Fuzzy Sets, Multisets, and Rough Approximations

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Abstract. Multisets alias bags are similar to fuzzy sets but essentially different in basic concepts and operations. We overview multisets together with basics of fuzzy sets in order to observe differences between the two. We then introduce fuzzy multisets and the combination of the both concepts. There is another concept of real-valued multisets as a generalization of multisets. Rough approximations of multisets and fuzzy multisets are discussed which uses a natural projection of the universal set onto the set of equivalence classes.

1 Multisets

Let $X = \{x_1, \ldots, x_n\}$ is a universal set. As is well-known, a fuzzy set F of X is characterized by a membership function $\mu_A \colon X \to [0, 1]$, where $\mu_A(x)$ is the degree of relevance of x to a concept represented by set symbol A. In contrast, a multiset M of X is characterized by a count function $C_M \colon X \to \boldsymbol{N}$, where $\boldsymbol{N} = \{0, 1, \ldots\}$. $C_M(x) = m$ implies that x exists m times in multiset M. the inclusion, equality, union, and intersection of multisets are defined by the same relations as those of fuzzy sets except that the relations and operations are on $\boldsymbol{N}$ instead of $[0, 1]$. Multisets have the addition ($\oplus$) and minus ($\ominus$) operations which fuzzy sets do not have, while multisets do not have the complement while a fuzzy set has $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$. Note that $C_{M_1 \oplus M_2}(x) = C_{M_1}(x) + C_{M_2}(x)$ and $C_{M_1 \ominus M_2}(x) = \max\{0, C_{M_1}(x) - C_{M_2}(x)\}$.

Note 1. Multisets [1] are also called bags [7]. The name of multisets are used throughout this paper. Crisp multisets are simply called multisets.

2 Fuzzy Multisets

An easy generalization of multisets is real-valued multiset which generalizes $C_M \colon X \to \boldsymbol{N}$ to $C_M \colon X \to [0, +\infty]$ which includes the point of $+\infty$. We omit the details of real-valued multisets here but there are important properties in multi-relations and its algebra (see, e.g., [4]).

Another well-known generalization is fuzzy multisets which originally have been proposed by Yager [7]. Later the author [2] proposed another set of operations as the union and intersection of fuzzy multisets. A fuzzy multiset is a multiset of $X \times [0, 1]$ [7].

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Example 1. Let $X = \{a, b, c, d\}$. A crisp multiset M is expressed as $\{a, b, d, a, a, b\} = \{a, a, a, b, b, d\}$, i.e., $C_M(a) = 3$, $C_M(b) = 2$, $C_M(c) = 0$, and $C_M(d) = 1$. An example of fuzzy multiset A of X is

$$A = \{(a, 0.1), (b, 0.5), (c, 0.6), (c, 0.6), (a, 0.2), (c, 0.8)\}.$$

A membership sequence is the collection of memberships for a particular element of X which is arranged into decreasing order. In the above example, the membership sequence form of the above A is

$$A = \{(0.2, 0.1)/a, (0.5)/b, (0.8, 0.6.0.6)/c\}.$$

The ith member of membership sequence is denoted by $\mu_A^i(x)$. Thus $\mu_A^2(a) = 0.1$, $\mu_A^3(c) = 0.6$, etc. The inclusion, union, and intersection are then defined as follows:

inclusion: $A \subseteq B \iff \mu_A^i(x) \le \mu_B^i(x),\ \forall x \in X,\ i = 1, 2, \dots,$
union: $\mu_{A\cup B}^i(x) = \max\{\mu_A^i(x), \mu_B^i(x)\},\ i = 1, 2, \dots,$
intersection: $\mu_{A\cap B}^i(x) = \min\{\mu_A^i(x), \mu_B^i(x)\},\ i = 1, 2, \dots,$
minus: $\mu_{A\ominus B}^i(x) = \max\{0, \mu_A^i(x) - \mu_B^i(x)\},\ i = 1, 2, \dots,$

while addition is defined without the use of the membership sequence, by simply gathering members of the two fuzzy multisets.

Example 2. Let

$$\begin{aligned} B &= \{(a, 0.1), (b, 0.5), (c, 0.6), (a, 0.1), (c, 0.7)\} \\ &= \{(0.1, 0.1)/a, (0.5)/b, (0.7, 0.6)/c\} \end{aligned}$$

and

$$\begin{aligned} C &= \{(a, 0.3), (a, 0.1), (c, 0.5), (a, 0.2), (c, 0.9), (d, 0.7)\} \\ &= \{(0.3, 0.2, 0.1)/a, (0.9, 0.5)/c, (0.7)/d\}. \end{aligned}$$

Then we have $B \subseteq A$ and

$$\begin{aligned} A \cup C &= \{(0.3, 0.2, 0.1)/a, (0.5)/b, (0.9, 0.6, 0.6)/c, (0.7)/d\} \\ A \cap C &= \{(0.2, 0.1)/a, (0.8, 0.5)/c\} \\ A \oplus C &= \{(a, 0.1), (b, 0.5), (c, 0.6), (c, 0.6), (a, 0.2), (c, 0.8), \\ &\qquad (a, 0.3), (a, 0.1), (c, 0.5), (a, 0.2), (c, 0.9), (d, 0.7)\}. \end{aligned}$$

Let X and $Y = \{y_1, \dots, y_l\}$ be two universal sets and $f\colon X \to Y$. Suppose G is an ordinary set: since $f(G) = \bigcup_{x\in A} \{f(x)\}$, the extension principle of fuzzy sets is derived. Thus we have

$$C_{f(M)}(y) = \max_{x\in X, f(x)=y} C_M(x).$$

In the same way, we have

$$\mu_{f(A)}^i(y) = \max_{x\in X, f(x)=y} \mu_A^i(x),$$

as the extension principle of fuzzy multisets, which is used below.

3 Rough Approximations

Let us assume that an equivalence relation R is given on X, in order to consider rough approximations [5,6]. In other words, relation xRy classifies X into equivalence classes $U_1, \ldots, U_K$. For convenience, let $\mathcal{U} = \{U_1, \ldots, U_K\}$. A natural projection of X onto $\mathcal{U}$ is given by $g_R \colon X \to \mathcal{U}$:

$$g_R(x) = [x]_R = U_i \iff x \in U_i.$$

Let us recall that, given an ordinary set G of X, its upper approximation and lower approximation are respectively given by

$$R^*(G) = \bigcup\{U_i \colon U_i \cap G \neq \emptyset\},$$
$$R_*(G) = \bigcup\{U_i \colon U_i \subseteq G\}.$$

Moreover the rough boundary is $B(G) = R^*(G) - R_*(G)$.

The authors already defined rough approximations of fuzzy sets [3]:

$$\mu_{R^*(F)}(x) = \max_{x' \in U_i} \mu_F(x'), \iff x \in U_i,$$
$$\mu_{R_*(F)}(x) = \min_{x' \in U_i} \mu_F(x'), \iff x \in U_i.$$

We use the natural projection here to define the rough approximations of multisets and fuzzy multisets.

Note first that

$$R^*(G) = g_R^{-1}(g_R(G)) = (g_R^{-1} \circ g_R)(G),$$

where G is an arbitrary crisp set. In contrast, $R_*(G)$ is expressed as follows: Let

$$\hat{G} = g_R^{-1}(g_R(G)) - G.$$

Then

$$B(G) = g_R^{-1}(g_R(\hat{G}))$$

and

$$R_*(G) = g_R^{-1}(g_R(G)) - g_R^{-1}(g_R(\hat{G})).$$

Rough approximation of a multiset M is then straightforward:

$$\mu_{R^*(M)}(x) = \max_{x' \in U_i} C_M(x'), \iff x \in U_i,$$
$$\mu_{R_*(M)}(x) = \min_{x' \in U_i} C_M(x'), \iff x \in U_i.$$

by using the natural projection g_R. In a similar way, we can show that

$$\mu^i_{R^*(A)}(x) = \max_{x' \in U_i} \mu^i_A(x'), \iff x \in U_i,$$
$$\mu^i_{R_*(A)}(x) = \min_{x' \in U_i} \mu^i_A(x'), \iff x \in U_i.$$

for a fuzzy multiset A.

4 Conclusion

We briefly overviewed the theory of multisets and fuzzy multisets as well as their rough approximations. In this paper we focused upon the way how rough approximations are derived using the natural projection. In contrast, many fundamental topics of multisets have been omitted here, but more about basics of multisets and fuzzy multisets will be explained in the talk with illustrations so that readers with no sufficient background will understand their fundamental concepts and basic theory. Readers interested in this theory could refer to, e.g., [3,4,7]. Applications of fuzzy multisets will also be mentioned.

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