Chapter 5
Multiscale Properties of Tempered Stable Lévy Processes

In this chapter we characterize the multiscale properties of p-tempered α-stable Lévy processes. Specifically, let $X = \{X_t : t \geq 0\}$ be a p-tempered α-stable Lévy process. We will show when there exist deterministic function $a_t > 0$ and $b_t \in \mathbb{R}^d$ and a random variable Y not concentrated at a point such that

$$a_t X_t - b_t \stackrel{d}{\to} Y \text{ as } t \to c \tag{5.1}$$

for $c \in \{0, \infty\}$. When $c = \infty$ this is called **long time behavior** and when $c = 0$ it is called **short time behavior**.

From Lemma 2.5 it follows that in both cases Y must follow some β-stable distribution. Further, by Theorem 4.12 it must have a distribution in ETS^p_α. The only β-stable distributions in ETS^p_α are those with $\beta \in [\alpha, 2]$ if $\alpha \in (0, 2)$ and those with $\beta \in (0, 2]$ if $\alpha \leq 0$. Thus, these are the only possible limiting distributions.

An important consequence of long and short time behavior is that it can be extended to convergence at the level of processes. For $h > 0$ consider the time rescaled process $X^h = \{X_{th} : t \geq 0\}$. Theorem 15.17 in [41] implies that, if (5.1) holds, then there exist processes $\tilde{X}^h \stackrel{d}{=} X^h$ such that for all $t \geq 0$

$$\sup_{s \leq t} |a_h \tilde{X}^h_s - b_h - Y_s| \stackrel{p}{\to} 0 \text{ as } h \to c, \tag{5.2}$$

where $\{Y_t : t \geq 0\}$ is a Lévy process with $Y_1 \stackrel{d}{=} Y$. Thus, in a sense, long time behavior corresponds to what the process looks like when we "zoom out" and short time behavior corresponds to what the process looks like when we "zoom in" on it. When the long and short time behavior of a process are different, the process is multiscaling: it behaves differently in a long time frame from how it behaves in a short time frame.

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5.1 Long and Short Time Behavior

In this section we characterize the long and short time behavior of tempered stable Lévy processes. The proofs are deferred until Section 5.2. First note that if $\{X_t : t \geq 0\}$ is a Lévy process with $X_1 \sim TS^p_\alpha(R, b)$, then by Proposition 3.5 for any $a_t > 0$ and $b_t \in \mathbb{R}^d$ the distribution of $a_t X_t - b_t$ is given by $TS^p_\alpha(R_t, \eta_t)$, where

$$R_t(A) = t \int_{\mathbb{R}^d} 1_A(a_t x) R(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d) \tag{5.3}$$

and η_t is given by

$$ta_t b + ta_t(1 - a_t^2) \int_{\mathbb{R}^d} \int_0^\infty \frac{x|x|^2}{(1 + |x|^2 r^2 a_t^2)(1 + |x|^2 r^2)} e^{-r^p} r^{2-\alpha} \mathrm{d}r R(\mathrm{d}x) - b_t. \tag{5.4}$$

We begin with the case where the limiting distribution is β-stable with $\beta \in (0 \vee \alpha, 2)$. From Proposition 3.12 it follows that all such β-stable distributions belong to the class TS^p_α and have a Rosiński measure given by R^β_σ as in (3.19). Note that, by Theorem 4.12 and Remark 4.7, for the long (or short) time behavior of μ to be β-stable it is necessary that

$$R_t \xrightarrow{v} R^\beta_\sigma \text{ on } \bar{\mathbb{R}}^d_0 \text{ as } t \to c,$$

where $c = \infty$ (or $c = 0$). We will show that this is also sufficient and that it is equivalent to the regular variation of R at c. For $\alpha \neq 0$ a version of this result was given in [30]. Our proof, which we defer until Section 5.2, allows for the case $\alpha = 0$ and is shorter and simpler.

Theorem 5.1. *Fix $c \in \{0, \infty\}$, $\alpha < 2$, $p > 0$, $\beta \in (0 \vee \alpha, 2)$, and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t : t \geq 0\}$ be a p-tempered α-stable Lévy Process with $X_1 \sim TS^p_\alpha(R, b)$ and let $Y \sim S_\beta(\sigma, 0)$. There exist non-stochastic functions $a_t > 0$ and $b_t \in \mathbb{R}^d$ such that*

$$a_t X_t - b_t \xrightarrow{d} Y \text{ as } t \to c \tag{5.5}$$

if and only if $R \in RV^c_{-\beta}(\sigma)$. Moreover, in this case, $a_\bullet \in RV^c_{-1/\beta}$,

$$a_t \sim K^{1/\beta}/V^{\leftarrow}(t) \text{ as } t \to c, \tag{5.6}$$

where $K = \beta^{-1}\sigma(\mathbb{S}^{d-1})$ and $V(t) = 1/R(|x| > t)$, and $b_\bullet$ is such that, if $\eta_\bullet$ is as given by (5.4), then $\eta_t \to 0$ as $t \to c$.

We now turn to the case when $\alpha \in (0, 2)$ and the limiting stable distribution has the same index of stability as the one being tempered. In this case, instead of the Rosiński measure or the extended Rosiński measure, we prefer to work with

$$\nu^1(\mathrm{d}x) = |x|^\alpha R(\mathrm{d}x),$$

which we call the **modified Rosiński measure**. Theorem 3.3 implies that this is a finite measure if and only if R is the Rosiński measure of a proper p-tempered α-stable distribution.

Theorem 5.2. *Fix $c \in \{0, \infty\}$, $\alpha \in (0, 2)$, $p > 0$, and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. Let $\{X_t : t \geq 0\}$ be a p-tempered α-stable Lévy Process with $X_1 \sim TS^p_\alpha(R, b)$ and let $Y \sim S_\alpha(\sigma, 0)$. There exist non-stochastic functions $a_t > 0$ and $b_t \in \mathbb{R}^d$ such that*

$$a_t X_t - b_t \xrightarrow{d} Y \text{ as } t \to c \tag{5.7}$$

if and only if $\nu^1 \in RV^c_0(\sigma)$, where $\nu^1(\mathrm{d}x) = |x|^\alpha R(\mathrm{d}x)$. Moreover, in this case, $a_\bullet \in RV^c_{-1/\alpha}$ with

$$a_t \sim K^{1/\alpha}/V^{\leftarrow}(t) \text{ as } t \to c, \tag{5.8}$$

where $K = \sigma(\mathbb{S}^{d-1})$ and $V(t) = t^\alpha/\nu^1(|x| > t)$, and $b_\bullet$ is such that, if $\eta_\bullet$ is as given by (5.4), then $\eta_t \to 0$ as $t \to c$.

Combining this with facts about the domains of attraction of infinite variance stable distribution given in, e.g., [30] we get the following result, which extends Theorem 3.18.

Corollary 5.3. *Fix $\alpha \in (0, 2)$, $p > 0$, and let $\mu = TS^p_\alpha(R, b)$. If M is the Lévy measure of μ and $\nu^1(\mathrm{d}x) = |x|^\alpha R(\mathrm{d}x)$, then*

$$\mu \in RV^\infty_{-\alpha}(\sigma) \iff M \in RV^\infty_{-\alpha}(\sigma) \iff \nu^1 \in RV^\infty_0(\sigma). \tag{5.9}$$

It turns out that when $c = 0$ and X_1 has a proper p-tempered α-stable distribution the result of Theorem 5.2 always holds. In this case Theorem 3.3 implies that ν^1 is a finite measure, and hence $\nu^1 \in RV^0_0(\sigma)$ with

$$\sigma(B) = \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right) \nu^1(\mathrm{d}x) = \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right) |x|^\alpha R(\mathrm{d}x), \qquad B \in \mathfrak{B}(\mathbb{S}^{d-1}).$$

In this case

$$V(t) \sim t^\alpha/K \text{ as } t \downarrow 0$$

and by Proposition 2.6

$$a_t \sim t^{-1/\alpha} \text{ as } t \downarrow 0.$$

Thus Theorem 5.2 implies that if $Y \sim S_\alpha(\sigma, 0)$, then for properly chosen b_t

$$\lim_{t\downarrow 0} \left(t^{-1/\alpha} X_t - b_t\right) \stackrel{d}{\to} Y \text{ as } t \downarrow 0. \tag{5.10}$$

This is not surprising because by Remark 3.5 all proper p-tempered α-stable distributions with $\alpha \in (0, 2)$ belong to the class of generalized tempered stable distributions, and, for this class, results analogous to (5.10) are given in [66].

We conclude this section by turning to the case where the limiting distribution is Gaussian, i.e. where it is a β-stable distribution with $\beta = 2$.

Theorem 5.4. *Fix $c \in \{0, \infty\}$, $\alpha < 2$, $p > 0$, and let $B \neq 0$ be a symmetric nonnegative-definite matrix. Let $\{X_t : t \geq 0\}$ be a p-tempered α-stable Lévy process with $X_1 \sim TS^p_\alpha(R, b)$ and let*

$$A_t = \int_{|x|\leq t} xx^T R(\mathrm{d}x). \tag{5.11}$$

There exist non-stochastic functions $a_t > 0$ and $b_t \in \mathbb{R}^d$ such that

$$a_t X_t - b_t \stackrel{d}{\to} N(0, B) \text{ as } t \to c \tag{5.12}$$

if and only if $A_\bullet \in MRV^c_0(B/\mathrm{tr}B)$. Moreover, in this case, $a_\bullet \in RV^c_{-1/2}$ and

$$a_t \sim K^{-1/2}/V^{\leftarrow}(t) \text{ as } t \to c, \tag{5.13}$$

where $K = \int_0^\infty s^{1-\alpha} e^{-s^p} \mathrm{d}s/\mathrm{tr}B$ and $V(t) = t^2/\int_{|x|\leq t} |x|^2 R(\mathrm{d}x)$, and $b_\bullet$ is such that, if $\eta_\bullet$ is as given by (5.4), then $\eta_t \to 0$ as $t \to c$.

Note that in the case $\int_{\mathbb{R}^d} |x|^2 R(\mathrm{d}x) < \infty$ dominated convergence implies that $A_\bullet \in MRV^\infty_0(B/\mathrm{tr}B)$ where $B = \int_{\mathbb{R}^d} xx^T R(\mathrm{d}x)$. Combining Theorem 5.4 with facts about the domain of attraction of the multivariate Gaussian given in [29] gives the following.

Corollary 5.5. *Fix $c \in \{0, \infty\}$, let $\mu = TS^p_\alpha(R, b)$, and let M be the Lévy measure of μ. There exists a nonnegative definite matrix $B \neq 0$ with*

$$\int_{|x|\leq \bullet} xx^T R(\mathrm{d}x) \in MRV^c_0(B) \tag{5.14}$$

if and only if

$$\int_{|x|\leq \bullet} xx^T M(\mathrm{d}x) \in MRV^c_0(B). \tag{5.15}$$

Further, if $c = \infty$ and one of (5.14) or (5.15) holds, then there is a nonnegative definite matrix $B' \neq 0$ (possible different from B) such that

$$\int_{|x|\leq \bullet} xx^T \mu(\mathrm{d}x) \in MRV_0^\infty(B').$$

5.2 Proofs

In this section we prove the results of Section 5.1. We begin with several lemmas.

Lemma 5.6. *Fix $c \in \{0, \infty\}$, let Y be a random variable whose distribution is not concentrated at a point, let $a_\bullet$ be a positive function, and let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(A, M, b)$ and $M \neq 0$. Assume that there exists a deterministic function $\xi_\bullet$ taking values in $\mathbb{R}^d$ such that*

$$\lim_{t\to c} a_t X_t - \xi_t \xrightarrow{d} Y.$$

1. *If $c = 0$, then $\lim_{t\downarrow 0} a_t = \infty$ and $a_{1/t} \sim a_{1/(t+1)}$ as $t \to \infty$.*
2. *If $c = \infty$, then $\lim_{t\to\infty} a_t = 0$ and $a_t \sim a_{t+1}$ as $t \to \infty$.*

Proof. First assume $c = 0$. Let $\ell := \liminf_{t\downarrow 0} a_t$ and assume for the sake of contradiction that $\ell < \infty$. This means that there is a sequence of positive real numbers $\{t_n\}$ converging to 0 such that $\lim_{n\to\infty} a_{t_n} = \ell$. Consider a further subsequence $\{t_{n_i}\}$ such that $\lim_{i\to\infty} \xi_{t_{n_i}}$ exists (although we allow it to be infinite). Stochastic continuity of Lévy processes implies that $X_t \xrightarrow{p} 0$ as $t \downarrow 0$, thus Slutsky's Theorem implies that

$$Y = \text{d-}\lim_{i\to\infty}(a_{t_{n_i}} X_{t_{n_i}} - \xi_{t_{n_i}}) \stackrel{d}{=} \ell 0 - \lim_{i\to\infty} \xi_{t_{n_i}},$$

which contradicts the assumption that the distribution of Y is not concentrated at a point. Thus $\lim_{t\downarrow 0} a_t = \infty$.

Let $C_{X_1}(\bullet)$ be the cumulant generating function of X_1. The characteristic function of $a_{1/t}X_{1/t} - \xi_{1/t}$ is $\exp\left(\frac{1}{t} C_{X_1}(a_{1/t}z) - i\langle z, \xi_{1/t}\rangle\right)$. If $\hat{\mu}_Y(z)$ is the characteristic function of Y, then

$$\begin{aligned}
\hat{\mu}_Y(z) &= \lim_{t\to\infty} \exp\left(\frac{1}{t} C_{X_1}(a_{1/t}z) - i\langle z, \xi_{1/t}\rangle\right) \\
&= \lim_{t\to\infty} \exp\left(\frac{1}{t+1} C_{X_1}(a_{1/t}z) - i\langle z, \frac{t}{t+1}\xi_{1/t}\rangle\right),
\end{aligned}$$

which implies that

$$Y \stackrel{d}{=} \operatorname*{d\text{-lim}}_{t\to\infty} \left(a_{1/t} X_{1/(t+1)} - \frac{t}{t+1} \xi_{1/t} \right)$$
$$\stackrel{d}{=} \operatorname*{d\text{-lim}}_{t\to\infty} \left(\frac{a_{1/t}}{a_{1/(t+1)}} \left(a_{1/(t+1)} X_{1/(t+1)} - \xi_{1/(t+1)} \right) + \frac{a_{1/t}}{a_{1/(t+1)}} \xi_{1/(t+1)} - \frac{t}{t+1} \xi_{1/t} \right).$$

Since $\left(a_{1/(t+1)} X_{1/(t+1)} - \xi_{1/(t+1)}\right) \stackrel{d}{\to} Y$ as $t \to \infty$, the result follows by the Convergence of Types Theorem, see, e.g., Lemma 13.10 in [69].

Now assume that $c = \infty$. Let M_t be the Lévy measure of $a_t X_t - \xi_t$ and note that $M_t(\bullet) = tM(\bullet/a_t)$. By Lemma 2.5 Y has a stable distribution. Let M' be its Lévy measure and note that $M'(|x| = s) = 0$ for all $s > 0$. From here Propositions 4.8 and 4.4 imply that for any $s > 0$

$$\lim_{t\to\infty} tM(|x| > s/a_t) = \lim_{t\to\infty} M_t(|x| > s) = M'(|x| > s) < \infty,$$

where the finiteness follows from the fact that M' is a Lévy measure. This implies that $a_t \to 0$. Now let $X' \stackrel{d}{=} X_1$ be independent of $\{X_t : t \geq 0\}$. By Slutsky's Theorem $a_t X' \stackrel{p}{\to} 0$ as $t \to \infty$ and

$$Y \stackrel{d}{=} \operatorname*{d\text{-lim}}_{t\to\infty} (a_{t+1} X_{t+1} - \xi_{t+1})$$
$$\stackrel{d}{=} \operatorname*{d\text{-lim}}_{t\to\infty} \left(a_{t+1} X_t + a_{t+1} X' - \xi_{t+1}\right)$$
$$\stackrel{d}{=} \operatorname*{d\text{-lim}}_{t\to\infty} \left(\frac{a_{t+1}}{a_t} (a_t X_t - \xi_t) + \frac{a_{t+1}}{a_t} \xi_t - \xi_{t+1} \right).$$

Combining this with the fact that $(a_t X_t - \xi_t) \stackrel{d}{\to} Y$ as $t \to \infty$ and another application of the Convergence of Types Theorem gives the result. □

Lemma 5.7. *Fix $c \in \{0, \infty\}$. Let M be a Borel measure on $\mathbb{R}^d$ satisfying* (2.2)*. Fix $\alpha, \beta \geq 0$ with $\alpha + \beta \in (0, 2)$ and define $M^1(\mathrm{d}x) = |x|^\alpha M(\mathrm{d}x)$. If $M^1 \in RV^c_{-\beta}(\sigma)$ for some $\sigma \neq 0$ and*

$$M_t(D) = t \int_{\mathbb{R}^d} 1_D(a_t x) M(\mathrm{d}x), \qquad D \in \mathfrak{B}(\mathbb{R}^d),$$

where $a_t \sim k^{1/(\beta+\alpha)}/V^{\leftarrow}(t)$ for some $k > 0$ and $V(t) = t^\alpha / M^1(|x| > t)$, then

$$\lim_{s\to 0} \limsup_{t\to c} \int_{|x|\leq s} |x|^2 M_t(\mathrm{d}x) = 0.$$

Further, if for some $\eta \in [0, \beta + \alpha)$

$$\int_{|x|>1} |x|^\eta M(\mathrm{d}x) < \infty, \text{ then } \lim_{s\to\infty} \limsup_{t\to c} \int_{|x|>s} |x|^\eta M_t(\mathrm{d}x) = 0$$

and if $\alpha = 0$ *and*

$$\int_{|x|>1} \log|x| M(\mathrm{d}x) < \infty, \text{ then } \lim_{s\to\infty} \limsup_{t\to c} \int_{|x|>s} \log|x| M_t(\mathrm{d}x) = 0.$$

Note that when $c = \infty$ Proposition 2.12 implies that if $M^1 \in RV^\infty_{-\beta}(\sigma)$, then $\int_{|x|>1} |x|^\eta M(\mathrm{d}x) < \infty$ for any $\eta < \alpha + \beta$. However, a similar result does not hold when $c = 0$.

Proof. Define

$$U(u) := \int_{|x|>u} |x|^\alpha M(\mathrm{d}x) \text{ and } U^t(u) := \int_{|x|>u} |x|^\alpha M_t(\mathrm{d}x) = ta_t^\alpha U(u/a_t).$$

Note that (2.16) implies that $U \in RV^c_{-\beta}$ and (2.8) implies that $a_\bullet \in RV^c_{-1/(\beta+\alpha)}$ and hence by Proposition 2.6 $\lim_{t\to c} a_t = 1/c$. By Proposition 2.6 it follows that as $t \to c$

$$t \sim V(V^\leftarrow(t)) \sim \frac{k^{\alpha/(\alpha+\beta)}}{a_t^\alpha M^1(|x| > k^{1/(\beta+\alpha)}/a_t)} \sim \frac{k}{a_t^\alpha U(1/a_t)}.$$

Combining this with Fubini's Theorem gives

$$\begin{aligned}
\lim_{t\to c} \int_{|x|\le s} |x|^2 M_t(\mathrm{d}x) &= \lim_{t\to c} (2-\alpha) \int_{|x|\le s} \int_0^{|x|} u^{1-\alpha} \mathrm{d}u |x|^\alpha M_t(\mathrm{d}x) \\
&= \lim_{t\to c} \left[(2-\alpha) \int_0^s u^{1-\alpha} U^t(u) \mathrm{d}u - s^{2-\alpha} U^t(s) \right] \\
&= \lim_{t\to c} ta_t^\alpha \left[(2-\alpha) \int_0^s u^{1-\alpha} U(u/a_t) \mathrm{d}u - s^{2-\alpha} U(s/a_t) \right] \\
&= \lim_{t\to c} k \left[(2-\alpha) \frac{\int_0^s u^{1-\alpha} U(u/a_t) \mathrm{d}u}{U(1/a_t)} - s^{2-\alpha} \frac{U(s/a_t)}{U(1/a_t)} \right] \\
&= \lim_{t\to c} k(2-\alpha) \frac{a_t^{2-\alpha} \int_0^{s/a_t} u^{1-\alpha} U(u) \mathrm{d}u}{U(1/a_t)} - ks^{2-\alpha-\beta} \\
&= \lim_{t\to c} k(2-\alpha) \frac{\int_0^{s/a_t} u^{1-\alpha} U(u) \mathrm{d}u}{(s/a_t)^{2-\alpha} U(s/a_t)} s^{2-\alpha-\beta} - ks^{2-\alpha-\beta} \\
&= k \frac{2-\alpha}{2-\alpha-\beta} s^{2-\alpha-\beta} - ks^{2-\alpha-\beta},
\end{aligned}$$

which approaches 0 as $s \to 0$. In the above the fifth equality follows by change of variables and the seventh by Karamata's Theorem (Theorem 2.7). The proofs of the other parts are similar. We just need to note that by Fubini's Theorem for $\eta \in [0, \beta + \alpha)$ and $s > 0$ we have

$$\int_{|x|>s} |x|^\eta M_t(\mathrm{d}x) = (\eta - \alpha) \int_s^\infty u^{\eta-\alpha-1} U^t(u)\mathrm{d}u + s^{\eta-\alpha} U^t(s)$$

and for $\alpha = 0$ and $s > 1$ we have

$$\int_{|x|>s} \log |x| M_t(\mathrm{d}x) = \int_s^\infty u^{-1} U^t(u)\mathrm{d}u + U^t(s) \log(s).$$

This completes the proof. □

Proof (Proof of Theorem 5.1). Note that $a_t X_t - b_t \sim TS_\alpha^p(R_t, \eta_t)$, where R_t is given by (5.3) and η_t is given by (5.4). If (5.5) holds, then Lemma 5.6 implies that $\lim_{t\to c} a_t = 1/c$ and, by Theorem 4.12 and Remark 4.7, $\lim_{t\to c} \eta_t = 0$ and $R_t \xrightarrow{v} R_\sigma^\beta$ on $\bar{\mathbb{R}}_0^d$ as $t \to c$. Since, for all $b \geq 0$, $R_\sigma^\beta(|x| = b) = 0$, for any $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ the Portmanteau Theorem (Proposition 4.4) implies that

$$\begin{aligned}
\lim_{t\to c} tR\left(|x| > b/a_t, \frac{x}{|x|} \in D\right) &= \lim_{t\to c} R_t\left(|x| > b, \frac{x}{|x|} \in D\right) \\
&= R_\sigma^\beta\left(|x| > b, \frac{x}{|x|} \in D\right) \\
&= \int_D \int_b^\infty r^{-1-\beta} \mathrm{d}r\sigma(\mathrm{d}u) \\
&= \beta^{-1}\sigma(D) b^{-\beta}.
\end{aligned}$$

Thus, by Proposition 2.11, $R \in RV_{-\beta}^\infty(\sigma)$, $a_\bullet \in RV_{-1/\beta}^\infty$, and (5.6) holds.

Conversely, assume that $R \in RV_{-\beta}^\infty(\sigma)$. Let R_t be as in (5.3) and a_t as in (5.6). By Proposition 2.11, for any $b > 0$ and $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\begin{aligned}
\lim_{t\to c} R_t\left(|x| > b, \frac{x}{|x|} \in D\right) &= \lim_{t\to c} tR\left(|x| > b/a_t, \frac{x}{|x|} \in D\right) \\
&= \beta^{-1}\sigma(D) b^{-\alpha} \\
&= \int_D \int_b^\infty r^{-1-\beta} \mathrm{d}r\sigma(\mathrm{d}u) \\
&= R_\sigma^\beta\left(|x| > b, \frac{x}{|x|} \in D\right).
\end{aligned}$$

Since, for all $b \geq 0$, $R_\sigma^\beta(|x| = b) = 0$ we can use Lemma 4.9 to get $R_t \xrightarrow{v} R_\sigma^\beta$ on $\bar{\mathbb{R}}_0^d$ as $t \to \infty$. From here the result follows by applying Lemma[1] 5.7 and Remarks 4.6 and 4.7. □

Proof (Proof of Theorem 5.2). By Proposition 2.11 $\nu^1 \in RV_0^c(\sigma)$ if and only if there is a function $a_\bullet$ with $\lim_{t\to c} a_t = 1/c$ such that for all $s \in (0,\infty)$ and all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\lim_{t\to c} t a_t^\alpha \nu^1 \left(|x| > s/a_t, \frac{x}{|x|} \in D \right) = \sigma(D). \tag{5.16}$$

When this holds $a_\bullet \in RV_{-1/\alpha}^\infty$ and a_t is as in (5.8). Thus, it suffices to show that (5.7) holds if and only if (5.16) holds.

Let ν be the extended Rosiński measure of X_1, let ν_Y be the extended Rosiński measure of Y, and let R_t and ν_t be, respectively, the Rosiński measure and the extended Rosiński measure of $a_t X_t - b_t$. For $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $s \in (0,\infty)$ define $A_D^s := \{|x| > s, \xi(x) \in D\}$ and note that $\nu_Y(\partial A_D^s) = 0$.

First assume that (5.7) holds. Lemma 5.6 implies that $\lim_{t\to c} a_t = 1/c$ and Theorem 4.12 implies that $\nu_t \xrightarrow{v} \nu_Y$ on $\bar{\mathbb{R}}_0^d$ as $t \to c$. By the Portmanteau Theorem (Proposition 4.4)

$$\lim_{t\to c} \nu_t(A_D^s) = \nu_Y(A_D^s) = \nu_Y(\infty D) = \sigma(D). \tag{5.17}$$

When $s \geq 1$

$$\begin{aligned} t a_t^\alpha \nu^1 \left(A_D^{s/a_t} \right) &= t a_t^\alpha \int_{|x|>s/a_t} 1_D(\xi(x))|x|^\alpha R(\mathrm{d}x) \\ &= \int_{|x|>s} 1_D(\xi(x))|x|^\alpha R_t(\mathrm{d}x) = \nu_t(A_D^s), \end{aligned}$$

and similarly when $s \in (0,1)$

$$\begin{aligned} t a_t^\alpha \nu^1 \left(A_D^{s/a_t} \right) &= t a_t^\alpha \nu^1 \left(A_D^{1/a_t} \right) + t a_t^\alpha \nu^1 \left(A_D^{s/a_t} \right) - t a_t^\alpha \nu^1 \left(A_D^{1/a_t} \right) \\ &= \nu_t(A_D^1) + \int_{1\geq|x|>s} 1_D(\xi(x))|x|^\alpha R_t(\mathrm{d}x). \end{aligned}$$

Now observe that by (5.17) when $s \in (0,1)$ we have

$$\begin{aligned} \lim_{t\to c} \int_{1\geq|x|>s} 1_D(\xi(x))|x|^\alpha R_t(\mathrm{d}x) &\leq \lim_{t\to c} s^{-(2-\alpha)} \int_{1\geq|x|>s} 1_D(\xi(x))|x|^2 R_t(\mathrm{d}x) \\ &= \lim_{t\to c} s^{-(2-\alpha)} \left[\nu_t(A_D^s) - \nu_t(A_D^1) \right] \\ &= s^{-(2-\alpha)} [\sigma(D) - \sigma(D)] = 0. \end{aligned}$$

[1] It should be noted that the parameter α means different things in Theorem 5.1 and in Lemma 5.7.

Putting everything together implies that for any $s \in (0, \infty)$

$$\lim_{t\to c} ta_t^\alpha \nu^1 \left(A_D^{s/a_t}\right) = \lim_{t\to c} \nu_t(A_D^{s\vee 1}) = \sigma(D),$$

and (5.16) holds as required.

Now assume that (5.16) holds. By Proposition 2.11 $a_\bullet$ satisfies (5.8) and $a_\bullet \in RV^c_{-1/\alpha}$. As in the previous case, for $s \geq 1$ we have

$$\nu_t(A_D^s) = ta_t^\alpha \nu^1 \left(A_D^{s/a_t}\right),$$

and for $s \in (0, 1)$ we have

$$\nu_t(A_D^s) = \nu_t(A_D^1) + \nu_t(A_D^s) - \nu_t(A_D^1) = ta_t^\alpha \nu^1 \left(A_D^{1/a_t}\right) + \int_{1\geq |x|>s} 1_D(\xi(x))|x|^2 R_t(\mathrm{d}x).$$

Now observe that (5.16) implies that for $s \in (0, 1)$

$$\begin{aligned}\lim_{t\to c} \int_{1\geq|x|>s} 1_D(\xi(x))|x|^2 R_t(\mathrm{d}x) &\leq \lim_{t\to c} \int_{1\geq|x|>s} 1_D(\xi(x))|x|^\alpha R_t(\mathrm{d}x)\\ &= \lim_{t\to c}\left[ta_t^\alpha \nu^1(A_D^{s/a_t}) - ta_t^\alpha \nu^1(A_D^{1/a_t})\right]\\ &= \sigma(D) - \sigma(D) = 0.\end{aligned}$$

This implies that for all $s \in (0, \infty)$

$$\lim_{t\to c} \nu_t(A_D^s) = \lim_{t\to c} ta_t^\alpha \nu^1(A_D^{(s\vee 1)/a_t}) = \sigma(D),$$

and by Lemma 4.9 it follows that $\nu_t \stackrel{v}{\to} \nu_Y$ on $\bar{\mathbb{R}}^d_0$ as $t \to c$. Thus we have convergence of the extended Rosiński measures. It remains to show convergence of the shifts and Gaussian parts. The convergence of the shifts is equivalent to the condition that $\eta_t \to 0$ as $t \to c$. By (4.23) the limit will have no Gaussian part so long as

$$\limsup_{t\to c} \nu_t(|x| < 1) = 0,$$

which follows immediately from Lemma 5.7. This concludes the proof. □

To prove results for convergence to the Gaussian we need a few additional Lemmas.

Lemma 5.8. *Fix $\alpha < 2$, $p > 0$, and let $\{R_n\}$ be a sequence of measures on $\mathbb{R}^d$ satisfying* (2.2). *If, for any $s > 0$, $\lim_{n\to\infty} R_n(|x| > s) = 0$ and if for some $\kappa > 0$*

$$\sup_n \int_{|x|\leq\kappa} |x|^2 R_n(\mathrm{d}x) < \infty,$$

then for any $a, b, c \in (0, \infty)$

$$\lim_{n\to\infty}\left(\int_{|x|\le a} xx^T \int_0^{c/|x|} t^{1-\alpha}e^{-t^p}\mathrm{d}tR_n(\mathrm{d}x) - \zeta\int_{|x|\le b} xx^T R_n(\mathrm{d}x)\right) = 0,$$

where $\zeta = \int_0^\infty t^{1-\alpha}e^{-t^p}\mathrm{d}t$.

Proof. Fix $\epsilon > 0$ and let $C = \sup_n \int_{|x|\le\kappa}|x|^2R_n(\mathrm{d}x)$. By dominated convergence $\lim_{u\to\infty}\int_0^u t^{1-\alpha}e^{-t^p}\mathrm{d}t = \zeta$, which implies that there exists a $u' \in (0,\infty)$ such that if $u \ge u'$, then $\left|\zeta - \int_0^u t^{1-\alpha}e^{-t^p}\mathrm{d}t\right| < \frac{\epsilon}{C}$. Fix $a' > 0$ such that $a' < \min\{c/u', a, b, \kappa\}$. For any $1 \le i, j \le d$ we have

$$\begin{aligned}
&\left|\int_{|x|\le a} x_ix_j\int_0^{c/|x|} t^{1-\alpha}e^{-t^p}\mathrm{d}tR_n(\mathrm{d}x) - \zeta\int_{|x|\le b} x_ix_jR_n(\mathrm{d}x)\right| \\
&\quad\le \left|\int_{|x|\le a'} x_ix_j\int_0^{c/|x|} t^{1-\alpha}e^{-t^p}\mathrm{d}tR_n(\mathrm{d}x) - \zeta\int_{|x|\le a'} x_ix_jR_n(\mathrm{d}x)\right| \\
&\qquad + \left|\int_{a'<|x|\le a} x_ix_j\int_0^{c/|x|} t^{1-\alpha}e^{-t^p}\mathrm{d}tR_n(\mathrm{d}x)\right| \\
&\qquad + \left|\zeta\int_{a'<|x|\le b} x_ix_jR_n(\mathrm{d}x)\right| \\
&\quad =: A_{1,n} + A_{2,n} + A_{3,n}.
\end{aligned}$$

Further,

$$A_{2,n} \le \zeta\int_{a'<|x|\le a}|x|^2R_n(\mathrm{d}x) \le \zeta a^2R_n(|x| > a') \to 0,$$

$$A_{3,n} \le \zeta\int_{a'<|x|\le b}|x|^2R_n(\mathrm{d}x) \le \zeta b^2R_n(|x| > a') \to 0,$$

and

$$\begin{aligned}
A_{1,n} &\le \int_{|x|\le a'}|x|^2\left|\zeta - \int_0^{c/|x|} t^{1-\alpha}e^{-t^p}\right|\mathrm{d}tR_n(\mathrm{d}x) \\
&\le \int_{|x|\le a'}|x|^2\left|\zeta - \int_0^{c/a'} t^{1-\alpha}e^{-t^p}\right|\mathrm{d}tR_n(\mathrm{d}x) \\
&< \frac{\epsilon}{C}\sup_n\int_{|x|\le a'}|x|^2R_n(\mathrm{d}x) \le \epsilon,
\end{aligned}$$

which completes the proof. ◻

Lemma 5.9. *Fix $c \in \{0, \infty\}$. Let M be a measure on $\mathbb{R}^d$ satisfying (2.2) and let $A_u = \int_{|x|\le u} xx^T M(\mathrm{d}x)$. If $A_\bullet \in MRV_0^c(B)$ for some $B \neq 0$ and*

$$M_t(D) = t \int_{\mathbb{R}^d} 1_D(a_t x) M(\mathrm{d}x), \qquad D \in \mathfrak{B}(\mathbb{R}^d),$$

where $a_t \sim k^{-1/2}/V^{\leftarrow}(t)$ for some $k > 0$ and $V(t) = t^2 / \int_{|x|\le t} |x|^2 M(\mathrm{d}x)$, then the following hold.

1. There exists a $\delta > 0$ such that if $B_c^\delta = (0, \delta)$ when $c = 0$ and $B_c^\delta = (1/\delta, \infty)$ when $c = \infty$, then

$$\sup_{t \in B_c^\delta} \int_{|x|\le 1} |x|^2 M_t(\mathrm{d}x) < \infty.$$

2. If, for $\eta \in [0, 2)$,

$$\int_{|x|>1} |x|^\eta M(\mathrm{d}x) < \infty, \tag{5.18}$$

then $\lim_{t\to c} \int_{|x|>s} |x|^\eta M_t(\mathrm{d}x) = 0$ for all $s > 0$. Moreover, when $c = \infty$ (5.18) holds for every $\eta \in [0, 2)$.

3. If $\int_{|x|>1} \log|x| M(\mathrm{d}x) < \infty$, then $\lim_{s\to\infty} \limsup_{t\to c} \int_{|x|>s} \log|x| M_t(\mathrm{d}x) = 0$.

Proof. Let

$$U(u) := \int_{|x|\le u} |x|^2 M(\mathrm{d}x) = \mathrm{tr} A_u \text{ and } U^t(u) := \int_{|x|\le u} |x|^2 M_t(\mathrm{d}x) = t a_t^2 U(u/a_t).$$

From Definition 2.8, (2.8), and Proposition 2.6 it follows that $U \in RV_0^c$, $a_\bullet \in RV_{-1/2}^c$, $\lim_{t\to c} a_t = 1/c$, and $t \sim V(1/(a_t\sqrt{k})) = [k a_t^2 U(1/(a_t\sqrt{k}))]^{-1} \sim [k a_t^2 U(1/a_t)]^{-1}$ as $t \to c$. Part 1 follows from the fact that

$$\lim_{t\to c} \int_{|x|\le 1} |x|^2 M_t(\mathrm{d}x) = \lim_{t\to c} t a_t^2 \int_{|x|\le 1/a_t} |x|^2 M(\mathrm{d}x) = \lim_{t\to c} \frac{U(1/a_t)}{kU(1/a_t)} = 1/k < \infty.$$

Now to show Part 2. By Fubini's Theorem it follows that for any $s > 0$

$$\int_{|x|>s} |x|^\eta M_t(\mathrm{d}x) = (2-\eta) \int_s^\infty u^{\eta-3} U^t(u) \mathrm{d}u - s^{\eta-2} U^t(s).$$

When $c = \infty$ the right side is finite by Lemma 2 on Page 277 in [23], and hence the left side must be finite as well. Further, we have

$$\begin{aligned}
\lim_{t\to c}\int_{|x|>s}|x|^\eta M_t(\mathrm{d}x) &= \lim_{t\to c} ta_t^2\left[(2-\eta)\int_s^\infty u^{\eta-3}U(u/a_t)\mathrm{d}u - s^{\eta-2}U(s/a_t)\right]\\
&= \lim_{t\to c} k^{-1}\left[(2-\eta)\frac{\int_s^\infty u^{\eta-3}U(u/a_t)\mathrm{d}u}{U(1/a_t)} - s^{\eta-2}\frac{U(s/a_t)}{U(1/a_t)}\right]\\
&= \lim_{t\to c} k^{-1}(2-\eta)\frac{\int_{s/a_t}^\infty u^{\eta-3}U(u)\mathrm{d}u}{U(s/a_t)(s/a_t)^{\eta-2}}s^{\eta-2} - k^{-1}s^{\eta-2}\\
&= k^{-1}\left(s^{\eta-2}-s^{\eta-2}\right) = 0,
\end{aligned}$$

where the third equality follows by change of variables and the fourth by Karamata's Theorem (Theorem 2.7). We now turn to Part 3. First consider the case $c=\infty$. The fact that $\log|x| \le |x|$ (see, e.g., 4.1.36 in [2]) and the result of Part 2 gives

$$0 \le \lim_{s\to\infty}\limsup_{t\to\infty}\int_{|x|>s}\log|x| M_t(\mathrm{d}x) \le \lim_{s\to\infty}\limsup_{t\to\infty}\int_{|x|>s}|x|M_t(\mathrm{d}x) = 0.$$

Now assume that $c=0$. In this case $a_t\to\infty$ as $t\to 0$ and we have

$$\begin{aligned}
\limsup_{t\to 0}\int_{|x|>s}\log|x|M_t(\mathrm{d}x) &= \limsup_{t\to 0} t\int_{|x|>s/a_t}\log|xa_t|M(\mathrm{d}x)\\
&= \limsup_{t\to 0}\frac{\int_{|x|>s/a_t}\log|xa_t|M(\mathrm{d}x)}{ka_t^2U(1/a_t)}\\
&= \limsup_{t\to 0}\left[\frac{\int_{|x|>1}\log|xa_t|M(\mathrm{d}x)}{ka_t^2U(1/a_t)}\right.\\
&\qquad\left. + \frac{\int_{1\ge|x|>s/a_t}\log|xa_t|M(\mathrm{d}x)}{ka_t^2U(1/a_t)}\right]\\
&=: \limsup_{t\to 0}\left[I_1(t)+I_2(s,t)\right].
\end{aligned}$$

Define

$$f(u) = \frac{a+b\log|u|}{kU(1/u)}u^{-2},$$

where

$$a = \int_{|x|>1}\log|x|M(\mathrm{d}x) \text{ and } b = \int_{|x|>1}M(\mathrm{d}x),$$

and note that, by assumption, $a, b \in (0, \infty)$. The fact that $U \in RV_0^0$ implies that $f \in RV_{-2}^\infty$ and thus by Proposition 2.6

$$\lim_{t\to 0} I_1(t) = \lim_{t\to 0} f(a_t) = \lim_{t\to\infty} f(t) = 0.$$

Using the inequality $\log|x| \le |x|$ again and Fubini's Theorem gives

$$\begin{aligned}\int_{1\ge|x|>s/a_t} \log|xa_t| M(\mathrm{d}x) &\le a_t \int_{1\ge|x|>s/a_t} |x| M(\mathrm{d}x)\\ &= a_t \int_{s/a_t}^\infty u^{-2} \int_{s/a_t<|x|\le(u\wedge 1)} |x|^2 M(\mathrm{d}x)\mathrm{d}u\\ &\le a_t \int_{s/a_t}^\infty u^{-2} \int_{|x|\le(u\wedge 1)} |x|^2 M(\mathrm{d}x)\mathrm{d}u\\ &= a_t \int_{s/a_t}^1 u^{-2} U(u)\mathrm{d}u + a_t U(1)\\ &= a_t \int_1^{a_t/s} U(1/u)\mathrm{d}u + a_t U(1),\end{aligned}$$

where the final line follows by change of variables. This implies that

$$I_2(s,t) \le \frac{\int_1^{a_t/s} U(1/u)\mathrm{d}u}{ka_t U(1/a_t)} + \frac{U(1)}{ka_t U(1/a_t)} =: I_{21}(s,t) + I_{22}(t).$$

By Karamata's Theorem (Theorem 2.7) and the fact that $U(1/\bullet) \in RV_0^\infty$ we have

$$\lim_{s\to\infty} \limsup_{t\to 0} I_{21}(s,t) = \lim_{s\to\infty} \limsup_{t\to 0} \frac{\int_1^{a_t/s} U(1/u)\mathrm{d}u}{k(a_t/s)U(s/a_t)} s^{-1} = \lim_{s\to\infty} \frac{1}{k} s^{-1} = 0.$$

Finally, note that the function of u given by $\frac{U(1)}{kuU(1/u)}$ is an element of RV_{-1}^∞, which implies that $\lim_{t\to 0} I_{22}(t) = 0$. □

Proof (Proof of Theorem 5.4). Note that $a_t X_t - b_t \sim TS_\alpha^p(R_t, \eta_t)$, where R_t is given by (5.3) and η_t is given by (5.4). Before proceeding set $\zeta = \int_0^\infty s^{1-\alpha} e^{-s^p} \mathrm{d}s$.

First assume that $A_\bullet \in MRV_0^c(B/\mathrm{tr}B)$ and that a_t is given by (5.13). This implies that $a_\bullet \in RV_{-1/2}^c$. Further, Lemma 5.9 implies that the assumptions of Lemma 5.8 hold. Using this lemma gives

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\lim_{t\to c} \int_{|x|\le\sqrt{\epsilon}} xx^T \int_0^{\epsilon/|x|} s^{1-\alpha} e^{-s^p} \mathrm{d}s R_t(\mathrm{d}x) &= \lim_{t\to c} \zeta \int_{|x|\le 1/\sqrt{K}} xx^T R_t(\mathrm{d}x)\\ &= \zeta \lim_{t\to c} t a_t^2 \int_{|x|\le 1/(\sqrt{K}a_t)} xx^T R(\mathrm{d}x)\end{aligned}$$

$$= \zeta \lim_{t \to c} K^{-1} \frac{\int_{|x| \le 1/(\sqrt{K} a_t)} x x^T R(\mathrm{d}x)}{\int_{|x| \le 1/(\sqrt{K} a_t)} |x|^2 R(\mathrm{d}x)}$$

$$= \mathrm{tr} B \lim_{t \to c} \frac{\int_{|x| \le 1/(\sqrt{K} a_t)} x x^T R(\mathrm{d}x)}{\int_{|x| \le 1/(\sqrt{K} a_t)} |x|^2 R(\mathrm{d}x)} = B.$$

From here the result will follow by Theorem 4.12. We just need to show that the extended Rosiński measure goes to zero, which follows by Remark 4.7 and Lemma 5.9.

Now assume that (5.12) holds. Theorem 4.12 implies that for every $s > 0$

$$\lim_{t \to c} R_t(|x| > s) = 0$$

and

$$\lim_{\epsilon \downarrow 0} \lim_{t \to c} \int_{|x| \le \sqrt{\epsilon}} x x^T \int_0^{\epsilon/|x|} s^{1-\alpha} e^{-s^p} \mathrm{d}s R_t(\mathrm{d}x) = B. \tag{5.19}$$

This means that there exist an $\epsilon' > 0$ and a $\delta > 0$ such that

$$\infty > \sup_{t \in B_c^\delta} \int_{|x| \le \sqrt{\epsilon'}} |x|^2 \int_0^{\epsilon'/|x|} s^{1-\alpha} e^{-s^p} \mathrm{d}s R_t(\mathrm{d}x)$$

$$\ge \sup_{t \in B_c^\delta} \int_{|x| \le \sqrt{\epsilon'}} |x|^2 R_t(\mathrm{d}x) \int_0^{\sqrt{\epsilon'}} s^{1-\alpha} e^{-s^p} \mathrm{d}s,$$

where $B_c^\delta = (0, \delta)$ if $c = 0$ and $B_c^\delta = (1/\delta, \infty)$ if $c = \infty$. Hence

$$\sup_{t \in B_c^\delta} \int_{|x| \le \sqrt{\epsilon'}} |x|^2 R_t(\mathrm{d}x) < \infty$$

and we can use Lemma 5.8, which combined with (5.19) tells us that for any $s > 0$

$$\zeta \lim_{t \to c} t a_t^2 \int_{|x| \le s/a_t} x x^T R(\mathrm{d}x) = \zeta \lim_{t \to c} \int_{|x| \le s} x x^T R_t(\mathrm{d}x) = B.$$

Thus, for any $s > 0$,

$$\zeta \lim_{t \to c} t a_t^2 U(s/a_t) = \mathrm{tr} B,$$

where $U(t) = \int_{|x|\leq t} |x|^2 R(\mathrm{d}x)$. Lemma 5.6 implies that the sequential criterion for regular variation of monotone functions (see Proposition 2.6) holds and thus that $U \in RV_0^c$. The fact that

$$\lim_{t\to c} \frac{\int_{|x|\leq t} xx^T R(\mathrm{d}x)}{\int_{|x|\leq t} |x|^2 R(\mathrm{d}x)} = \lim_{t\to c} \frac{\zeta t a_t^2 \int_{|x|\leq 1/a_t} xx^T R(\mathrm{d}x)}{\zeta t a_t^2 \int_{|x|\leq 1/a_t} |x|^2 R(\mathrm{d}x)} = \frac{B}{\mathrm{tr}B}$$

shows that $A_\bullet \in MRV_0^c(B/\mathrm{tr}B)$ as required. □