A Parameter-Uniform First Order Convergent Numerical Method for a System of Singularly Perturbed Second Order Delay Differential Equations

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Abstract In this paper, a boundary value problem for a system of two singularly perturbed second order delay differential equations is considered on the interval $[0, 2]$. The components of the solution of this system exhibit boundary layers at $x = 0$ and $x = 2$ and interior layers at $x = 1$. A numerical method composed of a classical finite difference scheme applied on a piecewise uniform Shishkin mesh is suggested to solve the problem. The method is proved to be first order convergent in the maximum norm uniformly in the perturbation parameters. Numerical illustration provided support the theory.

1 Introduction

Delay differential equations are common in the mathematical modelling of various physical, biological phenomena and control theory. A subclass of these equations consists of singularly perturbed ordinary differential equations with a delay. Such type of equations arise frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of human pupil-light reflex [6], models of HIV infection [1], the study of bistable devices in digital electronics [2], variational problems in control theory [3], first exit time problems in modelling of activation of neuronal variability [5], evolutionary biology [8], mathematical ecology [4] and in a variety of models for physiological processes [7].

Investigation of boundary value problems for singularly perturbed linear second-order differential-difference equations was initiated by Lange and Miura [5].

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The singularly perturbed boundary value problem for a system of delay differential equations under consideration is

$$\mathbf{Lu}(x) \;=\; -E\,\mathbf{u}''(x) + A(x)\,\mathbf{u}(x) + B(x)\,\mathbf{u}(x-1) = \mathbf{f}(x) \text{ on } (0,2) \tag{1}$$

$$\text{with } \mathbf{u} = \boldsymbol{\phi} \text{ on } [-1,0] \text{ and } \mathbf{u}(2) = \mathbf{l}, \tag{2}$$

where $\boldsymbol{\phi}(x) = (\phi_1(x), \phi_2(x))^T$ is sufficiently smooth on $[-1,0]$. For all $x \in [0,2], \mathbf{u}(x) = (u_1(x), u_2(x))^T$ and $\mathbf{f}(x) = (f_1(x), f_2(x))^T$. E, $A(x)$ and $B(x)$ are 2×2 matrices. $E = \text{diag}(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ with $0 < \varepsilon_1 < \varepsilon_2 << 1$, $B(x) = \text{diag}(\mathbf{b}(x))$, $\mathbf{b}(x) = (b_1(x), b_2(x))$. For all $x \in [0,2]$, it is also assumed that the entries $a_{ij}(x)$ of $A(x)$ and the components $b_i(x)$ of $B(x)$ satisfy

$$b_i(x),\ a_{ij}(x) \le 0 \text{ for } 1 \le i \ne j \le 2, \quad a_{ii}(x) > \sum_{i\ne j} |a_{ij}(x) + b_i(x)| \tag{3}$$

$$\text{and} \qquad 0 < \alpha < \min_{\substack{x\in[0,2]\\ i=1,2}} \Big(\sum_{j=1}^{2} a_{ij}(x) + b_i(x)\Big), \text{ for some } \alpha. \tag{4}$$

Further, the functions $f_i(x)$, $a_{ij}(x)$ and $b_i(x)$, $1 \le i,j \le 2$ are assumed to be in $C^2([0,2])$. The above assumptions ensure that $\mathbf{u} \in \mathscr{C} = C^0([0,2]) \cap C^1((0,2)) \cap C^2((0,1)\cup(1,2))$.

The problem (1)–(2) can be rewritten as

$$\mathbf{L}_1\mathbf{u}(x) = -E\,\mathbf{u}''(x) + A(x)\,\mathbf{u}(x) = \mathbf{f}(x) - B(x)\,\boldsymbol{\phi}(x-1) = \mathbf{g}(x) \text{ on } (0,1), \tag{5}$$

$$\mathbf{L}_2\mathbf{u}(x) = -E\,\mathbf{u}''(x) + A(x)\,\mathbf{u}(x) + B(x)\,\mathbf{u}(x-1) = \mathbf{f}(x) \text{ on } (1,2), \tag{6}$$

$$\mathbf{u}(0) = \boldsymbol{\phi}(0),\ \ \mathbf{u}(2) = \mathbf{l}, \mathbf{u}(1-) = \mathbf{u}(1+) \text{ and } \mathbf{u}'(1-) = \mathbf{u}'(1+). \tag{7}$$

The reduced problem corresponding to (5), (6) and (7) is defined by

$$A(x)\,\mathbf{u}_0(x) = \mathbf{g}(x) \text{ on } (0,1), \tag{8}$$

$$A(x)\,\mathbf{u}_0(x) + B(x)\,\mathbf{u}_0(x-1) = \mathbf{f}(x) \text{ on } (1,2). \tag{9}$$

For any vector-valued function $\mathbf{y}$ on $[0,2]$ the following norms are introduced: $\| \mathbf{y}(x) \| = \max_i |y_i(x)|$ and $\| \mathbf{y} \| = \sup\{\| \mathbf{y}(x) \| : x \in [0,2]\}$. For any mesh function $\mathbf{V}$ on $\overline{\Omega}^N = \{x_j\}_{j=0}^N$ the following discrete maximum norms are introduced: $\| \mathbf{V}(x_j) \| = \max_i |V_i(x_j)|$ and $\| \mathbf{V} \| = \max\{\| \mathbf{V}(x_j) \| : x_j \in \overline{\Omega}^N\}$.

For any function $\boldsymbol{\psi}$ the jump at x is $[\boldsymbol{\psi}](x) = \boldsymbol{\psi}(x+) - \boldsymbol{\psi}(x-)$.

Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

2 Analytical Results

This section presents some analytical results related to the problem (5), (6) and (7) which include maximum principle, stability result and the estimates of the derivatives.

Lemma 1 *Let conditions (3) and (4) hold. Let* $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$ *be any function in* $\mathscr{C}$ *such that* $\boldsymbol{\psi}(0) \geq \mathbf{0}$ $\boldsymbol{\psi}(2) \geq \mathbf{0}$, $\mathbf{L}_1\boldsymbol{\psi} \geq \mathbf{0}$ *on* $(0,1)$, $\mathbf{L}_2\boldsymbol{\psi} \geq \mathbf{0}$ *on* $(1,2)$ *and* $[\boldsymbol{\psi}](1) = \mathbf{0}, [\boldsymbol{\psi}'](1) \leq \mathbf{0}$ *then* $\boldsymbol{\psi} \geq \mathbf{0}$ *on* $[0,2]$.

Proof Let i^*, x^* be such that $\psi_{i^*}(x^*) = \min\limits_{i=1,2,\, x\in[0,2]} \psi_i(x)$. If $\psi_{i^*}(x^*) \geq 0$, there is nothing to prove. Suppose therefore that $\psi_{i^*}(x^*) < 0$. Then $x^* \notin \{0,2\}$, $\psi''_{i^*}(x^*) \geq 0$. If $x^* \in (0,1)$ then $(\mathbf{L}_1\boldsymbol{\psi})_{i^*}(x^*) < 0$, which is a contradiction. And if $x^* \in (1,2)$ then $(\mathbf{L}_2\boldsymbol{\psi})_{i^*}(x^*) < 0$, which is also a contradiction.

Because of the boundary values, the only other possibility is that $x^* = 1$. In this case, the argument depends on whether or not $\boldsymbol{\psi}_{i^*}$ is differentiable at $x = 1$. If $\psi'_{i^*}(1)$ does not exist then $[\psi'_{i^*}](1) \neq 0$ and since $\psi'_{i^*}(1-) \leq 0$, $\psi'_{i^*}(1+) \geq 0$, it is clear that $[\psi'_{i^*}](1) > 0$, which is a contradiction. On the other hand, let ψ_{i^*} be differentiable at $x = 1$. As $\sum\limits_{j=1}^{2} a_{i^*j}(x)\psi_j(x) < 0$ and all the entries of $A(x)$ and $\psi_j(x)$ are in $C([0,2])$, there exist an interval $[1-h, 1)$ on which $\sum\limits_{j=1}^{2} a_{i^*j}(x)\psi_j(x) < 0$. If $\psi''_{i^*}(\hat{x}) \geq 0$ at any point $\hat{x} \in [1-h, 1)$, then $(\mathbf{L}_1\boldsymbol{\psi})_{i^*}(\hat{x}) < 0$, which is a contradiction. Thus we can assume that $\psi''_{i^*}(x) < 0$ on $[1-h, 1)$. But this implies that $\psi'_{i^*}(x)$ is strictly decreasing on $[1-h, 1)$. Already we know that $\psi'_{i^*}(1) = 0$ and $\psi'_{i^*} \in C((0,2))$, so $\psi'_{i^*}(x) > 0$ on $[1-h, 1)$. Consequently the continuous function $\psi_{i^*}(x)$ cannot have a minimum at $x = 1$, which contradicts the assumption that $x^* = 1$. □

As a consequence of the maximum principle, there is established the stability result for the problem (1)–(2) in the following

Lemma 2 *Let conditions (3) and (4) hold. Let* $\boldsymbol{\psi}$ *be any function in* $\mathscr{C}$, *such that* $[\boldsymbol{\psi}](1) = \mathbf{0}$ *and* $[\boldsymbol{\psi}'](1) = \mathbf{0}$, *then for each* $i = 1,2$ *and* $x \in [0,2]$,

$$|\psi_i(x)| \leq \max\left\{\| \boldsymbol{\psi}(0) \|, \| \boldsymbol{\psi}(2) \|, \tfrac{1}{\alpha} \| \mathbf{L}_1\boldsymbol{\psi} \|, \tfrac{1}{\alpha} \| \mathbf{L}_2\boldsymbol{\psi} \|\right\}.$$

Proof Let $M = \max\{\| \boldsymbol{\psi}(0) \|, \| \boldsymbol{\psi}(2) \|, \frac{1}{\alpha} \| \mathbf{L}_1\boldsymbol{\psi} \|, \frac{1}{\alpha} \| \mathbf{L}_2\boldsymbol{\psi} \|\}$. Define two functions $\boldsymbol{\theta}^{\pm}(x) = M\mathbf{e} \pm \boldsymbol{\psi}(x)$ where $\mathbf{e} = (1,1)^T$. Using the properties of $A(x)$ and $B(x)$ it is not hard to verify that $\boldsymbol{\theta}^{\pm}(0) \geq \mathbf{0}$, $\boldsymbol{\theta}^{\pm}(2) \geq \mathbf{0}, \mathbf{L}_1\boldsymbol{\theta}^{\pm}(x) \geq \mathbf{0}$ on $(0,1)$ and $\mathbf{L}_2\boldsymbol{\theta}^{\pm}(x) \geq \mathbf{0}$ on $(1,2)$. Moreover $[\boldsymbol{\theta}^{\pm}](1) = \pm[\boldsymbol{\psi}](1) = \mathbf{0}$ and $[\boldsymbol{\theta}^{\pm'}](1) = \pm[\boldsymbol{\psi}'](1) = \mathbf{0}$. It follows from Lemma 1 that $\boldsymbol{\theta}^{\pm}(x) \geq \mathbf{0}$ on $[0,2]$. □

Standard estimates of the solution of (1)–(2) and its derivatives are contained in the following

Lemma 3 *Let conditions (3) and (4) hold. Let* $\mathbf{u}$ *be the solution of (1)–(2). Then for all* $x \in [0, 2]$ *and* $i = 1, 2,$

$$|u_i^{(k)}(x)| \leq C\,\varepsilon_i^{-\frac{k}{2}}\ (||\mathbf{u}(0)|| + ||\mathbf{u}(2)|| + ||\mathbf{f}||)\,,\ \text{for } k = 0, 1 \textit{ and}$$

$$|u_i^{(k)}(x)| \leq C\,\varepsilon_1^{-\frac{(k-2)}{2}}\ \varepsilon_i^{-1}\,(||\mathbf{u}(0)|| + ||\mathbf{u}(2)|| + ||\mathbf{f}|| + \varepsilon_1^{\frac{k-2}{2}}\ ||\mathbf{f}^{(k-2)}||),\ \text{for } k = 2, 3, 4.$$

Proof The proof is by the method of steps. First, the bounds of $\mathbf{u}$ and its derivatives are estimated in [0, 1]. Next, these bounds of $\mathbf{u}$ and its derivatives are used to get the estimates in [1, 2]. Applying Lemma 3 of [9], the estimates of derivatives of $\mathbf{u}$ on [0, 1] follow and using the procedure adopted in the proof of Lemma 3 of [9], it is not hard to derive estimates of derivatives of $\mathbf{u}$ on [1, 2]. □

The Shishkin decomposition of the solution $\mathbf{u}$ of (1)–(2) is $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where the smooth component $\mathbf{v}$ is the solution of

$$\mathbf{L}_1\mathbf{v} = \mathbf{g}\ \text{ on } (0, 1),\ \ \mathbf{v}(0) = \mathbf{u}_0(0),\ \ \mathbf{v}(1-) = (A(1))^{-1}(\mathbf{f}(1) - B(1)\,\boldsymbol{\phi}(0)), \quad (10)$$

$$\mathbf{L}_2\mathbf{v} = \mathbf{f}\ \text{ on } (1, 2),\ \mathbf{v}(1+) = (A(1))^{-1}(\mathbf{f}(1) - B(1)\,\mathbf{u}_0(0)),\ \mathbf{v}(2) = \mathbf{u}_0(2) \quad (11)$$

and the singular component $\mathbf{w}$ is the solution of

$$\mathbf{L}_1\,\mathbf{w} = \mathbf{0} \text{ on } (0, 1),\ \mathbf{L}_2\,\mathbf{w} = \mathbf{0} \text{ on } (1, 2)\ \text{ with}$$

$$\mathbf{w}(0) = \mathbf{u}(0) - \mathbf{v}(0),\ \mathbf{w}(2) = \mathbf{u}(2) - \mathbf{v}(2),\ [\mathbf{w}](1) = -[\mathbf{v}](1) \text{ and } [\mathbf{w}'](1) = -[\mathbf{v}'](1). \quad (12)$$

The singular component is given a further decomposition

$$\mathbf{w}(x) = \tilde{\mathbf{w}}(x) + \hat{\mathbf{w}}(x) \quad (13)$$

where $\tilde{\mathbf{w}}$ is the solution of

$$-E\tilde{\mathbf{w}}''(x) + A(x)\tilde{\mathbf{w}}(x) = \mathbf{0} \text{ on } (0, 1),\ \tilde{\mathbf{w}}(0) = \mathbf{w}(0), \tilde{\mathbf{w}}(1) = \mathbf{K}_1, \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1, 2]$$

and $\hat{\mathbf{w}}$ is the solution of

$$-E\hat{\mathbf{w}}''(x) + A(x)\hat{\mathbf{w}}(x) + B(x)\hat{\mathbf{w}}(x-1) = \mathbf{0} \text{ on } (1, 2),$$
$$\hat{\mathbf{w}}(1) = \mathbf{K}_2,\ \hat{\mathbf{w}}(2) = \mathbf{w}(2), \hat{\mathbf{w}} = \mathbf{0} \text{ on } [0, 1).$$

Here, $\mathbf{K}_1$ and $\mathbf{K}_2$ are vector constants to be chosen in such a way that the jump conditions at $x = 1$ are satisfied.

Bounds on the smooth component and its derivatives are contained in the following

Lemma 4 *Let conditions (3) and (4) hold. Then for* $i = 1, 2$ *and for all* $x \in [0, 2]$, $|v_i^{(k)}(x)| \le C$, *for* $k = 0, 1, 2$ *and* $|v_i^{(k)}(x)| \le C(1 + \varepsilon_i^{1-\frac{k}{2}})$, *for* $k = 3, 4$.

Proof The proof is by the method of steps. Applying Lemma 4 of [9], the estimates of derivatives of $\mathbf{v}$ on $[0, 1-]$ follow. Now consider $[1+, 2]$. On this interval $\mathbf{v}$ satisfies $\mathbf{L}_2\mathbf{v}(x) = \mathbf{f}(x)$ or $\mathbf{L}_1\mathbf{v}(x) = \mathbf{f}(x) - B(x)\,\mathbf{v}(x-1)$. Using the bounds of $\mathbf{v}$ and its derivatives on $[0, 1-]$ and the procedure adopted in the proof of Lemma 4 of [9] for the operator $\mathbf{L}_1$, it is not hard to derive the estimates of derivatives of $\mathbf{v}$ on $[1+, 2]$. □

The layer functions $B^l_{1,i}, B^r_{1,i}, B^l_{2,i}, B^r_{2,i}, B_{1,i}, B_{2,i}, i = 1, 2$, associated with the solution $\mathbf{u}$, of (1)–(2), are defined by

$$B^l_{1,i}(x) = e^{-x\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B^r_{1,i}(x) = e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{1,i}(x) = B^l_{1,i}(x) + B^r_{1,i}(x),\ \text{on}\ [0, 1],$$
$$B^l_{2,i}(x) = e^{-(x-1)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B^r_{2,i}(x) = e^{-(2-x)\sqrt{\alpha}/\sqrt{\varepsilon_i}},\ B_{2,i}(x) = B^l_{2,i}(x) + B^r_{2,i}(x),$$
$$\text{on}\ [1, 2].$$

Definition 1 For $B^l_{1,1}, B^l_{1,2}$, let x^* be the point defined by $\dfrac{B^l_{1,1}(x^*)}{\varepsilon_1} = \dfrac{B^l_{1,2}(x^*)}{\varepsilon_2}$.

$$\text{Then}\quad \frac{B^r_{1,1}(1-x^*)}{\varepsilon_1} = \frac{B^r_{1,2}(1-x^*)}{\varepsilon_2},\quad \frac{B^l_{2,1}(1+x^*)}{\varepsilon_1} = \frac{B^l_{2,2}(1+x^*)}{\varepsilon_2}$$
$$\text{and}\quad \frac{B^r_{2,1}(2-x^*)}{\varepsilon_1} = \frac{B^r_{2,2}(2-x^*)}{\varepsilon_2}.$$

The existence, uniqueness and the properties of x^* can be verified as in [9, 10].

Bounds on the singular component $\mathbf{w}$ of $\mathbf{u}$ and its derivatives are contained in the following

Lemma 5 *Let conditions (3) and (4) hold. Then there exists a constant C such that for* $i = 1, 2$ *and for* $x \in [0, 1]$,

$$|w_i(x)| \le C B_{1,2}(x),\quad |w_i'(x)| \le C\sum_{q=i}^{2}\frac{B_{1,q}(x)}{\sqrt{\varepsilon_q}},\quad |w_i''(x)| \le C\sum_{q=i}^{2}\frac{B_{1,q}(x)}{\varepsilon_q},$$
$$|w_i^{(3)}(x)| \le C\sum_{q=1}^{2}\frac{B_{1,q}(x)}{\varepsilon_q^{\frac{3}{2}}},\quad |\varepsilon_i w_i^{(4)}(x)| \le C\sum_{q=1}^{2}\frac{B_{1,q}(x)}{\varepsilon_q}$$

$$\textit{and for } x \in [1,2], \quad |w_i(x)| \le C B_{2,2}(x), \quad |w_i'(x)| \le C \sum_{q=i}^{2} \frac{B_{2,q}(x)}{\sqrt{\varepsilon_q}},$$

$$|w_i''(x)| \le C \sum_{q=i}^{2} \frac{B_{2,q}(x)}{\varepsilon_q}, \ |w_i^{(3)}(x)| \le C \sum_{q=1}^{2} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{3}{2}}}, \ |\varepsilon_i w_i^{(4)}(x)| \le C \sum_{q=1}^{2} \frac{B_{2,q}(x)}{\varepsilon_q}.$$

Proof The proof is by the method of steps. First, the bounds of $\mathbf{w}$ and its derivatives are estimated in $[0, 1]$. Next, these bounds of $\mathbf{w}$ and its derivatives are used to get the estimates in $[1, 2]$.

We now derive the bound on $\mathbf{w}$ on $[0, 1]$. By using the barrier functions $\boldsymbol{\theta}^{\pm}(x) = C B_{1,2}(x)\mathbf{e} \pm \mathbf{w}(x)$, where $\mathbf{e} = (1, 1)^T$, and Lemma 1 of [9] to the operator $\mathbf{L}_1$, the estimates of $\mathbf{w}$ on $[0, 1]$ follow. By using the mean value theorem it is easy to find that $|w_1'(x)| \le C\varepsilon_1^{-1/2} B_{1,2}(x)$ and $|w_2'(x)| \le C\varepsilon_2^{-1/2} B_{1,2}(x)$. In particular $|w_i'(0)| \le C\varepsilon_i^{-1/2}$ and $|w_i'(1)| \le C\varepsilon_i^{-1/2}, i = 1, 2$.

By using the barrier functions $\boldsymbol{\theta}^{\pm}(x) = \begin{bmatrix} C_1\left(\varepsilon_1^{\frac{-1}{2}} B_{1,1}(x) + \varepsilon_2^{\frac{-1}{2}} B_{1,2}(x)\right) \\ C_2\varepsilon_2^{\frac{-1}{2}} B_{1,2}(x) \end{bmatrix} \pm \mathbf{w}'(x)$ and Lemma 1 of [9] to the operator $\mathbf{L}_1$, the estimates of $\mathbf{w}'$ on $[0, 1]$ follow. The bounds on $\mathbf{w}''$, $\mathbf{w}^{(3)}$ and $\mathbf{w}^{(4)}$ are derived by similar arguments. By using these same techniques and the bounds of $\mathbf{w}$ and its derivatives on $[0, 1]$, the bounds on $\mathbf{w}$ and its derivatives are derived on $[1, 2]$. □

3 Improved Estimates

In the following lemma sharper estimates of the smooth component are presented.

Lemma 6 *Let conditions (3) and (4) hold. Then the smooth component* $\mathbf{v}$ *of the solution* $\mathbf{u}$ *of* (1)–(2) *satisfies for* $i = 1, 2, \ k = 0, 1, 2, 3$ *and for* $x \in [0, 1]$,

$$|v_i^{(k)}(x)| \le C\left(1 + \sum_{q=i}^{2} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right) \textit{and for } x \in [1,2], \ |v_i^{(k)}(x)| \le C\left(1 + \sum_{q=i}^{2} \frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right).$$

Proof Here also the proof is by the method of steps. Applying Lemma 6 of [9], the estimates of the derivatives of $\mathbf{v}$ on $[0, 1]$ follow. Next for $x \in [1, 2]$, the bounds on the derivatives of $\mathbf{v}$ are derived using the procedure adopted in the proof of Lemma 6 of [9] and the bounds of the derivatives of $\mathbf{v}$ in the interval $[0, 1]$. □

4 The Shishkin Mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed on $[0,2]$ as follows. Let $\Omega^N = {\Omega_1}^N \cup {\Omega_2}^N$ where ${\Omega_1}^N = \{x_j\}_{j=1}^{\frac{N}{2}-1}$, ${\Omega_2}^N = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$ and $x_{\frac{N}{2}} = 1$. Then $\overline{\Omega_1}^N = \{x_j\}_{j=0}^{\frac{N}{2}}$, $\overline{\Omega_2}^N = \{x_j\}_{j=\frac{N}{2}}^{N}$, $\overline{\Omega_1}^N \cup \overline{\Omega_2}^N = \overline{\Omega}^N = \{x_j\}_{j=0}^{N}$ and $\Gamma^N = \{0,2\}$. As the solution exhibits overlapping layers at $x=0$ and $x=2$ and interior overlapping layers at $x=1$, a Shishkin mesh is constructed to resolve these layers. The interval $[0,1]$ is subdivided into 5 sub-intervals as follows $[0,\tau_1]\cup(\tau_1,\tau_2]\cup(\tau_2,1-\tau_2]\cup(1-\tau_2,1-\tau_1]\cup(1-\tau_1,1]$. The parameters $\tau_r, r=1,2,$ which determine the points separating the uniform meshes, are defined by $\tau_2 = \min\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_2}}{\sqrt{\alpha}}\ln N\}$ and $\tau_1 = \min\{\frac{\tau_2}{2}, \frac{2\sqrt{\varepsilon_1}}{\sqrt{\alpha}}\ln N\}$.

On the sub-interval $(\tau_2, 1-\tau_2]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $[0,\tau_1]$, $(\tau_1,\tau_2]$, $(1-\tau_2, 1-\tau_1]$ and $(1-\tau_1, 1]$, a uniform mesh of $\frac{N}{16}$ mesh points is placed. Similarly, the interval $(1,2]$ is also divided into 5 sub-intervals $(1,1+\tau_1]$, $(1+\tau_1, 1+\tau_2]$, $(1+\tau_2, 2-\tau_2]$, $(2-\tau_2, 2-\tau_1]$ and $(2-\tau_1, 2]$, using the same parameters τ_1 and τ_2. In particular, when both the parameters τ_1 and τ_2 take on their lefthand value, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh throughout from 0 to 2. In practice, it is convenient to take $N = 16k$, $k \geq 2$. From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau_r, 1-\tau_r, 1+\tau_r, 2-\tau_r\}, r=1,2,$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}, h_{j+1} = x_{j+1} - x_j$ and if $x_j = \tau_r$ then $h_j^- = x_j - x_{j-1}, h_j^+ = x_{j+1} - x_j$, $J = \{x_j : h_j^+ \neq h_j^-\}$.

5 The Discrete Problem

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1)–(2) which is shown later to be essentially first order parameter-uniform convergent.

The discrete two-point boundary value problem is now defined to be

$$\begin{gathered}\mathbf{L}^N\mathbf{U}(x_j) = -E\,\delta^2\mathbf{U}(x_j) + A(x_j)\,\mathbf{U}(x_j) + B(x_j)\,\mathbf{U}(x_j-1) = \mathbf{f}(x_j) \text{ on } \Omega^N, \\ \mathbf{U}(x_j-1) = \boldsymbol{\phi}(x_j-1) \text{ for } x_j \in {\Omega_1}^N \text{ and } \mathbf{U} = \mathbf{u} \text{ on } \Gamma^N.\end{gathered} \tag{14}$$

The problem (14) can be rewritten as,

$$\begin{gathered}\mathbf{L}_1^N\mathbf{U}(x_j) = -E\,\delta^2\mathbf{U}(x_j) + A(x_j)\,\mathbf{U}(x_j) = \mathbf{g}(x_j) \text{ on } {\Omega_1}^N, \\ \mathbf{L}_2^N\mathbf{U}(x_j) = -E\,\delta^2\mathbf{U}(x_j) + A(x_j)\,\mathbf{U}(x_j) + B(x_j)\,\mathbf{U}(x_j-1) = \mathbf{f}(x_j) \text{ on } {\Omega_2}^N, \\ \mathbf{U} = \mathbf{u} \text{ on } \Gamma^N, \ \ D^-\mathbf{U}(x_{N/2}) = D^+\mathbf{U}(x_{N/2}),\end{gathered} \tag{15}$$

where $\delta^2\mathbf{Y}(x_j) = \dfrac{2}{x_{j+1} - x_{j-1}}\left\{D^+\mathbf{Y}(x_j) - D^-\mathbf{Y}(x_j)\right\}$, $D^+\mathbf{Y}(x_j) = \dfrac{\mathbf{Y}(x_{j+1}) - \mathbf{Y}(x_j)}{x_{j+1} - x_j}$ and $D^-\mathbf{Y}(x_j) = \dfrac{\mathbf{Y}(x_j) - \mathbf{Y}(x_{j-1})}{x_j - x_{j-1}}$.

This is used to compute numerical approximations to the solution of (1)–(2). The following discrete results are analogous to those for the continuous case.

Lemma 7 *Let conditions (3) and (4) hold. Then, for any mesh function* $\mathbf{Y}$, *the inequalities* $\mathbf{Y} \geq \mathbf{0}$ *on* $\Gamma^N, \mathbf{L}_1^N\mathbf{Y} \geq \mathbf{0}$ *on* ${\Omega_1}^N$, $\mathbf{L}_2^N\mathbf{Y} \geq \mathbf{0}$ *on* ${\Omega_2}^N$ *and* $D^+\mathbf{Y}(x_{N/2}) - D^-\mathbf{Y}(x_{N/2}) \leq \mathbf{0}$ *imply that* $\mathbf{Y} \geq \mathbf{0}$ *on* $\overline{\Omega}^N$.

Proof Let i^*, j^* be such that $Y_{i^*}(x_{j^*}) = \min\limits_{i,j} Y_i(x_j)$ and assume that the lemma is false. Then $Y_{i^*}(x_{j^*}) < 0$. From the hypotheses it is clear that $j^* \neq 0, N$. Suppose $x_{j^*} \in {\Omega_1}^N$. $Y_{i^*}(x_{j^*}) - Y_{i^*}(x_{j^*-1}) \leq 0$, $Y_{i^*}(x_{j^*+1}) - Y_{i^*}(x_{j^*}) \geq 0$, so $\delta^2 Y_{i^*}(x_{j^*}) \geq 0$. It follows that $(\mathbf{L}_1^N\mathbf{Y})_{i^*}(x_{j^*}) < 0$, which is a contradiction. If $x_{j^*} \in {\Omega_2}^N$, a similar argument shows that $(\mathbf{L}_2^N\mathbf{Y})_{i^*}(x_{j^*}) < 0$, which is a contradiction. Because of the boundary values, the only other possibility is that $x_{j^*} = x_{N/2}$. Then $D^-Y_{i^*}(x_{N/2}) \leq 0 \leq D^+Y_{i^*}(x_{N/2}) \leq D^-Y_{i^*}(x_{N/2})$, by the hypothesis. Then $(\mathbf{L}_1^N\mathbf{Y})_{i^*}(x_{\frac{N}{2}-1}) < 0$, a contradiction. □

An immediate consequence of this is the following discrete stability result.

Lemma 8 *Let conditions (3) and (4) hold. Then, for any mesh function* $\mathbf{Y}$ *satisfying* $D^+\mathbf{Y}(x_{N/2}) = D^-\mathbf{Y}(x_{N/2})$, $|Y_i(x_j)| \leq \max\{||\mathbf{Y}(x_0)||, ||\mathbf{Y}(x_N)||, \frac{1}{\alpha} \parallel \mathbf{L}_1^N\mathbf{Y} \parallel_{{\Omega_1}^N}, \frac{1}{\alpha} \parallel \mathbf{L}_2^N\mathbf{Y} \parallel_{{\Omega_2}^N}\}$, *for each* $i = 1, 2$ *and* $0 \leq j \leq N$.

Proof Let $M = \max\{||\mathbf{Y}(x_0)||, ||\mathbf{Y}(x_N)||, \frac{1}{\alpha} \parallel \mathbf{L}_1^N\mathbf{Y} \parallel_{{\Omega_1}^N}, \frac{1}{\alpha} \parallel \mathbf{L}_2^N\mathbf{Y} \parallel_{{\Omega_2}^N}\}$. Define two functions $\mathbf{Z}^{\pm}(x_j) = M\mathbf{e} \pm \mathbf{Y}(x_j)$ where $\mathbf{e} = (1, 1)^T$. Using the properties of $A(x_j)$ and $B(x_j)$, it is not hard to find that $\mathbf{Z}^{\pm}(x_j) \geq \mathbf{0}$ for $j = 0, N$, $\mathbf{L}_1^N\mathbf{Z}^{\pm}(x_j) \geq \mathbf{0}$ for $x_j \in {\Omega_1}^N$ and $\mathbf{L}_2^N\mathbf{Z}^{\pm}(x_j) \geq \mathbf{0}$ for $x_j \in {\Omega_2}^N$. At $j = \frac{N}{2}$, $D^+\mathbf{Z}^{\pm}(x_{N/2}) - D^-\mathbf{Z}^{\pm}(x_{N/2}) = \pm(D^+\mathbf{Y}(x_{N/2}) - D^-\mathbf{Y}(x_{N/2})) = \mathbf{0}$. Hence by Lemma 7, $\mathbf{Z}^{\pm} \geq \mathbf{0}$ on $\overline{\Omega}^N$. □

6 Error Estimate

Analogous to the continuous case, the discrete solution $\mathbf{U}$ can be decomposed into $\mathbf{V}$ and $\mathbf{W}$ which are defined to be the solutions of the following discrete problems

$$\mathbf{L}_1^N\mathbf{V}(x_j) = \mathbf{g}(x_j),\ x_j \in {\Omega_1}^N,\ \mathbf{V}(0) = \mathbf{v}(0),\ \mathbf{V}(x_{N/2-1}) = \mathbf{v}(1-),$$
$$\mathbf{L}_2^N\mathbf{V}(x_j) = \mathbf{f}(x_j),\ x_j \in {\Omega_2}^N,\ \mathbf{V}(x_{N/2+1}) = \mathbf{v}(1+),\ \mathbf{V}(2) = \mathbf{v}(2)$$

and

$$\mathbf{L}_1^N\mathbf{W}(x_j) = \mathbf{0},\ x_j \in {\Omega_1}^N,\ \mathbf{W}(0) = \mathbf{w}(0),\ \ \mathbf{L}_2^N\mathbf{W}(x_j)=\mathbf{0},\ x_j \in {\Omega_2}^N,\ \mathbf{W}(2)=\mathbf{w}(2),$$
$$D^-\mathbf{V}(x_{N/2}) + D^-\mathbf{W}(x_{N/2}) = D^+\mathbf{V}(x_{N/2}) + D^+\mathbf{W}(x_{N/2}).$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $\mathbf{e}(x_j) = \mathbf{U}(x_j) - \mathbf{u}(x_j)$. Then the local truncation error $\mathbf{L}^N\mathbf{e}(x_j)$, for $j \neq N/2$, has the decomposition $\mathbf{L}^N\mathbf{e}(x_j) = \mathbf{L}^N(\mathbf{V}-\mathbf{v})(x_j) + \mathbf{L}^N(\mathbf{W}-\mathbf{w})(x_j)$. The error in the smooth and singular components are bounded in the following

Theorem 1 *Let conditions (3) and (4) hold. If* $\mathbf{v}$ *denotes the smooth component of the solution of (1)–(2) and* $\mathbf{V}$ *the smooth component of the solution of the problem (15), then, for* $i = 1, 2,\ j \neq N/2$,

$$|(\mathbf{L}_1^N(\mathbf{V}-\mathbf{v}))_i(x_j)| \le C\,(N^{-1}\,\ln N)^2,\ \ 0 \le j \le N/2 - 1, \tag{16}$$

$$|(\mathbf{L}_2^N(\mathbf{V}-\mathbf{v}))_i(x_j)| \le C\,(N^{-1}\,\ln N)^2,\ \ N/2 + 1 \le j \le N. \tag{17}$$

If $\mathbf{w}$ *denotes the singular component of the solution of (1)–(2) and* $\mathbf{W}$ *the singular component of the solution of the problem (15), then, for* $i = 1, 2,\ j \neq N/2$,

$$|(\mathbf{L}_1^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,(N^{-1}\,\ln N)^2,\ \ 0 \le j \le N/2 - 1, \tag{18}$$

$$|(\mathbf{L}_2^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,(N^{-1}\,\ln N)^2,\ \ N/2 + 1 \le j \le N. \tag{19}$$

Proof As the expression derived for the local truncation error in $\mathbf{V}$ and $\mathbf{W}$ and the estimates for the derivatives of the smooth and singular components are exactly in the form found in [9], the required bounds hold good. □

Define, for $i = 1, 2$, a set of discrete barrier functions on $\overline{\Omega}^N$ by

$$\omega_i(x_j) = \begin{cases} \dfrac{\Pi_{k=1}^{j}(1 + (\sqrt{\alpha}h_k/\sqrt{2\varepsilon_i}))}{\Pi_{k=1}^{N/2}(1 + (\sqrt{\alpha}h_k/\sqrt{2\varepsilon_i}))}, & 0 \le j \le N/2, \\ \\ \dfrac{\Pi_{k=j}^{N-1}(1 + (\sqrt{\alpha}h_{k+1}/\sqrt{2\varepsilon_i}))}{\Pi_{k=N/2}^{N-1}(1 + (\sqrt{\alpha}h_{k+1}/\sqrt{2\varepsilon_i}))}, & N/2 \le j \le N. \end{cases} \tag{20}$$

It is not hard to see that,

$$\omega_i(0) = 0,\ \ \omega_i(1) = 1,\ \ \omega_i(2) = 0 \text{ and } \omega_1(x_j) < \omega_2(x_j) \text{ for any } 0 \le j \le N, \tag{21}$$

$$(\mathbf{L}_1^N\boldsymbol{\omega})_i(x_j) > -\alpha\ \omega_i(x_j) + \sum_{l=1}^{i} a_{il}(x_j)\,\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j), \tag{22}$$

$$(\mathbf{L}_2^N \boldsymbol{\omega})_i(x_j) \geq -\alpha\ \omega_i(x_j)\ + \sum_{l=1}^{i} a_{il}(x_j)\,\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j) + b_i(x_j), \tag{23}$$

$$\text{and}\ \ (D^+ - D^-)\omega_i(x_{N/2}) \leq -\frac{C}{\sqrt{\varepsilon_i}}. \tag{24}$$

It is to be noted that

$$|(D^+ - D^-)e_i(x_{\frac{N}{2}})|\ \leq\ C\,\frac{h^*}{\varepsilon_i} \tag{25}$$

where $h^* = h^-_{N/2} = h^+_{N/2}$.

We now state and prove the main theoretical result of this paper.

Theorem 2 *Let* $\mathbf{u}(x_j)$ *be the solution of the problem (1)–(2) and* $\mathbf{U}(x_j)$ *be the solution of the discrete problem (14). Then,*

$$\|\ \mathbf{U}(x_j) - \mathbf{u}(x_j)\ \| \leq C N^{-1} \ln N,\ \ 0 \leq j \leq N.$$

Proof Consider $Z_i(x_j) = C_1 N^{-1} \ln N + C_2 \dfrac{h^*}{\sqrt{\varepsilon_i}} \omega_i(x_j) \pm e_i(x_j),\ i = 1, 2,\ 0 \leq j \leq N,$ where C_1 and C_2 are constants. Then,

$$[\mathbf{L}_1^N \mathbf{Z}]_i(x_j)\ =\ C_1 \sum_{l=1}^{2} a_{il}(x_j) N^{-1} \ln N + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} [\mathbf{L}_1^N \boldsymbol{\omega}\,]_i(x_j) \pm [\mathbf{L}_1^N \mathbf{e}\,]_i(x_j). \tag{26}$$

Using (22) in (26) and Theorem 1,

$$\begin{aligned}
&[\mathbf{L}_1^N \mathbf{Z}]_i(x_j)\\
&\geq\ C_1 \sum_{l=1}^{2} a_{il}(x_j) N^{-1} \ln N + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \left[-\alpha\ \omega_i(x_j) + \sum_{l=1}^{i} a_{il}(x_j)\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j) \right]\\
&\quad \pm C N^{-1}\ \ln N\\
&=\ C_1 \sum_{l=1}^{2} a_{il}(x_j) N^{-1} \ln N + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \left[\sum_{l=1}^{i} a_{il}(x_j) - \alpha \right] \omega_i(x_j) + C_2 \frac{h^*}{\sqrt{\varepsilon_i}} \sum_{l=i+1}^{2} a_{il}(x_j)\\
&\quad \pm C N^{-1}\ \ln N.
\end{aligned}$$

Let $\lambda_i(x_j) = \left(\sum_{l=1}^{i} a_{il}(x_j) - \alpha\right)\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j),\ i = 1, 2$. Then choosing $C_1 > C_2||\boldsymbol{\lambda}|| + C,\ [\mathbf{L}_1^N\mathbf{Z}]_i(x_j) \geq 0,$ on $\Omega_1^N,$ for $i = 1, 2$. For $x_j \in \Omega_2^N$,

$$[\mathbf{L}_2^N\mathbf{Z}]_i(x_j) = C_1\left(\sum_{l=1}^{2} a_{il}(x_j) + b_i(x_j)\right)N^{-1}\ln N + C_2\frac{h^*}{\sqrt{\varepsilon_i}}[\mathbf{L}_2^N\boldsymbol{\omega}]_i(x_j) \pm [\mathbf{L}_2^N\mathbf{e}]_i(x_j). \tag{27}$$

Using (23) in (27) and Theorem 1,

$$\begin{aligned}
&[\mathbf{L}_2^N\mathbf{Z}]_i(x_j)\\
&\geq C_1\left(\sum_{l=1}^{2} a_{il}(x_j) + b_i(x_j)\right)N^{-1}\ln N\\
&\quad + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[-\alpha\,\omega_i(x_j) + \sum_{l=1}^{i} a_{il}(x_j)\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j) + b_i(x_j)\right] \pm CN^{-1}\ln N\\
&= C_1\left(\sum_{l=1}^{2} a_{il}(x_j) + b_i(x_j)\right)N^{-1}\ln N\\
&\quad + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[\sum_{l=1}^{i} a_{il}(x_j) - \alpha\right]\omega_i(x_j) + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\left[\sum_{l=i+1}^{2} a_{il}(x_j) + b_i(x_j)\right] \pm CN^{-1}\ln N.
\end{aligned}$$

Let $\mu_i(x_j) = \left(\sum_{l=1}^{i} a_{il}(x_j) - \alpha\right)\omega_i(x_j) + \sum_{l=i+1}^{2} a_{il}(x_j) + b_i(x_j),\ i = 1, 2$. Then choosing $C_1 > C_2||\boldsymbol{\mu}|| + C,\ [\mathbf{L}_2^N\mathbf{Z}]_i(x_j) \geq 0,$ on $\Omega_2^N,$ for $i = 1, 2$. Further,

$$\begin{aligned}
D^+Z_i(1) - D^-Z_i(1) &\leq -C_2\frac{Ch^*}{\varepsilon_i} \pm C\frac{h^*}{\varepsilon_i}, \text{ using (24) and (25)}\\
&\leq 0.
\end{aligned}$$

Also, using (21), for $i = 1, 2,\ Z_i(0) \geq 0,\ Z_i(2) \geq 0$. Therefore, using Lemma 7 for $\mathbf{Z}$, it follows that $Z_i(x_j) \geq 0$ for $i = 1, 2,\ 0 \leq j \leq N$. As, from (21), $\omega_i(x_j) \leq 1$ for $i = 1, 2,\ 0 \leq j \leq N,$ for N sufficiently large, $\| \mathbf{U} - \mathbf{u} \| \leq CN^{-1}\ln N$. □

7 Numerical Illustration

The parameter-uniform convergence of the numerical method proposed in this paper is illustrated through an example presented in this section.

Example Consider the BVP

$$-E\mathbf{u}''(x) + A(x)\mathbf{u}(x) + B(x)\mathbf{u}(x-1) = \mathbf{f}(x), \text{ for } x \in (0, 2),$$

$\mathbf{u}(x) = \mathbf{1}$, for $x \in [-1, 0]$, $\mathbf{u}(2) = \mathbf{1}$, where $E = \text{diag}(\varepsilon_1, \varepsilon_2)$, $A(x) = \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix}$, $B(x) = \text{diag}(-0.5, -0.5)$, $\mathbf{f}(x) = (1, 1)^T$.

The maximum pointwise errors and the rate of convergence for this BVP are presented in Table 1. The solution of this problem for $\varepsilon_1 = 2^{-13}$, $\varepsilon_2 = 2^{-11}$ and $N = 2048$ is portraited in Fig. 1.

Table 1 Values of maximum pointwise errors D_ε^N and D^N, order of convergence p^N, error constant C_p^N, order of $\boldsymbol{\varepsilon}$ -uniform convergence p^* and $\boldsymbol{\varepsilon}$ -uniform error constant $C_{p^*}^N$ for $\varepsilon_1 = \frac{\eta}{16}$, $\varepsilon_2 = \frac{\eta}{4}$ and $\alpha = 2.4999$

	Number of mesh points N				
η	128	256	512	1024	2048
2^0	0.398E−03	0.202E−03	0.102E−03	0.510E−04	0.255E−04
2^{-3}	0.146E−02	0.738E−03	0.369E−03	0.185E−03	0.923E−04
2^{-6}	0.442E−02	0.204E−02	0.104E−02	0.522E−03	0.261E−03
2^{-9}	0.407E−02	0.290E−02	0.178E−02	0.102E−02	0.572E−03
2^{-12}	0.407E−02	0.290E−02	0.178E−02	0.102E−02	0.572E−03
D^N	0.442E−02	0.290E−02	0.178E−02	0.102E−02	0.572E−03
p^N	0.606E+00	0.702E+00	0.799E+00	0.841E+00	
C_p^N	0.244E+00	0.244E+00	0.228E+00	0.200E+00	0.170E+00
Computed order of $\boldsymbol{\varepsilon}$-uniform convergence, $p^* = 0.6064957$					
Computed $\boldsymbol{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.2441407$					

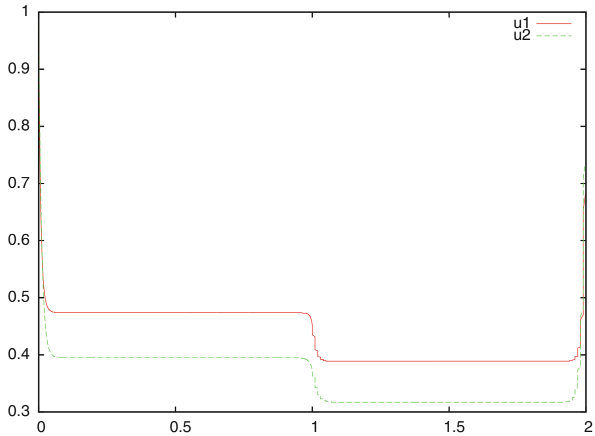

Fig. 1 Solution profile

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