Chapter 1
Classes of Modulars

1.1 Modulars Versus Metrics

In order to motivate the notion of *modular* on a set, we begin by recalling the notion of *metric*. Let X be a nonempty set.

A function $d : X \times X \to \mathbb{R}$ is said to be a *metric* on X if, for all elements (conventionally called points) $x, y, z \in X$, it satisfies the following three conditions (axioms):

(d.1) $x = y$ if and only if $d(x, y) = 0$ (nondegeneracy);
(d.2) $d(x, y) = d(y, x)$ (symmetry);
(d.3) $d(x, y) \le d(x, z) + d(z, y)$ (triangle inequality).

The pair (X, d) is called a *metric space*. (Actually, two axioms suffice to define a metric, because conditions (d.1)–(d.3) are equivalent to (d.1), and (d.3) written in the 'strong' form as $d(x, y) \le d(x, z) + d(y, z)$: in fact, putting $z = x$ and then interchanging x and y, we obtain (d.2). Traditionally, the symmetry property of d is introduced explicitly.)

Clearly, a metric d assumes nonnegative (and finite) values, and if X has at least two elements (which is tacitly assumed throughout), then $d \not\equiv 0$ on $X \times X$. If the value ∞ is allowed for d satisfying (d.1)–(d.3), then d is called an *extended metric* on X, and the pair (X, d) is called an *extended metric space*.

If $d : X \times X \to [0, \infty]$ satisfies (d.2), (d.3) and (only) a weaker condition

(d.1′) $d(x, x) = 0$ for all $x \in X$,

then d is called a *pseudometric* on X, and it is called an *extended pseudometric* on X if the value 'infinity' is allowed for d.

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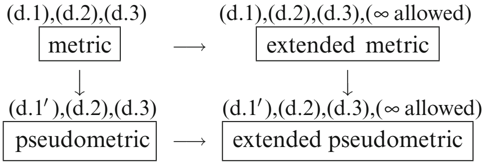

Fig. 1.1 Variants of metric notions

The notion of metric reflects our *geometric* intuition of what a distance function on a set should be: to any two points $x, y \in X$, a number

$$0 \le d(x,y) < \infty \qquad \text{(the } \textit{distance} \text{ between } x \text{ and } y\text{)}$$

is assigned, satisfying properties (d.1)–(d.3). The implications between the above four metric notions are presented in Fig. 1.1.

The idea of modular w on X can be expressed in *physical* terms as follows: to any parameter $\lambda > 0$, interpreted as time, and any two points $x, y \in X$, a quantity

$$0 \le w_\lambda(x,y) \le \infty \quad \text{(the } \textit{velocity} \text{ between } x \text{ and } y \text{ in time } \lambda\text{)}$$

is assigned, satisfying three axioms to be discussed below, and the one-parameter family $w = \{w_\lambda : \lambda > 0\} \equiv \{w_\lambda\}_{\lambda>0}$ of functions of the form $w_\lambda : X \times X \to [0,\infty]$ is a (generalized, nonlinear) *velocity field* on X.

Now we address the axioms of a modular. By a *scaling* of time $\lambda > 0$ we mean any value $h(\lambda) > 0$ such that the function $\lambda \mapsto \lambda/h(\lambda)$ is nonincreasing in λ (e.g., given $p \ge 1$, $h(\lambda) = \lambda^p$, or $h(\lambda) = \exp(\lambda^p) - 1$, or $h(\lambda) = \lambda e^\lambda$, etc.). Let (X, d) be a metric space, and $x, y \in X$. Consider the quantity

$$w_\lambda(x,y) = \frac{d(x,y)}{h(\lambda)} = \frac{\lambda}{h(\lambda)} \cdot \frac{d(x,y)}{\lambda}\,, \tag{1.1.1}$$

which is a *scaled mean velocity* between x and y in time λ. For $h(\lambda) = \lambda$, this is the *mean* (or *uniform*) velocity, and so, in order to cover the distance $d(x, y)$, it takes time λ to move between x and y with velocity $w_\lambda(x,y) = d(x,y)/\lambda$.

The following natural-looking properties of quantity (1.1.1) hold.

(i) Two points x and y from X coincide (and $d(x,y) = 0$) if and only if any time $\lambda > 0$ will do in order to move from x to y with velocity $w_\lambda(x,y) = 0$ (that is, no movement is needed at any time). Formally, given $x, y \in X$, we have:

$$x = y \text{ if and only if } w_\lambda(x,y) = 0 \text{ for all } \lambda > 0 \text{ (nondegeneracy).}$$

(ii) For any time $\lambda > 0$, the mean velocity during the movement from point x to point y is equal to the mean velocity in the opposite direction, i.e., given $x, y \in X$,

$$w_\lambda(x,y) = w_\lambda(y,x) \text{ for all } \lambda > 0 \text{ (symmetry).}$$

(iii) The third property of quantity (1.1.1), which is, in a sense, a counterpart of the triangle inequality (for velocities!), is new and the most important. Suppose that movements from x to y happen to be made in two ways, but the *duration of time is the same* in each of the cases: (a) passing through a third point $z \in X$, or (b) moving directly from x to y. If λ is the time needed to move from x to z and μ is the time needed to move from z to y, then the corresponding mean velocities are equal to $w_\lambda(x, z)$ and $w_\mu(z, y)$. The total time spent during the movement in case (a) is equal to $\lambda + \mu$. It follows that the mean velocity in case (b) should be equal to $w_{\lambda+\mu}(x, y)$. It may become clear from the physical intuition that the velocity $w_{\lambda+\mu}(x, y)$ does not exceed at least one of the velocities $w_\lambda(x, z)$ or $w_\mu(z, y)$. This is expressed as

$$w_{\lambda+\mu}(x, y) \le \max\{w_\lambda(x, z), w_\mu(z, y)\} \le w_\lambda(x, z) + w_\mu(z, y) \qquad (1.1.2)$$

for all points $x, y, z \in X$ and times $\lambda, \mu > 0$. These inequalities can be verified rigorously: since $(\lambda + \mu)/h(\lambda + \mu) \le \lambda/h(\lambda)$, it follows from (1.1.1) and (d.3) that

$$\begin{aligned} w_{\lambda+\mu}(x, y) &= \frac{d(x, y)}{h(\lambda + \mu)} \le \frac{d(x, z)+d(z, y)}{h(\lambda+\mu)} \le \frac{\lambda}{\lambda+\mu} \cdot \frac{d(x, z)}{h(\lambda)} + \frac{\mu}{\lambda+\mu} \cdot \frac{d(z, y)}{h(\mu)} \\ &= \frac{\lambda}{\lambda+\mu}\, w_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\, w_\mu(z, y) \le w_\lambda(x, z) + w_\mu(z, y). \qquad (1.1.3) \end{aligned}$$

By (in)equality (1.1.3), conditions $w_\lambda(x, z) < w_{\lambda+\mu}(x, y)$ and $w_\mu(z, y) < w_{\lambda+\mu}(x, y)$ cannot hold simultaneously, which proves the left-hand side inequality in (1.1.2).

A *modular* on a set X is any one-parameter family $w = \{w_\lambda\}_{\lambda>0}$ of functions w_λ mapping $X \times X$ into $[0, \infty]$ and satisfying properties (i), (ii), and (iii) meaning (1.1.2). (The interpretation of modular as a generalized nonlinear mean velocity field has been chosen as the most intuitive and accessible; there are different interpretations of modular such as a double joint generalized variation of two mappings x and y.)

Even on a metric space (X, d), modulars may look unusual: given $\lambda > 0$ and $x, y \in X$, set $w_\lambda(x, y) = \infty$ if $\lambda \le d(x, y)$, and $w_\lambda(x, y) = 0$ if $\lambda > d(x, y)$.

The difference between a metric (= distance function) and a modular (= velocity field) on a set is now clearly seen: a modular depends on a positive parameter λ and may assume infinite values (to say nothing of the axioms). The equality $w_\lambda(x, y) = \infty$ may be thought of as there is no possibility (or there is a prohibition) to move from x to y in time λ. For instance, the distance $d(x, y) = 10{,}000$ km between two cities x and y cannot be covered physically in $\lambda = 1, 2, \ldots, 100$ s; however, for times λ large enough, a certain finite velocity will do.

The essential property of a modular w (e.g., (1.1.1)) is that the velocity $w_\lambda(x, y)$ is *nonincreasing* as a function of time $\lambda > 0$.

A modular w on X gives rise to a *modular space* around a (chosen) point $x^\circ \in X$—this is the set

$$X_w^* = \{x \in X : w_\lambda(x, x^\circ) \text{ is finite for some } \lambda = \lambda(x) > 0\}$$

of those points x, which are *reachable* from x° with a finite velocity. The knowledge of (mean) velocities $w_\lambda(x, y)$ for all $\lambda > 0$ and $x, y \in X$ provides more information than simply the knowledge of distances $d(x, y)$ between points x and y. In fact, if w satisfies (i), (ii) and the left-hand side (in)equality in (1.1.3), then the modular space X_w^* is metrizable by the following (implicit, or limit case) metric:

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \le 1\}.$$

Naturally, the pair (X_w^*, d_w^*) is called a *metric modular space*. For instance, the original metric space (X, d) is restored via the (mean velocity) modular (1.1.1) with $h(\lambda) = \lambda$ as the 'limit case' in that $X_w^* = X$ and $d_w^*(x, y) = d(x, y)$ for all $x, y \in X$.

This book is intended as the general study of modulars, modular spaces and metric modular spaces generated by modulars. Since the metric space theory is a well-established and rich theory, the main emphasis of this exposition is focused (where it is possible) on non-metric features of modulars and modular spaces.

Essentially, modulars serve two important purposes:

- to define new metric spaces, such as (X_w^*, d_w^*) and others, in a unified and general manner, and
- to present a new type of convergence in the modular space X_w^*, the so called *modular convergence*, whose topology is weaker (coarser) than the d_w^*-metric topology and, in general, is non-metrizable.

1.2 The Classification of Modulars

In the sequel, we study functions w of the form $w : (0, \infty) \times X \times X \to [0, \infty]$, where X is a fixed nonempty set (with at least two elements). Due to the disparity of the arguments, we may (and will) write $w_\lambda(x, y) = w(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$. In this way, $w = \{w_\lambda\}_{\lambda>0}$ is a one-parameter family of functions $w_\lambda : X \times X \to [0, \infty]$. On the other hand, given $x, y \in X$, we may set $w^{x,y}(\lambda) = w(\lambda, x, y)$ for all $\lambda > 0$, so that $w^{x,y} : (0, \infty) \to [0, \infty]$. In the latter case, the usual terminology of Real Analysis can be applied to $w = \{w^{x,y}\}_{x,y\in X}$. For instance, the function w is called nonincreasing (right/left continuous, etc.) on $(0, \infty)$ if the function $w^{x,y}$ is such for all $x, y \in X$.

Definition 1.2.1. A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a *metric modular* (or simply *modular*) on X if it satisfies the following three axioms:

(i) given $x, y \in X$, $x = y$ if and only if $w_\lambda(x, y) = 0$ for all $\lambda > 0$;
(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Weaker and stronger versions of conditions (i) and (iii) will be of importance. If, instead of (i), the function w satisfies (only) a weaker condition

(i′) $w_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$,

then w is said to be a *pseudomodular* on X. Furthermore, if, instead of (i), the function w satisfies (i′) and a stronger condition

($\mathrm{i_s}$) given $x, y \in X$ with $x \neq y$, $w_\lambda(x,y) \neq 0$ for all $\lambda > 0$,

then w is called a *strict modular* on X.

A modular (or pseudomodular, or strict modular) w on X is said to be *convex* if, instead of (iii), it satisfied the (stronger) inequality (iv):

(iv) $w_{\lambda+\mu}(x,y) \leq \dfrac{\lambda}{\lambda+\mu}\, w_\lambda(x,z) + \dfrac{\mu}{\lambda+\mu}\, w_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

A few remarks concerning this definition are in order.

Remark 1.2.2. (a) The assumption $w : (0,\infty)\times X\times X \to (-\infty,\infty]$ in the definition of a pseudomodular does not lead to a greater generality: in fact, setting $y = x$ and $\mu = \lambda > 0$ in (iii) and taking into account (i′) and (ii), we find

$$0 = w_{2\lambda}(x,x) \leq w_\lambda(x,z) + w_\lambda(z,x) = 2w_\lambda(x,z),$$

and so, $w_\lambda(x,z) \geq 0$ or $w_\lambda(x,z) = \infty$ for all $\lambda > 0$ and $x, z \in X$.

(b) If $w_\lambda(x,y) = w_\lambda$ is independent of $x, y \in X$, then, by (i′), $w \equiv 0$. Note that $w \equiv 0$ is only a pseudomodular on X (by virtue of (i)).

If $w_\lambda(x,y) = w(x,y)$ does not depend on $\lambda > 0$, then axioms (i)–(iii) mean that w is an extended metric (extended pseudometric if (i) is replaced by (i′)) on X; w is a metric on X if, in addition, it assumes finite values.

(c) Axiom (i) can be written as $(x = y) \Leftrightarrow (w^{x,y} \equiv 0)$, and part ($\mathrm{i}_{\Leftarrow}$) in it—as $(x \neq y) \Rightarrow (w^{x,y} \not\equiv 0)$. Condition ($\mathrm{i_s}$) says that $(x \neq y) \Rightarrow (w^{x,y}(\lambda) \neq 0$ for all $\lambda > 0)$, and so, it implies ($\mathrm{i}_{\Leftarrow}$). In other words, ($\mathrm{i_s}$) means that if $w_\lambda(x,y) = 0$ for some $\lambda > 0$ (and not necessarily for all $\lambda > 0$ as in ($\mathrm{i}_{\Leftarrow}$)), then $x = y$. Thus, (i′)+($\mathrm{i_s}$) $\Rightarrow$ (i) $\Rightarrow$ (i′). Clearly, (iv) $\Rightarrow$ (iii). Thus, a (convex) strict modular on X is a (convex) modular on X, and so, it is a (convex) pseudomodular on X. These implications are shown in Fig. 1.2, and it will be seen later that none of them can be reversed.

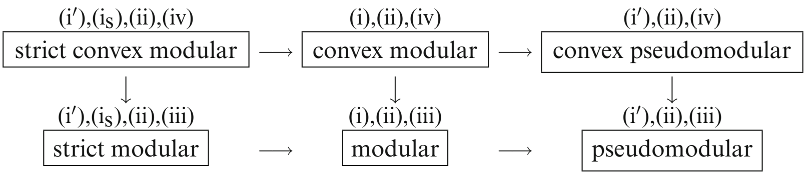

Fig. 1.2 Classification of modulars

(d) Rewriting (iv) in the form $(\lambda + \mu)w_{\lambda+\mu}(x, y) \le \lambda w_\lambda(x, z) + \mu w_\mu(z, y)$, we see that the function w is a *convex* (pseudo)modular on X if and only if the function $\hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ is simply a (pseudo)modular on X. This somewhat unusual observation on the convexity of w will be justified later (see Sect. 1.3.3).

The *essential property* of a pseudomodular w on X is its monotonicity: given $x, y \in X$, the function $w^{x,y} : (0, \infty) \to [0, \infty]$ is *nonincreasing* on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then axioms (iii) (with $z = x$) and (i′) imply

$$w_\lambda(x, y) = w_{(\lambda-\mu)+\mu}(x, y) \le w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y). \tag{1.2.1}$$

As a consequence, given $x, y \in X$, at each point $\lambda > 0$ the limit from the right

$$(w_{+0})_\lambda(x, y) \equiv w_{\lambda+0}(x, y) = \lim_{\mu\to\lambda+0} w_\mu(x, y) = \sup\{w_\mu(x, y) : \mu > \lambda\} \tag{1.2.2}$$

and the limit from the left

$$(w_{-0})_\lambda(x, y) \equiv w_{\lambda-0}(x, y) = \lim_{\mu\to\lambda-0} w_\mu(x, y) = \inf\{w_\mu(x, y) : 0 < \mu < \lambda\} \tag{1.2.3}$$

exist in $[0, \infty]$, and the following inequalities hold, for all $0 < \mu < \lambda$:

$$w_{\lambda+0}(x, y) \le w_\lambda(x, y) \le w_{\lambda-0}(x, y) \le w_{\mu+0}(x, y) \le w_\mu(x, y) \le w_{\mu-0}(x, y). \tag{1.2.4}$$

To see this, by the monotonicity of w, for any $0 < \mu < \mu_1 < \lambda_1 < \lambda$, we have:

$$w_\lambda(x, y) \le w_{\lambda_1}(x, y) \le w_{\mu_1}(x, y) \le w_\mu(x, y),$$

and it remains to pass to the limits as $\lambda_1 \to \lambda - 0$ and $\mu_1 \to \mu + 0$.

Proposition 1.2.3. *Let w be a convex pseudomodular on X, and $x, y \in X$. We have:*

(a) *functions $\lambda \mapsto w_\lambda(x, y)$ and $\lambda \mapsto \lambda w_\lambda(x, y)$ are nonincreasing on $(0, \infty)$, and*

$$w_\lambda(x, y) \le (\mu/\lambda)w_\mu(x, y) \le w_\mu(x, y) \quad \textit{for all} \quad 0 < \mu \le \lambda; \tag{1.2.5}$$

(b) *if $w^{x,y} \not\equiv 0$ (e.g., w is a convex modular and $x \ne y$), then* $\lim_{\mu\to+0} w_\mu(x, y) = \infty$;
(c) *if $w^{x,y} \not\equiv \infty$, then* $\lim_{\lambda\to\infty} w_\lambda(x, y) = 0$.

Proof. (a) is a consequence of (1.2.1) and Remark 1.2.2(d) concerning $\hat{w}$.
(b) follows from the fact that $w_\lambda(x, y) = w^{x,y}(\lambda) \in (0, \infty]$ for some $\lambda > 0$ and the left-hand side inequality in (1.2.5): $w_\mu(x, y) \ge (1/\mu)\lambda w_\lambda(x, y)$ for all $0 < \mu < \lambda$. In particular, by Remark 1.2.2(c), if w is a convex modular and $x \ne y$, then $w^{x,y} \not\equiv 0$.
(c) Since $w_\mu(x, y) < \infty$ for some $\mu > 0$, the assertion is a consequence of the left-hand side inequality in (1.2.5). □

Definition 1.2.4. Functions $w_{+0}, w_{-0} : (0,\infty)\times X\times X \to [0,\infty]$, defined in (1.2.2) and (1.2.3), are called the *right* and *left regularizations* of w, respectively.

Proposition 1.2.5. *Let w be a pseudomodular on X, possibly having additional properties shown in* Fig. 1.2 *on* p. 5*. Then w_{+0} and w_{-0} are also pseudomodulars on X having the same additional properties as w. Moreover, w_{+0} is continuous from the right and w_{-0} is continuous from the left on* $(0,\infty)$.

Proof. Since properties (i′), (ii), (iii), and (iv) are clear for w_{+0} and w_{-0}, we verify only ($\mathrm{i}_{\Leftarrow}$), the strictness, and one-sided continuities.

Suppose w is a modular. Let $x, y \in X$, and $(w_{+0})_\mu(x,y) = 0$ for all $\mu > 0$. Given $\lambda > 0$, choose μ such that $0 < \mu < \lambda$. Then (1.2.4) and (1.2.2) yield $0 \le w_\lambda(x,y) \le w_{\mu+0}(x,y) = 0$, and so, by axiom (i), $x = y$. Now, assume that $(w_{-0})_\lambda(x,y) = 0$ for all $\lambda > 0$. Since $w_\lambda(x,y) \le w_{\lambda-0}(x,y) = 0$, axiom (i) implies $x = y$.

Let w be strict, and $x, y \in X$, $x \ne y$. By condition ($\mathrm{i_s}$) and (1.2.4),

$$0 \ne w_\lambda(x,y) \le w_{\lambda-0}(x,y) \le w_{\mu+0}(x,y), \qquad 0 < \mu < \lambda,$$

and so, $(w_{-0})_\lambda(x,y) \ne 0$ for all $\lambda > 0$ and $(w_{+0})_\mu(x,y) \ne 0$ for all $\mu > 0$.

Let us show that w_{+0} is continuous from the right on $(0,\infty)$ (the left continuity of w_{-0} is treated similarly). Since w_{+0} is a pseudomodular on X, it is nonincreasing on $(0,\infty)$, and so, if $\mu > 0$, $x, y \in X$, and $\gamma = (w_{+0})_{\mu+0}(x,y)$, we have, by (1.2.4), $\gamma \le (w_{+0})_\mu(x,y)$. In order to obtain the reverse inequality, we may assume that γ is finite. For any $\varepsilon > 0$ there exists $\mu_0 = \mu_0(\varepsilon) > \mu$ such that, if $\mu < \mu' \le \mu_0$, we have $(w_{+0})_{\mu'}(x,y) < \gamma + \varepsilon$. Given λ with $\mu < \lambda \le \mu_0$, choosing μ' such that $\mu < \mu' < \lambda$, we find, by virtue of (1.2.4), that $w_\lambda(x,y) \le w_{\mu'+0}(x,y) < \gamma + \varepsilon$. Passing to the limit as $\lambda \to \mu + 0$, we get the inequality $(w_{+0})_\mu(x,y) \le \gamma + \varepsilon$ for all $\varepsilon > 0$. □

Remark 1.2.6. In the above proof, we have shown that $(w_{+0})_{+0} = w_{+0}$ (as well as $(w_{-0})_{-0} = w_{-0}$), and one can show that $(w_{-0})_{+0} = w_{+0}$ and $(w_{+0})_{-0} = w_{-0}$.

1.3 Examples of Modulars

In order to get a better feeling of the notion of modular, we ought to have a sufficiently large reservoir of them. This section serves this purpose (to begin with). Where a metric notion is needed, we prefer a metric space context; generalizations to extended metrics and pseudometrics can then be readily obtained in a parallel manner. Instead of referring to the family $w = \{w_\lambda\}_{\lambda>0}$, it is often convenient and nonambiguous to term a (pseudo)modular the value $w_\lambda(x,y)$.

1.3.1 Separated Variables

Let (X, d) be a metric space, and $g : (0, \infty) \to [0, \infty]$ be an extended (nonnegative) valued function. We set

$$w_\lambda(x, y) = g(\lambda) \cdot d(x, y), \qquad \lambda > 0, \quad x, y \in X, \tag{1.3.1}$$

with the convention that $\infty \cdot 0 = 0$, and $\infty \cdot a = \infty$ for all $a > 0$.

By (d.1) and (d.2), the family $w = \{w_\lambda\}_{\lambda>0}$ satisfies axioms (i′) and (ii). It follows that the modular classes of w on X (cf. Fig. 1.2) are characterized as follows:

(A) w is a pseudomodular on X iff axiom (iii) is satisfied;
(B) w is a modular on X iff conditions ($\mathrm{i}_\Leftarrow$) and (iii) are satisfied;
(C) w is a strict modular on X iff ($\mathrm{i_s}$) and (iii) are satisfied.

Replacing (iii) by (iv) in the right-hand sides of (A)–(C), we get the characterization of w to be a convex pseudomodular / modular / strict modular.

Properties ($\mathrm{i}_\Leftarrow$), ($\mathrm{i_s}$), (iii), and (iv) of w from (1.3.1) are expressed as follows.

Proposition 1.3.1. (a) ($\mathrm{i}_\Leftarrow$) *is equivalent to* $g \not\equiv 0$;
(b) ($\mathrm{i_s}$) *if and only if* $g(\lambda) \neq 0$ *for all* $\lambda > 0$;
(c) (iii) *if and only if* g *is nonincreasing on* $(0, \infty)$;
(d) (iv) *if and only if* $\lambda \mapsto \lambda g(\lambda)$ *is nonincreasing on* $(0, \infty)$.

Proof. Let $\lambda, \mu > 0$ and $x, y, z \in X$.

(a) ($\Rightarrow$) To show that $g \not\equiv 0$, choose $x \neq y$, so that $d(x, y) \neq 0$. If $g \equiv 0$, then $w_\lambda(x, y) = 0$ for all $\lambda > 0$, and, by virtue of ($\mathrm{i}_\Leftarrow$), $x = y$, which is a contradiction.

($\Leftarrow$) Let $w_\lambda(x, y) = 0$ for all $\lambda > 0$. Since $g \not\equiv 0$, there exists $\lambda_0 > 0$ such that $g(\lambda_0) \neq 0$, and since $g(\lambda_0)d(x, y) = 0$, we have $d(x, y) = 0$, and so, $x = y$.

(b) If $x \neq y$, then $d(x, y) \neq 0$, and so, $w_\lambda(x, y) = g(\lambda)d(x, y) \neq 0$ for all $\lambda > 0$ if and only if $g(\lambda) \neq 0$ for all $\lambda > 0$.

(c) ($\Rightarrow$) By (iii) for w as above, $g(\lambda + \mu)d(x, y) \le g(\lambda)d(x, z) + g(\mu)d(z, y)$. Choosing $x \neq y = z$, we find $g(\lambda + \mu)d(x, y) \le g(\lambda)d(x, y)$, i.e., $g(\lambda + \mu) \le g(\lambda)$.

($\Leftarrow$) The triangle inequality (d.3) and inequality $g(\lambda + \mu) \le g(\lambda)$ imply

$$w_{\lambda+\mu}(x, y) = g(\lambda+\mu)d(x, y) \le g(\lambda)d(x, z)+g(\mu)d(z, y) = w_\lambda(x, z)+w_\mu(z, y).$$

(d) Apply (c) to the function $\hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ (see Remark 1.2.2(d)). □

Let us consider some particular cases of modulars (1.3.1). Note that $w \equiv 0$ is the only pseudomodular on X (corresponding to $g \equiv 0$), which is not a modular on X.

Example 1.3.2. (1) Setting $g(\lambda) = 1/\lambda$ in (1.3.1), we get the convex strict modular $w_\lambda(x, y) = d(x, y)/\lambda$ (the mean velocity between x and y in time λ from Sect. 1.1), called the *canonical modular* on the metric space (X, d). Another natural strict modular $\mathsf{w}_\lambda(x,y) = d(x,y)$ on X, corresponding to $g \equiv 1$, is nonconvex. Due to this, the canonical modular admits more adequate properties in order to 'embed' the metric space theory into the modular space theory.

More generally, $w_\lambda(x,y) = d(x,y)/\lambda^p$ ($p \geq 0$) is a strict modular on X, which is convex if and only if $p \geq 1$ (here λ^p may be replaced by $\exp(\lambda^p) - 1$, or λe^λ, etc.).

(2) The modular w given by $w_\lambda(x,y) = d(x,y)/\lambda$ if $0 < \lambda < 1$, and $w_\lambda(x,y) = 0$ if $\lambda \geq 1$, is convex and nonstrict. If we replace the first equality by $w_\lambda(x,y) = d(x,y)$ if $0 < \lambda < 1$, then the resulting modular w on X is nonconvex and nonstrict.

(3) Given a set X, denote by δ the *discrete metric* on X (i.e., $\delta(x,y) = 0$ if $x = y$, and $\delta(x,y) = 1$ if $x \neq y$), and let $d = \delta$ in (1.3.1).

If $g \equiv \infty$, we get the *infinite modular* on X, which is strict and convex:

$$w_\lambda(x,y) = \infty \cdot \delta(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \infty & \text{if } x \neq y, \end{cases} \qquad \text{for all} \quad \lambda > 0.$$

Let $\lambda_0 > 0$, and $a > 0$ or $a = \infty$. Define $g(\lambda)$ by: $g(\lambda) = a$ if $0 < \lambda < \lambda_0$, and $g(\lambda) = 0$ if $\lambda \geq \lambda_0$. The *step-like modular* w on X is of the form:

$$w_\lambda(x,y) = g(\lambda) \cdot \delta(x,y) = \begin{cases} 0 & \text{if } x = y \text{ and } \lambda > 0, \\ a & \text{if } x \neq y \text{ and } 0 < \lambda < \lambda_0, \\ 0 & \text{if } x \neq y \text{ and } \lambda \geq \lambda_0. \end{cases}$$

It is nonstrict, convex if $a = \infty$, and nonconvex if $a > 0$ (is finite).

1.3.2 Families of Extended (Pseudo)metrics

A generalization of previous considerations in Sect. 1.3.1 is as follows.

Given $\lambda > 0$, let $d_\lambda : X \times X \to [0, \infty]$ be an extended pseudometric on X. Setting $w = \{w_\lambda\}_{\lambda>0}$ with $w_\lambda(x,y) = d_\lambda(x,y)$, $x, y \in X$, we find that w satisfies (i$'$) and (ii). So, modular classes of w on X (including convex w) are characterized as in assertions (A)–(C) of Sect. 1.3.1, where properties ($\mathrm{i}_\Leftarrow$), (i_s), (iii), and (iv) are given in terms of functions $w^{x,y}$ (in place of the function g in Proposition 1.3.1):

(a) ($\mathrm{i}_\Leftarrow$) $\iff$ $w^{x,y} \not\equiv 0$ for all $x, y \in X$ with $x \neq y$

$\iff$ condition '$d_\lambda(x,y) = 0$ for all $\lambda > 0$' implies $x = y$;

(b) (i_s) $\iff$ $w^{x,y}(\lambda) \neq 0$ for all $\lambda > 0$ and $x, y \in X$ with $x \neq y$

$\iff$ d_λ is an extended metric on X for all $\lambda > 0$;

(c) (iii) $\iff$ $\lambda \mapsto d_\lambda(x,y)$ is nonincreasing on $(0, \infty)$ for all $x, y \in X$;

(d) (iv) $\iff$ $\lambda \mapsto \lambda d_\lambda(x,y)$ is nonincreasing on $(0, \infty)$ for all $x, y \in X$.

Note only that in establishing assertion (c)($\Leftarrow$) we have: since $\lambda \mapsto d_\lambda(x,y)$ is nonincreasing, the triangle inequality for $w_{\lambda+\mu} = d_{\lambda+\mu}$ implies

$$w_{\lambda+\mu}(x,y) \le d_{\lambda+\mu}(x,z) + d_{\lambda+\mu}(z,y) \le d_\lambda(x,z) + d_\mu(z,y) = w_\lambda(x,z) + w_\mu(z,y),$$

which proves the inequality in axiom (iii).

Now, we expose two particular cases of families of (extended) metrics.

Example 1.3.3. (1) Let (X,d) be a metric space, and $h : (0,\infty) \to (0,\infty)$ be a nondecreasing function. Setting

$$w_\lambda(x,y) = \frac{d(x,y)}{h(\lambda) + d(x,y)}, \qquad \lambda > 0, \quad x,y \in X, \tag{1.3.2}$$

we find that $w = \{w_\lambda\}_{\lambda>0}$ is a family of metrics on X such that the function $\lambda \mapsto w_\lambda(x,y)$ is nonincreasing on $(0,\infty)$, and so, by (b) and (c) above, w is a strict modular on X. For instance, the triangle inequality for w_λ is obtained as follows: the function

$$f(t) = \frac{t}{h(\lambda)+t} = 1 - \frac{h(\lambda)}{h(\lambda)+t}, \qquad t > -h(\lambda),$$

is increasing in $t \ge 0$, which together with the triangle inequality for d gives

$$\begin{aligned} w_\lambda(x,y) &= f(d(x,y)) \le f\big(d(x,z) + d(z,y)\big) = \frac{d(x,z)+d(z,y)}{h(\lambda)+d(x,z)+d(z,y)} \\ &\le \frac{d(x,z)}{h(\lambda)+d(x,z)} + \frac{d(z,y)}{h(\lambda)+d(z,y)} = w_\lambda(x,z) + w_\lambda(z,y). \end{aligned}$$

The modular w is nonconvex: this is a consequence of Proposition 1.2.3(b) and the fact that $w_\lambda(x,y)$ tends to $d(x,y)/(h(+0)+d(x,y)) \le 1$ as $\lambda \to +0$ for all $x,y \in X$.

(2) Let $T \subset [0,\infty)$, (M,d) be a metric space, and $X = M^T$ be the set of all mappings $x : T \to M$ from T into M. If w is defined by

$$w_\lambda(x,y) = \sup_{t\in T} \mathrm{e}^{-\lambda t} d(x(t),y(t)), \qquad \lambda > 0, \quad x,y \in X,$$

then w_λ is an extended metric on X, for which the function $\lambda \mapsto w_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Hence, $w = \{w_\lambda\}_{\lambda>0}$ is a strict modular on X. Let us show that w is nonconvex. Choose $x_0, y_0 \in M$, $x_0 \ne y_0$, and set $x(t) = x_0$ and $y(t) = y_0$ for all $t \in T$. Then $x \ne y$, and $w_\lambda(x,y) = \exp(-\lambda \inf T) d(x_0,y_0)$. It follows that, as $\lambda \to +0$, we have $w_\lambda(x,y) \to d(x_0,y_0) < \infty$. It remains to refer to Proposition 1.2.3(b).

1.3.3 Classical Modulars on Real Linear Spaces

Let X be a real linear space. A functional $\rho : X \to [0, \infty]$ is said to be a *classical modular* on X in the sense of H. Nakano, J. Musielak and W. Orlicz if it satisfies the following four conditions:

(ρ.1) $\rho(0) = 0$;
(ρ.2) if $x \in X$, and $\rho(\alpha x) = 0$ for all $\alpha > 0$, then $x = 0$;
(ρ.3) $\rho(-x) = \rho(x)$ for all $x \in X$;
(ρ.4) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$, and $x, y \in X$.

If, instead of the inequality in (ρ.4), ρ satisfies

(ρ.5) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$,

then it is said to be a classical *convex modular* on X.

An example of a classical convex modular on X is the usual *norm* (i.e., a functional $\|\cdot\| : X \to [0, \infty)$ with properties: $\|x\| = 0 \Leftrightarrow x = 0$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, and $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$).

In the next two Propositions, we show that modulars in the sense of Definition 1.2.1 are extensions of classical modulars on linear spaces.

Proposition 1.3.4. *Given a functional $\rho : X \to [0, \infty]$, we set*

$$w_\lambda(x, y) = \rho\Big(\frac{x - y}{\lambda}\Big), \qquad \lambda > 0, \quad x, y \in X. \tag{1.3.3}$$

Then, we have: ρ is a classical (convex) modular on the linear space X if and only if w is a (convex) modular on the set X.

Proof. Since the assertions (ρ.1) $\Leftrightarrow$ (i′), (ρ.2) $\Leftrightarrow$ ($\mathrm{i}_{\Leftarrow}$), and (ρ.3) $\Leftrightarrow$ (ii) are clear, we show only that (ρ.5) $\Leftrightarrow$ (iv) (the equivalence (ρ.4) $\Leftrightarrow$ (iii) is established similarly).

(ρ.5) $\Rightarrow$ (iv). Given $\lambda, \mu > 0$ and $x, y, z \in X$, we have

$$\frac{x - y}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu} \cdot \frac{x - z}{\lambda} + \frac{\mu}{\lambda + \mu} \cdot \frac{z - y}{\mu} = \alpha x' + \beta y', \tag{1.3.4}$$

where

$$\alpha = \frac{\lambda}{\lambda + \mu} > 0, \quad \beta = \frac{\mu}{\lambda + \mu} > 0, \quad \alpha + \beta = 1, \quad x' = \frac{x - z}{\lambda} \quad \text{and} \quad y' = \frac{z - y}{\mu}.$$

By virtue of (1.3.3) and (ρ.5), we obtain the inequality in axiom (iv) as follows:

$$\begin{aligned} w_{\lambda+\mu}(x, y) &= \rho\Big(\frac{x - y}{\lambda + \mu}\Big) = \rho(\alpha x' + \beta y') \le \alpha\rho(x') + \beta\rho(y') \\ &= \frac{\lambda}{\lambda + \mu} w_\lambda(x, z) + \frac{\mu}{\lambda + \mu} w_\mu(z, y). \end{aligned}$$

(iv) ⇒ (ρ.5). Assume that $\alpha > 0$, $\beta > 0$, and $\alpha + \beta = 1$ (otherwise, (ρ.5) is obvious). Taking into account (1.3.3) and (iv), for $x, y \in X$, we get

$$\begin{aligned}\rho(\alpha x + \beta y) &= \rho\Big(\frac{\alpha x - (-\beta y)}{\alpha + \beta}\Big) = w_{\alpha+\beta}(\alpha x, -\beta y)\\ &\le \frac{\alpha}{\alpha+\beta}\, w_\alpha(\alpha x, 0) + \frac{\beta}{\alpha+\beta}\, w_\beta(0, -\beta y)\\ &= \alpha\rho\Big(\frac{\alpha x}{\alpha}\Big) + \beta\rho\Big(\frac{\beta y}{\beta}\Big) = \alpha\rho(x) + \beta\rho(y). \qquad \square\end{aligned}$$

Proposition 1.3.5. *Suppose the function* $w : (0,\infty) \times X \times X \to [0,\infty]$ *satisfies the following two conditions:*

(I) $w_\lambda(x + z, y + z) = w_\lambda(x, y)$ *for all* $\lambda > 0$ *and* $x, y, z \in X$;
(II) $w_\lambda(\mu x, 0) = w_{\lambda/\mu}(x, 0)$ *for all* $\lambda, \mu > 0$ *and* $x \in X$.

Given $x \in X$, *we set* $\rho(x) = w_1(x, 0)$. *Then, we have:*

(a) *equality* (1.3.3) *holds;*
(b) w *is a* (*convex*) *modular on the set* X *if and only if* ρ *is a classical* (*convex*) *modular on the real linear space* X.

Proof. (a) By virtue of assumptions (I) and (II), we find

$$w_\lambda(x, y) = w_\lambda(x - y, y - y) = w_1\Big(\frac{x-y}{\lambda}, 0\Big) = \rho\Big(\frac{x-y}{\lambda}\Big).$$

(b) As in the proof of Proposition 1.3.4, we verify only that (iv) ⇔ (ρ.5).

(iv) ⇒ (ρ.5). Given $\alpha, \beta > 0$ with $\alpha + \beta = 1$, and $x, y \in X$, equalities (1.3.3), (I), and (II), and condition (iv) imply

$$\begin{aligned}\rho(\alpha x + \beta y) = w_1(\alpha x, -\beta y) &\le \frac{\alpha}{\alpha+\beta}\, w_\alpha(\alpha x, 0) + \frac{\beta}{\alpha+\beta}\, w_\beta(0, -\beta y)\\ &= \alpha w_{\alpha/\alpha}(x, 0) + \beta w_{\beta/\beta}(y, 0) = \alpha\rho(x) + \beta\rho(y).\end{aligned}$$

(ρ.5) ⇒ (iv). Taking into account equality (1.3.3), this is established as the corresponding implication in the proof of Proposition 1.3.4. $\square$

Remark 1.3.6. Proposition 1.3.4 provides tools for further examples of metric modulars w, generating them from classical modulars by means of formula (1.3.3). In view of Proposition 1.3.5, modulars w on a real linear space X, not satisfying conditions (I) or (II), may be nonclassical (e.g., modulars (1.3.1) and (1.3.2)).

Example 1.3.7 (the generalized Orlicz modular). Suppose (Ω, Σ, μ) is a measure space with measure μ and $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ is a function satisfying the following two conditions: (a) for every $t \in \Omega$, the function $\varphi(t,\cdot) = [u \mapsto \varphi(t,u)]$

is nondecreasing and continuous on $[0, \infty)$, $\varphi(t, u) = 0$ iff $u = 0$, and $\lim_{u\to\infty} \varphi(t, u) = \infty$; (b) for all $u \geq 0$, the function $\varphi(\cdot, u) = [t \mapsto \varphi(t, u)]$ is Σ-measurable. Let X be the set of all real- (or complex-)valued functions on Ω, which are Σ-measurable and finite μ-almost everywhere (with equality μ-almost everywhere). Then, for every $x \in X$, the function $t \mapsto \varphi(t, |x(t)|)$ is Σ-measurable on Ω, and

$$\rho(x) = \int_\Omega \varphi(t, |x(t)|)\, d\mu \quad \text{is a classical modular on } X,$$

known as the *generalized Orlicz modular* (note that $\rho(x) = 0$ iff $x = 0$).

1.3.4 φ-Generated Modulars

Let $\varphi : [0, \infty) \to [0, \infty]$ be a nondecreasing function such that $\varphi(0) = 0$ and $\varphi \not\equiv 0$.

Given a *normed space* $(X, \|\cdot\|)$ (i.e., X is a linear space and $\|\cdot\|$ is a norm on it), the functional $\rho(x) = \varphi(\|x\|)$, $x \in X$, is a classical modular on X; in addition, ρ is a convex modular on X if and only if φ is convex on $[0, \infty)$. Since $d(x, y) = \|x - y\|$ is a metric on X, taking into account equality (1.3.3), we proceed as follows.

Proposition 1.3.8. *Let (X, d) be a metric space. Set*

$$w_\lambda(x, y) = \varphi\Big(\frac{d(x, y)}{\lambda}\Big), \qquad \lambda > 0, \quad x, y \in X. \tag{1.3.5}$$

Then w is a modular on X. Moreover, if φ is convex, then w is a convex modular, and if $\varphi(u) \neq 0$ for all $u > 0$, then w is strict.

Proof. We shall verify some of the properties of w directly (with no reference to ρ).

To see that ($\mathrm{i}_\Leftarrow$) holds, suppose $x, y \in X$, and $w_\lambda(x, y) = 0$ for all $\lambda > 0$. If $x \neq y$, then $d(x, y) > 0$, and so, given $u > 0$, setting $\lambda_u = d(x, y)/u$, we find that $\varphi(u) = \varphi(d(x, y)/\lambda_u) = w_{\lambda_u}(x, y) = 0$. Since $\varphi(0) = 0$, we have $\varphi \equiv 0$ on $[0, \infty)$, which is in contradiction with the assumption on φ. Thus, $x = y$.

In checking axiom (iii) for w, the following observation plays a key role. Given $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, and $u_1, u_2 \geq 0$, we have

$$(\alpha + \beta) \min\{u_1, u_2\} \leq \alpha u_1 + \beta u_2 \leq \max\{u_1, u_2\},$$

and so, since φ is nondecreasing on $[0, \infty)$,

$$\varphi(\alpha u_1 + \beta u_2) \leq \max\{\varphi(u_1), \varphi(u_2)\} \leq \varphi(u_1) + \varphi(u_2). \tag{1.3.6}$$

Now, if $\lambda, \mu > 0$ and $x, y, z \in X$, the triangle inequality for d implies

$$w_{\lambda+\mu}(x,y) = \varphi\Big(\frac{d(x,y)}{\lambda+\mu}\Big) \le \varphi\Big(\frac{\lambda}{\lambda+\mu}\cdot\frac{d(x,z)}{\lambda} + \frac{\mu}{\lambda+\mu}\cdot\frac{d(z,y)}{\mu}\Big) \tag{1.3.7}$$
$$\le \varphi\Big(\frac{d(x,z)}{\lambda}\Big) + \varphi\Big(\frac{d(z,y)}{\mu}\Big) = w_\lambda(x,z) + w_\mu(z,y).$$

If, in addition, φ is convex, then we proceed from (1.3.7) as follows:

$$w_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\,\varphi\Big(\frac{d(x,z)}{\lambda}\Big) + \frac{\mu}{\lambda+\mu}\,\varphi\Big(\frac{d(z,y)}{\mu}\Big)$$
$$= \frac{\lambda}{\lambda+\mu}\,w_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\,w_\mu(z,y). \qquad \square \tag{1.3.8}$$

Example 1.3.9. Let $\varphi(u) = 0$ if $0 \le u \le 1$, and $\varphi(u) = a$ if $u > 1$, where $a > 0$ or $a = \infty$. Then modular (1.3.5), called the $(a,0)$-*modular*, is of the form:

$$w_\lambda(x,y) = \begin{cases} a & \text{if } \ 0 < \lambda < d(x,y), \\ 0 & \text{if } \qquad \lambda \ge d(x,y). \end{cases} \tag{1.3.9}$$

It is nonstrict, convex if $a = \infty$, and nonconvex if $a > 0$.

If $\varphi(u) = \infty$ for all $u > 0$, we get the infinite modular from Example 1.3.2(3).

Let $(M, \|\cdot\|)$ be a normed space and $X = M^{\mathbb{N}}$ the set of all sequences $x : \mathbb{N} \to M$ equipped with the componentwise operations of addition and multiplication by scalars. As usual, given $x \in X$, we set $x_n = x(n)$ for $n \in \mathbb{N}$, and so, x is also denoted by $\{x_n\}_{n=1}^\infty \equiv \{x_n\}$. The functional $\rho(x) = \sum_{n=1}^\infty \|x_n\|^p$ $(p \ge 1)$ is a classical convex modular on the linear space X.

This gives an idea to replace the function $u \mapsto u^p$, defining ρ, by the function φ as above and consider the following more general construction.

Example 1.3.10. Let (M, d) be a metric space, $X = M^{\mathbb{N}}$, and $h : [0,\infty) \to [0,\infty)$ be a superadditive function (see Appendix A.1). Define $w : (0,\infty) \times X \times X \to [0,\infty]$ by

$$w_\lambda(x,y) = \sum_{n=1}^\infty \varphi\Big(\frac{d(x_n,y_n)}{h(\lambda)}\Big), \qquad \lambda > 0, \quad x, y \in X. \tag{1.3.10}$$

Then w is a modular on X. For axioms (iii) and (iv), it is to be noted only that, by virtue of (1.3.6), we have (instead of (1.3.7))

$$\varphi\Big(\frac{d(x_n,y_n)}{h(\lambda+\mu)}\Big) \le \varphi\Big(\frac{h(\lambda)}{h(\lambda+\mu)}\cdot\frac{d(x_n,z_n)}{h(\lambda)} + \frac{h(\mu)}{h(\lambda+\mu)}\cdot\frac{d(z_n,y_n)}{h(\mu)}\Big)$$
$$\le \varphi\Big(\frac{d(x_n,z_n)}{h(\lambda)}\Big) + \varphi\Big(\frac{d(z_n,y_n)}{h(\mu)}\Big). \tag{1.3.11}$$

This example can be further generalized if we allow d, φ and h to depend on n, i.e., $d(x,y) = d_n(x,y)$, $\varphi(u) = \varphi_n(u)$, and $h(\lambda) = h_n(\lambda)$.

1.3.5 Pseudomodulars on the Power Set

Given a set X, we denote by $\mathscr{P}(X) \equiv 2^X$ the family of all subsets of X, also called the *power set* of X. We employ the convention that $\sup \varnothing = 0$ and $\inf \varnothing = \infty$.

Let w be a (pseudo)modular on a set X (in the sense of (i′), (i)–(iii)). Following the idea of construction of the Hausdorff distance (see Appendix A.2), we are going to introduce a pseudomodular W on the power set $\mathscr{P}(X)$, induced by w.

Given $\lambda > 0$ and nonempty sets $A, B \in \mathscr{P}(X)$, we put

$$E_\lambda(A,B) = \sup_{x \in A} \inf_{y \in B} w_\lambda(x,y) \in [0,\infty]. \tag{1.3.12}$$

Furthermore, we set

$$E_\lambda(\varnothing, B) = 0 \quad \text{for all } \lambda > 0 \text{ and } B \in \mathscr{P}(X), \tag{1.3.13}$$

and

$$E_\lambda(A, \varnothing) = \infty \quad \text{for all } \lambda > 0 \text{ and } A \in \mathscr{P}(X),\ A \neq \varnothing. \tag{1.3.14}$$

Proposition 1.3.11. *The function* $E : (0,\infty) \times \mathscr{P}(X) \times \mathscr{P}(X) \to [0,\infty]$ *is well-defined and has the following two properties:*

(a) $E_\lambda(A,B) = 0$ *for all* $\lambda > 0$ *and* $A \subset B \subset X$;
(b) $E_{\lambda+\mu}(A,C) \le E_\lambda(A,B) + E_\mu(B,C)$ *for all* $\lambda, \mu > 0$ *and* $A, B, C \in \mathscr{P}(X)$.

Proof. (a) If $A = \varnothing$, then the assertion follows from (1.3.13), and if $A \neq \varnothing$, then, given $x \in A$ (so that $x \in B$), we have, by (i′), $0 \le \inf_{y \in B} w_\lambda(x,y) \le w_\lambda(x,x) = 0$. Since $x \in A$ is arbitrary, (1.3.12) implies $E_\lambda(A,B) = 0$.

(b) If at least one of the sets A, B, or C is empty, then we have the possibilities shown in Table 1.1.

Now, assume that A, B and C are nonempty and apply (1.3.12). Given $x \in A$, $y \in B$, and $\lambda, \mu > 0$, by virtue of (iii) for w, we have

$$\inf_{z \in C} w_{\lambda+\mu}(x,z) \le w_{\lambda+\mu}(x,z_1) \le w_\lambda(x,y) + w_\mu(y,z_1) \quad \text{for all } z_1 \in C. \tag{1.3.15}$$

Taking the infimum over all $z_1 \in C$, we get, for all $y \in B$,

$$\inf_{z \in C} w_{\lambda+\mu}(x,z) \le w_\lambda(x,y) + \inf_{z_1 \in C} w_\mu(y,z_1) \le w_\lambda(x,y) + E_\mu(B,C).$$

Table 1.1 Inequality (b) when at least one of the sets A, B, or C is empty

Sets A, B, C	$E_{\lambda+\mu}(A,C)$	$E_\lambda(A,B)$	$E_\mu(B,C)$	Apply
$A=\varnothing,\ B=\varnothing,\ C=\varnothing$	0	0	0	(1.3.13)
$A\ne\varnothing,\ B=\varnothing,\ C=\varnothing$	∞	∞	0	(1.3.14), (1.3.13)
$A=\varnothing,\ B\ne\varnothing,\ C=\varnothing$	0	0	∞	(1.3.13), (1.3.14)
$A=\varnothing,\ B=\varnothing,\ C\ne\varnothing$	0	0	0	(1.3.13)
$A\ne\varnothing,\ B\ne\varnothing,\ C=\varnothing$	∞	$\cdots$	∞	(1.3.14), (1.3.12)
$A\ne\varnothing,\ B=\varnothing,\ C\ne\varnothing$	$\cdots$	∞	0	(1.3.12), (1.3.14), (1.3.13)
$A=\varnothing,\ B\ne\varnothing,\ C\ne\varnothing$	0	0	$\cdots$	(1.3.13), (1.3.12)

Now, taking the infimum over all $y \in B$, we find, for all $x \in A$,

$$\inf_{z\in C} w_{\lambda+\mu}(x,z) \le \inf_{y\in B} w_\lambda(x,y) + E_\mu(B,C) \le E_\lambda(A,B) + E_\mu(B,C),$$

and it remains to take the supremum over all $x \in A$. □

Definition 1.3.12. The function $W : (0,\infty)\times\mathscr{P}(X)\times\mathscr{P}(X)\to[0,\infty]$, defined by

$$W_\lambda(A,B) = \max\{E_\lambda(A,B), E_\lambda(B,A)\}, \qquad \lambda>0, \quad A,B\in\mathscr{P}(X),$$

has the following properties, for all $\lambda,\mu>0$ and $A,B,C\in\mathscr{P}(X)$:

(A) $W_\lambda(A,A)=0$;
(B) $W_\lambda(A,B)=W_\lambda(B,A)$;
(C) $W_{\lambda+\mu}(A,C)\le W_\lambda(A,B)+W_\mu(B,C)$.

Thus, W is (only) a pseudomodular on the power set $\mathscr{P}(X)$, called the *Hausdorff pseudomodular*, induced by w.

Note that $W_\lambda(\varnothing,\varnothing)=0$, while $W_\lambda(A,\varnothing)=\infty$ if $\varnothing\ne A\in\mathscr{P}(X)$ and $\lambda>0$. If w is a convex (pseudo)modular on X, then applying axiom (iv) in (1.3.15) instead of (iii), we find that W is a convex pseudomodular on $\mathscr{P}(X)$.

Further properties of W will be presented below (see Theorem 2.2.13, Example 3.3.11, and Theorem 4.1.3).

1.4 Bibliographical Notes and Comments

Section 1.1. An exposition of the theory of metric spaces can be found in many monographs and textbooks, e.g., Aleksandrov [2], Copson [33], Kaplansky [51], Kolmogorov and Fomin [54], Kumaresan [57], Kuratowski [58], Schwartz [97], Shirali and Vasudeva [98] (to mention a few). A good source of metric and distance notions is a recent book by Deza and Deza [36]. The ‘strong’ form of the triangle inequality is due to Lindenbaum [64]. The classical reference on pseudometric spaces is Kelley's book [52]. Extended metrics, also called generalized metrics, were studied by Jung [50] and Luxemburg [67] in connection with an extension of Banach's Fixed Point Theorem from [4].

The interpretation of a modular as a generalized velocity field was initiated by Chistyakov in [26, 28].

Section 1.2. Definition 1.2.1 of (metric) modular w on a set X appeared implicitly in Chistyakov [18, 19] in connection with the studies of (bounded variation and the like) selections of set-valued mappings, and multivalued superposition operators. Explicitly and axiomatically, (pseudo)modulars were introduced in Chistyakov [22], and their main properties were established by the author in [23–25]. The strictness condition ($\mathrm{i_s}$) and modular regularizations $w_{\pm 0}$ were defined in Chistyakov [28].

Section 1.3. Examples of (pseudo)modulars relevant for specific purposes are contained in [18–29]. In Sect. 1.3 and furtheron, we add some new and more general ones. An extended metric as in Example 1.3.3(2) was first defined by Bielecki [8] in order to obtain global solutions of ordinary differential equations (see also Goebel and Kirk [41, Sect. 2]).

The term *modular* on a real linear space X, extending the notion of norm, was introduced by Nakano [80, 81], where he developed the theory of modular spaces. Nakano's axioms [81, Sect. 78] of a modular $\rho : X \to [0, \infty]$ include (ρ.1)–(ρ.3), (ρ.5), and (ρ.6) $\rho(x) = \sup\{\rho(\alpha x) : 0 \le \alpha < 1\}$ for all $x \in X = X_\rho^*$ (see Sect. 1.3.3 and Remark 2.3.4(1)), i.e., ρ is a left-continuous convex semimodular on X in the sense of Musielak [75, Sect. 1].

In the special case of φ-integrable functions on [0, 1] and φ-summable sequences, a theory (of not necessarily convex) modulars was initiated by Mazur and Orlicz [71], and a general theory of modular spaces was developed by Musielak and Orlicz [77]. The key axiom in the nonconvex case is axiom (ρ.4).

Propositions 1.3.4 and 1.3.5 are taken from Chistyakov [22, 24]. They show that our approach to (metric) modulars on arbitrary sets X is an extension of the classical approach of Nakano, Musielak and Orlicz applied to modulars on linear spaces. In particular, classical modulars are metric modulars via (1.3.3). The same situation holds for *function modulars* on linear spaces developed by Kozlowski [55].

For $h(\lambda) = \lambda$, modular (1.3.2) can be obtained from the classical nonconvex modular $\rho(x) = |x|/(1+|x|)$, $x \in \mathbb{R}$, by means of (1.3.3) (cf. Maligranda [68, p. 8]).

In Example 1.3.7, we follow Musielak [75, Chap. II,Sect. 7].

The material of Sects. 1.3.4 and 1.3.5 is new.