1-\cos 2\pi y)$. The streamlines of the velocity converge to a sink at the center of D along trajectories that become parallel to the diagonal that joins the left upper corner with the right lower corner. The initial condition $u(x,y)$ represents a transition from $u(0,0) = 0$ to $u(1,1) = 1$ according to the rule

$$u(x,y) = \begin{cases} 0 & \text{if } \xi < 0, \\ \frac{1}{2}(1-\cos\pi\xi), & 0 \le \xi \le 1, \\ 1 & \text{if } 1 < \xi, \end{cases} \tag{35}$$

where $\xi = x + y - 1/2$. The Dirichlet boundary conditions $c(\cdot,t) = u(\cdot)$ are imposed for all $0 \le t \le T$. The forcing term $f = 0$, the diffusion coefficient $\varepsilon = 0.001$ and the reaction term $\alpha = 0$. The finite element spaces used in this example are: $X_h = \{v_h \in C^0(\overline{D}) : v_h|_K \in P_2(K)\}$ and $G_h = \{q_h \in L^2(D) : q_h|_K \in P_0(K)\}$, and $\tau_K = h^2$. Figure 1 represents the cross section $u(x,1/2,1)$ calculated in the mesh $h = 1/32$ and with the time step $\Delta t = h/2$. Comparing this figure with Figure 6 of [5], where the same cross sections of the solutions calculated by the conventional LG method and the Euler implicit-quadratic finite element method are represented, we see that at least for this example the LPS-LG method yields much better results than those methods because much of the spurious oscillations have been killed and the interior boundary layer is well resolved even with a relatively coarse mesh. The amplitudes of the overshoot and undershoot, which appear in the figure, are ± 0.031 respectively.

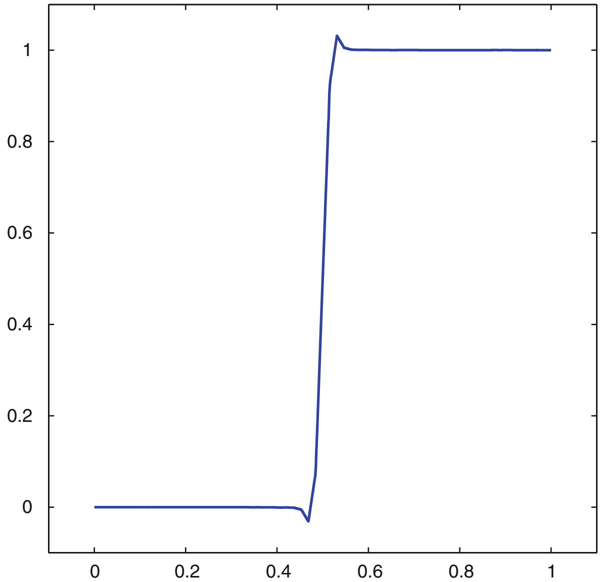

Fig. 1 Section $c_h(x, y = 1/2, t = 1)$ for $h = 1/32$ and $\Delta t = h/2$

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