Construction of Associative Functions for Several Fuzzy Logics via the Ordinal Sum Theorem

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Abstract. In this report, the ordinal sum theorem of semigroups is applied to construct logical operations for several fuzzy logics. The generalized form of ordinal sum for fuzzy logics on [0, 1] is defined in order to uniformly express several families of logical operations. Then, the conditions in ordinal sums for various properties of logical operations are presented: for examples, the monotonicity, the location of the unit element, the left/right-continuity, or and/or-likeness. Finally, some examples to construct pseudo-uninorms by the proposed method are illustrated.

Keywords: Ordinal sum · Pseudo-t-norms · Pseudo t-conorms · Pseudo-uninorms

1 Introduction

The concept of ordinal sums has been originated by Climescu [3], and then has been generalized by Clifford [1], [2] to a method for constructing a new semigroup from a given linearly-ordered system of semigroups. In the history of the research on fuzzy logical connectives as t-norms, t-conorms, and uninorms, the ordinal sum has been often appeared as representations of such operations [4]-[14].

In this paper, the authors challenge to reform the ordinal sum on [0,1] to a more general scheme as a common platform to construct fuzzy logical connectives in the broader sense: including non-commutative ones besides t-norms, t-conorms and uninorms. The results of this work would be useful for obtaining an associative operation suitable for human thinking/evaluation, in several applications such as information aggregation in diagnoses systems, construction of metrics based on fuzzy relation, constraint satisfaction in multicriteria decision making, and so on.

2 Origin of Ordinal Sum Theorem

Climescu [3] has introduced the original concept of ordinal sums which is a method to construct a new semigroup from a family of semigroups. According to Schweizer et al. [13], his definition of an ordinal sum is expressed as follows.

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Ordinal Sum Theorem by Climescu [3] and Schweizer et al. [13]
Let (A,F) and (B,F) be semigroups. If the sets A and B are disjoint and if H is the mapping defined on $(A \cup B) \times (A \cup B)$ by

$$H(x,y) = \begin{cases} F(x,y), & x \in A,\, y \in A, \\ x, & x \in A,\, y \in B, \\ y, & x \in B,\, y \in A, \\ G(x,y), & x \in B,\, y \in B, \end{cases} \tag{1}$$

then $(A \cup B, H)$ is a semigroup.

On the other hand, Clifford [1], [2] has introduced a more generalized definition of the same concept, and has named it an ordinal sum. The following theorem is the reformatted version by Klement et al.

Ordinal Sum Theorem by Clifford [1], [2], and Klement et al. [11]
Let $(A,\leq)$ with $A \neq \varnothing$ be a linearly ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta}$ is both the unit element of G_α and the annihilator of G_β, and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \beta < \alpha. \end{cases} \tag{2}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Here, G is called the ordinal sum of $(G_\alpha)_{\alpha\in A}$, and each G_α is called a summand.

3 A Generalization of Ordinal Sums on the Unit Interval [0, 1]

In this research work, let us restrict the linearly ordered set *A*, mentioned in Section 2, to be finite. One of the main ideas proposed here is to give an indexing independently from ordering to the set of summands $(G_\alpha)_{\alpha\in A}$ by introducing a bijection as a correspondence between them.

Definition 1. *Consider a permutation* σ *on* $A=\{1,2,..,n\}$, *i.e. a bijection* $\sigma:\{1,2,..n\}\to\{1,2,..n\}$, *then define a linear order* $\le$ *in the family of sets* $\{X_i\}_{i\in\{1,2,\dots,n\}}$ *as follows*:

$$X_i \le X_j \overset{def.}{\Leftrightarrow} \sigma(i)\le\sigma(j) \quad \text{for } \forall i,j\in\{1,2,\dots,n\}. \tag{3}$$

Example 1. If $n=6$, and a permutation σ is given as

$$\sigma=\begin{pmatrix}1&2&3&4&5&6\\1&3&5&6&4&2\end{pmatrix}=\begin{pmatrix}1&6&2&5&3&4\\1&2&3&4&5&6\end{pmatrix},$$

then we get the linear order in $\{X_i\}_{i\in\{1,2,\dots,6\}}$ as $X_1\le X_6\le X_2\le X_5\le X_3\le X_4$. This permutation σ works to locate an element X_i with index i at the $\sigma(i)$-th position.

Hereafter, we treat the case that $X=\bigcup_{i=1}^{n}X_i=[0,1]$ and X_i $(i=1,2,\dots n)$ are disjoint each another, in order to apply the ordinal sum theorem for constructing various logical connectives defined in $[0,1]$.

Definition 2. *Let* $I=\{I_i\}_{i=1,2,\dots,n}$ *be a partition by a finite number of non-empty subintervals of* $[0,1]$, *i.e.* $\bigcup_{i=1}^{n}I_i=[0,1]$ *and* $I_i\cap I_j=\varnothing$ $(i\ne j)$ *hold. Also, we denote* $a_i=\inf I_i$, $b_i=\sup I_i$.

There exists the linear order relation among pairwise disjoint real subintervals according to the real number order. Thus, the permutation σ in Definition 1 gives the indexing as $I_i\le I_j \Rightarrow \sigma(i)\le\sigma(j)$. In other words, the subinterval at *k*-th position is indexed as $I_{\sigma^{-1}(k)}$.

Example 2. If $n=4$, and a partition I and a permutation σ are given as

$$I=\{[0,0.25],\ ]0.25,0.5],\ ]0.5,0.75],\ ]0.75,1]\} \text{ and}$$

$$\sigma=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}=\begin{pmatrix}3&1&4&2\\1&2&3&4\end{pmatrix},$$

respectively, then we have the following indexing for subintervals :

$$I_3=[0,0.25], I_1=]0.25,0.5], I_4=]0.5,0.75], I_2=]0.75,1];$$

$$I_3\le I_1\le I_4\le I_2.$$

Definition 3. *Let* I' *be a subset of* $I = \{I_i\}_{i=1,2,\ldots,n}$. I' *is called to be ascending ordered if* $I_i \leq I_j$ *holds for* $\forall I_i, I_j \in I' \ (i \leq j)$. *Similarly,* I' *is called to be descending ordered if* $I_i \leq I_j$ *holds for* $\forall I_i, I_j \in I' \ (j \leq i)$.

Definition 4. *Two subsets* $\underset{\sim}{I}$ *and* $\tilde{I}$ *of* $I = \{I_i\}_{i=1,2,\ldots,n}$ *are defined as follows*:

$$\underset{\sim}{I} \overset{def.}{=} \left\{ I_i \middle| I_i \leq I_n \right\}, \quad \tilde{I} \overset{def.}{=} \left\{ I_i \middle| I_n \leq I_i \right\}.$$

Example 3. When $n = 6$ and a permutation σ is given as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 5 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 6 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$

we obtain the indexing of $I = \{I_i\}_{i=1,2,\ldots,6}$ as $I_1 \leq I_3 \leq I_5 \leq I_6 \leq I_4 \leq I_2$, and we have $\underset{\sim}{I} = \left\{ I_1, I_3, I_5, I_6 \right\}$ and $\tilde{I} = \left\{ I_6, I_4, I_2 \right\}$. Here, $\underset{\sim}{I}$ is ascending ordered, and $\tilde{I}$ descending ordered.

Definition 5. *Assign each binary operation* $H_i : [0,1]^2 \to [0,1]$ *to each direct product* $I_i \times I_i$ *of a subinterval* $I_i \in I$. *Then, we define the binary operation* $H : [0,1]^2 \to [0,1]$ *as the following ordinal sum*:

$$H(x,y) = \begin{cases} a_i + (b_i - a_i) H_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{if } (x,y) \in I_i \times I_i \\ x & \text{if } (x,y) \in I_i \times I_j \text{ and } i < j \\ y & \text{if } (x,y) \in I_i \times I_j \text{ and } j < i. \end{cases} \quad (4)$$

4 Construction of Logical Connectives on [0, 1]

4.1 Properties Required for Fuzzy Logical Connectives

Conjunctive/disjunctive operations on [0,1] used in various fuzzy logic are defined by combining some of the following properties.

Associativity:
(A) $a * (b * c) = (a * b) * c$

Commutativity:
(C) $a * b = b * a$

Existence of the unit element:
(U1) $a * 1 = 1 * a = a$
(U0) $a * 0 = 0 * a = a$
(UE) $a * e = e * a = a \quad (e \in]0,1[)$

Boundary conditions:
(Bmin) $a * b \le \min(a, b)$
(Bmax) $a * b \ge \max(a, b)$

Monotonicity:
(M) $a \le b \Rightarrow a * c \le b * c,\; c * a \le c * b$

Left-continuity, right-continuity:
(LC) $\lim_{x \to b-0} a * x = a * b, \quad \lim_{x \to a-0} x * b = a * b$
(RC) $\lim_{x \to b+0} a * x = a * b, \quad \lim_{x \to a+0} x * b = a * b$

And-like, or-like [6]:
(AL) $0 * 1 = 1 * 0 = 0$
(OL) $0 * 1 = 1 * 0 = 1$

The definitions of already-known logical operations are expressed by the combinations of the above-mentioned properties as follows.

- t-Norms: (A), (C), (U1), (M)
- t-Conorms: (A), (C), (U0), (M)
- Uninorms: (A), (C), (UE), (M)

- Pseudo-t-norms [5]: (A), (U1), (M)
- Pseudo-t-conorms: (A), (U0), (M)
- Pseudo-uninorms [10]: (A), (UE), (M)

- t-Subnorms [8], [9]: (A), (C), (Bmin), (M)
- t-Subconorms: (A), (C), (Bmax), (M)

Here, the authors introduce the notion of pseudo-t-sub(co)norms as follows.

- Pseudo-t-subnorms: (A), (Bmin), (M)
- Pseudo-t-subconorms: (A), (Bmax), (M)

4.2 Realizations of the Properties in the Framework of Ordinal Sum

We obtain the following theorems regarding to a binary operation $H:[0,1]^2 \to [0,1]$ defined in Definition 5.

Theorem 1. (Clifford [1], [2])
(i) *H is associative if and only if all summands H_i $(i=1,\ldots,n)$ are associative.*
(ii) *H is commutative if and only if all summands H_i $(i=1,\ldots,n)$ are commutative.*

Theorem 2
If $\underset{\sim}{I}$ is ascending ordered, $\tilde{I}$ is descending ordered, H_i for $\sigma(i)<\sigma(n)$ satisfy the boundary condition (Bmin), H_i for $\sigma(n)<\sigma(i)$ satisfy the boundary condition (Bmax), and all summands including H_n are monotone-increasing (M), then H is monotone-increasing (M).

See Appendix for the detailed proof of Theorem 2.

Theorem 3
(i) *If I_n is right-closed (i.e. there exists $\max I_n$) and H_n satisfies (U1) (i.e. it has the unit element 1), then $e=\max I_n = b_n$ is the unit element of H.*
(ii) *If I_n is left-closed (i.e. there exists $\min I_n$) and H_n satisfies (U0) (i.e. it has the unit element 0), then $e=\min I_n = a_n$ is the unit element of H.*
(iii) *If H_n satisfies (UE) (i.e. it has the unit element $e' \in]0,1[$), then $e = a_n + e'(b_n - a_n)$ is the unit element of H.*

Corollary of Theorem 3
(i) *If $\sigma(n)=n$ and H_n satisfies (U1) (i.e. it has the unit element 1), then H satisfies (U1).*
(ii) *If $\sigma(n)=1$ and H_n satisfies (U0) (i.e. it has the unit element 0), then H satisfies (U0).*

Theorem 4
(i) *Let the subinterval including 0 be closed, and the other subintervals be left-open and right-closed as $I_i =]a_i, b_i]$. Then, H is left-continuous (LC) if and only if all summands H_i $(i=1,\ldots,n)$ are left-continuous (LC).*
(ii) *Let the subinterval including 1 be closed, and the other subintervals be left-closed and right-open as $I_i = [a_i, b_i[$. Then, H is right-continuous (RC) if and only if all summands H_i $(i=1,\ldots,n)$ are right-continuous (RC).*

Theorem 5
Suppose that the indices $i, j \in \{1,2,\ldots,n\}$ *satisfy* $\sigma(i)=1$ *and* $\sigma(j)=n$.
(i) *If* $i<j$, *then* H *is and-like* (AL).
(ii) *If* $j<i$, *then* H *is or-like* (OL).

5 Applications

Example 4. Let us consider the case to construct "a left-continuous t-norm." We can obtain it by applying the following conditions to eq. (4):

Theorem 1 (i), (ii)	for associativity and commutativity,
Theorem 2	for monotonicity,
Corollary of Theorem 3 (i)	for unit element $e=1$, and
Theorem 4 (i)	for left-continuity.

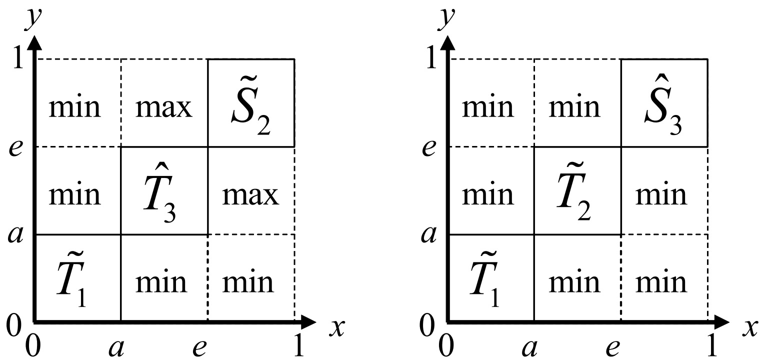
The above result is a finite version of Jenei's method [8], [9], to construct a left-continuous t-norm.

Example 5. Also, we can construct various kinds of pseudo-uninorms through applying the following conditions to eq. (4):

Theorem 1 (i)	for associativity,
Theorem 2	for monotonicity,
Theorem 3	for unit element $e \in [0,1]$,
Theorem 4	for left-continuity/right-continuity, and
Theorem 5	for and-likeness/or-likeness.

Fig.1 (a) illustrates a case of left-continuous and-like pseudo-uninorms, where $n=3$, $\sigma=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}=\begin{pmatrix}1&3&2\\1&2&3\end{pmatrix}$ and $I=\{[0,a],]a,e],]e,1]\}$. All summands $\tilde{T}_1$, $\hat{T}_3$ and $\tilde{S}_2$ are associative, monotone increasing and left-continuous. Since $\sigma(1)=1$ and $\sigma(2)=3$, Th.5(i) is applicable. Also, Th.3(i) is applicable because $\hat{T}_3$ is a pseudo-t-norm and $I_3=(a,e]$ is right-closed, thus $e=\max I_3$ is the unit element.

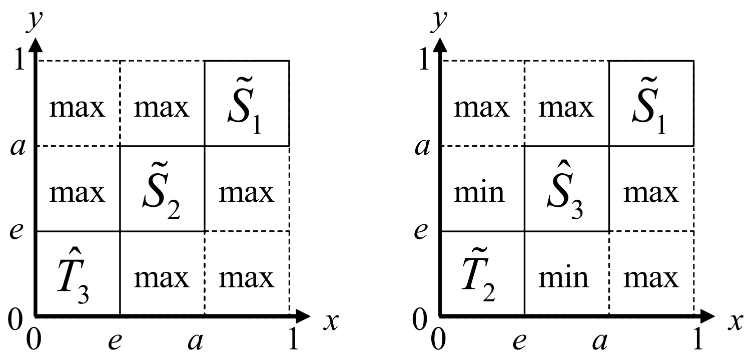
Fig.1 (b) illustrates a case of right-continuous and-like pseudo-uninorms, where $n=3$, $\sigma=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ and $I=\{[0,a[,[a,e[,[e,1]\}$. All summands $\tilde{T}_1$, $\tilde{T}_2$ and $\hat{S}_3$ are associative, monotone increasing and right-continuous. Since $\sigma(1)=1$ and $\sigma(3)=3$, Th.5(i) is applicable. Also, Th.3(ii) is applicable because $\hat{S}_3$ is a pseudo-t-conorm and $I_3=[a,e)$ is left-closed, thus $e=\min I_3$ is the unit element.



(a) left-continuous (LC) case
$I = \{[0,a],]a,e],]e,1]\}$
$\tilde{T}_1$: LC pseudo-t-subnorm
$\tilde{S}_2$: LC pseudo-t-subconorm
$\hat{T}_3$: LC pseudo-t-norm

(b) right-continuous (RC) case
$I = \{[0,a[, [a,e[, [e,1]\}$
$\tilde{T}_1$: RC pseudo-t-subnorm
$\tilde{T}_2$: RC pseudo-t-subnorm
$\hat{S}_3$: RC pseudo-t-conorm

Fig. 1. Examples of and-like pseudo-uninorms $H(x,y)$ ($n=3$, e: unit element)



(a) left-continuous (LC) case
$I = \{[0,e],]e,a],]a,1]\}$
$\tilde{S}_1$: LC pseudo-t-subconorm
$\tilde{S}_2$: LC pseudo-t-subconorm
$\hat{T}_3$: LC pseudo-t-norm

(b) right-continuous (RC) case
$I = \{[0,e[, [e,a[, [a,1]\}$
$\tilde{S}_1$: RC pseudo-t-subconorm
$\tilde{T}_2$: RC pseudo-t-subnorm
$\hat{S}_3$: RC pseudo-t-conorm

Fig. 2. Examples of or-like pseudo-uninorms $H(x,y)$ ($n=3$, e: unit element)

Fig. 2 (a) illustrates a case of left-continuous or-like pseudo-uninorms, where $n=3$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ and $I = \{[0,e],]e,a],]a,1]\}$. All summands $\hat{T}_3$, $\tilde{S}_2$ and $\tilde{S}_1$ are associative, monotone increasing and left-continuous. Since $\sigma(3)=1$ and $\sigma(1)=3$, Th. 5(ii) is applicable. Also, Th. 3(i) is applicable because $\hat{T}_3$ is a pseudo-t-norm and $I_3 = [0,e]$ is right-closed, thus $e = \max I_3$ is the unit element.

Fig. 2 (b) illustrates a case of right-continuous or-like pseudo-uninorms, where $n=3$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ and $I = \{[0,e[, [e,a[, [a,1]\}$. All summands $\tilde{T}_2$, $\hat{S}_3$ and $\tilde{S}_1$ are associative, monotone increasing and right-continuous. Since $\sigma(2)=1$ and $\sigma(1)=3$, Th. 5(ii) is applicable. Also, Th. 3(ii) is applicable because $\hat{S}_3$ is a pseudo-t-conorm and $I_3 = [e,a)$ is left-closed, thus $e = \min I_3$ is the unit element.

6 Concluding Remarks

In this paper, the authors proposed a general method to construct various fuzzy logical connectives on $[0,1]$ by the ordinal sum scheme which generates a new semigroup from a system of semigroups. Through our proposed method, we can generate various fuzzy logical connectives as t-norms, t-conorms, uninorms, and also non- commutative ones such as pseudo-t-norms, pseudo-t-conorms, pseudo-uninorms, by combining the conditions corresponding to the required properties, and by choosing adequate summands from already-known operations.

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Appendix

Proof of Theorem 2

(a) $x \in I_i \quad (\sigma(i) < \sigma(n))$

Let us consider a partition of $[0,1]$ as follows:

$$\bigcup_{1\le\sigma(j)<\sigma(i)} I_j < I_i < \bigcup_{\sigma(i)<\sigma(j)\le\sigma(n)} I_j < \bigcup_{\substack{\sigma(n)<\sigma(j)\le n,\\ i<j}} I_j < \bigcup_{\substack{\sigma(n)<\sigma(j)\le n,\\ j<i}} I_j .$$

Then we have

$$H(x,y) = \begin{cases} y = \min(x,y) & \text{if } y \in \bigcup_{1\le\sigma(j)<\sigma(i)} I_j \\ a_i + (b_i - a_i)H_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) & \text{if } y \in I_i \\ x = \min(x,y) & \text{if } y \in \bigcup_{\sigma(i)<\sigma(j)\le\sigma(n)} I_j \\ x = \min(x,y) & \text{if } y \in \bigcup_{\sigma(n)<\sigma(j)\le n,\, i<j} I_j \\ y = \max(x,y) & \text{if } y \in \bigcup_{\sigma(n)<\sigma(j)\le n,\, j<i} I_j . \end{cases}$$

Since H_i satisfies (Bmin) and (M), $\min(x,a_i) \le a_i \le H(x,y) \le \min(x,b_i)$ for any $y \in I_i$. Besides min, max and H_i satisfy (M). Thus, $H(x,y)$ is monotone increasing w.r.t. $y \in [0,1]$.

(b) $x \in I_n$

Let us consider a partition of $[0,1]$ as follows:

$$\bigcup_{1 \le \sigma(j) < \sigma(n)} I_j < I_n < \bigcup_{\sigma(n) < \sigma(j) \le n} I_j .$$

Then we have

$$H(x,y) = \begin{cases} y = \min(x,y) & \text{if } y \in \bigcup_{1 \le \sigma(j) < \sigma(n)} I_j \\ a_n + (b_n - a_n) H_n\left(\dfrac{x - a_n}{b_n - a_n}, \dfrac{y - a_n}{b_n - a_n} \right) & \text{if } y \in I_n \\ y = \max(x,y) & \text{if } y \in \bigcup_{\sigma(n) < \sigma(j) \le n} I_j . \end{cases}$$

Since $\min(x,a_n) \le a_n \le H(x,y) \le b_n \le \max(x,b_n)$ for any $y \in I_n$ and min, max and H_n satisfy (M), $H(x,y)$ is monotone increasing w.r.t. $y \in [0,1]$.

(c) $x \in I_i \quad (\sigma(n) < \sigma(i))$

Let us consider a partition of $[0,1]$ as follows:

$$\bigcup_{\substack{1 \le \sigma(j) < \sigma(n), \\ j<i}} I_j < \bigcup_{\substack{1 \le \sigma(j) < \sigma(n), \\ i<j}} I_j < \bigcup_{\sigma(n) \le \sigma(j) < \sigma(i)} I_j < I_i < \bigcup_{\sigma(i) < \sigma(j) \le n} I_j .$$

Then we have

$$H(x,y) = \begin{cases} y = \min(x,y) & \text{if } y \in \bigcup_{\substack{1 \le \sigma(j) < \sigma(n), \\ j<i}} I_j \\ x = \max(x,y) & \text{if } y \in \bigcup_{\substack{1 \le \sigma(j) < \sigma(n), \\ i<j}} I_j \\ x = \max(x,y) & \text{if } y \in \bigcup_{\sigma(n) \le \sigma(j) < \sigma(i)} I_j \\ a_i + (b_i - a_i) H_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{if } y \in I_i \\ y = \max(x,y) & \text{if } y \in \bigcup_{\sigma(i) < \sigma(j) \le n} I_j . \end{cases}$$

Since H_i satisfies (Bmax) and (M), $\max(x,a_i) \le H(x,y) \le b_i \le \max(x,b_i)$ for any $y \in I_i$. Besides min, max and H_i satisfy (M). Thus, $H(x,y)$ is monotone increasing w.r.t. $y \in [0,1]$.

From (a), (b) and (c), for any $x \in [0,1]$, $H(x,y)$ is monotone increasing w.r.t. $y \in [0,1]$. The similar discussion is valid for the case w.r.t. x. Therefore, H is monotone increasing. □