Chapter 30
Numerical Solution of Stochastic Differential Equations

Stochastic differential equations (SDEs) including the geometric Brownian motion are widely used in natural sciences and engineering. In finance they are used to model movements of risky asset prices and interest rates. The solutions of SDEs are of a different character compared with the solutions of classical ordinary and partial differential equations in the sense that the solutions of SDEs are stochastic processes. Thus it is a nontrivial matter to measure the efficiency of a given algorithm for finding numerical solutions. In this chapter we introduce two methods for numerically solving stochastic differential equations. For more details consult [50, 92].

30.1 Discretization of Stochastic Differential Equations

Given an SDE

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t\ , \quad X_0 = x_0$$

defined on a time interval $[t_0, T]$, we consider its corresponding time discretization

$$Y_{n+1} = Y_n + a(t_n, Y_n)\,\Delta_n + b(t_n, Y_n)\,\Delta W_n\ , \quad n = 0, 1, \cdots, N-1$$

where $t_0 < t_1 < \cdots < t_N = T$, $\Delta_n = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$, and study iterative algorithms to find numerical solutions.

To plot a sample path for Y_t on an interval $t \in [t_0, T]$ we plot a piecewise linear function defined by

$$Y_t = Y_{t_n} + \frac{t - t_n}{t_{n+1} - t_n}(Y_{t_{n+1}} - Y_{t_n})\ , \quad t_n \le t \le t_{n+1}\ ,$$

which reflects the nondifferentiability of a sample path of X_t.

Definition 30.1 (Strong Convergence) Let $\delta = \max_n |t_{n+1} - t_n|$. Suppose that an SDE for X_t has a discrete solution Y_n such that there exists $\delta_0 > 0$, $\gamma > 0$ and $K > 0$ for which

$$\mathbb{E}[|X_T - Y_N|] \le K\delta^\gamma \quad \text{for every } 0 < \delta < \delta_0 .$$

Then we say that Y converges to X in the *strong* sense and call γ the order of strong convergence.

Definition 30.2 (Weak Convergence) Let $\delta = \max_n |t_{n+1} - t_n|$. Suppose that an SDE for X_t has a discrete solution Y_n such that there exists $\delta_0 > 0$, $\beta > 0$ and $K > 0$ for which

$$|\mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_N)]| \le K\delta^\beta \quad \text{for every } 0 < \delta < \delta_0$$

for an arbitrary nice function g such as a polynomial or a piecewise linear function. Then we say that Y converges to X in the *weak sense* and call β the order of weak convergence.

Remark 30.1 (i) If we take $g(x) = x$ and $g(x) = (x - \mathbb{E}[X_T])^2$ in the previous definition of weak convergence, we can obtain the average and variance of X_T, respectively.

  (ii) Let $\phi(x)$ be a convex function. Then, by Jensen's inequality we have $\mathbb{E}[\phi(X)] \ge \phi(\mathbb{E}[X])$. If we take $\phi(x) = |x|$, then we obtain

$$\mathbb{E}[|X_T - Y_N|] \ge |\mathbb{E}[X_T - Y_N]| = |\mathbb{E}[X_T] - \mathbb{E}[Y_N]| .$$

 (iii) In many applications we need not find the values of X_t for all $0 \le t \le T$. For example, to compute the price of a European option where C_T denotes the payoff function at maturity T it suffices to know $C_T(X_T)$. That is, it is enough to consider the weak convergence speed.

 (iv) Since the root mean square of ΔW_n is not δ but $\delta^{1/2}$, the discrete approximate solution of an SDE has a smaller order of convergence than the discrete approximate solution of an ordinary differential equation, in general.

  (v) Consider a computer simulation for strong convergence where $t_0 = 0$. We take the time step $\delta = \frac{T}{N}$, and obtain a discrete solution Y and its values at T, denoted by Y_T^j, $1 \le j \le J$, and compute

$$\varepsilon(\delta) = \frac{1}{J} \sum_{j=1}^{J} |X_T^j - Y_N^j|$$

and finally plot the graph of $-\log \varepsilon(\delta)$ against $-\log \delta$ for the values $\delta = 2^{-3}, 2^{-4}, 2^{-5}$, and so on.

From now on we treat only the case when the functions a and b do not depend on t.

30.2 Stochastic Taylor Series

In this section we present the Taylor series expansion of a stochastic process given by an SDE $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$. If $a(t, x) = a(t)$ and $b(t, x) = 0$, then we have the usual Taylor series expansion. In the following discussion, for the sake of notational simplicity, we consider the case when $a(t, x)$ and $b(t, x)$ are functions of x only.

30.2.1 Taylor Series for an Ordinary Differential Equation

Given a sufficiently smooth function $a(x) : \mathbb{R} \to \mathbb{R}$, we consider a one-dimensional autonomous ordinary differential equation $\frac{d}{dt} X_t = a(X_t)$ on the time interval $[t_0, T]$ which has a solution X_t with an initial data X_{t_0}. We may rewrite the equation as

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds \; . \tag{30.1}$$

Given a C^1 function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{d}{dt}(f(X_t)) = f'(X_t)\frac{d}{dt}X_t = a(X_t)\frac{\partial}{\partial x}f(X_t)$$

by the chain rule. If we let L be a differential operator defined by

$$L = a(x)\frac{\partial}{\partial x} \; ,$$

then

$$\frac{d}{dt}(f(X_t)) = Lf(X_t) \; .$$

Equivalently,

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} Lf(X_s)\, ds \; . \tag{30.2}$$

If we substitute $f(x) = a(x)$ in (30.2), then we obtain

$$a(X_t) = a(X_{t_0}) + \int_{t_0}^t La(X_s)\, ds \; . \tag{30.3}$$

Substituting (30.3) back into (30.1), we have

$$X_t = X_{t_0} + \int_{t_0}^t \left(a(X_{t_0}) + \int_{t_0}^s La(X_z)dz \right) ds$$

$$= X_{t_0} + a(X_{t_0})\,(t - t_0) + \int_{t_0}^t \int_{t_0}^s La(X_z)\, dz\, ds \; . \tag{30.4}$$

Similarly, if we substitute $f = La$ in (30.2) then we obtain

$$La(X_t) = La(X_{t_0}) + \int_{t_0}^t L^2 a(X_u)\, du \; , \tag{30.5}$$

and, by substituting (30.5) back into (30.4), we obtain

$$X_t = X_{t_0} + a(X_{t_0}) \int_{t_0}^t ds + \int_{t_0}^t \int_{t_0}^s \left(La(X_{t_0}) + \int_{t_0}^z L^2 a(X_u)du \right) dz\, ds$$

$$= X_{t_0} + a(X_{t_0})\,(t - t_0) + \frac{1}{2}La(X_{t_0})\,(t - t_0)^2 + R(t_0; t) \; , \tag{30.6}$$

where

$$R(t_0; t) = \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^2 a(X_u)\, du\, dz\, ds \; .$$

The idea is to keep on substituting the nth order approximation of X_t into the original equation (30.1) to obtain the $(n + 1)$-st order approximation.

30.2.2   Taylor Series for a Stochastic Differential Equation

Now we consider the Taylor series expansion for an SDE

$$dX_t = a(X_t)\, dt + b(X_t)\, dW_t \; . \tag{30.7}$$

By Itô's lemma, we have

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial f}{\partial x}(X_s) + \frac{1}{2}b^2(X_s)\frac{\partial^2 f}{\partial x^2}(X_s) \right) ds$$

$$+ \int_{t_0}^{t} b(X_s)\frac{\partial f}{\partial x}(X_s)\, dW_s$$

$$= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)\, ds + \int_{t_0}^{t} L^1 f(X_s)\, dW_s \tag{30.8}$$

where

$$L^0 = a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2} \; , \quad L^1 = b\frac{\partial}{\partial x} \; .$$

If we take $f(x) = x$ in (30.8), we have

$$L^0 f = a \; , \quad L^1 f = b$$

and recover the original equation (30.7) rewritten in the integral form as

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)\, ds + \int_{t_0}^{t} b(X_s)\, dW_s \; . \tag{30.9}$$

We take

$$f = a \; , \quad f = b$$

in (30.8) and obtain

$$a(X_t) = a(X_{t_0}) + \int_{t_0}^{t} L^0 a(X_s)\, ds + \int_{t_0}^{t} L^1 a(X_s)\, dW_s \tag{30.10}$$

and

$$b(X_t) = b(X_{t_0}) + \int_{t_0}^{t} L^0 b(X_s)\, ds + \int_{t_0}^{t} L^1 b(X_s)\, dW_s \; . \tag{30.11}$$

We substitute (30.10) and (30.11) into (30.9), and obtain

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_u)\, du + \int_{t_0}^{s} L^1 a(X_u)\, dW_u \right) ds$$

$$+ \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_u)\, du + \int_{t_0}^{s} L^1 b(X_u)\, dW_u \right) dW_s$$

$$= X_{t_0} + a(X_{t_0})\,(t - t_0) + b(X_{t_0})\,(W_t - W_{t_0}) + R(t_0; t) \tag{30.12}$$

where

$$R(t_0; t) = \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_u) \, du \, ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_u) \, dW_u \, ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_u) \, du \, dW_s + \int_{t_0}^{t} \int_{t_0}^{s} L^1 b(X_u) \, dW_u \, dW_s .$$

From (30.8) we have

$$f(X_t) = f(X_{t_0}) + L^0 f(X_{t_0}) \int_{t_0}^{t} ds + L^1 f(X_{t_0}) \int_{t_0}^{t} dW_s$$

$$+ c(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s_2} dW_{s_1} dW_{s_2} + R(f, t_0; t)$$

where

$$c(x) = b(x) \left\{ b(x) f''(x) + b'(x) f'(x) \right\} . \qquad (30.13)$$

Note that

$$\int_{t_0}^{t} \int_{t_0}^{s_2} dW_{s_1} dW_{s_2} = \int_{t_0}^{t} (W_{s_2} - W_{t_0}) \, dW_{s_2}$$

$$= \int_{t_0}^{t} W_{s_2} \, dW_{s_2} - W_{t_0} \int_{t_0}^{t} dW_{s_2}$$

$$= \frac{1}{2} \{ W_t^2 - W_{t_0}^2 - (t - t_0) \} - W_{t_0} (W_t - W_{t_0})$$

$$= \frac{1}{2} \{ (W_t - W_{t_0})^2 - (t - t_0) \} .$$

Hence (30.13) becomes

$$f(X_t) = f(X_{t_0}) + L^0 f(X_{t_0})(t - t_0) + L^1 f(X_{t_0})(W_t - W_{t_0})$$

$$+ \frac{1}{2} c(X_{t_0}) \{ (W_t - W_{t_0})^2 - (t - t_0) \} + R(f, t_0; t) . \qquad (30.14)$$

If we take $f(x) = x$ in (30.14), then

$$X_t = X_{t_0} + a(X_{t_0}) (t - t_0) + b(X_{t_0}) (W_t - W_{t_0})$$

$$+ \frac{1}{2} b(X_{t_0}) b'(X_{t_0}) \{ (W_t - W_{t_0})^2 - (t - t_0) \} + R(f, t_0; t). \qquad (30.15)$$

30.3   The Euler Scheme

Consider an SDE

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t\,, \quad X_0 = x_0$$

defined on the time interval $[t_0, T]$. Here, for the sake of notational simplicity, we consider the case when $a(t, x)$ and $b(t, x)$ are functions of x only. The *Euler scheme* is a numerical method based on the approximation given by (30.12), after truncation of the remainder term $R(t_0; t)$, to find a numerical solution of

$$Y_{n+1} = Y_n + a(Y_n)\Delta_n + b(Y_n)\Delta W_n\,, \quad n = 0, 1, \ldots, N-1$$

at $0 = t_0 < t_1 < \cdots < t_N = T$, where $Y_0 = x_0$, $\Delta_n = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. The increment ΔW_n has normal distribution with average 0 and variance Δ_n. The increments ΔW_n are independent of each other and obtained by random number generators in computer simulations.

If a and b are bounded and Lipschitz continuous, then the Euler scheme has strong order $\gamma = 0.5$. On the other hand, the strong order of a discretized numerical solution of an ordinary differential equation is equal to 1. The weak order of the Euler scheme is equal to $\beta = 1$.

In Fig. 30.1 is plotted the speed of numerical approximation by the Euler scheme for geometric Brownian motion

$$\mathrm{d}S_t = \mu\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t$$

with $\mu = 0.5$, $\sigma = 0.6$. The points $(n, -\log \varepsilon_n)$, $4 \le n \le 10$, are plotted. In this case, since we have a closed form solution, we can compare the numerical solution obtained by the Euler scheme with the theoretical solution, where we take time step $\delta = 2^{-n}$, $4 \le n \le 10$, and sample size 10^4.

In the case of strong convergence the error satisfies $\varepsilon \approx K\delta^\gamma$, and hence we have

$$-\log_2 \varepsilon \approx -\log_2 K + \gamma\,n\,.$$

Thus the slope of the regression line is approximately equal to γ if we plot $-\log_2 \varepsilon$ for each n. In the case of weak convergence the slope is approximately equal to β. In Fig. 30.1 we observe that the slope in the first graph is close to 0.5 and in the second graph the slope is approximately equal to 1, and thus the speed of convergence to zero is exponential.
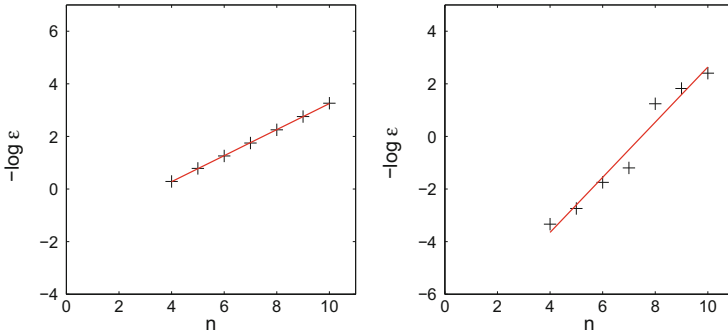
Fig. 30.1 The Euler Scheme: speeds of strong convergence (*left*) and weak convergence (*right*)
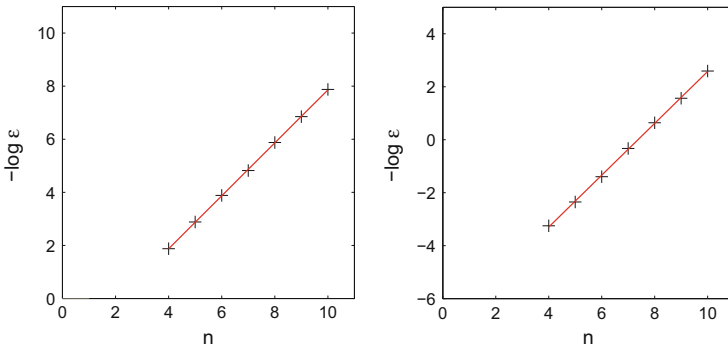
Fig. 30.2 The Milstein Scheme: speeds of strong convergence (*left*) and weak convergence (*right*)

30.4   The Milstein Scheme

The *Milstein scheme* is a numerical method based on the approximation given by (30.15), after truncation of the remainder term $R(f, t_0; t)$, to find a numerical solution of

$$Y_{n+1} = Y_n + a(Y_n)\Delta_n + b(Y_n)\Delta W_n + \frac{1}{2}b(Y_n)\,b'(Y_n)\left\{(\Delta W_n)^2 - \Delta_n\right\}$$

for $n = 0, 1, \ldots, N-1$ at $0 = t_0 < t_1 < \cdots < t_N = T$, where $b'(x)$ denotes the derivative of $b(x)$ with respect to x. It was named after Grigori N. Milstein [69].

If $\mathbb{E}[(X_0)^2] < \infty$ and if a and b are twice continuously differentiable and the second order derivatives are Lipschitz continuous, then the order of strong convergence of the Milstein scheme is $\gamma = 1.0$. The order of weak convergence is also $\beta = 1.0$.

Figure 30.2 displays numerical results from the Milstein scheme for geometric Brownian motion with $\mu = 0.5$, $\sigma = 0.6$. The sample size is 10^4. Observe that the slopes are approximately equal to 1 in both cases.

30.5   Computer Experiments

Simulation 30.1 (Euler Scheme: Weak Convergence)

We test the speed of weak convergence of the Euler scheme for solving the geometric Brownian motion when we take $g(x) = (x - \mathbb{E}[X_T])^2$.

```
T = 1;
N = [2^4,2^5,2^6,2^7,2^8,2^9,2^10];
J = 10^4;
mu = 0.5;
sigma = 0.6;
X_0 = 10;

X_T=zeros(1,J);
Y_N=zeros(1,J);
for n=1:length(N)
    dt = T/N(n);
    for j=1:J
        W(1) = 0;
        Y(1) = X_0;
        for i=1:N(n)
            dW = sqrt(dt)*randn;
            W(i+1) = W(i) + dW;
            Y(i+1) = Y(i) + mu*Y(i)*dt + sigma*Y(i)*dW;
        end
        Y_N(j) = Y(N(n)+1);
        X_T(j) = X_0*exp((mu - 0.5*sigma^2)*T + sigma*W(N(n)+1));
    %epsilon(n) = abs(mean(X_T) - mean(Y_T));
    epsilon(n) = abs(var(X_T) - var(Y_N));
    end
end
line_fit = polyfit(log(N)/log(2),-log(epsilon)/log(2),1)

plot(log(N)/log(2),-log(epsilon)/log(2),'+')
hold on
plot(log(N)/log(2),line_fit(1)*log(N)/log(2) + line_fit(2), ':')
```

Simulation 30.2 (Milstein Scheme: Strong Convergence)

We test the speed of strong convergence of the Milstein scheme for solving the geometric Brownian motion.

```
T = 1;
N = [2^4,2^5,2^6,2^7,2^8,2^9,2^10];
J = 10^4;
Error=zeros(1,J);
mu = 0.5;
sigma = 0.6;

X_0 = 10;
for n=1:length(N)
    dt = T/N(n);
    t = [0:dt:T];
        for j=1:J
```

```
                W(1) = 0;
                Y(1) = X_0;
                for i=1:N(n)
                dW = sqrt(dt)*randn;
                W(i+1) = W(i)+dW;
                Y(i+1) =Y(i)+mu*Y(i)*dt+sigma*Y(i)*dW+sigma^2/2*Y(i)*(dW^2-dt);
                end
            X_T = X_0*exp((mu - 0.5*sigma^2)*T + sigma*W(N(n)+1));
            Error(j) = abs(X_T - Y(N(n)+1));
            end
        epsilon(n) = mean(Error);
    end
    line_fit = polyfit(log(N)/log(2),-log(epsilon)/log(2),1)

    plot(log(N)/log(2),-log(epsilon)/log(2),'+')
    hold on
    plot(log(N)/log(2),line_fit(1)*log(N)/log(2) + line_fit(2), ':')
```

Exercises

30.1 Let $\delta t = \frac{T}{L}$ and $t_i = i\,\delta t$. Consider the geometric Brownian motion

$$\frac{S(t_{i+1}) - S(t_i)}{S(t_i)} = \mu\,\delta t + \sigma\sqrt{\delta t}\,Y_i$$

where μ, σ are positive constants and $Y_0, Y_1, Y_2, \ldots$ are independent standard normal variables.

 (i) What is the distribution of $\log\left(\dfrac{S(t)}{S_0}\right)$? Justify your answer.

(ii) Find

$$\lim_{\delta t \to 0+} \frac{1}{\delta t}\,\mathbb{E}\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^2\right].$$

30.2 Compare the exact solution obtained in Problem 12.3 for the SDE

$$dX_t = dt + 2\sqrt{X_t}\,dW_t$$

with a numerical solution obtained by the Milstein scheme.

30.3 Compare the exact solution obtained in Problem 12.4 for the SDE

$$dX_t = -X_t(2\log X_t + 1)dt + 2X_t\sqrt{-\log X_t}\,dW_t$$

with a numerical solution obtained by the Milstein scheme.