Chapter 3
Tempered Stable Distributions

In this chapter we formally define tempered stable distributions and discuss many properties. These distributions were first introduced in [65]. From here the class was expanded in several directions in [9, 51, 66], and [27]. Our discussion mainly follows [27].

3.1 Definitions and Basic Properties

Fix $\alpha \in (0, 2)$, let σ be a finite Borel measure on $\mathbb{S}^{d-1}$, and recall that the Lévy measure of an α-stable distribution with spectral measure σ is given by

$$L(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}u), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{3.1}$$

Now, fix $p > 0$ and define a new Lévy measure of the form

$$M(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) q(r^p, u) r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}u), \quad A \in \mathfrak{B}(\mathbb{R}^d), \tag{3.2}$$

where $q : (0, \infty) \times \mathbb{S}^{d-1} \mapsto (0, \infty)$ is a Borel function such that, for all $u \in \mathbb{S}^{d-1}$, $q(\cdot, u)$ is completely monotone and satisfies

$$\int_0^1 r^{1-\alpha} q(r^p, u) \mathrm{d}r < \infty, \quad \int_1^\infty r^{-1-\alpha} q(r^p, u) \mathrm{d}r < \infty, \tag{3.3}$$

and

$$\lim_{r\to\infty} q(r, u) = 0. \tag{3.4}$$

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The conditions in (3.3) guarantee that this is a valid Lévy measure, while the fact that (3.4) holds implies that the tails of M are lighter than those of L. This implies that the tails of the associated infinitely divisible distribution are lighter as well.

The complete monotonicity[1] of $q(\cdot, u)$ means that, for each $u \in \mathbb{S}^{d-1}$, the function $q(r, u)$ is infinitely differentiable in r and

$$(-1)^n \frac{\partial^n}{\partial r^n} q(r, u) \geq 0. \tag{3.5}$$

In particular, this implies that $q(\cdot, u)$ is a monotonely decreasing function for each $u \in \mathbb{S}^{d-1}$. By (3.4) and Bernstein's Theorem (see, e.g., Theorem 1a in Section XIII.4 of [23] or Remark 3.2 in [6]) it follows that there exists a measurable family[2] $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ of Borel measures on $(0, \infty)$ with

$$q(r^p, u) = \int_{(0,\infty)} e^{-r^p s} Q_u(\mathrm{d}s). \tag{3.6}$$

From here it follows that, so long as $Q_u \neq 0$, we have $q(r^p, u) > 0$ for all $r > 0$.

Note that, under the given conditions on the function q, (3.2) defines a valid Lévy measure even for α outside of the interval $(0, 2)$. However, since $q(\cdot, u)$ is a decreasing function for each $u \in \mathbb{S}^{d-1}$, when $\alpha \geq 2$ condition (3.3) holds only with $q \equiv 0$. For this reason, we only consider the case $\alpha \in (-\infty, 2)$. This leads to the following definition.

Definition 3.1. Fix $\alpha < 2$ and $p > 0$. An infinitely divisible probability measure μ is called a **p-tempered α-stable distribution** if it has no Gaussian part and its Lévy measure is given by (3.2), where σ is a finite Borel measure on $\mathbb{S}^{d-1}$ and $q : (0, \infty) \times \mathbb{S}^{d-1} \mapsto (0, \infty)$ is a Borel function such that for all $u \in \mathbb{S}^{d-1}$, $q(\cdot, u)$ is completely monotone and satisfies (3.3) and (3.4). We denote the class of p-tempered α-stable distributions by TS_α^p.

We use the term **tempered stable distributions** to refer to the class of all p-tempered α-stable distributions with all $\alpha < 2$ and $p > 0$.

Remark 3.1. Under appropriate integrability conditions, one can define Lévy measures of the form (3.2) with $p \leq 0$. The case $p = 0$ corresponds to the class of α-stable distributions and only makes sense for $\alpha \in (0, 2)$. The case $p < 0$ has significantly different behavior from the case $p > 0$ and will not be considered here.

Remark 3.2. From Theorem 15.10 in [69] it follows that p-tempered α-stable distributions belong to the class of self-decomposable distributions if and only if $q(r^p, u) r^{-\alpha}$ is a decreasing function of r for every $u \in \mathbb{S}^{d-1}$. This always holds when

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[1] A general reference on completely monotone functions is [72].

[2] The measurability of the family means that for any Borel set A the function $f(u) = Q_u(A)$ is measurable.

$\alpha \in [0, 2)$, but it may fail when $\alpha < 0$. Thus, when $\alpha \in [0, 2)$, p-tempered α-stable distributions possess all properties of self-decomposable distributions. In particular, if they are nondegenerate, then they have a density with respect to Lebesgue measure in d-dimensions and when $d = 1$ they are unimodal.

In Definition 3.1, the case when $\alpha \leq 0$ no longer corresponds to the idea of modifying the tails of a stable distribution. Nevertheless, such distributions serve to make the class richer and more robust. It should be added that, even in the case when $\alpha \in (0, 2)$ we may no longer have a Lévy measure that looks much like that of an α-stable distribution. For that to hold, we would need the function q to be close to 1 in some region near zero. This leads to the following definition.

Definition 3.2. Fix $p > 0$ and $\alpha < 2$. Let μ be a p-tempered α-stable distribution with Lévy measure M. If M can be represented in the form (3.2) where

$$\lim_{r \downarrow 0} q(r, u) = 1 \text{ for every } u \in \mathbb{S}^{d-1}, \tag{3.7}$$

then μ is called a **proper p-tempered α-stable distribution**.

Proper p-tempered α-stable distributions with $\alpha \in (0, 2)$ are the ones that correspond to the original motivation of modifying the tails of stable distributions to make them lighter.

Remark 3.3. In [9, 65], and [27] proper p-tempered α-stable distributions are defined to be ones where M is of the form (3.2) and (3.7) holds. However, it may happen that q does not satisfy (3.7), but that there is a Borel function $c : \mathbb{S}^{d-1} \mapsto (0, \infty)$ such that $q'(r, u) = q(r, u)/c(u)$ satisfies (3.7). In this case we can take $\sigma'(\mathrm{d}u) = c(u)\sigma(\mathrm{d}u)$ and write M as (3.2) but with q' and σ' in place of q and σ. In this case we still want to consider M to be a proper p-tempered α-stable distribution. For this reason we need the somewhat more subtle formulation given in Definition 3.2.

Remark 3.4. Assume that $q(r, u)$ satisfies (3.6). The Monotone Convergence Theorem implies that $q(r, u)$ satisfies (3.7) if and only if (3.6) holds with Q_u being a *probability* measure for every $u \in \mathbb{S}^{d-1}$.

Remark 3.5. When $\alpha \in (0, 2)$ and $p > 0$, the class of proper p-tempered α-stable distributions belongs to the class of Generalized Tempered Stable Distributions introduced in [66].

It is somewhat artificial to work with the family of measures $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ and the measure σ separately. Ideally, we would like to combine these into one object. Toward this end, let Q be a Borel measure on $\mathbb{R}^d$ given by

$$Q(A) = \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} 1_A(ru) Q_u(\mathrm{d}r)\sigma(\mathrm{d}u), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{3.8}$$

and note that $Q(\{0\}) = 0$. Now define a Borel measure R on $\mathbb{R}^d$ by

$$R(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^{1+1/p}}\right) |x|^{\alpha/p} Q(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{3.9}$$

and again note that $R(\{0\}) = 0$. To get the inverse transformation we have

$$Q(A) = \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|^{p+1}}\right) |x|^{\alpha} R(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d). \tag{3.10}$$

From here it follows that

$$Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} |x|^{\alpha} R(\mathrm{d}x). \tag{3.11}$$

We now write the Lévy measure M in terms of R. By (3.2) and (3.6) for any $A \in \mathfrak{B}(\mathbb{R}^d)$ we have

$$\begin{aligned} M(A) &= \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} \int_0^\infty 1_A(ru) r^{-\alpha-1} e^{-r^p s} \mathrm{d}r Q_u(\mathrm{d}s) \sigma(\mathrm{d}u) \\ &= \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} \int_0^\infty 1_A(ts^{-1/p}u) t^{-1-\alpha} e^{-t^p} \mathrm{d}t s^{\alpha/p} Q_u(\mathrm{d}s) \sigma(\mathrm{d}u) \\ &= \int_{\mathbb{R}^d} \int_0^\infty 1_A\left(t\frac{x}{|x|^{1+1/p}}\right) t^{-1-\alpha} e^{-t^p} \mathrm{d}t |x|^{\alpha/p} Q(\mathrm{d}x), \end{aligned}$$

where the second equality follows by the substitution $t = rs^{1/p}$. From here (3.10) gives

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx) t^{-1-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x), \quad A \in \mathfrak{B}(\mathbb{R}^d). \tag{3.12}$$

This is the form of the Lévy measure that tends to be the most convenient to work with.

This representation raises several questions: If we are given a measure of the form (3.12), under what conditions will it be a Lévy measure? When it is a Lévy measure, is it necessarily the Lévy measure of a p-tempered α-stable distribution? Is there a one-to-one relationship between the measures M and R? The answers are provided by the following.

Theorem 3.3. *1. Fix $p > 0$ and let M be given by (3.12). M is the Lévy measure of an infinitely divisible distribution if and only if either $\alpha \in \mathbb{R}$ and $R = 0$ or $\alpha < 2$,*

$$R(\{0\}) = 0, \tag{3.13}$$

and

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\alpha\right) R(\mathrm{d}x) < \infty \text{ if } \alpha \in (0,2),$$

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge [1 + \log^+ |x|]\right) R(\mathrm{d}x) < \infty \quad \text{if } \alpha = 0, \tag{3.14}$$

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) R(\mathrm{d}x) < \infty \quad \text{if } \alpha < 0.$$

2. *Fix $p > 0$, $\alpha < 2$, and let M be given by (3.12). If R satisfies (3.13) and (3.14), then M is the Lévy measure of a p-tempered α-stable distribution and it uniquely determines R. Moreover, M is the Lévy measure of a proper p-tempered α-stable distribution if and only if*

$$\int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x) < \infty. \tag{3.15}$$

Proof. We begin with Part 1. By (2.2) M is a Lévy measure if and only if $M(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)M(\mathrm{d}x) < \infty$. Assume $R \neq 0$, since the other case is trivial. For any $\alpha \in \mathbb{R}$

$$M(\{0\}) = \int_{\mathbb{R}^d}\int_0^\infty 1_{\{0\}}(tx)t^{-\alpha-1}e^{-t^p}\mathrm{d}tR(\mathrm{d}x) = \int_{\{0\}}\int_0^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x),$$

which equals zero if and only if $R(\{0\}) = 0$.

Now assume that $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)M(\mathrm{d}x) < \infty$. We will show that this implies that $\alpha < 2$ and that R satisfies (3.14). Fix $\epsilon > 0$ and note that

$$\begin{aligned}\infty > \int_{|x|\le 1} |x|^2 M(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &\ge \int_{|x|\le 1/\epsilon} |x|^2 \int_0^\epsilon t^{1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x) \ge e^{-\epsilon^p}\int_{|x|\le 1/\epsilon}|x|^2R(\mathrm{d}x)\int_0^\epsilon t^{1-\alpha}\mathrm{d}t.\end{aligned}$$

Since $R \neq 0$, for this be finite for all $\epsilon > 0$ it is necessary that $\alpha < 2$. Taking $\epsilon = 1$ gives the necessity of $\int_{|x|\le 1}|x|^2R(\mathrm{d}x) < \infty$. Observing that

$$\begin{aligned}\infty > \int_{|x|\ge 1} M(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &\ge \int_1^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}t\int_{|x|\ge 1}R(\mathrm{d}x) + e^{-1}\int_{|x|\ge 1}\int_{|x|^{-1}}^1 t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x)\end{aligned}$$

gives the necessity of $\int_{|x|\ge 1} R(\mathrm{d}x) < \infty$ and $\int_{|x|\ge 1}\int_{|x|^{-1}}^1 t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x) < \infty$. When $\alpha < 0$ we are done. When $\alpha = 0$ we have

$$\int_{|x|\ge 1}\int_{|x|^{-1}}^1 t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x) = \int_{|x|\ge 1} \log|x|R(\mathrm{d}x),$$

and when $\alpha \in (0,2)$ we have

$$\int_{|x|\ge 1}\int_{|x|^{-1}}^1 t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x) = \frac{1}{\alpha}\int_{|x|\ge 1} \left(|x|^\alpha - 1\right) R(\mathrm{d}x),$$

which together with the necessity of $\int_{|x|\ge 1} R(\mathrm{d}x) < \infty$ gives (3.14).

Now assume that $\alpha < 2$ and that R satisfies (3.14). We have

$$\begin{aligned}\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &\le \int_{|x|\le 1}|x|^2R(\mathrm{d}x)\int_0^\infty t^{1-\alpha}e^{-t^p}\mathrm{d}t + \int_{|x|>1}|x|^2\int_0^{|x|^{-1}} t^{1-\alpha}\mathrm{d}tR(\mathrm{d}x)\\ &= p^{-1}\Gamma\left(\frac{2-\alpha}{p}\right)\int_{|x|\le 1}|x|^2R(\mathrm{d}x) + (2-\alpha)^{-1}\int_{|x|>1}|x|^\alpha R(\mathrm{d}x),\end{aligned}$$

which is finite. Now let $D = \sup_{t\ge 1} t^{2-\alpha}e^{-t^p}$ and note that

$$\begin{aligned}\int_{|x|\ge 1} M(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &\le D\int_{|x|\le 1}\int_{|x|^{-1}}^\infty t^{-3}\mathrm{d}tR(\mathrm{d}x) + \int_{|x|>1}\int_{|x|^{-1}}^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &= .5D\int_{|x|\le 1}|x|^2R(\mathrm{d}x) + \int_{|x|>1}\int_{|x|^{-1}}^1 t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &\qquad + \int_1^\infty t^{-1-\alpha}e^{-t^p}\mathrm{d}t\int_{|x|>1}R(\mathrm{d}x),\end{aligned}$$

which is finite since the second integral is bounded by $\int_{|x|>1}\frac{|x|^\alpha-1}{\alpha}R(\mathrm{d}x)$ when $\alpha \ne 0$ and by $\int_{|x|>1}\log|x|R(\mathrm{d}x)$ when $\alpha = 0$.

We now turn to Part 2. First we show that M is, necessarily, the Lévy measure of a p-tempered α-stable distribution. From R define Q by (3.10) and note that $Q(\{0\}) = 0$. By a straightforward extension of Lemma 2.1 in [6], Q has a polar decomposition, i.e. there exists a finite Borel measure σ on $\mathbb{S}^{d-1}$ and a measurable family of Borel measures $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ on $(0,\infty)$ such that $Q(A) = \int_{\mathbb{S}^{d-1}}\int_{(0,\infty)} 1_A(ru)Q_u(\mathrm{d}r)\sigma(\mathrm{d}u)$ for $A \in \mathfrak{B}(\mathbb{R}^d)$. Define $q(s,u) := \int_{(0,\infty)} e^{-sr}Q_u(\mathrm{d}r)$ and note that (3.14) implies that for every $\alpha < 2$

$$\begin{aligned}\infty > \int_{\mathbb{R}^d}\left(|x|^2 \wedge |x|^\alpha\right) R(\mathrm{d}x) &= \int_{\mathbb{R}^d}\left(|x|^{-(2-\alpha)/p} \wedge 1\right) Q(\mathrm{d}x)\\ &= \int_{\mathbb{S}^{d-1}}\int_{(0,\infty)}\left(r^{-(2-\alpha)/p} \wedge 1\right) Q_u(\mathrm{d}r)\sigma(\mathrm{d}u),\end{aligned}$$

which means that for σ a.e. u the function $q(s,u)$ is finite for every $s > 0$. For $A \in \mathfrak{B}(\mathbb{R}^d)$ we have

$$\begin{aligned}M(A) &= \int_{\mathbb{R}^d}\int_0^\infty 1_A(xt)t^{-1-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_A(tx|x|^{-1-1/p})t^{-1-\alpha}e^{-t^p}\mathrm{d}t|x|^{\alpha/p}Q(\mathrm{d}x)\\ &= \int_{\mathbb{S}^{d-1}}\int_{(0,\infty)}\int_0^\infty 1_A(tur^{-1/p})t^{-1-\alpha}e^{-t^p}\mathrm{d}tr^{\alpha/p}Q_u(\mathrm{d}r)\sigma(\mathrm{d}u)\\ &= \int_{\mathbb{S}^{d-1}}\int_{(0,\infty)}\int_0^\infty 1_A(us)s^{-1-\alpha}e^{-s^pr}\mathrm{d}sQ_u(\mathrm{d}r)\sigma(\mathrm{d}u)\\ &= \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_A(us)q(s^p,u)s^{-1-\alpha}\mathrm{d}s\sigma(\mathrm{d}u),\end{aligned} \tag{3.16}$$

which means that this is the Lévy measure of a p-tempered α-stable distribution.

Now to show the uniqueness of R. Assume that two measures R^1 and R^2 satisfy (3.12), (3.13), and (3.14). For each $i = 1,2$ define Q^i by (3.10), let $\{Q^i_u\}_{u\in\mathbb{S}^{d-1}}$ and σ^i be a polar decomposition of Q^i, and define $q^i(s,u) := \int_{(0,\infty)} e^{-sr}Q^i_u(\mathrm{d}r)$. From (3.16) it follows that we can decompose M into polar coordinates in two ways. First as $\{q^1(s^p,u)s^{-1-\alpha}\mathrm{d}s\}_{u\in\mathbb{S}^{d-1}}$ and σ^1 and second as $\{q^2(s^p,u)s^{-1-\alpha}\mathrm{d}s\}_{u\in\mathbb{S}^{d-1}}$ and σ^2. By the uniqueness of polar decompositions (see Lemma 2.1 in [6]) there exists a Borel function $c(u)$ such that $0 < c(u) < \infty$,

$$\sigma^1(\mathrm{d}u) = c(u)\sigma^2(\mathrm{d}u),$$

and

$$c(u)q^1(s^p,u)s^{-1-\alpha}\mathrm{d}s = q^2(s^p,u)s^{-1-\alpha}\mathrm{d}s \text{ for } \sigma^1 \text{ a.e. } u.$$

By Theorem 16.10 in [10] and the continuity in s of $q^i(s,u)$ for $i = 1,2$ this implies that for σ^1 a.e. u

$$c(u)q^1(s^p,u) = q^2(s^p,u), \quad s > 0$$

which can be rewritten as

$$\int_0^\infty e^{-s^p t} c(u) Q_u^1(\mathrm{d}t) = \int_0^\infty e^{-s^p t} Q_u^2(\mathrm{d}t), \quad s > 0 \text{ for } \sigma^1 \text{ a.e. } u.$$

Since Laplace transforms uniquely determine measures we have $c(u)Q_u^1(\cdot) = Q_u^2(\cdot)$ for σ^1 a.e. u. Thus for any $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned} Q^1(A) &= \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} 1_A(ru) Q_u^1(\mathrm{d}r)\sigma^1(\mathrm{d}u) \\ &= \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} 1_A(ru) c(u) Q_u^1(\mathrm{d}r) \frac{1}{c(u)} \sigma^1(\mathrm{d}u) \\ &= \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} 1_A(ru) Q_u^2(\mathrm{d}r)\sigma^2(\mathrm{d}u) = Q^2(A). \end{aligned}$$

By (3.9) this implies that $R^1(A) = R^2(A)$ as well.

We now consider the case of proper p-tempered α-stable distributions. Let Q be given by (3.10). From Remark 3.4 it follows that Q corresponds to a proper p-tempered α-stable distribution if and only if there is a polar decomposition of Q into $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ and σ such that Q_u is a probability measure for each $u \in \mathbb{S}^{d-1}$ and σ is a finite Borel measure on $\mathbb{S}^{d-1}$. Such a polar decomposition of Q exists if and only if Q is finite. From here the result follows by (3.11). □

Definition 3.4. Fix $\alpha < 2, p > 0$, and let $\mu \in TS_\alpha^p$. Then $\mu = ID(0, M, b)$ for some $b \in \mathbb{R}^d$ and some Lévy measure M, which can be written in the form (3.12) for a unique measure R. We call R the **Rosiński measure** of μ and we write $TS_\alpha^p(R, b)$ to denote this distribution.

An important property of p-tempered α-stable distributions is that they are closed under shifting, scaling, and convolution. Specifically, from (3.12) and (2.1) we get the following.

Proposition 3.5. *Fix $\alpha < 2$ and $p > 0$. 1. If $X \sim TS_\alpha^p(R, b)$ and $a \in \mathbb{R}$, then $aX \sim TS_\alpha^p(R_a, b_a)$, where*

$$R_a(A) = \int_{\mathbb{R}^d} 1_{A\setminus\{0\}}(ax) R(\mathrm{d}x), \qquad A \in (\mathbb{R}^d)$$

and

$$\begin{aligned} b_a &= ab + \int_{\mathbb{R}^d} \int_0^\infty \left(\frac{ax}{1 + a^2t^2|x|^2} - \frac{ax}{1 + t^2|x|^2} \right) t^{-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x) \\ &= ab + a(1 - a^2) \int_{\mathbb{R}^d} \int_0^\infty \frac{x|x|^2}{(1 + a^2t^2|x|^2)(1 + t^2|x|^2}) t^{2-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x). \end{aligned}$$

2. *If* $X_1 \sim TS^p_\alpha(R_1, b_1)$ *and* $X_2 \sim TS^p_\alpha(R_2, b_2)$ *are independent and* $b \in \mathbb{R}^d$*, then*

$$X_1 + X_2 + b \sim TS^p_\alpha(R_1 + R_2, b_1 + b_2 + b),$$

where $R_1 + R_2$ *is the Borel measure defined by* $(R_1 + R_2)(B) = R_1(B) + R_2(B)$ *for any* $B \in \mathfrak{B}(\mathbb{R}^d)$.

For proper p-tempered α-stable distributions we can recover the representation of the Lévy measure given by (3.2) as follows.

Proposition 3.6. *Fix* $\alpha < 2$, $p > 0$*, and let* M *be the Lévy measure of a proper* p*-tempered* α*-stable distribution with Rosiński measure* R. M *can be represented by* (3.2) *with* $q(r, u)$ *satisfying* (3.7) *and*

$$\sigma(B) = \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right)|x|^\alpha R(\mathrm{d}x), \qquad B \in \mathfrak{B}(\mathbb{S}^{d-1}). \tag{3.17}$$

If, in addition, $\alpha \in (0, 2)$*, then the Lévy measure of an* α*-stable distribution with spectral measure* σ *is given by*

$$L(B) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(tx)\, t^{-\alpha-1}\mathrm{d}tR(\mathrm{d}x), \qquad B \in \mathfrak{B}(\mathbb{R}^d).$$

Proof. Let Q be derived from R by (3.10). Remark 3.4 implies that there is a finite Borel measure σ on $\mathbb{S}^{d-1}$ and a measurable family of probability measures $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ such that Q can be represented in terms of σ and $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ as in (3.8) and that M can be represented by (3.2) where $q(r, u) = \int_{(0,\infty)} e^{-sr}Q_u(\mathrm{d}r)$. From here it follows that $q(r, u)$ satisfies (3.7) by the Monotone Convergence Theorem and the fact that Q_u is a probability measure for each $u \in \mathbb{S}^{d-1}$. Further, for any $A \in \mathfrak{B}(\mathbb{S}^{d-1})$

$$\begin{aligned}\int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|}\right)|x|^\alpha R(\mathrm{d}x) &= \int_{\mathbb{R}^d} 1_A\left(\frac{x}{|x|}\right)Q(\mathrm{d}x)\\ &= \int_A\int_{(0,\infty)} Q_u(\mathrm{d}s)\sigma(\mathrm{d}u) = \sigma(B).\end{aligned}$$

The second part follows from the first and the fact that for any $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned}L(A) &= \int_{\mathbb{S}^{d-1}}\int_0^\infty 1_B(su)s^{-\alpha-1}\mathrm{d}s\sigma(\mathrm{d}u)\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_B(sx/|x|)s^{-1-\alpha}\mathrm{d}s|x|^\alpha R(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_B(tx)t^{-\alpha-1}\mathrm{d}tR(\mathrm{d}x),\end{aligned}$$

where the third equality follows by the substitution $t = s/|x|$. □

3.2 Identifiability and Subclasses

In Theorem 3.3 we saw that for fixed $p > 0$ and $\alpha < 2$ there is a one-to-one relationship between the Rosiński measure R and the Lévy measure M. We may further ask whether all of the parameters are jointly identifiable. Unfortunately, the answer is negative. In fact, even for fixed $p > 0$, the parameters α and R are not jointly identifiable. However, for fixed $p > 0$, in the subclass of proper tempered stable distribution, they are jointly identifiable. On the other hand, for fixed $\alpha < 2$, even in the subclass of proper tempered stable distributions, the parameters p and R are not jointly identifiable. These facts will be verified in this section. We begin with a lemma.

Lemma 3.7. *Fix $\alpha < 2$, $p > 0$, and let M be the Lévy measure of a p-tempered α-stable distribution with Rosiński measure $R \neq 0$.*

1. *The map $s \mapsto s^\alpha M(|x| > s)$ is decreasing and* $\lim_{s\to\infty} s^\alpha M(|x| > s) = 0$.
2. *If $\alpha \in (0, 2)$, then*

$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \frac{1}{\alpha}\int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x)$$

and if $\alpha \leq 0$, then

$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \infty.$$

3. *If $\alpha < 0$, then*

$$\lim_{s\downarrow 0} s^\alpha M(|x| < s) = \frac{1}{|\alpha|}\int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x)$$

and if $\alpha \in [0, 2)$, then for all $s > 0$

$$M(|x| < s) = \infty.$$

Proof. We begin with the first part. Since

$$\begin{aligned} s^\alpha M(|x| > s) &= s^\alpha \int_{\mathbb{R}^d}\int_{s|x|^{-1}}^\infty t^{-1-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty t^{-1-\alpha} e^{-(st)^p} \mathrm{d}t R(\mathrm{d}x), \end{aligned} \tag{3.18}$$

the map $s \mapsto s^\alpha M(|x| > s)$ is decreasing. For large enough s, the integrand in (3.18) is bounded by $1_{t>1/|x|} t^{-1-\alpha} e^{-t^p}$, which is integrable. Thus by dominated convergence $\lim_{s\to\infty} s^\alpha M(|x| > s) = 0$.

For the second part, by (3.18) and the Monotone Convergence Theorem

$$\lim_{s\downarrow 0} s^{\alpha} M(|x| > s) = \int_{\mathbb{R}^d}\int_{|x|^{-1}}^{\infty} t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x).$$

Thus if $\alpha \in (0,2)$, then

$$\lim_{s\downarrow 0} s^{\alpha} M(|x| > s) = \frac{1}{\alpha}\int_{\mathbb{R}^d} |x|^{\alpha} R(\mathrm{d}x),$$

and if $\alpha \le 0$, then

$$\lim_{s\downarrow 0} s^{\alpha} M(|x| > s) = \infty.$$

We now show the third part. If $\alpha \in [0,2)$, then for all $s > 0$

$$\begin{aligned} M(|x| < s) &= \int_{\mathbb{R}^d}\int_0^{s|x|^{-1}} t^{-1-\alpha} e^{-t^p}\mathrm{d}tR(\mathrm{d}x) \\ &\ge \int_{\mathbb{R}^d} e^{-(s/|x|)^p}\int_0^{s|x|^{-1}} t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x) = \infty, \end{aligned}$$

and if $\alpha < 0$, then

$$\begin{aligned} \lim_{s\downarrow 0} s^{\alpha} M(|x| < s) &= \lim_{s\downarrow 0} s^{\alpha}\int_{\mathbb{R}^d}\int_0^{s|x|^{-1}} t^{-1-\alpha} e^{-t^p}\mathrm{d}tR(\mathrm{d}x) \\ &= \lim_{s\downarrow 0}\int_{\mathbb{R}^d}\int_0^{|x|^{-1}} t^{-1-\alpha} e^{-(st)^p}\mathrm{d}tR(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d}\int_0^{|x|^{-1}} t^{-1-\alpha}\mathrm{d}tR(\mathrm{d}x) = \frac{1}{|\alpha|}\int_{\mathbb{R}^d} |x|^{\alpha} R(\mathrm{d}x), \end{aligned}$$

where the third line follows by the Monotone Convergence Theorem. □

Combining Lemma 3.7 with (3.15) gives the following.

Proposition 3.8. *In the subclass of proper tempered stable distributions with parameter $p > 0$ fixed, the parameters R and α are jointly identifiable.*

However, in general, the parameters α and p are not identifiable. This will become apparent from the following results.

Proposition 3.9. *Fix $\alpha < 2$, $\beta \in (\alpha, 2)$, and let $K = \int_0^{\infty} s^{\beta-\alpha-1} e^{-s^p}\mathrm{d}s$. If $\mu = TS^p_{\beta}(R, b)$ and*

$$R'(A) = K^{-1}\int_{\mathbb{R}^d}\int_0^1 1_A(ux)u^{-\beta-1}\left(1-u^p\right)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x),$$

then R' is the Rosiński measure of a p-tempered α-stable distribution and $\mu = TS_\alpha^p(R', b)$.

Proof. We begin by verifying that R' is the Rosiński measure of some p-tempered α-stable distribution. Let $C = \max_{u\in[0,.5]}(1-u^p)^{(\beta-\alpha)/p-1}$. We have

$$\begin{aligned}
K\int_{|x|\le 1}|x|^2R'(\mathrm{d}x) &= \int_{\mathbb{R}^d}|x|^2\int_0^{1\wedge|x|^{-1}} u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x)\\
&\le \int_{|x|\le 2}|x|^2R(\mathrm{d}x)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\\
&\quad +C\int_{|x|>2}|x|^2\int_0^{|x|^{-1}} u^{1-\beta}\mathrm{d}uR(\mathrm{d}x)\\
&= \int_{|x|\le 2}|x|^2R(\mathrm{d}x)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\\
&\quad +\frac{C}{2-\beta}\int_{|x|\ge 2}|x|^\beta R(\mathrm{d}x) < \infty.
\end{aligned}$$

If $\alpha \in (0,2)$, then

$$\begin{aligned}
K\int_{|x|>1}|x|^\alpha R'(\mathrm{d}x) &= \int_{|x|\ge 1}|x|^\alpha\int_{|x|^{-1}}^1 u^{\alpha-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x)\\
&\le \int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^{1/2} u^{\alpha-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x)\\
&\quad +\int_{|x|\ge 1}|x|^\alpha\int_{1/2}^1 u^{\alpha-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x)\\
&\le C\int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^\infty u^{\alpha-\beta-1}\mathrm{d}uR(\mathrm{d}x)\\
&\quad +\int_{|x|\ge 1}|x|^\beta R(\mathrm{d}x)\int_{1/2}^1 u^{\alpha-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u,
\end{aligned}$$

which is finite since the first integral equals $\frac{C}{\beta-\alpha}\int_{|x|\ge 2}|x|^\beta R(\mathrm{d}x) < \infty$. Now assume $\alpha = 0$ and fix $\epsilon \in (0,\beta)$. By 4.1.37 in [2] there exists a $C_\epsilon > 0$ such that for all $u > 0$, $\log u \le C_\epsilon u^\epsilon$. Thus

$$K\int_{|x|>1}\log|x|R'(\mathrm{d}x) \le KC_\epsilon\int_{|x|>1}|x|^\epsilon R'(\mathrm{d}x),$$

which is finite by arguments similar to the previous case. When $\alpha < 0$

$$K\int_{|x|>1} R'(\mathrm{d}x) = \int_{|x|\geq 1}\int_{|x|^{-1}}^{1} u^{-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}uR(\mathrm{d}x)$$

$$\leq C\int_{|x|\geq 2}\int_{|x|^{-1}}^{1} u^{-\beta-1}\mathrm{d}uR(\mathrm{d}x)$$

$$+\int_{|x|\geq 1} R(\mathrm{d}x)\int_{1/2}^{1} u^{-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u,$$

which is finite since for $\beta \neq 0$ the first integral is $\frac{C}{\beta}\int_{|x|>2}\left(|x|^{\beta}-1\right)R(\mathrm{d}x) < \infty$ and for $\beta = 0$ it is $\int_{|x|>2}\log|x|R(\mathrm{d}x) < \infty$. Now, let M' be the Lévy measure of $TS^p_\alpha(R', b)$. By (3.12) for $A \in \mathfrak{B}(\mathbb{R}^d)$ we have

$$\begin{aligned} M'(A) &= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^1 1_A(utx)t^{-1-\alpha}e^{-t^p}u^{-\beta-1}\left(1-u^p\right)^{\frac{\beta-\alpha}{p}-1}\mathrm{d}u\mathrm{d}tR(\mathrm{d}x)\\ &= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^t 1_A(vx)t^{\beta-\alpha-1}e^{-t^p}v^{-\beta-1}\left(1-v^p/t^p\right)^{\frac{\beta-\alpha}{p}-1}\mathrm{d}v\mathrm{d}tR(\mathrm{d}x)\\ &= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_v^\infty 1_A(vx)t^{p-1}e^{-t^p}v^{-\beta-1}\left(t^p-v^p\right)^{\frac{\beta-\alpha}{p}-1}\mathrm{d}t\mathrm{d}vR(\mathrm{d}x)\\ &= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)e^{-v^p}v^{-\beta-1}\mathrm{d}vR(\mathrm{d}x)\int_0^\infty e^{-s^p}s^{\beta-\alpha-1}\mathrm{d}s\\ &= \int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)e^{-v^p}v^{-\beta-1}\mathrm{d}vR(\mathrm{d}x), \end{aligned}$$

where the second line follows by the substitution $v = ut$ and the fourth by the substitution $s^p = t^p - v^p$. □

To show a similar result for the parameter p we need some additional notation. For $r \in (0,1)$, let f_r be a probability density with $f_r(x) = 0$ for $x < 0$ and

$$\int_0^\infty e^{-tx}f_r(x)\mathrm{d}x = e^{-t^r}.$$

Such a density exists, and is, in fact, the density of a certain type of r-stable distribution, see Proposition 1.2.12 in [68]. The only case where an explicit formula is known is

$$f_{.5}(s) = \frac{1}{2\sqrt{\pi}}e^{-1/(4s)}s^{-3/2}1_{[s>0]}$$

(see Examples 2.13 and 8.11 in [69]). From Theorem 5.4.1 in [78] it follows that if $\beta \geq 0$, then

$$\int_0^\infty s^{-\beta}f_r(s)\mathrm{d}s < \infty.$$

Proposition 3.10. *Fix $\alpha < 2$ and $0 < p < q$. If $\mu = TS^p_\alpha(R,b)$ and*

$$R'(A) = \int_{\mathbb{R}^d}\int_0^\infty 1_A(s^{-1/q}x)s^{\alpha/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x),$$

then R' is the Rosiński measure of a q-tempered α-stable distribution and $\mu = TS^q_\alpha(R',b)$. Moreover, μ is a proper p-tempered α-stable distribution if and only if it is a proper q-tempered α-stable distribution.

This implies that, for fixed α, the parameters p and R are not jointly identifiable even within the subclass of proper tempered stable distributions.

Proof. We begin by verifying that R' is, in fact, the Rosiński measure of a q-tempered α-stable distribution. We have

$$\begin{aligned}\int_{|x|\le 1}|x|^2R'(\mathrm{d}x) &= \int_{\mathbb{R}^d}|x|^2\int_{|x|^q}^\infty s^{-(2-\alpha)/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)\\ &\le \int_{|x|\le 1}|x|^2\int_0^\infty s^{-(2-\alpha)/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)\\ &\quad + \int_{|x|>1}|x|^\alpha R(\mathrm{d}x)\int_0^\infty f_{p/q}(s)\mathrm{d}s < \infty.\end{aligned}$$

If $\alpha \neq 0$ and $\beta = \alpha \vee 0$, then

$$\begin{aligned}\int_{|x|>1}|x|^\beta R'(\mathrm{d}x) &= \int_{\mathbb{R}^d}|x|^\beta\int_0^{|x|^q} s^{-(\beta-\alpha)/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)\\ &\le \int_{|x|\le 1}|x|^2\int_0^\infty s^{-(2-\alpha)/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)\\ &\quad + \int_{|x|>1}|x|^\beta\int_0^\infty s^{-(\beta-\alpha)/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x) < \infty.\end{aligned}$$

If $\alpha = 0$, then

$$\begin{aligned}\int_{|x|>1}\log|x|R'(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_0^{|x|^q}\log|xs^{-1/q}|f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)\\ &\le .5\int_{|x|\le 1}|x|^2R(\mathrm{d}x)\int_0^\infty s^{-2/q}f_{p/q}(s)\mathrm{d}s\\ &\quad + \int_{|x|>1}\log|x|R(\mathrm{d}x)\int_0^\infty f_{p/q}(s)\mathrm{d}s\\ &\quad + \int_{|x|>1}R(\mathrm{d}x)\int_0^\infty s^{-1/q}f_{p/q}(s)\mathrm{d}s < \infty,\end{aligned}$$

where the inequality uses the fact that $\log|x| \le |x|$ (see 4.1.36 in [2]).

If M' is the Lévy measure of $TS^q_\alpha(R', b)$, then by (3.12) for any $A \in \mathfrak{B}(\mathbb{R}^d)$

$$M'(A) = \int_{\mathbb{R}^d}\int_0^\infty\int_0^\infty 1_A(s^{-1/q}tx)t^{-1-\alpha}e^{-t^q}\mathrm{d}ts^{\alpha/q}f_{p/q}(s)\mathrm{d}sR(\mathrm{d}x)$$
$$= \int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)v^{-1-\alpha}\int_0^\infty e^{-v^q s}f_{p/q}(s)\mathrm{d}s\mathrm{d}vR(\mathrm{d}x)$$
$$= \int_{\mathbb{R}^d}\int_0^\infty 1_A(vx)v^{-1-\alpha}e^{-v^p}\mathrm{d}vR(\mathrm{d}x),$$

where $v = s^{-1/q}t$. The last part follows from (3.15) and the fact that

$$\int_{\mathbb{R}^d}|x|^\alpha R'(\mathrm{d}x) = \int_{\mathbb{R}^d}|x|^\alpha R(\mathrm{d}x)\int_0^\infty s^{-\alpha/q}s^{\alpha/q}f_{p/q}(s)\mathrm{d}s = \int_{\mathbb{R}^d}|x|^\alpha R(\mathrm{d}x).$$

This concludes the proof. □

Propositions 3.9 and 3.10 give a constructive proof of the following.

Proposition 3.11. *Fix* $\alpha < 2$, $p > 0$, *and let* $\mu \in TS^p_\alpha$.

1. *For any* $q \geq p$, $\mu \in TS^q_\alpha$.
2. *For any* $\beta \leq \alpha$, $\mu \in TS^p_\beta$.

We now characterize when a p-tempered α-stable distribution is β-stable for some $\beta \in (0, 2)$.

Proposition 3.12. *Fix* $\alpha < 2$, $p > 0$, *and* $\beta \in (0, 2)$. *Let* $\mu = S_\beta(\sigma, b)$, *where* $\sigma \neq 0$. *If* $\beta \leq \alpha$, *then* $\mu \notin TS^p_\alpha$. *If* $\beta \in (0 \vee \alpha, 2)$, *then* $\mu = TS^p_\alpha(R^\beta_\sigma, b)$ *and*

$$R^\beta_\sigma(A) = K^{-1}\int_{\mathbb{S}^{d-1}}\int_0^\infty 1_A(ru)r^{-1-\beta}\mathrm{d}r\sigma(\mathrm{d}u), \quad A \in \mathfrak{B}(\mathbb{R}^d), \qquad (3.19)$$

where $K = \int_0^\infty t^{\beta-\alpha-1}e^{-t^p}\mathrm{d}t$.

Combining (3.15) with the fact that

$$\int_{\mathbb{R}^d}|x|^\alpha R^\beta_\sigma(\mathrm{d}x) = K^{-1}\sigma(\mathbb{S}^{d-1})\int_0^\infty r^{-(\beta-\alpha)-1}\mathrm{d}r = \infty,$$

shows that no stable distributions belong to the subclass of proper p-tempered α-stable distributions.

Proof. If $\mu \in TS^p_\alpha$, then its Lévy measure can be written as (3.2). By uniqueness of the polar decomposition of Lévy measures (see Lemma 2.1 in [6]) there exists a nonnegative Borel function $c(u)$ with $\sigma(\{u : c(u) > 0\}) > 0$ such that $q(r, u) = c(u)r^{(\alpha-\beta)/p}$. This does not satisfy (3.4) when $\beta \leq \alpha$.

Now assume that $\beta > \alpha$. In this case $R_\sigma^\beta(\{0\}) = 0$ and for any $\gamma \in [0, \beta)$

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge |x|^\gamma\right) R_\sigma^\beta(\mathrm{d}x) = K^{-1}\sigma(\mathbb{S}^{d-1}) \int_0^\infty (r^{1-\beta} \wedge r^{\gamma-\beta-1})\mathrm{d}r < \infty.$$

Thus, by Theorem 3.3, R_σ^β is the Rosiński measure of a p-tempered α-stable distribution. If M is the Lévy measure of $TS_\alpha^p(R_\sigma^\beta, b)$, then for any $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\begin{aligned} M(A) &= K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(rtu) t^{-1-\alpha} e^{-t^p} \mathrm{d}t r^{-1-\beta} \mathrm{d}r\sigma(\mathrm{d}u) \\ &= K^{-1} \int_0^\infty t^{\beta-\alpha-1} e^{-t^p} \mathrm{d}t \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) r^{-1-\beta} \mathrm{d}r\sigma(\mathrm{d}u) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(ru) r^{-1-\beta} \mathrm{d}r\sigma(\mathrm{d}u), \end{aligned}$$

which is the Lévy measure of μ. □

Recall that a probability measure μ is called compound Poisson if its characteristic function can be written as

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1\right) M(\mathrm{d}x)\right\}, \qquad z \in \mathbb{R}^d,$$

where M is a finite Lévy measure. To classify when tempered stable distributions are compound Poisson we begin with a lemma.

Lemma 3.13. *Let M be given by (3.12). M is finite if and only if either $R = 0$ or $\alpha < 0$ and R is a finite measure.*

Proof. Observing that

$$\begin{aligned} R(\mathbb{R}^d)e^{-1} \int_0^1 t^{-1-\alpha}\mathrm{d}t &\le \int_{\mathbb{R}^d} \int_0^\infty e^{-t^p} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\ &\le R(\mathbb{R}^d) \left(\int_0^1 t^{-1-\alpha}\mathrm{d}t + \int_1^\infty e^{-t^p} t^{-1-\alpha}\mathrm{d}t \right) \end{aligned}$$

gives the result. □

This immediately gives the following.

Proposition 3.14. *If $\mu = TS_\alpha^p(R, b)$, then μ is compound Poisson if and only if either $R = 0$ or $\alpha < 0$, R is a finite measure, and $b = \int_{\mathbb{R}^d} \int_0^\infty \frac{x}{1+t^2|x|^2} t^{-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x)$.*

3.3 Tails of Tempered Stable Distributions

Since the motivation for introducing p-tempered α-stable distributions is to get models with tails lighter than those of α-stable distributions, it is important to understand how the tails behave. One of the easiest ways to describe the tails of a distribution is to characterize which moments are finite. Toward this end we present several results that were proved in [27]. Throughout this section we adopt the convention that $0^0 = 1$.

Theorem 3.15. *Fix* $\alpha < 2$, $p > 0$, *and let* $\mu = TS^p_\alpha(R, b)$.

1. *If* $\alpha \in (0, 2)$ *and* $q_1, \dots, q_d \geq 0$ *with* $q := \sum_{j=1}^d q_j < \alpha$, *then*

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(\mathrm{d}x) \leq \int_{\mathbb{R}^d} |x|^q \mu(\mathrm{d}x) < \infty.$$

2. *If* $\alpha \in (0, 2)$, *then*

$$\int_{\mathbb{R}^d} |x|^\alpha \mu(\mathrm{d}x) < \infty \iff \int_{|x|>1} |x|^\alpha \log |x| R(\mathrm{d}x) < \infty.$$

Additionally, if $q_1, \dots, q_d \geq 0$ *with* $\sum_{j=1}^d q_j = \alpha$, *then*

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(\mathrm{d}x) < \infty$$

if and only if

$$\int_{|x|>1} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \log |x| R(\mathrm{d}x) < \infty. \tag{3.20}$$

3. *If* $q > (\alpha \vee 0)$, *then*

$$\int_{\mathbb{R}^d} |x|^q \mu(\mathrm{d}x) < \infty \iff \int_{|x|>1} |x|^q R(\mathrm{d}x) < \infty.$$

Additionally, if $q_1, \dots, q_d \geq 0$ *with* $\sum_{j=1}^d q_j > (\alpha \vee 0)$, *then*

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{r_j} \right) \mu(\mathrm{d}x) < \infty \text{ for all } r_k \in [0, q_k],\ k = 1, \dots, d$$

if and only if

$$\int_{|x|>1}\left(\prod_{j=1}^{d}|x_j|^{r_j}\right)R(\mathrm{d}x)<\infty \text{ for all } r_k\in[0,q_k],\ k=1,\dots,d. \tag{3.21}$$

Further, we can find explicit formulas for the moments and the mixed moments. However, these formulas can get quite complicated. When working with infinitely divisible distribution it is often easier to find the cumulants instead. Recall that for any infinitely divisible distribution μ the function C_μ given by (2.1) is called the cumulant generating function. This name is explained by the following. Let $k = (k_1, k_2, \dots, k_d)$ be a d-dimensional vector of nonnegative integers and let

$$c_k = (-i)^{\sum k_i}\frac{\partial^{\sum k_i}}{\partial z_d^{k_d}\cdots\partial z_1^{k_1}}C_\mu(z)\Big|_{z=0},$$

whenever the derivative exists and is continuous in a neighborhood of zero. We call this the cumulant of order k. The cumulants can be uniquely expressed in terms of the moments, see, e.g., [73]. In particular let $X\sim\mu$. When $k_i = 1$ and $k_j = 0$ for all $j\neq i$ then $c_k = \mathrm{E}[X_i]$, when $k_i = 2$ and $k_j = 0$ for all $j\neq i$ then $c_k = \mathrm{var}(X_i)$, and when for some $i\neq j$ we have $k_i = k_j = 1$ and $k_\ell = 0$ for all $\ell\neq i,j$ then $c_k = \mathrm{cov}(X_i, X_j)$. The following is given in [27].

Theorem 3.16. *Fix $\alpha < 2$, $p > 0$, and let $\mu = TS^p_\alpha(R, b)$. Let $q_1,\dots,q_d$ be nonnegative integers and let $q_+ = \sum_{i=1}^d q_i$. Further, if $q_+ = \alpha = 1$, assume that (3.20) holds and if $q_+ > \alpha$, that (3.21) holds. If $q_i = q_+ = 1$ for some i, then*

$$c_{(q_1,\dots,q_d)} = b_i + \int_{\mathbb{R}^d}\int_0^\infty x_i\frac{|x|^2}{1+|x|^2t^2}t^{2-\alpha}e^{-t^p}\mathrm{d}tR(\mathrm{d}x).$$

If $q_+\geq 2$, then

$$c_{(q_1,\dots,q_d)} = p^{-1}\Gamma\left(\frac{q_+-\alpha}{p}\right)\int_{\mathbb{R}^d}\left(\prod_{j=1}^{d}x_j^{q_j}\right)R(\mathrm{d}x).$$

We now turn to the question of exponential moments.

Theorem 3.17. *Fix $\alpha < 2$, $p\in(0,1]$, and $\theta > 0$. Let $\mu = TS^p_\alpha(R, b)$.*

1. *If $\alpha\in(0,2)$, then $\int_{\mathbb{R}^d} e^{\theta|x|^p}\mu(\mathrm{d}x)<\infty$ if and only if*

$$R(|x| > \theta^{-1/p}) = 0.$$

2. *If $\alpha = 0$, then $\int_{\mathbb{R}^d} e^{\theta|x|^p}\mu(\mathrm{d}x)<\infty$ if and only if*

$$R(|x|\geq\theta^{-1/p}) = 0 \text{ and } \int_{0<|x|^{-p}-\theta<1}|\log(|x|^{-p}-\theta)|\,R(\mathrm{d}x)<\infty.$$

3. If $\alpha < 0$*, then* $\int_{\mathbb{R}^d} e^{\theta|x|^p}\mu(\mathrm{d}x) < \infty$ *if and only if*

$$R(|x| \ge \theta^{-1/p}) = 0 \text{ and } \int_{0<|x|^{-p}-\theta<1} (|x|^{-p} - \theta)^{\alpha/p} R(\mathrm{d}x) < \infty.$$

Further, from Theorem 4 in [27] it follows that if $p > 1$ and there exists an $\epsilon > 0$ such that

$$\int_{|x|>1} e^{|x|^{\epsilon+p/(p-1)}} |x|^{-\alpha/(p-1)} R(\mathrm{d}x) < \infty, \tag{3.22}$$

then

$$\int_{\mathbb{R}^d} e^{\theta|x|}\mu(\mathrm{d}x) < \infty \text{ for all } \theta \ge 0. \tag{3.23}$$

However, the tails cannot be too light and if $R \ne 0$, then

$$\int_{\mathbb{R}^d} e^{\theta|x|\log|x|}\mu(\mathrm{d}x) = \infty \text{ for all } \theta > 0.$$

When the exponential moments exist, we can evaluate them. Specifically, if $\mu = TS^p_\alpha(R, b)$ and $z \in \mathbb{C}^d$ is such that $\int_{\mathbb{R}^d} e^{\langle x, \Re z\rangle}\mu(\mathrm{d}x) < \infty$, then Theorem 25.17 in [69] implies that $\int_{\mathbb{R}^d} |e^{\langle x,z\rangle}|\mu(\mathrm{d}x) < \infty$ and that $\int_{\mathbb{R}^d} e^{\langle x,z\rangle}\mu(\mathrm{d}x)$ is given by

$$\exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{\langle x,z\rangle t} - 1 - \frac{t\langle x, z\rangle}{1+|x|^2}\right) t^{-1-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x) + \langle z, b\rangle\right\}. \tag{3.24}$$

For the case $p = 1$ more explicit formulas will be given in Section 3.5.

Another way to analyze the tails of a probability measure is to ask when they are regularly varying. First consider the case where $\mu = TS^p_\alpha(R, b)$ with $\alpha \in (0, 2)$. Theorem 3.15 implies that $\int_{\mathbb{R}^d} |x|^\eta \mu(\mathrm{d}x) < \infty$ for all $\eta \in [0, \alpha)$, and hence, by Proposition 2.12, μ cannot have regularly varying tails with tail index $|\gamma| < \alpha$. However, other tail indices are possible. The following result from [27] characterizes this.

Theorem 3.18. *Fix* $\alpha < 2$ *and* $p > 0$*. Let* $\mu = TS^p_\alpha(R, b)$ *and let* M *be the Lévy measure of* μ*. If* $\gamma < (-\alpha) \wedge 0$*, then*

$$\mu \in RV^\infty_\gamma(\sigma) \iff M \in RV^\infty_\gamma(\sigma) \iff R \in RV^\infty_\gamma(\sigma).$$

Moreover, if $M \in RV^\infty_\gamma(\sigma)$*, then for all* $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ *with* $\sigma(\partial D) = 0$ *and* $\sigma(D) > 0$

$$\lim_{r\to\infty} \frac{R\left(|x| > r, x/|x| \in D\right)}{M\left(|x| > r, x/|x| \in D\right)} = \frac{p}{\Gamma\left(\frac{|\gamma|-\alpha}{p}\right)}.$$

Now recall that for $\beta \in (0, 2)$ a probability measure μ belongs to the domain of attraction of a β-stable distribution with spectral measure $\sigma \neq 0$ if and only if $\mu \in RV^{\infty}_{-\beta}(\sigma)$. See, e.g., [67] or [54] although they make the additional assumption that the limiting stable distribution is full. This leads to the following.

Corollary 3.19. *Fix $\alpha < 2$, $p > 0$, let $\mu = TS^p_\alpha(R, b)$, and let $\sigma \neq 0$ be a finite Borel measure on $\mathbb{S}^{d-1}$. If $\beta \in (0 \vee \alpha, 2)$, then μ belongs to the domain of attraction of a β-stable distribution with spectral measure σ if and only if $R \in RV^{\infty}_{-\beta}(\sigma)$.*

3.4 Tempered Stable Lévy Processes

Fix $\alpha < 2$ and $p > 0$. A Lévy process $\{X_t : t \geq 0\}$ is called a **p-tempered α-stable Lévy process** if $X_1 \sim TS^p_\alpha(R, b)$. In this section we discuss properties of such processes.

Proposition 3.20. *Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, b)$, and assume that $R \neq 0$.*

1. *The paths of $\{X_t : t \geq 0\}$ are discontinuous a.s.*
2. *The paths of $\{X_t : t \geq 0\}$ are piecewise constant a.s. if and only if $\alpha < 0$, R is a finite measure, and $b = \int_{\mathbb{R}^d} \int_0^\infty \frac{x}{1+t^2|x|^2} t^{-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x)$.*
3. *If $\alpha < 0$ and R is a finite measure, then, almost surely, jumping times are infinitely many and countable in increasing order. The first jumping time has an exponential distribution with mean $1/a$, where $a = R(\mathbb{R}^d) p^{-1} \Gamma(|\alpha|/p)$.*
4. *If $\alpha \geq 0$ or R is an infinite measure, then, almost surely, jumping times are countable and dense in $[0, \infty)$.*

Proof. Part 1 follows by Theorem 21.1 in [69]. Part 2 follows by Theorem 21.2 in [69] and Proposition 3.14. Parts 3 and 4 follow by Theorem 21.3 in [69] and Lemma 3.13. □

A useful index that determines many properties of Lévy processes was introduced by Blumenthal and Getoor [12]. It is defined as follows.

Definition 3.21. Let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0, M, b)$. The number

$$\beta = \inf \left\{ \gamma > 0 : \int_{|x| \leq 1} |x|^\gamma M(\mathrm{d}x) < \infty \right\}$$

is called the **Blumenthal-Getoor index**.

From the definition of a Lévy measure, it is clear that the Blumenthal-Getoor index is a number in $[0, 2]$.

Lemma 3.22. *Fix $p > 0$, $\alpha < 2$, and let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS_\alpha^p(R, b)$. If $R \neq 0$, then the Blumenthal-Getoor index of this process is*

$$\beta = \alpha \vee r, \tag{3.25}$$

where

$$r = \inf\left\{\gamma > 0 : \int_{|x|\leq 1} |x|^\gamma R(\mathrm{d}x) < \infty\right\}.$$

This follows immediately from the following.

Lemma 3.23. *Fix $\alpha < 2$, $p > 0$, let R be the Rosiński measure of a p-tempered α-stable distribution, and let M be the corresponding Lévy measure. If $R \neq 0$, then for any $q \in (-\infty, 2)$*

$$\int_{|x|\leq 1} |x|^q M(\mathrm{d}x) < \infty \Longleftrightarrow \alpha < q \text{ and } \int_{|x|\leq 1} |x|^q R(\mathrm{d}x) < \infty.$$

Proof. First assume that $\int_{|x|\leq 1} |x|^q M(\mathrm{d}x) < \infty$ and choose $r > 0$ such that $R(|x| \leq r) > 0$. We have

$$\begin{aligned}
\infty &> \int_{|x|\leq 1} |x|^q M(\mathrm{d}x) \\
&\geq \int_{|x|\leq r} |x|^q \int_0^{|x|^{-1}} t^{q-\alpha-1} e^{-t^p} \mathrm{d}t R(\mathrm{d}x) \\
&\geq e^{-r^{-p}} \int_{|x|\leq r} |x|^q R(\mathrm{d}x) \int_0^{r^{-1}} t^{q-\alpha-1} \mathrm{d}t,
\end{aligned}$$

which implies that $\alpha < q$ and $\int_{|x|\leq 1} |x|^q R(\mathrm{d}x) < \infty$. Now assume that $\alpha < q$ and $\int_{|x|\leq 1} |x|^q R(\mathrm{d}x) < \infty$. We have

$$\begin{aligned}
\int_{|x|\leq 1} |x|^q M(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^q \int_0^{|x|^{-1}} t^{q-\alpha-1} e^{-t^p} \mathrm{d}t R(\mathrm{d}x) \\
&\leq \int_{|x|\leq 1} |x|^q R(\mathrm{d}x) \int_0^\infty t^{q-\alpha-1} e^{-t^p} \mathrm{d}t + \int_{|x|>1} |x|^q \int_0^{|x|^{-1}} t^{q-\alpha-1} \mathrm{d}t R(\mathrm{d}x) \\
&\leq \int_{|x|\leq 1} |x|^q R(\mathrm{d}x) \int_0^\infty t^{q-\alpha-1} e^{-t^p} \mathrm{d}t + (q-\alpha)^{-1} \int_{|x|>1} |x|^\alpha R(\mathrm{d}x),
\end{aligned}$$

which is finite. □

Combining Lemma 3.22 with (3.15) tells us that the Blumenthal-Getoor index of a proper p-tempered α-stable Lévy processes with $\alpha \in (0,2)$ is α. It may be interesting to note that α is also the Blumenthal-Getoor index of any α-stable Lévy process, see, e.g., [12]. We now discuss several properties that are characterized by this index.

Let $X = \{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim TS^p_\alpha(R, b)$ and let β be given by (3.25). From [12] it follows that, with probability 1,

$$\limsup_{t\to 0} t^{-1/\gamma}|X_t| = \begin{cases} \infty & \text{if } \gamma < \beta \\ 0 & \text{if } \gamma > \beta \end{cases}.$$

Now, fix $0 \leq a < b < \infty$, $\gamma > 0$, and define

$$V_\gamma(X; a, b) = \sup \sum_{j=1}^{n} |X_{t_i} - X_{t_{i-1}}|^\gamma,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of the interval $[a, b]$. This is called the **γ-variation** of X. From [12] and [56] it follows that for any $0 \leq a < b < \infty$ with probability 1

$$V_\gamma(X; a, b) \begin{cases} = \infty & \text{if } \gamma < \beta \\ < \infty & \text{if } \gamma > \beta \end{cases}. \tag{3.26}$$

Finiteness of γ-variation gives useful results about how one can define stochastic integrals with respect to these processes. It is well known that if a process has finite 1-variation, then one can define a Stieltjes integral with respect to it. When the 1-variation is infinite, under certain assumptions about the finiteness of γ-variation for some $\gamma > 0$, one can define generalizations of Stieltjes integrals, see [22] for details.

We sometimes refer to 1-variation as simply **variation**. Thus (3.26) and Lemma 3.22 imply that a p-tempered α-stable Lévy process has finite variation if and only if $\alpha < 1$ and $\int_{|x|\leq 1} |x| R(\mathrm{d}x) < \infty$. In particular, in light of (3.15), all proper p-tempered α-stable Lévy processes with $\alpha < 1$ have finite variation. We now turn to a related concept.

A one-dimensional Lévy process, which is nondecreasing almost surely is called a **subordinator**. Such a process necessarily has finite variation. Further, by combining the above discussion with Theorems 21.5 and 21.9 in [69] we can fully characterize when a p-tempered α-stable Lévy process is a subordinator.

Proposition 3.24. *Let $\{X_t : t \geq 0\}$ be a one-dimensional Lévy process with $X_1 \sim TS^p_\alpha(R, b)$ with $R \neq 0$. The process is a subordinator if and only if $\alpha < 1$, $R((-\infty, 0)) = 0$, $\int_{(0,1)} xR(\mathrm{d}x) < \infty$, and $b \geq \int_{(0,\infty)} \int_0^\infty \frac{x}{1+t^2x^2} t^{-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x)$.*

Remark 3.6. A Lévy process is a subordinator if and only if the distribution of X_t has its support contained in $[0, \infty)$ for every t. Further, if $\{X_t : t \geq 0\}$ is a subordinator with $X_1 \sim TS^p_\alpha(R, b)$ and $R \neq 0$ then, by Theorem 24.10 in [69], the support of the distribution of X_t is given by $[t\zeta, \infty)$, where $\zeta = b - \int_{(0,\infty)} \int_0^\infty \frac{x}{1+t^2x^2} t^{-\alpha} e^{-t^p} \mathrm{d}t R(\mathrm{d}x)$.

We conclude this section by discussing when the distribution of a proper p-tempered α-stable Lévy process (with $\alpha \in (0,2)$) is absolutely continuous with respect to the distribution of the α-stable Lévy process that is being tempered. Our presentation follows [66] closely. Let $\Omega = D([0,\infty), \mathbb{R}^d)$ be the space of mappings $\omega(\cdot)$ from $[0,\infty)$ into $\mathbb{R}^d$ that are right-continuous with left limits. Let $X = \{X_t : t \geq 0\}$ be the collection of functions from Ω into $\mathbb{R}^d$ with $X_t(\omega) = \omega(t)$. Assume that Ω be equipped with the σ-algebra $\mathscr{F} = \sigma(X_s : s \geq 0)$ and the right-continuous natural filtration $(\mathscr{F}_t)_{t\geq 0}$ where $\mathscr{F}_t = \bigcap_{s>t} \sigma(X_u : u \leq s)$. In this case X is called **the canonical process**. The distribution of this process is completely determined by a probability measure P on $(\Omega, \mathscr{F})$. Let $P_{|\mathscr{F}_t}$ denote the restriction of P to the σ-algebra $\mathscr{F}_t$.

Theorem 3.25. *Fix $\alpha \in (0,2)$ and $p > 0$. In the above setting, consider two probability measures P_0 and P on $(\Omega, \mathscr{F})$ and let $X = \{X_t : t \geq 0\}$ be the canonical process. Assume that, under P, X is a Lévy process with $X_1 \sim TS^p_\alpha(R, b)$, where R satisfies (3.15).*[3] *Derive σ from R by (3.17) and let $q(u,r)$ be as in Proposition 3.6. If, under P_0, X is a Lévy process with $X_1 \sim S_\alpha(a,\sigma)$, then*

1. *$P_{0|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{\mathbb{S}^{d-1}} \int_0^1 [1 - q(r^p, u)]^2 r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}u) < \infty \tag{3.27}$$

and

$$b - a = \int_{\mathbb{R}^d} \int_0^\infty \frac{x}{1 + |x|^2 t^2} t^{-\alpha} (e^{-t^p} - 1) \mathrm{d}t R(\mathrm{d}x). \tag{3.28}$$

2. *If $P_{0|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$ are not mutually absolutely continuous for some $t > 0$, then they are singular for all $t > 0$.*
3. *If (3.27) and (3.28) hold, then for every $t > 0$*

$$\frac{\mathrm{d}P_{|\mathscr{F}_t}}{\mathrm{d}P_{0|\mathscr{F}_t}} = e^{U_t}, \ P_0 \ a.s.$$

where

$$U_t = \lim_{\epsilon \downarrow 0} \left\{ \sum_{\{s \in (0,t] : |\Delta X_s| > \epsilon} \log q \left(|\Delta X_s|^p, \frac{\Delta X_s}{|\Delta X_s|} \right) \right.$$
$$\left. + t \int_{\mathbb{S}^{d-1}} \int_\epsilon^\infty [1 - q(r^p, u)] r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}u) \right\},$$

[3] This implies that X_1 has a proper p-tempered α-stable distribution.

and the convergence is uniform in t on any bounded interval, P_0 a.s. Further, $\{U_t : t \geq 0\}$ is a one-dimensional Lévy process defined on the probability space $(\Omega, \mathscr{F}, P_0)$. It satisfies $U_1 \sim ID(0, M_U, b_U)$, where

$$M_U(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_{A\setminus\{0\}}(\log[q(r^p, u)]) r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u), \qquad A \in \mathfrak{B}(\mathbb{R})$$

and

$$b_U = -\int_{-\infty}^0 \left(e^y - 1 - \frac{y}{1+|y|^2}\right) M_U(\mathrm{d}y).$$

Note that Proposition 3.6 implies that $q(r^p, u) \in (0, 1]$, and hence that M_U satisfies $M_U([0, \infty)) = 0$.

Proof. By Remark 3.5 all proper p-tempered α-stable distributions with $\alpha \in (0, 2)$ belong to the class of generalized tempered α-stable distributions. For these, analogues of Parts 1 and 2 are given in Theorem 4.1 of [66]. In [66] the analogue of (3.27) is actually

$$\int_{\mathbb{S}^{d-1}} \int_0^1 (1 - [q(r^p, u)]^{1/2})^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u) < \infty.$$

As observed in [65], this is equivalent to (3.27) since for any $x \in [0, 1]$

$$.25(1-x)^2 \leq (1-\sqrt{x})^2 \leq (1-x)^2,$$

and $q(r^p, u) \in [0, 1]$ for all r and u. For any Lévy process, an analogue of Part 3 is given in Theorem 33.2 of [69]. To specialize it to our situation we just need to apply (3.2). □

Under additional conditions, representations of the process U_t in terms of certain extensions of γ-variation can be given, see [24]. As pointed out in [65] condition (3.27) fails when the function $q(r^p, u)$ decreases too quickly near zero. In other words when there is too much tempering near zero. This is illustrated by the following.

Corollary 3.26. *Fix $\alpha \in (0, 2)$ and $p > 0$. Let P_0, P, and $\{X_t : t \geq 0\}$ be as in Theorem 3.25. If $p \leq \alpha/2$, then $P_{0|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$ are mutually singular for all $t > 0$.*

Proof. By Remark 3.4 we can write $q(r^p, u) = \int_{(0,\infty)} e^{-r^p s} Q_u(\mathrm{d}s)$ for some measurable family of probability measures $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$. Since $1 - e^{-x} \geq \frac{x}{1+x}$ for any $x \geq 0$ (see, e.g., 4.2.32 in [2]) it follows that

$$\int_{\mathbb{S}^{d-1}} \int_0^1 [1 - q(r^p, u)]^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)$$
$$= \int_{\mathbb{S}^{d-1}} \int_0^1 \left[\int_{(0,\infty)} (1 - e^{-r^p s}) Q_u(\mathrm{d}s)\right]^2 r^{-\alpha-1} \mathrm{d}r\sigma(\mathrm{d}u)$$

$$\geq \int_{\mathbb{S}^{d-1}} \int_0^1 \left[\int_{(0,\infty)} \frac{r^p s}{1 + r^p s} Q_u(\mathrm{d}s) \right]^2 r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}u)$$

$$\geq \int_{\mathbb{S}^{d-1}} \left[\int_{(0,\infty)} \frac{s}{1 + s} Q_u(\mathrm{d}s) \right]^2 \sigma(\mathrm{d}u) \int_0^1 r^{2p-\alpha-1} \mathrm{d}r,$$

which equals infinity when $p \leq \alpha/2$. From here the result follows by Part 2 of Theorem 3.25. □

3.5 Exponential Moments When $p = 1$

A representation for the exponential moments of p-tempered α-stable distributions is given by (3.24). In this section we derive significantly simpler formulas for the case[4] where $p = 1$. Throughout this section we use the principle branch of the complex logarithm, i.e. we make a cut along the negative real axis. This implies that for $z \in \mathbb{C}$ with $\Re z > 0$ we have $\log(z) = \log|z| + i \arctan(\Im z/\Re z)$, where arctan refers to the branch of the arctangent whose image is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We begin with a lemma.

Lemma 3.27. *Fix $\alpha < 2$, $p = 1$, and $\mu = TS^1_\alpha(R, b)$. Let $X \sim \mu$, let S be the support[5] of R, and fix $z \in \mathbb{C}^d$. When $\alpha \in (0, 2)$ we have*

$$\mathrm{E}\left|e^{\langle z,X\rangle}\right| < \infty \tag{3.29}$$

if and only if $\sup_{x \in S} \Re\langle z, x\rangle \leq 1$. When $\alpha \leq 0$ a sufficient[6] condition for (3.29) is $\sup_{x \in S} \Re\langle z, x\rangle < 1$.

Proof. We will need the following fact from 6.1.1 in [2]. When $\alpha < 0$ and $w \in \mathbb{C}$ with $\Re w > 0$ we have

$$\int_0^\infty e^{-wt} t^{-\alpha-1} dt = w^\alpha \Gamma(-\alpha). \tag{3.30}$$

By Theorem 25.17 in [69] (3.29) is equivalent to $\int_{|x|>1} e^{\langle c,x\rangle} M(\mathrm{d}x) < \infty$ where $c = \Re z$ and M is the Lévy measure of μ. When $c = 0$ this always holds so assume that $c \neq 0$. By (3.12) we have

[4] The only other case where reasonable representations are known is when $p = 2$ and $\alpha \in (0, 2)$. In this case [9] gives formulas in terms of confluent hypergeometric functions.

[5] This means that S is the smallest closed subset of $\mathbb{R}^d$ with $R(S^c) = 0$.

[6] In light of Theorem 3.17, it is clear that this is not a necessary condition when $\alpha \leq 0$.

$$
\begin{aligned}
\int_{|x|>1} e^{\langle c,x\rangle} M(\mathrm{d}x) &= \int_S \int_{|x|^{-1}}^\infty e^{\langle c,x\rangle t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&= \int_{S\cap[|x|\le 1/(2|c|)]} \int_{|x|^{-1}}^\infty e^{\langle c,x\rangle t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&\qquad \int_{S\cap[|x|>1/(2|c|])} \int_{|x|^{-1}}^\infty e^{\langle c,x\rangle t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&=: I_1(\alpha) + I_2(\alpha).
\end{aligned}
$$

Let $K := \sup_{t\ge 2|c|} e^{-t/2} t^{2-\alpha}$ and note that for every $\alpha < 2$ we have

$$
\begin{aligned}
I_1(\alpha) &\le \int_{|x|\le 1/(2|c|)} \int_{|x|^{-1}}^\infty e^{t/2} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&\le K \int_{|x|\le 1/(2|c|)} \int_{|x|^{-1}}^\infty t^{-3} \mathrm{d}t R(\mathrm{d}x) = .5K \int_{|x|\le 1/(2|c|)} |x|^2 R(\mathrm{d}x) < \infty.
\end{aligned}
$$

Thus, finiteness of the exponential moment is determined by $I_2(\alpha)$. Define $\theta = \sup_{x\in S}\langle c,x\rangle$. We begin with the case $\alpha \in (0,2)$. If $\theta \le 1$, then

$$
\begin{aligned}
I_2(\alpha) &\le \int_{S\cap[|x|>1/(2|c|])} \int_{|x|^{-1}}^\infty e^{\theta t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&\le \int_{|x|>1/(2|c|)} \int_{|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) = \alpha^{-1} \int_{|x|>1/(2|c|)} |x|^\alpha \mathrm{d}t R(\mathrm{d}x) < \infty.
\end{aligned}
$$

On the other hand, if $\theta > 1$, then there is an $\epsilon > 0$ and a Borel set $S_\epsilon \subset S \cap [|x| > 1/(2|c|)]$ with $R(S_\epsilon) > 0$ such that for every $x \in S_\epsilon$ we have $\langle x,c\rangle \ge 1+\epsilon$. This implies that

$$
\begin{aligned}
I_2(\alpha) &\ge \int_{S_\epsilon} \int_{|x|^{-1}}^\infty e^{\langle c,x\rangle t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&\ge \int_{S_\epsilon} \int_{2|c|}^\infty e^{(1+\epsilon)t} e^{-t} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&= R(S_\epsilon) \int_{2|c|}^\infty e^{\epsilon t} t^{-1-\alpha} \mathrm{d}t = \infty.
\end{aligned}
$$

Now assume that $\alpha \le 0$ and $\theta < 1$. For $\alpha < 0$ we can use (3.30) to get

$$
\begin{aligned}
I_2(\alpha) &\le \int_{|x|>1/(2|c|)} \int_0^\infty e^{-t(1-\theta)} t^{-1-\alpha} \mathrm{d}t R(\mathrm{d}x) \\
&= (1-\theta)^{-|\alpha|} \Gamma(|\alpha|) \int_{|x|>1/(2|c|)} R(\mathrm{d}x) < \infty,
\end{aligned}
$$

and for $\alpha = 0$ we get

$$I_2(0) \le \int_{|x|>1/(2|c|)} \int_{|x|^{-1}}^{2|c|} t^{-1}\mathrm{d}tR(\mathrm{d}x) + R\left(|x| > \frac{1}{2|c|}\right)\int_{2|c|}^{\infty} e^{-t(1-\theta)}t^{-1}\mathrm{d}t$$
$$= \int_{|x|>1/(2|c|)} \log\left(2|c||x|\right) R(\mathrm{d}x) + R\left(|x| > \frac{1}{2|c|}\right)\int_{2|c|}^{\infty} e^{-t(1-\theta)}t^{-1}\mathrm{d}t < \infty,$$

which completes the proof. □

We now give the main result of this section.

Theorem 3.28. *Fix $\alpha < 2$, $p = 1$, $\mu = TS^1_\alpha(R, b)$, and let $X \sim \mu$. Let S be the support of R and fix $z \in \mathbb{C}^d$ such that either a) $\sup_{x\in S} \Re\langle z, x\rangle < 1$ or b) $\Im z = 0$, $\sup_{x\in S} \Re\langle z, x\rangle \le 1$, and $\alpha \in (0, 2)$. In both cases (3.29) holds and we have:*

1. If $\int_{\mathbb{R}^d} |x|\mu(\mathrm{d}x) < \infty$, then

$$\mathrm{E}e^{\langle z,X\rangle} = \exp\left\{\int_{\mathbb{R}^d} \psi_\alpha(\langle z, x\rangle)R(\mathrm{d}x) + \langle z, b_1\rangle\right\}, \tag{3.31}$$

where

$$b_1 = b + \int_{\mathbb{R}^d}\int_0^\infty x\frac{|x|^2}{1+|x|^2t^2}t^{2-\alpha}e^{-t}\mathrm{d}tR(\mathrm{d}x) \tag{3.32}$$

and

$$\psi_\alpha(s) = \begin{cases} \Gamma(-\alpha)[(1-s)^\alpha - 1 + \alpha s] & \alpha \ne 0, 1 \\ -\log(1-s) - s & \alpha = 0 \\ (1-s)\log(1-s) + s & \alpha = 1 \end{cases}. \tag{3.33}$$

In particular this holds when $1 < \alpha < 2$, or

$$\alpha = 1 \quad \text{and} \quad \int_{|x|>1} |x|\log|x|R(\mathrm{d}x) < \infty,$$

or

$$\alpha < 1 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|R(\mathrm{d}x) < \infty.$$

2. If $\alpha < 1$ and $\int_{|x|\le 1} |x|R(\mathrm{d}x) < \infty$, then

$$\mathrm{E}e^{\langle z,X\rangle} = \exp\left\{\int_{\mathbb{R}^d} \psi^0_\alpha(\langle z, x\rangle)R(\mathrm{d}x) + \langle z, b_0\rangle\right\}, \tag{3.34}$$

where

$$b_0 = b - \int_{\mathbb{R}^d}\int_0^\infty \frac{x}{1+|x|^2t^2} t^{-\alpha} e^{-t} \mathrm{d}t R(\mathrm{d}x) \tag{3.35}$$

and

$$\psi^0_\alpha(s) = \begin{cases} \Gamma(-\alpha)[(1-s)^\alpha - 1] & \alpha \neq 0 \\ -\log(1-s) & \alpha = 0 \end{cases}. \tag{3.36}$$

In particular, this holds if μ is a proper TS^1_α distribution with $\alpha < 1$.

In the above we take $\Psi_1(1) = 1$, which is the limiting value of the function $\Psi_\alpha(s)$ in both s and α. A simple way to ensure that the assumption of Theorem 3.28 holds is as follows. Fix $\theta > 0$. If $R(|x| > \theta^{-1}) = 0$, then for any $z \in \mathbb{C}^d$ with $|\Re z| < \theta$ we have $\sup_{x\in S}\langle \Re z, x\rangle \le \sup_{x\in S}|\langle \Re z, x\rangle| \le \sup_{x\in S}|\Re z||x| < \theta/\theta = 1$. The vectors b_1 and b_0 given above have the following interpretations. When $\int_{\mathbb{R}^d}|x|\mu(\mathrm{d}x) < \infty$ we have $b_1 = \int_{\mathbb{R}^d} x\mu(\mathrm{d}x)$, and when $\alpha < 1$ and $\int_{|x|\le 1}|x|R(\mathrm{d}x) < \infty$ the vector b_0 is the drift.

Proof. Our proof will use the following. If $t \in (0,1)$, $s \in \mathbb{C}$, and $\alpha \le 1$, then

$$\begin{aligned} |(e^{st} - 1 - st)t^{-\alpha-1}e^{-t}| &\le \sum_{n=2}^\infty \frac{|st|^n}{n!} t^{-\alpha-1}e^{-t} \\ &= t^{1-\alpha}e^{-t}|s|^2 \sum_{n=2}^\infty \frac{|st|^{n-2}}{n(n-1)(n-2)!} \\ &\le e^{-t}|s|^2 \sum_{n=2}^\infty \frac{|st|^{n-2}}{(n-2)!} \\ &= e^{t(|s|-1)}|s|^2. \end{aligned} \tag{3.37}$$

Lemma 3.27 implies that we can use (3.24) to get a representation for the exponential moment. We begin with the case $\int_{\mathbb{R}^d}|x|\mu(\mathrm{d}x) < \infty$. In this case b_1 is definable as a vector in $\mathbb{R}^d$ and from (3.24) it follows that

$$\begin{aligned} \mathrm{E}e^{\langle z,X\rangle} &= \exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{\langle x,z\rangle t} - 1 - \langle x,z\rangle t\right) t^{-1-\alpha}e^{-t}\mathrm{d}t R(\mathrm{d}x) + \langle z, b_1\rangle\right\} \\ &= \exp\left\{\int_S\int_0^\infty \left(e^{\langle x,z\rangle t} - 1 - \langle x,z\rangle t\right) t^{-1-\alpha}e^{-t}\mathrm{d}t R(\mathrm{d}x) + \langle z, b_1\rangle\right\}. \end{aligned}$$

Fix $x \in S$. For simplicity of notation let $s = \langle x,z\rangle$ and note that, by assumption, when $\alpha \le 0$ we have $\Re s < 1$ and $\Re(1-s) > 0$ and when $\alpha \in (0,2)$ we have $\Re s \le 1$ and $\Re(1-s) \ge 0$. When $\alpha < 0$ we can use (3.30) to get

$$\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t = \int_0^\infty (e^{-(1-s)t} - e^{-t} - se^{-t}t)t^{-\alpha-1}\mathrm{d}t$$
$$= \Gamma(-\alpha)[(1-s)^\alpha - 1 + \alpha s].$$

When $\alpha = 0$ we can use l'Hôpital's rule[7] to get

$$\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-1}\mathrm{d}t = \int_0^\infty \lim_{\alpha\uparrow 0}(e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t$$
$$= \lim_{\alpha\uparrow 0}\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t$$
$$= \lim_{\alpha\uparrow 0}\Gamma(-\alpha)[(1-s)^\alpha - 1 + \alpha s]$$
$$= \lim_{\alpha\uparrow 0}\frac{\Gamma(1-\alpha)[(1-s)^\alpha - 1 + \alpha s]}{-\alpha}$$
$$= \lim_{\alpha\uparrow 0}\frac{(1-s)^\alpha - 1 + \alpha s}{-\alpha}$$
$$= -\log(1-s) - s,$$

where we can interchange limit and integral using dominated convergence. Specifically, for $\alpha \in (-1, 0)$ if $t \in (0, 1)$, then (3.37) gives a bound that is integrable on $(0, 1)$ and if $t \geq 1$, then

$$|(e^{st} - 1 - st)t^{-\alpha-1}e^{-t}| \leq e^{-t(1-\Re s)} + (1 + |s|t)e^{-t},$$

which is integrable on $[1, \infty)$ since $\Re s < 1$.

Now assume that $\alpha \in (0, 1)$. For any $v, w \in \mathbb{C}$ with w satisfying $\Re w > 0$ and v satisfying either $\Re v > 0$ or $v = 0$ integration by parts and (3.30) give

$$\int_0^\infty \left(e^{-vt} - e^{-wt}\right)t^{-1-\alpha}\mathrm{d}t = \Gamma(-\alpha)(v^\alpha - w^\alpha), \tag{3.38}$$

which implies

$$\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t = \int_0^\infty (e^{-(1-s)t} - e^{-t})t^{-\alpha-1}\mathrm{d}t - s\int_0^\infty e^{-t}t^{(1-\alpha)-1}\mathrm{d}t$$
$$= \Gamma(-\alpha)[(1-s)^\alpha - 1 + s\alpha].$$

[7] We can use l'Hôpital's rule because the denominator is real. However, in general, l'Hôpital's rule may fail for complex valued functions of real numbers, see [18].

Now assume that $\alpha \in (1, 2)$. For any $v, w \in \mathbb{C}$ with w satisfying $\Re w > 0$ and v satisfying either $\Re v > 0$ or $v = 0$ integration by parts and (3.38) give

$$\int_0^\infty \left(e^{-vt} - e^{-wt} + (v-w)t\right) t^{-1-\alpha} \mathrm{d}t = \Gamma(-\alpha)(v^\alpha - w^\alpha),$$

which implies

$$\begin{aligned}\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t &= \int_0^\infty (e^{-(1-s)t} - e^{-t} - st)t^{-\alpha-1}\mathrm{d}t \\ &\quad + s\int_0^\infty (1-e^{-t})t^{(1-\alpha)-1}\mathrm{d}t \\ &= \Gamma(-\alpha)[(1-s)^\alpha - 1 + s\alpha].\end{aligned}$$

Now consider the case $\alpha = 1$. By l'Hôpital's rule

$$\begin{aligned}\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-2}\mathrm{d}t &= \int_0^\infty \lim_{\alpha\uparrow 1}(e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t \\ &= \lim_{\alpha\uparrow 1}\int_0^\infty (e^{st} - 1 - st)e^{-t}t^{-\alpha-1}\mathrm{d}t \\ &= \lim_{\alpha\uparrow 1}\Gamma(-\alpha)[(1-s)^\alpha - 1 + s\alpha] \\ &= \lim_{\alpha\uparrow 1}\frac{\Gamma(2-\alpha)}{(\alpha-1)\alpha}[(1-s)^\alpha - 1 + s\alpha] \\ &= (1-s)\log(1-s) + s,\end{aligned}$$

where the second line follows by dominated convergence. Specifically, for $\alpha \in (.5, 1)$ if $t \in (0, 1)$, then (3.37) gives a bound that is integrable on $(0, 1)$, and for $t \geq 1$ we have

$$|(e^{st} - 1 - st)t^{-\alpha-1}e^{-t}| \leq e^{-t(1-\Re s)}t^{-.5-1} + (1+|s|)e^{-t},$$

which is integrable on $[1, \infty)$ since $\Re s \leq 1$.

We now turn to the case when $\int_{|x|\leq 1}|x|R(\mathrm{d}x) < \infty$ and $\alpha < 1$. In this case b_0 is definable as a vector in $\mathbb{R}^d$ and (3.24) implies that

$$\mathrm{E}e^{\langle z,X\rangle} = \exp\left\{\int_{\mathbb{R}^d}\int_0^\infty \left(e^{\langle x,z\rangle t} - 1\right)t^{-1-\alpha}e^{-t}\mathrm{d}tR(\mathrm{d}x) + \langle z, b_0\rangle\right\}.$$

The fact that $\int_0^\infty \left(e^{\langle x,z\rangle t} - 1\right)t^{-1-\alpha}e^{-t}\mathrm{d}t$ has the required form can be shown in a similar way to the previous part. The conditions to guarantee $\int_{\mathbb{R}^d}|x|\mu(\mathrm{d}x) < \infty$ follow from Theorem 3.15, while the fact that $\int_{|x|\leq 1}|x|R(\mathrm{d}x) < \infty$ for all proper TS^1_α distribution with $\alpha < 1$ follows by Theorem 3.3. □

Note that the assumption of Theorem 3.28 always holds when $\Re z = 0$. This gives the following representation for the characteristic function.

Corollary 3.29. *Fix $\alpha < 2$, $p = 1$, and let $\mu = TS^1_\alpha(R, b)$.*

1. *If $\int_{\mathbb{R}^d} |x|\mu(\mathrm{d}x) < \infty$, then*

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d} \psi_\alpha(i\langle z, x\rangle)R(\mathrm{d}x) + i\langle z, b_1\rangle\right\}, \qquad z \in \mathbb{R}^d,$$

   where b_1 is given by (3.32) and ψ_α is given by (3.33).
2. *If $\alpha < 1$ and $\int_{|x|\le 1} |x|R(\mathrm{d}x) < \infty$, then the characteristic function is given by*

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d} \psi^0_\alpha(i\langle z, x\rangle)R(\mathrm{d}x) + i\langle z, b_0\rangle\right\}, \qquad z \in \mathbb{R}^d,$$

   where b_0 is given by (3.35) and ψ^0_α is given by (3.36).

Now consider the case when $X \sim TS^1_\alpha(R, b)$ is a one-dimensional random variable with $R((-\infty, 0)) = 0$. In this case the support, S, of R satisfies $S \subset [0, \infty)$. Thus for all $z \in \mathbb{R}$ with $z \le 0$ we have $\sup_{x\in S}(zx) < 1$ and we can use Theorem 3.28 to get the following representation for the Laplace transform.

Corollary 3.30. *Fix $\alpha < 2$, $p = 1$, let $\mu = TS^1_\alpha(R, b)$ be a 1-tempered α-stable distribution on $\mathbb{R}$ with $R((-\infty, 0)) = 0$, and let $X \sim \mu$.*

1. *If $\mathrm{E}|X| < \infty$, then*

$$\mathrm{E}[e^{-zX}] = \exp\left\{\int_{(0,\infty)} \psi_\alpha(-zx)R(\mathrm{d}x) - zb_1\right\}, \quad z \ge 0,$$

   where b_1 is given by (3.32) and ψ_α is given by (3.33).
2. *If $\alpha < 1$ and $\int_{|x|\le 1} |x|R(\mathrm{d}x) < \infty$, then*

$$\mathrm{E}[e^{-zX}] = \exp\left\{\int_{(0,\infty)} \psi^0_\alpha(-zx)R(\mathrm{d}x) - zb_0\right\}, \quad z \ge 0,$$

   where b_0 is given by (3.35) and ψ^0_α is given by (3.36).