Restricted Four-Valued Logic for Default Reasoning

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Abstract. In Reiter's default logic, it is possible that no useful information can be brought from inconsistent knowledge or no extension of incoherent default theories exists. In this paper, based on Belnap's four-valued logic, we propose a new variant of default logic called the restricted four-valued default logic to tolerate inconsistency and incoherency of knowledge in default reasoning. Our proposal can maintain both the expressive power of full default logic and the ability of default reasoning. Moreover, we present a transformation-based approach to compute the restricted four-valued extensions.

1 Introduction

Reiter's default logic [20] is a widely studied nonmonotonic logic. Despite that, default logic has its own shortcomings. Some default theories have only one trivial extension, which contains everything as its conclusion. Even the existence of extensions is not always guaranteed. Such incoherences may happen when contradictions occur in defaults or between defaults and facts.

To deal with incoherences, some variants of default logic were introduced. Some researchers treat incoherences as illegal. With this viewpoint, they focus on finding characterizations of default theories which have extensions, such as normal default theories [20] and ordered default theories [18] among others. These fragments of default logic are strictly weak and, as a result, lost full expressive power of default logic.

Another approach to handle incoherences is to modify the definition of extensions. For instances, the justified default extensions [16], the constrained default extensions [21] and the cumulative default extensions [12] are all guaranteed to exist for every default theory. However, these extensions have different semantics from Reiter's, even when Reiter's default extensions exist and are consistent.

To deal with inconsistencies, an approach is to transform inconsistent default theories into consistent ones, but still hold some useful conclusions. In [10], the authors handle inconsistencies by default logic itself. Another approach takes advantage of paraconsistent logics, which do not infer everything from contradictions, such as the question marked logic [1], the four-valued default logic [23],

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the bi-default logic [14], the fault-tolerant default logic [15], and the annotated default logic [22]. However, some of these attempts cannot handle all inconsistent and incoherent problems, or have different semantics from Reiter's.

In this paper we introduce a novel extension of default logic named the restricted four-valued default logic, based on Belnap's four-valued logic [3,8, 9], which is a multi-valued paraconsistent logic, to handle both inconsistencies and incoherences. Not like in [23], which is also based on Belnap's four-valued logic, we ensure that every default theory has at least one nontrivial extension. Moreover, we keep our extensions as similar as Reiter's original ones. Finally, we proved that Reiter's default logic is a special case of our logic on consistent and coherent default theories. Interestingly but not too surprisingly, we also show that our default logic is indeed an expansion of the preferred four-valued logic.

The paper is structured as follows. First we review preliminaries in section 2. Our main contributions are presented in sections 3 and 4, in which we describe our restricted four-valued default logic from underlying logic to extensions, together with comparison with default logic and four-valued logic. To calculate the restricted four-valued extensions, we present an approach of the formula transformation in section 5. We compare our results with related works in section 6, and summarize in section 7 as conclusion.

2 Preliminaries

In the rest of this paper we denote $\mathcal{L}$ as a propositional language, $\mathcal{A}$ as the set of all atoms, $\models_2$ as the classical propositional consequence relation and Th as the consequence operator. The propositional constants $\mathbf{t}$ and $\mathbf{f}$ are interpreted as true and false in all interpretations respectively.

2.1 Default Logic

A *default* d is an inference rule of form $d = \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$, where $\alpha, \beta_1, \ldots, \beta_n, \gamma$ are all propositional formulas. We define $Pre(d) = \alpha$ as *prerequisite* of d, $Just(d) = \{\beta_1, \ldots, \beta_n\}$ as *justification* of d, and $Con(d) = \gamma$ as *consequence* of d. For a set of defaults D, define $Pre(D) = \{Pre(d)|d \in D\}$, $Just(D) = \bigcup\{Just(d)|d \in D\}$, and $Con(D) = \{Con(d)|d \in D\}$.

A *default theory* is a pair $T = (D, W)$, where D is a set of defaults and W is a set of formulas. For convenience, neither $\mathbf{t}$ nor $\mathbf{f}$ is permitted to be presented in D or W.

An *extension* of a default theory is defined as follows.

Definition 1 ([20]). *Let $T = (D, W)$ be a default theory. For any set of formulas E, let $\Gamma(E)$ be the smallest set of formulas such that:*

1. *$W \subseteq \Gamma(E)$;*
2. *$Th(\Gamma(E)) = \Gamma(E)$;*
3. *For any $d \in D$, if $\Gamma(E) \models_2 Pre(d)$ and $\neg\beta \notin E$ for all $\beta \in Just(d)$, then $\Gamma(E) \models_2 Con(d)$.*

A set of formulas E is an (default) extension *of T iff $\Gamma(E) = E$, i.e. E is a fixed point of the operator Γ.*

We say T skeptically entails *a formula set F, if all extensions of T entail F.*

A default theory may have none, one or many extensions. Sometimes the extension may be trivial, which means it contains all propositional formulas.

Example 1. Let $T_i = (D_i, W_i)(i = 1, 2, 3, 4)$ be a default theory, where

1. $D_1 = \{\frac{:p}{q}\}$, $W_1 = \{\neg p, r, \neg r\}$;
2. $D_2 = \{\frac{:p}{\neg p}\}$, $W_2 = \{q\}$;
3. $D_3 = \{\frac{:p}{q}, \frac{:p}{\neg q}\}$, $W_3 = \emptyset$;
4. $D_4 = \{\frac{:q}{\neg r}, \frac{:r}{\neg p}, \frac{:p}{\neg q}\}$, $W_4 = \emptyset$.

T_1 has a trivial extension, while, none of T_2, T_3, T_4 has any extension.

2.2 Four-Valued Logic

To deal with inconsistent and incomplete knowledge, Belnap's four-valued logic [3,8,9] is constructed on the bilattice structure $FOUR = \{t, f, \top, \bot\}$. The elements of $FOUR$ can also be represented by pairs of two-valued truth values: $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, $\bot = (0, 0)$. Intuitively, the truth values $\top$ and $\bot$ represent inconsistencies and lacking of information respectively.

The *set of designated elements* is chosen as $\mathcal{D} = \{t, \top\}$. A *four-valued valuation* is a function that assigns a truth value from $FOUR$ to each atomic formula. The truth operators on $\langle FOUR\rangle$ are defined as follows: $\neg(x, y) = (y, x)$, $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$, $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$ and $(x_1, y_1) \supset (x_2, y_2) = (\neg x_1 \vee x_2, x_1 \wedge y_2)$. For constants, let $v(\mathbf{t}) = t$ and $v(\mathbf{f}) = f$.

A valuation v *satisfies* a formula ϕ if $v(\phi) \in \mathcal{D}$. We say v is a *model* of a formula set S if v satisfies every formula in S. We use $\langle FOUR\rangle$ to denote the structure $FOUR$ together with $\mathcal{D}$. The consequence relation on $\langle FOUR\rangle$ are defined in the following.

Definition 2 ([3]). *Suppose that Γ and Δ are two sets of formulae. $\Gamma \models_4 \Delta$ if every model of Γ in $\langle FOUR\rangle$ is a model of some formula of Δ.*

Definition 3 ([3]). *Let u, v be four-valued valuations. u is more classical than v if $v(p) \in \{\top, \bot\}$ whenever $u(p) \in \{\top, \bot\}$.*

Suppose that Γ and Δ are two sets of formulas. $\Gamma \models^4_{cl} \Delta$ if every most classical model of Γ is a model of some formula of Δ.

As a nonmonotonic and paraconsistent consequence relation, $\models^4_{cl}$ is equivalent to classical logic on consistent theories. For more details, see [3].

3 Restricted Four-Valued Default Logic

3.1 Restricted Four-Valued Logic

In this section, we present a *restricted four-valued logic* as the underlying logic of our default logic. In our restricted four-valued logic, we focus on those valuations whose nonclassical values only occur in a given set of atoms. The idea of restricting paraconsistent atoms in a fixed subset can be inspired from Vasil'év's imaginary logic [7].

Definition 4. *Let S be a set of atoms. A four-valued valuation v is* restricted by S, *if* $\{a \in \mathcal{A} | v(a) \notin \{t, f\}\} \subseteq S$.

Definition 5. *Let S be a set of atoms, Γ, Σ be sets of formulas. A four-valued valuation v is a* four-valued model of Γ restricted by S *if v is a four-valued model of Γ and restricted by S.*

$\Gamma \models_S \Sigma$ if every four-valued model of Γ restricted by S is a four-valued model of Σ.

Denote $Th_S(\Gamma)$ as the consequence operator restricted by S: $Th_S(\Gamma) = \{\alpha | \Gamma \models_S \alpha\}$.

The motivation for restricting nonclassical values is a trade-off between classical reasoning power and paraconsistent properties. Obviously, classical logic and four-valued logic can be treated as two extreme cases of our restricted four-valued logic.

Proposition 1. *Let S be a set of atoms, Γ a set of formulas and ϕ a formula. $\Gamma \models_\emptyset \phi$ iff $\Gamma \models_2 \phi$, and $\Gamma \models_{\mathcal{A}} \phi$ iff $\Gamma \models_4 \phi$.*

In fact, our restricted four-valued logic can be expressed in four-valued logic.

Theorem 1. *Let S be a set of atoms, Γ a set of formulas and ϕ a formula. $\Gamma \models_S \phi$ iff $\Gamma \cup f_{\mathcal{A}}(S) \models_4 \phi$, where $f_{\mathcal{A}}(S) = \bigcup_{a \in \mathcal{A} \setminus S} \{a \vee \neg a, (a \wedge \neg a) \supset \mathbf{f}\}$.* [1]

Proof. For any four-valued valuation v and atom a, v satisfies $a \vee \neg a$ iff $v(a) \neq \bot$ and v satisfies $(a \wedge \neg a) \supset \mathbf{f}$ iff $v(a) \neq \top$. As a result, v satisfies $f_{\mathcal{A}}(S)$ iff $v(a) \in \{t, f\}$ for all $a \notin S$, i.e. v is restricted by S.

Therefore, the four-valued models of $\Gamma \cup f_{\mathcal{A}}(S)$ are exactly the four-valued models of Γ restricted by S. □

By Theorem 1, many properties of restricted four-valued logic can be proved by transforming them to four-valued logic, such as monotonicity.

Proposition 2 (Monotonicity). *Let Γ, Σ be sets of formulas, S a set of atoms and ϕ a formula. If $\Gamma \subseteq \Sigma$ and $\Gamma \models_S \phi$, then $\Sigma \models_S \phi$.*

Proof. $\Gamma \models_S \phi$ infers that $\Gamma \cup f_{\mathcal{A}}(S) \models_4 \phi$, where f is defined in Theorem 1. As four-valued logic is monotonic([3]) and $\Gamma \subseteq \Sigma$, we know that $\Sigma \cup f_{\mathcal{A}}(S) \models_4 \phi$, which is equivalent to $\Sigma \models_S \phi$ according to Theorem 1. □

[1] It may be argued that $f_{\mathcal{A}}(S)$ can cause infiniteness if $\mathcal{A}$ is not finite. In fact, $\mathcal{A}$ can be replaced by any atom set which contains all atoms occur in Γ and ϕ.

3.2 Restricted Four-Valued Extension

In this subsection, we introduce the restricted four-valued extensions based on the restricted four-valued logic. We include a restricting set as a part of an extension.

In Reiter's default logic, a justification is satisfiable in a formula set E if its negation is not in E. The corresponding concept in restricted four-valued extension has a subtle distinction since a formula may coexist with its negation in the same formula set but not be trivial in restricted four-valued logic. We use formula $\beta \supset \mathbf{f}$ as a stronger negation of β, since $\{\beta, \beta \supset \mathbf{f}\}$ is always unsatisfiable in restricted four-valued logic. We also need to ensure that restricted four-valued extension is satisfiable, since we should better enlarge the restricting set rather than accept it as an extension if it is not satisfiable. We define the restricted four-valued extension as follows.

Definition 6 (Restricted Four-Valued Extension). *Let $T = (D, W)$ be a default theory and S a set of atoms. For any set of formulas E, let $\Gamma_S(E)$ be the smallest set satisfying the following properties:*

1. $\Gamma_S(E) \not\models_S \mathbf{f}$;
2. $W \subseteq \Gamma_S(E)$;
3. $Th_S(\Gamma_S(E)) = \Gamma_S(E)$;
4. *For any $d \in D$, if $\Gamma_S(E) \models_S Pre(d)$ and $\beta_i \supset \mathbf{f} \notin E$ for any $\beta_i \in Just(d)$, then $\Gamma_S(E) \models_S Con(d)$.*

For any set of formulas E and set of atoms S, $\langle E, S\rangle$ is a restricted four-valued extension *iff $\Gamma_S(E) = E$, i.e. E is a fixed point of the operator Γ_S. We denote S as the restricting set of $\langle E, S\rangle$ and say that $\langle E, S\rangle$ is restricted by S.*

We review Example 1 to show that the restricted four-valued extensions follow our intuition and recover several useful conclusions which are lost in Reiter's.

Example 2. (Continuation of Example 1) Consider default theories in Example 1 which are all trouble in Reiter's default logic. In contrast, all these default theories have restricted four-valued extensions which are nontrivial and intuitive.

1. One restricted four-valued extension of T_1 is $\langle Th_{S_1}(W_1), S_1\rangle$, where $S_1 = \{r\}$. This extension keeps all information of W_1 but is not trivial. For instance, it rejects q as its conclusion.
2. One restricted four-valued extension of T_2 is $\langle Th_{S_2}(\{\neg p, q\}), S_2\rangle$, where $S_2 = \{p\}$. It means that we allow $\neg p$ in this extension, but leave p with suspicion in S_2. This extension has no doubt on formula q since it is independent of p.
3. One restricted four-valued extension of T_3 is $\langle Th_{S_3}(\{q, \neg q\}), S_3\rangle$, where $S_3 = \{q\}$. We keep two conflict default consequences q and $\neg q$ together with no explosion. Also $\neg p$ is not derivable, since we still treat p as a classical atom.

4. Three restricted four-valued extensions of T_4 are: $\langle Th_{S_4^1}(\{\neg p, \neg q\}), S_4^1\rangle$, $\langle Th_{S_4^2}(\{\neg q, \neg r\}), S_4^2\rangle$, and $\langle Th_{S_4^3}(\{\neg r, \neg p\}), S_4^3\rangle$, where $S_4^1 = \{p\}$, $S_4^2 = \{q\}$, $S_4^3 = \{r\}$. All these extensions entail two of $\{\neg p, \neg q, \neg r\}$, but none of them entail all these three formulas. Our intuition is that the three rules in D_4 cannot be executed together unless considering one of their justifications as troubled.

Restricted four-valued extensions inherit many properties of Reiter's extensions due to the monotonic property. Although most variants of default logic hold these properties naturally, it is not the same as paraconsistent ones, especially those whose underlying logic is nonmonotonic. For example, the following propositions hold and can be proved by the same way of Reiter's original proofs.

Proposition 3. *Let $T = (D, W)$ be a default theory, and let S be a set of atoms. For any set of formulas E, $\langle E, S\rangle$ is a restricted four-valued extension iff $E \not\models_S \mathbf{f}$ and $E = \bigcup_{i=0}^{\infty} E_i$, where:*

1. *$E_0 = W$;*
2. *For all $i \geq 0$, $E_{i+1} = Th_S(E_i) \cup \{\gamma \in Con(d) | d \in D, where Pre(d) \in E_i and \beta \supset \mathbf{f} \notin E for all \beta \in Just(d)\}$.*

Proposition 4. *Let $T = (D, W)$ be a default theory. Suppose $\langle E, S\rangle$ is a restricted four-valued extension of T, then $E = Th_S(W \cup Con(GD(E, T)))$, where $GD(E, T) = \{d \in D | Pre(d) \in E, \beta \supset \mathbf{f} \notin E for any \beta \in Just(d)\}$.*

Our restricted four-valued default logic can ensure that the extensions of any default theory always exist.

Theorem 2. *Every default theory has restricted four-valued extensions.*

Proof. Let $T = (D, W)$ be a default theory and S the set of all atoms occurs in T. Let $E = \bigcup_{i=0}^{\infty} E_i$, where:

1. $E_0 = W$;
2. For all $i \geq 0$, $E_{i+1} = Th_S(E_i) \cup \{\gamma \in Con(d) | d \in D, Pre(d) \in E_i\}$.

Let v be the valuation with $v(a) = \top$ for all $a \in \mathcal{A}$. v is a four-valued model of E while only classical connectives occurs in E([3]). Because $v(\mathbf{f}) = \mathbf{f}$ which is not a designated value, $E \not\models_S \mathbf{f}$. For any $\beta \in Just(D)$, $v(\beta) = \top$ implies $v(\beta \supset \mathbf{f}) = \mathbf{f}$, so $E \not\models_S \beta \supset \mathbf{f}$. Compare with Proposition 3, we have proved that $\langle E, S\rangle$ is a restricted four-valued extension of T. $\square$

3.3 Preferred Restricted Four-Valued Extension

Although we guarantee that every default theory has at least one restricted four-valued extension, it is still too tolerant to permit all of them.

Example 3. (Continuation of Example 2) Considering default theory T_1 in Example 2, all restricted four-valued extensions of T_1 are:

1. $E_1 = \langle Th_{S_1^1}(W_1), S_1^1\rangle$, where $S_1^1 = \{r\}$. This is our intuitive extension.
2. $E_2 = \langle Th_{S_1^2}(W_1 \cup \{q\}), S_1^2\rangle$, where $S_1^2 = \{r, p\}$. The unnecessary atom p in restricting set causes $\neg p$ not enough to prevent applying of the only default rule and causes q be included as a counter-intuitive conclusion.
3. $E_3 = \langle Th_{S_1^3}(W_1), S_1^3\rangle$, where $S_1^3 = \{r, q\}$. The unnecessary atom q in restricting set weakens reasoning power. For example, E_1 entails $\neg q \to s$, which does not hold in E_3.
4. $E_4 = \langle Th_{S_1^4}(W_1 \cup \{q\}), S_1^4\rangle$, where $S_1^4 = \{r, p, q\}$. This is an even worse extension since it merges both shortcomings of E_2 and E_3.
5. We have more extensions if we add other atoms which are not present in our language to any restricting sets above.

As we can see in the above example, adding redundant atoms to restricting set would cause unwanted or/and weaker conclusions. We prefer to extensions which have only necessary atoms in their restricting sets.

Definition 7 (Preferred Restricted Four-Valued Extension). *Let T be a default theory. A restricted four-valued extension $\langle E, S\rangle$ of T is a* preferred restricted four-valued extension *of T, if there is no restricted four-valued extension of T restricted by R and $R \subsetneq S$.*

Example 4. The restricted four-valued extensions mentioned in Example 2 are whole preferred restricted four-valued extensions of their corresponding default theories respectively. As explained before, they are all conform to our intuition.

Similarly, we also ensure the existence of preferred extensions.

Theorem 3. *Every default theory has at least one preferred restricted four-valued extension.*

Proof. Let T be a default theory. T has a restricted four-valued extension $\langle E, S\rangle$ by Theorem 2. Since the atom set S is finite, there is a minimal atom set R which restricts a restricted four-valued extension $\langle E', R\rangle$ of T and is a subset of S. $\langle E', R\rangle$ is also a preferred restricted four-valued extension. □

4 Discussions

4.1 Connection with Reiter's Default Logic

Restricted four-valued default logic enhances the flexibility of default logic. On the other hand, if default extensions are consistent, we should not make any of them invalid. It is even better if we do not accept any out of them either.

Fortunately, the restricted four-valued extensions have the classical property, which can be formalized by the following theorem.

Theorem 4. *Let T be a default theory. E is a consistent default extension of T iff $\langle E, \emptyset\rangle$ is a restricted four-valued extension of T.*

Proof. According to the difference between default extension and restricted four-valued extension, we need to prove:

1. E is consistent iff $E \not\models_{\emptyset} \mathbf{f}$;
2. $Th_{\emptyset}(E) = Th(E)$;
3. for any formula α and formula set Σ, $\Sigma \models_2 \alpha$ iff $\Sigma \models_{\emptyset} \alpha$; and
4. for any formula β and formula set Σ, $E \not\models_2 \neg\beta$ iff $E \not\models_{\emptyset} \beta \supset \mathbf{f}$.

which are all corollaries of Proposition 1. □

Now we can see why the preferred restricted four-valued extensions are intuitive: the classical extensions are always preferred, and the only preferred.

Corollary 1. *Let T be a default theory which has a consistent default extension. E is a default extension of T iff $\langle E, \emptyset\rangle$ is a preferred restricted four-valued extension of T.*

Proof. Combine Theorem 4 and the following fact:

if $\langle E, \emptyset\rangle$ is a restricted four-valued extension of T, then all preferred restricted four-valued extension of T are restricted by $\emptyset$. □

Theorem 4 and Corollary 1 show that (preferred) restricted four-valued default logic is an expansion to Reiter's default logic. In fact, Reiter's default extensions are only distinguishable with preferred restricted four-valued default extensions when there is no nontrivial default extension. So we can safely replace Reiter's default extensions with preferred restricted four-valued extensions.

4.2 Connection with Preferred Four-Valued Logic

Restricted four-valued default logic is not only a default logic, but also four-valued. The following theorem reveals that the four-valued consequence relation $\models^4_{cl}$ can be treated as a special case of restricted four-valued skeptical entailment.

Theorem 5. *$W \models^4_{cl} \phi$ iff all preferred restricted four-valued extensions of default theory $T = (W, \emptyset)$ entail ϕ.*

Proof. Denote M as the model set of all minimal four-valued model of W.

For any model $m \in M$, let $S(m) = \{a \in \mathcal{A} | m(a) \notin \{\mathbf{t}, \mathbf{f}\}\}$, i.e. m is exactly restricted by $S(m)$. We denote $M'(m) = \{n \in M | S(n) = S(m)\}$ as the set of minimal models which share the same restricted set as m. Since $m \in M'(m)$, we know that $M = \bigcup_{m \in M} M'(m)$. As a result, $W \models^4_{cl} \phi$ iff $M'(m) \models \phi$ for all $m \in M$.

Note that m is already one four-valued model of W and restricted by $S(m)$, we know that $W \not\models_{S(m)} \mathbf{f}$. According to the definition of restricted four-valued extension, $\langle W, S(m)\rangle$ is a restricted four-valued extension of T. We call this extension be generated by m and denote it as $E(m)$.

In fact, all models of W restricted by $S(m)$ are minimal models. Otherwise, there would be a minimal model n restricted by R and $R \subsetneq S(m)$, which contradicts with m is minimal. So $M'(m) = \{n | n \mathrm{is a model of} W \mathrm{restricted by} S(m)\}$. As a result, $E(m) \models \phi$ iff $M'(m) \models \phi$.

Now we want to show the equivalent relation between preferred restricted four-valued extensions and generated extensions. We prove it in two directions:

1. $E(m)$ is preferred for any $m \in M$. If $E(m)$ is not preferred, then there is a restricted four-valued extension $\langle E, R\rangle$ of default theory T and $R \subsetneq S(m)$. Since $E \not\models_R \mathbf{f}$, E has a four-valued model m' restricted by R. Because $R \subsetneq S(m)$, m' is a four-valued model of W and is more consistent than m, which contradicts with m is minimal.
2. Every preferred restricted four-valued extension $\langle E, S\rangle$ is generated by some $m \in M$. Since $E \not\models_S \mathbf{f}$, there is a four-valued model n of E restricted by S. If $n \in M$, let m be n itself. Otherwise, there is a minimal model $m \in M$ which is more consistent than n. In both case m is restricted by S. So $S(m) \subseteq S$. Because both $\langle E, S\rangle$ and $E(m) = \langle W, S(m)\rangle$ are preferred restricted four-valued extensions, we also know that $S(m) \subsetneq S$. Therefore, $S = S(m)$.

Altogether, we show that $W \models^4_{cl} \phi$ iff $M'(m) \models \phi$ for all $m \in M$, iff $E(m) \models \phi$ for all $m \in M$, iff all preferred restricted four-valued extensions entail ϕ. $\square$

5 Calculate Restricted Four-Valued Extensions

To compute the restricted four-valued extensions, we introduce the formula transformation proposed in [4]. The main purpose of this approach is to simulate four-valued reasoning by classical reasoning, which can be achieved by separating the truth relation of a formula and its negation. The technique details have been explained in [2,4,5].

Definition 8. *For any atom $p \in \Sigma$ and formula $\phi, \psi \in L$, define inductively:*

- $\overline{\mathbf{t}}^+ = \mathbf{t}$, $\overline{\mathbf{t}}^- = \mathbf{f}$, $\overline{\mathbf{f}}^+ = \mathbf{f}$, $\overline{\mathbf{f}}^- = \mathbf{t}$;
- $\overline{p}^+ = p^+$, $\overline{p}^- = p^-$;
- $\overline{\neg\phi}^+ = \overline{\phi}^-$, $\overline{\neg\phi}^- = \overline{\phi}^+$;
- $\overline{\phi \vee \psi}^+ = \overline{\phi}^+ \vee \overline{\psi}^+$, $\overline{\phi \vee \psi}^- = \overline{\phi}^- \wedge \overline{\psi}^-$;
- $\overline{\phi \wedge \psi}^+ = \overline{\phi}^+ \wedge \overline{\psi}^+$, $\overline{\phi \wedge \psi}^- = \overline{\phi}^- \vee \overline{\psi}^-$;
- $\overline{\phi \supset \psi}^+ = \neg\overline{\phi}^+ \vee \overline{\psi}^+$, $\overline{\phi \supset \psi}^- = \overline{\phi}^+ \wedge \overline{\psi}^-$;

Theorem 6 ([4]). $\Sigma \models_4 \phi$ *iff* $\overline{\Sigma}^+ \models \overline{\phi}^+$.

The following theorem is a restricted four-valued version of Theorem 6.

Theorem 7. *For any formula set E and formula ϕ, let $\overline{E}^+_S = \{\overline{\phi}^+ | \phi \in E\} \cup \{p^+ \leftrightarrow \neg p^- | p \notin S\})$. $E \models_S \phi$ iff $\overline{E}^+_S \models \overline{\phi}^+$.*

Proof. According to Theorem 1 and 6, we have

$E \models_S \phi$ iff $E \cup \bigcup_{p \notin S}\{p \vee \neg p, (p \wedge \neg p) \supset \mathbf{f}\} \models_4 \phi$, iff $\overline{E}^+ \bigcup_{p \notin S}\{\overline{p}^+ \vee \overline{p}^-, \neg\overline{p}^+ \vee \neg\overline{p}^-\} \models \overline{\phi}^+$, iff $\overline{E}^+_S \models \overline{\phi}^+$. □

In [11], they construct their paraconsistent logic by transforming proposition theories to default theories after applying signed transformation. In contrast, we want to apply our signed transformation on default theories.

Definition 9. *For any default rule* $d = \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$*, let* $\overline{d}^+ = \frac{\overline{\alpha}^+:\overline{\beta_1}^+,\ldots,\overline{\beta_n}^+}{\overline{\gamma}^+}$.

Let $T = (D, W)$ *be a default theory. The transformed default theory* $\overline{T}^+_S$ *of* T *restricted by* S*, is defined as* $\overline{T}^+_S = (\overline{D}^+, \overline{W}^+_S)$*, where* $\overline{D}^+ = \{\overline{d}^+ | d \in D\}$.

Theorem 8. *Let* $T = (D, W)$ *be a default theory.* E *is a restricted four-valued extension of* T *restricted by* S*, iff* $\overline{E}^+_S$ *is a consistent extension of* $\overline{T}^+$.

Proof. According to Theorem 7 and the definition of restricted four-valued extension, we only need to prove that $E \not\models_S \beta \supset \mathbf{f}$ iff $\overline{E}^+_S \not\models \neg\overline{\beta}^+$, which is also proved by Theorem 7. □

Theorem 8 represents a feasible approach to convert a restricted four-valued default logic problem to the corresponding default logic problem.

6 Related Works

As an important nonmonotonic logic, Reiter's default logic has been widely used in knowledge representation. In [11], their signed system is paraconsistent by using default logic to restore information from inconsistent theories. In [6], they also use default logic to process inconsistent knowledge. Conversely, we introduce paraconsistency to default logic. In [17], they develop a novel framework to deal with default reasoning with fuzzy and uncertain information. In this paper, we focus on handling inconsistent and incoherent information.

Reiter's default logic has many variants presented by different researchers. In justified default extensions [16], constrained default extensions [21] and cumulative default extensions [12], they modify the definition of extensions to ensure their existences. However, they have different semantics from default logic and still cannot deal with inconsistencies.

To take advantage of tolerance on inconsistencies, paraconsistent variants of default logic are represented by several researchers. Among these, question marked logic [1] is a generalization of the inconsistent default logic [19] which is based on Da Costa's paraconsistent logic [13]. The basic idea is annotating formulas with a hierarchy of meta-levels by question marks, and preventing trivialization by paraconsistent logic. Also, its semantics is different from Reiter's.

The bi-default logic [14] is based on a signed system and proposed for handling inconsistencies by splitting default theories to two consistent parts. The four-valued default logic [23] is based on Belnap's four-valued logic and can be treated as an expansion of four-valued logic in k-minimally reasoning. However, these approaches focus on eliminating inconsistencies but not on preventing incoherences. Also, our preferred extensions can infer stronger consequences than the k-minimal models. For example, the law of excluded middle can be infered from the only preferred restricted four-valued extension of default theory $(\emptyset, \emptyset)$, but cannot be concluded in its k-minimal models.

The fault-tolerant default logic [15] is constructed on its own paraconsistent reasoning relation $\vdash_{mc}$, and succeeds in handling inconsistencies and incoherences simultaneously. Unfortunately, it still needs to be clarified that how to compute its extensions. In contrast, we have provided a transformation from our logic to classical default logic.

By using a nonmonotonic underlying logic based on a 16-valued lattice, the annotated default logic [22] also guarantees the existence of nontrivial extensions and characterizes Reiter's default extensions in its extensions. By contrast, our default logic does not only take these advantages, but also keeps our underlying logic monotonic. As a result, our default logic holds some useful properties such like Proposition 4, which do not hold if the underlying logic is nonmonotonic.

An approach to the trivial extension problem by transforming default theories with minimally unsatisfiable subformulas is also presented in [10]. The transformed default theories still hold some information from original ones. Despite that, this approach does not handle incoherences, and some propositions only hold on normal default theories but not general ones. Even more, some information may be lost in transformation. As a comparison, our extensions are based on general default theories. We also ensure that the facts W of a default theory $T = (D, W)$ always hold in every restricted four-valued extensions.

7 Conclusion

In this paper we present our restricted four-valued default logic based on the monotonic restricted four-valued logic. In our default logic, we guarantee the existence of nontrivial extensions of default theories with inconsistent or incoherent knowledge. We also have showed that our default logic is an expansion of both default logic and preferred four-valued logic. To compute restricted four-valued extensions, a signed formula transformation is also presented.

In future, we would consider other features of restricted four-valued default logic and try to extend our work in first-order logic.

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