A Generalization of Independence and Multivariate Student's *t*-distributions

Monta Sakamoto[1] and Hiroshi Matsuzoe[2(✉)]

[1] Engineering School of Information and Digital Technologies, Efrei, 30-32 Avenue de la Republique, 94800 Villejuif, France

[2] Department of Computer Science and Engineering Graduate School of Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya 466-8555, Japan

matsuzoe@nitech.ac.jp

Abstract. In anomalous statistical physics, deformed algebraic structures are important objects. Heavily tailed probability distributions, such as Student's t-distributions, are characterized by deformed algebras. In addition, deformed algebras cause deformations of expectations and independences of random variables. Hence, a generalization of independence for multivariate Student's t-distribution is studied in this paper. Even if two random variables which follow to univariate Student's t-distributions are independent, the joint probability distribution of these two distributions is not a bivariate Student's t-distribution. It is shown that a bivariate Student's t-distribution is obtained from two univariate Student's t-distributions under q-deformed independence.

Keywords: Deformed exponential family · Deformed independence · Statistical manifold · Tsallis statistics · Information geometry

1 Introduction

In the theory of complex systems, heavily tailed probability distributions are important objects. Power law tailed probability distributions and their related probability distributions have been studied in anomalous statistical physics ([6,12,15]). One of an important probability distribution in anomalous statistical physics is a q-Gaussian distribution. It is a noteworthy fact that a q-Gaussian distribution coincides with a Student's t-distribution in statistics. Hence we can discuss Student's t-distributions from the viewpoint of anomalous statistical physics. Though Student's t-distributions have been studied by many authors (cf. [3,7]), our motivation is quite different from the others.

Heavily tailed probability distributions including Student's t-distributions are represented using deformed exponential functions (cf. [11,12]). However, these functions do not satisfy the law of exponents. Hence deformed algebraic

H. Matsuzoe—This work was partially supported by MEXT KAKENHI Grant Numbers 15K04842 and 26108003.

F. Nielsen and F. Barbaresco (Eds.): GSI 2015, LNCS 9389, pp. 740–749, 2015.
DOI: 10.1007/978-3-319-25040-3_79

structures are naturally introduced (cf. [4,6]). Once such a deformed algebra is introduced, the sample space can be regarded as a some deformed algebraic space, not the standard Euclidean space (cf. [11]). Hence it is natural to introduce suitable deformed expectations and independences of random variables. In fact, we find that the duality of exponential and logarithm can express the notion of independence of random variables. Hence we can generalize the independence using deformed exponential and deformed logarithm functions [9].

In this paper, we summarize such deformed algebraic structures, then we apply these deformed algebras to multivariate Student's t-distributions. Even if two independent random variables follow to univariate Student's t-distributions, the joint probability distribution is not a bivariate Student's t-distribution. Hence we show that a bivariate Student's t-distribution can be obtained from two univariate Student's t-distributions under q-deformed independence with a suitable normalization.

We remark that deformed algebraic structures for statistical models and generalization of independence are discussed in information geometry. (cf. [5,9,11]. See also [1].) Though normalizations of positive densities are necessary in the arguments of generalized independence, statistical manifold structures are changed by normalizations of positive densities. In particular, generalized conformal equivalence relations for statistical manifolds are needed (cf. [9,10]). Hence a statistical manifold of the set of bivariate Student's t-distributions with q-independent random variables is not equivalent to a product statistical manifold of two sets of univariate Student's t-distributions.

2 Deformed Exponential Families

In this paper, we assume that all objects are smooth for simplicity. Let us begin by reviewing the foundations of deformed exponential functions and deformed exponential families (cf. [9,12]).

Let χ be a strictly increasing function from $(0, \infty)$ to $(0, \infty)$. We define a *χ-logarithm function* or a *deformed logarithm function* by

$$\ln_\chi s := \int_1^s \frac{1}{\chi(t)} dt.$$

The inverse of $\ln_\chi s$ is called a *χ-exponential function* or a *deformed exponential function*, which is defined by

$$\exp_\chi t := 1 + \int_0^t u(s) ds,$$

where the function $u(s)$ is given by $u(\ln_\chi s) = \chi(s)$.

From now on, we suppose that χ is a power function, that is, $\chi(t) = t^q$. Then the deformed logarithm and the deformed exponential are defined by

$$\ln_q s := \frac{s^{1-q} - 1}{1-q}, \qquad (s > 0),$$
$$\exp_q t := (1 + (1-q)t)^{\frac{1}{1-q}}, \qquad (1 + (1-q)t > 0).$$

We say that $\ln_q s$ is a *q-logarithm function* and $\exp_q t$ is a *q-exponential function.* By taking a limit $q \to 1$, these functions coincide with the standard logarithm $\ln s$ and the standard exponential $\exp t$, respectively. In this paper, we focus on q-exponential case. However, many of arguments for q-exponential family can be generalized for χ-exponential family ([9,11]).

A statistical model S_q is called a *q-exponential family* if

$$S_q = \left\{ p(x,\theta) \,\middle|\, p(x;\theta) = \exp_q \left[\sum_{i=1}^{n} \theta^i F_i(x) - \psi(\theta) \right],\ \theta \in \Theta \subset \mathbf{R}^n \right\},$$

where $F_1(x), \ldots, F_n(x)$ are functions on the sample space Ω, $\theta = {}^t(\theta^1, \ldots, \theta^n)$ is a parameter, and $\psi(\theta)$ is the normalization with respect to the parameter θ.

Example 1 (Student's t-distribution). Fix a number q $(1 < q < 1 + 2/d,\ d \in \mathbf{N})$, and set $\nu = -d - 2/(1-q)$. We define an *n-dimensional Student's t-distribution with degree of freedom ν or a q-Gaussian distribution* by

$$p_q(x;\mu,\Sigma) := \frac{\Gamma\left(\frac{1}{q-1}\right)}{(\pi\nu)^{\frac{d}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\det(\Sigma)}} \left[1 + \frac{1}{\nu}{}^t(x-\mu)\Sigma^{-1}(x-\mu)\right]^{\frac{1}{1-q}},$$

where $X = {}^t(X_1, \ldots, X_d)$ is a random vector on $\mathbf{R}^d$, $\mu = {}^t(\mu^1, \ldots, \mu^d)$ is a location vector on $\mathbf{R}^d$ and Σ is a scale matrix on $\mathrm{Sym}^+(d)$. For simplicity, we assume that Σ is invertible. Otherwise, we should choose a suitable basis $\{v^\alpha\}$ on $\mathrm{Sym}^+(d)$ such that $\Sigma = \sum_\alpha w_\alpha v^\alpha$. Then, the set of all Student's t-distributions is a q-exponential family. In fact, set

$$z_q = \frac{(\pi\nu)^{\frac{d}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\det(\Sigma)}}{\Gamma\left(\frac{1}{q-1}\right)}, \quad \tilde{R} = \frac{z_q^{q-1}}{(1-q)d+2}\Sigma^{-1}, \quad \text{and} \quad \theta = 2\tilde{R}\mu. \tag{1}$$

Then we have

$$\begin{aligned} p_q(x;\mu,\Sigma) &= \frac{1}{z_q}\left[1 + \frac{1}{\nu}{}^t(x-\mu)\Sigma^{-1}(x-\mu)\right]^{\frac{1}{1-q}} \\ &= \left[\left(\frac{1}{z_q}\right)^{1-q} - \frac{1-q}{(1-q)d+2}\left(\frac{1}{z_q}\right)^{1-q} {}^t(x-\mu)\Sigma^{-1}(x-\mu)\right]^{\frac{1}{1-q}} \\ &= \exp_q\left[-{}^t(x-\mu)\tilde{R}(x-\mu) + \ln_q \frac{1}{z_q}\right] \\ &= \exp_q\left[\sum_{i=1}^{d}\theta^i x_i - \sum_{i=1}^{d}\tilde{R}_{ii}x_i^2 - 2\sum_{i<j}\tilde{R}_{ij}x_i x_j - \frac{1}{4}{}^t\theta\tilde{R}^{-1}\theta + \ln_q\frac{1}{z_q}\right]. \end{aligned}$$

Since $\theta \in \mathbf{R}^d$ and $\tilde{R} \in \mathrm{Sym}^+(d)$, the set of all Student's t-distributions is a $d(d+3)/2$-dimensional q-exponential family. The normalization $\psi(\theta)$ is given by

$$\psi(\theta) = \frac{1}{4}{}^t\theta\tilde{R}^{-1}\theta - \ln_q\frac{1}{z_q}.$$

3 Statistical Manifold Structures Based on *q*-Fisher Metric

In this section we give a brief review of statistical manifold structures on a q-exponential family. We consider a q-Fisher metric in this paper. However, it is known that a q-exponential family naturally has three kinds of statistical manifold structures. See [2,9,11] for more details.

Let S_q be a q-exponential family. The normalization $\psi(\theta)$ on S_q is convex, but may not be strictly convex. Hence we assume that ψ is strictly convex throughout this paper. In fact, we obtain the following proposition.

Proposition 1. *Let $S_q = \{p(x;\theta)\}$ be a q-exponential family. Then the normalization function $\psi(\theta)$ is convex.*

Proof. Set $u(x) = \exp_q x$ and $\partial_i = \partial/\partial\theta^i$. Then we have

$$\partial_i p(x;\theta) = u'\left(\sum \theta^k F_k(x) - \psi(\theta)\right)(F_i(x) - \partial_i\psi(\theta)),$$

$$\partial_i\partial_j p(x;\theta) = u''\left(\sum \theta^k F_k(x) - \psi(\theta)\right)(F_i(x) - \partial_i\psi(\theta))(F_j(x) - \partial_j\psi(\theta)) \\ -u'\left(\sum \theta^k F_k(x) - \psi(\theta)\right)\partial_i\partial_j\psi(\theta).$$

Since $\partial_i \int_\Omega p(x;\theta)dx = \int_\Omega \partial_i p(x;\theta)dx = 0$ and $\int_\Omega \partial_i\partial_j p(x;\theta)dx = 0$, we have

$$Z_q(p) = \int_\Omega \{(p(x;\theta)\}^q dx = \int_\Omega u'\left(\sum \theta^k F_k(x) - \psi(\theta)\right)dx,$$

$$\partial_i\partial_j\psi(\theta) = \frac{1}{Z_q(p)}\int_\Omega u''\left(\sum \theta^k F_k(x) - \psi(\theta)\right) \\ \times(F_i(x) - \partial_i\psi(\theta))(F_j(x) - \partial_j\psi(\theta))dx.$$

For an arbitrary vector $c = {}^t(c^1, c^2, \ldots . c^n) \in \mathbf{R}^n$, since $Z_q(p) > 0$ and $u''(x) > 0$, we have

$$\sum_{i,j=1}^n c^i c^j(\partial_i\partial_j\psi(\theta)) = \frac{1}{Z_q(p)}\int_\Omega u''\left(\sum_{k=1}^n \theta^k F_k(x) - \psi(\theta)\right) \\ \times\left\{\sum_{i=1}^n c^i(F_i(x) - \partial_i\psi(\theta))\right\}^2 dx \ \geq \ 0.$$

This implies that the Hessian matrix $(\partial_i\partial_j\psi(\theta))$ is semi-positive definite. □

From the assumption for $\psi(\theta)$, we can define the *q-Fisher metric* and the *q-cubic form* by

$$g_{ij}(\theta) = \partial_i\partial_j\psi(\theta), \qquad C_{ijk}(\theta) = \partial_i\partial_j\partial_k\psi(\theta),$$

respectively. For a fixed real number α, set

$$g\left(\nabla^{q(\alpha)}_X Y, Z\right) = g\left(\nabla^{q(0)}_X Y, Z\right) - \frac{\alpha}{2}C\left(X, Y, Z\right),$$

where $\nabla^{q(0)}$ is the Levi-Civita connection with respect to g. Since g is a Hessian metric, from standard arguments in Hessian geometry [13], $\nabla^{q(e)} := \nabla^{q(1)}$ and $\nabla^{q(m)} := \nabla^{q(-1)}$ are flat affine connections and mutually dual with respect to g. Hence the quadruplet $(S_q, g, \nabla^{q(e)}, \nabla^{q(m)})$ is a dually flat space.

Next, we consider deformed expectations for q-exponential families. We define the *escort distribution* $P_q(x;\theta)$ of $p(x;\theta) \in S_q$ and the *normalized escort distribution* $P_q^{esc}(x;\theta)$ by

$$
\begin{aligned}
P_q(x;\theta) &= \{p(x;\theta)\}^q, \\
P_q^{esc}(x;\theta) &= \frac{1}{Z_q(p)}\{p(x;\theta)\}^q, \quad \text{where} \quad Z_q(p) = \int_\Omega \{p(x;\theta)\}^q dx,
\end{aligned}
$$

respectively. Let $f(x)$ be a function on Ω. The *q-expectation* $E_{q,p}[f(x)]$ and the *normalized q-expectation* $E_{q,p}^{esc}[f(x)]$ are defined by

$$
E_{q,p}[f(x)] \;=\; \int_\Omega f(x)P_q(x;\theta)dx, \qquad E_{q,p}^{esc}[f(x)] \;=\; \int_\Omega f(x)P_q^{esc}(x;\theta)dx,
$$

respectively. Under q-expectations, we have the following proposition. (cf. [8])

Proposition 2. *For S_q a q-exponential family, (1) set $\phi(\eta) = E_{q,p}^{esc}[\log_q p(x;\theta)]$, then $\phi(\eta)$ is the potential of g with respect to $\{\eta_i\}$. (2) Set $\eta_i = E_{q,p}^{esc}[F_i(x)]$. Then $\{\eta_i\}$ is a $\nabla^{q(m)}$-affine coordinate system such that*

$$
g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\eta_j}\right) = \delta_i^j.
$$

□

We define an *α-divergence* $D^{(\alpha)}$ with $\alpha = 1 - 2q$ and a *q-relative entropy* (or a *normalized Tsallis relative entropy*) D_q^T by

$$
\begin{aligned}
D^{(1-2q)}(p(x), r(x)) = \frac{1}{q}E_{q,p}[\log_q p(x) - \log_q r(x)] \;&=\; \frac{1 - \int_\Omega p(x)^q r(x)^{1-q}dx}{q(1-q)}, \\
D_q^T(p(x), r(x)) = E_{q,p}^{esc}[\log_q p(x) - \log_q r(x)] \;&=\; \frac{1 - \int_\Omega p(x)^q r(x)^{1-q}dx}{(1-q)Z_q(p)},
\end{aligned}
$$

respectively. It is known that the α-divergence $D^{(1-2q)}(r,p)$ induces a statistical manifold structure $(S_q, g^F, \nabla^{(2q-1)})$, where g^F is the Fisher metric on S_q and $\nabla^{(2q-1)}$ is the α-connection with $\alpha = 2q-1$, and the q-relative entropy $D_q^T(r,p)$ induces $(S_q, g, \nabla^{q(e)})$.

Proposition 3 (cf. [10]). *For a q-exponential family S_q, the two statistical manifolds $(S_q, g^F, \nabla^{(2q-1)})$ and $(S_q, g, \nabla^{q(e)})$ are 1-conformally equivalent.* □

We remark that the difference of a α-divergence and a q-relative entropy is only the normalization $q/Z_q(p)$. Hence a normalization for probability density imposes a generalized conformal change for a statistical model.

4 Generalization of Independence

In this section, we review the notions of q-deformed product and generalization of independence. For more details, see [9,11].

Let us introduce the q-deformed algebras since q-exponential functions and q-logarithm functions do not satisfy the law of exponent. Let $\exp_q x$ be a q-exponential function and $\ln_q y$ be a q-logarithm function. For a fixed number q, we suppose that

$$1 + (1-q)x_1 > 0, \quad 1 + (1-q)x_2 > 0, \quad y_1^{1-q} + y_2^{1-q} - 1 > 0, \tag{2}$$
$$y_1 > 0, \quad y_2 > 0. \tag{3}$$

We define the *q-sum* $\tilde{\oplus}^q$ and the *q-product* $\otimes_q$ by the following formulas [4]:

$$\begin{aligned} x_1 \tilde{\oplus}^q x_2 &:= \ln_q \left[\exp_q x_1 \cdot \exp_q x_2\right] \\ &= x_1 + x_2 + (1-q)x_1 x_2, \\ y_1 \otimes_q y_2 &:= \exp_q \left[\ln_q y_1 + \ln_q y_2\right] \\ &= \left[y_1^{1-q} + y_2^{1-q} - 1\right]^{\frac{1}{1-q}}. \end{aligned}$$

Since the base of an exponential function and the argument of a logarithm functions must be positive, conditions (2) and (3) are necessary. We then obtain q-deformed law of exponents as follows.

$$\exp_q(x_1 \,\tilde{\oplus}^q\, x_2) = \exp_q x_1 \cdot \exp_q x_2, \qquad \ln_q(y_1 \cdot y_2) = \ln_q y_1 \,\tilde{\oplus}^q\, \ln_q y_2,$$
$$\exp_q(x_1 + x_2) = \exp_q x_1 \otimes_q \exp_q x_2, \quad \ln_q(y_1 \otimes_q y_2) = \ln_q y_1 + \ln_q y_2.$$

We remark that the q-sum works on the domain of a q-exponential function and a q-product works on the target space. This implies that the domain of q-exponential function (i.e. the total sample space Ω) may not be a standard Euclidean space.

Let us recall the notion of independence of random variables. Suppose that X and Y are random variables which follow to probabilities $p_1(x)$ and $p_2(y)$, respectively. We say that two random variables are *independent* if the joint probability $p(x,y)$ is given by the product of $p_1(x)$ and $p_2(y)$:

$$p(x,y) = p_1(x)p_2(y).$$

Hence $p_1(x)$ and $p_2(y)$ are marginal distributions of $p(x,y)$. When $p_1(x) > 0$ and $p_2(y) > 0$, the independence is equivalent to the additivity of information:

$$\ln p(x,y) = \ln p_1(x) + \ln p_2(y).$$

Let us generalize the notion of independence based on q-products. Suppose that X and Y are random variables which follow to probabilities $p_1(x)$ and $p_2(y)$, respectively. We say that X and Y are *q-independent with e-normalization* (or *exponential normalization*) if a probability density $p(x,y)$ is decomposed by

$$p(x,y) = p_1(x) \otimes_q p_2(y) \otimes_q (-c),$$

where c is the normalization defined by

$$\iint_{Supp(p(x,y))\subset \Omega_X\times\Omega_Y} p_1(x)\otimes_q p_2(y)\otimes_q (-c)\, dxdy = 1.$$

We say that X and Y are *q-independent with m-normalization* (or *mixture normalization*) if a probability density $p(x,y)$ is decomposed by

$$p(x,y) = \frac{1}{Z(p_1,p_2)} p_1(x)\otimes_q p_2(y),$$

where $Z(p_1,p_2)$ is the normalization defined by

$$Z(p_1,p_2) := \iint_{Supp(p(x,y))\subset \Omega_X\times\Omega_Y} p_1(x)\otimes_q p_2(y)\, dxdy.$$

In the case of q-exponential families, including the standard exponential families, we can change normalizations from exponential type to mixture type and vice versa. (See the calculation in Example 1.) Hence we can carry out e- and m-normalization simultaneously. However, e- and m-normalizations are different in general [14].

In some problems, the normalization of probability density is not necessary. In this case, we say that X and Y are *q-independent* if a positive function $f(x,y)$ is decomposed by a q-product of two probability densities $p_1(x)$ and $p_2(y)$:

$$f(x,y) = p_1(x)\otimes_q p_2(y).$$

The function $f(x,y)$ is not necessary to be a probability density. In addition, the total integral of $f(x,y)$ may diverge.

5 *q*-independence and Student's *t*-distributions

In this section, we consider relations between univariate and bivariate Student's t-distributions

Suppose that X_1 and X_2 are random variables which follow to univariate Student's t-distributions $p_1(x_1)$ and $p_2(x_2)$, respectively. Even if X_1 and X_2 are independent, the joint probability $p_1(x_1)p_2(x_2)$ is not a bivariate Student's t-distribution [7]. We show that q-deformed algebras work for bivariate Student's t-distributions.

Theorem 1. *Suppose that X_1 and X_2 are random variables which follow to univariate Student's t-distributions $p_1(x_1)$ and $p_2(x_2)$, respectively, with same parameter q $(1 < q < 2)$. Then there exist a bivariate Student's t-distribution $p(x_1,x_2)$ such that X_1 and X_2 are q-independent with e-normalization.*

Proof. Suppose that X_1 follows to a univariate Student's t-distribution (or a q-Gaussian distribution) given by

$$p(x_1;\mu_1,\sigma_1) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\sqrt{\pi}\sqrt{\frac{3-q}{q-1}}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)\sigma_1}\left[1-(1-q)\frac{(x_1-\mu_1)^2}{(3-q)\sigma_1^2}\right]^{\frac{1}{1-q}},$$

where μ_1 $(-\infty < \mu < \infty)$ is a location parameter, and σ_1 $(0 < \sigma < \infty)$ is a scale parameter. Similarly, suppose that X_2 follows to $p(x_2; \mu_2, \sigma_2)$. By setting

$$z_q(\sigma_1) = \frac{\sqrt{\pi}\sqrt{\frac{3-q}{q-1}}\Gamma\left(\frac{3-q}{2(q-1)}\right)\sigma_1}{\Gamma\left(\frac{1}{q-1}\right)} = \sqrt{\frac{3-q}{q-1}}\mathrm{Beta}\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)\sigma_1,$$

we obtain a q-exponential representation as follows:

$$p(x_1; \mu_1, \sigma_1) = \exp_q\left[\theta^1 x_1 - \theta^{11}{x_1}^2 - \frac{(\theta^1)^2}{4\theta^{11}} + \ln_q \frac{1}{z_q(\sigma_1)}\right],$$

where θ^1 and θ^{11} are natural parameters defined by

$$\theta^1 = \frac{2\mu_1\{z_q(\sigma_1)\}^{q-1}}{(3-q)\sigma_1^2}, \quad \theta^{11} = \frac{\{z_q(\sigma_1)\}^{q-1}}{(3-q)\sigma_1^2}.$$

We remark that the normalization $z_q(\sigma_1)$ can be determined by the parameter θ^{11}. Therefore, $p(x_1; \mu_1, \sigma_1)$ is uniquely determined from natural parameters θ^1 and θ^{11}. Set θ^2 and θ^{22} by changing parameters to μ_2 and σ_2. Then we obtain a positive density by

$$\begin{aligned} & p(x_1; \mu_1, \sigma_1) \otimes_q p(x_2; \mu_2, \sigma_2) \\ = \ & \exp_q\left[\theta^1 x_1 + \theta^2 x_2 - \theta^{11}{x_1}^2 - \theta^{22}{x_2}^2 - \frac{(\theta^1)^2}{4\theta^{11}} - \frac{(\theta^2)^2}{4\theta^{22}} + A(\theta)\right], \end{aligned} \quad (4)$$

where $A(\theta)$ is given by

$$A(\theta) = \ln_q \frac{1}{z_q(\sigma_1)} + \ln_q \frac{1}{z_q(\sigma_2)}.$$

Recall that $p(x_1; \mu_1, \sigma_1) \otimes_q p(x_2; \mu_2, \sigma_2)$ is not a probability distribution. Set the e-normalization function c by

$$c = A(\theta) - \ln_q \frac{1}{z_q} = \left(\ln_q \frac{1}{z_q(\sigma_1)} + \ln_q \frac{1}{z_q(\sigma_2)}\right) - \ln_q \frac{1}{z_q}, \quad (5)$$

where z_q is the m-normalization function of bivariate Student's t-distribution.

As a consequence, we have

$$\begin{aligned} p(x_1, x_2) &= p(x_1; \mu_1, \sigma_1) \otimes_q p(x_2; \mu_2, \sigma_2) \otimes_q (-c) \\ &= \exp_q\left[\theta^1 x_1 + \theta^2 x_2 - \theta^{11}{x_1}^2 - \theta^{22}{x_2}^2 - \frac{(\theta^1)^2}{4\theta^{11}} - \frac{(\theta^2)^2}{4\theta^{22}} + \ln_q \frac{1}{z_q}\right]. \end{aligned}$$

This implies that X_1 and X_2 are q-independent with e-normalization, and the joint positive measure $p(x_1, x_2)$ is a bivariate Student's t-distribution. □

Let us give the normalization function z_q in θ-coordinate, explicitly. Using a property of gamma function, we have

$$\frac{\Gamma\left(\frac{1}{1-q}\right)}{\nu\Gamma\left(\frac{\nu}{2}\right)} = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\nu\Gamma\left(\frac{\nu}{2}\right)} = \frac{1}{2}.$$

Hence the m-normalization function of bivariate Student's t-distribution is simply given by

$$z_q = 2\pi\sqrt{\det \Sigma}.$$

From Equation (1) and (4), the constant z_q should be given by

$$z_q = 2\pi\left(\frac{4(2-q)^2}{(2\pi)^{2q-2}}\det \tilde{R}\right)^{\frac{1}{2(q-2)}} = \left(\frac{2-q}{\pi}\right)^{\frac{1}{q-2}}(\theta^{11}\theta^{22})^{\frac{1}{2q-4}}.$$

6 Concluding Remarks

In this paper, we showed that a bivariate Student's t-distribution can be obtained from two univariate Student's t-distributions using e- and m-normalizations. Recall that statistical manifold structures of statistical models are changed by their normalizations. Hence a statistical manifold structure of a bivariate Student's t-distribution does not coincide with the product manifold structure of two univariate Student's t-distributions.

References

1. Amari, S., Nagaoka, H.: Method of Information Geometry. American Mathematical Society, Providence. Oxford University Press, Oxford (2000)
2. Amari, S., Ohara, A., Matsuzoe, H.: Geometry of deformed exponential families: invariant, dually-flat and conformal geometry. Phys. A **391**, 4308–4319 (2012)
3. Berg, C., Vignat, C.: On the density of the sum of two independent Student t-random vectors. Statist. Probab. Lett. **80**, 1043–1055 (2010)
4. Borgesa, E.P.: A possible deformed algebra and calculus inspired in nonextensive thermostatistics. Phys. A **340**, 95–101 (2004)
5. Fujimoto, Y., Murata, N.: A generalization of independence in naive bayes model. In: Fyfe, C., Tino, P., Charles, D., Garcia-Osorio, C., Yin, H. (eds.) IDEAL 2010. LNCS, vol. 6283, pp. 153–161. Springer, Heidelberg (2010)
6. Kaniadakis, G.: Theoretical foundations and mathematical formalism of the power-law tailed statistical distributions. Entropy **15**, 3983–4010 (2013)
7. Kotz, S., Nadarajah, S.: Multivariate t Distributions and Their Applications. Cambridge University Press, New York (2004)
8. Matsuzoe, H.: Statistical manifolds and geometry of estimating functions. In: Prospects of Differential Geometry and its Related Fields, World Scientific Publishing, pp. 187–202 (2013)
9. Matsuzoe, H., Henmi, M.: Hessian structures and divergence functions on deformed exponential families. In: Nielsen, F. (ed.) Geometric Theory of Information. Signals and Communication Technology. Springer, Switzerland (2014)
10. Matsuzoe, H.., Ohara, A..: Geometry for q-exponential families. In: Recent Progress in Differential Geometry and its Related Fields, World Scientific Publishing, pp. 55–71 (2011)

11. Matsuzoe, H., Wada, T.: Deformed algebras and generalizations of independence on deformed exponential families. Entropy **17**, 5729–5751 (2015)
12. Naudts, J.: Generalised Thermostatistics. Springer, London (2011)
13. Shima, H.: The Geometry of Hessian Structures. World Scientific Publishing, Singapore (2007)
14. Takatsu, A.: Behaviors of φ-exponential distributions in Wasserstein geometry and an evolution equation. SIAM J. Math. Anal. **45**, 2546–2556 (2013)
15. Tsallis, C.: Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World. Springer, New York (2009)