Chapter 7
Nonlocal BVPs and the Discrete Fractional Calculus

7.1 Introduction

In this chapter we discuss the concept of a nonlocal boundary value problem in the context of the discrete fractional calculus. More generally, we discuss how the nonlocal structure of the discrete fractional difference and sum operators affect their interpretation and analysis. In particular, we recall from earlier chapters that the fractional difference and sum contain *de facto* nonlocalities. For example, in the case of the discrete fractional forward difference, we have that $\Delta_a^\nu y(t)$, for $N-1 < \nu \leq N$ with $N \in \mathbb{N}$, depends not only on the value $y(t+\nu-N)$ but also on the entire collection of values $\{y(a), y(a+1), \ldots, y(t+\nu-N)\}$, for each $t \in \mathbb{Z}_{a+N-\nu}$. This means that the discrete fractional operator in some sense possesses a memory-like property, wherein the operator at a point is influenced by a linear combination of values of y back to the initial time point $t = a$ itself.

The nonlocal nature described in the preceding paragraph seriously complicates the study of many potentially fundamental properties of fractional sums and differences. For example, there is, at present, no satisfactory understanding of the geometrical properties of the fractional difference. Contrast this with the integer-order setting, i.e., $\nu \in \mathbb{N}$, in which there is a complete understanding of the various geometrical implications of the sign of the fractional difference. Thus, while it is trivial to prove that $\Delta y(t) > 0$ for $t \in \mathbb{Z}$ implies that y is strictly increasing on $\mathbb{Z}$, it is very *nontrivial* to decide how monotonicity is connected to the positivity or negativity of the fractional difference. Similarly, while it is equally trivial to prove that $\Delta^2 y(t) > 0$ for $t \in \mathbb{Z}$ implies that Δy is strictly increasing on $\mathbb{Z}$ and thus that y satisfies a convexity-type property, the analogue of this sort of result in the discrete fractional setting is much more difficult to obtain, and we only explore these properties to a limited extent in Sects. 7.2 and 7.3 in the sequel. And, of course, issues of monotonicity and convexity are hardly the only properties affected

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by the nonlocal structure of the fractional difference. The analysis of boundary value problems, for example, is also widely affected and complicated by this inherent nonlocality.

All in all, in this section we provide a collection of applications involving the nonlocal structure of the fractional difference. We also consider the problem of analyzing nonlocal boundary value problems within the context of the discrete fractional calculus. While the latter is not necessarily explicitly affected by the nonlocal nature of the fractional sum and difference operators, it does demonstrate in what way the existence of explicit nonlocalities in a boundary value problem can complicate the analysis of the problem, much as the implicit nonlocalities in the discrete fractional operates complicate the analysis of the geometrical properties of these operators.

7.2 A Monotonicity Result for Discrete Fractional Differences

The first result we present demonstrates that the discrete fractional difference satisfies a particular monotonicity condition—note that the results of this section can mostly be found in Dahal and Goodrich [67]. Roughly stated, see Theorem 7.1 for a precise statement, the main result of this section can be summarized as follows: Given $\nu \in (1, 2)$ and a map $y : \mathbb{N}_0 \to \mathbb{R}$ satisfying

- $y(t) \geq 0$, for each $t \in \mathbb{N}_0$;
- $\Delta y(0) \geq 0$; and
- $\Delta^\nu y(t) \geq 0$, for each $t \in \mathbb{N}_{2-\nu}$;

then it holds that y is increasing on its domain.

Recalling that

$$\Delta_a^\nu y(t) := \Delta^N \Delta^{\nu-N} y(t) = \Delta^N \underbrace{\left[\frac{1}{\Gamma(N-\nu)} \sum_{s=a}^{t+\nu-N} (t-s-1)^{\underline{N-\nu-1}} y(s) \right]}_{:=\Delta^{\nu-N} y(t)},$$

this result does not seem to be immediately apparent from the definition of the fractional difference. Hence, it is not obvious that the fractional order difference behaves in this way, and it highlights one of the consequences of the nonlocal structure of the fractional difference operator. In addition, that this monotonicity result holds implies some other nontrivial consequences, and we shall detail a few of these toward the conclusion of this section.

We now state and prove the monotonicity result. Observe that the proof of this result is based upon the principle of strong induction. Moreover, the reader should observe the way in which the nonlocal structure of Δ_0^ν is explicitly utilized in the proof.

Theorem 7.1. *Let* $y : \mathbb{N}_0 \to \mathbb{R}$ *be a nonnegative function satisfying* $y(0) = 0$. *Fix* $\nu \in (1, 2)$ *and suppose that* $\Delta_0^\nu y(t) \geq 0$ *for each* $t \in \mathbb{N}_{2-\nu}$. *Then* y *is increasing on* $\mathbb{N}_0$.

Proof. We prove this result by the principle of strong induction. To this end, observe that the base case holds somewhat trivially since we calculate

$$y(1) - y(0) = y(1) \geq 0,$$

due to the fact that $y(0) = 0$, by assumption, and the fact that $y(1) \geq 0$, also by assumption.

Now, to complete the induction step fix $k \in \mathbb{N}$ and suppose that

$$\Delta y(j-1) = y(j) - y(j-1) \geq 0,$$

for each $1 \leq j \leq k-1$. Recall that $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$, which by means of Lemma 2.33 implies that

$$\begin{aligned}
-\Delta_0^\nu y(2-\nu) &= \nu y(1) - y(2) \leq 0 \\
-\Delta_0^\nu y(3-\nu) &= \frac{1}{2}\nu(1-\nu)y(1) + \nu y(2) - y(3) \leq 0 \\
-\Delta_0^\nu y(4-\nu) &= \frac{1}{6}\nu(1-\nu)(2-\nu)y(1) + \frac{1}{2}\nu(1-\nu)y(2) + \nu y(3) - y(4) \\
&\leq 0 \\
&\vdots \\
-\Delta_0^\nu y(k-\nu) &= \frac{1}{(k-1)!}\nu(1-\nu)\cdots(k-2-\nu)y(1) + \cdots \\
&\quad + \nu y(k-1) - y(k) \leq 0,
\end{aligned} \tag{7.1}$$

for fixed $k \in \mathbb{N}$; note that in (7.1) we have used the assumption that $y(0) = 0$ to simplify suitably $\Delta^\nu y(j-\nu)$ for each j. In particular, (7.1) implies that

$$y(k) \geq \frac{1}{(k-1)!}\nu(1-\nu)\cdots(k-2-\nu)y(1) + \cdots + \nu y(k-1) \tag{7.2}$$

for fixed $k \in \mathbb{N}$. Inequality (7.2) shall be used repeatedly in the sequel.

We claim that for the value of k fixed at the beginning of the preceding paragraph

$$y(k) - y(k-1) \geq 0, \tag{7.3}$$

which will complete the induction step. To prove (7.3) we first calculate, by using estimate (7.2),

$$
\begin{aligned}
&y(k) - y(k-1) \\
&\geq \nu y(k-1) + \frac{1}{2}\nu(1-\nu)y(k-2) + \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-3) \\
&\quad + \cdots + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(1) - y(k-1) \\
&= (\nu-1)y(k-1) + \frac{1}{2}\nu(1-\nu)y(k-2) + \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-3) \\
&\quad + \cdots + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(1) \\
&= (\nu-1)y(k-1) \\
&\quad + \left(\frac{1}{2}\nu(1-\nu)y(k-2) - \frac{1}{2}\nu(1-\nu)y(k-1)\right) + \frac{1}{2}\nu(1-\nu)y(k-1) \\
&\qquad + \left(\frac{1}{6}\nu(1-\nu)(2-\nu)y(k-3) - \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-1)\right) \\
&\qquad\quad + \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-1) \\
&\quad \vdots \\
&\quad + \left(\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(1)\right. \\
&\qquad \left. - \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(k-1)\right) \\
&\qquad\qquad + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(k-1).
\end{aligned}
\tag{7.4}
$$

On the other hand, invoking the induction hypothesis implies that

$$
\begin{aligned}
\underbrace{\frac{1}{2}\nu(1-\nu)}_{<0}\underbrace{(-y(k-1)+y(k-2))}_{\leq 0} &\geq 0 \\
\underbrace{\frac{1}{6}\nu(1-\nu)(2-\nu)}_{<0}\underbrace{(-y(k-1)+y(k-3))}_{\leq 0} &\geq 0 \\
&\vdots \\
\underbrace{\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)}_{<0}\underbrace{(-y(k-1)+y(1))}_{\leq 0} &\geq 0.
\end{aligned}
\tag{7.5}
$$

Observe that in (7.5) we are using the fact that since $y(k-1) \geq y(k-2)$ it follows that $y(k-1) - y(k-3) \geq y(k-2) - y(k-3) \geq 0$, and so forth. In any case, putting the $k-2$ estimates in (7.5) into inequality (7.4) yields

$$\begin{aligned} & y(k) - y(k-1) \\ & \geq \left[(\nu - 1) + \frac{1}{2}\nu(1-\nu) + \frac{1}{6}\nu(1-\nu)(2-\nu) + \dots \right. \\ & \qquad \left. + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu) \right] y(k-1). \end{aligned}$$

Since $y(k-1) \geq 0$ by assumption, to complete the proof it suffices to show that

$$\begin{aligned} & (\nu - 1) + \frac{1}{2}\nu(1-\nu) + \frac{1}{6}\nu(1-\nu)(2-\nu) \\ & \qquad + \cdots + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu) \geq 0, \end{aligned}$$

for each $1 < \nu < 2$. To complete this final step define the $(k-1)$-th degree polynomial function $P_{k-1} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} P_{k-1}(\nu) := & (\nu - 1) + \frac{1}{2}\nu(1-\nu) + \frac{1}{6}\nu(1-\nu)(2-\nu) + \dots \\ & + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu). \end{aligned}$$

Then, for example,

$$\frac{2P_{k-1}(\nu)}{1-\nu} = (\nu - 2) + \frac{1}{3}\nu(2-\nu) + \cdots + \frac{2}{(k-1)!}\nu(2-\nu)\cdots(k-2-\nu).$$

And, moreover,

$$\begin{aligned} \frac{6P_{k-1}(\nu)}{(1-\nu)(2-\nu)} = & (\nu - 3) + \frac{1}{4}\nu(3-\nu) + \cdots \\ & + \frac{6}{(k-1)!}\nu(3-\nu)\cdots(k-2-\nu). \end{aligned}$$

Continuing in this fashion we eventually arrive at

$$\frac{(k-1)!P_{k-1}(\nu)}{(1-\nu)(2-\nu)\cdots(k-2-\nu)} = -(k-1-\nu),$$

whence

$$P_{k-1}(\nu) = \frac{-1}{(k-1)!}(1-\nu)(2-\nu)\cdots(k-2-\nu)(k-1-\nu). \tag{7.6}$$

But then (7.6) implies that

$$\begin{aligned} P_{k-1}(\nu) &= -\frac{1}{(k-1)!}(1-\nu)(2-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &= \frac{1}{(k-1)!}(-1)^2(\nu-1)(2-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &= \frac{1}{(k-1)!}(-1)^3(\nu-1)(\nu-2)(3-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &\ \vdots \\ &= \frac{1}{(k-1)!}(-1)^k(\nu-1)(\nu-2)\cdots(\nu-k+2)(\nu-k+1). \end{aligned} \tag{7.7}$$

The factorization of P_{k-1} given by (7.7) implies that P_{k-1} has $k-1$ distinct zeros and these zeros are, in particular, $\nu = 1, 2, \ldots, k-1$. In particular, observe that when k is even, it follows that $P_{k-1}(\nu) > 0$, for each $\nu \in (1, 2)$, since $(-1)^k > 0$ and the other factors will constitute a product of $k-2$ negative numbers and exactly one positive number. On the other hand, when k is odd, it follows that $P_{k-1}(\nu) > 0$, for each $\nu \in (1, 2)$, since $(-1)^k < 0$ and the other factors will once again constitute a product of $k-2$ negative numbers and exactly one positive number.

We conclude that for each $k \in \mathbb{N}$ it follows that $P_{k-1}(\nu) > 0$ whenever $\nu \in (1, 2)$. And this implies that (7.3) holds. Since this completes the induction step, we obtain that y is increasing for $k \in \mathbb{N}$ and $\nu \in (1, 2)$, and this completes the proof. □

Having proved the case where $y(0) = 0$, it is easy to generalize this to the case where $y(0) \geq 0$. We state this generalization as Corollary 7.2. It turns out that this generalization will be useful in the next section when we consider concavity and convexity properties of the fractional difference operator.

Corollary 7.2. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function. Fix $\nu \in (1, 2)$ and suppose that $\Delta_0^\nu y(t) \geq 0$, for each $t \in \mathbb{N}_{2-\nu}$. If $\Delta y(0) \geq 0$, then y is increasing on $\mathbb{N}_0$.*

Proof. Define the function $\tilde{y} : \mathbb{Z}_{-1} \to \mathbb{R}$ by $\tilde{y}(t) := y(t)$, if $t \neq -1$, and $\tilde{y}(-1) := 0$. Then we may apply Theorem 7.1 to $\tilde{y}$ on $\mathbb{Z}_{-1}$ and obtain that $\tilde{y}$ is increasing on $\mathbb{Z}_{-1}$; observe that it can be shown that $\Delta_{-1}^\nu \tilde{y} \equiv \Delta_0^\nu y$ so that $\Delta_{-1}^\nu \tilde{y}(t) \geq 0$ holds. Thus, y is increasing on $\mathbb{N}_0$, as desired. □

Remark 7.3. It is important to point out that the original version of the paper by Dahal and Goodrich [67], in which the monotonicity results for discrete fractional operators first appeared, contained a minor error in one result. In particular, [67, Corollary 2.3] is missing the hypothesis that $\Delta y(0) \geq 0$ holds.

As it may be instructive to see why this error occurs, let us briefly explain the problem. So, to see why we cannot deduce the monotonicity of y from the hypothesis $\Delta_0^\nu y(t) \geq 0$, $t \in \mathbb{Z}_{2-\nu}$, alone, we recall that the proof of the corollary in the original paper employed the map $\tilde{y} : \mathbb{Z}_{-1} \to \mathbb{R}$ defined by

$$\tilde{y}(t) := \begin{cases} y(t), & t \in \mathbb{N}_0 \\ 0, & t \in \{-1\} \end{cases}. \tag{7.8}$$

Note that in (7.8) the map y is the same map as in the statement of Corollary 7.2 above. The goal of the proof in [67, Corollary 2.3] was to show that $\Delta_{-1}^{\nu}\tilde{y}(t) \geq 0$, for each $t \in \mathbb{Z}_{1-\nu}$, so that we could use [67, Theorem 2.2] to conclude that $\tilde{y}$ and, hence, y was increasing.

To see why this does not work quite as intended, observe that

$$\begin{aligned}
\Delta_{-1}^{\nu}\tilde{y}(1-\nu) &= \frac{1}{\Gamma(-\nu)} \sum_{s=-1}^{1} ((1-\nu)-s-1)^{\underline{-\nu-1}}\tilde{y}(s) \\
&= \frac{1}{\Gamma(-\nu)}\Big[(-\nu+1)^{\underline{-\nu-1}}\tilde{y}(-1) + (-\nu)^{\underline{-\nu-1}}\tilde{y}(0) \\
&\qquad\qquad + (-\nu-1)^{\underline{-\nu-1}}\tilde{y}(1)\Big] \\
&= -\frac{1}{2}\nu(-\nu+1)\tilde{y}(-1) - \nu\tilde{y}(0) + \tilde{y}(1) \\
&= -\nu y(0) + y(1).
\end{aligned} \tag{7.9}$$

Thus, (7.9) shows us that without additional information about $y(0)$ and $y(1)$, we cannot deduce that $\Delta_{-1}^{\nu}\tilde{y}(1-\nu) \geq 0$. Moreover, while it is true that $\Delta_{-1}^{\nu}\tilde{y}(k-\nu) \equiv \Delta_0^\nu y(k-\nu)$, for each $k \in \mathbb{N}_2$, this does not force $\Delta_{-1}^{\nu}\tilde{y}(1-\nu)$ to be nonnegative, as shown by (7.9) above. This is the basis of the error.

Finally, suppose that $\Delta y(0) \geq 0$, which is the necessary additional hypothesis as noted above. By a calculation similar to (7.9) we find that

$$\Delta_0^\nu y(2-\nu) = -\frac{1}{2}\nu(-\nu+1)y(0) - \nu y(1) + y(2) \geq 0. \tag{7.10}$$

Thus, combining (7.10) with the fact that $\Delta y(0) \geq 0$ we estimate

$$\Delta y(1) \geq (\nu-1)y(1) + \frac{1}{2}\nu(1-\nu)y(0) \geq (\nu-1)\left[1-\frac{1}{2}\nu\right]y(0). \tag{7.11}$$

Since the map $\nu \mapsto (\nu-1)\left[1-\frac{1}{2}\nu\right]$ is nonnegative for $\nu \in (1,2)$, we obtain from (7.11) that $\Delta y(1) \geq 0$. Finally, proceeding inductively from inequality (7.11), we obtain the monotonicity of $\tilde{y}$ on $\mathbb{Z}_{-1}$ and thus of y on $\mathbb{N}_0$.

Recall that in the statement of both Theorem 7.1 and Corollary 7.2 we have that y is nonnegative. It is easy to obtain a result similar to Theorem 7.1 in case y is instead nonpositive. In addition, as with Theorem 7.1, we may obtain a corollary dual to Corollary 7.2, but we omit its statement.

Theorem 7.4. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonpositive function satisfying $y(0) = 0$. Fix $\nu \in (1,2)$ and suppose that $\Delta_0^\nu y(t) \le 0$, for each $t \in \mathbb{N}_{2-\nu}$. If it also holds that $\Delta y(0) \le 0$, then y is decreasing on $\mathbb{N}_0$.*

Proof. Let y be as in the statement of this theorem. Put $z := -y$. Then $z(0) = 0$, z is nonnegative on its domain, and (by the linearity of the fractional difference operator) $\Delta_0^\nu z(t) \ge 0$ for each $t \in \mathbb{N}_{2-\nu}$. Consequently, each of the hypotheses of Theorem 7.1 is satisfied. Therefore, we conclude that z is increasing, whence $-z = y$ is decreasing at each $t \in \mathbb{N}_0$. And this completes the proof. □

We mention next a couple of representative consequences of Theorems 7.1 and 7.4. We begin by providing a result regarding a discrete fractional IVP, which is Corollary 7.5, and then a result about a discrete fractional BVP with (possibly) inhomogeneous boundary conditions, which is Corollary 7.6.

Corollary 7.5. *Let $h : [1, +\infty)_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a nonnegative, continuous function, and let A, $B \in \mathbb{R}$ be nonnegative constants. Then the unique solution of the discrete fractional IVP*

$$\begin{aligned} \Delta_0^\nu y(t) &= h(t+\nu-1, y(t+\nu-1)),\ t \in [2-\nu, +\infty)_{\mathbb{Z}_{2-\nu}} \\ y(0) &= A,\ \Delta y(0) = B \end{aligned}$$

is increasing (and nonnegative).

Proof. Simply note that the proof of Theorem 7.1 reveals that one may replace the hypothesis that y is nonnegative on its domain with the hypothesis that $y(1) \ge y(0) \ge 0$. Since $A, B \ge 0$, the result follows. □

Corollary 7.6. *Let $h : \mathbb{Z}_{\nu-2} \times \mathbb{R} \to \mathbb{R}$ be a nonpositive function, and let A, $B \in \mathbb{R}$ be nonpositive constants. Then the unique solution of the discrete fractional IVP*

$$\begin{aligned} \Delta_{\nu-2}^\nu y(t) &= h(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) &= A \\ \Delta y(\nu-2) &= B \end{aligned}$$

is decreasing.

Our final consequence of Theorem 7.1 deserves special mention. To contextualize the result, let us consider the problem

$$\begin{aligned} \left(\Delta_{\alpha-1}^\alpha u\right)(t) &= \lambda u(t+\alpha-1) + f(t+\alpha-1, u(t+\alpha-1)),\ t \in [0, T-1]_{\mathbb{N}_0} \\ u(\alpha-1) &= u(\alpha-1+T), \end{aligned} \tag{7.12}$$

which was studied by Ferreira and Goodrich [83]. Supposing that f satisfied superlinear growth at 0 and $+\infty$ (uniformly for t), they proved that problem (7.12) has at least one positive solution for a range of values of the parameter λ, even if f is nonnegative. Assuming $\lambda > 0$, such an occurrence is clearly impossible in case $\alpha = 1$, for then (7.12) implies that u is strictly increasing, contradicting the periodic boundary conditions. So, this result demonstrates that the fractional difference can behave in unexpected ways, due precisely to its nonlocal structure.

With this somewhat aberrant result in mind, we might wonder if it is possible for the problem

$$\begin{aligned} -\Delta^{\nu}_{\nu-2}y(t) &= f(t+\nu-1, y(t+\nu-1)),\ t \in [0, b+1]_{\mathbb{N}_0} \\ y(\nu-2) &= 0 \\ y(\nu+b+1) &= 0 \end{aligned} \tag{7.13}$$

to have at least one positive solution if f is nonpositive. Now, when $\nu = 2$, the nonpositivity of f implies that $\Delta^2 y(t) \geq 0$, for each admissible t, from which it follows at once that if y is a solution of problem (7.13), then $y(t) \leq 0$ for each t. This is a simple consequence of the geometrical implications of $\Delta^2 y(t) \geq 0$ together with the Dirichlet boundary conditions. However, as the discussion regarding problem (7.12) demonstrates, in the fractional setting one cannot be so sure. In fact, it would not be entirely unreasonable to suspect that perhaps that nonlocal structure of $\Delta^{\nu}_{\nu-2}$ somehow allows for a positive solution to exist in spite of the nonpositivity of f. Corollary 7.7 demonstrates conclusively that this particular geometric aberration is forbidden.

Corollary 7.7. *Let $f : [\nu-1, \nu+b]_{\mathbb{Z}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ be continuous and nonpositive and $b \in \mathbb{N}$ a given constant. Then the discrete fractional boundary value problem* (7.13) *has no positive solution.*

We would also like to mention that it is necessary to impose *some* additional restriction beyond the positivity of the fractional difference if we hope to deduce the monotonicity of y. For example, in Corollary 7.2 we impose the condition $\Delta y(0) \geq 0$, which, as was explained earlier, was inadvertently omitted from the statement of the corresponding result in [67], though **all** of the other results in that paper are correct as stated. In any case, to demonstrate that the positivity of the fractional difference is *not sufficient*, we provide the following example.

Example 7.8. Let $f(t) = 2^{-t}$, $t \in \mathbb{N}_0$, and assume that $\frac{2+\sqrt{2}}{2} < \nu < 2$. We will show that $\Delta^{\nu}_0 f(t) \geq 0$, $t \in \mathbb{N}_{2-\nu}$, $f(t) \geq 0$ on $\mathbb{N}_0$, but $f(t)$ is not increasing on $\mathbb{N}_1$. Clearly $f(t) \geq 0$ on $\mathbb{N}_0$.

For $t = 2-\nu+k$, $k \geq 0$, we have

$$\begin{aligned} \Delta^{\nu}_0 f(t) &= \int_0^{3+k} h_{-\nu-1}(2-\nu+k, \tau+1) f(\tau)\, \Delta\tau \\ &= \sum_{i=0}^{k+2} h_{-\nu-1}(2-\nu+k, i+1) 2^{-i}. \end{aligned}$$

For $0 \le i \le k+2$, $1 < \nu < 2$, we have

$$\begin{aligned} h_{-\nu-1}(2-\nu+k, i+1) &= \frac{(1-\nu+k-i)^{\underline{-\nu-1}}}{\Gamma(-\nu)} \\ &= \frac{\Gamma(2-\nu+k-i)}{\Gamma(3+k-i)\Gamma(-\nu)} \\ &= \frac{(-\nu+1+k-i)\cdots(-\nu+1)(-\nu)}{(2+k-i)!}. \end{aligned} \tag{7.14}$$

It follows from (7.14) that if $k-i \ge 1$, then $h_{-\nu-1}(2-\nu+k, i+1) > 0$. When $i = k, k+1, k+2$, we have

$$h_{-\nu-1}(2-\nu+k, k+1) = \frac{\Gamma(2-\nu)}{2!\Gamma(-\nu)} = \frac{(-\nu+1)(-\nu)}{2}, \tag{7.15}$$

$$h_{-\nu-1}(2-\nu+k, k+2) = \frac{\Gamma(1-\nu)}{\Gamma(-\nu)} = -\nu, \tag{7.16}$$

$$h_{-\nu-1}(2-\nu+k, k+3) = \frac{\Gamma(-\nu)}{\Gamma(-\nu)} = 1, \tag{7.17}$$

respectively. So from (7.15), (7.16), (7.17), and the fact that $\frac{2+\sqrt{2}}{2} < \nu < 2$, we get that

$$\begin{aligned} \Delta_0^\nu f(t) &\ge \sum_{i=k}^{k+2} h_{-\nu-1}(2-\nu+k, i+1)2^{-i} \\ &= \frac{(-\nu+1)(-\nu)}{2} \cdot \frac{1}{2^k} - \frac{\nu}{2^{k+1}} + \frac{1}{2^{k+2}} \\ &= \frac{2\nu^2 - 4\nu + 1}{2^{k+2}} > 0. \end{aligned}$$

But since $f(t)$ is obviously decreasing, it follows that the some additional condition is necessary above and beyond the positivity of the fractional difference.

It should be noted that the above example is just a special case of a more general result, which we illustrate with the following example.

Example 7.9. Let $f(t) = \alpha^{-t}$, $t \in \mathbb{N}_0$, and assume that $\alpha > 1$. If we proceed as in the example above, we obtain the estimate

$$\begin{aligned}\Delta_0^\nu f(t) &\geq \sum_{i=k}^{k+2} h_{-\nu-1}(2-\nu+k, i+1)\alpha^{-i}\\ &= \frac{(-\nu+1)(-\nu)}{2}\cdot\frac{1}{\alpha^k} - \frac{\nu}{\alpha^{k+1}} + \frac{1}{\alpha^{k+2}}\\ &= \frac{\alpha^2\nu^2 - \nu\alpha^2 - 2\nu\alpha + 2}{2\alpha^{k+2}}.\end{aligned}$$

Let us define

$$g(\nu) := \nu^2 - \frac{\nu(2+\alpha)}{\alpha} + \frac{2}{\alpha^2}.$$

We see, therefore, that $\Delta_0^\nu f(t) > 0$ if $g(\nu) > 0$. If we solve the quadratic inequality $g(\nu) > 0$, we find that for all $\nu_0 < \nu < 2$, we have $\Delta_0^\nu f(t) > 0$, where $\nu_0 := \frac{\alpha+2+\sqrt{\alpha^2+4\alpha-4}}{2\alpha}$. Since $\nu_0 < 2$ holds because $\alpha > 1$, it follows that we obtain a family of functions $f(t) := \alpha^{-t}$ which are decreasing but for which $\Delta_0^\nu f(t) > 0$. Finally, the following table illustrates how the interval $(\nu_0, 2)$ varies with differing choices for $\alpha \in (1, \infty)$.

α	1.2	2	3	8
ν_0	1.950	1.707	1.510	1.224

We conclude by presenting a monotonicity theorem of a different color. The interesting point regarding this result is that we do ***not*** necessarily suppose that $\Delta f(a) \geq 0$. Rather, we replace this condition with a weaker condition on the "initial growth" of the map $t \mapsto y(t)$. Thus, this result improves the monotonicity result that was proved earlier. This new result was proved by Baoguo, Erbe, Goodrich, and Peterson, and it relies in a special way on a useful difference inequality discovered by Baoguo, Erbe, and Peterson. Thus, we first state the aforementioned technical lemma and then state and prove the existence theorem; we mention that for the interested reader, the proof of Lemma 7.10 may be found in [47]. (For readers who have read Chap. 3, it is seen that Lemma 7.10 is very much related, in the nabla setting, to Theorem 3.115.)

Lemma 7.10. *Assume that $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2-\nu}$, with $1 < \nu < 2$. Then*

$$\Delta f(a+k+1) \geq -h_{-\nu}(a+k+2-\nu, a)f(a) - \sum_{\tau=a}^{a+k} h_{-\nu}(a+k+2-\nu, \sigma(\tau))\Delta f(\tau)$$

for each $k \in \mathbb{N}_0$, where

$$h_{-\nu}(t,\sigma(\tau)) = \frac{(t-\tau)^{\underline{-\nu}}}{(t-\nu-\tau)!(a+k+2-\tau)!} < 0,$$

for $t \in \mathbb{N}_{a+2-\nu}$, $a \le \sigma(\tau) \le t+\nu-1$.

Theorem 7.11. *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *and that* $\Delta_a^\nu f(t) \ge 0$*, for each* $t \in \mathbb{N}_{a+2-\nu}$*, with* $1 < \nu < 2$*. If*

$$f(a+1) \ge \frac{\nu}{k+2} f(a)$$

for each $k \in \mathbb{N}_0$*, then* $\Delta f(t) \ge 0$*, for* $t \in \mathbb{N}_{a+1}$.

Proof. We prove that $\Delta f(a+k+1) \ge 0$, for each $k \ge 0$, by the principle of strong induction. From Lemma 7.10, in case $k = 0$, together with the hypothesis $f(a+1) \ge \frac{\nu}{2} f(a)$, we estimate

$$\begin{aligned}
\Delta f(a+1) &\ge -h_{-\nu}(a+2-\nu,a)f(a) - h_{-\nu}(a+2-\nu,a+1)\Delta f(a) \\
&= -\left[\frac{\Gamma(3-\nu)}{\Gamma(3)\Gamma(-\nu+1)} f(a) + \frac{\Gamma(2-\nu)}{\Gamma(2)\Gamma(-\nu+1)} \Delta f(a)\right] \\
&= -\frac{\Gamma(2-\nu)}{\Gamma(-\nu+1)} \left[\frac{1}{2}(2-\nu) f(a) + \Delta f(a)\right] \\
&\ge -\frac{\Gamma(2-\nu)}{\Gamma(-\nu+1)} \underbrace{\left[\frac{1}{2}(2-\nu) + \frac{\nu}{2} - 1\right]}_{=0} f(a) \\
&= 0.
\end{aligned}$$

Suppose next that $k \ge 1$ and $\Delta f(a+i) \ge 0$, for $i \in \mathbb{N}_1^k$. From Lemma 7.10 together with the hypothesis $f(a+1) \ge \frac{\nu}{k+2} f(a)$ for each $k \in \mathbb{N}_0$, we estimate

$$\begin{aligned}
&\Delta f(a+k+1) \\
&\quad \ge -f(a)h_{-\nu}(a+k+2-\nu,a) - \sum_{\tau=a}^{a+k} h_{-\nu}(a+k+2-\nu,\sigma(\tau))\Delta f(\tau) \\
&\quad \ge -f(a)h_{-\nu}(a+k+2-\nu,a) - h_{-\nu}(a+k+2-\nu,a+1)\Delta f(a) \\
&\quad = -\frac{\Gamma(k+3-\nu)}{\Gamma(k+3)\Gamma(-\nu+1)} f(a) - \frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)} \Delta f(a) \\
&\quad = \underbrace{-\frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)}}_{>0} \left[\frac{k+2-\nu}{k+2} f(a) + \Delta f(a)\right]
\end{aligned}$$

$$\geq -\frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)}\underbrace{\left[\frac{k+2-\nu}{k+2}+\frac{\nu}{k+2}-1\right]}_{=0}f(a)$$

$$= 0.$$

As this inequality implies that f is monotone increasing, the proof is thus complete. □

Remark 7.12. We would like to point out that very recently (i.e., as of late 2015) there has been quite a bit of progress made in extending the results of this section, which were deduced by Dahal and Goodrich in late 2013. Correspondingly, there has been much activity very recently (again, as of late 2015) extending the concavity/convexity results of the next section. In addition to the preceding theorem, additional generalizations have been produced. We direct the interested reader to the forthcoming papers by Baoguo, Erbe, Goodrich, and Peterson [53, 54].

7.3 Concavity and Convexity Results for Fractional Difference Operators

In the previous section we demonstrated that the discrete fractional difference operator satisfied a particular monotonicity condition, and we saw how this was a direct consequence of the implicit nonlocality in the construction of the fractional difference. In this section we present some convexity and concavity results for discrete fractional difference operators. We note that the results of this section can mostly be found in a paper by Goodrich [114].

We begin by stating the main result of this section and then proceed to state and discuss several consequences of this result. Essentially, the result states that if

- $y(0) = \Delta y(0) = \cdots = \Delta^{N-3}y(0) = 0$;
- $\Delta^{N-2}y(0) \geq 0$;
- $\Delta^{N-1}y(0) \geq 0$; and
- $\Delta_0^{\mu}y(t) \geq 0$, for each $t \in \mathbb{N}_{N-\mu}$;

then $\Delta^{N-1}y(t) \geq 0$, for each $t \in \mathbb{N}_0$. In particular, if we fix $N = 3$, then we obtain a suitable convexity result. In this special case we obtain that if $y(0) = 0$, $\Delta y(0)$ is nonnegative, $\Delta^2 y(0)$ is also nonnegative, and $\Delta_0^{\mu}y(t)$ is nonnegative for each admissible t, then the map $t \mapsto y(t)$ is concave on its domain. So, this result essentially demonstrates that if y has a bit of initial convexity, so to speak, then this is propagated provided that the sufficient auxiliary conditions are in force, as described precisely above. In some sense, this is not quite what one would expect, since it would be preferable not to have to require the condition $\Delta^2 y(0) \geq 0$. At the end of this section we shall suggest how this may be improved.

With these considerations in mind, we now state and prove the convexity result. Observe that the proof of this result and its corollaries are strongly based on an application of the monotonicity result. Thus, this collection of results gives us yet another nontrivial application of the monotonicity result.

Theorem 7.13. *Fix $\mu \in (N-1, N)$, for $N \in \mathbb{N}_3$ given, and let $y : \mathbb{N}_0 \to \mathbb{R}$ be a given function satisfying $\Delta^j y(0) = 0$ for each $j \in \{0, 1, 2, \ldots, N-3\}$, $\Delta^{N-2} y(0) \geq 0$, and $\Delta_0^{\mu} y(t) \geq 0$ for each $t \in \mathbb{N}_{N-\mu}$. If it also holds that $\Delta^{N-1} y(0) \geq 0$, then $\Delta^{N-1} y(t) \geq 0$, for each $t \in \mathbb{N}_0$.*

Proof. Define the function $w : \mathbb{N}_0 \to \mathbb{R}$ by

$$w(t) := \Delta^{N-2} y(t).$$

We show that w satisfies the monotonicity theorem—namely, Corollary 7.2. To this end, put $\nu := \mu - N + 2 \in (1, 2)$. On the one hand, by Theorem 2.51 we obtain

$$\begin{aligned}
\Delta_0^{\nu} w(t) = \Delta_0^{\nu} \Delta_0^{N-2} y(t) = \Delta_{2-\nu}^{N-2} \Delta_0^{\nu} y(t) & \\
- \underbrace{\sum_{j=0}^{N-3} \frac{\Delta_0^j y(0)}{\Gamma(-\nu - N + j + 3)} t^{\underline{-\nu-N+2+j}}}_{=0} & \\
= \Delta_{2-\nu}^{N-2} \Delta_0^{\nu} y(t) & \\
= \Delta_0^{\nu+N-2} y(t) & \\
= \Delta_0^{\mu} y(t) & \\
\geq 0, &
\end{aligned}$$

for each $t \in \mathbb{N}_{2-\nu} = \mathbb{N}_{N-\mu}$. We also observe that

$$w(0) = \Delta^{N-2} y(0) \geq 0.$$

By Corollary 7.2 it follows that $t \mapsto w(t)$ is increasing at $t = 0$. That is to say, it holds that $\Delta^{N-1} y(0) \geq 0$. We also note that $\Delta^{N-1} y(0) = \Delta w(0) \geq 0$. Finally, by repeatedly applying Corollary 7.2 we obtain that $\Delta^{N-1} y(t) \geq 0$ for each $t \in \mathbb{N}_0$. And this completes the proof. □

Remark 7.14. Observe that in the proof of Theorem 7.13 we repeatedly apply Corollary 7.2 at the end of the argument. In fact, it is worth noting that one can strengthen Corollary 7.2 in precisely this way—namely, it is sufficient that $y(0) \geq 0$. In particular, one need not know *a priori* that y is nonnegative, merely that y is "initially" nonnegative. The nonnegativity is, in fact, then propagated. A careful proof of this assertion is left to the reader.

We now demonstrate that the hypotheses of Theorem 7.13 can be altered somewhat.

Corollary 7.15. *Fix* $\mu \in (N-1, N)$*, for* $N \in \mathbb{N}_3$ *given, and assume that* $\Delta_0^{\mu} y(t) \geq 0$ *for each* $t \in \mathbb{N}_{N-\mu}$*. In case* N *is odd, assume that*

$$\begin{cases} \Delta^j y(0) < 0, & j = 0, 2, \ldots, N-3 \\ \Delta^j y(0) > 0, & j = 1, 3, \ldots, N-4 \end{cases},$$

whereas in case N *is even, assume that*

$$\begin{cases} \Delta^j y(0) > 0, & j = 0, 2, \ldots, N-4 \\ \Delta^j y(0) < 0, & j = 1, 3, \ldots, N-3 \end{cases}.$$

If in addition it holds both that $\Delta^{N-2} y(0) \geq 0$ *and that* $\Delta^{N-1} y(0) \geq 0$*, then* $\Delta^{N-1} y(t) \geq 0$*, for each* $t \in \mathbb{N}_0$*.*

Proof. Observe that by the calculation in the proof of Theorem 7.13, it follows that $\Delta_0^{\nu} w(t) \geq 0$ if and only if

$$\sum_{j=0}^{N-3} \frac{\Delta^j y(0)}{\Gamma(-\nu - N + j + 3)} t^{\underline{-\nu - N + 2 + j}} \leq 0; \tag{7.18}$$

recall here that the inequality $\Delta_{2-\nu}^{N-2} \Delta_0^{\nu} y(t) = \Delta_0^{\mu} y(t) \geq 0$ is still assumed. It then follows from (7.18) that

$$\frac{t(t-1)\cdots(-\mu+1)}{(t+\mu)!} y(0) + \cdots + \frac{t(t-1)\cdots(-\mu+N-2)}{(t+3+\mu-N)!} \Delta^{N-3} y(0) \leq 0$$

must hold for each $t \in \mathbb{N}_{N-\mu}$.

For notational convenience we next define the map $C_j : \mathbb{N}_3 \times \mathbb{N}_{N-\mu} \to \mathbb{R}$, for $j \in \{0, 1, \ldots, N-3\}$, by

$$\begin{aligned} C_j(N, t) &:= \frac{1}{\Gamma(-\nu - N + j + 3)} t^{\underline{-\nu - N + 2 + j}} \\ &= \frac{\Gamma(t+1)}{\Gamma(-\nu - N + j + 3)\Gamma(t + \nu + N - 1 - j)}. \end{aligned}$$

Observe that $C_j(N, t)$ is nothing more than the coefficient of $\Delta^j y(0)$ in (7.18). Recalling that $t \in \mathbb{N}_{N-\mu}$, we may simplify the ratio of gamma functions appearing in the definition of C_j. In particular, it is then apparent that if j is even, then $C_j(N, t) < 0$ if N is even, whereas $C_j(N, t) > 0$ if N is odd; moreover, this holds for each $t \in \mathbb{N}_{N-\mu}$ as a simple calculation reveals. The sign relationship is reversed if j is odd, and, once again, the relationship holds for each admissible t. We then see that (7.18) holds provided that $\Delta^j y(0)$ satisfies the sign condition, for each admissible j, presented in the statement of the theorem. And this completes the proof. □

We next present three examples to illustrate the application of Corollary 7.15 in the cases where $N = 3$, 4, or 5.

Example 7.16. Suppose that $N = 3$. Then Corollary 7.15 demonstrates that $\Delta^2 y(t) \geq 0$, for example, provided that

$$\begin{aligned} y(0) &< 0 \\ \Delta y(0) &\geq 0 \\ \Delta^2 y(0) &\geq 0 \\ \Delta_0^\mu y(t) &\geq 0 \text{ for some } \mu \in (2,3). \end{aligned}$$

Example 7.17. Suppose that $N = 4$. Then Corollary 7.15 implies that $\Delta^3 y(t) \geq 0$ if it holds that

$$\begin{aligned} y(0) &> 0 \\ \Delta y(0) &\leq 0 \\ \Delta^2 y(0) &\geq 0 \\ \Delta^3 y(0) &\geq 0 \\ \Delta_0^\mu y(t) &\geq 0 \text{ for some } \mu \in (3,4). \end{aligned}$$

Example 7.18. Suppose that $N = 5$. Then Corollary 7.15 implies that $\Delta^4 y(t) \geq 0$ if it holds that

$$\begin{aligned} y(0) &< 0 \\ \Delta y(0) &\geq 0 \\ \Delta^2 y(0) &\leq 0 \\ \Delta^3 y(0) &\geq 0 \\ \Delta^4 y(0) &\geq 0 \\ \Delta_0^\mu y(t) &\geq 0 \text{ for some } \mu \in (4,5). \end{aligned}$$

The next corollary is immediate and provides a geometrical interpretation of Theorem 7.13 in case $\mu \in (2,3)$ and thus $N = 3$; in particular, it provides for a convexity-type result.

Corollary 7.19. *If* $2 < \mu < 3$ *and* $y : \mathbb{N}_0 \to \mathbb{R}$ *satisfies* $y(0) = 0$, $\Delta y(0) \geq 0$, $\Delta^2 y(0) \geq 0$, *and* $\Delta_0^{\mu} y(t) \geq 0$, *for each* $t \in \mathbb{Z}_{3-\mu}$, *then* $\Delta^2 y(t) \geq 0$, *for each* $t \in \mathbb{N}_0$.

We can also obtain an alternative version of Theorem 7.13.

Corollary 7.20. *Fix* $\mu \in (N-1, N)$, *for* $N \in \mathbb{N}$ *given, and let* $y : \mathbb{N}_0 \to \mathbb{R}$ *be a given function satisfying* $\Delta^j y(0) = 0$ *for each* $j \in \{0, 1, 2, \ldots, N-3\}$, $\Delta^{N-2} y(0) \leq 0$, *and* $\Delta_0^{\mu} y(t) \leq 0$ *for each* $t \in \mathbb{Z}_{N-\mu}$. *If it also holds that* $\Delta^{N-1} y(0) \leq 0$, *then* $\Delta^{N-1} y(t) \leq 0$, *for each* $t \in \mathbb{N}_0$.

Proof. Put $z \equiv -y$ and apply Theorem 7.13 to the function z. □

Finally, as a specific application of this result and to demonstrate a nontrivial consequence of Theorem 7.13 we consider the theorem in case $N = 3$. To this end, consider the following FBVP, and observe that this is a special case of the so-called $(N-1, 1)$ problem for the case $N = 3$; see [124] for additional results on a class of discrete fractional $(N-1, 1)$ problems.

$$\begin{aligned} \Delta_{\mu-3}^{\mu} y(t) &= f(t+\mu-1, y(t+\mu-1)),\ t \in \mathbb{N}_0^{b+1} =: \{0, 1, \ldots, b+1\} \\ y(\mu-3) &= 0 = \Delta y(\mu-3) \\ y(\mu+b+1) &= 0 \end{aligned} \tag{7.19}$$

Corollary 7.21. *If the continuous function* $f : \mathbb{N}_{\mu-1}^{\mu+b} \times \mathbb{R} \to \mathbb{R}$ *is nonnegative and* $2 < \mu < 3$, *then problem* (7.19) *has no nontrivial positive solution.*

Proof. We begin by noting that from the boundary conditions we clearly have both that $y(\mu-3) = 0$ and that $\Delta y(\mu-3) \geq 0$. Furthermore, we compute

$$\Delta^2 y(\mu-3) = y(\mu-1) - 2y(\mu-2) + y(\mu-3) = y(\mu-1).$$

Therefore, supposing for contradiction that y is a fictitious positive solution of problem (7.19), the above calculation demonstrates that $\Delta^2 y(\mu-3) \geq 0$. Note, in addition, that by the form of the difference equation in (7.19) together with the assumption on the function f we may also conclude that $\Delta_{\mu-3}^{\mu} y(t) \geq 0$ for each $t \in \mathbb{N}_0$. Thus, we may invoke Theorem 7.13 to deduce that $\Delta^2 y(t) \geq 0$ for each $t \in \mathbb{Z}_{\mu-3}$.

Now, by the contradiction assumption we have that y is nontrivial, and so, it follows that for some time $t_0 \in \mathbb{Z}_{\mu-2}^{\mu+b}$ it holds that $y(t_0) > 0$. But then $\Delta y(t_0) > 0$. Since Theorem 7.13 has shown that $\Delta^2 y(t) \geq 0$ for all t, it follows that $\Delta y(t) \geq 0$ for each $t \in \mathbb{Z}_{t_0}^{\mu+b}$. Thus, $y(\mu+b+1) > 0$, which violates the boundary condition at the right endpoint, and so, a contradiction is obtained. Consequently, (7.19) cannot have a nontrivial positive solution, as claimed. □

Remark 7.22. For obvious geometrical reasons, problem (7.19) cannot have a positive solution in case $\mu = 3$. However, lacking this simple geometric intuition when $\mu \notin \mathbb{N}$, it does not appear to be plainly obvious that problem (7.19) maintains this similar solution structure. Accordingly, Corollary 7.21 demonstrates that no such aberrant or otherwise pathological behavior occurs in the fractional case.

In fact, this is of some interest since the nonlocal structure of the fractional difference can be responsible for aberrant behavior. As mentioned in the previous section, it has been shown by Ferreira and Goodrich [83, Theorem 3.13] that this nonlocality can contribute to certain boundary value problems possessing positive solutions even in the case where their integer-order counterpart does not possess nontrivial, positive solutions. Thus, Corollary 7.21 demonstrates that no such aberration occurs with respect to the fractional $(2, 1)$ problem studied above.

The reader should observe that it is certainly possible to write numerous analogues of Corollary 7.21 by repeatedly applying Theorem 7.13 for different choice of N. It is instructive to do this in a few cases to obtain a better sense of the implications of the result. However, we leave this as an optional exercise.

Remark 7.23. Due to the minor error in [67, Corollary 2.3], there are subsequently some minor errors in [114]. Fortunately, the changes required to that paper are very minor. In particular, the following minor changes must be made. It should be noted that other than including a single additional hypothesis, no changes are required to the proofs in [114]; the proofs are otherwise correct.

- In [114, Theorem 2.6, Corollary 2.8] the hypothesis $\Delta^{N-1}y(0) \geq 0$ must be added, whereas in [114, Corollary 2.11] the hypothesis $\Delta^{N-1}y(0) \leq 0$ must be added.
- In [114, Example 2.9] the hypothesis $\Delta^2 y(0) \geq 0$ must be added in the first part of the example, whereas in the second part of the example the hypothesis $\Delta^3 y(0) \geq 0$ must be added.
- In [114, Corollary 2.10] the hypothesis $\Delta^2 y(0) \geq 0$ must be added.

We would like to conclude this section, much as we did in Sect. 7.2, by pointing out that due to a flurry of recent work in the area, the basic convexity and concavity results presented earlier in this section have been able to be substantively extended in a variety of directions. As one such representative result, we present the following theorem, which was recently proved by Baoguo, Erbe, Goodrich, and Peterson; it will appear in a forthcoming paper [53], and we direct the reader to the paper for further details on this and other related results. In particular, we omit the proof of the result, but instead focus on its relationship to the results presented earlier in this section, and the way in which it improves them.

Theorem 7.24. *Fix $\nu \in (2, 3)$ and suppose that $\Delta_a^\nu f(t) \geq 0$ for each $t \in \mathbb{N}_{3+a-\nu}$. If for each $k \in \mathbb{N}_{-1}$ it holds that*

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+2+k}{(\nu-1)(3+k)} f(a+1) - \frac{\nu}{(3+k)(4+k)} f(a) \leq 0, \tag{7.20}$$

then $\Delta^2 f(t) \geq 0$ for each $t \in \mathbb{N}_{a+1}$.

Proof. Omitted—see [53]. □

Remark 7.25. Observe that inequality (7.20) does *not* necessarily imply that $\Delta^2 f(a) \geq 0$. For example, if we put $f(a) = 0, f(a+1) = 1$, and $f(a+2) = 1.9$ and we also fix $\nu = \frac{5}{2} \in (2, 3)$, then we calculate

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+1}{2(\nu-1)} f(a+1) - \frac{\nu}{6} f(a) = -\frac{2}{3} \cdot 1.9 + \frac{7}{6} \cdot 1 - \frac{5}{12} \cdot 0 = -\frac{1}{10} < 0,$$

which shows that inequality (7.20) is satisfied in case $k = -1$; in fact, it can be shown that (7.20) is satisfied for each $k \in \mathbb{N}_{-1}$. Yet we calculate $\Delta^2 f(a) = -\frac{1}{10} < 0$.

Similarly, if we put $g(a) = 1$, $g(a+1) = 2.8$, and $g(a+2) = 4.5$ as well as again taking $\nu = \frac{5}{2}$, then we see that $\Delta^2 g(a) = -\frac{1}{10} < 0$. Yet at the same time we calculate

$$\frac{1}{-\nu+1} g(a+2) + \frac{\nu+1}{2(\nu-1)} g(a+1) - \frac{\nu}{6} g(a) = -\frac{2}{3} \cdot 4.5 + \frac{7}{6} \cdot 2.8 - \frac{5}{12} \cdot 1 = -\frac{3}{20} < 0,$$

which shows that inequality (7.20) is satisfied in case $k = -1$, and, as can be easily shown, it holds for $k \in \mathbb{N}_{-1}$.

All in all, then, we see that condition (7.20) may be satisfied even if the map $t \mapsto f(t)$ is not convex "at" $t = a$. In particular, this means that Theorem 7.24 does *not* require that the map $t \mapsto f(t)$ be "initially convex."

7.4 Analysis of a Three-Point Boundary Value Problem

In the preceding two sections we demonstrated that under certain conditions the discrete fractional difference operator satisfies both monotonicity and convexity properties. We thus focused on the nonlocal structure implicit to the fractional operators. As mentioned in the introduction to this chapter, one can also study explicitly nonlocal boundary value problems. In this and the succeeding sections of this chapter, we examine a few specific examples of these so-called nonlocal boundary value problems.

We begin by examining a three-point problem in this section. This is a special case of the so-called *m*-point problem, wherein our boundary value problem has a boundary condition of the form, say,

$$y(0) = \sum_{j=1}^{m} a_j y\left(\xi_j\right),$$

where, for each j, the number ξ_j is both nonzero and an element of the domain of y. In particular, then, the value of y at $t = 0$ depends on the values of y at time points other than $t = 0$. This then gives rise to an explicit nonlocal boundary condition. Now, in the integer-order case determining the Green's function and its properties for such a problem can be tedious, but is usually not too overly taxing. However, in the fractional case, as this section demonstrates, determining explicitly the Green's function and its properties for even the three-point problem is extremely technical. In particular, the problem we study in this section is

$$\begin{aligned} -\Delta^{\nu} y(t) &= f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) &= 0 \\ \alpha y(\nu+K) &= y(\nu+b). \end{aligned}$$

Finally, as we shall note later in this section, if we remove the nonlocal boundary condition element by simply putting $\alpha = 0$, then we recover the Green's function for the conjugate problem as is easily checked by the reader. Moreover, most of the results in this section can be found in the paper by Goodrich [104].

So, with this context in mind, we first deduce the Green's function for the operator $-\Delta^{\nu}$ together with the boundary conditions $y(\nu-2) = 0$ and $\alpha y(\nu+K) = y(\nu+b)$, where $0 \le \alpha \le 1$ and $K \in [-1, b-1]_{\mathbb{Z}}$. We first present a preliminary lemma, which will prove to be rather useful in what follows. The lemma can be found in a paper by Goodrich [88].

Lemma 7.26. *Fix $k \in \mathbb{N}$ and let $\{m_j\}_{j=1}^{k}, \{n_j\}_{j=1}^{k} \subseteq (0, +\infty)$ such that*

$$\max_{1\le j\le k} m_j \le \min_{1\le j\le k} n_j$$

and that for at least one j_0, $1 \le j_0 \le k$, we have that $m_{j_0} < n_{j_0}$. Then for fixed $\alpha_0 \in (0,1)$, it follows that

$$\left(\frac{n_1}{n_1+\alpha_0}\cdot\dots\cdot\frac{n_k}{n_k+\alpha_0}\right)\left(\frac{m_1+\alpha_0}{m_1}\cdot\dots\cdot\frac{m_k+\alpha_0}{m_k}\right) > 1.$$

Proof. Fix an index j_0, where j_0 is one of the indices, of which there exists at least one, for which $n_{j_0} > m_{j_0}$. Notice that as $n_{j_0} > m_{j_0}$ and $\alpha_0 > 0$, it follows that $n_{j_0}\alpha_0 > m_{j_0}\alpha_0$, whence $m_{j_0}n_{j_0} + n_{j_0}\alpha_0 > m_{j_0}n_{j_0} + m_{j_0}\alpha_0$, so that

$$\frac{m_{j_0}+\alpha_0}{m_{j_0}} > \frac{n_{j_0}+\alpha_0}{n_{j_0}},$$

whence

$$\frac{n_{j_0}}{n_{j_0}+\alpha_0}\cdot\frac{m_{j_0}+\alpha_0}{m_{j_0}} > 1.$$

But now the claim follows at once by repeating the above steps for each of the remaining $j_0 - 1$ terms and observing that the product of j terms, each of which is at least unity and at least one of which exceeds unity, is greater than unity. □

In addition, for reference in the sequel and to simplify the rather formidable notational burden associated with this problem, let us make the following declarations; note that we provide the domains of these maps in the statement of Theorem 7.27.

$$
\begin{aligned}
g_1(t,s) &:= \frac{1}{\Gamma(\nu)}\Bigg[-(t-s-1)^{\underline{\nu-1}} \\
&\qquad + \frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]\Bigg] \\
g_2(t,s) &:= \frac{1}{\Gamma(\nu)}\left[\frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]\right] \\
g_3(t,s) &:= \frac{1}{\Gamma(\nu)}\left[-(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right] \\
g_4(t,s) &:= \frac{1}{\Gamma(\nu)}\left[\frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right] \\
\Omega_0 &:= (b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}
\end{aligned}
\tag{7.21}
$$

Theorem 7.27. *Let $h : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be given. The unique solution of the problem*

$$
\begin{aligned}
-\Delta^{\nu} y(t) &= h(t+\nu-1) \\
y(\nu-2) &= 0 \\
\alpha y(\nu+K) &= y(\nu+b)
\end{aligned}
\tag{7.22}
$$

is the function

$$y(t) = \sum_{s=0}^{b} G(t,s)h(s+\nu-1),$$

where $G(t,s)$ is the Green's function for the operator $-\Delta^{\nu}$ together with the boundary conditions in (7.22), and where

$$
G(t,s) := \begin{cases} g_1(t,s), & 0 \le s \le \min\{t-\nu, K\} \\ g_2(t,s), & 0 \le t-\nu < s \le K \le b \\ g_3(t,s), & 0 < K < s \le t-\nu \le b \\ g_4(t,s), & \max\{t-\nu, K\} < s \le b \end{cases},
$$

with $g_i(t,s)$, $1 \le i \le 4$, are as defined in (7.21) above.

Proof. Omitted—see [104]. □

Remark 7.28. It is easy to observe that in case $\alpha = 0$, not only does problem (7.22) reduce to the usual conjugate FBVP that was considered in [31], but, moreover, the Green's function given by Theorem 7.27 reduces to the Green's function derived in [31].

We now wish to prove that the Green's function $(t, s) \mapsto G(t, s)$ in Theorem 7.27 satisfies certain properties that will prove to be of use in the sequel and are also of independent interest. We first prove an easy preliminary lemma.

Lemma 7.29. *Let Ω_0 be as defined in (7.21). Then for each $K \in [-1, b-1]_{\mathbb{Z}}$, $\nu \in (1, 2]$, and $b \in \mathbb{N}$, we find that $\Omega_0 > 0$.*

Proof. Recall from (7.21) that $\Omega_0 = (b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}$. It is evident that this function is decreasing in α for each fixed K, ν, and b, and so, it suffices to show that $\Omega_0 > 0$ when $\alpha = 1$. To this end, note that $t^{\underline{\mu}}$ is increasing in t, whenever $0 < \mu < 1$. Since $b + \nu > K + \nu$, it immediately follows that

$$\Omega_0\big|_{\alpha=1} = (b+\nu)^{\underline{\nu-1}} - (K+\nu)^{\underline{\nu-1}} > 0,$$

which proves the claim. We observe that this same result holds even in the case where $\nu = 2$. □

Theorem 7.30. *Let G be the Green's function given in the statement of Theorem 7.27. Then for each $(t, s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0, b]_{\mathbb{N}_0}$, we find that $G(t, s) \geq 0$.*

Proof. As was mentioned at the beginning of this section, the proof of this result may be found in its entirety in [104]. However, we include the proof here for its instructive value. In particular, it shall give the reader a sense of the delicacy that is involved in arguing the properties of Green's functions associated with fractional difference operators—delicacy that is obviated if we pass to the integer-order case. Moreover, this will also give the reader a general sense for certain of the techniques that may be utilized in these sorts of arguments.

With this in mind, our program to complete the proof is to show that for each i, $1 \leq i \leq 4$, it holds that $g_i(t, s) > 0$ for each admissible pair (t, s). To complete this program, we begin by showing both that $g_2(t, s) > 0$ and that $g_4(t, s) > 0$, as these are the easier cases. In the case of $g_2(t, s)$, observe that it suffices to show that the inequality

$$(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}} > 0 \tag{7.23}$$

holds. Showing that (7.23) is true is equivalent to showing that

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}}{\alpha(K+\nu-s-1)^{\underline{\nu-1}}} > 1. \tag{7.24}$$

But to see that (7.24) is true for each admissible pair (t, s) and each $\alpha \in (0, 1]$, notice that $t^{\underline{\mu}}$ is increasing in t provided that $\mu \in (0, 1)$. Consequently, we obtain that

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}}{\alpha(K+\nu-s-1)^{\underline{\nu-1}}} \geq \frac{(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu-s-1)^{\underline{\nu-1}}} > 1,$$

which proves (7.24) and thus (7.23). On the other hand, we note that by the form of g_4 given in (7.21), we obtain at once that $g_4(t, s) > 0$ since $\Omega_0 > 0$ by Lemma 7.29 and $b+\nu-s-1 > 0$ in this case. Thus, we conclude that both g_2 and g_4 are positive on each of their respective domains.

We next consider the function g_3. Recall from (7.21) that

$$g_3(t, s) = \frac{1}{\Gamma(\nu)}\left[-(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}\right].$$

Evidently, to prove that $g_3(t, s) > 0$, we may instead just prove that $\Gamma(\nu)g_3(t, s) > 0$. Now, it is clear that g_3 is increasing in α, for as α increases, Ω_0 clearly decreases. In particular, then, we deduce that

$$\Gamma(\nu)g_3(t, s) \geq -(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}. \tag{7.25}$$

Note that (7.25) implies that $g_3(t, s) > 0$ if and only if

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{7.26}$$

To prove that (7.26) holds, recall that on the domain of g_3 it holds that $t \geq s+\nu > K+\nu$. Consequently, given a fixed but arbitrary $s_0 > K$, we have that $t = s_0+\nu+j$, for some $0 \leq j \leq b-s_0$ with $j \in \mathbb{N}_0$. But then for this number s_0, we may recast the left-hand side of (7.26) as

$$\begin{aligned}
&\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} \\
&= \frac{\Gamma(t+1)\Gamma(b+\nu-s_0)\Gamma(t-s_0-\nu+1)\Gamma(b+2)}{\Gamma(t-\nu+2)\Gamma(b-s_0+1)\Gamma(t-s_0)\Gamma(b+\nu+1)} \\
&= \frac{\Gamma(s_0+\nu+j+1)\Gamma(b+\nu-s_0)\Gamma(j+1)\Gamma(b+2)}{\Gamma(s_0+j+2)\Gamma(b-s_0+1)\Gamma(\nu+j)\Gamma(b+\nu+1)} \\
&= \frac{j!(b+1)!\,[(\nu+j+s_0)\cdots(\nu+j)]}{(s_0+j+1)!\,(b-s_0)!\,[(b+\nu)\cdots(b+\nu-s_0)]} \\
&= \frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s_0)}\cdot\frac{(\nu+j+s_0)\cdots(\nu+j)}{(s_0+j+1)\cdots(j+1)}.
\end{aligned} \tag{7.27}$$

Now, observe that each of the fractions on the right-hand side of (7.27) has exactly $s_0 + 1$ factors in each of its numerator and denominator. In addition, by putting $\alpha_0 := \nu - 1 > 0$, we observe that this expression satisfies the hypotheses of Lemma 7.26. (Of course, some repetition of factors may occur between the two fractions on the right-hand side of (7.27), but these may always be canceled to obtain the form required by Lemma 7.26. Thus, we may safely ignore the existence of any possible repetition.) Consequently, we deduce from this lemma that

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} = \frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s)} \cdot \frac{(\nu+j+s_0)\cdots(\nu+j)}{(s_0+j+1)\cdots(j+1)} > 1,$$

whence (7.26) holds. But as (7.26) holds for each admissible pair (t, s), it follows at once that (7.25) holds, too, so that $g_3(t, s) > 0$, as claimed.

Finally, we show that $g_1(t, s) > 0$ on its domain, which we recall is $0 \le s \le \min\{t - \nu, K\}$. Recall from (3.1) that

$$\Gamma(\nu)g_1(t,s) = -(t-s-1)^{\underline{\nu-1}} + \frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right],$$

where we again use the fact that $g_1(t, s)$ is positive if and only if $\Gamma(\nu)g_1(t, s)$ is positive. Let us pause momentarily to notice that

$$(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}} > 0, \tag{7.28}$$

which is evidently an important condition. Note that (7.28) just follows from (7.23) above.

Notice that $g_1(t, s) > 0$ only if

$$\frac{t^{\underline{\nu-1}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right] > (t-s-1)^{\underline{\nu-1}}. \tag{7.29}$$

We begin by proving that the function $F : [0, 1] \to \mathbb{R}$ defined by

$$F(\alpha) := \frac{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}} \tag{7.30}$$

is increasing in α for $0 \le \alpha \le 1$. Note that a straightforward calculation shows that $F(\alpha)$ is increasing in α if and only if

$$\frac{(b+\nu-s-1)^{\underline{\nu-1}}(K+\nu)^{\underline{\nu-1}}}{(K+\nu-s-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{7.31}$$

To see that (7.31) holds, let s_0 be arbitrary but fixed such that each of $s_0 \in [0, b]_{\mathbb{N}_0}$ and $0 \le s_0 \le \min\{t - \nu, K\}$ holds. So, it follows that the left-hand side of (7.31) above satisfies

$$\frac{(b+\nu-s_0-1)^{\underline{\nu-1}}(K+\nu)^{\underline{\nu-1}}}{(K+\nu-s_0-1)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} = \frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s_0)} \cdot \frac{(K+\nu)\cdots(K+\nu-s_0)}{(K+1)\cdots(K-s_0+1)}. \tag{7.32}$$

But it is easy to check that by putting $\alpha_0 := \nu - 1 > 0$, we may apply Lemma 7.26 to the right-hand side of (7.32) to conclude that (7.31) holds. Thus, the map $\alpha \mapsto F(\alpha)$ is increasing in α, as desired. In particular, this implies that to prove that (7.29) is true, it suffices to check its truth in case $\alpha = 0$. In this case, we find that proving (7.29) reduces to proving that

$$\frac{t^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-1}}} > 1 \tag{7.33}$$

holds. Observe that the same proof that was used to show that (7.26) held can be used to show that (7.33) holds, too. Thus, as (7.29) holds in case $\alpha = 0$, the result of (7.30)–(7.33) implies that (7.29) holds for each admissible α. Consequently, we conclude that $g_1(t, s) > 0$, from which it follows that $g_i(t, s) > 0$ for each i, $1 \le i \le 4$. Hence, it follows that $G(t, s) \ge 0$ for each admissible pair (t, s). And this concludes the proof. □

Theorem 7.31. *Let G be the Green's function given in the statement of Theorem 7.27. In addition, suppose that for given $K \in [-1, b-1]_{\mathbb{Z}}$ and $1 < \nu \le 2$, we have that α satisfies the inequality*

$$0 \le \alpha \le \min_{(t,s)\in[s+\nu,\nu+b]_{\mathbb{N}_{\nu-1}}\times[0,b]_{\mathbb{N}_0}} \left\{ \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}} \right\}. \tag{7.34}$$

Then for each $s \in [0, b]_{\mathbb{N}_0}$ it holds that

$$\max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} G(t, s) = G(s+\nu-1, s). \tag{7.35}$$

Proof. Our strategy is to show that $\Delta_t g_i(t, s) > 0$ for each $i = 2, 4$, and that $\Delta_t g_i(t, s) < 0$ for $i = 1, 3$. From this the claim will follow at once. To this end, we first show that the former claim holds, as this is the easier of the two cases. Note, for example, that when $i = 2$, we find by direct computation that

$$\Gamma(\nu)\Delta_t g_2(t, s) = \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]. \tag{7.36}$$

So, it is clear from (7.36) that $\Delta_t g_2(t,s) > 0$ if and only if

$$(b+\nu-s-1)^{\underline{\nu-1}} > \alpha(K+\nu-s-1)^{\underline{\nu-1}}.$$

But as this immediately follows from (7.23)–(7.24), we deduce that

$$\Delta_t g_2(t,s) > 0,$$

as desired. On the other hand, the estimate $\Delta_t g_4(t,s) > 0$ evidently holds considering that $\Delta_t g_4(t,s) = \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}}$. And this concludes the analysis of $\Delta_t g_i(t,s)$ in case i is even.

We next attend to $g_3(t,s)$. In particular, we claim that $\Delta_t g_3(t,s) < 0$ for each admissible pair (t,s). To see that this claim holds, note that

$$\Gamma(\nu)\Delta_t g_3(t,s) = -(\nu-1)(t-s-1)^{\underline{\nu-2}} + \frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}(b+\nu-s-1)^{\underline{\nu-1}},$$

where we have used the fact that $\Delta_t(t-s-1)^{\underline{\nu-1}} = (\nu-1)(t-s-1)^{\underline{\nu-2}}$, which may be easily verified from the definition. So, if $\Delta_t g_3$ is to be a nonpositive function, then it must hold that

$$\frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{\Omega_0} < (t-s-1)^{\underline{\nu-2}}. \tag{7.37}$$

Notice that (7.37) is true if and only if $(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}} > \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-2}}}$ is true. But this latter inequality is true only if

$$-\alpha > \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-2}}(K+\nu)^{\underline{\nu-1}}} - \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} \tag{7.38}$$

is true. From (7.38) we see that by requiring α to satisfy, for each admissible K and ν, the estimate

$$\begin{aligned} 0 &\le \alpha \\ &\le \min_{(t,s)\in[s+\nu,\nu+b]_{\mathbb{N}_{\nu-1}}\times[0,b]_{\mathbb{N}_0}} \left\{ \frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}} - \frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}} \right\}, \end{aligned} \tag{7.39}$$

it follows that (7.37) is true—that is, that $g_3(t,s) > 0$ for each admissible pair (t,s). Note that restriction (7.39) above is precisely restriction (7.34), which was given in the statement of this theorem. Thus, with restriction (7.34) in place, we conclude that the map $(t,s) \mapsto \Delta_t g_3(t,s)$ will be nonpositive on its domain, as desired.

Finally, we claim that $\Delta_t g_1(t,s) < 0$ on its domain. Observe that by the definition of g_1 given in (7.21), we must argue that

$$\begin{aligned} &-(\nu-1)(t-s-1)^{\underline{\nu-2}} \\ &+\frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}}-\alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]<0. \end{aligned} \tag{7.40}$$

But observe that

$$\begin{aligned} &-(\nu-1)(t-s-1)^{\underline{\nu-2}} \\ &+\frac{(\nu-1)t^{\underline{\nu-2}}}{\Omega_0}\left[(b+\nu-s-1)^{\underline{\nu-1}}-\alpha(K+\nu-s-1)^{\underline{\nu-1}}\right] \\ &\leq -(\nu-1)(t-s-1)^{\underline{\nu-2}}+\frac{(\nu-1)t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{\Omega_0}. \end{aligned}$$

So, we deduce that if

$$-(\nu-1)(t-s-1)^{\underline{\nu-2}}+\frac{(\nu-1)t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{\Omega_0}<0, \tag{7.41}$$

then inequality (7.40) holds. Now, note that we can solve for α in (7.41) to obtain an upper bound on α. As this calculation is exactly the same as the one given earlier in the argument, we do not repeat it here. Instead we point out that the restriction (7.41) implies that

$$0\leq\alpha\leq\frac{(b+\nu)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}}-\frac{t^{\underline{\nu-2}}(b+\nu-s-1)^{\underline{\nu-1}}}{(K+\nu)^{\underline{\nu-1}}(t-s-1)^{\underline{\nu-2}}}.$$

Note that the right-hand side of (7.41) is precisely restriction (7.34). So, by assuming (7.34) we also get that (7.40) holds. Consequently, the preceding analysis shows that (7.40) holds, from which it follows that $\Delta_t g_1(t,s) > 0$ on its domain. Thus, we deduce that (7.35) holds, which completes the proof. □

Before presenting our final theorem in this section regarding the map $(t,s) \mapsto G(t,s)$, we make some definitions for convenience.

$$\gamma_1 := \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}$$

$$\gamma_2 := \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-1)^{\underline{\nu-1}}}\right]$$

$$\gamma_3 := \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}\right]$$

We will make use of these constants in the sequel.

Theorem 7.32. *Let G be the Green's function given in the statement of Theorem 7.27. Let γ_i, $1 \le i \le 3$, be defined as above. Then it follows that for each $s \in [0,b]_{\mathbb{N}_0}$*

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \ge \gamma \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \gamma G(s+\nu-1,s), \tag{7.42}$$

where

$$\gamma := \min\{\gamma_1, \gamma_3\}, \tag{7.43}$$

and γ satisfies the inequality $0 < \gamma < 1$.

Proof. To simplify the notation used in this proof, let us put, for each $1 \le i \le 4$,

$$\tilde{g}_i(t,s) := \begin{cases} \frac{g_i(t,s)}{g_2(s+\nu-1,s)}, & i = 1,2 \\ \frac{g_i(t,s)}{g_4(s+\nu-1,s)}, & i = 3,4 \end{cases}.$$

Observe that for $s \ge t-\nu+1$ and $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$, it holds that

$$\tilde{g}_2(t,s) = \tilde{g}_4(t,s) = \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} \ge \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}, \tag{7.44}$$

whence from (7.44) it is clear that in the case where both $s \ge t-\nu+1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, it follows that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \ge \gamma_1 G(s+\nu-1,s).$$

On the other hand, suppose that $s < t - \nu + 1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$. Then we consider two cases depending upon whether or not the pair (t, s) lives in the domain of $\tilde{g}_1$ or $\tilde{g}_3$. In the case where (t, s) lives in the domain of $\tilde{g}_3$, we note that by definition

$$\begin{aligned}
&\tilde{g}_3(t,s)\\
&= \frac{-(t-s-1)^{\underline{\nu-1}}\Omega_0}{(s+\nu-1)^{\underline{\nu-1}}(b+\nu-s-1)^{\underline{\nu-1}}} + \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}\\
&= \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[t^{\underline{\nu-1}} - \frac{(t-s-1)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}}}\right]\\
&\geq \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\\
&\quad\times\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-1)^{\underline{\nu-1}}}\right].
\end{aligned} \tag{7.45}$$

So, it is clear from (7.45) that in case $s < t - \nu + 1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, we get that $\min_{t\in\left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma_2 G(s+\nu-1, s)$.

Finally, suppose that $s < t - \nu + 1$, $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$, and that the pair (t, s) lives in the domain of $\tilde{g}_1$. By using a similar calculation as in (7.45) together with the definition of $\tilde{g}_1$, we obtain the lower bound

$$\begin{aligned}
&\tilde{g}_1(t,s)\\
&= \frac{-(t-s-1)^{\underline{\nu-1}}\Omega_0}{(s+\nu-1)^{\underline{\nu-1}}\left[(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}\right]}\\
&\qquad\qquad + \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}\\
&\geq \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\\
&\quad\times\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}\right].
\end{aligned} \tag{7.46}$$

We now need to focus on the quotient $\frac{(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}$ appearing on the right-hand side of (7.46). We claim that this is a decreasing function of α.

To prove this claim, let us put

$$g(\alpha) := \frac{(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}, \tag{7.47}$$

where for each fixed but arbitrary b, s, ν, and K, we have that $g : [0,1] \to [0,+\infty)$. Now, let the map $\alpha \mapsto F(\alpha)$ be defined as in (7.30) above. Note from (7.47) that

$$g(\alpha) = \frac{1}{F(\alpha)}.$$

Recall that in case $0 \le \alpha \le 1$ we have already argued that F is increasing in α. So, straightforward computations demonstrate that g is decreasing in α, for $0 \le \alpha \le 1$, as desired.

Since g is decreasing in α, we conclude that

$$\begin{aligned}
&\tilde{g}_1(t,s) \\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \\
&\quad \times \left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\left[(b+\nu)^{\underline{\nu-1}} - \alpha(K+\nu)^{\underline{\nu-1}}\right]}{(b+\nu-s-1)^{\underline{\nu-1}} - \alpha(K+\nu-s-1)^{\underline{\nu-1}}}\right] \\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-s-1)^{\underline{\nu-1}}}\right] \\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}\right].
\end{aligned}$$

Thus, we observe that in this case it holds that $\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \ge \gamma_3 G(s+\nu-1,s)$.

Finally, note that since $\gamma_2 \ge \gamma_3$, it must hold that $\min\{\gamma_1,\gamma_2,\gamma_3\} = \min\{\gamma_1,\gamma_3\}$. Thus, we can put $\gamma := \min\{\gamma_1,\gamma_3\}$ as in (7.43). The previous part of the proof then shows that for each $s \in [0,b]_{\mathbb{N}_0}$ it holds that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} G(t,s) \ge \gamma \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \gamma G(s+\nu-1,s), \tag{7.48}$$

and as (7.48) is (7.42), the first part of the proof is complete.

To complete the proof, it remains to show that γ, as defined in (7.43), satisfies $0 < \gamma < 1$. We first observe that $\gamma_1 < 1$. This follows from the fact that $t^{\underline{\nu-1}}$ is an increasing function in t whenever $\nu \in (1, 2]$. To see that this latter claim is true, simply observe that

$$\Delta\left[t^{\underline{\nu-1}}\right] = (\nu - 1) \cdot \frac{\Gamma(t+1)}{\Gamma(t-\nu+3)} > 0.$$

Thus, as $\frac{b+\nu}{4} > b+\nu$ and $\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}, (b+\nu)^{\underline{\nu-1}} \neq 0$, the claim follows. In particular, this demonstrates that

$$\gamma = \min\{\gamma_1, \gamma_3\} \leq \gamma_1 < 1. \tag{7.49}$$

On the other hand, observe that $\gamma_1 > 0$. So, it only remains to show that $\gamma_3 > 0$. Note that γ_3 is strictly positive if and only if

$$\frac{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}(b+\nu-1)^{\underline{\nu-1}}}{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-1}}} > 1. \tag{7.50}$$

But (7.50) is true if and only if

$$\frac{(b+1)\left(\frac{3(b+\nu)}{4}\right)}{(b+\nu)\left(\frac{3(b+\nu)}{4}-\nu+1\right)} > 1 \tag{7.51}$$

holds for each admissible b and ν—that is, each $b \in [2, +\infty)_{\mathbb{N}}$ and $\nu \in (1, 2]$.

We claim that (7.51) is true for each $b \in [2, +\infty)$ and each $\nu \in (1, 2]$. To see this, for each fixed and admissible b, put

$$H_b(\nu) := \frac{(b+1)\left(\frac{3(b+\nu)}{4}\right)}{(b+\nu)\left(\frac{3(b+\nu)}{4}-\nu+1\right)}, \tag{7.52}$$

which is the left-hand side of inequality (7.51), and note that each of

$$H_b(1) = 1 \tag{7.53}$$

and

$$H_b(2) = \frac{(b+1)\left(\frac{3}{4}b+\frac{3}{2}\right)}{(b+2)\left(\frac{3}{4}b+\frac{1}{2}\right)} = \frac{3b+3}{3b+2} \tag{7.54}$$

holds. Now, $H_b(2) > 1$ is evidently true for each admissible b. Moreover, a straightforward computation shows that

$$H_b'(\nu) = \frac{3(b+1)}{(3b-\nu+4)^2}. \tag{7.55}$$

But then (7.55) demonstrates that for each b, it holds that the map $\nu \mapsto H_b(\nu)$ is strictly increasing in ν. Therefore, as $H_b(1) = 1$ and $H_b(2) > 1$, we obtain at once that

$$H_b(\nu) > 1 \tag{7.56}$$

for each $\nu \in (1, 2]$ and $b \in [2, +\infty)_{\mathbb{N}}$. But then from (7.56) we deduce that (7.51) holds, as desired.

In summary, (7.50)–(7.56) demonstrate that $\gamma_3 > 0$. But we then find that

$$\gamma = \min\{\gamma_1, \gamma_3\} > 0. \tag{7.57}$$

Putting (7.49) and (7.57) together implies that $\gamma \in (0, 1)$, as claimed. And this completes the proof. □

Remark 7.33. Note that in case $\alpha = 0$, the result of Theorem 7.32 reduces to the results obtained in [31], as the reader may easily check.

Remark 7.34. For a brief investigation of the properties of the set of admissible values of α generated by condition (7.34) above, one may consult [104].

Remark 7.35. Once we have the preceding properties of the Green's function G in hand, it then is standard to provide some basic existence result for the FBVP

$$\begin{aligned} -\Delta^{\nu} y(t) &= f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) &= 0 \\ \alpha y(\nu+K) &= y(\nu+b), \end{aligned}$$

where $f : [0, b]_{\mathbb{N}_0} \times \mathbb{R} \to [0, +\infty)$ is a continuous map. However, since we complete this sort of analysis in the somewhat more general case of (potentially) nonlinear boundary conditions in the next section, we will not present existence theorems for the three-point problem studied in this section. We instead direct the interested reader to [104, §5] where results of this sort may be found for the three-point problem studied in this section.

7.5 A Nonlocal BVP with Nonlinear Boundary Conditions

In the previous section we saw how a three-point problem can be analyzed. In particular, notice that the boundary condition in that setting is linear in the sense that if we define the boundary operator B defined by

$$By := \alpha y(t+\nu) + y(t+\nu+b), \quad y \in \mathbb{R}^m,$$

then B is linear map from $\mathbb{R}^m$ into $\mathbb{R}$, for some $m > 1$ with $m \in \mathbb{N}$. However, there is no requirement that the boundary conditions for a given BVP be linear. In fact, if the boundary conditions are *nonlinear*, then the mathematical analysis of the problem can be very interesting and potentially challenging. For one thing, one cannot generally approach the problem in the same way—namely by determining an appropriate Green's function. Rather, an alternative but viable approach in this setting is to instead construct a new operator by taking the operator associated with the linear boundary condition problem and then suitably perturbing it. This approach will be seen in this section. In particular, we wish to consider a modification of the BVP considered in the previous section; namely, we consider in this section the problem

$$\begin{aligned} -\Delta^{\nu} y(t) &= f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) &= g(y) \\ y(\nu+b) &= 0 \end{aligned}$$

in the case where the map $y \mapsto g(y)$ is potentially nonlinear. The results of this section may be found largely in Goodrich [92].

We begin by providing a lemma, which essentially recasts the above BVP as an appropriate summation operator. Studying the existence of solutions to the BVP will then be reduced to demonstrating the existence of nontrivial fixed points of the associated summation operator.

Theorem 7.36. *Let $h : [\nu-1,\dots,\nu+b-1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ and $g : \mathbb{R}^{b+3} \to \mathbb{R}$ be given. A function y is a solution of the discrete FBVP*

$$\begin{aligned} -\Delta^{\nu} y(t) &= h(t+\nu-1) \\ y(\nu-2) &= g(y) \\ y(\nu+b) &= 0 \end{aligned} \tag{7.58}$$

where $t \in [0,b]_{\mathbb{N}_0}$, if and only if $y(t)$, for each $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$, has the form

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1) \\ &+ t^{\underline{\nu-1}}\left[\frac{1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}h(s+\nu-1)\right. \\ &\left.-\frac{g(y)}{(b+2)\Gamma(\nu-1)}\right] + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}g(y). \end{aligned} \tag{7.59}$$

Proof. Using the results from earlier in this text, we find that a general solution for (7.58) is the function

$$y(t) = -\Delta^{-\nu} h(t+\nu-1) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}}, \tag{7.60}$$

where $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. On the one hand, applying the boundary condition at $t = \nu - 2$ in (7.58) implies at once that

$$c_2 = \frac{1}{\Gamma(\nu-1)} g(y). \tag{7.61}$$

Applying the boundary condition at $t = \nu + b$ in (7.58) yields

$$\begin{aligned} 0 &= y(\nu+b) \\ &= [-\Delta^{-\nu} h(t)]_{t=\nu+b} + c_1(\nu+b)^{\underline{\nu-1}} + \frac{(\nu+b)^{\underline{\nu-2}}}{\Gamma(\nu-1)} g(y) \\ &= -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-s-1)^{\underline{\nu-1}} h(s+\nu-1) + c_1(\nu+b)^{\underline{\nu-1}} \\ &+ \frac{(\nu+b)^{\underline{\nu-2}}}{\Gamma(\nu-1)} g(y), \end{aligned} \tag{7.62}$$

whence (7.62) implies that

$$\begin{aligned} c_1 &= \frac{1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-s-1)^{\underline{\nu-1}} h(s+\nu-1) \\ &\quad - \frac{(\nu+b)^{\underline{\nu-2}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu-1)} g(y) \qquad (7.63) \\ &= \frac{1}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-s-1)^{\underline{\nu-1}} h(s+\nu-1) \\ &\quad - \frac{1}{(b+2)\Gamma(\nu-1)} g(y). \end{aligned}$$

Consequently, using (7.60)–(7.63), we deduce that for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ it holds that y has the form given in (7.59) above. And this shows that if (7.58) has a solution, then it can be represented by (7.59) and that every function of the form (7.59) is a solution of (7.58). And this completes the proof of the theorem. □

We now recall an additional lemma that will prove to be useful later in this section.

Lemma 7.37. *For t and s for which both* $(t-s-1)^{\underline{\nu}}$ *and* $(t-s-2)^{\underline{\nu}}$ *are defined, we find that*

$$\Delta_s\left[(t-s-1)^{\underline{\nu}}\right] = -\nu(t-s-2)^{\underline{\nu-1}}.$$

Proof. Omitted—see [89, Lemma 2.4]. □

Finally, for $\nu \in (1,2]$ given, we provide the following lemma, which will also be of importance later in this section.

Lemma 7.38. *The map*

$$t \mapsto \frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right]$$

is strictly decreasing in t, for $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. *In addition, it holds both that*

$$\min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left[\frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right]\right] = 0$$

and that

$$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left[\frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right]\right] = 1.$$

Proof. Note that

$$\Delta_t\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right] = (\nu-2)t^{\underline{\nu-3}} - \frac{\nu-1}{b+2}t^{\underline{\nu-2}} < 0, \tag{7.64}$$

where the inequality in (7.64) follows from the observation that $(\nu-2)(b+2) - (t-\nu+3)(\nu-1) < 0$. It follows that the map

$$t \mapsto \frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right]$$

is strictly decreasing in t as well. Furthermore, notice both that

$$\frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{1}{b+2}t^{\underline{\nu-1}}\right]_{t=\nu-2} = \frac{1}{\Gamma(\nu-1)}\left[\Gamma(\nu-1) - \frac{0}{b+2}\right] = 1 \tag{7.65}$$

and that

$$\frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}}-\frac{1}{b+2}t^{\underline{\nu-1}}\right]_{t=\nu+b}=0.$$

In particular, as a consequence of (7.64)–(7.65) we see that the second claim in the statement of the theorem follows. And this completes the proof of the lemma. □

We now wish to show that under certain conditions, problem (7.58) has at least one solution. We observe that problem (7.58) may be recast as an equivalent summation equation. In particular, y is a solution of (7.58) if and only if y is a fixed point of the operator $T : \mathbb{R}^{b+3} \to \mathbb{R}^{b+3}$, where

$$\begin{aligned}(Ty)(t) :=& -\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s+\nu-1, y(s+\nu-1))\\ &+\frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}f(s+\nu-1, y(s+\nu-1))\\ &-\frac{t^{\underline{\nu-1}}g(y)}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}g(y),\end{aligned} \tag{7.66}$$

for $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$; this observation follows from Theorem 7.36. We now use this fact to state and prove our first existence theorem.

Theorem 7.39. *Suppose that the maps $(t, y) \mapsto f(t, y)$ and $y \mapsto g(y)$ are Lipschitz in y. That is, there exist α, $\beta > 0$ such that $|f(t, y_1) - f(t, y_2)| \le \alpha|y_1 - y_2|$ whenever $y_1, y_2 \in \mathbb{R}$, and $|g(y_1) - g(y_2)| \le \beta\|y_1 - y_2\|$ whenever $y_1, y_2 \in \mathcal{C}([\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$. Then it follows that problem (7.58) has a unique solution provided that the condition*

$$2\alpha\prod_{j=1}^{b}\left(\frac{\nu+j}{j}\right)+\beta<1 \tag{7.67}$$

holds.

Proof. We will show that under the hypotheses in the statement of this theorem T is a contraction mapping. To this end, we notice that for each admissible y_1 and y_2 it holds that

$$
\begin{aligned}
&\|Ty_1 - Ty_2\| \\
&\le \alpha\|y_1 - y_2\| \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left[\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\right] \\
&\quad + \alpha\|y_1 - y_2\| \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left[\frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\right] \\
&\quad + \beta\|y_1 - y_2\| \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left|-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right|.
\end{aligned}
\tag{7.68}
$$

We now analyze each of the three terms on the right-hand side of (7.68).

We first notice, by an application of Lemma 7.37, that

$$
\begin{aligned}
&\alpha\|y_1 - y_2\|\left[\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\right] = \frac{\alpha\|y_1 - y_2\|}{\Gamma(\nu)}\left[-\frac{1}{\nu}(t-s)^{\underline{\nu}}\right]_{s=0}^{t-\nu+1} \\
&= \alpha\|y_1 - y_2\|\left[\frac{\Gamma(t+1)}{\Gamma(t-\nu+1)\Gamma(\nu+1)}\right] \le \alpha\|y_1 - y_2\|\left[\frac{\Gamma(\nu+b+1)}{\Gamma(b+1)\Gamma(\nu+1)}\right] \\
&= \alpha\prod_{j=1}^{b}\left(\frac{\nu+j}{j}\right)\|y_1 - y_2\|.
\end{aligned}
\tag{7.69}
$$

So, this estimates the first term on the right-hand side of (7.68). Then another application of Lemma 7.37 reveals that

$$
\begin{aligned}
&\alpha\|y_1 - y_2\|\left[\frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\right] \\
&\le \frac{\alpha\|y_1 - y_2\|}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}} = \frac{\alpha\|y_1 - y_2\|}{\Gamma(\nu)}\left[-\frac{1}{\nu}(\nu+b-s)^{\underline{\nu}}\right]_{s=0}^{b+1} \\
&= \alpha\|y_1 - y_2\|\prod_{j=1}^{b}\left(\frac{\nu+j}{j}\right),
\end{aligned}
\tag{7.70}
$$

which provides an upper bound for the second term appearing on the right-hand side of (7.66). Finally, we may estimate the third term on the right-hand side of (7.68) by employing Lemma 7.38 and observing that

$$
\beta\|y_1 - y_2\|\left|-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right| \le \beta\|y_1 - y_2\|. \tag{7.71}
$$

Putting (7.69)–(7.71) into the right-hand side of (7.68), we conclude at once that

$$\|Ty_1 - Ty_2\| \leq \left\{ 2\alpha \prod_{j=1}^{b} \left(\frac{\nu + j}{j} \right) + \beta \right\} \|y_1 - y_2\|.$$

So, by requiring condition (7.67) to hold, we find that (7.58) has a unique solution. And this completes the proof. □

By weakening the conditions imposed on the functions f and g, we can still deduce the existence of at least one solution to (7.58). We shall appeal to the Brouwer theorem to accomplish this.

Theorem 7.40. *Suppose that there exists a constant $M > 0$ such that $f(t, y)$ satisfies the inequality*

$$\max_{(t,y)\in[\nu-1,\nu+b-1]_{\mathbb{N}_{\nu-1}}\times[-M,M]} |f(t, y)| \leq \frac{M}{\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)} + 1} \tag{7.72}$$

and $g(y)$ satisfies the inequality

$$\max_{0\leq\|y\|\leq M} |g(y)| \leq \frac{M}{\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)} + 1}. \tag{7.73}$$

Then (7.58) has at least one solution, say y_0, satisfying $|y_0(t)| \leq M$, for all $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$.

Proof. Consider the Banach space $\mathcal{B} := \left\{ y \in \mathbb{R}^{b+3} : \|y\| \leq M \right\}$. Let T be the operator defined in (7.66). It is clear that T is a continuous operator. Therefore, the main objective in establishing this result is to show that $T : \mathcal{B} \to \mathcal{B}$—that is, whenever $\|y\| \leq M$, it follows that $\|Ty\| \leq M$. Once this is established, the Brouwer theorem will be invoked to deduce the conclusion.

To this end, assume that inequalities (7.72)–(7.73) hold for given f and g. For notational convenience in the sequel, let us put

$$\Omega_0 := \frac{M}{\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)} + 1}, \tag{7.74}$$

which is a positive constant. Using the notation introduced previously in (7.74), observe that

$$\begin{aligned} &\|Ty\| \\ &\leq \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} |f(s + \nu - 1), y(s + \nu - 1))| \\ &\quad + \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left\{ \frac{t^{\underline{\nu-1}}}{(\nu + b)^{\underline{\nu-1}}\Gamma(\nu)} \right. \end{aligned}$$

$$\times \sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}|f(s+\nu-1,y(s+\nu-1))|\Bigg\}$$

$$+\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left|-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu)}\right||g(y)|$$

$$\leq \Omega_0 \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left[\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\right.$$

$$\left.+\sum_{s=0}^{b}\frac{t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\right]$$

$$+\Omega_0\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left|-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu)}\right|. \tag{7.75}$$

Now, much as in the proof of Theorem 7.39 we can simplify the expression on the right-hand side of inequality (7.75). In particular, we observe that

$$\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}+\frac{t^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}$$

$$\leq \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}$$

$$\leq \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}$$

$$=\frac{2}{\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}, \tag{7.76}$$

where to obtain inequality (7.76) we have used the fact that the map $t\mapsto t^{\underline{\nu-1}}$ is increasing in t since $\nu>1$. Furthermore, it holds that

$$\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}=\left[-\frac{1}{\nu}(\nu+b-s)^{\underline{\nu}}\right]_{s=0}^{b+1}=\frac{\Gamma(\nu+b+1)}{\nu\Gamma(b+1)}. \tag{7.77}$$

In addition we may estimate the second term on the right-hand side of inequality (7.75) by using Lemma 7.38, which implies that

$$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left|-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu)}\right|=1. \tag{7.78}$$

If we now put (7.75)–(7.78) together, then we find that

$$\begin{aligned}\|Ty\| &\leq \Omega_0\left[\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)}\right]+\Omega_0\\ &= \Omega_0\left[\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)}+1\right].\end{aligned} \tag{7.79}$$

Finally, by the definition of Ω_0 given earlier in (7.74), we deduce that (7.79) implies that

$$\|Ty\| \leq \Omega_0\left[\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu)\Gamma(b+1)}+1\right]=M. \tag{7.80}$$

Thus, from (7.80) we conclude that $T : \mathcal{B} \to \mathcal{B}$, as desired. Consequently, it follows at once by the Brouwer theorem that there exists a fixed point of the map T, say $y_0 \in \mathcal{B}$. But this function y_0 is a solution of (7.58). Moreover, y_0 satisfies the bound $|y_0(t)| \leq M$, for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. Thus, the proof is complete. □

We next we wish to deduce the existence of at least one positive solution to problem (7.58). To this end, we first need recall some facts about the Green's function for the problem

$$\begin{aligned}-\Delta^\nu y(t) &= f(t+\nu-1, y(t+\nu-1))\\ y(\nu-2) &= 0\\ y(\nu+b) &= 0.\end{aligned}$$

In particular, we recall the following result.

Lemma 7.41. *Let* $1 < \nu \leq 2$. *The unique solution of the FBVP*

$$\begin{aligned}-\Delta^\nu y(t) &= h(s+\nu-1)\\ y(\nu-2) &= 0\\ y(\nu+b) &= 0\end{aligned}$$

is given by the map $y : [\nu-2, \nu+b]_{\mathbb{Z}_{\nu-2}} \to \mathbb{R}$ *defined by*

$$y(t) = \sum_{s=0}^{b+1} G(t,s)h(s+\nu-1),$$

where the Green's function $G : [\nu-2, \nu+b]_{\mathbb{Z}_{\nu-2}} \times [0,b]_{\mathbb{N}_0} \to \mathbb{R}$ *is defined by*

$$G(t,s) := \frac{1}{\Gamma(\nu)}\begin{cases}\frac{t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}} - (t-s-1)^{\underline{\nu-1}}, & (t,s) \in T_2\\ \frac{t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-1}}}, & (t,s) \in T_2\end{cases},$$

where

$$T_1 := \{(t,s) \in [\nu-2, \nu+b]_{\mathbb{Z}_{\nu-2}} \times [0,b]_{\mathbb{N}_0} \ : \ 0 \le s < t-\nu+1 \le b+1\}$$

and

$$T_2 := \{(t,s) \in [\nu-2, \nu+b]_{\mathbb{Z}_{\nu-2}} \times [0,b]_{\mathbb{N}_0} \ : \ 0 \le t-\nu+1 \le s \le b+1\}.$$

Lemma 7.42. *The Green's function G defined in Lemma 7.41 satisfies the following conditions:*

(i) $G(t,s) > 0$ *for* $t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}$ *for* $s \in [0,b]_{\mathbb{N}}$*;*
(ii) $\max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu-1,s)$ *for* $s \in [0,b]_{\mathbb{N}}$*; and*
(iii) *There exists a number* $\gamma \in (0,1)$ *such that*

$$\min_{\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}} G(t,s) \ge \gamma \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s) = \gamma G(s+\nu-1,s),$$

for $s \in [0,b]_{\mathbb{N}_0}$.

Remark 7.43. The proof of both Lemmas 7.41 and 7.42 are simple modifications of the proofs of [31, Theorem 3.1] and [31, Theorem 3.2], respectively. Hence, we omit the proofs.

Before defining the cone that we shall use to prove our existence theorems, we need a preliminary lemma.

Lemma 7.44. *If the map* $y \mapsto g(y)$ *is nonnegative, then there exists a constant* $\tilde{\gamma} \in (0,1)$ *with the property that*

$$\begin{aligned}
&\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} \sum_{s=0}^{b} G(t,s)f(s+\nu-1, y(s+\nu-1)) \\
&\qquad + \min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y) \\
&\ge \tilde{\gamma} \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b} G(t,s)f(s+\nu-1, y(s+\nu-1)) \\
&\qquad + \tilde{\gamma} \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y). \qquad (7.81)
\end{aligned}$$

Proof. To see that this is true, observe first that by Lemma 7.42 we find $\gamma \in (0,1)$ such that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,y(s+\nu-1))$$
$$\geq \gamma \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,y(s+\nu-1)). \tag{7.82}$$

Now, recall from Lemma 7.38 that the map

$$t\mapsto \frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}}-\frac{1}{b+2}t^{\underline{\nu-1}}\right]$$

is strictly decreasing in t and, furthermore, is strictly positive for $t < b+\nu$. In particular, from this observation we deduce the existence of a number $M > 0$ such that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]=M. \tag{7.83}$$

Note that we assume here that there exists a point $\tilde{t}\in\mathbb{N}_{\nu-2}$ such that $\frac{b+\nu}{4}\leq\tilde{t}\leq\frac{3(b+\nu)}{4}$. Additionally, we recall from Lemma 7.38 that

$$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}}-\frac{1}{b+2}t^{\underline{\nu-1}}\right]=1. \tag{7.84}$$

In particular, then, (7.83)–(7.84) imply that by putting

$$\gamma_0 := M, \tag{7.85}$$

where γ_0 is clearly strictly positive, it follows from (7.83)–(7.85) that

$$\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]g(y)$$
$$=\gamma_0\cdot\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]g(y).$$

Finally, define $\tilde{\gamma}$ by

$$\tilde{\gamma} := \min\{\gamma,\gamma_0\}. \tag{7.86}$$

Evidently, definition (7.86) implies that $\tilde{\gamma}\in(0,1)$. Moreover, inequality (7.82) implies that

$$
\begin{aligned}
&\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,y(s+\nu-1))\\
&\qquad+\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]g(y)\\
&\geq\tilde{\gamma}\max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,y(s+\nu-1))\\
&\qquad+\tilde{\gamma}\max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}}\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)}+\frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]g(y),
\end{aligned}
\tag{7.87}
$$

which since (7.87) is (7.81) completes the proof of the lemma. □

Now, let us put

$$
\eta:=\frac{1}{\sum_{s=0}^{b}G(s+\nu-1,s)}
$$

and

$$
\lambda:=\frac{1}{\sum_{s=\left\lceil\frac{\nu+b}{4}-\nu+1\right\rceil}^{\left\lfloor\frac{3(\nu+b)}{4}-\nu+1\right\rfloor}\tilde{\gamma}G\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu,s\right)}.
\tag{7.88}
$$

In addition, define the set $\mathcal{K}\subseteq\mathcal{C}\left([\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}},\mathbb{R}\right)$ by

$$
\mathcal{K}:=\left\{y\ :\ [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\to\mathbb{R}\ :\ y(t)\geq 0,\right.
\left.\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}y(t)\geq\tilde{\gamma}\|y(t)\|\right\},
\tag{7.89}
$$

which is a cone in the Banach space $\mathcal{C}\left([\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}},\mathbb{R}\right)$, where the number $\tilde{\gamma}$ in (7.88)–(7.89) is the same number as given in Lemma 7.44 above. Moreover, we will also need in the sequel the constant

$$
\eta^*:=\frac{1}{2}.
$$

Finally, we introduce some conditions that will be helpful in the sequel; these conditions place some control on the growth of the nonlinearity f as well as the functional g appearing in (7.58).

F1: There exists a number $r > 0$ such that $f(t,y) \leq \frac{1}{2}\eta r$ whenever $0 \leq y \leq r$.
F2: There exists a number $r > 0$ such that $f(t,y) \geq \lambda r$ whenever $\tilde{\gamma} r \leq y \leq r$, where $\tilde{\gamma}$ is the number provided in Lemma 7.44.
G1: There exists a number $r > 0$ such that $g(y) \leq \eta^* r$ whenever $0 \leq \|y\| \leq r$.

Remark 7.45. The operator T defined in (7.66) may be written in the form

$$(Ty)(t) = \sum_{s=0}^{b} G(t,s) f(s+\nu-1, y(s+\nu-1)) + \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y), \tag{7.90}$$

where G is the Green's function from Lemma 7.41. This observation is important since it allows us to use the known properties of the map $(t,s) \mapsto G(t,s)$ to obtain useful estimates in the existence argument.

With these declarations in hand, we proceed with proving an existence theorem. We begin with a preliminary lemma, however, to establish separately that T in fact maps $\mathcal{K}$ into itself.

Lemma 7.46. *Let T be defined as in (7.90) and $\mathcal{K}$ as in (7.89). Assume in addition that both f and g are nonnegative. Then $T(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof. Let T be the operator defined in (7.90). Observe that

$$\begin{aligned}
&\min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} (Ty)(t) \\
&\geq \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} \sum_{s=0}^{b} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\
&\qquad + \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y) \\
&\geq \tilde{\gamma} \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\
&\qquad + \tilde{\gamma} \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y) \\
&\geq \tilde{\gamma} \|Ty\|,
\end{aligned} \tag{7.91}$$

where $\tilde{\gamma}$ is as defined in (7.86). Since it is obvious that $(Ty)(t) \geq 0$ for all t whenever $y \in \mathcal{K}$, it follows that (7.91) establishes that $T(\mathcal{K}) \subseteq \mathcal{K}$, as desired. □

Theorem 7.47. *Suppose that there exists two distinct numbers r_1 and r_2, with $r_1, r_2 > 0$, such that conditions (F1) and (G1) hold at r_1 and condition (F2) holds at r_2. Finally, assume that each of f and g is nonnegative. Then problem (7.58) has a positive solution, whose norm lies between r_1 and r_2.*

Proof. Let T be the operator defined in (7.90). It is clear that T is completely continuous, and Lemma 7.46 establishes that $T(\mathcal{K}) \subseteq \mathcal{K}$. Without loss of generality, suppose that $0 < r_1 < r_2$. Define the set Ω_1 by

$$\Omega_1 := \{y \in \mathcal{C}([\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}},\mathbb{R}) \ : \ \|y\| < r_1\}.$$

Then we have that for $y \in \partial\Omega_1 \cap \mathcal{K}$

$$\begin{aligned}
\|Ty\| \le& \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\
&\qquad + \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left\{\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y)\right\} \\
\le& \frac{r_1\eta}{2}\sum_{s=0}^{b} G(s+\nu-1,s) \\
&\qquad + g(y) \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{1}{\Gamma(\nu-1)}\left[t^{\underline{\nu-2}} - \frac{t^{\underline{\nu-1}}}{b+2}\right] \\
\le& \frac{r_1}{2} + \frac{r_1}{2} \\
=& \|y\|.
\end{aligned} \tag{7.92}$$

So, from (7.92) we conclude that $\|Ty\| \le \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$.

Conversely, define the set Ω_2 by

$$\Omega_2 := \{y \in \mathcal{C}([\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}},\mathbb{R}) \ : \ \|y\| < r_2\}.$$

Then using Lemma 7.42, for $y \in \partial\Omega_2 \cap \mathcal{K}$ we estimate

$$\begin{aligned}
&(Ty)\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu\right) \\
&\ge \sum_{s=0}^{b} G\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\
&\ge \lambda r_2 \sum_{s=\lceil\frac{\nu+b}{4}-\nu+1\rceil}^{\lfloor\frac{3(\nu+b)}{4}-\nu+1\rfloor} \tilde{\gamma} G\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu, s\right) \ge r_2 = \|y\|.
\end{aligned} \tag{7.93}$$

Consequently, from (7.93) we conclude that $\|Ty\| \geq \|y\|$ whenever $y \in \partial\Omega_2 \cap \mathcal{K}$. But then by an application of the well-known Krasnosel'skiĭ fixed point theorem we conclude that T has a fixed point, say, $y_0 \in \mathcal{K}$. This map $t \mapsto y_0(t)$ is a positive solution to problem (7.58) since $y_0 \in \mathcal{K}$ satisfies $r_1 < \|y_0\| < r_2$. Thus, the proof is complete. □

We now provide a second result that yields the existence of at least one positive solution. In what follows, we shall assume that f has the special form $f(t, y) \equiv F_1(t)F_2(y)$. Moreover, to facilitate this result, we introduce the following additional conditions on F_2 and g.

F3: The function F_2 satisfies $\lim_{y\to 0^+} \frac{F_2(y)}{y} = 0$.

F4: The function F_2 satisfies $\lim_{y\to\infty} \frac{F_2(y)}{y} = +\infty$.

G2: The function g satisfies $\lim_{\|y\|\to 0^+} \frac{g(y)}{\|y\|} = 0$.

Remark 7.48. Observe that there are many nontrivial functionals $y \mapsto g(y)$ satisfying condition (G2). For example, the functional defined by $g(y) := [y(\nu+1)]^3$ clearly satisfies (G2).

Theorem 7.49. *Suppose that conditions (F3)–(F4) and (G2) hold. Moreover, assume that each of F_1, F_2, and g is nonnegative. Then problem* (7.58) *has at least one positive solution.*

Proof. Because of condition (F3), there exists a number $\alpha_1 > 0$ sufficiently small such that

$$F_2(y) \leq \eta_1 y, \tag{7.94}$$

for each $y \in (0, \alpha_1]$, and where we choose η_1 sufficiently small so that

$$\eta_1 \sum_{s=0}^{b} G(s+\nu-1, s)F_1(s) \leq \frac{1}{2} \tag{7.95}$$

holds. Similarly, condition (G2) implies that there exists a number $\alpha_2 > 0$ such that

$$g(y) \leq \eta_2 \|y\| \tag{7.96}$$

whenever $\|y\| \in (0, \alpha_2]$, and where η_2 is chosen so that

$$\eta_2 \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left\{ -\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)} \right\} \leq \eta_2 \leq \frac{1}{2}. \tag{7.97}$$

Now, put $\alpha^* := \min\{\alpha_1, \alpha_2\}$ and define the set Ω_1 by

$$\Omega_1 := \{y \in \mathcal{K} : \|y\| < \alpha^*\}.$$

Then it follows that for all $y \in \mathcal{K} \cap \partial\Omega_1$ inequalities (7.94)–(7.97) imply that

$$\begin{aligned}\|Ty\| &\leq \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b} G(t,s)F_1(s+\nu-1)F_2(y(s+\nu-1))\\ &\quad + \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \left\{\left[\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y)\right\}\\ &\leq \eta_1\|y\| \sum_{s=0}^{b} G(s+\nu-1,s)F_1(s+\nu-1) + \eta_2\|y\|\\ &\leq \left[\frac{1}{2}+\frac{1}{2}\right]\|y\|\\ &= \|y\|, \end{aligned} \tag{7.98}$$

whence (7.98) implies that $\|Ty\| \leq \|y\|$.

On the other hand, condition (F4) implies the existence of a number $\alpha_3 > 0$ such that

$$F_2(y) \geq \eta_3 y \tag{7.99}$$

whenever $y \geq \alpha_3$. Furthermore, we can choose η_3 sufficiently large such that

$$\eta_3 \sum_{s=\lceil \frac{\nu+b}{4}-\nu+1\rceil}^{\lfloor \frac{3(\nu+b)}{4}-\nu+1\rfloor} \tilde{\gamma} G\left(\left\lfloor \frac{b+1}{2}\right\rfloor + \nu, s\right) F_1(s+\nu-1) \geq 1. \tag{7.100}$$

Put

$$\alpha^{**} := \max\left\{2\alpha^*, \frac{\alpha_3}{\tilde{\gamma}}\right\} \tag{7.101}$$

and observe that for $\|y\| = \alpha^{**}$ we estimate

$$\min_{\frac{b+\nu}{4}\leq t\leq \frac{3(b+\nu)}{4}} y(t) \geq \tilde{\gamma}\|y\| \geq \alpha_3. \tag{7.102}$$

Now, define the set Ω_2 by

$$\Omega_2 := \{y \in \mathcal{K} \ : \ \|y\| < \alpha^{**}\}.$$

Recall from the proof of Lemma 7.38 that

$$-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)} \geq 0, \tag{7.103}$$

for each $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$. And from this it follows that

$$\left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] g(y) \geq 0, \tag{7.104}$$

for each $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$. Thus, putting (7.99)–(7.104) together, we find that for $y \in \partial\Omega_2 \cap \mathcal{K}$,

$$\begin{aligned}
&(Ty)\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu\right) \\
&= \sum_{s=0}^{b} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) F_1(s+\nu-1) F_2(y(s+\nu-1)) \\
&\qquad + \left[-\frac{t^{\underline{\nu-1}}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]_{t=\left\lfloor \frac{b+1}{2} \right\rfloor + \nu} g(y) \\
&\geq \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) F_1(s+\nu-1) F_2(y(s+\nu-1)) \\
&\geq \eta_3 \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) F_1(s+\nu-1) y(s+\nu-1) \\
&\geq \eta_3 \|y\| \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} \tilde{\gamma} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) F_1(s+\nu-1) \geq \|y\|.
\end{aligned} \tag{7.105}$$

So, from (7.105) we conclude that $\|Ty\| \geq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_2$. Consequently, we deduce that T has a fixed point in the set $\left(\mathcal{K} \cap \overline{\Omega_2}\right) \setminus \Omega_1$. Since this fixed point is a positive solution to (7.58), the claim follows. □

Remark 7.50. Observe that in case $\nu = 2$, both Theorems 7.47 and 7.49 provide results for the existence of a positive solution to the integer-order nonlocal BVP given by (7.58).

We conclude this section by providing two examples of certain of the theorems presented in this section. We begin with an example illustrating Theorem 7.39 followed by an example illustrating Theorem 7.40.

Example 7.51. Suppose that $\nu = \frac{11}{10}$ and $b = 10$. In addition, let us suppose that $f(t, y) := \frac{\sin y}{30+t^2}$ and that $g(y) := \frac{1}{50}\,[y(\nu+1) + y(\nu+2)]$. We consider the FBVP

$$
\begin{aligned}
-\Delta^{\frac{11}{10}} y(t) &= \frac{\sin\left(y\left(t+\frac{1}{10}\right)\right)}{30+\left(t+\frac{1}{10}\right)^2} \\
y(\nu-2) &= \frac{1}{50}\left[y(\nu+1)+y(\nu+2)\right] \\
y(\nu+b) &= 0.
\end{aligned}
\tag{7.106}
$$

Now, in this case inequality (7.67) is

$$
2\alpha \prod_{j=1}^{b}\left(\frac{\nu+j}{j}\right)+\beta \leq 26.851\alpha+\beta<1. \tag{7.107}
$$

But it is not difficult to prove that each of f and g is Lipschitz with Lipschitz constants $\alpha = \frac{1}{30}$ and $\beta = \frac{1}{25}$, respectively. So, for these choices of α and β, inequality (7.107) is satisfied. Therefore, we deduce from Theorem 7.39 that problem (7.106) has a unique solution.

Example 7.52. Suppose that $\nu = \frac{3}{2}$, $b = 10$, and $M = 1000$. Also suppose that $f(t,y) := \frac{1}{10}te^{-\frac{1}{100}t|y|}$ and that $g(y) := \sum_{i=1}^{n} c_i y\left(t_i\right)$, where $\{t_i\}_{i=1}^{n} \subseteq [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ is a strictly increasing sequence satisfying $\nu-2 \leq t_1 < t_2 < \cdots < t_n \leq \nu+b$ with $t_i \in \mathbb{N}_{\nu-1}$ for each i. (Clearly, we must take $n \leq b+3$ here.) Thus, in this case problem (7.58) becomes

$$
\begin{aligned}
-\Delta^{\frac{3}{2}} y(t) &= \frac{1}{10}\left(t+\frac{1}{2}\right) e^{-\frac{1}{100}\left(t+\frac{1}{2}\right)\left|y\left(t+\frac{1}{2}\right)\right|} \\
y(\nu-2) &= \sum_{i=1}^{n} c_i y\left(t_i\right) \\
y(\nu+b) &= 0.
\end{aligned}
\tag{7.108}
$$

Furthermore, note that in this setting the Banach space $\mathcal{B}$ assumes the form $\mathcal{B} := \left\{y \in \mathbb{R}^{13} \;:\; \|y\| \leq 1000\right\}$.

We claim that (7.108) has at least one solution. So, to check that the hypotheses of Theorem 7.40 hold, we note that

$$
\frac{M}{\frac{2\Gamma(\nu+b+1)}{\Gamma(\nu+1)\Gamma(b+1)}+1} = \frac{1000}{\frac{2\Gamma\left(\frac{3}{2}+10+1\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma(11)}+1} \approx 11.614.
$$

It is evident that $|f(t,y)| \leq \frac{23}{20} < 11.614$ whenever $y \in [-1000, 1000]$. On the other hand, if we require, say, the condition

$$
\sum_{i=1}^{n} |c_i| \leq \frac{1}{100}, \tag{7.109}
$$

then (7.109) implies that for each $y \in \mathcal{C}([\nu-2,\nu+b],\mathbb{R})$ satisfying the condition $\|y\| \le 1000$ it holds that

$$|g(y)| \le \sum_{i=1}^{n} |c_i|\,|y(t_i)| \le 1000 \sum_{i=1}^{n} |c_i| \le 10 < 11.350$$

so that g satisfies condition (7.73). Thus, given restriction (7.109), we conclude from Theorem 7.40 that (7.108) has at least one solution. In particular, by the conclusion of Theorem 7.40 we deduce that this solution, say y_0, satisfies

$$|y_0(t)| \le 1000, \text{ for } t \in \left[-\frac{1}{2}, \frac{23}{2}\right]_{\mathbb{Z}_{-\frac{1}{2}}}.$$

7.6 Discrete Sequential Fractional Boundary Value Problems

In this section we emphasize a different property of the discrete fractional difference and see how it can give rise to a suitably nonlocal problem. In particular, we consider the concept of a so-called sequential fractional boundary value problem. Recall that for fractional differences it does *not* necessarily hold that $\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t) = \Delta^{\nu+\mu}_{a}f(t)$, as was discussed in Chap. 2. Consequently, we may consider a so-called discrete sequential FBVP. In this case, we consider the discrete fractional boundary value problem

$$\begin{aligned} -\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) &= f\left(t+\mu_1+\mu_2+\mu_3-1, y\left(t+\mu_1+\mu_2+\mu_3-1\right)\right) \\ y(0) &= 0 \\ y(b+2) &= 0, \end{aligned} \tag{7.110}$$

for

$$t \in [2-\mu_1-\mu_2-\mu_3, b+2-\mu_1-\mu_2-\mu_3]_{\mathbb{Z}_{2-\mu_1-\mu_2-\mu_3}},$$

and where throughout we make the assumptions that $\mu_i \in (0,1)$, for each $i = 1,2,3$, and that each of $1 < \mu_2+\mu_3 < 2$ and $1 < \mu_1+\mu_2+\mu_3 < 2$ holds. The potential interest in problem (7.110) is that the sequence of fractional difference $\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t)$ is not necessarily equivalent to the non-sequential difference $\Delta^{\mu_1+\mu_2+\mu_3}y(t)$. Consequently, we have in the fractional setting a situation that cannot occur in the integer-order setting since $\Delta^{k_1}\Delta^{k_2}y(t) = \Delta^{k_1+k_2}y(t)$, for each k_1, $k_2 \in \mathbb{N}$. Moreover, this dissimilarity is a direct consequence of the implicit nonlocal structure of the fractional difference. We note that the results of this section can be found in Goodrich [99].

We begin by proving a simple proposition. This realization will be important later in this section.

Proposition 7.53. *Let* $y : \mathbb{N}_0 \to \mathbb{R}$ *with* $\mu \in (0,1]$. *Then we find that*

$$\Delta^{\mu-1}y(1-\mu) = y(0).$$

Proof. To see that this is true, observe that $\mu - 1 \leq 0$ since $\mu \in (0,1]$. By definition, then, it follows that

$$\begin{aligned}
\Delta^{\mu-1}y(1-\mu) &= \left[\frac{1}{\Gamma(1-\mu)}\sum_{s=0}^{t+\mu-1}(t-s-1)^{\underline{-\mu}}y(s)\right]_{t=1-\mu} \\
&= \frac{1}{\Gamma(1-\mu)}\sum_{s=0}^{0}(-\mu-s)^{\underline{-\mu}}y(s) \\
&= \frac{1}{\Gamma(1-\mu)}\cdot\Gamma(1-\mu)y(0) \\
&= y(0),
\end{aligned}$$

as claimed. □

We now provide an analysis of problem (7.110). We begin by repeatedly applying the composition rules for fractional differences to derive a representation of a solution to (7.110) as the fixed point of an appropriate operator. In the sequel, the Banach space $\mathcal{B}$ is the set of (continuous) real-valued maps from $[0, b+2]_{\mathbb{N}_0}$ when equipped with the usual maximum norm, $\|\cdot\|$. Moreover, henceforth we also put

$$\tilde{\mu} := \mu_1 + \mu_2 + \mu_3,$$

for notational convenience. Recall that in what follows we assume both that $\mu_1 + \mu_2 \in (1,2)$ and that $\tilde{\mu} \in (1,2)$. Finally, we give the following notation, which will also be useful in the sequel.

$$T_1 := \Big\{(t,s) \in [0, b+2]_{\mathbb{N}_0} \times [2-\tilde{\mu}, b+2-\tilde{\mu}]_{\mathbb{N}_{2-\tilde{\mu}}} : \\ 0 \leq s < t - \tilde{\mu} + 1 \leq b+2\Big\}$$

$$T_2 := \Big\{(t,s) \in [0, b+2]_{\mathbb{N}_0} \times [2-\tilde{\mu}, b+2-\tilde{\mu}]_{\mathbb{N}_{2-\tilde{\mu}}} : \\ 0 \leq t - \tilde{\mu} + 1 \leq s \leq b+2\Big\}$$

Theorem 7.54. *Let the operator* $T : \mathcal{B} \to \mathcal{B}$ *be defined by*

$$(Ty)(t) := \alpha(t)y(1) + \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G(t,s)f\left(s+\tilde{\mu}-1, y\left(s+\tilde{\mu}-1\right)\right), \tag{7.111}$$

where $\alpha : [0, b+2]_{\mathbb{N}_0} \to \mathbb{R}$ *is defined by*

$$\alpha(t) := \frac{(t-2+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{\Gamma(\mu_2+\mu_3)} - \frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}\Gamma(\mu_2+\mu_3)}(t+\tilde{\mu}-2)^{\underline{\tilde{\mu}-1}} \tag{7.112}$$

and $G : [0, b+2]_{\mathbb{N}_0} \times [-\tilde{\mu}+2, -\tilde{\mu}+b+2]_{\mathbb{N}_{2-\tilde{\mu}}} \to \mathbb{R}$ *is the Green's function for the non-sequential conjugate problem given by*

$$G(t,s) := \begin{cases} \frac{(t+\tilde{\mu}-2)^{\underline{\tilde{\mu}-1}}(b+1-s)^{\underline{\tilde{\mu}-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}} - (t-s-1)^{\underline{\tilde{\mu}-1}}, & (t,s) \in T_1 \\ \frac{(t+\tilde{\mu}-2)^{\underline{\tilde{\mu}-1}}(b+1-s)^{\underline{\tilde{\mu}-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}}, & (t,s) \in T_2 \end{cases}. \tag{7.113}$$

Then whenever $y \in \mathcal{B}$ *is a fixed point of* T*, it follows that* y *is a solution of problem* (7.110)*.*

Proof. To begin the proof notice that by the operational properties deduced in Chap. 2 we may write

$$\begin{aligned}
&\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) \\
&= \Delta^{\mu_1}\left[\Delta^{\mu_2+\mu_3}y(t) - \frac{y(0)}{\Gamma(-\mu_2)}(t-1+\mu_3)^{\underline{-\mu_2-1}}\right] \\
&= \Delta^{\mu_1}\left[\Delta^{\mu_2+\mu_3}y(t)\right] - \frac{y(0)}{\Gamma(-\mu_2)}\Delta^{\mu_1}\left[(t-1+\mu_3)^{\underline{-\mu_2-1}}\right] \\
&= \Delta^{\tilde{\mu}}y(t) - \frac{y(0)}{\Gamma(-\mu_2)}\cdot\frac{\Gamma(-\mu_2)}{\Gamma(-\mu_2-\mu_1)}(t-1+\mu_3)^{\underline{-\mu_2-\mu_1-1}}] \\
&\quad - \sum_{j=0}^{1}\left[\frac{\Delta^{j-2+\mu_2+\mu_3}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1-2+j+1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-2+j}}\right] \\
&= \Delta^{\tilde{\mu}}y(t) - \frac{\Delta^{\mu_2+\mu_3-2}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1-1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-2}} \\
&\quad - \frac{\Delta^{\mu_2+\mu_3-1}y(2-\mu_2-\mu_3)}{\Gamma(-\mu_1)}(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}} \\
&\qquad - \frac{y(0)}{\Gamma(-\mu_2-\mu_1)}(t-1+\mu_3)^{\underline{-\mu_2-\mu_1-1}}.
\end{aligned} \tag{7.114}$$

Now, the same argument as in Proposition 7.53 shows that

$$\Delta^{\mu_2+\mu_3-2} y\,(2-\mu_2-\mu_3) = y(0). \tag{7.115}$$

On the other hand, note that by the definition of the fractional sum, keeping in mind that $\mu_2 + \mu_3 - 2 < 0$, we obtain that

$$\begin{aligned}
\Delta^{\mu_2+\mu_3-1} y(t) &= \Delta\Delta^{\mu_2+\mu_3-2} y(t) \\
&= \Delta_t \left[\frac{1}{\Gamma(2-\mu_2-\mu_3)} \sum_{s=0}^{t-2+\mu_2+\mu_3} (t-s-1)^{\underline{1-\mu_2-\mu_3}} y(s) \right] \\
&= \frac{1}{\Gamma(2-\mu_2-\mu_3)} \sum_{s=0}^{t-1+\mu_2+\mu_3} (t-s)^{\underline{1-\mu_2-\mu_3}} y(s) \\
&\quad - \frac{1}{\Gamma(2-\mu_2-\mu_3)} \sum_{s=0}^{t-2+\mu_2+\mu_3} (t-s-1)^{\underline{1-\mu_2-\mu_3}} y(s).
\end{aligned} \tag{7.116}$$

So, from (7.116), we obtain

$$\begin{aligned}
&\Delta^{\mu_2+\mu_3-1} y\,(2-\mu_2-\mu_3) \\
&= \frac{1}{\Gamma(2-\mu_2-\mu_3)} \sum_{s=0}^{1} (2-\mu_2-\mu_3-s)^{\underline{1-\mu_2-\mu_3}}\, y(s) \\
&\quad - \frac{1}{\Gamma(2-\mu_2-\mu_3)} \sum_{s=0}^{0} (1-\mu_2-\mu_3-s)^{\underline{1-\mu_2-\mu_3}}\, y(s) \\
&= \frac{1}{\Gamma(2-\mu_2-\mu_3)} y(0) \left[(2-\mu_2-\mu_3)^{\underline{1-\mu_2-\mu_3}} - (1-\mu_2-\mu_3)^{\underline{1-\mu_2-\mu_3}} \right] \\
&\quad + \frac{1}{\Gamma(2-\mu_2-\mu_3)} (1-\mu_2-\mu_3)^{\underline{1-\mu_2-\mu_3}}\, y(1).
\end{aligned} \tag{7.117}$$

Putting (7.115) and (7.117) into (7.114), we deduce that

$$\begin{aligned}
&\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3} y(t) \\
&= \Delta^{\tilde{\mu}} y(t) - \frac{[y(1) + (1-\mu_2-\mu_3)\,y(0)]}{\Gamma(-\mu_1)} (t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}} \\
&\quad - \frac{y(0)}{\Gamma(-\mu_1-1)} (t-2+\mu_2+\mu_3)^{\underline{-\mu_1-2}} \\
&\quad - \frac{y(0)}{\Gamma(-\mu_2-\mu_1)} (t-1+\mu_3)^{\underline{-\mu_2-\mu_1-1}},
\end{aligned} \tag{7.118}$$

where we have made some routine simplifications. Now, by the boundary conditions in (7.110) we find that (7.118) reduces to

$$\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}y(t) = \Delta^{\tilde{\mu}}y(t) - \frac{(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}}}{\Gamma(-\mu_1)}y(1). \tag{7.119}$$

Inverting the problem (7.110), we find by means of (7.119) that

$$\begin{aligned} y(t) = &-\Delta^{-\tilde{\mu}}\left[-\frac{(t-2+\mu_2+\mu_3)^{\underline{-\mu_1-1}}}{\Gamma(-\mu_1)}y(1)\right] \\ &- \Delta^{-\tilde{\mu}}f\left(t+\tilde{\mu}-1, y\left(t+\tilde{\mu}-1\right)\right) \\ &+ c_1\left(t+\tilde{\mu}-2\right)^{\underline{\tilde{\mu}-1}} + c_2\left(t+\tilde{\mu}-2\right)^{\underline{\tilde{\mu}-2}} \end{aligned} \tag{7.120}$$

holds.

Now, continuing from (7.120), it is clear that the boundary condition $y(0) = 0$ implies that $c_2 = 0$. On the other hand, the boundary condition $y(b+2) = 0$, implies that

$$\begin{aligned} 0 = &\, c_1\left(b+\tilde{\mu}\right)^{\underline{\tilde{\mu}-1}} + \frac{y(1)}{\Gamma(\mu_2+\mu_3)}\left(b+\mu_2+\mu_3\right)^{\underline{\mu_2+\mu_3-1}} \\ &- \frac{1}{\Gamma(\tilde{\mu})}\sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}}\left(b+1-s\right)^{\underline{\tilde{\mu}-1}}f\left(s+\tilde{\mu}-1, y\left(s+\tilde{\mu}-1\right)\right) \end{aligned} \tag{7.121}$$

From (7.121), we deduce that

$$\begin{aligned} c_1 = &-\frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}\,\Gamma(\mu_2+\mu_3)}y(1) \\ &+ \frac{1}{\Gamma(\tilde{\mu})}\sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}}\frac{(b+1-s)^{\underline{\tilde{\mu}-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}}f\left(s+\tilde{\mu}-1, y\left(s+\tilde{\mu}-1\right)\right). \end{aligned}$$

At last, substituting the values of c_1 and c_2 into (7.120), we conclude that

$$y(t) = \alpha(t)y(1) + \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G(t,s)f\left(s+\tilde{\mu}-1, y\left(s+\tilde{\mu}-1\right)\right), \tag{7.122}$$

where α is as defined in (7.112) above and the map $(t,s) \mapsto G(t,s)$ is as defined in (7.113) above. Now, if $(Ty)(t)$ is defined by the right-hand side of (7.122), i.e., we define $T : \mathcal{B} \to \mathcal{B}$ as in the statement of this theorem, then it is clear that T satisfies the boundary value problem (7.110). And this completes the proof. □

We next state an easy proposition regarding the Green's function, $(t,s)\mapsto G(t,s)$, appearing in the operator T, as defined above.

Proposition 7.55. *The Green's function* $(t,s)\mapsto G(t,s)$ *given in Theorem 7.54 satisfies:*

(i) $G(t,s)\geq 0$ *for each* $(t,s)\in[0,b+2]_{\mathbb{N}_0}\times[2-\tilde{\mu},b+2-\tilde{\mu}]_{\mathbb{N}_{2-\tilde{\mu}}}$;
(ii) $\max_{t\in[0,b+2]_{\mathbb{N}_0}} G(t,s)=G(s+\tilde{\mu}-1,s)$ *for each* $s\in[2-\tilde{\mu},b+2-\tilde{\mu}]_{\mathbb{N}_{2-\tilde{\mu}}}$; *and*
(iii) *there exists a number* $\gamma\in(0,1)$ *such that*

$$\min_{\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}_0}} G(t,s)\geq\gamma\max_{t\in[0,b+2]_{\mathbb{N}_0}} G(t,s)=\gamma G(s+\tilde{\mu}-1,s),$$

for $s\in[2-\tilde{\mu},b+2-\tilde{\mu}]_{\mathbb{N}_{2-\tilde{\mu}}}$.

Proof. Omitted—see [99] for details. □

We next require a preliminary lemma regarding the behavior of α appearing in (7.112) above.

Lemma 7.56. *Let* α *be defined as in* (7.112)*. Then* $\alpha(0)=\alpha(b+2)=0$*. Moreover,* $\|\alpha\|\in(0,1)$.

Proof. That $\alpha(0)=\alpha(b+2)=0$ is obvious. On the other hand, to show that $0<\|\alpha\|<1$, we argue as follows.

We show first that $\alpha(t)>0$, for all $t\in[1,b+1]_{\mathbb{N}}$. To this end, let us first note that

$$\begin{aligned}
&\alpha(t)\\
&=\frac{(t-2+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{\Gamma(\mu_2+\mu_3)}-\frac{(b+\mu_2+\mu_3)^{\underline{\mu_2+\mu_3-1}}}{(b+\tilde{\mu})^{\underline{\tilde{\mu}-1}}\Gamma(\mu_2+\mu_3)}(t+\tilde{\mu}-2)^{\underline{\tilde{\mu}-1}}\\
&=\frac{\Gamma(t+\mu_2+\mu_3-1)}{\Gamma(t)\Gamma(\mu_2+\mu_3)}-\frac{\Gamma(b+\mu_2+\mu_3+1)\Gamma(t+\tilde{\mu}-1)}{\Gamma(b+\tilde{\mu}+1)\Gamma(\mu_2+\mu_3)\Gamma(t)}\\
&=\frac{\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\tilde{\mu}+1)-\Gamma(t+\tilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)}{\Gamma(t)\Gamma(\mu_2+\mu_3)\Gamma(b+\tilde{\mu}+1)}.
\end{aligned}\tag{7.123}$$

Therefore, $\alpha(t)>0$, for each $t\in[1,b+1]_{\mathbb{N}}$, if and only if

$$\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\tilde{\mu}+1)>\Gamma(t+\tilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)\tag{7.124}$$

for each $t\in[1,b+1]_{\mathbb{N}}$. Now, (7.124) is equivalent to

$$\frac{\Gamma(t+\mu_2+\mu_3-1)\Gamma(b+\tilde{\mu}+1)}{\Gamma(t+\tilde{\mu}-1)\Gamma(b+\mu_2+\mu_3+1)}>1.$$

But since

$$\frac{\Gamma(t+\mu_2+\mu_3-1)\,\Gamma(b+\tilde{\mu}+1)}{\Gamma(t+\tilde{\mu}-1)\,\Gamma(b+\mu_2+\mu_3+1)}=\frac{(b+\tilde{\mu})\cdots(t+\tilde{\mu}-1)}{(b+\mu_2+\mu_3)\cdots(t+\mu_2+\mu_3-1)} \tag{7.125}$$

and the right-hand side of (7.125) is clearly greater than unity, it follows that (7.124) holds, and so, we conclude from (7.123)–(7.125) that $\alpha(t)>0$, for $t\in[1,b+1]_{\mathbb{N}}$, as claimed.

On the other hand, to argue that $\alpha(t)<1$, for $t\in[0,b+2]_{\mathbb{N}_0}$, we begin by recasting $\alpha(t)$ in a different form. In particular, define $\mu_0\in(1,2)$ by

$$\mu_0:=\mu_2+\mu_3. \tag{7.126}$$

Then it follows that

$$\tilde{\mu}=\mu_0+\mu_1. \tag{7.127}$$

Therefore, putting (7.126)–(7.127) into the definition of α provided in (7.112) we conclude that

$$\alpha(t)=\frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)}-\frac{(b+\mu_0)^{\underline{\mu_0-1}}(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}\,\Gamma(\mu_0)}. \tag{7.128}$$

Now, consider the map

$$t\mapsto\frac{(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}} \tag{7.129}$$

appearing in the second addend on the right-hand side of (7.128). Since

$$\begin{aligned}&\frac{(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}}\\&\quad=\frac{(b+1)\cdots(t+1)(t)}{(b+\mu_0+\mu_1)\cdots(t+\mu_0+\mu_1)\,(t+\mu_0+\mu_1-1)},\end{aligned} \tag{7.130}$$

we see from (7.130) that for each fixed but arbitrary b, t, and μ_0, the map defined in (7.129) decreases as μ_1 increases. Consequently, for fixed but arbitrary b, t, and μ_0 we conclude that

$$\begin{aligned}\alpha(t)&<\frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)}-\left[\frac{(b+\mu_0)^{\underline{\mu_0-1}}(t+\mu_0+\mu_1-2)^{\underline{\mu_0+\mu_1-1}}}{(b+\mu_0+\mu_1)^{\underline{\mu_0+\mu_1-1}}\,\Gamma(\mu_0)}\right]_{\mu_1=1}\\&=\frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)}-\frac{(b+\mu_0)^{\underline{\mu_0-1}}(t+\mu_0-1)^{\underline{\mu_0}}}{(b+\mu_0+1)^{\underline{\mu_0}}\,\Gamma(\mu_0)}\end{aligned}$$

$$= \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \frac{\Gamma(b+\mu_0+1)\,\Gamma(t+\mu_0)\,\Gamma(b+2)}{\Gamma(b+2)\Gamma(t)\Gamma(\mu_0)\,\Gamma(b+\mu_0+2)}$$

$$= \frac{(t-2+\mu_0)^{\underline{\mu_0-1}}}{\Gamma(\mu_0)} - \frac{\Gamma(t+\mu_0)}{(b+\mu_0+1)\,\Gamma(t)\Gamma(\mu_0)}. \tag{7.131}$$

Now, from (7.131), we see that $\alpha(t) < 1$ if and only if

$$\frac{\Gamma(t+\mu_0-1)}{\Gamma(\mu_0)\,\Gamma(t)} - \frac{\Gamma(t+\mu_0)}{(b+\mu_0+1)\,\Gamma(t)\Gamma(\mu_0)} \le 1, \tag{7.132}$$

which is itself equivalent to

$$\frac{(b+\mu_0+1)\,\Gamma(t+\mu_0-1)\,\Gamma(t)\Gamma(\mu_0)}{\Gamma(\mu_0)\,\Gamma(t)\,[(b+\mu_0+1)\,\Gamma(\mu_0)\,\Gamma(t)+\Gamma(t+\mu_0)]} \le 1. \tag{7.133}$$

Inequality (7.133) is equivalent to

$$\frac{(b+\mu_0+1)\,\Gamma(t+\mu_0-1)}{(b+\mu_0+1)\,\Gamma(\mu_0)\,\Gamma(t)+\Gamma(t+\mu_0)} \le 1. \tag{7.134}$$

We claim that (7.134) holds for each triple $(b, t, \mu_0) \in \mathbb{N} \times [1, b+1]_{\mathbb{N}_0} \times (1, 2)$.

To prove this latter claim, we rewrite left-hand side of inequality (7.134) in the following way:

$$\frac{(b+\mu_0+1)\,\Gamma(t+\mu_0-1)}{(b+\mu_0+1)\,\Gamma(\mu_0)\,\Gamma(t)+\Gamma(t+\mu_0)} = \frac{\Gamma(t+\mu_0-1)}{\Gamma(\mu_0)\,\Gamma(t)+\frac{\Gamma(t+\mu_0)}{b+\mu_0+1}}$$

$$= \frac{1}{\frac{\Gamma(\mu_0)\Gamma(t)}{\Gamma(t+\mu_0-1)}+\frac{t+\mu_0-1}{b+\mu_0+1}}.$$

Then inequality (7.134) is equivalent to

$$\frac{\Gamma(\mu_0)\,\Gamma(t)}{\Gamma(t+\mu_0-1)} + \frac{t+\mu_0-1}{b+\mu_0+1} \ge 1. \tag{7.135}$$

Now, each of the addends on the left-hand side of (7.135) is nonnegative. In addition, we observe that

$$\frac{\Gamma(\mu_0)\,\Gamma(t)}{\Gamma(t+\mu_0-1)} \ge 1, \tag{7.136}$$

for each admissible t and μ_0 since $t > t+\mu_0-1$, noting that in the case where $\mu_0 = 1$ we get equality in (7.136). But then (7.136) implies (7.135), which in turn implies that (7.132) holds.

In summary, for each admissible triple (b, t, μ_0), we conclude that $\alpha(t) < 1$. Moreover, based on the discussion regarding μ_1 given in (7.129)–(7.130), we have actually shown something stronger—namely, that for each fixed but arbitrary b, t, and μ_0, it holds that

$$\sup_{\mu_1 \in (0,1)} \alpha\,(t; b, \mu_0) < 1. \tag{7.137}$$

Thus, (7.137) implies that $\alpha(t) < 1$, for each fixed but arbitrary 4-tuple $(b, t, \mu_0, \mu_1) \in \mathbb{N} \times [1, b+2]_{\mathbb{N}} \times (1, 2) \times (0, 1)$. Since we earlier showed that $\alpha(t) > 0$ whenever $t \neq 0, b+2$, we conclude that

$$\|\alpha\| < 1,$$

as desired. And this completes the proof. □

Remark 7.57. As we mentioned in the introduction to this section, note that Theorem 7.54 shows that problem (7.110) is not necessarily the same as the conjugate problem studied in [31]. In fact, there is a *de facto* nonlocal nature to problem (7.110) as evidenced by the explicit appearance of $y(1)$ in the operator T, as defined above. As remarked above, this is an interesting complication that cannot occur in the integer-order setting.

As an application of the preceding analysis, we now provide a typical existence theorem for problem (7.110). The basic argument is similar to those presented elsewhere in this book—e.g., Sect. 7.7. However, the appearance of the term $y(1)$ in the operator T does add some interest.

So, let us next provide some standard assumptions on the nonlinearity. For simplicity's sake, we assume that $f(t, y) := a(t)g(y)$; here, it is assumed that a is continuous and not zero identically on $[0, b+2]_{\mathbb{N}_0}$. We also assume (H1) and (H2) below. These assumptions are standard superlinear growth assumptions on g at both 0 and $+\infty$.

H1: We find that $\lim_{y \to 0^+} \frac{g(y)}{y} = 0$.

H2: We find that $\lim_{y \to \infty} \frac{g(y)}{y} = +\infty$.

We also need to define a suitable cone in which to look for fixed points of T. In particular, we consider the cone $\mathcal{K} \subseteq \mathcal{B}$, defined by

$$\mathcal{K} := \left\{ y \in \mathcal{B} \; : \; y \geq 0, \min_{t \in \left[\frac{b}{4}, \frac{3b}{4}\right]_{\mathbb{N}}} y(t) \geq \gamma^* \|y\| \right\}, \tag{7.138}$$

where $\gamma^* \in (0, 1)$ is a constant to be determined later. Note that in (7.138) the constant γ^* is ***not*** the same as the constant γ appearing in part 3 of Proposition 7.55. However, it does satisfy $0 < \gamma^* < 1$, as will be demonstrated in the proof of

Lemma 7.58 below. We first show that the cone $\mathcal{K}$ is invariant under the operator T. We then argue that conditions (H1)–(H2) imply, as is well known in the integer-order case (e.g., [77]), that problem (7.110) has at least one positive solution.

Lemma 7.58. *Let T be the operator defined in* (7.111) *and $\mathcal{K}$ the cone defined in* (7.138). *Then $T(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof. Evidently when $y \in \mathcal{K}$, it follows that $(Ty)(t) \geq 0$, for each t. On the other hand, we observe that

$$\begin{aligned}
&\min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} (Ty)(t) \\
&\geq \gamma_0 y(1)\|\alpha\| + \gamma \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G(s+\tilde{\mu}-1,s) f(s+\tilde{\mu}-1, y(s+\tilde{\mu}-1)) \\
&\geq \gamma^* \left[y(1)\|\alpha\| + \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G(s+\tilde{\mu}-1,s) f(s+\tilde{\mu}-1, y(s+\tilde{\mu}-1)) \right] \\
&\geq \gamma^* \|Ty\|,
\end{aligned} \tag{7.139}$$

where the number γ appearing in (7.139) is the same number γ as in part 3 of Proposition 7.55. Furthermore, the number $\gamma_0 > 0$ appearing in (7.139) is defined by

$$\gamma_0 := \frac{\min_{t\in\left[\frac{b}{4},\frac{3b}{4}\right]_{\mathbb{N}}} \alpha(t)}{\|\alpha\|}.$$

We may then define γ^* by

$$\gamma^* := \min\{\gamma_0, \gamma\},$$

where $0 < \gamma^* < 1$. Thus, whenever $y \in \mathcal{K}$, it follows that $Ty \in \mathcal{K}$, as desired. And this completes the proof. □

Theorem 7.59. *Assume that f satisfies conditions (H1)–(H2). Then problem* (7.110) *has at least one positive solution.*

Proof. First of all, note that T is trivially completely continuous in this setting. Second of all, recall from Lemma 7.56 that $\alpha(t) < 1$, for all $t \in [0, b+2]_{\mathbb{N}_0}$. Therefore, we may select $\varepsilon > 0$ so that $\alpha(t) < \varepsilon < 1$ holds for all admissible t. Given this ε, we may, by way of condition (H1), select $\eta_1 > 0$ sufficiently small so that both

$$g(y) \leq \eta_1 y \tag{7.140}$$

and

$$\eta_1 \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G\left(s+\tilde{\mu}-1, s\right) a(s) \leq 1-\varepsilon \tag{7.141}$$

hold for all $0 < y < r_1$, where $r_1 := r_1\left(\eta_1\right)$. Next put

$$\Omega_1 := \{y \in \mathcal{B} \ : \ \|y\| < r_1\}.$$

Let $y \in \partial\Omega_1 \cap \mathcal{K}$ be arbitrary but fixed. Then upon combining (7.140)–(7.141) we estimate

$$\begin{aligned}
\|Ty\| &\leq y(1) \max_{t\in[0,b+2]_{\mathbb{N}_0}} \alpha(t) + \max_{t\in[0,b+2]_{\mathbb{N}_0}} \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G(t,s)a(s)g\left(y\left(s+\tilde{\mu}-1\right)\right) \\
&< \varepsilon y(1) + \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G\left(s+\tilde{\mu}-1, s\right) a(s)\eta_1 y(s) \\
&\leq \varepsilon\|y\| + \|y\| \cdot \eta_1 \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} G\left(s+\tilde{\mu}-1, s\right) a(s) \\
&\leq \|y\|,
\end{aligned} \tag{7.142}$$

whence (7.142) implies that T is a cone contraction on $\partial\Omega_1 \cap \mathcal{K}$.

On the other hand, from condition (H2) we may select a number $\eta_2 > 0$ such that both

$$\eta_2 \sum_{s=-\tilde{\mu}+2}^{b+2-\tilde{\mu}} \gamma^* G\left(s+\tilde{\mu}-1, s\right) a(s) > 1$$

and

$$g(y) > \eta_2 y$$

hold whenever $y > r_2 > 0$, for some sufficiently large number $r_2 := r_2\left(\eta_2\right) > 0$. Define the number $r_2^* > 0$ by

$$r_2^* := \left\{2r_1, \frac{r_2}{\gamma^*}\right\}$$

and put

$$\Omega_2 := \{y \in \mathcal{B} \ : \ \|y\| < r_2^*\}.$$

Recall that for $y \in \mathcal{K}$, we must have $y(1) \geq 0$, and that from Lemma 7.56 we know also that $\alpha(t) \geq 0$, for all $t \in [0, b+2]_{\mathbb{N}_0}$. Then it is not difficult to show (see, for example, a similar argument in [94]) that

$$\|Ty\| \geq \|y\|,$$

whenever $y \in \partial\Omega_2 \cap \mathcal{K}$, so that T is a cone expansion on $\partial\Omega_2 \cap \mathcal{K}$.

In summary, by once again appealing to Krasnosel'skiĭ's fixed point theorem we obtain the existence of a function $y_0 \in \mathcal{K} \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ such that $Ty_0 = y_0$, where y_0 is a positive solution to problem (7.110). And this completes the proof. □

We now briefly comment on a couple of extensions of the preceding results. In particular, let us consider the following sequential fractional difference

$$\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t),$$

where $\mu_j \in (0,1)$ for each $j = 1, \ldots, n$, under a couple of different additional assumptions on the collection $\{\mu_j\}_{j=1}^n$. For notational simplicity in the sequel, we define

$$\tilde{\mu}_j^+ := \sum_{k=1}^{j} \mu_k$$

and

$$\tilde{\mu}_j^- := \sum_{k=n-j}^{n-1} \mu_k.$$

We continue to use the symbol $\tilde{\mu}$ to denote the sum $\sum_{j=1}^n \mu_j$.

Proposition 7.60. *Assume that* $0 < \sum_{j=1}^{n-1} \mu_j < 1$ *and* $1 < \sum_{j=1}^n \mu_j < 2$. *Then it follows that*

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t) \\
&= \Delta^{\tilde{\mu}_n^+} y(t) \\
&\quad - \left[\frac{\left(t-1+\tilde{\mu}_{n-1}^+\right)^{\underline{-\mu_n-1}}}{\Gamma(-\mu_n)} - \sum_{j=1}^{n-2} \frac{\left(t-1+\tilde{\mu}_j^+\right)^{\underline{-\tilde{\mu}_{n-j+1}^- - \mu_n - 1}}}{\Gamma\left(-\tilde{\mu}_{n-j+1}^- - \mu_n\right)} \right] y(0).
\end{aligned}$$

Proof. We note first that

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_3}\left[\Delta^{\mu_2}\Delta^{\mu_1} y(t)\right] \\
&= \Delta^{\mu_n} \cdots \Delta^{\mu_3}\left[\Delta^{\tilde{\mu}_2^+} y(t) - \frac{\Delta^{\mu_1-1} y(1-\mu_1)}{\Gamma(-\mu_2)}(t-1+\mu_1)^{\underline{-\mu_2-1}}\right] \\
&= \Delta^{\mu_n} \cdots \Delta^{\mu_4}\Bigg[\Delta^{\tilde{\mu}_3^+} y(t) \\
&\quad - \frac{\Delta^{\mu_1+\mu_2-1} y(1-\mu_1-\mu_2)}{\Gamma(-\mu_3)}(t-1+\mu_1+\mu_2)^{\underline{-\mu_3-1}} \\
&\quad - \frac{\Delta^{\mu_1-1} y(1-\mu_1)}{\Gamma(-\mu_2-\mu_3)}(t-1+\mu_1)^{\underline{-\mu_2-\mu_3-1}}\Bigg].
\end{aligned}$$

Now, inductively repeating this process results in the following equality:

$$\begin{aligned}
&\Delta^{\mu_{n-1}} \cdots \Delta^{\mu_1} y(t) \\
&= \Delta^{\tilde{\mu}_{n-1}^+} y(t) - \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^+ -1} y\left(1-\tilde{\mu}_j^+\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^-\right)}\left(t-1+\tilde{\mu}_j^+\right)^{\underline{-\tilde{\mu}_{n-j-1}^- -1}}\right].
\end{aligned}$$

So, it follows that

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t) \\
&= \Delta^{\mu_n}\left\{\Delta^{\tilde{\mu}_{n-1}^+} y(t) - \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^+ -1} y\left(1-\tilde{\mu}_j^+\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^-\right)}\left(t-1+\tilde{\mu}_j^+\right)^{\underline{-\tilde{\mu}_{n-j-1}^- -1}}\right]\right\} \\
&= \Delta^{\tilde{\mu}_n^+} y(t) - \frac{\Delta^{-1+\tilde{\mu}_{n-1}^+} y\left(1-\tilde{\mu}_{n-1}^+\right)}{\Gamma(-\mu_n)}\left(t-1+\tilde{\mu}_{n-1}^+\right)^{\underline{-\mu_n-1}} \\
&\quad + \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^+ -1} y\left(1-\tilde{\mu}_j^+\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^-\right)} \cdot \frac{\Gamma\left(-\tilde{\mu}_{n-j-1}^-\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^- - \mu_n\right)}\left(t-1+\tilde{\mu}_j^+\right)^{\underline{-\tilde{\mu}_{n-j-1}^- -\mu_n-1}}\right] \\
&= \Delta^{\tilde{\mu}_n^+} y(t) \\
&\quad - \left[\frac{\left(t-1+\tilde{\mu}_{n-1}^+\right)^{\underline{-\mu_n-1}}}{\Gamma(-\mu_n)} - \sum_{j=1}^{n-2}\frac{\left(t-1+\tilde{\mu}_j^+\right)^{\underline{-\tilde{\mu}_{n-j+1}^- -\mu_n-1}}}{\Gamma\left(-\tilde{\mu}_{n-j+1}^- - \mu_n\right)}\right] y(0),
\end{aligned}$$

as claimed, which completes the proof. □

Our next proposition provides for a more direct generalization of problem (7.110) considered earlier.

Proposition 7.61. *Suppose that* $0 < \sum_{j=1}^{n-2} \mu_j < 1$, $1 < \sum_{j=1}^{n-1} \mu_j < 2$, *and* $1 < \sum_{j=1}^{n} \mu_j < 2$. *Then we find that*

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t)\\
&= \Delta^{\tilde{\mu}} y(t) - \frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)} y(1)\\
&\quad - \sum_{j=1}^{n-2}\left[\frac{1}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}-\mu_n\right)}\left(t-1+\tilde{\mu}_j^{+}\right)^{\underline{-\tilde{\mu}_{n-j-1}^{-}-\mu_n-1}}\right] y(0)\\
&\quad - \left[\frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}\left(1-\tilde{\mu}_{n-1}^{+}\right) - \frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)}\right] y(0).
\end{aligned}$$

Proof. We first write

$$\begin{aligned}
&\Delta^{\mu_n} \cdots \Delta^{\mu_1} y(t)\\
&= \Delta^{\mu_n}\left\{\Delta^{\tilde{\mu}_{n-1}^{+}} y(t) - \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^{+}-1} y\left(1-\tilde{\mu}_j^{+}\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}\right)}\left(t-1+\tilde{\mu}_j^{+}\right)^{\underline{-\tilde{\mu}_{n-j-1}^{-}-1}}\right]\right\}\\
&= \Delta^{\mu_n} \Delta^{\tilde{\mu}_{n-1}^{+}} y(t)\\
&\quad - \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^{+}-1} y\left(1-\tilde{\mu}_j^{+}\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}\right)} \Delta^{\mu_n}\left[\left(t-1+\tilde{\mu}_j^{+}\right)^{\underline{-\tilde{\mu}_{n-j-1}^{-}-1}}\right]\right]\\
&= \Delta^{\tilde{\mu}} y(t) - \sum_{k=0}^{1} \frac{\Delta^{j-2+\tilde{\mu}_{n-1}^{+}} y\left(2-\tilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n-1+j\right)}\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2+j}}\\
&\quad - \sum_{j=1}^{n-2}\left[\frac{\Delta^{\tilde{\mu}_j^{+}-1} y\left(1-\tilde{\mu}_j^{+}\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}\right)}\right.\\
&\quad \left.\times \frac{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}\right)}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}-\mu_n\right)}\left(t-1+\tilde{\mu}_j^{+}\right)^{\underline{-\tilde{\mu}_{n-j-1}^{-}-\mu_n-1}}\right].
\end{aligned}$$

Now notice both that

$$\frac{\Delta^{-2+\tilde{\mu}_{n-1}^{+}} y\left(2-\tilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n-1\right)}\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}} = \frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)} y(0)$$

and that

$$\begin{aligned}&\frac{\Delta^{-1+\tilde{\mu}_{n-1}^{+}}y\left(2-\tilde{\mu}_{n-1}^{+}\right)}{\Gamma\left(-\mu_n\right)}\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}\\&=\frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}\left[\left(1-\tilde{\mu}_{n-1}^{+}\right)y(0)+y(1)\right].\end{aligned}$$

So, we conclude that

$$\begin{aligned}&\Delta^{\mu_n}\cdots\Delta^{\mu_1}y(t)\\&=\Delta^{\tilde{\mu}}y(t)-\frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}y(1)\\&\quad-\sum_{j=1}^{n-2}\left[\frac{1}{\Gamma\left(-\tilde{\mu}_{n-j-1}^{-}-\mu_n\right)}\left(t-1+\tilde{\mu}_j^{+}\right)^{\underline{-\tilde{\mu}_{n-j-1}^{-}-\mu_n-1}}\right]y(0)\\&\quad-\left[\frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-1}}}{\Gamma\left(-\mu_n\right)}\left(1-\tilde{\mu}_{n-1}^{+}\right)-\frac{\left(t-2+\tilde{\mu}_{n-1}^{+}\right)^{\underline{-\mu_n-2}}}{\Gamma\left(-\mu_n-1\right)}\right]y(0).\end{aligned}$$

And this completes the proof. □

Propositions 7.60 and 7.61 again demonstrate that the sequential problems are (potentially) different than the non-sequential problems and, in particular, isolate these differences. Furthermore, with Propositions 7.60 and 7.61 in hand, we can write down a number of existence results for sequential discrete FBVPs. But we omit their statements here.

7.7 Systems of FBVPs with Nonlinear, Nonlocal Boundary Conditions

In this section we shall demonstrate how we can apply our analysis of nonlocal discrete fractional boundary value problems to systems of such problems. Essentially, other than modifying the Banach space and associated cone in which we work, the analysis is very similar. In particular, we are interested in the system

$$\begin{aligned}-\Delta^{\nu_1}y_1(t)&=\lambda_1a_1\left(t+\nu_1-1\right)f_1\left(y_1\left(t+\nu_1-1\right),y_2\left(t+\nu_2-1\right)\right)\\-\Delta^{\nu_2}y_2(t)&=\lambda_2a_2\left(t+\nu_2-1\right)f_2\left(y_1\left(t+\nu_1-1\right),y_2\left(t+\nu_2-1\right)\right),\end{aligned}\tag{7.143}$$

for $t\in[0,b]_{\mathbb{N}_0}$, subject to the boundary conditions

$$y_1(\nu_1-2)=\psi_1(y_1),\ y_2(\nu_2-2)=\psi_2(y_2)$$
$$y_1(\nu_1+b)=\phi_1(y_1),\ y_2(\nu_2+b)=\phi_2(y_2), \tag{7.144}$$

where $\lambda_i>0$, $a_i:\mathbb{R}\to[0,+\infty)$, $\nu_i\in(1,2]$ for each $1\le i\le 2$, and for each i we have that $\psi_i,\phi_i:\mathbb{R}^{b+3}\to\mathbb{R}$ are given functionals. We shall also assume that $f_i:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous for each admissible i. One point of interest to which we wish to draw the reader's attention is the fact that because it may well occur that $\nu_1\ne\nu_2$, it follows, due to the inherent domain shifting of the operator Δ_0^ν, that the two functions y_1 and y_2 appearing in (7.143) may be defined on different domains. Evidently, this cannot occur in the integer-order problem—i.e., when $\nu_1,\nu_2\in\mathbb{N}$. Problem (7.143)–(7.144) was originally studied by Goodrich [94], and the results of this section may be found in that paper.

We now wish to fix our framework for the study of problem (7.143)–(7.144). First of all, we let $\mathcal{B}_i$ represent the Banach space of all maps from $[\nu_i-2,\dots,\nu_i+b]_{\mathbb{N}_{\nu_i-2}}$ into $\mathbb{R}$ when equipped with the usual maximum norm, $\|\cdot\|$. We shall then put

$$\mathcal{X}:=\mathcal{B}_1\times\mathcal{B}_2.$$

By equipping $\mathcal{X}$ with the norm

$$\|(y_1,y_2)\|:=\|y_1\|+\|y_2\|,$$

it follows that $(\mathcal{X},\|\cdot\|)$ is a Banach space, too—see, for example, [74].

Next we wish to develop a representation for a solution of (7.143)–(7.144) as the fixed point of an appropriate operator on $\mathcal{X}$. To accomplish this we present some adaptations of results from [31] that will be of use here. Because the proofs of these lemmas are straightforward, we omit them.

Lemma 7.62 ([31]). *Let $1<\nu\le 2$ and $h:[\nu-1,\nu+b-1]_{\mathbb{N}_{\nu-1}}\to\mathbb{R}$ be given. The unique solution of the FBVP $-\Delta^\nu y(t)=h(t+\nu-1)$, $y(\nu-2)=0=y(\nu+b)$ is given by $y(t)=\sum_{s=0}^{b}G(t,s)h(s+\nu-1)$, where $G:[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\times[0,b]_{\mathbb{N}_0}\to\mathbb{R}$ is defined by*

$$G(t,s):=\begin{cases}\frac{t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(\nu+b)^{\underline{\nu-1}}}-(t-s-1)^{\underline{\nu-1}}, & 0\le s<t-\nu+1\le b\\ \frac{t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(\nu+b)^{\underline{\nu-1}}}, & 0\le t-\nu+1\le s\le b\end{cases}.$$

Lemma 7.63 ([31]). *The Green's function G given in Lemma 7.62 satisfies:*

(i) $G(t,s)\ge 0$ *for each* $(t,s)\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\times[0,b]_{\mathbb{N}_0}$;
(ii) $\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}G(t,s)=G(s+\nu-1,s)$ *for each* $s\in[0,b]_{\mathbb{N}_0}$; *and*
(iii) *there exists a number* $\gamma\in(0,1)$ *such that*

$$\min_{\frac{b+\nu}{4}\le t\le\frac{3(b+\nu)}{4}}G(t,s)\ge\gamma\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}G(t,s)=\gamma G(s+\nu-1,s),$$

for $s\in[0,b]_{\mathbb{N}_0}$.

Now consider the operator $S : \mathcal{X} \to \mathcal{X}$ defined by

$$S(y_1, y_2)(t_1, t_2) := (S_1(y_1, y_2)(t_1), S_2(y_1, y_2)(t_2)), \tag{7.145}$$

where we define $S_1 : \mathcal{X} \to \mathcal{B}_1$ by

$$\begin{aligned} S_1(y_1, y_2)(t_1) &\\ := \alpha_1(t_1)\psi_1(y_1) &+ \beta_1(t_1)\phi_1(y_1) \\ &+ \lambda_1 \sum_{s=0}^{b} G_1(t_1, s) a_1(s + \nu_1 - 1) f_1(y_1(s + \nu_1 - 1), y_2(s + \nu_2 - 1)) \end{aligned}$$

and $S_2 : \mathcal{X} \to \mathcal{B}_2$ by

$$\begin{aligned} S_2(y_1, y_2)(t_2) &\\ := \alpha_2(t_2)\psi_2(y_2) &+ \beta_2(t_2)\phi_2(y_2) \\ &+ \lambda_2 \sum_{s=0}^{b} G_2(t_2, s) a_2(s + \nu_2 - 1) f_2(y_1(s + \nu_1 - 1), y_2(s + \nu_2 - 1)); \end{aligned}$$

note that, for $j = 1, 2$, we define the maps $\alpha_j, \beta_j : [\nu_j - 2, \nu_j + b]_{\mathbb{Z}_{\nu_j - 2}} \to \mathbb{R}$ by

$$\begin{aligned} \alpha_j(t) &:= \frac{1}{\Gamma(\nu_j - 1)} \left[t^{\underline{\nu_j - 2}} - \frac{1}{b+2} t^{\underline{\nu_j - 1}} \right] \\ \beta_j(t) &:= \frac{t^{\underline{\nu_j - 1}}}{(\nu + b)^{\underline{\nu_j - 1}}}, \end{aligned}$$

which occur in the definitions of S_1 and S_2 above. Moreover, the map $(t, s) \mapsto G_j(t, s)$ is precisely the map $(t, s) \mapsto G(t, s)$ as given in Lemma 7.62 with ν replaced by ν_j. We claim that whenever $(y_1, y_2) \in \mathcal{X}$ is a fixed point of the operator S, it follows that the pair of functions y_1 and y_2 is a solution to problem (7.143)–(7.144).

Theorem 7.64. *Let* $f_j : \mathbb{R}^2 \to [0, +\infty)$ *and*

$$\psi_j, \phi_j \in \mathcal{C}\left([\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j - 2}}, \mathbb{R}\right)$$

be given, for $j = 1, 2$. *If* $(y_1, y_2) \in \mathcal{X}$ *is a fixed point of* S, *then the pair of functions* y_1 *and* y_2 *is a solution to problem* (7.143)–(7.144).

Proof. Omitted—see [94]. □

The following lemma and its associated corollary are of particular importance in the sequel. Because the proofs of each of these are straightforward, we omit them.

Lemma 7.65. *For each* $j = 1, 2$*, the function* $t_j \mapsto \alpha_j\left(t_j\right)$ *is decreasing in* t_j*, for* $t_j \in \left[\nu_j - 2, \nu_j + b\right]_{\mathbb{N}_{\nu_j-2}}$*. Also, it holds both that*

$$\min_{t_j \in \left[\nu_j-2,\nu_j+b\right]_{\mathbb{N}_{\nu_j-2}}} \alpha_j\left(t_j\right) = 0$$

and

$$\max_{t_j \in \left[\nu_j-2,\nu_j+b\right]_{\mathbb{N}_{\nu_j-2}}} \alpha_j\left(t_j\right) = 1.$$

On the other hand, for each $j = 1, 2$*, the function* $t_j \mapsto \beta_j\left(t_j\right)$ *is strictly increasing in* t_j*, for* $t_j \in \left[\nu_j - 2, \nu_j + b\right]_{\mathbb{N}_{\nu_j-2}}$*. In addition, it holds that*

$$\min_{t_j \in \left[\nu_j-2,\nu_j+b\right]_{\mathbb{N}_{\nu_j-2}}} \beta_j\left(t_j\right) = 0$$

and that

$$\max_{t_j \in \left[\nu_j-2,\nu_j+b\right]_{\mathbb{N}_{\nu_j-2}}} \beta_j\left(t_j\right) = 1.$$

Corollary 7.66. *Let* $j = 1, 2$ *be given. Put* $I_j := \left[\frac{b+\nu_j}{4}, \frac{3(b+\nu_j)}{4}\right]$*. Then there exist constants* $M_{\alpha_j}, M_{\beta_j} \in (0, 1)$ *such that*

$$\min_{t_j \in I_j} \alpha_j\left(t_j\right) = M_{\alpha_j} \|\alpha_j\|$$

and

$$\min_{t_j \in I_j} \beta_j\left(t_j\right) = M_{\beta_j} \|\beta_j\|.$$

Let us conclude this section with a remark.

Remark 7.67. Observe that unlike in the case of the integer-order problem (i.e., when $\nu_1 = \nu_2 = 2$), in the fractional-order problem we encounter a significant problem with respect to the domains of the various operators insofar as it may occur that $\mathbb{Z}_{\nu_1-2} \neq \mathbb{Z}_{\nu_2-2}$. As has been noted with different problems in previous sections, this complication arises in the discrete fractional calculus due to the domain shifting of the fractional forward difference and sum operators.

We now present the first of two theorems for the existence of at least one positive solution to problem (7.143)–(7.144). Note that for this first existence result we shall not assume that either $\psi_i\left(y_i\right)$ or $\phi_i\left(y_i\right)$, with $i = 1, 2$, is nonnegative for all $y_i \geq 0$. Rather, we shall make some other assumptions about these functionals.

So, let us now present the conditions that we shall assume henceforth. We note that conditions (F1) and (F2) are essentially the same conditions given by Henderson et al. [121]. Moreover, condition (L1) is essentially the same condition (up to a constant multiple) as given in [121, Theorem 3.1].

F1: There exist numbers f_1^* and f_2^*, with $f_1^*, f_2^* \in (0, +\infty)$, such that

$$\lim_{y_1+y_2\to 0^+} \frac{f_1(y_1, y_2)}{y_1 + y_2} = f_1^* \text{ and } \lim_{y_1+y_2\to 0^+} \frac{f_2(y_1, y_2)}{y_1 + y_2} = f_2^*.$$

F2: There exist numbers f_1^{**} and f_2^{**}, with $f_1^{**}, f_2^{**} \in (0, +\infty)$, such that

$$\lim_{y_1+y_2\to +\infty} \frac{f_1(y_1, y_2)}{y_1 + y_2} = f_1^{**} \text{ and } \lim_{y_1+y_2\to +\infty} \frac{f_2(y_1, y_2)}{y_1 + y_2} = f_2^{**}.$$

G1: For each $j = 1, 2$, the functionals ψ_j and ϕ_j are linear. In particular, we assume both that

$$\psi_j\left(y_j\right) = \sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} y_j(i)$$

and that

$$\phi_j\left(y_j\right) = \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} y_j(k),$$

for constants $c^j_{i-\nu_j+2}, d^j_{k-\nu_j+2} \in \mathbb{R}$.

G2: For each $j = 1, 2$, we have both that

$$\sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} G_j(i, s) \geq 0$$

and that

$$\sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} G_j(k, s) \geq 0,$$

for each $s \in [0, b]_{\mathbb{N}_0}$, and in addition that

$$\sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} + \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} \leq \frac{1}{4}.$$

G3: We have that each of $\psi_i(\alpha_i)$, $\psi_i(\beta_i)$, $\phi_i(\alpha_i)$, and $\phi_i(\beta_i)$ is nonnegative for each admissible i—that is, $i = 1, 2$.

L1: The constants λ_1 and λ_2 satisfy

$$\Lambda_1 < \lambda_i < \Lambda_2,$$

for each i, where

$$\Lambda_1 := \max\left\{\frac{1}{2}\left[\sum_{s=0}^{b}\gamma G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_1, s\right) a_1(s+\nu_1-1) f_1^{**}\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\sum_{s=0}^{b}\gamma G_2\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_2, s\right) a_2(s+\nu_2-1) f_2^{**}\right]^{-1}\right\}$$

and

$$\Lambda_2 := \min\left\{\frac{1}{4}\left[\sum_{s=0}^{b} G_1(s+\nu_1-1, s)\, a_1(s+\nu_1-1) f_1^{*}\right]^{-1},\right.$$
$$\left.\frac{1}{4}\left[\sum_{s=0}^{b} G_2(s+\nu_2-1, s)\, a_2(s+\nu_2-1) f_2^{*}\right]^{-1}\right\},$$

where $\gamma \in (0, 1)$ is a constant defined by

$$\gamma := \min\left\{M_{\alpha_1}, M_{\alpha_2}, M_{\beta_1}, M_{\beta_2}, \gamma_1, \gamma_2\right\},$$

where M_{α_1}, M_{α_2}, M_{β_1}, and M_{β_2} each comes from Corollary 7.66 and γ_1 and γ_2 are associated by Lemma 7.63 with G_1 and G_2, respectively. Recall that these are defined on possibly different time scales.

In what follows we shall also make use of the cone

$$\begin{aligned}\mathcal{K} := \Bigg\{ &(y_1, y_2) \in \mathcal{X} \ : \ y_1, y_2 \geq 0, \\ &\min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4}, \frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4}, \frac{3(b+\nu_2)}{4}\right]} [y_1(t_1) + y_2(t_2)] \geq \gamma \|(y_1, y_2)\|, \\ &\psi_j(y_j) \geq 0,\ \phi_j(y_j) \geq 0, \text{ for each } j = 1, 2\Bigg\},\end{aligned} \tag{7.146}$$

where γ is defined exactly as in the statement of condition (L1) above. This cone is essentially a modification of the type of cone introduced by Infante and Webb [159]. Clearly, we have that $\mathcal{K} \subseteq \mathcal{X}$. In order to show that S has a fixed point in $\mathcal{K}$, we must first demonstrate that $\mathcal{K}$ is invariant under S—that is, $S(\mathcal{K}) \subseteq \mathcal{K}$. This we now show.

Lemma 7.68. *Let $S : \mathcal{X} \to \mathcal{X}$ be the operator defined as in (7.145). Then $S : \mathcal{K} \to \mathcal{K}$.*

Proof. Suppose that $(y_1, y_2) \in \mathcal{K}$. We show first that

$$\min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} \left[S_1\left(y_1, y_2\right)\left(t_1\right) + S_2\left(y_1, y_2\right)\left(t_2\right)\right]$$
$$\geq \gamma \|S\left(y_1, y_2\right)\|,$$

whenever $(y_1, y_2) \in \mathcal{K}$.

So note that

$$\begin{aligned}
&\min_{t_1\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]} S_1\left(y_1, y_2\right)\left(t_1\right)\\
&\geq M_{\alpha_1}\|\alpha_1\|\phi_1\left(y_1\right) + M_{\beta_1}\|\beta_1\|\psi_1\left(y_1\right)\\
&\quad + \lambda_1 \sum_{s=0}^{b} G_1\left(t_1, s\right) a_1\left(s+\nu_1-1\right) f_1\left(y_1\left(s+\nu_1-1\right), y_2\left(s+\nu_1-1\right)\right)\\
&\geq \tilde{\gamma}_1 \max_{t_1\in[\nu_1-2,\nu_1+b]} \Bigg[\alpha_1\left(t_1\right)\phi_1\left(y_1\right) + \beta_1\left(t_1\right)\psi_1\left(y_1\right)\\
&\quad + \lambda_1 \sum_{s=0}^{b} G_1\left(t_1, s\right) a_1\left(s+\nu_1-1\right) f_1\left(y_1\left(s+\nu_1-1\right), y_2\left(s+\nu_1-1\right)\right)\Bigg]\\
&= \tilde{\gamma}_1\|S_1\left(y_1, y_2\right)\|,
\end{aligned} \tag{7.147}$$

where $\widetilde{\gamma_1} := \min\left\{M_{\alpha_1}, M_{\beta_1}, \gamma_1\right\}$, whence

$$\min_{t_1\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]} S_1\left(y_1, y_2\right)\left(t_1\right) \geq \tilde{\gamma}_1\|S_1\left(y_1, y_2\right)\|, \tag{7.148}$$

as desired. In an entirely similar manner to (7.147), we deduce that

$$\min_{t_2\in\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} S_2\left(y_1, y_2\right)\left(t_2\right) \geq \tilde{\gamma}_2\|S_2\left(y_1, y_2\right)\|, \tag{7.149}$$

where $\tilde{\gamma}_2 := \min\left\{M_{\alpha_2}, M_{\beta_2}, \gamma_2\right\}$.

Now, put $\gamma := \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$. Consequently, from (7.148)–(7.149) it follows that

$$\begin{aligned}
&\min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} \left[S_1(y_1,y_2)(t_1) + S_2(y_1,y_2)(t_2)\right] \\
&\geq \min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} S_1(y_1,y_2)(t_1) \\
&+ \min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} S_2(y_1,y_2)(t_2) \\
&\geq \left(\tilde{\gamma}_1\|S_1(y_1,y_2)\| + \tilde{\gamma}_2\|S_2(y_1,y_2)\|\right) \\
&\geq \left(\gamma\|S_1(y_1,y_2)\| + \gamma\|S_2(y_1,y_2)\|\right) \\
&= \gamma\|\left(S_1(y_1,y_2), S_2(y_1,y_2)\right)\| \\
&= \gamma\|S(y_1,y_2)\|. \qquad (7.150)
\end{aligned}$$

So, from (7.150) we conclude that whenever $(y_1, y_2) \in \mathcal{X}$, we find that

$$\begin{aligned}
&\min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4},\frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4},\frac{3(b+\nu_2)}{4}\right]} \left[S_1(y_1,y_2)(t_1) + S_2(y_1,y_2)(t_2)\right] \\
&\geq \gamma\|S(y_1,y_2)\|,
\end{aligned}$$

as desired.

We next show that for each $j = 1, 2$ we have $\psi_j\left(S_j(y_1,y_2)\right) \geq 0$ whenever $(y_1, y_2) \in \mathcal{K}$. Indeed, first note that

$$\begin{aligned}
&\psi_j\left(S_j(y_1,y_2)\right) \\
&= \lambda_j \sum_{s=0}^{b} \sum_{i=\nu_j-2}^{\nu_j+b} \Bigg\{ c^j_{i-\nu_j+2} G_j(i,s) a_j\left(s+\nu_j-1\right) \\
&\times f_j\left(y_1(s+\nu_1-1), y_2(s+\nu_2-1)\right) + \psi_j\left(\alpha_j\right)\psi_j\left(y_j\right) + \psi_j\left(\beta_j\right)\phi_j\left(y_j\right) \Bigg\}. \qquad (7.151)
\end{aligned}$$

But by assumptions (G2) and (G3) together with the nonnegativity of $f_j(y_1,y_2)$ and the fact that $(y_1, y_2) \in \mathcal{K}$, we find from (7.151) that

$$\psi_j\left(S_j(y_1,y_2)\right) \geq 0,$$

for each $j = 1, 2$. An entirely dual argument, which we omit, shows that

$$\phi_j\left(S_j(y_1,y_2)\right) \geq 0,$$

too, whenever $(y_1, y_2) \in \mathcal{K}$ and $j = 1, 2$.

Finally, it is clear from the definitions of both S_1 and S_2 that

$$S_1(y_1, y_2)(t_1) \geq 0 \text{ and } S_2(y_1, y_2)(t_2) \geq 0,$$

for each t_1 and t_2, whenever $(y_1, y_2) \in \mathcal{K}$. Therefore, we conclude that whenever $(y_1, y_2) \in \mathcal{K}$, it follows that $S(y_1, y_2) \in \mathcal{K}$. Thus, $S : \mathcal{K} \to \mathcal{K}$, as desired. And this completes the proof. □

We now prove the first of our two main existence theorems, which we label Theorem 7.69.

Theorem 7.69. *Suppose that conditions (F1)–(F2), (G1)–(G3), and (L1) hold. Then problem* (7.143)–(7.144) *has at least one positive solution.*

Proof. We have already shown in Lemma 7.68 that $S : \mathcal{K} \to \mathcal{K}$. Furthermore, it is evident that S is completely continuous.

We begin by observing that by condition (L1) there exists a number $\varepsilon > 0$ such that each of

$$\max\left\{\frac{1}{2}\left[\sum_{s=0}^{b}\gamma G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu_1, s\right)a_1(s+\nu_1-1)\left(f_1^{**}-\varepsilon\right)\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\sum_{s=0}^{b}\gamma G_2\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu_2, s\right)a_2(s+\nu_2-1)\left(f_2^{**}-\varepsilon\right)\right]^{-1}\right\} \leq \lambda_1, \lambda_2 \tag{7.152}$$

and

$$\lambda_1, \lambda_2 \leq \min\left\{\frac{1}{4}\left[\sum_{s=0}^{b}G_1(s+\nu_1-1, s)\,a_1(s+\nu_1-1)\left(f_1^{*}+\varepsilon\right)\right]^{-1},\right.$$
$$\left.\frac{1}{4}\left[\sum_{s=0}^{b}G_2(s+\nu_2-1, s)\,a_2(s+\nu_2-1)\left(f_2^{*}+\varepsilon\right)\right]^{-1}\right\}. \tag{7.153}$$

holds. Now, given this number ε, by condition (F1) it follows that there exists some number $r_1^* > 0$ such that

$$f_1(y_1, y_2) \leq \left(f_1^* + \varepsilon\right)(y_1 + y_2), \tag{7.154}$$

whenever $\|(y_1, y_2)\| < r_1$. Similarly, by condition (F2) and for the same number ε, there exists a number $r_1^{**} > 0$ such that

$$f_2(y_1, y_2) \leq \left(f_2^* + \varepsilon\right)(y_1 + y_2), \tag{7.155}$$

whenever $\| (y_1, y_2) \| < r_2$. Then by putting $r_1 := \min \left\{r_1^*, r_1^{**}\right\}$, we find that each of (7.154) and (7.155) is true whenever $\| (y_1, y_2) \| < r_1$. This suggests defining the set $\Omega_1 \subseteq \mathcal{X}$ by

$$\Omega_1 := \{(y_1, y_2) \in \mathcal{X} \ : \ \| (y_1, y_2) \| < r_1\}, \tag{7.156}$$

which we shall use momentarily.

Now, let Ω_1 be as in (7.156) above. Then for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we find that

$$\begin{aligned}
&\|S_1 (y_1, y_2) \| \\
&= \max_{t_1 \in [\nu_1 - 2, \nu_1 + b]_{\mathbb{N}_{\nu_1 - 2}}} \Bigg| \alpha_1 (t_1) \phi_1 (y_1) + \beta_1 (t_1) \psi_1 (y_1) \\
&\quad + \lambda_1 \sum_{s=0}^{b} G_1 (t_1, s) a_1 (s + \nu_1 - 1) f_1 (y_1 (s + \nu_1 - 1), y_2 (s + \nu_2 - 1)) \Bigg| \\
&\le r_1 \left[\sum_{i=\nu_1-2}^{\nu_1+b} c^1_{i-\nu_1+2} + \sum_{k=\nu_1-2}^{\nu_1+b} d^1_{k-\nu_1+2} \right] \\
&\quad + \lambda_1 \sum_{s=0}^{b} G_1 (s + \nu_1 - 1, s) a_1 (s + \nu_1 - 1) \left(f_1^* + \varepsilon\right) \| (y_1, y_2) \| \\
&\le \| (y_1, y_2) \| \left[\frac{1}{4} + \lambda_1 \sum_{s=0}^{b} G_1 (s + \nu_1 - 1, s) a_1 (s + \nu_1 - 1) \left(f_1^* + \varepsilon\right) \right],
\end{aligned} \tag{7.157}$$

where we use the fact that $S_1 (y_1, y_2)$ is nonnegative whenever $(y_1, y_2) \in \mathcal{K}$. However, by the choice of λ_1 as given in (7.152)–(7.153), we deduce from (7.157) that

$$\|S_1 (y_1, y_2) \| \le \frac{1}{2} \| (y_1, y_2) \|. \tag{7.158}$$

We note that by an entirely dual argument we may estimate

$$\|S_2 (y_1, y_2) \| \le \frac{1}{2} \| (y_1, y_2) \|. \tag{7.159}$$

Thus, by combining estimates (7.152)–(7.159) we deduce that for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we have

$$\|S (y_1, y_2) \| \le \frac{1}{2} \| (y_1, y_2) \| + \frac{1}{2} \| (y_1, y_2) \| = \| (y_1, y_2) \|.$$

Now, let $\varepsilon > 0$ be the same number selected at the beginning of this proof. Then by means of condition (F2) we can find a number $\tilde{r}_2 > 0$ such that

$$f_1(y_1, y_2) \geq \left(f_1^{**} - \varepsilon\right)(y_1 + y_2) \tag{7.160}$$

and

$$f_2(y_1, y_2) \geq \left(f_2^{**} - \varepsilon\right)(y_1 + y_2), \tag{7.161}$$

whenever $y_1 + y_2 \geq \tilde{r}_2$. Put

$$r_2 := \max\left\{2r_1, \frac{\tilde{r}_2}{\gamma}\right\}, \tag{7.162}$$

where, as before, we take

$$\gamma := \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}.$$

Moreover, define the set $\Omega_2 \subseteq \mathcal{X}$ by

$$\Omega_2 := \{(y_1, y_2) \in \mathcal{X} : \|(y_1, y_2)\| < r_2\}. \tag{7.163}$$

Note that if $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$, then it follows that

$$\begin{aligned} y_1(t_1) + y_2(t_2) &\geq \min_{(t_1,t_2)\in\left[\frac{b+\nu_1}{4}, \frac{3(b+\nu_1)}{4}\right]\times\left[\frac{b+\nu_2}{4}, \frac{3(b+\nu_2)}{4}\right]} [y_1(t_1) + y_2(t_2)] \\ &\geq \gamma\|(y_1, y_2)\| \\ &\geq \tilde{r}_2. \end{aligned} \tag{7.164}$$

Now, define the numbers $0 < \sigma_1 < \sigma_2$ by

$$\sigma_1 := \max\left\{\left\lceil \frac{\nu_1 + b}{4} - \nu_1 + 1 \right\rceil, \left\lceil \frac{\nu_2 + b}{4} - \nu_2 + 1 \right\rceil\right\}$$

and

$$\sigma_2 := \min\left\{\left\lfloor \frac{3(\nu_1 + b)}{4} - \nu_1 + 1 \right\rfloor, \left\lfloor \frac{3(\nu_2 + b)}{4} - \nu_2 + 1 \right\rfloor\right\};$$

we assume in the sequel that b is sufficiently large so that $[\sigma_1, \sigma_2] \cap \mathbb{N}_0 \neq \varnothing$. Then for each $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$ we estimate

$$
\begin{aligned}
&S_1(y_1,y_2)\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu_1\right)\\
&=\sum_{i=\nu_1-2}^{\nu_1+b} c^1_{i-\nu_1+2} y_1(k)+\sum_{k=\nu_1-2}^{\nu_1+b} d^1_{k-\nu_1+2} y_1(k)\\
&\quad+\lambda_1\sum_{s=0}^{b}\left[G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu_1,s\right)\right.\\
&\quad\left.\times a_1(s+\nu_1-1) f_1(y_1(s+\nu_1-1),y_2(s+\nu_2-1))\right]\\
&\geq \lambda_1\sum_{s=\sigma_1}^{\sigma_2} G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu_1,s\right) a_1(s+\nu_1-1)\left(f_1^{**}-\epsilon\right)\gamma\left[\|y_1\|+\|y_2\|\right]\\
&\geq \frac{1}{2}\|(y_1,y_2)\|,
\end{aligned}
\tag{7.165}
$$

where to arrive at the first inequality in (7.165) we have used the positivity assumption imposed on each of ψ_1 and ϕ_1 whenever $(y_1, y_2) \in \mathcal{K}$. Thus, we conclude from (7.165) that

$$\|S_1(y_1,y_2)\| \geq \frac{1}{2}\|(y_1,y_2)\|. \tag{7.166}$$

In a completely similar way, it can be shown that

$$\|S_2(y_1,y_2)\| \geq \frac{1}{2}\|(y_1,y_2)\|. \tag{7.167}$$

Consequently, (7.160)–(7.167) imply that

$$\|S(y_1,y_2)\| \geq \|(y_1,y_2)\|, \tag{7.168}$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$.

Finally, notice that (7.160) implies that the operator S is a cone compression on $\mathcal{K} \cap \partial\Omega_1$, whereas (7.168) implies that S is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. Consequently we conclude that S has a fixed point, say $\left(y_1^*, y_2^*\right) \in \mathcal{K}$. As $\left(y_1^*, y_2^*\right)$ is a positive solution of (7.143)–(7.144), the theorem is proved. □

Remark 7.70. Note that in the preceding arguments it is important that each of γ_1 and γ_2 (and thus γ) is a constant. That γ is constant here is a reflection of the fact that the Green's function G satisfies a sort of Harnack-like inequality. Interestingly, however, in the continuous fractional setting, this may (see [90]) or may not (see [46]) be true. This is one of the differences one may observe between the discrete and continuous fractional calculus.

Remark 7.71. It is clear that Theorem 7.69 could be readily extended to the case of n equations and $2n$ boundary conditions.

We now wish to present an alternative method for deducing the existence of at least one positive solution to problem (7.143)–(7.144). In particular, instead of using the cone given in (7.146), we shall now revert to a more traditional cone, whose use can be found in innumerable papers. An advantage of this approach is that it shall allow us to weaken hypothesis (G1). However, we achieve this increased generality at the expense of having to assume *a priori* the positivity of each of these functionals for all $y \geq 0$. In particular, for the second existence result we make the following hypotheses.

G4: For $i = 1, 2$ we have that

$$\lim_{\|y_i\| \to 0^+} \frac{\psi_i\left(y_i\right)}{\|y_i\|} = 0.$$

G5: For each $i = 1, 2$ we have that

$$\lim_{\|y_i\| \to 0^+} \frac{\phi_i\left(y_i\right)}{\|y_i\|} = 0.$$

G6: For each $i = 1, 2$ we have that $\psi_i\left(y_i\right)$ and $\phi_i\left(y_i\right)$ are nonnegative for all $y_i \geq 0$.

L2: The constants λ_1 and λ_2 satisfy

$$\Lambda_1 < \lambda_i < \Lambda_2,$$

for each $i = 1, 2$, where

$$\Lambda_1 := \max\left\{\frac{1}{2}\left[\sum_{s=0}^{b} \gamma G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_1, s\right) a_1\left(s+\nu_1-1\right) f_1^{**}\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\sum_{s=0}^{b} \gamma G_2\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_2, s\right) a_2\left(s+\nu_2-1\right) f_2^{**}\right]^{-1}\right\}$$

and

$$\Lambda_2 := \min\left\{\frac{1}{3}\left[\sum_{s=0}^{b} G_1\left(s+\nu_1-1, s\right) a_1\left(s+\nu_1-1\right) f_1^{*}\right]^{-1},\right.$$
$$\left.\frac{1}{3}\left[\sum_{s=0}^{b} G_2\left(s+\nu_2-1, s\right) a_2\left(s+\nu_2-1\right) f_2^{*}\right]^{-1}\right\},$$

where f_i^* and f_i^{**} retain their earlier meaning from conditions (F1)–(F2), for each $i = 1, 2$. Moreover, γ is defined just as it was earlier in this section.

Remark 7.72. Observe that there do exist nontrivial functionals satisfying conditions (G4) and (G5). For example, consider the functional given by

$$\phi(y) := [y(t_0)]^6,$$

where t_0 is some number in the domain of y. Then it is clear that

$$0 \leq \lim_{\|y\| \to 0+} \frac{[y(t_0)]^6}{\|y\|} \leq \lim_{\|y\| \to 0+} \frac{[y(t_0)]^6}{y(t_0)} = \lim_{\|y\| \to 0+} [y(t_0)]^5 = 0,$$

from which it follows that ϕ satisfies conditions (G4)–(G5); this specifically relies upon the fact that $\phi(y)$ is nonnegative for all $y \geq 0$.

We now present our second existence theorem of this section.

Theorem 7.73. *Suppose that conditions (F1)–(F2), (G4)–(G6), and (L2) hold. Then problem* (7.143)–(7.144) *has at least one positive solution.*

Proof. Begin by noting that by condition (L2) that there is $\varepsilon > 0$ such that

$$\max\left\{\frac{1}{2}\left[\sum_{s=0}^{b} G_1\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_1, s\right) a_1(s+\nu_1-1)\left(f_1^{**}-\varepsilon\right)\right]^{-1},\right.$$
$$\left.\frac{1}{2}\left[\sum_{s=0}^{b} G_2\left(\left\lfloor\frac{b+1}{2}\right\rfloor + \nu_2, s\right) a_2(s+\nu_2-1)\left(f_2^{**}-\varepsilon\right)\right]^{-1}\right\} \leq \lambda_1, \lambda_2 \tag{7.169}$$

and

$$\lambda_1, \lambda_2 \leq \min\left\{\frac{1}{3}\left[\sum_{s=0}^{b} G_1(s+\nu_1-1, s)\, a_1(s+\nu_1-1)\left(f_1^{*}+\varepsilon\right)\right]^{-1},\right.$$
$$\left.\frac{1}{3}\left[\sum_{s=0}^{b} G_2(s+\nu_2-1, s)\, a_2(s+\nu_2-1)\left(f_2^{*}+\varepsilon\right)\right]^{-1}\right\}. \tag{7.170}$$

Now, for the number ε determined by (7.169)–(7.170), it follows from conditions (G4)–(G5) there exists a number $\eta_1 > 0$ such that

$$\phi_1(y_1) \leq \varepsilon\|y_1\|, \tag{7.171}$$

whenever $\|y_1\| \le \eta_1$, and there exists a number $\eta_2 > 0$ such that

$$\psi_1(y_2) \le \varepsilon \|y_1\|, \tag{7.172}$$

whenever $\|y_1\| \le \eta_2$. Put

$$\eta^* := \min\{\eta_1, \eta_2\}.$$

We conclude that whenever $\|(y_1, y_2)\| < \eta^*$, each of (7.171) and (7.172) holds.

Now, for the same $\varepsilon > 0$ given in the first paragraph of this proof, we find that there exists a number η_3 such that

$$f_1(y_1, y_2) \le \left(f_1^* + \epsilon\right)(y_1 + y_2), \tag{7.173}$$

whenever $\|(y_1, y_2)\| < \eta_3$. Thus, by putting

$$\eta^{**} := \min\{\eta^*, \eta_3\},$$

we get that (7.171), (7.172), and (7.173) are collectively true.

So, define the set $\Omega_1 \subseteq \mathcal{X}$ by

$$\Omega_1 := \{(y_1, y_2) \in \mathcal{X} \ : \ \|(y_1, y_2)\| < \eta^{**}\}.$$

Then whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we have, for any $t_1 \in [\nu_1 - 2, \nu_1 + b]_{\mathbb{N}_{\nu_1 - 2}}$,

$$\begin{aligned} & S_1(y_1, y_2)(t_1) \\ & \le \varepsilon\|y_1\| + \varepsilon\|y_1\| \\ & \qquad + \lambda_1 \sum_{s=0}^{b} G_1(t_1, s)\, a_1(s + \nu_1 - 1) f_1(y_1(s + \nu_1 - 1), y_2(s + \nu_2 - 1)) \\ & \le 2\varepsilon\|y_1\| + \lambda_1 \sum_{s=0}^{b} G_1(s + \nu_1 - 1, s)\, a_1(s + \nu_1 - 1)\left(f_1^* + \varepsilon\right)\|(y_1, y_2)\| \\ & \le \left[2\varepsilon + \frac{1}{3}\right]\|(y_1, y_2)\|, \end{aligned} \tag{7.174}$$

where we have used condition (L2) together with (7.170). An entirely dual argument reveals that

$$S_2(y_1, y_2)(t) \le \left[2\varepsilon + \frac{1}{3}\right]\|(y_1, y_2)\|. \tag{7.175}$$

Therefore, putting (7.174)–(7.175) together we conclude that

$$\|S(y_1, y_2)\| \le \|(y_1, y_2)\|,$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ and ε is chosen sufficiently small, which may be assumed without loss of generality.

To complete the proof, we may give an argument essentially identical to the second half of the proof of Theorem 7.69. We omit this, and so, the proof is complete. □

Remark 7.74. As with Theorem 7.69, it is clear how the results of this section can be extended to the case in which (7.143) is replaced with n equations and boundary conditions (7.144) are extended to $2n$ boundary conditions in the obvious way. As with the corresponding generalization of Theorem 7.69, however, we omit the details of this extension.

We conclude by providing an explicit numerical example in order to illustrate the application of Theorem 7.69. This is the same example as the one presented in [94].

Example 7.75. Consider the boundary value problem

$$\begin{aligned} -\Delta^{1.3} y_1(t) &= \lambda_1 a_1\left(t + \frac{3}{10}\right) f_1\left(y_1\left(t + \frac{3}{10}\right), y_2\left(t + \frac{7}{10}\right)\right) \\ -\Delta^{1.7} y_2(t) &= \lambda_2 a_2\left(t + \frac{7}{10}\right) f_2\left(y_1\left(t + \frac{3}{10}\right), y_2\left(t + \frac{7}{10}\right)\right), \end{aligned} \tag{7.176}$$

subject to the boundary conditions

$$\begin{aligned} y_1\left(\frac{-7}{10}\right) &= \frac{1}{12} y_1\left(\frac{13}{10}\right) - \frac{1}{25} y_1\left(\frac{53}{10}\right) \\ y_1\left(\frac{213}{10}\right) &= \frac{1}{30} y_1\left(\frac{83}{10}\right) - \frac{1}{100} y_1\left(\frac{73}{10}\right) \\ y_2\left(-\frac{3}{10}\right) &= \frac{1}{40} y_2\left(\frac{17}{10}\right) - \frac{1}{150} y_2\left(\frac{77}{10}\right) \\ y_2\left(\frac{217}{10}\right) &= \frac{1}{17} y_2\left(\frac{47}{20}\right) - \frac{1}{30} y_2\left(\frac{107}{20}\right), \end{aligned} \tag{7.177}$$

where we take

$$\begin{aligned} a_1(t) &:= e^{t-4}, \\ a_2(t) &:= e^{t-4}, \end{aligned}$$

$$f_1(y_1,y_2) := 5000e^{-y_2}(y_1+y_2)+(y_1+y_2)\,,\text{ and}$$
$$f_2(y_1,y_2) := 7500e^{-y_1}(y_1+y_2)+(y_1+y_2)\,,$$

with $f_1, f_2 \ : \ [0,+\infty)\times[0,+\infty) \to [0,+\infty)$. It is clear from the statement of problem (7.176)–(7.177) that we have made the following declarations.

$$\begin{aligned}\psi_1(y_1) &:= \frac{1}{12}y_1\left(\frac{13}{10}\right) - \frac{1}{25}y_1\left(\frac{53}{10}\right)\\ \phi_1(y_1) &:= \frac{1}{30}y_1\left(\frac{83}{10}\right) - \frac{1}{100}y_1\left(\frac{73}{10}\right)\\ \psi_2(y_2) &:= \frac{1}{40}y_2\left(\frac{17}{10}\right) - \frac{1}{150}y_2\left(\frac{77}{10}\right)\\ \phi_2(y_2) &:= \frac{1}{17}y_2\left(\frac{47}{20}\right) - \frac{1}{30}y_2\left(\frac{107}{20}\right)\end{aligned} \tag{7.178}$$

Note, in addition, that y_1 is defined on the set

$$\left\{-\frac{7}{10},\frac{3}{10},\ldots,\frac{213}{10}\right\}\subseteq\mathbb{Z}_{-\frac{7}{10}}^{\frac{213}{10}},$$

whereas y_2 is defined on the set

$$\left\{-\frac{3}{10},\frac{7}{10},\ldots,\frac{217}{10}\right\}\subseteq\mathbb{Z}_{-\frac{3}{10}}^{\frac{217}{10}},$$

and we note that $\mathbb{Z}_{-\frac{7}{10}}^{\frac{213}{10}} \cap \mathbb{Z}_{-\frac{3}{10}}^{\frac{217}{10}} = \varnothing$, as, toward the beginning of this section, we indicated could occur in the study of problem (7.143)–(7.144). In particular, we have chosen $\nu_1 = \frac{13}{10}$, $\nu_2 = \frac{17}{10}$, and $b = 20$. We shall select λ_1 and λ_2 below.

We next check that each of conditions (F1)–(F2), (G1)–(G3), and (L1) holds. It is easy to check that (F1)–(F2) hold. On the other hand, since (7.178) reveals that each of the functionals is linear in y_1 and y_2, we conclude at once that (G1) holds. On the other hand, to see that conditions (G2)–(G3) hold, observe both that

$$\frac{1}{12}+\frac{1}{25}+\frac{1}{30}+\frac{1}{100}=\frac{1}{6}\le\frac{1}{4}$$

and that

$$\frac{1}{40}+\frac{1}{150}+\frac{1}{17}+\frac{1}{30}=\frac{421}{3400}\le\frac{1}{4}.$$

Furthermore, additional calculations show both that

$$\sum_{i=\nu_j-2}^{\nu_j+b} c_{i-\nu_j+2}^{j} G_j(i,s) \geq 0$$

and that

$$\sum_{k=\nu_j-2}^{\nu_j+b} d_{k-\nu_j+2}^{j} G_j(k,s) \geq 0,$$

for each $j = 1, 2$. So, we conclude that condition (G2) holds. Finally, one can compute the following estimates.

$$\begin{aligned}
\psi_1(\alpha_1) &\approx 0.012\\
\psi_1(\beta_1) &\approx 0.012\\
\psi_2(\alpha_2) &\approx 0.012\\
\psi_2(\beta_2) &\approx 0.0012\\
\phi_1(\alpha_1) &\approx 0.00091\\
\phi_1(\beta_1) &\approx 0.018\\
\phi_2(\alpha_2) &\approx 0.015\\
\phi_2(\beta_2) &\approx 0.000099
\end{aligned}$$

Consequently, condition (G3) is satisfied.

Finally, we check condition (L1) to determine the admissible range of the parameters, λ_i for $i = 1, 2$. To this end, recall from [31, Theorem 3.2] that the constant γ in Lemma 7.63 is

$$\gamma := \min\left\{ \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \times \left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4} - 2\right)^{\underline{\nu-1}} (\nu+b+1)^{\underline{\nu-1}}}{(\nu+b-1)^{\underline{\nu-1}}} \right], \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}} \right\}. \tag{7.179}$$

Thus, using the definition of γ provided by (7.179), we estimate that

$$\begin{aligned}
\Lambda_1 &\approx \max\left\{ f_1^{**} \cdot 3.288 \times 10^{-7}, f_2^{**} \cdot 1.0322 \times 10^{-7} \right\}\\
&= \max\left\{ 3.288 \times 10^{-7}, 1.0322 \times 10^{-7} \right\}\\
&= 3.337 \times 10^{-7},
\end{aligned}$$

whereas

$$\begin{aligned}\Lambda_2 &\approx \min\left\{f_1^* \cdot 1.871 \times 10^{-9}, f_2^* \cdot 1.363 \times 10^{-9}\right\}\\ &= \min\left\{5001 \cdot 1.871 \times 10^{-9}, 7501 \cdot 1.363 \times 10^{-9}\right\}\\ &= \min\left\{9.357 \times 10^{-6}, 1.022 \times 10^{-5}\right\}\\ &= 9.357 \times 10^{-6}.\end{aligned}$$

So, suppose that

$$\lambda_1, \lambda_2 \in \left[3.337 \times 10^{-7}, 9.357 \times 10^{-6}\right].$$

Then we conclude from Theorem 7.69 that problem (7.176)–(7.177) has at least one positive solution. And this completes the example.

Remark 7.76. A similar example could be provided for Theorem 7.73.

Remark 7.77. We note that a class of functions satisfying conditions (F1)–(F2) are given by the function $f : \overline{\mathbb{R}^n_+} \to [0, +\infty)$ defined by

$$f(\mathbf{x}) := C_1 e^{-g(\mathbf{x})} \nabla \cdot \mathbf{H}(\mathbf{x}),$$

where $g : \overline{\mathbb{R}^n_+} \to [0, +\infty)$, $C_1 > 0$ is a constant, and $\mathbf{H} : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{R}^n_+}$ is the vector field defined by

$$\mathbf{H}(\mathbf{x}) := \sum_{i=1}^{n} \frac{1}{2} x_i^2 \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the i-th vector in the standard ordered basis for $\mathbb{R}^n$; note that by the notation $\overline{\mathbb{R}^n_+}$ we mean the closure of the open positive cone in $\mathbb{R}^n$—i.e., we put

$$\overline{\mathbb{R}^n_+} := \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for each } 1 \leq i \leq n\} \subseteq \mathbb{R}^n.$$

More trivially, we remark that the collection of functions defined $L(y_1, y_2) = ay_1 + ay_2$, for $a > 0$, satisfies (F1)–(F2).

7.8 Concluding Remarks

In this chapter we have demonstrated several ways in which nonlocal elements may occur in the discrete fractional calculus. Such elements may arise explicitly, as is the case in the nonlocal BVP setting. On the other hand, the fractional sum and difference themselves contain nonlocal elements, and this considerably complicates the analysis and interpretation of fractional operators.

In closing we wish to draw attention to the fact that due to this implicit nonlocal structure, there are many open questions regarding the interpretation, particularly geometric, of the discrete fractional difference. For instance, as alluded to earlier, only recently has there been any development in our understanding of how the sign of, say $\Delta_0^\nu y(t)$ is related, for various ranges of ν, to the behavior of y itself. Yet in spite of these recent developments, it seems that there is likely much to be discovered in this arc of research.

Moreover, above and beyond pure geometrical implications, the nonlocal structure embedded within $\Delta_0^\nu y$ affects negatively the analysis of boundary and initial value problems insofar as the attendant analysis is much more complicated and there still remain some very basic open questions. For example, as we have seen in this chapter, even the elementary problem of analyzing a particular Green's function associated with a given boundary value operator is very nontrivial, often requiring arguments that while elementary are nonetheless technical. Furthermore, many fundamental areas of study in the integer-order difference calculus presently do not possess satisfactory analogues in the fractional-order setting. Among these is oscillation theory, which has no satisfactory fractional-order analogue. On the one hand, this is rather remarkable in recognition of the centrality of such results in the integer-order theory. On the other hand, however, given the tremendous complexity that the nonlocal structure of $\Delta_0^\nu y$ creates, perhaps it is unsurprising that such gaps exist. As with some of the other questions surrounding the discrete fractional calculus, it is unclear at present whether this gap can ultimately be filled in an at once satisfactory and elegant manner.

All in all, this section has shown a few of the ways in which nonlocalities may arise in the setting of boundary value problems. Moreover, we have seen how the implicit nonlocal structure of the discrete fractional sum and difference complicate in surprising ways their analysis. Finally, we hope that the reader has gained a sense of some of the open and unanswered questions in the discrete fractional calculus, questions whose solutions appear to be at once greatly complicated and substantively enriched by the nonlocal structure of fractional operators. As a concluding point, we wish to note that the interested reader may consult any of the following references for additional information on not only local and nonlocal boundary value problems, but also on other related topics in the discrete calculus that we have touched upon in this and other chapters [1, 2, 5, 6, 8–12, 14–30, 44, 45, 48, 51, 55–61, 68–73, 75, 79–82, 84–86, 97, 98, 100–103, 105–113, 115–118, 120, 122, 126–130, 132, 133, 136, 138, 140–144, 148–151, 154–158, 160–166, 168–172]:

7.9 Exercises

7.1. Prove the result mentioned in Remark 7.14.