Maximum Lower Bound Estimation of Fuzzy Priority Weights from a Crisp Comparison Matrix

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Abstract. In Interval AHP, our uncertain judgments are denoted as interval weights by assuming a comparison as a ratio of the real values in the corresponding interval weights. Based on the same concept as Interval AHP, this study denotes uncertain judgments as fuzzy weights which are the extensions of the interval weights. In order to obtain the interval weight for estimating a fuzzy weight, Interval AHP is modified by focusing on the lower bounds of the interval weights similarly to the viewpoint of belief function in evidence theory. It is reasonable to maximize the lower bound since it represents the weight surely assigned to one of the alternatives. The sum of the lower bounds of all alternatives is considered as a membership value and then the fuzzy weight is estimated. The more consistent comparisons are given as a result of the higher-level sets of fuzzy weights in a decision maker's mind.

Keywords: Interval AHP · Fuzzy weight · Interval weight · Membership function · Uncertainty

1 Introduction

AHP (Analytic Hierarchy Process) is a useful tool to extract a decision maker's preference from his/her intuitive judgments [9]. When a decision maker gives the comparisons of all pairs of alternatives, his/her preferences are obtained as the weights of alternatives. It is easy for a decision maker to give his/her intuitive judgments as pairwise comparisons since s/he focuses on comparing a pair of alternatives without caring for the other alternatives. As a result, an alternative is compared several times and the given comparisons are seldom consistent each other.

The inconsistency of the given comparisons is well-known and discussed a lot in AHP. One of the ways to treat inconsistency is to introduce the consistency index and distinguish whether the given comparisons are too inconsistent [1,8]. On the other hand, Interval AHP [10,11] takes the comparisons possibly into consideration, instead of distinguishing them. It is based on the idea that a

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decision maker does not perceive a precise weight of an alternative but a range of its weight in his/her mind. In Interval AHP, a comparison is considered as a part of the rational decision so that the interval weights are obtained so as to include the given comparisons. In short, AHP and Interval AHP induce plausible and possible preferences from the given judgments, respectively.

Based on the same idea as Interval AHP, this paper assumes that our judgments are uncertain and obtains the possible preferences reflecting the uncertainty. Instead of the interval weights, such uncertain judgments are denoted as the fuzzy weights. Therefore, the method to estimate the fuzzy weights from the given crisp comparisons is proposed. In some fuzzy approaches in AHP [2,4,7], a fuzzy or interval comparison matrix is used. From the viewpoint of uncertainty, several models of Interval AHP have been proposed [6]. This paper modifies Interval AHP by focusing on the lower bound of an interval weight since it represents the weight surely assigned to an alternative similarly to the viewpoint of belief function in evidence theory [5]. The left weight is considered as ignorance since it is common of some alternatives and possible to be assigned to more than two alternatives. The upper bound of an interval weight includes such a weight as possibly assigned to an alternative. It is reasonable to assign the weight to one of the alternatives as much as possible so that the lower bounds are maximized. Then, the sum of the lower bounds of all alternatives is considered as a membership value of a fuzzy weight in a decision maker's mind. The more consistent the given comparisons are, the higher the membership value becomes. The decision maker gives the comparisons based on this certain level sets of the fuzzy weights of alternatives.

The given comparison is represented as the ratio of the weights of the corresponding alternatives. In Interval AHP, the inclusion relation between a comparison and the interval ration of its corresponding interval weights are used to obtain the interval weights. In the proposed model, the relation between a comparison and a ratio of weights is reconsidered. It assumes that the weight of an alternative is estimated by the corresponding comparisons and the weights of the other alternatives. The interval weight of an alternative is estimated by the other alternatives and the relation between the weight and its estimations are used to obtain the interval weights. Since the proposed model is based on the lower bound, the estimations by the lower bounds of the others are used. Such lower bounds of estimations are compared to the upper bound of an interval weight so that the interval weight and its estimations are common.

In order to estimate fuzzy weights from their certain level sets which are interval weights focusing on their lower bounds, it assumes some membership values, i.e., the sums of the lower bounds from 0 to 1, in addition. The relation between an interval weight and its estimations are modified depending on whether the membership value is higher than the certain level or not. In the higher case, the weight is forced to be assigned to one of the alternatives so that a weight and its estimations may not be common and the deficiency is minimized. While, in the lower case, they are always common so that their differences can be minimized. In each case, the interval weights are obtained based on the proposed model as the respective level sets of the fuzzy weight. Then, the fuzzy weight is estimated such interval weights as its representative level sets.

This paper is organized as follows. In the next section, as the modification of Interval AHP, the lower bound based Interval AHP which focuses on the lower bounds is proposed. Then, in section 3, the fuzzy weights are estimated from some representative interval weights by the proposed model as their level sets. Section 4 shows two numerical examples and discusses the results. The last section is the conclusion.

2 Lower Bound Based Interval AHP

A decision maker gives the pairwise comparisons on n alternatives as follows.

$$A = \begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & 1 \end{bmatrix}, \tag{1}$$

where a_{ij} is his/her intuitive judgment on the importance ratio of alternative i to that of alternative j. The comparisons are identical and reciprocal as $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$. The comparisons are consistent if and only if

$$a_{ij} = a_{il}a_{lj}, \forall i, j, l. \tag{2}$$

However, (2) is seldom satisfied since an alternative is compared to the other $(n-1)$ alternatives.

In AHP, the weights of alternatives are obtained from (1) by eigenvector method $A\boldsymbol{w} = \lambda\boldsymbol{w}$, where $\boldsymbol{w} = (w_1, \ldots, w_n)^T$ is the eigenvector corresponding to principal eigenvalue. The weights are normalized such that $\sum_i w_i = 1$. The weight is assigned to one of the alternatives without ignorance so that the plausible preferences are obtained by AHP.

In Interval AHP, the interval weight $W_i = [w_i^L, w_i^R]$ which includes the given comparisons is obtained by the following LP problem [10,11]. It is assumed that the given comparisons are inconsistent since the weights of alternatives in a decision maker's mind are uncertain. A decision maker may use a real value in interval weight W_i in giving comparison a_{ij}, where $j > i$.

$$\begin{array}{rl} \min & \sum_i (w_i^R - w_i^L), \\ \text{s.t.} & \sum_{i \neq j} w_i^R + w_j^L \geq 1, \forall j, \\ & \sum_{i \neq j} w_i^L + w_j^R \leq 1, \forall j, \\ & \dfrac{w_i^L}{w_j^R} \leq a_{ij} \leq \dfrac{w_i^R}{w_j^L}, \forall i, j, j > i, \\ & w_i^L \geq \varepsilon, \forall i, \end{array} \tag{3}$$

where the first two kinds of constraints are for the normalization of intervals based on interval probability [3,12]. They are the interval counterparts of the ordinal crisp probability. When the weights are real values as $w_i^R = w_i^L = w_i, \forall i$, two inequalities are replaced into $\sum_i w_i = 1$. The redundancy of the intervals to make the sum of any real values in the intervals be 1 is excluded. For instance, the 1st inequality for j requires w_j^L not to be too small. The next inequalities for a_{ij} are the inclusion constraints. They require the obtained interval weights to include the given comparisons as

$$a_{ij} \in \frac{W_i}{W_j} = \frac{[w_i^L, w_i^R]}{[w_j^L, w_j^R]} = \left[\frac{w_i^L}{w_j^R}, \frac{w_i^R}{w_j^L}\right], \tag{4}$$

where the fraction of intervals is defined as its maximum range. By minimizing the widths of the interval weights, both bounds in the right side of (4) become the closest to the give comparison in the left side. In other words, the primal objective is to minimize uncertainty of the interval weight. If the comparisons are perfectly consistent as in (2), the weights are obtained as real values $w_i^R = w_i^L, \forall i$ and they equal to those by eigenvector or geometric mean method in AHP. On the other hand, the more inconsistent the given comparisons are, the wider the obtained interval weights become. The lower bound of the interval weight is considered as the weight surely assigned to one of the alternatives. While, its upper bound includes the possibly assigned weight in addition and such weight is a common weight of some alternatives. As the surely assigned weight decreases, the ignorance which is the possibly assigned weight increases.

In the same concept as Interval AHP, this paper assumes that our judgments are often uncertain and then they are represented as fuzzy weights of alternatives, instead of interval weights. The fuzzy weight can be considered as a set of interval weights. In order to estimate a fuzzy weight, some representative interval weights are used. The core interval weights, based on which a decision maker gives the comparisons, are obtained as follows. We revisit Interval AHP by focusing on the lower bound of interval weight. It is reasonable to focus more on the weight surely assigned to one of the alternatives than the weight assigned to more than two alternatives. Interval AHP by (3) is modified so as to be suitable for fuzzy weight estimation and we name the proposed model lower bound based Interval AHP. In Interval AHP, the inclusion relation as in (4) is considered based on the relation between the given comparison and the corresponding weights as $a_{ij} = \frac{w_i}{w_j}$. In the proposed lower bound based Interval AHP, based on the same relation, the weight of an alternative is estimated by the weight of the other alternative as $w_i = a_{ij} w_j$. In case of the crisp weights, there are $n-1$ estimations of the weight of alternative i as $w_i' = a_{ij} w_j, \forall j \neq i$. When the weights are extended to interval $W_i = [w_i^L, w_i^R]$, its estimations are intervals $W_i' = [a_{ij} w_j^L, a_{ij} w_j^R], \forall j \neq i$. Since the proposed model is based on the lower bound of the interval weight, the estimations by the lower bounds $a_{ij} w_j^L, \forall j \neq i$ are used. The relation between the interval weight of alternative i, W_i, and its estimation, W_i', by the other alternative $j \neq i$ is as follows.

$$a_{ij} w_j^L \leq w_i^R, \forall j \neq i, \tag{5}$$

which are satisfied if the interval weight and its estimation have at least a value in common.

By replacing the inclusion relation (4) in (3) into the estimation (5) and maximizing the lower bound w_i^L, the lower bound based Interval AHP is formulated as the following LP problem.

$$\begin{aligned} &\alpha = \max \textstyle\sum_i w_i^L, \\ &s.t.\ \textstyle\sum_{i \neq j} w_i^R + w_j^L \geq 1, \forall j, \\ &\qquad \textstyle\sum_{i \neq j} w_i^L + w_j^R \leq 1, \forall j, \\ &\qquad a_{ij} w_j^L \leq w_i^R, \forall i, j, j \neq i, \end{aligned} \tag{6}$$

where the optimal objective function value α represents the weight surely assigned to one of the alternatives and the left weight $1 - \alpha$ may be assigned to some alternatives. The lower and upper bounds of the interval weight, w_i^L and w_i^R, represent the weights surely and possibly assigned to alternative i, respectively. In this way, the possible preferences are obtained by lower bound based Interval AHP.

Let us denote the maximum surplus of the upper bound of alternative i from its estimations by the other alternatives $\forall j \neq i$ in (5) as p_i. The surplus should be minimized and (6) is rewritten as follows.

$$\begin{aligned} &\max\ (\textstyle\sum_i w_i^L - \varepsilon \sum_i p_i), \\ s.t.\ &\textstyle\sum_{i \neq j} w_i^R + w_j^L \geq 1, \forall j, \\ &\textstyle\sum_{i \neq j} w_i^L + w_j^R \leq 1, \forall j, \\ &0 \leq w_i^R - a_{ij} w_j^L \leq p_i, \forall i, j, j \neq i, \end{aligned} \tag{7}$$

where ε is a small positive value so that the surplus of the interval weight from its estimations is minimized secondarily.

When the given comparisons are perfectly consistent as in (2), the optimal solutions of (7) are $w_i = w_i^R = w_i^L, \forall i$ and then $\alpha = 1$. The more inconsistent the comparisons are, the less α becomes.

3 Estimating Fuzzy Weight

The weight of alternative i in a decision maker's mind is denoted as fuzzy weight $\tilde{W}_i$. Let us denote the membership function of fuzzy weight $\tilde{W}_i$ as $\mu_{\tilde{W}_i}$. In this section, the fuzzy weight is estimated by its α-level set obtained by (7) in the previous section. A decision maker gives the comparisons based on α-level sets of the fuzzy weights in his/her mind. The sum of the weights of all alternatives surely assigned to one of the alternatives represents a membership value of a fuzzy weight. It means that $\alpha = \mu_{\tilde{W}_i}(w_i^L) = \mu_{\tilde{W}_i}(w_i^R), \forall i$, i.e., α-level sets of fuzzy weights $\tilde{W}_i, \forall i$ are intervals $[w_i^L, w_i^R], \forall i$ by (7). They are core interval weights to estimate the fuzzy weights. The fuzzy weight consists of some representative interval weights which are their level sets. As far as $\mu_{\tilde{W}_i}$ is less than $\alpha = \sum_i w_i^L$ by (7), the relation between an interval weight and its estimations satisfy (5), however, in the other cases the relation is not satisfied. Therefore, we assume β_k

and γ_k, where $\alpha = \beta_0 < \beta_1 < \ldots < \beta_m = 1$ and $\alpha = \gamma_0 > \gamma_1 > \ldots > \gamma_l \geq 0$, respectively, for given m and l. As m and l increase, the more precise estimations can be done.

First, let assume $\mu_{\tilde{W}_i} = \beta_k$, which requires that the weight surely assigned to an alternative is more than β_k. Because of $\alpha \leq \beta_k$, they cannot satisfy (5). The maximum deficiency of interval weight W_i from its estimations, q_i, should be minimized.

$$\begin{aligned} & \min \textstyle\sum_i q_i, \\ s.t.\ & \beta_k \leq \textstyle\sum_i w_i^{L\beta_k}, \\ & w_i^{L\beta_{k-1}} \leq w_i^{L\beta_k}, w_i^{R\beta_k} \leq w_i^{R\beta_{k-1}}, \forall i \\ & \textstyle\sum_{i \neq j} w_i^{R\beta_k} + w_j^{L\beta_k} \geq 1, \forall j, \\ & \textstyle\sum_{i \neq j} w_i^{L\beta_k} + w_j^{R\beta_k} \leq 1, \forall j, \\ & -q_i \leq w_i^{R\beta_k} - w_j^{L\beta_k} a_{ij}, \forall i, j, j \neq i, \\ & q_i \geq 0, \forall i, \end{aligned} \tag{8}$$

where $[w_i^{L\beta_k}, w_i^{R\beta_k}], \forall i$ are the variables and $[w_i^{L\beta_{k-1}}, w_i^{R\beta_{k-1}}], \forall i$ are obtained previously. By repeating solving (8) from $k = 1$ to m, sequentially m representative interval weights are obtained. They are higher-level sets of the fuzzy weight than α-level one by (7).

In case of $m = 1$, (8) is reduced to the following problem where $\beta_1 = \mu_{\tilde{W}_i} = 1$ indicates crisp weights as $w_i = w_i^{L\beta_1} = w_i^{R\beta_1}, \forall i$. The weight is surely assigned to one of the alternatives so that there is no ignorance which is possibly assigned weight to an alternative.

$$\begin{aligned} & \min \textstyle\sum_i q_i, \\ s.t.\ & \textstyle\sum_i w_i = 1, \\ & w_i^L \leq w_i \leq w_i^R, \forall i, \\ & -q_i \leq w_i - a_{ij} w_j, \forall i, j, j \neq i, \\ & q_i \geq 0, \forall i, \end{aligned} \tag{9}$$

where $w_i^L = w_i^{L\beta_0}, \forall i$ and $w_i^R = w_i^{R\beta_0}, \forall i$ are the optimal solutions of (7). The weight is forced to be assigned to one of the alternatives without ignorance, instead of allowing some estimations of an alternative to be more than its crisp weight. The crisp weights $w_i, \forall i$ make their estimations be the closest to them and be included in α-level set of the fuzzy weights denoted as interval weights $W_i, \forall i$ by (7).

Next, let assume $\mu_{\tilde{W}_i} = \gamma_k$, which requires that the weight surely assigned to one of the alternatives is at most γ_k. Because of $\alpha \geq \gamma_k$, (5) is satisfied and an interval weight and its estimations are always common. Since the upper bound of the interval weight of an alternative is always more than the estimations by the lower bounds of the others, the constraint of the sum of the lower bounds is added into (7). The problem to obtained interval weights $[w_i^{L\gamma_k}, w_i^{R\gamma_k}], \forall i$, is formulated as follows.

$$\begin{aligned} &\max\, (\textstyle\sum_i w_i^{L\gamma_k} - \varepsilon \sum_i p_i),\\ s.t.\ &\textstyle\sum_i w_i^{L\gamma_k} \leq \gamma_k,\\ &\varepsilon \leq w_i^{L\gamma_k} \leq w_i^{L\gamma_{k-1}}, w_i^{R\gamma_{k-1}} \leq w_i^{R\gamma_k}, \forall i\\ &\textstyle\sum_{i\neq j} w_i^{R\gamma_k} + w_j^{L\gamma_k} \geq 1, \forall j,\\ &\textstyle\sum_{i\neq j} w_i^{L\gamma_k} + w_j^{R\gamma_k} \leq 1, \forall j,\\ &0 \leq w_i^{R\gamma_k} - w_j^{L\gamma_k} a_{ij} \leq p_i, \forall i, j, j \neq i, \end{aligned} \tag{10}$$

where $p_i, \forall i$ are the surpluses of the weights from their estimations and in the same way as (8), $[w_i^{L\gamma_{k-1}}, w_i^{R\gamma_{k-1}}], \forall i$ are obtained previously. By repeating solving (10) from $k = 1$ to l, sequentially l representative interval weights are obtained and they are lower-level sets of the fuzzy weight than its α-level set.

In case of $\gamma_l = 0$, the lower bounds of 0-level sets of the fuzzy weights are 0 as $w_i^{L\gamma_l} = w_i^{L0} = 0, \forall i$. As a result, the estimations are also 0, as $a_{ji} w_i^{L0} = 0, \forall i$, whatever a_{ij} is. In order not to ignore and to reflect the given comparison a_{ij} to the weight of alternative i, its lower bound is assumed as $w_i^{L\gamma_l} = w_i^{L0} = \varepsilon, \forall i$, where ε is a small positive number so that $\gamma_l = n\varepsilon$. Then, in case of $l = 1$, (10) is reduced to the following problem to obtain the upper bounds $= w_i^{R\gamma_1} = w_i^{R0}, \forall i$.

$$\begin{aligned} &\min \textstyle\sum_i w_i^{R0} - \varepsilon \sum_i r_i,\\ s.t.\ &w_i^R \leq w_i^{R0}, \forall i\\ &\textstyle\sum_{i\neq j} w_i^{R0} + \varepsilon \geq 1, \forall j,\\ &\textstyle\sum_{i\neq j} \varepsilon + w_j^{R0} \leq 1, \forall j,\\ &0 \leq w_i^{R0} - \varepsilon a_{ij} \leq p_i, \forall i, j, j \neq i, \end{aligned} \tag{11}$$

where $w_i^{L\gamma_1} = w_i^{L0}, \forall i$ are replaced into ε and the obtained interval weights $[\varepsilon, w_i^{R0}], \forall i$ should include α-level sets of the fuzzy weights $[w_i^L, w_i^R], \forall i$ by (7). The weight surely assigned to an alternative is reduced to $n\varepsilon \leq \alpha$ and the ignorance is increased to $1 - n\varepsilon$.

For simplicity, assuming $m = l = 1$, the membership function of fuzzy weight $\tilde{W}_i$ is illustrated as in Figure 1. It is estimated by three representative interval weights such as its 0-level set $[\varepsilon, w_i^{R0}]$, its α-level set $[w_i^L, w_i^R]$ as a core interval weight, and its 1-level set w_i as follows.

$$\mu_{\tilde{W}_i}(x) = \begin{cases} \dfrac{\alpha}{w_i^L} x, & 0 < x \leq w_i^L\\ \dfrac{1-\alpha}{w_i - w_i^L}(x - w_i^L) + \alpha, & w_i^L \leq x \leq w_i\\ \dfrac{\alpha - 1}{w_i^R - w_i}(x - w_i) + 1, & w_i \leq x \leq w_i^R\\ \dfrac{-\alpha}{w_i^{R0} - w_i^R}(x - w_i^R) + \alpha, & w_i^R \leq x \leq w_i^{R0}\\ 0, & w_i^{R0} \leq x \end{cases}. \tag{12}$$

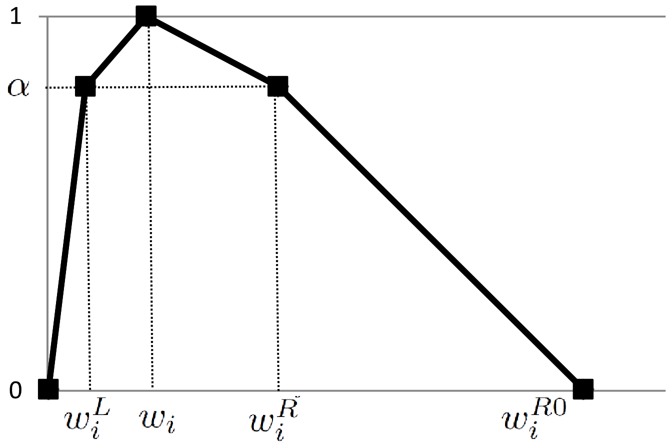

Fig. 1. Membership function of fuzzy weight of alternative i

The interval weights by any level-sets of the fuzzy weights by (12) are normalized so that they are interval probabilities and satisfy the 1st and 2nd constraints in (3). For instance, the 1st constraint in case of $\alpha \leq \beta \leq 1$ is verified as follows. Assume interval weights $[w_i^{L\beta}, w_i^{R\beta}], \forall i$, where $\mu_{\tilde{W}_i}(w_i^{L\beta}) = \mu_{\tilde{W}_i}(w_i^{R\beta}) = \beta$. Their bounds are denoted by the 2nd and 3rd functions in (12) as follows.

$$\beta = \frac{1-\alpha}{w_i - w_i^L}(w_i^{L\beta} - w_i^L) + \alpha \leftrightarrow w_i^{L\beta} = \frac{1-\beta}{1-\alpha}w_i^L + \frac{\beta-\alpha}{1-\alpha}w_i, \forall i,$$
$$\beta = \frac{\alpha-1}{w_i^R - w_i}(w_i^{R\beta} - w_i) + 1 \leftrightarrow w_i^{R\beta} = \frac{1-\beta}{1-\alpha}w_i^R + \frac{\beta-\alpha}{1-\alpha}w_i, \forall i,$$

where $[w_i^L, w_i^R], \forall i$ by (7) satisfy the 1st constraint in (3) so that $\sum_{i\neq j} w_i^R + w_j^L \geq 1, \forall j$ and $w_i, \forall i$ by (9) satisfy $\sum_i w_i = 1$. The 1st constraint in (3) for β-level sets of the fuzzy weights $[w_i^{L\beta}, w_i^{R\beta}], \forall i$ is verified as follows.

$$\begin{aligned}
&\textstyle\sum_{i\neq j} w_i^{R\beta} + w_j^{L\beta} \\
&\quad = \textstyle\sum_{i\neq j}\left(\frac{1-\beta}{1-\alpha}w_i^R + \frac{\beta-\alpha}{1-\alpha}w_i\right) + \left(\frac{1-\beta}{1-\alpha}w_j^L + \frac{\beta-\alpha}{1-\alpha}w_j\right) \\
&\quad = \frac{1-\beta}{1-\alpha}(\textstyle\sum_{i\neq j} w_i^R + w_j^L) + \frac{\beta-\alpha}{1-\alpha}(\textstyle\sum_i w_i) \geq 1, \forall j.
\end{aligned} \tag{13}$$

Similarly, the 2nd constraint in (3) for β-level sets is verified. In case of $0 < \gamma \leq \alpha$, in the same way, it is verified that γ-level sets of the fuzzy weights satisfy the 1st and 2nd constraints in (3). Therefore, any-level sets of the fuzzy weights which are denoted as interval weights are normalized from the viewpoint of interval probability.

In general, the membership function of fuzzy weight $\tilde{W}_i$ is denoted by the representative interval weights by (8) and (10) as follows.

$$\mu_{\tilde{W}_i}(x) = \begin{cases} \dfrac{\gamma_k - \gamma_{k-1}}{w_i^{L\gamma_{k-1}} - w_i^{L\gamma_k}}(x - w_i^{L\gamma_k}) + \gamma_k, & w_i^{L\gamma_k} \leq x \leq w_i^{L\gamma_{k-1}},\ k = l, \ldots, 1 \\ \dfrac{\beta_k - \beta_{k-1}}{w_i^{L\beta_k} - w_i^{L\beta_{k-1}}}(x - w_i^{L\beta_{k-1}}) + \beta_{k-1}, & w_i^{L\beta_{k-1}} \leq x \leq w_i^{L\beta_k},\ k = 1, \ldots, m \\ \dfrac{\beta_{k-1} - \beta_k}{w_i^{R\beta_{k-1}} - w_i^{R\beta_k}}(x - w_i^{R\beta_k}) + \beta_k, & w_i^{R\beta_k} \leq x \leq w_i^{R\beta_{k-1}},\ k = m, \ldots, 1 \\ \dfrac{\gamma_k - \gamma_{k-1}}{w_i^{R\gamma_k} - w_i^{R\gamma_{k-1}}}(x - w_i^{R\gamma_{k-1}}) + \gamma_{k-1}, & w_i^{R\gamma_{k-1}} \leq x \leq w_i^{R\gamma_k},\ k = 1, \ldots, l \\ 0, & w_i^{R\gamma_l} \leq x \end{cases} \tag{14}$$

The fuzzy weights denoted as in Figure 1 are obtained from a crisp comparison matrix A in (1) reflecting the uncertainty in A. They reflect the possibilities of the given information with membership values. When a rigid order of alternatives is needed, the interval weight by a high-level set of a fuzzy weight is used and a crisp weight is found as a focal point in case of 1-level set. While, when the possibility of an alternative is a concern, the low-level set is useful.

4 Numerical Examples

Two decision makers A and B give the pairwise comparison matrices on 4 alternatives as shown in Table 1. By (7), the core interval weights for fuzzy weight estimation are obtained and they are shown next to each matrix. The weight is assigned to each alternative as much as possible. The sum of lower bounds of the interval weights by decision maker B, $\alpha_B = 0.929$, is more than that by decision maker A, $\alpha_A = 0.813$, so that B gives the comparisons based on higher membership value than A. These interval weights are assumed as 0.813-level sets and 0.929-level sets of their fuzzy weights, respectively. For comparison, the interval weights by (3), where the widths are minimized, are shown at the right column of Table 1. Their sums of the lower bounds are shown at the 1st rows and they are less than those by (7). It mentions that the more weight is surely assigned to one of the alternatives by the proposed (7) than (3). Since the lower bound is more suitable to represent a membership value than the width, Interval AHP is modified by focusing on the lower bounds of the interval weights and the proposed model is used to estimate a fuzzy weight.

In addition, the representative interval weights by $\beta_{A1} = 0.9$, $\beta_{A2} = 1$, $\gamma_{A1} = 0.5$, and $\gamma_{A2} = n\varepsilon = 0.004$ for decision maker A and those by $\beta_{B1} = 1$, $\gamma_{B1} = 0.5$, and $\gamma_{B2} = 0.004$ for decision maker B are obtained by (8) and (10). Then, the fuzzy weights by decision makers A and B are estimated by (14) and illustrated as in Figures 2 and 3, respectively. It is noted that the numbers of representatives, m and l, are arbitrary and the more they are, the more precisely a fuzzy weight is estimated. For instance, we can obtain 11 representative interval weights assuming membership values, β or γ, as 0, 0.1, ..., 0.9, and 1.0.

Table 1. Two comparison matrices

	A1	A2	A3	A4	α_A=0.813 by (7)	0.785 by (3)
A1	1	1/3	7	1/7	[0.037, 0.224]	[0.025, 0.230]
A2	3	1	6	4	[0.671, 0.671]	[0.689, 0.689]
A3	1/7	1/6	1	3	[0.032, 0.219]	[0.033, 0.115]
A4	7	1/4	1/3	1	[0.073, 0.260]	[0.038, 0.172]

	B1	B2	B3	B4	α_B=0.929 by (7)	0.914 by (3)
B1	1	7	1	9	[0.463, 0.534]	[0.444, 0.444]
B2	1/7	1	1/7	1/9	[0.015, 0.066]	[0.015, 0.063]
B3	1	7	1	3	[0.392, 0.463]	[0.406, 0.444]
B4	1/9	9	1/3	1	[0.059, 0.131]	[0.049, 0.135]

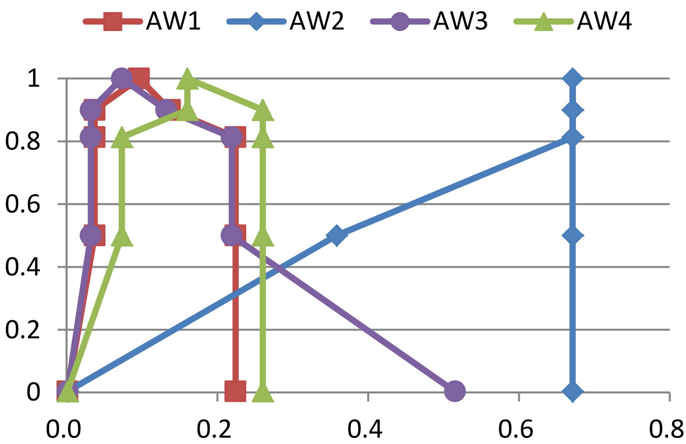

Fig. 2. Fuzzy weights by decision maker A

As for the fuzzy weights from the crisp comparisons by decision maker A, alternative A2 is apparently better than the other alternatives regardless of levels. Among the other alternatives, alternative A4 is a little better than alternatives A1 and A3 at 0.813-level. As higher the level becomes to 0.9 or 1, it becomes more apparent that A4 is better than them. The common weight of some alternatives is assigned more to alternative A4 than to alternatives A1 and A3. It may be because decision maker A potentially evaluates alternative A4 better but s/he is not very sure of it. While, at 0-level, where the weight is assigned to more than two alternatives, the possible weight of alternatives A3 increases, instead of the decrease of the weight surely assigned to alternative A2, because alternatives A2 and A3 are similar and substitutes in a sense.

As for the fuzzy weights from the crisp comparisons by decision maker B, at 0.929-level alternative B1 is better than B3, however, at 1-level their weights are the same. The weight surely assigned to B1 does not increase when the

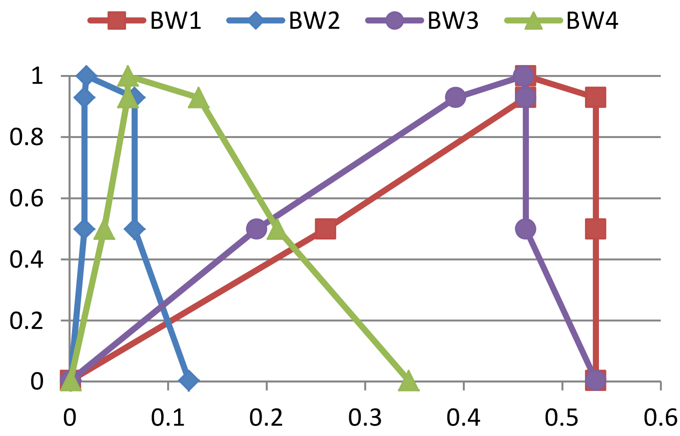

Fig. 3. Fuzzy weights by decision maker B

membership value is higher than 0.929. Most of the left weight, 0.069 of 0.071, is assigned to B3 at 1-level since decision maker B may not know alternative B3 well. S/he is sure of the weight of alternatives B1, B2 and B4 so that they are assigned little weight at 1-level sets of their fuzzy weights.

5 Conclusion

Based on the idea that the decision maker's judgments are uncertain, the fuzzy weights have been obtained from the given crisp comparisons. This paper proposed the lower bound based Interval AHP which obtains the interval weights of alternatives from the given crisp comparisons for fuzzy weight estimation. In the proposed model, the lower bound of an interval weight is focused on. It is reasonable to maximize the lower bound since it represents the weight surely assigned to one of the alternatives and an alternative is assigned the weight as much as possible. The left weight is considered as ignorance and it is common of some alternatives. The upper bound includes such a weight as possibly assigned to the alternative, in addition to the lower bound. For fuzzy weight estimation, the sum of the lower bounds of all alternatives is considered as a membership value of a fuzzy weight in a decision maker's mind. In other words, it is considered that the decision maker gives comparisons based on the certain level set of the fuzzy weight in his/her mind. The comparisons are more consistent each other when a decision maker gives them based on the higher membership values of his/her fuzzy weights. The sums of the lower bounds are assumed to some values from 0 to 1 so as to estimate a fuzzy weight. The relation between an interval weight and its estimations are modified depending on whether the value is higher than the certain level or not. For the higher-level set of the fuzzy weight, whose extreme case is a crisp weight, the maximum deficiency of a weight from its estimations is minimized, while for its lower-level set, where some weights are assigned to more than two alternatives, the maximum surplus of a weight from its estimations is minimized. Then, the fuzzy weight is estimated by the representative interval weights by the proposed lower bound based Interval AHP as its level sets.

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