Chapter 4
Multiple Curve Extensions of Libor Market Models (LMM)

This chapter deals with multiple curve models on a discrete tenor in the spirit of the Libor market models (LMM) and, somewhat differently from the previous two Chaps. 2 and 3, we present here basically just an overview of the major existing approaches.

The Libor market models that were originated by Miltersen et al. (1997) and Brace et al. (1997), were later further developed in several works by Mercurio and co-authors, as well as authors related to them. Consequently these authors were also among the first ones to extend the LMMs to a multi-curve setting. Starting from papers like Morini (2009) and Bianchetti (2010), where the latter uses the analogy with cross-currency modeling to develop a two-curve interest rate model, a series of papers have appeared extending the LMMs to a multi-curve setting, among them Mercurio (2009, 2010a, b, c), Mercurio and Xie (2012) and Ametrano and Bianchetti (2013). This series of papers, in particular Mercurio (2010a) and Mercurio and Xie (2012), which include the developments contained in previous papers authored/co-authored by Mercurio, form the first approach of which we give an overview in Sect. 4.1. We do not, however, enter into the details of the pricing formulas and the calibration examples, for which we therefore refer to the original papers. Related to the papers by Mercurio (2010a) and Mercurio and Xie (2012) is the paper Ametrano and Bianchetti (2013), where the authors deal in particular with the bootstrapping of various multiple-tenor yield curves, thereby considering essentially only linear interest rate derivatives; here too we simply refer to the original paper.

The other approach, that is alternative to the one just mentioned and that proposes a further theoretical development, is the one in Grbac et al. (2014) which concerns an affine Libor model with multiple curves. A description of this further approach is presented in Sect. 4.2.

The above papers concern mainly "clean valuation" approaches. A more comprehensive, multi-currency, multi-curve approach has been initiated in Fujii et al. (2010, 2011), see also Piterbarg (2010). In parallel, Henrard has developed a more practically oriented approach, for which we refer to his recent book Henrard (2014) that

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synthesizes his previous production, but it is not exactly in the context of an LMM that is our main concern in this chapter. On the other hand Henrard has considered, see e.g. Henrard (2010), multiplicative spreads as example of a form of spreads that are alternative to the additive spreads considered in the above papers and that may turn out to be advantageous in some situations. Multiplicative spreads form also the basis of the approach presented in the paper Cuchiero et al. (2015). Section 4.3 contains a brief overview of the approach in Henrard (2010), as well as that in Cuchiero et al. (2015).

4.1 Multi-curve Extended LMM

As mentioned above, this section is essentially a synthesis of work done by Mercurio and related authors and partly also by Ametrano and Bianchetti (2013).

The classical Libor market models are based on the joint evolution of consecutive forward Libor rates corresponding to a given tenor structure. We recall from the discussion in Chap. 1 that in the classical setup the Libor rate $L(t; T, S)$ was assumed to coincide with the corresponding forward OIS rate $F(t; T, S)$, but this assumption is no longer valid after the crisis.

Recalling the definition of the forward OIS rate in (1.16),

$$F(t; T, S) = E^{Q^S}\{F(T; T, S) \mid \mathcal{F}_t\} = \frac{1}{S-T}\left(\frac{p(t, T)}{p(t, S)} - 1\right) \tag{4.1}$$

notice that this rate is directly related to the OIS bond prices $p(t, T)$, namely to the discount curve and so the first issue concerns the proper modeling of the discount curve, which occasionally we shall also denote by $p_D(t, T)$. Notice, furthermore, that the discount curve intervenes also in other situations, for example we have that

(i) Swap rates can be represented as linear combinations of forward Libor rates, which are referred to as FRA rates in Mercurio (2010a), with coefficients that depend solely on the discount curve (cf. 1.28).
(ii) Pricing measures correspond to numéraires given by portfolios of OIS bonds and affect thus the drift correction in a measure change.

As already discussed in Chap. 1, a general choice of the discount curve is the OIS curve and we shall do so here as well. This choice is supported by various arguments (recall also Sect. 1.3.1). In particular, collaterals in cash are revalued daily at a rate equal or close to the overnight rate. Note, however, that collaterals can be based also on bonds or other assets, such as foreign currency. In the latter case appropriate adjustments have to be performed for the remuneration rate (see Fujii et al. 2010). The OIS curve is commonly accepted as discount curve and for possible situations, such as in exotics or different currencies, where different discounting is adopted, it can still be considered as a good proxy of the risk-free rate.

Remark 4.1 In this chapter the standing assumption will be that the OIS bonds are tradable assets, see for example Mercurio (2010a) and the comments in Sect. 1.3.1. Note that in the spirit of the Libor market models we do not assume the existence of the OIS short rate r derived from the OIS bond prices and the related martingale measure Q as in the previous chapters, but instead we work directly under the forward measures using the OIS bonds as numéraires.

Since the forward rate $F(t; T, S)$ can also be considered as the fair fixed rate at time $t \leq T$ of a forward rate agreement, where the floating rate received at S is $F(T; T, S)$, we shall call $F(t; T, S)$ the forward OIS curve (recall that in practice it can be stripped from OIS swap rates; see also the middle part of Remark 1.2).

Concerning the Libor rates, in the work by Mercurio and related authors an FRA rate is considered that is given as the fair fixed rate at $t \leq T$ to be exchanged at time S for the Libor rate $L(T; T, S)$, namely such that this swap has zero value at $t \leq T$. Denoting by Q^S the S-forward measure with numéraire $p(t, S)$, the FRA rate is then given by

$$FRA(t; T, S) = E^{Q^S}\{L(T; T, S) \mid \mathscr{F}_t\} \tag{4.2}$$

where E^{Q^S} denotes expectation with respect to Q^S. This definition, that corresponds to the forward Libor rate $L(t; T, S)$ as in Definition 1.2, has the following advantages

(i) The rates $FRA(t; T, S)$ coincide with the corresponding spot Libor rates at their reset times; they can thus generate any payoff depending on the Libor rates.
(ii) The rates $FRA(t; T, S)$ are martingales under the corresponding forward measures.
(iii) The fact that swap rates can be written as linear combinations of FRA rates with coefficients depending solely on the discount curve is convenient for bootstrapping purposes (see Mercurio 2010a).

In the sequel we shall continue consistently using the name forward Libor rates, keeping in mind that these are by definition the same as the FRA rates from Mercurio (2010a).

4.1.1 Description of the Model

According to a practice followed in the post-crisis setting, the forward Libor rate is mostly viewed as a sum of the forward OIS rate plus a basis/spread (thereby thinking of this basis as a factor driving the Libors in conjunction with the OIS curve). In line with this practice, in the more recent work of Mercurio and related authors an additive spread between the Libor and the OIS curves is considered.

We start with some notation keeping it in line with the above-mentioned papers. As in Sect. 1.3, for given a tenor x, let $\mathscr{T}^x = \{0 \leq T_0^x < \cdots < T_{M_x}^x\}$ be a tenor structure compatible with x and denote by δ_k^x the year fraction of the length of the generic kth interval $(T_{k-1}^x, T_k^x]$. Denote by $p(t, T_k^x)$ the price of the OIS bond maturing at T_k^x (discount curve) and set (see (1.16) or (4.1))

$$F_k^x(t) := F(t; T_{k-1}^x, T_k^x) = \frac{1}{\delta_k^x}\left[\frac{p(t, T_{k-1}^x)}{p(t, T_k^x)} - 1\right] \tag{4.3}$$

Furthermore, as mentioned at the end of the previous subsection, since the FRA rates, as given in (4.2), were introduced in relation to the FRAs with the underlying Libor rates, set

$$L_k^x(t) := FRA(t; T_{k-1}^x, T_k^x) \tag{4.4}$$

and call $L_k^x(t)$ the forward Libor rate.

Notice that the FRA rates as introduced in (4.2) correspond to what is called a standard (or text-book) FRA. This has to be contrasted with the so-called "market FRA", cf. Remark 1.3, which differ from the standard ones in that the payment is made at the beginning of the reference interval, discounted by the corresponding Libor rate. Hence, taking $T = T_{k-1}^x$, $S = T_k^x$ for a generic k, one has

$$P^{mFRA}(T_{k-1}^x; T_{k-1}^x, T_k^x, R, 1) = \frac{\delta_k^x \left(L(T_{k-1}^x; T_{k-1}^x, T_k^x) - R\right)}{1 + \delta_k^x L(T_{k-1}^x; T_{k-1}^x, T_k^x)} \tag{4.5}$$

implying that (see Appendix A in Mercurio 2010b), at $t < T_{k-1}^x$ one has,

$$R^{mFRA}(t; T_{k-1}^x, T_k^x) = \frac{1}{\delta_k^x}\left[\frac{1}{E^{Q^{T_{k-1}^x}}\left\{\frac{1}{1+\delta_k^x L(T_{k-1}^x; T_{k-1}^x, T_k^x)} \mid \mathscr{F}_t\right\}} - 1\right] \tag{4.6}$$

As recalled in Remark 1.3, Mercurio (2010b) (see also Ametrano and Bianchetti 2013) points out that the difference in the values $P^{mFRA}(T_{k-1}^x; T_{k-1}^x, T_k^x, R, 1)$ and $P^{FRA}(T_{k-1}^x; T_{k-1}^x, T_k^x, R, 1)$ is generally small, so that one can limit oneself to standard FRA rates also for what concerns a possible bootstrapping from market FRA rates.

As already mentioned, following the practice to build Libor curves at a spread over the OIS curve, Mercurio and the related authors consider additive spreads that can now be defined as

$$S_k^x(t) := L_k^x(t) - F_k^x(t) \tag{4.7}$$

Additive spreads have the advantage that, since L_k^x and F_k^x are martingales under $Q^{T_k^x}$, so is also S_k^x. To model their dynamics under $Q^{T_k^x}$ one thus needs to specify only the volatility and correlation structure.

Having now the three quantities $L_k^x(t)$, $F_k^x(t)$, $S_k^x(t)$, we need to introduce dynamics for them. Given the relationship (4.7), we need only the joint dynamics of two of the three quantities, which then induces the dynamics also for the third one. The aim thereby should be to achieve model tractability in view of interest rate derivative pricing, as well as a convenient setup for calibration and bootstrapping. In Mercurio (2010a, b) the author chooses to jointly model $F_k^x(t)$ and $S_k^x(t)$ that has as main advantage the direct modeling of the spread allowing thus to model its dynamics so that it remains positive. Such a choice is made also in Fujii et al. (2011). Notice furthermore that when, as required for a multi-curve setup, one has to model multiple tenors simultaneously, i.e. $F_k^{x_i}(t)$ and $S_k^{x_i}(t)$ for different values x_i of the tenor x, one has to properly account for possible no-arbitrage relations that have to hold across different time intervals. To this effect notice that only forward OIS rates with different tenors are constrained by no-arbitrage relations; the associated spreads are relatively free to move independently from one another. Given this freedom, one may try to derive models that preserve the tractability of the single tenor case, especially in view of pricing optional derivatives in closed form. In this sense, in Mercurio (2010a, b) an approach is presented by choosing the dynamics of the OIS rates and related spreads so that they are similar for all considered tenors. Furthermore, as the title in Mercurio and Xie (2012) puts it, the spread should by all means be stochastic. In fact, a relatively simple approach would be to elect a given forward OIS curve as reference curve and model all other curves at a deterministic spread over the reference curve. However, this is in contrast with the empirical evidence (see Figs. 1.3 (left) and 1.4) and furthermore, with a deterministic basis, the Libor-OIS swaption price would in some situations, e.g. OTM swaptions, turn out to be zero. Although the impact of a stochastic basis on the pricing of exotic products is difficult to assess a priori, in Mercurio and Xie (2012) it is shown that also with a suitably modeled stochastic basis one may achieve very good tractability.

4.1.2 Model Specifications

We mention here specific models considered in Mercurio (2010b) and Mercurio and Xie (2012) where, in line with the classical LMMs, log-normal and shifted log-normal models are considered, but with stochastic volatility in form of Heston or SABR.

For the multi-curve setup consider now, as in Sect. 1.3, different possible tenor values $x_1 < x_2 < \cdots < x_n$ and the associated tenor structures $\mathcal{T}^{x_i} = \{0 \le T_0^{x_i} < \cdots < T_{M_{x_i}}^{x_i}\}$, thereby assuming that $\mathcal{T}^{x_n} \subset \mathcal{T}^{x_{n-1}} \subset \cdots \subset \mathcal{T}^{x_1} \subseteq \mathcal{T}$.

Denote then by $L_k^{x_i}(t), F_k^{x_i}(t), S_k^{x_i}(t)$ the corresponding relevant quantities and let $\delta_k^{x_i}$ be the year fraction of $T_k^{x_i} - T_{k-1}^{x_i}$.

Starting from the forward OIS rates, in Mercurio (2010b) these are modeled according to the following shifted-type dynamics

$$dF_k^{x_i}(t) = \sigma_k^{x_i}(t) V^F(t) \left[\frac{1}{\delta_k^{x_i}} + F_k^{x_i}(t) \right] dZ_k^{F,x_i}(t) \tag{4.8}$$

where $\sigma_k^{x_i}$ are deterministic functions and Z_k^{F,x_i} are Wiener processes under the forward measures $Q^{T_k^{x_i}}$, for all $k = 1, \ldots, M_{x_i}$. The process $V^F(t)$ is a common factor process with $V^F(0) = 1$ and independent of all Z_k^{F,x_i}. As mentioned previously, the $F_k^{x_i}$ have to satisfy no-arbitrage consistency conditions and in Mercurio (2010b) it is shown that they are given by

$$\sigma_k^{x_i}(t) = \sum_{j=i_{k-1}+1}^{i_k} \sigma_j^{x_1}(t) \tag{4.9}$$

namely the volatility coefficient $\sigma_k^{x_i}$ of $F_k^{x_i}$ has to be equal to the sum of the volatility coefficients $\sigma_j^{x_1}$ of the rates $F_j^{x_1}$ for $j \in \{i_{k-1}+1, i_{k-1}+2, \ldots, i_k\}$. Here j correspond to the indices of the tenor dates $T_{k-1}^{x_i} = T_{i_{k-1}}^{x_1} < T_{i_{k-1}+1}^{x_1} < \cdots < T_{i_k}^{x_1} = T_k^{x_i}$ of the tenor structure $\mathscr{T}^{x_1}$ falling between the dates $T_{k-1}^{x_i}$ and $T_k^{x_i}$ of the tenor structure $\mathscr{T}^{x_i}$.

Coming now to the Libor-OIS spreads, Mercurio and Xie (2012) start from the following general model

$$S_k^{x_i}(t) = \phi_k^{x_i}(F_k^{x_i}(t), X_k^{x_i}(t)) \tag{4.10}$$

where $X_k^{x_i}$ are factor processes and the functions $\phi_k^{x_i}$ have to be chosen so that $S_k^{x_i}$ are martingales under $Q^{T_k^{x_i}}$. In particular, in Mercurio and Xie (2012) the functions $\phi_k^{x_i}$ are chosen to be affine functions with the advantage that the parameters in the affine specification can be explained in terms of correlations between OIS rates and basis spreads, as well as in terms of their variances. If also the forward OIS rates follow a convenient model, then caplets and swaptions can be priced in semi-closed form.

As for the factor processes $X_k^{x_i}$, things can be simplified without too much loss of flexibility by taking them to be independent of k, i.e. $X_k^{x_i}(t) = X^{x_i}(t)$, for all k, with $X^{x_i}(t)$ following a log-normal model. If, then, the OIS rate is, say, of the $G1^{++}$ form (i.e. one-factor Hull and White (1990) model with deterministic shift to be calibrated to the initial term structure), with the affine model choice for $\phi_k^{x_i}$ one obtains semi-analytic pricing formulas for caplets and swaptions in the sense that what is required is a one-dimensional integration of a closed-form function of the Black-Scholes type (see Mercurio and Xie 2012 for the case of a swaption).

Remaining always with the Libor-OIS spreads, Mercurio (2010b) assumes more specifically a model of the form

$$S_k^{x_i}(t) = S_k^{x_i}(0)\, M^{x_i}(t)\,, \quad k = 1, \ldots, M_{x_i} \tag{4.11}$$

where, analogously to the previous case where $X_k^{x_i}(t) = X^{x_i}(t)$, also M^{x_i} remains the same for all k and is defined by the following SABR-type process

$$\begin{cases} dM^{x_i}(t) = (M^{x_i}(t))^{\beta^{x_i}}\, V^{x_i}(t)\, dZ^{x_i}(t) \\ dV^{x_i}(t) = \varepsilon^{x_i}\, V^{x_i}(t)\, dW^{x_i}(t) \end{cases} \tag{4.12}$$

with $\beta^{x_i} \in (0, 1]$, $\varepsilon^{x_i} > 0$ and where Z^{x_i} and W^{x_i} are Wiener processes with respect to each forward measure $Q^{T_k^{x_i}}$, independent of the Wiener processes Z_k^{F,x_i} in (4.8), but which may be correlated, i.e. $dZ^{x_i} dW^{x_i} = \rho^{x_i} dt$ with $\rho^{x_i} \in [-1, 1]$. This model allows for convenient caplet and, in some particular cases, swaption pricing (see Mercurio 2010b). The fact that the rates and the spreads with different tenors x_i follow the same type of dynamics, leads to similar pricing formulas for caps and swaptions even if they are based on different tenors. This is particularly convenient for simultaneous option pricing across different tenors, as well as for calibration.

4.2 Affine Libor Models with Multiple Curves

In this section we present the model developed in Grbac et al. (2014), which is also a discrete-tenor model and is based on affine driving processes. This modeling approach has first been proposed in Keller-Ressel et al. (2013) in the single-curve case and further extended in Grbac et al. (2014) to the multiple curve setup. The main advantage of this framework is its ability to ensure positive interest rates and spreads by construction, and at the same time, produce semi-analytic caplet and swaption pricing formulas. In contrast to Mercurio (2010b), the OIS rates F_k^x and the Libor rates L_k^x are chosen as modeling quantities, which ensures straightforward pricing of caplets and at the same time the positivity of spreads S_k^x can be easily obtained. These features are due to a specific model construction, which relies on a family of parametrized martingales greater or equal to one and increasing with respect to the parameter. When such martingales are taken as building blocks of the model, as we shall see below, the positivity of interest rates and spreads follows simply by construction. The second important point is that affine processes are chosen as driving processes for this family of martingales, thus guaranteeing analytic tractability of the model and, consequently, semi-analytic pricing formulas for non-linear derivatives based on Fourier transform methods.

4.2.1 The Driving Process and Its Properties

In this section we shall fix the probability space and the driving process that we are going to work with. For sake of simplicity, we choose to work with affine diffusions in order to present the model in a concise and simple manner. Another reason is that affine diffusions were already used as driving processes in Chap. 2 and, hence, we can rely on the technical preliminaries from that chapter. Since the model construction requires a positive affine process as a driving process, this boils down to multidimensional CIR processes. However, we emphasize that the original paper of Grbac et al. (2014) is not limited to this class and the model is based on general positive affine processes allowing for jumps as well. This is especially important in view of model calibration, where the additional flexibility coming from the jumps is exploited to ensure a better fit to market data.

Let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ denote a complete stochastic basis, where $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ and T denotes some finite time horizon. Consider a stochastic process $X = (X^1, \ldots, X^d)$, where each component X^i solves the SDE

$$dX_t^i = (a^i - b^i X_t^i)dt + \sigma^i \sqrt{X_t^i}\, dw_t^i \tag{4.13}$$

with w^i a Wiener process such that all Wiener processes w^i, $i = 1, \ldots, d$, are mutually independent. The coefficients a^i, b^i and σ^i are positive and $a^i \geq \frac{(\sigma^i)^2}{2}$.

Denote

$$\mathscr{I}_T := \left\{u = (u^1, \ldots, u^d) \in \mathbb{R}^d : E_x\left\{e^{\langle u, X_T\rangle}\right\} < \infty\right\}$$

where E_x denotes the expectation conditional on $X_0 = x$. Note that this is a multi-dimensional analog of the set $\mathscr{I}_T$ defined in (2.12). Then according to Lemma 2.2, the conditional moment generating function of X_T has the following exponentially affine form:

$$E\left\{\exp\langle u, X_T\rangle \middle| \mathscr{F}_t\right\} = \exp\left(A^u(T-t) + \langle B^u(T-t), X_t\rangle\right) \tag{4.14}$$

for all $u \in \mathbb{R}_+^d \cap \mathscr{I}_T$ and $0 \leq t \leq T$. Here $A^u(T-t)$ and $B^u(T-t)$ are obtained by applying Lemma 2.2 to each component X^i of X and using independence. This yields

$$A^u(T-t) = \sum_{i=1}^d A^{u,i}(T-t), \qquad B^u(T-t) = (B^{u,1}(T-t), \ldots, B^{u,d}(T-t))$$

where, for each $i = 1, \ldots, d$, $A^{u,i}(T-t)$ and $B^{u,i}(T-t)$ correspond to $A(t,T) = A(T-t)$ and $-B(t,T) = -B(T-t)$ in Lemma 2.2 applied to the process X^i with $\gamma = 0$ and $K = -u$. In Eq. (4.14) $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^d$.

An essential ingredient in affine Libor models, as we shall see in the next subsection, is the construction of parametrized martingales which are greater than or equal to one and increasing in this parameter, see the review paper by Papapantoleon (2010). The following two lemmas, taken from Grbac et al. (2014) and Keller-Ressel et al. (2013), summarize the main ideas and properties on which the construction will be based.

Lemma 4.1 *Consider the affine process X defined above and let $u \in \mathbb{R}_+^d \cap \mathscr{I}_T$. Then the process $M^u = (M_t^u)_{t\in[0,T]}$ with*

$$M_t^u = E\left\{e^{\langle u, X_T\rangle} | \mathscr{F}_t\right\} = \exp\left(A^u(T-t) + \langle B^u(T-t), X_t\rangle\right) \tag{4.15}$$

is a P-martingale, greater than or equal to one, and the mapping $u \mapsto M_t^u$ is increasing, for every $t \in [0,T]$.

In the lemma below inequalities involving vectors are interpreted componentwise.

Lemma 4.2 *The functions $A^u(t)$ and $B^u(t)$ in (4.14) satisfy the following:*

1. $A^0(t) = B^0(t) = 0$ *for all* $t \in [0,T]$.
2. *For each $t \in [0,T]$, the functions $\mathscr{I}_T \ni u \mapsto A^u(t)$ and $\mathscr{I}_T \ni u \mapsto B^u(t)$ are (componentwise) convex.*

3. *$u \mapsto A^u(t)$ and $u \mapsto B^u(t)$ are order-preserving: let $(t,u),(t,v) \in [0,T] \times \mathscr{I}_T$, with $u \le v$. Then*

$$A^u(t) \le A^v(t) \quad \text{and} \quad B^u(t) \le B^v(t) \tag{4.16}$$

4. *$u \mapsto B^u(t)$ is* strictly *order-preserving: let $(t,u),(t,v) \in [0,T] \times \mathscr{I}_T$, with $u < v$. Then $B^u(t) < B^v(t)$.*

4.2.2 The Model

Consider again the tenor structures introduced in Sect. 1.3 and used in Sect. 4.1.2, $\mathscr{T}^{x_n} \subset \mathscr{T}^{x_{n-1}} \subset \cdots \subset \mathscr{T}^{x_1} \subseteq \mathscr{T}$, where $\mathscr{T}^{x_i} = \{0 \le T_0^{x_i} < \cdots < T_{M_{x_i}}^{x_i}\}$, for each $i = 1, \ldots, n$, and where $\delta_k^{x_i}$ denotes the year fraction of $T_k^{x_i} - T_{k-1}^{x_i}$. Recall that we assume that all $T_{M_{x_i}}^{x_i}$ coincide and denote by T_M the common terminal date for all tenor structures. As earlier, denote by $\mathscr{X} := \{x_1, \ldots, x_n\}$ the set of all tenors and for each tenor $x \in \mathscr{X}$, let $\mathscr{K}^x := \{1, 2, \ldots, M_x\}$ denote the collection of all subscripts related to the tenor structure $\mathscr{T}^x$. In the sequel we shall assume for ease of notation that each tenor structure is equidistant, i.e. $\delta_k^{x_i} = \delta^{x_i}$.

As earlier, we consider the OIS curve as discount curve. We denote by $p(t,T)$ the discount factor, i.e. the price of the OIS bond at time t for maturity T.

Moreover, let Q^{T_M} denote the terminal forward measure, i.e. the martingale measure associated with the numéraire $p(\cdot, T_M)$, which is supposed to be given. The corresponding expectation is denoted by E^{T_M}. Then, we introduce forward measures $Q^{T_k^x}$ associated to the numéraires $p(\cdot, T_k^x)$ for every tenor x and $k \in \mathscr{K}^x$. The corresponding expectation is denoted by $E^{T_k^x}$. The forward measures $Q^{T_k^x}$ are equivalent to Q^{T_M}, and defined in the usual way via

$$\left.\frac{dQ^{T_k^x}}{dQ^{T_M}}\right|_{\mathscr{F}_t} = \frac{p(0,T_M)}{p(0,T_k^x)} \frac{p(t,T_k^x)}{p(t,T_M)} \tag{4.17}$$

As seen in Sect. 4.1, the main modeling objects in the multiple curve LMM setting are the forward OIS rates F_k^x, the forward Libor rates L_k^x and the spreads S_k^x. Recalling again Mercurio (2010b), a good model for the dynamic evolution of the forward OIS and forward Libor rates, and thus also of their spread, should satisfy certain conditions which stem from economic reasoning, arbitrage requirements and the definitions of these rates. Grbac et al. (2014) formulate these conditions as model requirements:

(M1) $F_k^x(t) \ge 0$ and $F_k^x \in \mathscr{M}(Q^{T_k^x})$, for all $x \in \mathscr{X}$, $k \in \mathscr{K}^x$, $t \in [0, T_{k-1}^x]$.
(M2) $L_k^x(t) \ge 0$ and $L_k^x \in \mathscr{M}(Q^{T_k^x})$, for all $x \in \mathscr{X}$, $k \in \mathscr{K}^x$, $t \in [0, T_{k-1}^x]$.
(M3) $S_k^x(t) \ge 0$ and $S_k^x \in \mathscr{M}(Q^{T_k^x})$, for all $x \in \mathscr{X}$, $k \in \mathscr{K}^x$, $t \in [0, T_{k-1}^x]$.

Here $\mathscr{M}(Q^{T_k^x})$ denotes the set of $Q^{T_k^x}$-martingales.

The model presented in the sequel satisfies the above conditions by construction, while producing tractable dynamics for all three processes under all forward measures. The approach was first introduced by Keller-Ressel et al. (2013) and then extended to the multiple curve setup by Grbac et al. (2014). The first step is the construction of two families of parametrized Q^{T_M}-martingales driven by the process X defined in the previous subsection, which is assumed to be affine under the measure Q^{T_M}.

More precisely, assume that the process X starts at the canonical value $\mathbf{1} = (1, \dots, 1)$ and assume that two sequences of vectors $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^d_+ \cap \mathscr{I}_T$ are given. Then one constructs two families of parametrized Q^{T_M}-martingales following the method described in Lemma 4.1 by setting

$$M_t^{u_k} = \exp\left(A^{u_k}(T_M - t) + \langle B^{u_k}(T_M - t), X_t\rangle\right) \tag{4.18}$$

and

$$M_t^{v_k} = \exp\left(A^{v_k}(T_M - t) + \langle B^{v_k}(T_M - t), X_t\rangle\right) \tag{4.19}$$

By Lemma 4.1, $M_t^{u_k} \geq 1$ and $M_t^{v_k} \geq 1$, for all k. Note, moreover, that the ordering of parameters u_k and v_k carries over to the martingales related to them; for example, if $u_{k-1} \geq u_k$, then $M_t^{u_{k-1}} \geq M_t^{u_k}$ for all $t \geq 0$. Such families of martingales are then used to model the forward OIS and Libor rates.

Let us fix an arbitrary tenor x and the associated tenor structure $\mathscr{T}^x$. We begin by presenting the model for the OIS rates. This model is completely analogous to the single-curve model introduced by Keller-Ressel et al. (2013). In the first step, one notices that

$$1 + \delta^x F_k^x(t) = \frac{p(t, T_{k-1}^x)}{p(t, T_k^x)} = \frac{\frac{p(t,T_{k-1}^x)}{p(t,T_M)}}{\frac{p(t,T_k^x)}{p(t,T_M)}} \tag{4.20}$$

where the forward price process $\frac{p(\cdot,T_k^x)}{p(\cdot,T_M)}$ is a Q^{T_M}-martingale for any $k \in \mathscr{K}^x$. Therefore, to model the OIS rates F_k^x, one begins by postulating the dynamics of each forward price process $\frac{p(\cdot,T_k^x)}{p(\cdot,T_M)}$:

$$\frac{p(t, T_k^x)}{p(t, T_M)} = M_t^{u_k^x}, \qquad k \in \mathscr{K}^x,\ t \leq T_k^x \tag{4.21}$$

where $u_k^x \in \mathbb{R}^d_+ \cap \mathscr{I}_T$ is some vector. The second step follows from (4.20) and (4.21), namely

$$1 + \delta^x F_k^x(t) = \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} \tag{4.22}$$

and we note that this process is a $Q^{T_k^x}$-martingale since $M^{u_{k-1}^x}$ is a Q^{T_M}-martingale and $M^{u_k^x} = \frac{p(\cdot,T_k^x)}{p(\cdot,T_M)}$ is the density process for the measure change from Q^{T_M} to $Q^{T_k^x}$ up to a normalizing constant; compare with (4.17). Finally, the third step relies on the observation that, if the vectors u_k^x are chosen to be (componentwise) decreasing with respect to k, i.e. $u_{k-1}^x \geq u_k^x$, it follows $M^{u_{k-1}^x} \geq M^{u_k^x}$ and therefore $1 + \delta^x F_k^x(t) \geq 1$, or equivalently $F_k^x(t) \geq 0$.

Now on top of the model for the OIS rates, it remains to suitably specify the dynamics of the Libor rates L_k^x in order to completely specify the multiple curve model. To do so, a similar idea can be used. More precisely, one postulates that

$$1 + \delta^x L_k^x(t) = \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}} \tag{4.23}$$

for every $k = 2, \ldots, M_x$ and $t \in [0, T_{k-1}^x]$ and where $v_{k-1}^x \in \mathbb{R}_+^d \cap \mathscr{I}_T$ is some vector. Hence, the process $1 + \delta^x L_k^x$ is a $Q^{T_k^x}$-martingale by exactly the same arguments as above. In addition, if $v_{k-1}^x \geq u_k^x$, then $M^{v_{k-1}^x} \geq M^{u_k^x}$ and $1 + \delta^x L_k^x(t) \geq 1$, or equivalently $L_k^x(t) \geq 0$.

This procedure presents the main modeling idea. The questions that still have to be answered are, if such sequences of vectors (u_k^x) and (v_k^x) can be found for any given initial term structure of the forward OIS and the forward Libor rates. Moreover, in case of an affirmative answer, do these sequences possess the desired monotonicity properties? The following proposition, which summarizes the results shown by Grbac et al. (2014), describes the main properties of the model and explains how to construct it from a given initial term structure of OIS and Libor rates.

Proposition 4.1 *Consider the finest tenor structure $\mathscr{T}$, let $p(0, T_l)$, $l \in \mathscr{K}$, be the initial term structure of non-negative OIS bond prices and assume that*

$$p(0, T_1) \geq \cdots \geq p(0, T_M)$$

Moreover, for a fixed tenor x and the corresponding tenor structure $\mathscr{T}^x$, let $L_k^x(0)$, $k \in \mathscr{K}^x$, be the initial term structure of non-negative forward Libor rates and assume that for every $k \in \mathscr{K}^x$

$$L_k^x(0) \geq \frac{1}{\delta^x}\left(\frac{p(0, T_{k-1}^x)}{p(0, T_k^x)} - 1\right) = F_k^x(0) \tag{4.24}$$

Then the following statements are true:

1. *There exists a decreasing sequence $u_0 \geq u_1 \geq \cdots \geq u_M = 0$ in $\mathbb{R}_+^d \cap \mathscr{I}_T$, such that*

$$M_0^{u_l} = \frac{p(0, T_l)}{p(0, T_M)} \quad \textit{for all } l \in \mathscr{K} \tag{4.25}$$

   Furthermore, for each $k \in \mathscr{K}^x$, we set

$$u_k^x := u_l \tag{4.26}$$

where $l \in \mathcal{K}$ is such that $T_l = T_k^x$.

2. *There exists a sequence $v_0^x, v_2^x, \ldots, v_{M_x}^x = 0$ in $\mathbb{R}_+^d \cap \mathcal{I}_T$, such that $v_k^x \geq u_k^x$ and*

$$M_0^{v_k^x} = (1 + \delta^x L_{k+1}^x(0)) M_0^{u_{k+1}^x}, \quad \text{for all } k = 0, 1, \ldots, M_x - 1 \tag{4.27}$$

3. *If X is one-dimensional, the sequences $(u_k^x)_{k \in \mathcal{K}^x}$ and $(v_k^x)_{k \in \mathcal{K}^x}$ are unique.*
4. *If all initial forward OIS rates and initial spreads are positive, then the sequence (u_k^x) is strictly decreasing and $v_k^x > u_k^x$, for all $k = 0, 1, \ldots, M_x - 1$.*

Therefore, from the model construction it follows immediately:

1. F_k^x and L_k^x are $Q^{T_k^x}$-martingales, for every $k \in \mathcal{K}^x$.
2. $L_k^x(t) \geq F_k^x(t) \geq 0$, for every $k \in \mathcal{K}^x$ and $t \in [0, T_{k-1}^x]$.

Remark 4.2 The results of Proposition 4.1 confirm that, for given initial term structures, the affine Libor model with multiple curves can theoretically be constructed by choosing sequences (u_k^x) and (v_k^x) as described above. Regarding the practical implementation of the model, one notices that when the driving process is multidimensional (which will typically be the case in applications), the vector parameters (u_k^x) and (v_k^x) are not unique and it seems that there is no canonical choice for them. This in turn gives a freedom to devise special cases of the model by pre-choosing various suitable structures for these sequences and then fitting the initial structures. One such example is a factor model with common and idiosyncratic components for each OIS and Libor rate, which is presented in Sect. 8 of Grbac et al. (2014) and where this is achieved by setting some components of (u_k^x) and (v_k^x) to zero or mutually equal in order to exclude the effect of certain components of the driving process on each specific rate.

Remark 4.3 The multiple curve affine Libor model is constructed under the terminal forward measure Q^{T_M}. Grbac et al. (2014) show that the model structure is preserved under different forward measures. More precisely, the process X remains an affine process, although its 'characteristics' become time-dependent, under any forward measure other than Q^{T_M}. The affine property plays a crucial role in the derivation of tractable pricing formulas for interest rate derivatives in the next subsection. Note, furthermore, that the multiple curve affine LIBOR model fulfills requirements (M1)–(M3), which are consistent with the typical market observations of nonnegative interest rates and spreads. However, a phenomenon of negative values of various interest rates has been continually observed in the European markets starting from the second half of 2014 and thus, it is worthwhile mentioning that also negative interest rates can be easily accommodated in this setup by considering, for example, affine processes on $\mathbb{R}^d$ instead of $\mathbb{R}_+^d$ or 'shifted' positive affine processes where $\text{supp}(X) \in [a, \infty)^d$ with $a < 0$. A specification of the multiple curve affine Libor model allowing for negative rates and positive spreads is presented in Sect. 4.1 of that paper.

Remark 4.4 (*Connection to Libor market models*) Having a model of this Sect. 4.2 in which the dynamics of the OIS and the Libor rates are given by (4.22) and (4.23), it is natural to look for a possible relationship between this model and the multiple curve Libor market models of Sect. 4.1. This relationship has been established in Grbac et al. (2014). More precisely, starting from (4.22) and the definition of the martingales $M^{u_k^x}$ given in (4.18), as well as the definition of the process X in (4.13), and using Itô's formula and some algebraic manipulations, one arrives at the dynamics of the OIS rate F_k^x under the forward measure $Q^{T_k^x}$

$$\frac{dF_k^x(t)}{F_k^x(t)} = \Gamma_{x,k}^{\mathsf{T}}(t)\, dw_t^{x,k} \tag{4.28}$$

with the volatility structure $\Gamma_{x,k} = (\Gamma_{x,k}^1, \ldots, \Gamma_{x,k}^d) \in \mathbb{R}_+^d$ provided by

$$\Gamma_{x,k}^i(t) := \frac{1+\delta^x F_k^x(t)}{\delta^x F_k^x(t)} \left(B^{u_{k-1}^x,i}(T_M - t) - B^{u_k^x,i}(T_M - t)\right) \sqrt{X_t^i}\, \sigma^i \tag{4.29}$$

and the $Q^{T_k^x}$-Wiener process $w^{x,k} = (w^{x,k,1}, \ldots, w^{x,k,d})$ given by

$$\begin{aligned} w^{x,k,i} &:= w^i - \sum_{l=k+1}^{M_x} \int_0^{\cdot} \frac{\delta^x F_l^x(t)}{1+\delta^x F_l^x(t)} \Gamma_{x,l}^i(t)\, dt \\ &= w^i - \sum_{l=k+1}^{M_x} \int_0^{\cdot} \left(B^{u_{l-1}^x,i}(T_M - t) - B^{u_l^x,i}(T_M - t)\right) \sqrt{X_t^i}\, \sigma_i\, dt. \end{aligned} \tag{4.30}$$

Note from (4.29) that the volatility structure is determined by σ^i and by the driving process itself via $B^{u_{k-1}^x,i}(T_M - t)$ and $B^{u_k^x,i}(T_M - t)$, and also that there is a built-in shift in the model by construction. Furthermore, we notice that the dynamics of the OIS rates in Eq. (4.28) correspond to (4.8) in the Libor market model of Sect. 4.1.

In complete analogy, starting from the dynamics of the Libor rates in (4.23) and introducing the volatility structure $\Lambda_{x,k} = (\Lambda_{x,k}^1, \ldots, \Lambda_{x,k}^d) \in \mathbb{R}_+^d$

$$\Lambda_{x,k}^i(t) := \frac{1+\delta^x L_k^x(t)}{\delta^x L_k^x(t)} \left(B^{v_{k-1}^x,i}(T_M - t) - B^{u_k^x,i}(T_M - t)\right) \sqrt{X_t^i}\, \sigma^i \tag{4.31}$$

one obtains for L_k^x the following $Q^{T_k^x}$-dynamics in the spirit of the Libor market model

$$\frac{dL_k^x(t)}{L_k^x(t)} = \Lambda_{x,k}^{\mathsf{T}}(t)\, dw_t^{x,k} \tag{4.32}$$

where $w^{x,k}$ is the $Q^{T_k^x}$-Wiener process given above.

4.2.3 Pricing in the Multiple Curve Affine Libor Model

Thanks to its tractability under all forward measures, the multiple curve affine Libor model allows for semi-analytical pricing of interest rate derivatives. Below we present the main results from Grbac et al. (2014) and refer to the paper for details. In particular, the valuation of caplets based on Fourier transform methods is of the same complexity as the valuation of caplets in the single-curve affine Libor model. In order to obtain semi-closed pricing formulas for swaptions, Grbac et al. (2014) make use of the linear boundary approximation proposed by Singleton and Umantsev (2002) combined with Fourier transform methods.

Let us begin by considering linear derivatives, namely interest rate swaps and basis swaps. We use the definitions and the notation from Sects. 1.4.3 and 1.4.5, assuming for simplicity that the nominal is $N = 1$. Since the modeling objects in the multiple curve affine Libor model are directly the forward Libor rates, it is straightforward to conclude that the time-t value of the interest rate swap on the tenor structure $\mathscr{T}^x$ with fixed rate denoted by R is given by

$$P^{Sw}(t; \mathscr{T}^x, R) = \delta^x \sum_{k=1}^{M_x} p(t, T_k^x) \left(L_k^x(t) - R \right)$$

and the fair swap rate $R(t; \mathscr{T}^x)$ is therefore

$$R(t; \mathscr{T}^x) = \frac{\sum_{k=1}^{M_x} p(t, T_k^x) L_k^x(t)}{\sum_{k=1}^{M_x} p(t, T_k^x)} \tag{4.33}$$

Similarly, the time-t value of the basis swap defined on the tenor structures $\mathscr{T}^{x_1}$ and $\mathscr{T}^{x_2}$ with spread S is expressed as

$$P^{BSw}(t; \mathscr{T}^{x_1}, \mathscr{T}^{x_2}) = \sum_{i=1}^{M_{x_1}} \delta^{x_1} p(t, T_i^{x_1}) L_i^{x_1}(t) - \sum_{j=1}^{M_{x_2}} \delta^{x_2} p(t, T_j^{x_2}) \left(L_j^{x_2}(t) + S \right) \tag{4.34}$$

The fair basis swap spread $S^{BSw}(t; \mathscr{T}^{x_1}, \mathscr{T}^{x_2})$ is thus given by

$$S^{BSw}(t; \mathscr{T}^{x_1}, \mathscr{T}^{x_2}) = \frac{\sum_{i=1}^{M_{x_1}} \delta^{x_1} p(t, T_i^{x_1}) L_i^{x_1}(t) - \sum_{j=1}^{M_{x_2}} \delta^{x_2} p(t, T_j^{x_2}) L_j^{x_2}(t)}{\sum_{j=1}^{M_{x_2}} \delta^{x_2} p(t, T_j^{x_2})} \tag{4.35}$$

Passing now to non-linear derivatives, their pricing is based on the affine property of the driving process under all forward measures and an application of Fourier transform methods for option pricing. The Fourier transform methods for option pricing were discussed already in Chap. 3.

Let us first consider a caplet, as defined in Sect. 1.4.6. A straightforward application of the Fourier transform method yields the following pricing formula, which is proved in Proposition 6.1 in Grbac et al. (2014).

Proposition 4.2 *Consider a tenor x and a caplet with strike K and with payoff $\delta^x(L_k^x(T_{k-1}^x) - K)^+$ at time T_k^x. Its time-0 price is given by*

$$P^{Cpl}(0; T_k^x, K) = \frac{p(0, T_k^x)}{2\pi} \int_{\mathbb{R}} K_x^{1-\mathscr{R}+iw} \frac{\Theta_{\mathscr{W}_{k-1}^x}(\mathscr{R} - iw)}{(\mathscr{R} - iw)(\mathscr{R} - 1 - iw)} dw \tag{4.36}$$

for any $\mathscr{R} \in (1, \infty) \cap \mathscr{I}_k^x$, $K_x := 1 + \delta^x K$ and where

$$\mathscr{I}_k^x = \left\{ z \in \mathbb{R} : (1 - z)B^{u_k^x}(T_M - T_{k-1}^x) + zB^{v_{k-1}^x}(T_M - T_{k-1}^x) \in \mathscr{I}_T \right\} \tag{4.37}$$

The random variable $\mathscr{W}_{k-1}^x$ is defined as

$$\begin{aligned} \mathscr{W}_{k-1}^x &= \log \left(M_{T_{k-1}^x}^{v_{k-1}^x} / M_{T_{k-1}^x}^{u_k^x} \right) \\ &= A^{v_{k-1}^x}(T_M - T_{k-1}^x) - A^{u_k^x}(T_M - T_{k-1}^x) \\ &\quad + \left\langle B^{v_{k-1}^x}(T_M - T_{k-1}^x) - B^{u_k^x}(T_M - T_{k-1}^x), X_{T_{k-1}^x} \right\rangle \\ &=: A + \langle B, X_{T_{k-1}^x} \rangle \end{aligned} \tag{4.38}$$

with the moment generating function $\Theta_{\mathscr{W}_{k-1}^x}$ under the measure $Q^{T_k^x}$ given by

$$\Theta_{\mathscr{W}_{k-1}^x}(z) = E^{T_k^x}\left\{ e^{z\mathscr{W}_{k-1}^x} \right\} = E^{T_k^x}\left\{ \exp\left(z(A + \langle B, X_{T_{k-1}^x} \rangle) \right) \right\}$$

which is known explicitly thanks to the affine property of the model.

Note that the pricing formula (4.36) has an arbitrary $\mathscr{R} \in (1, \infty) \cap \mathscr{I}_k^x$ on the right-hand side. Theoretically, the value of the right-hand side does not depend on the specific choice of $\mathscr{R}$. However, different choices of $\mathscr{R}$ may affect the efficiency of the numerical implementation.

Regarding swaption pricing in the affine multiple curve Libor model, let us consider a swaption as defined in Sect. 1.4.7. The time-0 price of a swaption with exercise date T_0^x and swap rate R, written on an underlying swap with tenor structure $\mathscr{T}^x$, is given by

$$P^{Swn}(0; T_0^x, \mathscr{T}^x, R) = p(0, T_0^x)\, E^{Q^{T_0^x}} \left\{ \left(P^{Sw}(T_0^x; \mathscr{T}^x, R) \right)^+ \right\}$$

where $P^{Sw}(T_0^x; \mathscr{T}^x, R)$ is the price of the underlying swap at time T_0^x which can be expressed as follows

$$P^{Sw}(T_0^x; \mathscr{T}^x, R) = \delta^x \sum_{k=1}^{M_x} p(T_0^x, T_k^x)(L_k^x(T_0^x) - R) = \sum_{k=1}^{M_x} \frac{M_{T_0^x}^{v_{k-1}^x}}{M_{T_0^x}^{u_0^x}} - \sum_{k=1}^{M_x} R_x \frac{M_{T_0^x}^{u_k^x}}{M_{T_0^x}^{u_0^x}}$$

where $R_x := 1 + \delta^x R$. Here we have used (4.20), (4.22) and the telescopic product to obtain

$$p(T_0^x, T_k^x) = \frac{p(T_0^x, T_k^x)}{p(T_0^x, T_{k-1}^x)} \cdots \frac{p(T_0^x, T_1^x)}{p(T_0^x, T_0^x)} = \frac{M_{T_0^x}^{u_k^x}}{M_{T_0^x}^{u_0^x}} \tag{4.39}$$

and Eq. (4.23) for $L_k^x(T_0^x)$. Therefore, for the swaption price we have

$$\begin{aligned} P^{Swn}(0; T_0^x, \mathscr{T}^x, R) &= p(0, T_0^x)\, E^{Q^{T_0^x}} \left\{ \left(\sum_{k=1}^{M_x} \frac{M_{T_0^x}^{v_{k-1}^x}}{M_{T_0^x}^{u_0^x}} - \sum_{k=1}^{M_x} R_x \frac{M_{T_0^x}^{u_k^x}}{M_{T_0^x}^{u_0^x}} \right)^+ \right\} \\ &= p(0, T_M)\, E^{Q^{T_M}} \left\{ \left(\sum_{k=1}^{M_x} M_{T_0^x}^{v_{k-1}^x} - \sum_{k=1}^{M_x} R_x M_{T_0^x}^{u_k^x} \right)^+ \right\} \end{aligned} \tag{4.40}$$

where the second equality follows by a measure change from $Q^{T_0^x}$ to Q^{T_M}, cf. (4.17).

Evaluating the above expectation is a computationally demanding task, due to the high-dimensionality of the problem. However, in order to arrive at semi-closed pricing formulas based on the affine property of the model and the Fourier transform methods, an efficient and accurate linear boundary approximation developed in Singleton and Umantsev (2002) can be used. Numerical results for this approximation are reported in Grbac et al. (2014) and below we describe the method and cite the main result.

Firstly, one defines the probability measures $\overline{Q}^{T_k^x}$, for every $k \in \mathscr{K}^x$, by the Radon–Nikodym density

$$\left.\frac{d\overline{Q}^{T_k^x}}{dQ^{T_M}}\right|_{\mathscr{F}_t} = \frac{M_t^{v_k^x}}{M_0^{v_k^x}} \tag{4.41}$$

The process X, defined by its components in (4.13), is a time-inhomogeneous affine process under every $\overline{Q}^{T_k^x}$, which can be shown exactly in the same way as for the forward measures $Q^{T_k^x}$. The expectation with respect to the measure $\overline{Q}^{T_k^x}$ will be denoted by $\overline{E}^{T_k^x}$ below.

Next, starting from the second equality in (4.40) and using the definitions of martingales $M^{u_k^x}$ and $M^{v_{k-1}^x}$ given in (4.18) and (4.19), one defines the function $f : \mathbb{R}_+^d \to \mathbb{R}$ by

$$f(y) = \sum_{i=1}^{M_x} \exp\left(A^{v^x_{i-1}}(T_M - T^x_0) + \langle B^{v^x_{i-1}}(T_M - T^x_0), y\rangle\right)$$
$$- \sum_{i=1}^{M_x} R_x \exp\left(A^{u^x_i}(T_M - T^x_0) + \langle B^{u^x_i}(T_M - T^x_0), y\rangle\right) \tag{4.42}$$

This function determines the exercise boundary for the price of the swaption. Since the characteristic function of $f(X_{T^x_0})$ cannot be computed explicitly, the method of Singleton and Umantsev (2002) is used and f is approximated by a linear function. More precisely, one has

$$f(X_{T^x_0}) \approx \widetilde{f}(X_{T^x_0}) := \mathcal{C} + \langle \mathcal{D}, X_{T^x_0}\rangle \tag{4.43}$$

where the constants $\mathcal{C}$ and $\mathcal{D}$ are determined according to the linear regression procedure described in Singleton and Umantsev (2002, pp. 432–434). The line $\langle \mathcal{D}, X_{T^x_0}\rangle = -\mathcal{C}$ approximates the exercise boundary, hence $\mathcal{C}$, $\mathcal{D}$ are strike-dependent. Let $\Im(z)$ denote the imaginary part of a complex number $z \in \mathbb{C}$. Now, we have the following result.

Proposition 4.3 *Assume that $\mathcal{C}$, $\mathcal{D}$ are determined by the approximation (4.43). The price of the swaption with swap rate R, option maturity T^x_0, on a swap with tenor structure $\mathcal{T}^x$, is approximated by*

$$P^{Swn}(0; T^x_0, \mathcal{T}^x, K) \approx p(0, T_M) \sum_{i=1}^{M_x} M_0^{v^x_{i-1}} \left[\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\Im\left(\widetilde{\xi}^x_{i-1}(z)\right)}{z} dz\right]$$
$$- R_x \sum_{i=1}^{M_x} p(0, T^x_i) \left[\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\Im\left(\widetilde{\zeta}^x_i(z)\right)}{z} dz\right] \tag{4.44}$$

where $\widetilde{\zeta}^x_i$ and $\widetilde{\xi}^x_i$ approximate the characteristic functions

$$\zeta^x_i(z) := E^{T^x_i}\left\{\exp\left(\mathrm{i}zf(X_{T^x_0})\right)\right\} \quad \text{and} \quad \xi^x_i(z) := \overline{E}^{T^x_i}\left\{\exp\left(\mathrm{i}zf(X_{T^x_0})\right)\right\}$$

and are given by

$$\widetilde{\zeta}^x_i(z) := E^{T^x_i}\left\{\exp\left(\mathrm{i}z\widetilde{f}(X_{T^x_0})\right)\right\}$$
$$= \exp\Big(\mathrm{i}z\mathcal{C} + A^{B^{u^x_i}(T_M - T^x_0)+\mathrm{i}z\mathcal{D}}(T^x_0) - A^{B^{u^x_i}(T_M - T^x_0)}(T^x_0)$$
$$+ \left\langle B^{B^{u^x_i}(T_M - T^x_0)+\mathrm{i}z\mathcal{D}}(T^x_0) - B^{B^{u^x_i}(T_M - T^x_0)}(T^x_0), X_0\right\rangle\Big) \tag{4.45}$$

$$\widetilde{\xi}^x_i(z) := \overline{E}^{T^x_i}\left\{\exp\left(\mathrm{i}z\widetilde{f}(X_{T^x_0})\right)\right\}$$
$$= \exp\Big(\mathrm{i}z\mathcal{C} + A^{B^{v^x_i}(T_M - T^x_0)+\mathrm{i}z\mathcal{D}}(T^x_0) - A^{B^{v^x_i}(T_M - T^x_0)}(T^x_0)$$
$$+ \left\langle B^{B^{v^x_i}(T_M - T^x_0)+\mathrm{i}z\mathcal{D}}(T^x_0) - B^{B^{v^x_i}(T_M - T^x_0)}(T^x_0), X_0\right\rangle\Big) \tag{4.46}$$

Remark 4.5 Note that an approximate pricing formula for the price of a basis swaption defined in Remark 1.10 can be derived as well. For details we refer to Grbac et al. (2014).

Remark 4.6 (*Calibration*) Another important remark regarding the multiple curve affine model is its flexibility to calibrate to option market data. A specification of the model based on the CIR driving processes with jumps proves to fit very well the caplet data, simultaneously for multiple tenors. Since this issue is beyond the scope of this book, we refer the interested reader to Grbac et al. (2014) for all the details.

4.3 Multiplicative Spread Models

In this section we give an overview of the modeling approaches based on multiplicative spreads. The idea to consider the multiplicative spreads has been first proposed in Henrard (2007, 2010), see also the recent book Henrard (2014). The same choice for the modeling quantities has been made in the recent paper by Cuchiero et al. (2015). Recall from (1.36) of Sect. 1.4.4 (noting that $F(t; T, T + \Delta) = R^{OIS}(t; T, T + \Delta)$) that the multiplicative forward Libor-OIS spreads are defined as

$$\Sigma(t; T, T + \Delta) = \frac{1 + \Delta L(t; T, T + \Delta)}{1 + \Delta F(t; T, T + \Delta)} \tag{4.47}$$

where as usual

$$L(t; T, T + \Delta) = E^{Q^{T+\Delta}} \{L(T; T, T + \Delta) \mid \mathscr{F}_t\}$$

denotes the forward Libor rate (FRA rate) and the forward OIS rates are defined via relation (see (1.16))

$$1 + \Delta F(t; T, T + \Delta) = \frac{p(t, T)}{p(t, T + \Delta)}$$

Note that the notation $\Sigma(t; T, T + \Delta)$ in (4.47) corresponds to the notation $S^{\Delta}(t, T)$ in Cuchiero et al. (2015). As Henrard (2014) and Cuchiero et al. (2015) point out, the choice of multiplicative spreads as modeling quantities instead of the forward Libor rates is made for the convenience of modeling. Empirical findings on the positivity and monotonicity of the additive spreads with respect to the tenor Δ motivate one to model directly the spreads instead of the forward Libor rates in order to access more easily those two features, cf. also the comments in Sect. 4.1. Passing from the additive to the multiplicative spreads still serves the same purpose, while allowing for more analytical tractability in the model. Moreover, as noticed by Cuchiero et al. (2015), the multiplicative spreads are related to the forward exchange rates when the multiple curve market is considered in a foreign exchange analogy (see Sect. 3.3.1.2, Remark 3.8).

In the work of Henrard, expressions for the prices of various interest rate derivatives in terms of the multiplicative spreads have been developed. For the dynamics of the spreads, Henrard introduces some assumptions that, from a modeling point of view, appear to be rather restrictive. The assumptions are, in particular: the spreads are supposed to be independent from the forward OIS rates $F(t; T, T + \Delta)$, and for tractable pricing of optional derivatives, an assumption of the spreads being constant for each maturity is introduced in addition. As stated in Sect. 7.3 of Henrard (2014), this has the advantage of allowing to determine the price of any instrument in the post-crisis setting by directly applying the corresponding pre-crisis formula (in the case of optional derivatives one has only to scale the strike). To model the OIS bond price dynamics Henrard (2010) considers an HJM 1-factor Gaussian framework; cf. Sect. 2 therein.

The framework proposed in Cuchiero et al. (2015), that we shall describe in more detail below, allows for more modeling flexibility and, in fact, it can be shown that many of the existing modeling approaches can be recovered from their setting. To this effect the authors develop a general semimartingale HJM framework for the multiple curve term structure, which is inspired by Kallsen and Krühner (2013). Their approach is situated in between the HJM and the LMM approaches and in this sense is similar to the approach taken in Sect. 3.2. Concerning the model choice for the dynamics of the spreads, the affine specification of the framework by Cuchiero et al. (2015) can also be seen as a possible extension to continuous tenors of the model from Sect. 4.2 (see Remark 4.8).

Let us now give an overview of the framework proposed by Cuchiero et al. (2015). The modeling quantities are the OIS bonds $p(t, T)$ and the multiplicative spreads $\Sigma(t; T, T + \Delta)$. They consider a finite number of tenors denoted by $\Delta_1, \ldots, \Delta_m$ (corresponding to tenors from 1 day to 12 months). The framework allows to reproduce main features of the multiplicative spreads observed in the market (see (4.49)):

$$\Sigma(t; T, T + \Delta_i) \geq 1 \quad \text{and} \quad \Sigma(t; T, T + \Delta_i) \geq \Sigma(t; T, T + \Delta_j), \ \text{ for } \Delta_i \geq \Delta_j$$

The first property is equivalent to the positivity of additive spreads and the second one is the monotonicity with respect to the tenor. Moreover, the definition (4.47) of the spread $\Sigma(t; T, T + \Delta_i)$ implies that it has to be a Q^T-martingale because the forward Libor rate $1 + \Delta L(\cdot; T, T + \Delta)$ is a $Q^{T+\Delta}$-martingale and $\frac{dQ^T}{dQ^{T+\Delta}} |_{\mathcal{F}_t} = \frac{1+\Delta F(t;T,T+\Delta)}{1+\Delta F(0;T,T+\Delta)}$ by (1.15) and (1.16).

As stated above, to develop their framework, Cuchiero et al. (2015) make use of the classical HJM setup presented in the philosophy of Kallsen and Krühner (2013). The main idea behind this approach is to identify "canonical" assets which are the underlyings for the assets of interest and then obtain a convenient parametrization (a "codebook" as referred to in Kallsen and Krühner 2013) of the related term structures. In order to do so, one first specifies simple elementary models for the term structure of the canonical assets to understand the general relations that have to hold between the fundamental modeling quantities and thus obtains the codebooks; then one prescribes a stochastic evolution for the codebooks, which has to satisfy certain consistency conditions. Let us illustrate this approach on the first fundamental quantities in the framework of Cuchiero et al. (2015), which are the OIS bonds $p(t, T)$ (a similar procedure is then repeated for the second fundamental modeling quantities, i.e. the multiplicative spreads). The underlying canonical asset for the OIS bonds is the OIS short rate r. The idea is thus to exploit the connection of the OIS bond prices and the OIS bank account $B_t = \exp(\int_0^t r_s ds)$, supposing firstly that r is a deterministic short rate. This yields the relation $r_T = -\frac{\partial}{\partial T}\log(p(t, T))$, which is the codebook for the bond prices. Now, since market data indicate that $-\frac{\partial}{\partial T}\log(p(t, T))$ evolves randomly over time, this leads to instantaneous forward rates $f_t(T) = -\frac{\partial}{\partial T}\log(p(t, T))$, for which then a stochastic model is specified. Setting for the instantaneous forward rate $f_t(T) = -\frac{\partial}{\partial T}\log(p(t, T)) = -\eta_t(T)$ and $Z_t = -\log B_t = -\int_0^t r_s ds$ for the short rate r, Cuchiero et al. (2015) specify an HJM OIS bond price model given by a quintuple $(Z, \eta_0, \alpha, \sigma, X)$ such that

$$\frac{p(t, T)}{B_t} = e^{Z_t + \int_t^T \eta_t(u)du}$$

with

$$\eta_t(T) = \eta_0(T) + \int_0^t \alpha_s(T)ds + \int_0^t \sigma_s(T)dX_s \tag{4.48}$$

where (X, Z) is a general multidimensional semimartingale with absolutely continuous characteristics (Itô semimartingale) and the processes α and σ satisfy the implicit measurability and integrability conditions, together with a suitable HJM drift condition to ensure absence of arbitrage in the OIS bonds. This in particular yields

$$\frac{p(t, T)}{B_t} = E\left\{e^{Z_T} \mid \mathscr{F}_t\right\}$$

Similarly, for the multiplicative spread $\Sigma(t; T, T + \Delta_i)$, passing via a suitable codebook, Cuchiero et al. (2015) again obtain an HJM-type model given by a quintuple $(Z^i, \eta_0^i, \alpha^i, \sigma^i, X)$ such that

$$\Sigma(t; T, T + \Delta_i) = e^{Z_t^i + \int_t^T \eta_t^i(u)du}$$

with η^i having a dynamics similar to (4.48) with corresponding α^i and σ^i, which satisfy a drift condition ensuring the Q^T-martingale property of $\Sigma(\cdot; T, T + \Delta_i)$. The quantity $Z_t^i = \log(\Sigma(t; t, t + \Delta_i))$ can be seen as the log-spot spread and $-\eta_t^i(T) = -\frac{\partial}{\partial T}\log(\Sigma(t; T, T + \Delta_i))$ as the forward spread rate, by analogy to the OIS bond price model. The martingale property of $\Sigma(\cdot; T, T + \Delta_i)$ yields

$$\Sigma(t; T, T + \Delta_i) = E^{Q^T}\left\{e^{Z_T^i} \mid \mathscr{F}_t\right\}$$

In order to specify the model further, the semimartingales Z^i are assumed to be of the following form

$$Z_t^i = e^{\langle u_i, Y_t \rangle}$$

with a common n-dimensional Itô semimartingale Y for all i, and $u_1, \ldots, u_m$ given vectors in $\mathbb{R}^n$. A common driving process Y for all tenors Δ_i is a choice which allows to capture the interdependencies between the spreads associated to different tenors. The vectors u_i enable one to implement easily the ordered spreads $1 \leq \Sigma(t; T, T + \Delta_1) \leq \cdots \leq \Sigma(t; T, T + \Delta_m)$. More precisely, one would have to consider a process Y taking values in some cone $C \subset \mathbb{R}^n$ and vectors $u_i \in C^*$ such that $0 \prec u_1 \prec \cdots \prec u_m$, where C^* denotes the dual cone of C with the order relation $\prec$. This then easily implies

$$1 \leq \Sigma(t; T, T + \Delta_i) = E^{Q^T}\left\{ e^{\langle u_i, Y_T \rangle} \middle| \mathscr{F}_t \right\} \leq E^{Q^T}\left\{ e^{\langle u_j, Y_T \rangle} \middle| \mathscr{F}_t \right\} = \Sigma(t; T, T + \Delta_j) \tag{4.49}$$

for $\Delta_i < \Delta_j$.

Remark 4.7 Making use of relation (4.47), the payoffs of all linear and optional interest rate derivatives can be expressed as functions of the OIS bond prices $p(t, T)$ and the multiplicative spreads $\Sigma(t; T, T + \Delta_i)$, cf. Cuchiero et al. (2015) for explicit expressions in the general framework. The prices of linear derivatives can thus easily be expressed in terms of these modeling quantities.

Regarding optional derivatives, to obtain a tractable specification of the general framework, Cuchiero et al. (2015) suggest the class of affine processes as driving processes, which allows convenient pricing by standard techniques resorting to the Fourier transform.

Remark 4.8 Even though the approach proposed by Cuchiero et al. (2015) can be situated in the HJM framework, its affine specification can also be regarded as a continuous tenor extension of the affine Libor model from Sect. 4.2 with the difference that the quantities modeled here are not the forward Libor rates, but rather the multiplicative spreads which in this case are given by

$$\Sigma(t; T, T + \Delta_i) = \frac{M_t^{v(T, \Delta_i)}}{M_t^{u(T)}}$$

where $u(\cdot)$ and $v(\cdot, \Delta_i)$ are mappings from $[0, T]$ to $\mathbb{R}^d$. Similarly to Sect. 4.2, imposing conditions on these mappings allows to ensure positivity and monotonicity of the spreads in the model.