Chapter 2
Supersymmetric Almost-Commutative Geometries

Abstract We give a systematic analysis of the possibilities for almost-commutative geometries on a 4-dimensional, flat background to exhibit not only a particle content that is eligible for supersymmetry but also have a supersymmetric action. We come up with an approach in which we identify the basic 'building blocks' of potentially supersymmetric theories and the demands for their action to be supersymmetric. Examples that satisfy these demands turn out to be sparse.

2.1 Noncommutative Geometry and R-Parity

One of the key features of many supersymmetric theories is the notion of *R-parity*; particles and their superpartners are not only characterized by the fact that they are in the same representation of the gauge group and differ in spin by $\frac{1}{2}$, but in addition they have opposite R-parity values (cf. [9, Sect. 4.5]). As an illustration of this fact for the MSSM, see Table 2.1.

In this section we try to mimic such properties, providing an implementation of this concept in the language of noncommutative geometry:

Definition 2.1 *An R-extended, real, even spectral triple* is a real and even spectral triple $(\mathscr{A}, \mathscr{H}, D; \gamma, J)$ that is dressed with a grading $R : \mathscr{H} \to \mathscr{H}$ satisfying

$$[R, \gamma] = [R, J] = [R, a] = 0 \,\forall\, a \in \mathscr{A}.$$

We will simply write $(\mathscr{A}, \mathscr{H}, D; \gamma, J, R)$ for such an R-extended spectral triple.

Note that, as with any grading, R allows us to split the Hilbert space into an *R-even* and *R-odd* part:

$$\mathscr{H} = \mathscr{H}_{R=+} \oplus \mathscr{H}_{R=-}, \quad \mathscr{H}_{R=\pm} = \frac{1}{2}(1 \pm R)\mathscr{H}.$$

Consequently the Dirac operator splits in parts that (anti-)commute with R: $D = D_+ + D_-$ with $\{D_-, R\} = [D_+, R] = 0$. We anticipate what is coming in the next section by mentioning that in applying this notion to (the Hilbert space of) the MSSM,

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Table 2.1 The R-parity values for the various particles in the MSSM

Fermions	R-parity	Bosons	R-parity	Multiplicity
gauginos	-1	gauge bosons	$+1$	1
SM fermions	$+1$	sfermions	-1	3
higgsinos	-1	Higgs(es)	$+1$	1

In the left column are the fermions, in the right column the bosons. The SM fermions and their superpartners come in three generations each, whereas there is only one copy of the other particles. This statement presupposes that we view the up- and downtype Higgses and higgsinos as being distinct

elements of $\mathscr{H}_{R=+}$ should coincide with the SM particles and those of $\mathscr{H}_{R=-1}$ with the sfermions, gauginos and higgsinos.

Remark 2.2 In Krajewski diagrams we will distinguish between objects on which $R = 1$ and on which $R = -1$ in the following way:

- Representations in $\mathscr{H}_F$ on which $R = -1$ get a black fill, whereas those on which $R = +1$ get a white fill with a black stroke.
- Scalars (i.e. components of the Dirac operator) that commute with R are represented by a dashed line, whereas scalars that anti-commute with R get a solid line.

We immediately use the R-parity operator to make a refinement to the unimodularity condition (1.27). Instead of taking the trace over the full (finite) Hilbert space, we only take it over the part on which R equals 1, i.e. it now reads

$$\mathrm{tr}_{\mathscr{H}_{R=+}} A_\mu = 0. \tag{2.1}$$

Analogously, the definition (1.36) of the gauge group must then be modified to

$$SU(\mathscr{A}) := \{u \in U(\mathscr{A}), \det{}_{\mathscr{H}_{R=+}}(u) = 1\}. \tag{2.2}$$

We will justify this choice later, after Lemma 2.9.

Note that adjusting the unimodularity condition has no effect when applying it to the case of the NCSM, since all SM-fermions have R-parity $+1$ (Table 2.1).

2.2 Supersymmetric Spectral Triples

We give a classification of all almost-commutative geometries whose particle content and spectral action functional is supersymmetric. Throughout this section we characterize the finite spectral triples/almost-commutative geometries by their Krajewski diagrams as presented in Sect. 1.2.4. Since gravity is known to break global supersymmetry, we shall from the outset restrict ourselves to a canonical spectral triple on a flat background, i.e. all Christoffel symbols and consequently the Riemann tensor vanish.

For a given algebra A_F of the form

$$\mathscr{A}_F = \bigoplus_i^K M_{N_i}(\mathbb{C}), \tag{2.3}$$

we now look for supersymmetric 'building blocks'—made out of representations $\mathbf{N}_i \otimes \mathbf{N}_j^o$ ($i, j \in \{1, \ldots, K\}$) in the Hilbert space (fermions) and components of the finite Dirac operator (scalars)—that give a particle content and interactions eligible for supersymmetry. In particular, these building blocks should be 'irreducible'; they are the smallest extensions to a spectral triple that are necessary to retain a supersymmetric action. We underline that we do not require that the extra action associated to a building block is supersymmetric in itself. Rather, the building blocks will be defined such that the total action can remain supersymmetric, or can become it again.

2.2.1 First Building Block: The Adjoint Representation

For a finite algebra $\mathscr{A}_F = M_{N_j}(\mathbb{C})$ that consists of one component, the finite Hilbert space can be taken to be $\mathbf{N}_j \otimes \mathbf{N}_j^o \simeq M_{N_j}(\mathbb{C})$, the bimodule of the component $M_{N_j}(\mathbb{C})$ of the algebra. In order to reduce the fermionic degrees of freedom in the same way as in the NCSM, we need a finite spectral triple of KO-dimension 6, i.e. one that satisfies $\{J, \gamma\} = 0$. This requires at least two copies of this bimodule, both having a different value of the finite grading[1] and a finite real structure J_F that interchanges these copies (and simultaneously takes their adjoint):

$$J_F(m, n) := (n^*, m^*).$$

We call this

Definition 2.3 A *building block of the first type* $\mathscr{B}_j$ ($j \in \{1, \ldots, K\}$) consists of two copies of an adjoint representation $M_{N_j}(\mathbb{C})$ in the finite Hilbert space, having opposite values for the grading γ_F. It is denoted by

$$\mathscr{B}_j = (m, m', 0) \in M_{N_j}(\mathbb{C})_L \oplus M_{N_j}(\mathbb{C})_R \oplus \mathrm{End}(\mathscr{H}_F) \subset \mathscr{H}_F \oplus \mathrm{End}(\mathscr{H}_F).$$

As for the R-parity operator, we put $R|_{M_{N_j}(\mathbb{C})} = -1$. Since D_A maps between $R = -1$ representations the gauge field has $R = 1$, indeed opposite to the fermions. The Krajewski diagram that corresponds to this spectral triple is depicted in Fig. 2.1.

Via the inner fluctuations (1.17) of the canonical Dirac operator $\partial\!\!\!/_M$ (1.20) we obtain gauge fields that act on the $M_{N_j}(\mathbb{C})$ in the adjoint representation. If we write

[1] We will distinguish the copies by giving them subscripts L and R.

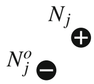

Fig. 2.1 The first building block consists of two copies in the adjoint representation $M_{N_j}(\mathbb{C})$, having opposite grading. The *solid fill* means that they have $R = -1$

$$(\lambda'_{jL}, \lambda'_{jR}) \in \mathscr{H}^+ = L^2(S_+ \otimes M_{N_j}(\mathbb{C})_L) \oplus L^2(S_- \otimes M_{N_j}(\mathbb{C})_R)$$

for the elements of the Hilbert space as they would appear in the inner product, we find for the fluctuated canonical Dirac operator (1.20) that:

$$\partial\!\!\!/_A(\lambda'_{jL}, \lambda'_{jR}) = i\gamma^\mu(\partial_\mu + \mathbb{A}_\mu)(\lambda'_{jL}, \lambda'_{jR}),$$

with $\mathbb{A}_\mu = -ig_j \operatorname{ad} A'_{\mu j}$. Here we have written $\operatorname{ad}(A'_{\mu j})\lambda'_{L,R} := A'_{\mu j}\lambda'_{L,R} - \lambda'_{L,R}A'_{\mu j}$ with $A'_{\mu j} \in \operatorname{End}(\Gamma(\mathscr{S}) \otimes u(N_j))$ self-adjoint and we have introduced a coupling constant g_j.

2.2.1.1 Matching Degrees of Freedom

In order for the gauginos to have the same number of finite degrees of freedom as the gauge bosons—an absolute necessity for supersymmetry—we can simply reduce their finite part $\lambda'_{jL,R}$ to $u(N_j)$, as described in [2, Sect. 4]. However, as is also explained in loc. cit., even though the finite part of the gauge field $A'_{\mu j}$ is initially also in $u(N_j)$, the trace part is invisible in the action since it acts on the fermions in the adjoint representation. To be explicit, writing $A'_{\mu j} = A_{\mu j} + \frac{1}{N_j}B_{\mu j}\operatorname{id}_{N_j}$, with $A_{\mu j}(x) \in su(N_j)$, $B_{\mu j}(x) \in u(1)$ (for conciseness we have left out coupling constants for the moment), we have

$$\operatorname{ad}(A'_{\mu j}) = \operatorname{ad}(A_{\mu j}).$$

This fact spoils the equality between the number of fermionic and bosonic degrees of freedom again. We observe however that upon splitting the fermions into a traceless and trace part, i.e.

$$\lambda'_{jL,R} = \lambda_{jL,R} + \lambda^0_{jL,R}\operatorname{id}_{N_j}, \tag{2.4}$$

the latter part is seen to fully decouple from the rest in the fermionic action:

$$\langle J_M\lambda'_{jL}, D_A\lambda'_{jR}\rangle = \langle J_M\lambda_{jL}, \partial\!\!\!/_A\lambda_{jR}\rangle + \langle J_M\lambda^0_{jL}, \partial\!\!\!/_M\lambda^0_{jR}\rangle.$$

We discard the trace part from the theory.

Remark 2.4 In particular, a building block of the first type with $N_j = 1$ does not yield an action since the bosonic interactions automatically vanish and all fermionic ones are discarded. This is remedied again in a set-up such as in the next section.

Note that applying the unimodularity condition (2.1) does not teach us anything here, for $\mathscr{H}_{R=+}$ is trivial.

One last aspect is hampering a theory with equal fermionic and bosonic degrees of freedom. There is a mismatch between the number of degrees of freedom for the theory *off shell*; the equations of motion for the gauge field and gaugino constrain a different number of degrees of freedom. This is a common issue in supersymmetry and is fixed by means of a non-propagating *auxiliary field*. We mimic this procedure by introducing a variable $G_j := G_j^a T_j^a \in C^\infty(M, su(N_j))$—with T_j^a the generators of $su(N_j)$—which appears in the action via[2]:

$$-\frac{1}{2n_j}\int_M \mathrm{tr}_{N_j}\, G_j^2 \sqrt{g}\mathrm{d}^4x. \tag{2.5}$$

The factor n_j stems from the normalization of the T_j^a, $\mathrm{tr}\, T_j^a T_j^b = n_j\delta^{ab}$, and is introduced so that in the action $(G^a)^2$ has coefficient $1/2$, as is customary. Typically $n_j = \frac{1}{2}$. Using the Euler-Lagrange equations we obtain $G_j = 0$, i.e. the auxiliary field does not propagate. This means that on shell the action corresponds to what the spectral action yields us. In proving the supersymmetry of the action, however, we will work with the off shell counterpart of the spectral action.

The action of the spectral triple associated to $\mathscr{B}_j$ has been determined before (e.g. [3–5]) and is given by

$$S_j[\lambda, \mathbb{A}] := \langle J_M\lambda'_{jR}, \partial\!\!\!/_A\lambda'_{jL}\rangle - \frac{f(0)}{24\pi^2}\int_M \mathrm{tr}_{\mathscr{H}_F}\, \mathbb{F}^j_{\mu\nu}\mathbb{F}^{j,\mu\nu} + \mathscr{O}(\Lambda^{-2}), \tag{2.6}$$

where we have written the fermionic terms as they would appear in the path integral (cf. [8, Sect. 16.3]).[3] Using the notation introduced in (1.32) we write $\mathbb{A}_\mu = -ig_j(A_{\mu j} - A^o_{\mu j})$ and find for the corresponding field strength (1.25)

$$\mathbb{F}_{\mu\nu} = -ig_j\big(F^j_{\mu\nu} - (F^j_{\mu\nu})^o\big),$$
$$\text{with}\quad F^j_{\mu\nu} = \partial_\mu(A_{\nu j}) - \partial_\nu(A_{\mu j}) - ig_j[A_{\mu j}, A_{\nu j}]$$

Hermitian. Consequently we have in the action

[2] This auxiliary field is commonly denoted by D. Since this letter already appears frequently in NCG, we instead take G to avoid confusion.

[3] It might seem that there are too many independent spinor degrees of freedom, but this is a characteristic feature for a theory on a Euclidean background, see e.g. [13–15] for details.

$$-\frac{f(0)}{24\pi^2}\int_M \mathrm{tr}_{\mathscr{H}_F}\,\mathbb{F}^j_{\mu\nu}\mathbb{F}^{j,\mu\nu} = \frac{1}{4}\frac{\mathscr{K}_j}{n_j}\int_M \mathrm{tr}_{N_j}\,F^j_{\mu\nu}F^{j,\mu\nu},$$
$$\text{with } \mathscr{K}_j = \frac{f(0)}{3\pi^2}n_j g_j^2(2N_j). \tag{2.7}$$

Here we have used that for $X \in M_{N_j}(\mathbb{C})$ traceless, $\mathrm{tr}_{M_{N_j}(\mathbb{C})}(X-X^o)^2 = 2N_j\,\mathrm{tr}_{N_j}\,X^2$ and there is an additional factor 2 since there are two copies of $M_{N_j}(\mathbb{C})$ in $\mathscr{H}_F$. The expression for $\mathscr{K}_j$ gets a contribution from each representation on which the gauge field $A_{\mu j}$ acts, see Remark 2.14 ahead. The factor n_j^{-1} in front of the gauge bosons' kinetic term anticipates the same factor arising when performing the trace over the generators of the gauge group. The same thing happens for the gauginos and since we want λ_j^a, rather than λ_j, to have a normalized kinetic term, we scale these according to

$$\lambda_j \to \frac{1}{\sqrt{n_j}}\lambda_j, \quad \text{where } \mathrm{tr}\, T_j^a T_j^b = n_j\delta_{ab}. \tag{2.8}$$

Discarding the trace part of the fermion, scaling the gauginos, introducing the auxiliary field G_j and working out the second term of (2.6) then gives us for the action

$$S_j[\lambda, \mathbb{A}, G_j] := \frac{1}{n_j}\langle J_M\lambda_{jL}, \partial\!\!\!/_A\lambda_{jR}\rangle + \frac{1}{4}\frac{\mathscr{K}_j}{n_j}\int_M \mathrm{tr}_{N_j}\,F^j_{\mu\nu}F^{j,\mu\nu} - \frac{1}{2n_j}\int_M \mathrm{tr}_{N_j}\,G_j^2, \tag{2.9}$$

with $\lambda_{jL,R} \in L^2(M, S_\pm \otimes su(N_j)_{L,R})$, $A_j \in \mathrm{End}(\Gamma(S) \otimes su(N_j))$, $G_j \in C^\infty(M, su(N_j))$.

For this action we have:

Theorem 2.5 *The action (2.9) of an R-extended almost-commutative geometry that consists of a building block $\mathscr{B}_j$ of the first type (Definition 2.3, with $N_j \geq 2$) is supersymmetric under the transformations*

$$\delta A_j = c_j\gamma^\mu\big[(J_M\varepsilon_R, \gamma_\mu\lambda_{jL})_{\mathscr{S}} + (J_M\varepsilon_L, \gamma_\mu\lambda_{jR})_{\mathscr{S}}\big], \tag{2.10a}$$
$$\delta\lambda_{jL,R} = c_j'\gamma^\mu\gamma^\nu F^j_{\mu\nu}\varepsilon_{L,R} + c_{G_j}' G_j\varepsilon_{L,R}, \tag{2.10b}$$
$$\delta G_j = c_{G_j}\big[(J_M\varepsilon_R, \partial\!\!\!/_A\lambda_{jL})_{\mathscr{S}} + (J_M\varepsilon_L, \partial\!\!\!/_A\lambda_{jR})_{\mathscr{S}}\big], \tag{2.10c}$$

with $c_j, c_j', c_{G_j}, c_{G_j}' \in \mathbb{C}$ iff

$$2ic_j' = -c_j\mathscr{K}_j, \quad c_{G_j} = -c_{G_j}'. \tag{2.11}$$

Proof The entire proof, together with the explanation of the notation, is given in the Appendix section 'First Building Block'.

We have now established that the building block of Definition 2.3 gives the super Yang-Mills action, which is supersymmetric under the transformations (2.10).[4] This building block is the NCG-analogue of a single vector superfield in the superfield formalism.

Note that we cannot define multiple copies of the same building block of the first type without explicitly breaking supersymmetry, since this would add new fermionic degrees of freedom but not bosonic ones. This exhausts all possibilities for a finite algebra that consists of one component.

2.2.2 Second Building Block: Adding Non-adjoint Representations

If the algebra (2.3) contains two summands, we can first of all have two *different* building blocks of the first type and find that the action is simply the sum of actions of the form (2.9) and thus still supersymmetric.

We have a second go at supersymmetry by adding the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ to the finite Hilbert space, corresponding to an off-diagonal vertex in a Krajewski diagram. This introduces non-gaugino fermions to the theory. A real spectral triple then requires us to also add its conjugate $\mathbf{N}_j \otimes \mathbf{N}_i^o$. To keep the spectral triple of KO-dimension 6, both representations should have opposite values of the finite grading γ_F. For concreteness we choose $\mathbf{N}_i \otimes \mathbf{N}_j^o$ to have value $+$ in this section, but the opposite sign works equally well with only minor changes in the various expressions. With only this content, the action corresponding to this spectral triple can never be supersymmetric for two reasons. First, it lacks the degrees of freedom of a bosonic (scalar) superpartner. Second, it exhibits interactions with gauge fields (via the inner fluctuations of $\partial\!\!\!/_M$) without having the necessary gaugino degrees to make the particle content supersymmetric. However, if we also add the building blocks $\mathscr{B}_i$ and $\mathscr{B}_j$ of the first type to the spectral triple, both the gauginos are present and a finite Dirac operator is possible, that might remedy this.

Lemma 2.6 *For a finite Hilbert space consisting of two building blocks $\mathscr{B}_i$ and $\mathscr{B}_j$ together with the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and its conjugate the most general finite Dirac operator on the basis*

$$\mathbf{N}_i \otimes \mathbf{N}_j^o \oplus M_{N_i}(\mathbb{C})_L \oplus M_{N_i}(\mathbb{C})_R \oplus M_{N_j}(\mathbb{C})_L \oplus M_{N_j}(\mathbb{C})_R \oplus \mathbf{N}_j \otimes \mathbf{N}_i^o. \quad (2.12)$$

is given by

[4] A similar result, without taking two copies of the adjoint representation, was obtained in [2].

$$D_F = \begin{pmatrix} 0 & 0 & A & 0 & B & 0 \\ 0 & 0 & M_i & 0 & 0 & JA^*J^* \\ A^* & M_i^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_j & JB^*J^* \\ B^* & 0 & 0 & M_j^* & 0 & 0 \\ 0 & JAJ^* & 0 & JBJ^* & 0 & 0 \end{pmatrix} \tag{2.13}$$

with $A : M_{N_i}(\mathbb{C})_R \to \mathbf{N}_i \otimes \mathbf{N}_j^o$ *and* $B : M_{N_j}(\mathbb{C})_R \to \mathbf{N}_i \otimes \mathbf{N}_j^o$.

Proof We start with a general 6×6 matrix for D_F. Demanding that $\{D_F, \gamma_F\} = 0$ already sets half of its components to zero, leaving 18 to fill. The first order condition (1.12) requires all components on the upper-right to lower-left diagonal of (2.13) to be zero, so 12 components are left. Furthermore, D_F must be self-adjoint, reducing the degrees of freedom by a factor two. The last demand $J_F D_F = D_F J_F$ links the remaining half components to the other half, but not for the components that map between the gauginos: because of the particular set-up they were already linked via the demand of self-adjointness. This leaves the four independent components A, B, M_i and M_j.

In this chapter we will set $M_i = M_j = 0$ since these components describe supersymmetry breaking gaugino masses. This will be the subject of the next chapter.

Lemma 2.7 *If the components A and B of* (2.13) *differ by only a complex number, then they generate a scalar field* $\widetilde{\psi}_{ij}$ *in the same representation of the gauge group as the fermion.*

Proof We write $D_{ij}{}^{ii} \equiv A$ and $D_{ij}{}^{jj} \equiv B$ in the notation of (1.33). First of all, $D_{ij}{}^{jj} : M_{N_j}(\mathbb{C}) \to \mathbf{N}_i \otimes \mathbf{N}_j^o$ constitutes of *left* multiplication with an element $C_{ijj}\,\eta_{ij}$, where $\eta_{ij} \in \mathbf{N}_i \otimes \mathbf{N}_j^o$ and $C_{ijj} \in \mathbb{C}$. Similarly, $D_{ij}{}^{ii} : M_{N_i}(\mathbb{C}) \to \mathbf{N}_i \otimes \mathbf{N}_j^o$ constitutes of *right* multiplication with an element in $\mathbf{N}_i \otimes \mathbf{N}_j^o$. If this differs from $D_{ij}{}^{jj}$ by only a complex factor, it is of the form $C_{iij}\eta_{ij}$, with $C_{iij} \in \mathbb{C}$.

Then the inner fluctuations (1.34) that $D_{ij}{}^{jj}$ develops, are of the form

$$D_{ij}{}^{jj} \to D_{ij}{}^{jj} + \sum_n (a_n)_i \big(D_{ij}{}^{jj}(b_n)_j - (b_n)_i D_{ij}{}^{jj}\big) \equiv C_{ijj}\widetilde{\psi}_{ij}, \tag{2.14}$$

with which we mean left multiplication by the element

$$\widetilde{\psi}_{ij} \equiv \eta_{ij} + \sum_n (a_n)_i[\eta_{ij}(b_n)_j - (b_n)_i\eta_{ij}]$$

times the coupling constant C_{ijj}. The demand $JD_F = D_F J$ (cf. Table 1.3) on D_F means that $D_{ki}{}^{ji} = JD_{ik}{}^{ij}J^* = J(D_{ij}{}^{ik})^*J^*$, from which we infer that the component $D_{ii}{}^{ji}$ constitutes of left multiplication with $C_{iij}\eta_{ij}$. Its inner fluctuations are of the form

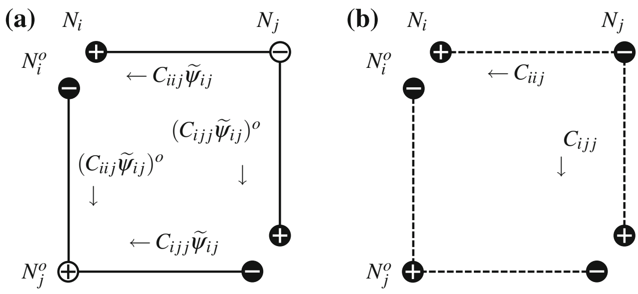

Fig. 2.2 After allowing for *off diagonal* representations we need a finite Dirac operator in order to have a chance at supersymmetry. The component A of (2.13) corresponds to the *upper* and *left lines*, whereas the component B corresponds to the *lower* and *right lines*. The off-diagonal vertex can have either $R = 1$ (*left image*) or $R = -1$ (*right image*). The R-value of the components of the finite Dirac operator changes accordingly, as is represented by the (*solid*/*dashed*) stroke of the edges

$$D_{ii}{}^{ji} \to D_{ii}{}^{ji} + \sum_n (a_n)_i \big(D_{ii}{}^{ji}(b_n)_j - (b_n)_i D_{ii}{}^{ji}\big) \equiv C_{iij}\widetilde{\psi}_{ij},$$

which coincides with (2.14). Furthermore, for $U = uJuJ^*$ with $u \in U(\mathscr{A})$ we find for these components (together with the inner fluctuations) that

$$UD_{ij}{}^{ii}U = u_i D_{ij}{}^{ii} u_j^*, \quad UD_{ij}{}^{jj}U = u_i D_{ij}{}^{jj} u_j^*,$$

establishing the result.

Since the diagonal vertices have an R-value of -1, the scalar field $\widetilde{\psi}_{ij}$ generated by D_F will always have an eigenvalue of R opposite to that of the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o \in \mathscr{H}_F$. This makes the off-diagonal vertices and these scalars indeed each other's superpartners, hence allowing us to call $\widetilde{\psi}_{ij}$ a sfermion. The Dirac operator (2.13) (together with the finite Hilbert space) is visualized by means of a Krajewski diagram in Fig. 2.2. Note that we can easily find explicit constructions for $R \in \mathscr{A}_F \otimes \mathscr{A}_F^o$. Requiring that the diagonal representations have an R-value of -1, we have the implementations $(1_{N_i}, -1_{N_j}) \otimes (-1_{N_i}, 1_{N_j})^o$ and $(1_{N_i}, 1_{N_j}) \otimes (-1_{N_i}, -1_{N_j})^o \in \mathscr{A}_F \otimes \mathscr{A}_F^o$, corresponding to the two possibilities of Fig. 2.2.

We capture this set-up with the following definition:

Definition 2.8 The *building block of the second type* $\mathscr{B}_{ij}^{\pm}$ consists of adding the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ (having γ_F-eigenvalue $\pm$) and its conjugate to a finite Hilbert space containing $\mathscr{B}_i$ and $\mathscr{B}_j$, together with maps between the representations $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and $\mathbf{N}_j \otimes \mathbf{N}_i^o$ and the adjoint representations that satisfy the prerequisites of Lemma 2.7. Symbolically it is denoted by

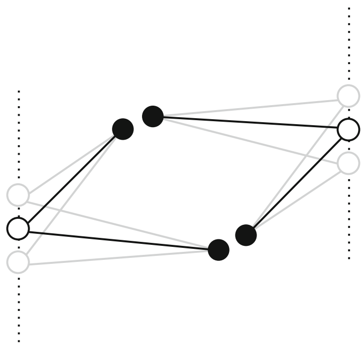

Fig. 2.3 An example of a building block of the second type for which the fermion has $R = 1$ and multiple generations

$$\mathscr{B}_{ij}^{\pm} = (e_i \otimes \bar{e}_j, e'_j \otimes \bar{e}'_i, D_{ii}{}^{ji} + D_{ij}{}^{jj}) \in \mathbf{N}_i \otimes \mathbf{N}_j^o \oplus \mathbf{N}_j \otimes \mathbf{N}_i^o \oplus \mathrm{End}(\mathscr{H}_F) \subset \mathscr{H}_F \oplus \mathrm{End}(\mathscr{H}_F).$$

When necessary, we will denote the chirality of the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ with a subscript L, R. Note that such a building block is always characterized by two indices and it can only be defined when $\mathscr{B}_i$ and $\mathscr{B}_j$ have previously been defined. In analogy with the building blocks of the first type and with the Higgses/higgsinos of the MSSM in the back of our minds we will require building blocks of the second type whose off-diagonal representation in $\mathscr{H}_F$ has $R = -1$ to have a maximal multiplicity of 1. In contrast, when the off-diagonal representation in the Hilbert space has $R = 1$ we can take multiple copies ('generations') of the same representation in $\mathscr{H}_F$, all having the *same* value of the grading γ_F. This also gives rise to an equal number of sfermions, keeping the number of fermionic and scalar degrees of freedom the same, which effectively entails giving the fermion/sfermion-pair a family structure. The C_{iij} and C_{ijj} are then promoted to $M \times M$ matrices acting on these copies. This situation is depicted in Fig. 2.3. We will always allow such a family structure when the fermion has $R = 1$, unless explicitly stated otherwise. There can also be two copies of a building block $\mathscr{B}_{ij}$ that have *opposite* values for the grading. We come back to this situation in Sect. 2.2.5.2.

Next, we compute the action corresponding to $\mathscr{B}_{ij}$. For a generic element ζ on the finite basis (2.12) we will write

$$\zeta = (\psi_{ijL}, \lambda'_{iL}, \lambda'_{iR}, \lambda'_{jL}, \lambda'_{jR}, \overline{\psi}_{ijR}) \in \mathscr{H}^+,$$

where the prime on the gauginos suggests that they still contain a trace-part (cf. (2.4)). To avoid notational clutter, we will write $\psi_L \equiv \psi_{ijL}$, $\overline{\psi}_R \equiv \overline{\psi}_{ijR}$ and $\widetilde{\psi} \equiv \widetilde{\psi}_{ijL}$ throughout the rest of this section. The *extra* action as a result of adding a building block $\mathscr{B}_{ij}^{+}$ of the second type (i.e. additional to that of (2.6) for $\mathscr{B}_i$ and $\mathscr{B}_j$) is given by

$$S_{ij}[\lambda'_i, \lambda'_j, \psi_L, \overline{\psi}_R, \mathbb{A}_i, \mathbb{A}_j, \widetilde{\psi}, \overline{\widetilde{\psi}}] \equiv S_{ij}[\zeta, \mathbb{A}, \widetilde{\zeta}] = S_{f,ij}[\zeta, \mathbb{A}, \widetilde{\zeta}] + S_{b,ij}[\mathbb{A}, \widetilde{\zeta}]. \tag{2.15}$$

The fermionic part of this action reads

$$\begin{aligned} S_{f,ij}[\zeta, \mathbb{A}, \widetilde{\zeta}] &= \tfrac{1}{2}\langle J(\psi_L, \overline{\psi}_R), \partial\!\!\!/_A(\psi_L, \overline{\psi}_R)\rangle \\ &\quad + \tfrac{1}{2}\langle J(\psi_L, \lambda'_{iL}, \lambda'_{iR}, \lambda'_{jL}, \lambda'_{jR}, \overline{\psi}_R), \gamma^5 \Phi(\psi_L, \lambda'_{i,L}, \lambda'_{iR}, \lambda'_{jL}, \lambda'_{jR}, \overline{\psi}_R)\rangle \\ &= \langle J_M \overline{\psi}_R, D_A \psi_L\rangle + \langle J_M \overline{\psi}_R, \gamma^5 \lambda'_{iR} C_{iij} \widetilde{\psi}\rangle + \langle J_M \overline{\psi}_R, \gamma^5 C_{ijj} \widetilde{\psi} \lambda'_{jR}\rangle \\ &\quad + \langle J_M \psi_L, \gamma^5 \overline{\widetilde{\psi}} C^*_{iij} \lambda'_{iL}\rangle + \langle J_M \psi_L, \gamma^5 \lambda'_{jL} \overline{\widetilde{\psi}} C^*_{ijj}\rangle\rangle, \end{aligned} \tag{2.16}$$

prior to scaling the gauginos according to (2.8). Here we have employed (2.14) and the symmetry of the inner product. The bosonic part of (2.15) is given by

$$S_{b,ij}[\mathbb{A}, \widetilde{\zeta}] = \int_M |\mathscr{N}_{ij} D_\mu \widetilde{\psi}|^2 + \mathscr{M}_{ij}(\widetilde{\psi}, \overline{\widetilde{\psi}}) \tag{2.17}$$

(cf. (1.24)) with $\mathscr{N}_{ij} = \mathscr{N}^*_{ij}$ the square root of the positive semi-definite $M \times M$–matrix

$$\mathscr{N}^2_{ij} = \frac{f(0)}{2\pi^2}(N_i C^*_{iij} C_{iij} + N_j C^*_{ijj} C_{ijj}), \tag{2.18}$$

where M is the number of particle generations, and

$$\mathscr{M}_{ij}(\widetilde{\psi}, \overline{\widetilde{\psi}}) = \frac{f(0)}{2\pi^2}\Big[N_i |C_{iij} \widetilde{\psi} \overline{\widetilde{\psi}} C^*_{iij}|^2 + N_j |\overline{\widetilde{\psi}} C^*_{ijj} C_{ijj} \widetilde{\psi}|^2 + 2|C_{iij}\widetilde{\psi}|^2 |C_{ijj}\widetilde{\psi}|^2\Big]. \tag{2.19}$$

The first term of this last equation corresponds to paths in the Krajewski diagram such as in the first example of Fig. 1.5, involving the vertex at (i, i). The second term corresponds to the same type of path but involving (j, j) and the third term consists of paths going in two directions such as the fourth example of Fig. 1.5.

2.2.2.1 Matching Degrees of Freedom

As far as the gauginos are concerned, there is a difference compared to the previous section; there the trace parts of the action fully decoupled from the rest of the action, but here this is not the case due to the fermion-sfermion-gaugino interactions in (2.15). At the same time, the gauge fields $A'_{\mu i}$ and $A'_{\mu j}$ do not act on $\mathbf{N}_i \otimes \mathbf{N}^o_j$ and $\mathbf{N}_j \otimes \mathbf{N}^o_i$ in the adjoint representation, causing their trace parts not to vanish either. We thus have fermionic and bosonic $u(1)$ fields, that are each other's potential superpartners.

We distinguish between two cases:

- In the left image of Fig. 2.2 $\mathscr{H}_{R=+} = \mathbf{N}_i \otimes \mathbf{N}_j^o \oplus \mathbf{N}_j \otimes \mathbf{N}_i^o$ and thus we can employ the unimodularity condition (2.1). This yields[5]

$$
\begin{aligned}
0 &= \mathrm{tr}_{\mathbf{N}_i \otimes \mathbf{N}_j^o}\, g_i' A_{i\mu}' + \mathrm{tr}_{\mathbf{N}_j \otimes \mathbf{N}_i^o}\, g_j' A_{j\mu}' \\
&= N_j g_{B_i} B_{i\mu} + N_i g_{B_j} B_{j\mu} \quad \Longrightarrow \quad B_{j\mu} = -(N_j g_{B_i}/N_i g_{B_j}) B_{i\mu},
\end{aligned}
$$

where we have first identified the independent gauge fields before introducing the coupling constants $g_{i,j}$, $g_{B_{i,j}}$ (cf. [7, Sect. 3.5.2]). Consequently the covariant derivative acting on the fermion ψ and scalar $\widetilde{\psi}$ and their conjugates is equal to $\partial\!\!\!/_A = i\gamma^\mu D_\mu$ with

$$
\begin{aligned}
D_\mu &= \nabla_\mu^S - i\Big(g_i A_{i\mu} + \frac{g_{B_i}}{N_i} B_i\Big) + i\Big(g_j A_{j\mu} + \frac{g_{B_j}}{N_j} B_j\Big)^o \\
&= \nabla_\mu^S - i g_i A_{i\mu} + i g_j A_{j\mu}^o - 2 i g_{B_i} \frac{B_i}{N_i}.
\end{aligned}
$$

This also means that the kinetic terms of the $u(1)$ gauge field now appear in the action. After applying the unimodularity condition, the kinetic terms of the gauge bosons, as acting on $\mathbf{N}_i \otimes \mathbf{N}_j^o$, are given by

$$
\begin{aligned}
&- \mathrm{tr}_{\mathbf{N}_i \otimes \mathbf{N}_j^o}\, \mathbb{F}'_{\mu\nu} \mathbb{F}'^{\mu\nu} \\
&\quad = \mathrm{tr}_{\mathbf{N}_i \otimes \mathbf{N}_j^o} \Big(g_i F_{\mu\nu}^i - g_j F_{\mu\nu}^{j\,o} + g_{B_i} \frac{2}{N_i} B_{\mu\nu}^i\Big)\Big(g_i F_i^{\mu\nu} - g_j F_j^{\mu\nu\,o} + g_{B_i} \frac{2}{N_i} B_i^{\mu\nu}\Big) \\
&\quad = N_j g_i^2\, \mathrm{tr}_{N_i}\, F_{\mu\nu}^i F_i^{\mu\nu} + N_i g_j^2\, \mathrm{tr}_{N_j}\, F_{\mu\nu}^j F_j^{\mu\nu} + 4 \frac{N_j}{N_i} g_{B_i}^2 B_{\mu\nu}^i B_i^{\mu\nu}, \qquad (2.20)
\end{aligned}
$$

with $B_{i\mu\nu} = \partial_{[\mu} B_{i\nu]}$. The contribution from $\mathbf{N}_j \otimes \mathbf{N}_i^o$ is the same and those from $\mathbf{N}_i \otimes \mathbf{N}_i^o$ and $\mathbf{N}_j \otimes \mathbf{N}_j^o$ have been given in the previous section.
We can use the supersymmetry transformations to also reduce the fermionic degrees of freedom:

Lemma 2.9 *Requiring the unimodularity condition (2.1) also for the supersymmetry transformations of the gauge fields, makes the traces of the gauginos proportional to each other.*

[5] When having multiple copies of the representations $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and $\mathbf{N}_j \otimes \mathbf{N}_i^o$ all expressions will be multiplied by the number of copies, since the gauge bosons act on each copy in the same way. This leaves the results unaffected, however.

Proof We introduce the notation $\lambda_{iL,R} = \lambda^a_{iL,R} \otimes T^a_i$, summed over the repeated index $a = 0, 1, \ldots, N_i^2 - 1$, where T^a_i are the generators of $u(N_i) \simeq u(1) \oplus su(N_i)$. Writing out the unimodularity condition (2.1) for the transformation (2.10a) of the gauge field reads in this case

$$0 = N_j(g_i \operatorname{tr} \delta A_{i\mu} + g_{B_i} \delta B_{i\mu}) + N_i(g_j \operatorname{tr} \delta A_{j\mu} + g_{B_j} \delta B_{j\mu}).$$

Putting in the expressions for the transformations and using that the $su(N_{i,j})$-parts of the gauginos are automatically traceless, we only retain the trace parts:

$$\begin{aligned} 0 &= N_j g_{B_i}\big[(J_M\varepsilon_R, \gamma_\mu \lambda^0_{iL}) + (J_M\varepsilon_L, \gamma_\mu\lambda^0_{iR})\big] \\ &\quad + N_i g_{B_j}\big[(J_M\varepsilon_R, \gamma_\mu \lambda^0_{jL}) + (J_M\varepsilon_L, \gamma_\mu\lambda^0_{jR})\big] \\ &= \big(J_M\varepsilon_R, \gamma_\mu(N_j g_{B_i}\lambda^0_{iL} + N_i g_{B_j}\lambda^0_{jL})\big) + (L \leftrightarrow R), \end{aligned} \tag{2.21}$$

where with '$(L \leftrightarrow R)$' we mean the expression preceding it, but everywhere with L and R interchanged. Since $\varepsilon = (\varepsilon_L, \varepsilon_R)$ can be any covariantly vanishing spinor, $(0, \varepsilon_R)$ with $\nabla^S\varepsilon_R = 0$ and $(\varepsilon_L, 0)$ with $\nabla^S\varepsilon_L = 0$ are valid solutions for which one of the terms in (2.21) vanishes, but the other does not. The term with left-handed gauginos is thus independent from that of the right-handed gauginos. Hence, for any ε_R,

$$\big(J_M\varepsilon_R, \gamma_\mu(N_j g_{B_i}\lambda^0_{iL} + N_i g_{B_j}\lambda^0_{jL})\big)$$

must vanish, establishing the result.

Via the transformation (2.10b) for the gaugino, we can also reduce one of the $u(1)$ parts of $G'_{i,j} = G^a_{i,j}T^a_{i,j} + H_{i,j} \in C^\infty(M, u(N_{i,j}))$.

This provides us a justification for the choice to take the trace in (2.1) only over $\mathscr{H}_F$. For if we had not, we would have been in a bootstrap-like situation in which the gaugino degrees of freedom would have contributed to the relation that we have employed to reduce them by.

- In the right image of Fig. 2.2 no constraint occurs due to the unimodularity condition because $\mathscr{H}_{R=+} = 0$ and the kinetic terms of the gauge bosons are given by:

$$\begin{aligned} &- \operatorname{tr}_{\mathbf{N}_i\otimes\mathbf{N}_j^o} \mathbb{F}'_{\mu\nu}\mathbb{F}'^{\mu\nu} \\ &\quad = \operatorname{tr}_{\mathbf{N}_i\otimes\mathbf{N}_j^o}\Big(g_i F^i_{\mu\nu} - g_j F^{j\,o}_{\mu\nu} + \frac{g_{B_i}}{N_i}B_{i\mu\nu} - \frac{g_{B_j}}{N_j}B_{j\mu\nu}\Big)^2 \\ &\quad = N_j g_i^2 \operatorname{tr}_{N_i} F^i_{\mu\nu}F_i^{\mu\nu} + N_i g_j^2 \operatorname{tr}_{N_j} F^j_{\mu\nu}F_j^{\mu\nu} \\ &\qquad + N_i N_j\Big(\frac{g_{B_i}B_i}{N_i} - \frac{g_{B_j}B_j}{N_j}\Big)_{\mu\nu}\Big(\frac{g_{B_i}B_i}{N_i} - \frac{g_{B_j}B_j}{N_j}\Big)^{\mu\nu}. \end{aligned} \tag{2.22}$$

Here for the second time we stumble upon problems with the fact that the spectral action gives us an on shell action only. The problem is twofold. First, there is—as in the case of $\mathscr{B}_i$ and $\mathscr{B}_j$—a mismatch in the degrees of freedom off shell between $\psi \equiv \psi_{ij}$ and $\widetilde{\psi} \equiv \widetilde{\psi}_{ij}$. We compensate for this by introducing a bosonic auxiliary field $F_{ij} \in C^\infty(M, \mathbf{N}_i \otimes \mathbf{N}_j^o)$ and its conjugate. They appear in the action via

$$S[F_{ij}, F_{ij}^*] = -\int_M \mathrm{tr}_{N_j}\, F_{ij}^* F_{ij} \sqrt{g}\mathrm{d}^4x. \tag{2.23}$$

From the Euler-Lagrange equations, it follows that $F_{ij} = F_{ij}^* = 0$, i.e. F_{ij} and its conjugate only have degrees of freedom off shell. Secondly, the four-scalar self-interaction of $\widetilde{\psi}$ poses an obstacle for a supersymmetric action; regardless of its specific form, a supersymmetry transformation of such a term must involve three scalars and one fermion, a term that cannot be canceled by any other. The standard solution is to rewrite these terms using the auxiliary fields G_i', G_j' that the building blocks of the first type provide us, such that we recover (2.17) on shell. The next lemma tells us that we can do this.

Lemma 2.10 *If $\mathscr{H}_{F,R=+} \neq 0$ then the four-scalar terms (2.19) of an almost-commutative geometry that consists of a single building block $\mathscr{B}_{ij}$ of the second type can be written in terms of auxiliary fields $G_{i,j} \in C^\infty(M, su(N_{i,j}))$ and $H \in C^\infty(M, u(1))$, as follows:*

$$\begin{aligned}\mathscr{L}(G_{i,j}, H, \widetilde{\psi}, \overline{\widetilde{\psi}}) = &-\frac{1}{2n_i}\,\mathrm{tr}\, G_i^2 - \frac{1}{2n_j}\,\mathrm{tr}\, G_j^2\\ &- \frac{1}{2}H^2 - \mathrm{tr}\, G_i \mathscr{P}_i' \widetilde{\psi}\overline{\widetilde{\psi}} - \mathrm{tr}\, G_j \overline{\widetilde{\psi}} \mathscr{P}_j' \widetilde{\psi} - H\,\mathrm{tr}\,\mathscr{Q}'\widetilde{\psi}\overline{\widetilde{\psi}},\end{aligned} \tag{2.24}$$

where in the terms featuring $G_{i,j}$ the trace is over the $N_{i,j} \times N_{i,j}$-matrices and with

$$\begin{aligned}\mathscr{P}_i' = \sqrt{\tfrac{f(0)}{\pi^2 n_i}N_i}C_{iij}^*C_{iij}, \quad \mathscr{P}_j' = \sqrt{\tfrac{f(0)}{\pi^2 n_j}N_j}C_{ijj}^*C_{ijj},\\ \mathscr{Q}' = \sqrt{\tfrac{f(0)}{\pi^2}}(C_{iij}^*C_{iij} + C_{ijj}^*C_{ijj})\end{aligned} \tag{2.25}$$

matrices on M-dimensional family space.

Proof Required for any building block $\mathscr{B}_{ij}$ of the second type are the building blocks $\mathscr{B}_i$ and $\mathscr{B}_j$ of the first type, initially providing auxiliary fields $G_{i,j} \equiv G_{i,j}^a T_{i,j}^a \in C^\infty(M, su(N_{i,j}))$ and $H_{i,j} \in C^\infty(M, u(1))$. Here the $T_{i,j}^a$ denote the generators of $su(N_{i,j})$ in the fundamental (defining) representation and are normalized according to $\mathrm{tr}\, T_{i,j}^a T_{i,j}^b = n_{i,j}\delta_{ab}$, where $n_{i,j}$ is the *constant of the representation*. After applying the unimodularity condition (2.1) in the case that $\mathscr{H}_{R=+} \neq 0$ (the left image of Fig. 2.2) for the gauge field and its transformation, only one $u(1)$ auxiliary field H remains. We thus consider the Lagrangian (2.24) with $\mathscr{P}_{i,j}'$, $\mathscr{Q}'$ self-adjoint. (These coefficients are written inside the trace since they may have family indices. However, the combinations $\mathscr{P}'_i\widetilde{\psi}\overline{\widetilde{\psi}}$ and $\overline{\widetilde{\psi}}\mathscr{P}'_j\widetilde{\psi}$ cannot have family-indices anymore, since G_i and G_j do not.) Applying the Euler-Lagrange equations to this Lagrangian yields

$$G_i^a = -\operatorname{tr} T_i^a\mathscr{P}'_i\widetilde{\psi}\overline{\widetilde{\psi}}, \qquad G_j^a = -\operatorname{tr} T_j^a\overline{\widetilde{\psi}}\mathscr{P}'_j\widetilde{\psi}, \qquad H = -\operatorname{tr}\mathscr{Q}'\widetilde{\psi}\overline{\widetilde{\psi}}$$

and consequently (2.24) equals *on shell*

$$\begin{aligned}\mathscr{L}(G_{i,j}, H, \widetilde{\psi}, \overline{\widetilde{\psi}}) &= \frac{1}{2}\operatorname{tr}(T_i^a\mathscr{P}'_i\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij})^2 + \frac{1}{2}\operatorname{tr}(T_j^a\overline{\widetilde{\psi}}_{ij}\mathscr{P}'_j\widetilde{\psi}_{ij})^2 + \frac{1}{2}\operatorname{tr}(\mathscr{Q}'\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij})^2\\ &= \frac{n_i}{2}\Big(|\mathscr{P}'_i\widetilde{\psi}\overline{\widetilde{\psi}}|^2 - \frac{1}{N_i}|\mathscr{P}'^{1/2}_i\widetilde{\psi}|^4\Big)\\ &\quad + \frac{n_j}{2}\Big(|\overline{\widetilde{\psi}}\mathscr{P}'_j\widetilde{\psi}|^2 - \frac{1}{N_j}|\mathscr{P}'^{1/2}_j\widetilde{\psi}|^4\Big) + \frac{1}{2}|\mathscr{Q}'^{1/2}\widetilde{\psi}|^4.\end{aligned}$$

Here we have employed the identity

$$(T^a_{i,j})_{mn}(T^a_{i,j})_{kl} = n_{i,j}\Big(\delta_{ml}\delta_{kn} - \frac{1}{N_{i,j}}\delta_{mn}\delta_{kl}\Big). \tag{2.26}$$

With the choices (2.25) we indeed recover the four-scalar terms (2.19) of the spectral action.

Even though in the case that $\mathscr{H}_{F,R=+} = 0$ (the right image of Fig. 2.2) the unimodularity condition cannot be used to relate the $u(1)$ fields H_i and H_j to each other, a similar solution is possible:

Corollary 2.11 *If $\mathscr{H}_{R=+} = 0$ then the four-scalar terms (2.19) of a building block $\mathscr{B}_{ij}$ of the second type can be written off shell using the Lagrangian*

$$\begin{aligned}\mathscr{L}(G_{i,j}, H_{i,j}, \widetilde{\psi}, \overline{\widetilde{\psi}}) &= -\frac{1}{2n_i}\operatorname{tr} G_i^2 - \frac{1}{2n_j}\operatorname{tr} G_j^2 - \frac{1}{2}H_i^2 - \frac{1}{2}H_j^2 - \operatorname{tr} G_i\mathscr{P}'_i\widetilde{\psi}\overline{\widetilde{\psi}}\\ &\quad - \operatorname{tr} G_j\overline{\widetilde{\psi}}\mathscr{P}'_j\widetilde{\psi} - H_i\operatorname{tr}\mathscr{Q}'_i\widetilde{\psi}\overline{\widetilde{\psi}} - H_j\operatorname{tr}\mathscr{Q}'_j\widetilde{\psi}\overline{\widetilde{\psi}},\end{aligned} \tag{2.27}$$

with

$$\mathscr{P}'_i = \sqrt{\tfrac{f(0)}{\pi^2 n_i}N_i}C^*_{iij}C_{iij}, \quad \mathscr{P}'_j = \sqrt{\tfrac{f(0)}{\pi^2 n_j}N_j}C^*_{ijj}C_{ijj},$$
$$\mathscr{Q}'_i = \mathscr{Q}'_j = \sqrt{\tfrac{f(0)}{2\pi^2}}(C^*_{iij}C_{iij} + C^*_{ijj}C_{ijj}),$$

not carrying a family-index.

In both cases we have obtained a system that has equal bosonic and fermionic degrees of freedom, both on shell and off shell.

2.2.2.2 The Final Action and Supersymmetry

We first turn to the case that $\mathscr{H}_{R=+} \neq 0$. Reducing the degrees of freedom by identifying half of the $u(1)$ fields with the other half and rewriting (2.15) to an off shell action we find the *extra* contributions

$$\begin{aligned}&\langle J_M \overline{\psi}_R, D_A\psi_L\rangle + \langle J_M\overline{\psi}_R, \gamma^5(\lambda'_{iR} C_{iij}\widetilde{\psi} + C_{ijj}\widetilde{\psi}\lambda'_{jR})\rangle\\ &\quad + \langle J_M \psi_L, \gamma^5(\overline{\widetilde{\psi}} C^*_{iij}\lambda'_{iL} + \lambda'_{jL}\overline{\widetilde{\psi}} C^*_{ijj})\rangle\\ &\quad + \int_M \Big[|\mathscr{N}_{ij} D_\mu \widetilde{\psi}|^2 - \mathrm{tr}_{N_i}\left(\mathscr{P}'_i \widetilde{\psi}\overline{\widetilde{\psi}} G_i\right) - \mathrm{tr}_{N_j}\left(\overline{\widetilde{\psi}}\mathscr{P}'_j\widetilde{\psi} G_j\right)\\ &\quad - H\,\mathrm{tr}_{N_i}\,\mathscr{Q}'\widetilde{\psi}\overline{\widetilde{\psi}} - \mathrm{tr}_{N_j^{\oplus M}}\, F^*_{ij}F_{ij}\Big]\end{aligned}$$

to the total action, with

$$\lambda'_i = \lambda_i + \lambda^0_i\,\mathrm{id}_{N_i}, \qquad \lambda'_j = \lambda_j - N_{j/i}\lambda^0_i\,\mathrm{id}_{N_j}$$

and $G_{i,j} \in C^\infty(M, su(N_{i,j}))$, $H \in C^\infty(M, u(1))$. For notational convenience we will suppress the subscripts in the traces when no confusion is likely to arise. In addition, adding a building block $\mathscr{B}_{ij}$ slightly changes the expressions for the prefactors of the kinetic terms of $A_{i\mu}$ and $A_{j\mu}$ (cf. Remark 2.14).

As a final step we scale the sfermion $\widetilde{\psi}_{ij}$ according to

$$\widetilde{\psi}_{ij} \to \mathscr{N}^{-1}_{ij}\widetilde{\psi}_{ij}, \qquad \overline{\widetilde{\psi}}_{ij} \to \overline{\widetilde{\psi}}_{ij}\mathscr{N}^{-1}_{ij}, \tag{2.28}$$

and the gauginos according to (2.8) to give us the correctly normalized kinetic terms for both:

$$\begin{aligned}&\langle J_M\overline{\psi}_R, D_A\psi_L\rangle + \langle J_M\overline{\psi}_R, \gamma^5[\lambda'_{iR}\widetilde{C}_{i,j}\widetilde{\psi} + \widetilde{C}_{j,i}\widetilde{\psi}\lambda'_{jR}]\rangle\\ &\quad + \langle J_M\psi_L, \gamma^5[\overline{\widetilde{\psi}}\widetilde{C}^*_{i,j}\lambda'_{iL} + \lambda'_{jL}\overline{\widetilde{\psi}}\widetilde{C}^*_{j,i}]\rangle\\ &\quad + \int_M \Big[|D_\mu\widetilde{\psi}|^2 - \mathrm{tr}\left(\mathscr{P}_i\widetilde{\psi}\overline{\widetilde{\psi}}G_i\right) - \mathrm{tr}\left(\overline{\widetilde{\psi}}\mathscr{P}_j\widetilde{\psi}G_j\right)\\ &\quad - \mathrm{tr}\, HQ\widetilde{\psi}\overline{\widetilde{\psi}} - \mathrm{tr}_{N_j^{\oplus M}}\, F^*_{ij}F_{ij}\Big].\end{aligned} \tag{2.29}$$

Here we have written

$$\begin{aligned}\widetilde{C}_{i,j} &:= \frac{C_{iij}}{\sqrt{n_i}}\mathscr{N}^{-1}_{ij}, & \widetilde{C}_{j,i} &:= \frac{C_{ijj}}{\sqrt{n_j}}\mathscr{N}^{-1}_{ij},\\ \mathscr{P}_{i,j} &:= \mathscr{N}^{-1}_{ij}\mathscr{P}'_{i,j}\mathscr{N}^{-1}_{ij}, & \mathscr{Q} &:= \mathscr{N}^{-1}_{ij}\mathscr{Q}'\mathscr{N}^{-1}_{ij}\end{aligned} \tag{2.30}$$

for the scaled versions of the parameters. For this action we have:

Theorem 2.12 *The total action that is associated to $\mathscr{B}_i \oplus \mathscr{B}_j \oplus \mathscr{B}_{ij}$, given by (2.9) and (2.29), is supersymmetric under the transformations (2.10),*

$$\delta\widetilde{\psi} = c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)_{\mathscr{S}}, \qquad \delta\overline{\widetilde{\psi}} = c^*_{ij}(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)_{\mathscr{S}}, \tag{2.31a}$$

$$\delta\psi_L = c'_{ij}\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R + d'_{ij}F_{ij}\varepsilon_L, \quad \delta\overline{\psi}_R = c'^*_{ij}\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L + d'^*_{ij}F^*_{ij}\varepsilon_R \tag{2.31b}$$

and

$$\delta F_{ij} = d_{ij}(J_M\varepsilon_R, \partial\!\!\!/_A\psi_L)_{\mathscr{S}} + d_{ij,i}(J_M\varepsilon_R, \gamma^5\lambda_{iR}\widetilde{\psi})_{\mathscr{S}} - d_{ij,j}(J_M\varepsilon_R, \gamma^5\widetilde{\psi}\lambda_{jR})_{\mathscr{S}}, \tag{2.32a}$$

$$\delta F^*_{ij} = d^*_{ij}(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_R)_{\mathscr{S}} + d^*_{ij,i}(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\lambda_{iL})_{\mathscr{S}} - d^*_{ij,j}(J_M\varepsilon_L, \gamma^5\lambda_{jL}\overline{\widetilde{\psi}})_{\mathscr{S}}, \tag{2.32b}$$

with c_{ij}, c'_{ij}, d_{ij}, d'_{ij}, $d_{ij,i}$ and $d_{ij,j}$ complex numbers, if and only if

$$\widetilde{C}_{i,j} = \varepsilon_{i,j}\sqrt{\frac{2}{\mathscr{K}_i}}g_i\,\mathrm{id}_M, \quad \widetilde{C}_{j,i} = \varepsilon_{j,i}\sqrt{\frac{2}{\mathscr{K}_j}}g_j\,\mathrm{id}_M, \quad \mathscr{P}_i^2 = \frac{g_i^2}{\mathscr{K}_i}\,\mathrm{id}_M, \quad \mathscr{P}_j^2 = \frac{g_j^2}{\mathscr{K}_j}\,\mathrm{id}_M, \tag{2.33}$$

for the unknown parameters of the finite Dirac operator (where id_M is the identity on family-space, which equals unity if ψ_{ij} has no family index) and

$$c'_{ij} = c^*_{ij} = \varepsilon_{i,j}\sqrt{2\mathscr{K}_i}c_i = -\varepsilon_{j,i}\sqrt{2\mathscr{K}_j}c_j,$$
$$d_{ij} = d'^*_{ij} = \varepsilon_{i,j}\sqrt{\frac{\mathscr{K}_i}{2}}\frac{d_{ij,i}}{g_i} = -\varepsilon_{j,i}\sqrt{\frac{\mathscr{K}_j}{2}}\frac{d_{ij,j}}{g_j}, \qquad c_{G_i} = \varepsilon_i\sqrt{\mathscr{K}_i}c_i,$$

with $\varepsilon_i, \varepsilon_{i,j}, \varepsilon_{j,i} \in \{\pm 1\}$ for the transformation constants.

Proof Since the action (2.9) is already supersymmetric by virtue of Theorem 2.5, we only have to prove that the same holds for the contribution (2.29) to the action from $\mathscr{B}_{ij}$. The detailed proof of this fact can be found in Appendix section 'Second Building Block'.

Then for C_{iij} and $\mathscr{P}_{i,j}$ that satisfy these relations (setting $\mathscr{K}_{i,j} = 1$), the supersymmetric action (but omitting the $u(1)$-terms for conciseness now) reads:

$$\begin{aligned}&\langle J_M\overline{\psi}_R, \partial\!\!\!/_A\psi_L\rangle + \sqrt{2}\langle J_M\overline{\psi}_R, \gamma^5(\varepsilon_{i,j}g_i\lambda_{iR}\widetilde{\psi} + \varepsilon_{j,i}\widetilde{\psi}g_j\lambda_{jR})\rangle\\ &\quad + \sqrt{2}\langle J_M\psi_L, \gamma^5(\varepsilon_{i,j}\overline{\widetilde{\psi}}g_i\lambda_{iL} + \varepsilon_{j,i}g_j\lambda_{jL}\overline{\widetilde{\psi}})\rangle\\ &\quad + \int_M \Big[|D_\mu\widetilde{\psi}|^2 - g_i\,\mathrm{tr}_{N_i}\big(\widetilde{\psi}\overline{\widetilde{\psi}}G_i\big) - g_j\,\mathrm{tr}_{N_j}\big(\overline{\widetilde{\psi}}\widetilde{\psi}G_j\big) - \mathrm{tr}_{N_j^{\oplus M}}F^*_{ij}F_{ij}\Big],\end{aligned} \tag{2.34}$$

i.e. we recover the pre-factors for the fermion-sfermion-gaugino and four-scalar interactions that are familiar for supersymmetry. The signs $\varepsilon_{i,j}$ and $\varepsilon_{j,i}$ above can be chosen freely.

Remark 2.13 In the case that $\mathscr{H}_{R=+} = 0$, there is an interaction

$$\propto \int_M B_{i\mu\nu} B_j^{\mu\nu} \tag{2.35}$$

present (see the last term of (2.22)). Transforming the gauge fields appearing in that interaction shows that the supersymmetry of the total action requires an interaction

$$\propto \langle J_M \lambda_i^0, \not{\partial}_M \lambda_j^0 \rangle,$$

a term that the fermionic action does not provide. Thus, a situation in which there are two different $u(1)$ fields that both act on the same representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ is an obstruction for supersymmetry. This is also the reason that a supersymmetric action with gauge groups $U(N_{i,j})$ is not possible in the presence of a representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$, since

$$\begin{aligned} -\operatorname{tr}_{\mathbf{N}_i \otimes \mathbf{N}_j^o} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} &= \operatorname{tr}_{\mathbf{N}_i \otimes \mathbf{N}_j^o} (g_i F^i_{\mu\nu} - g_j F^{j\,o}_{\mu\nu})(g_i F_i^{\mu\nu} - g_j F_j^{\mu\nu\,o}) \\ &= N_j g_i^2 \operatorname{tr} F^i_{\mu\nu} F_i^{\mu\nu} + N_i g_j^2 \operatorname{tr} F^j_{\mu\nu} F_j^{\mu\nu} - 2 g_i g_j \operatorname{tr} F^i_{\mu\nu} \operatorname{tr} F_j^{\mu\nu}, \end{aligned}$$

of which the last term spoils supersymmetry. Averting a theory in which two independent $u(1)$ gauge fields act on the same representation will be seen to put an important constraint on realistic supersymmetric models from noncommutative geometry.

Note that it is not per se the presence of an $R = -1$ off-diagonal fermion in the first place that is causing this; in a spectral triple that contains at least one $R = +1$ fermion the interaction (2.35) vanishes due to the unimodularity condition (2.1).

Remark 2.14 In the previous section we have compactly written

$$\mathscr{K}_i = \frac{f(0)}{3\pi^2} 2 N_i g_i^2 n_i$$

only partly for notational convenience. There are two other reasons. The first is that since the kinetic terms for the gauge bosons are normalized to $-1/4$, $\mathscr{K}_i$ must in the end have the value of 1. This puts a relation between $f(0)$ and g_i. This is the same as in the Standard Model [7, Sect. 17.1]. Secondly, the expression for $\mathscr{K}_i$ depends on the contents of the spectral triple. As (2.20) shows, when the Hilbert space is extended with $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and its opposite (both having $R = 1$), then (2.7) changes to

$$\begin{gathered} \mathscr{K}_i = \tfrac{f(0)}{3\pi^2} g_i^2 n_i (2N_i + M N_j), \quad \mathscr{K}_j = \tfrac{f(0)}{3\pi^2} g_j^2 n_j (M N_i + 2 N_j), \\ \mathscr{K}_B = \tfrac{4 f(0)}{3\pi^2} \tfrac{N_j}{N_i} M g_B^2. \end{gathered} \tag{2.36}$$

Here M denotes the number of generations that the fermion–sfermion pair comes in. In fact, the relation between the coupling constant(s) g_i and the function f should be evaluated only for the full spectral triple. In this case however, setting all three terms equal to one, implies the GUT-like relation

$$n_i(2N_i + MN_j)g_i^2 = n_j(2N_j + MN_i)g_j^2 = 4\frac{N_j}{N_i}Mg_B^2.$$

What remains, is to check whether there exist solutions for C_{iij} and C_{ijj} that satisfy the supersymmetry constraints (2.33).

Proposition 2.15 *Consider an almost-commutative geometry whose finite algebra is of the form $M_{N_i}(\mathbb{C}) \oplus M_{N_j}(\mathbb{C})$. The particle content and action associated to this almost-commutative geometry are both supersymmetric off shell if and only if it consists of two disjoint building blocks $\mathscr{B}_{i,j}$ of the first type, for which $N_i, N_j > 1$.*

Proof We will prove this by showing that the action of a single building block $\mathscr{B}_{ij}$ of the second type is not supersymmetric, falling back to Theorem 2.5 for a positive result. For the action of a $\mathscr{B}_{ij}$ of the second type to be supersymmetric requires the existence of parameters C_{iij} and C_{ijj} that—after scaling according to (2.30)—satisfy (2.33) both directly and indirectly via $\mathscr{P}_{i,j}$ of the form (2.25). To check whether they directly satisfy (2.33) we note that the pre-factor $\mathscr{N}_{ij}^2$ for the kinetic term of the sfermion $\widetilde{\psi}_{ij}$ appearing in (2.30) itself is an expression in terms of C_{iij} and C_{ijj}. We multiply the first relation of (2.33) with its conjugate and multiply with $\mathscr{N}_{ij}$ on both sides to get

$$C_{iij}^* C_{iij} = \frac{2}{\mathscr{K}_i} n_i g_i^2 \mathscr{N}_{ij}^2.$$

Inserting the expression (2.18) for $\mathscr{N}_{ij}^2$, we obtain

$$C_{iij}^* C_{iij} = g_i^2 n_i \frac{f(0)}{\pi^2} \frac{1}{\mathscr{K}_i} \Big[N_i C_{iij}^* C_{iij} + N_j C_{ijj}^* C_{ijj} \Big].$$

From (2.30) and (2.33) we infer that $C_{ijj}^* C_{ijj} = (n_j g_j^2 / n_i g_i^2) C_{iij}^* C_{iij}$, i.e. we require:

$$\mathscr{K}_i = \frac{f(0)}{\pi^2} \Big[g_i^2 n_i N_i + n_j g_j^2 N_j \Big].$$

If we use the expressions (2.36) for the pre-factors of the gauge bosons' kinetic terms to express the combinations $f(0) n_{i,j} g_{i,j}^2 / \pi^2$ in terms of $N_{i,j}$ and M, the requirement for consistency reads

$$1 = \left(\frac{3N_i}{2N_i + MN_j} + \frac{3N_j}{MN_i + 2N_j} \right).$$

The only solutions to this equation are given by $M = 4$ and $N_i = N_j$. However, inserting the solution (2.33) for $C^*_{iij}C_{iij}$ into the expression (2.25) for $\mathscr{P}_i$, $\mathscr{P}_j$ (necessary to write the action off shell) gives

$$\mathscr{P}_i^2 = 4\frac{f(0)}{\pi^2}N_i g_i^4 \frac{n_i}{\mathscr{K}_i^2}, \qquad \mathscr{P}_j^2 = 4\frac{f(0)}{\pi^2}N_j g_j^4 \frac{n_j}{\mathscr{K}_j^2},$$

with an id_M where appropriate. We again use Remark 2.14 to replace $f(0)g_i^2/(\pi^2\mathscr{K}_i)$ by an expression featuring $N_{i,j}$, M and $n_{i,j}$. This yields

$$\mathscr{P}_i^2 = \frac{12N_i}{2N_i + MN_j}\frac{g_i^2}{\mathscr{K}_i} = 2\frac{g_i^2}{\mathscr{K}_i}, \qquad \mathscr{P}_j^2 = \frac{12N_j}{2N_j + MN_i}\frac{g_j^2}{\mathscr{K}_j} = 2\frac{g_j^2}{\mathscr{K}_j}$$

for the values $M = 4$, $N_i = N_j$ that gave the correct fermion-sfermion-gaugino interactions. We thus have a contradiction with the demand on $\mathscr{P}_{i,j}^2$ from (2.33), necessary for supersymmetry.

We shortly pay attention to a case that is of similar nature but lies outside the scope of the above Proposition.

Remark 2.16 For $\mathscr{A}_F = \mathbb{C} \oplus \mathbb{C}$, a building block $\mathscr{B}_{ij}$ of the second type does not have a supersymmetric action either. In this case there are only $u(1)$ fields present in the theory and G_i, G_j and H are seen to coincide. It is possible to rewrite the four-scalar interaction of the spectral action off shell, but this set-up also suffers from a similar problem as in Proposition 2.15.

We can extend the result of Proposition 2.15 to components of the finite algebra that are defined over other fields than $\mathbb{C}$. For this, we first need the following lemma.

Lemma 2.17 *The inner fluctuations (1.17) of $\not{\partial}_M$ caused by a component of the finite algebra that is defined over $\mathbb{R}$ or $\mathbb{H}$, are traceless.*

Proof The inner fluctuations are of the form

$$i\gamma^\mu A_\mu^{\mathbb{F}}, \quad A_\mu^{\mathbb{F}} = \sum_i a_i\partial_\mu(b_i), \qquad \text{with} \quad a_i, b_i \in C^\infty(M, M_N(\mathbb{F})), \quad \mathbb{F} = \mathbb{R}, \mathbb{H}.$$

This implies that $A_\mu^{\mathbb{F}}$ is itself an $M_N(\mathbb{F})$-valued function. For the inner fluctuations to be self-adjoint, $A_\mu^{\mathbb{F}}$ must be skew-Hermitian. In the case that $\mathbb{F} = \mathbb{R}$ this implies that all components on the diagonal vanish and consequently so does the trace. In the case that $\mathbb{F} = \mathbb{H}$, all elements on the diagonal must themselves be skew-Hermitian. Since all quaternions are of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \alpha, \beta \in \mathbb{C},$$

this means that the diagonal of $A_\mu^{\mathbb{H}}$ consists of purely imaginary numbers that vanish pairwise. Its trace is thus also 0.

Then we have

Theorem 2.18 *Consider an almost-commutative geometry whose finite algebra is of the form* $M_{N_i}(\mathbb{F}_i) \oplus M_{N_j}(\mathbb{F}_j)$ *with* $\mathbb{F}_i, \mathbb{F}_j = \mathbb{R}, \mathbb{C}, \mathbb{H}$. *If the particle content and action associated to this almost-commutative geometry are both supersymmetric off shell, then it consists of two disjoint building blocks* $\mathscr{B}_{i,j}$ *of the first type, for which* $N_i, N_j > 1$.

Proof Not only do we have different possibilities for the fields $\mathbb{F}_{i,j}$ over which the components are defined, but we can also have various combinations for the values of the R-parity. We cover all possible cases one by one.

If $R = +1$ on the representations in the finite Hilbert space that describe the gauginos, then the gauginos and gauge bosons have the same R-parity and the particle content is not supersymmetric.

If $R = -1$ for these representations, and $R = +1$ on the off-diagonal representations, suppose at least one of the $\mathbb{F}_i, \mathbb{F}_j$ is equal to $\mathbb{R}$ or $\mathbb{H}$. Then using Lemma 2.17 we see that after application of the unimodularity condition (2.1) there is no $u(1)$-valued gauge field left. Lemma 2.9 then also causes the absence of a $u(1)$-auxiliary field that is needed to write the four-scalar action off shell as in Lemma 2.10. If both $\mathbb{F}_i$ and $\mathbb{F}_j$ are equal to $\mathbb{C}$ we revert to Proposition 2.15 to show that there is no supersymmetric solution for M and $N_{i,j}$ that satisfies the demands for $\widetilde{C}_{i,j}, \widetilde{C}_{j,i}$ and $\mathscr{P}_{i,j}$ from supersymmetry.

In the third case $R = -1$ on the off-diagonal representations in $\mathscr{H}_F$. If both $\mathbb{F}_{i,j}$ are equal to $\mathbb{R}$ or $\mathbb{H}$ then there is no $u(1)$ gauge field and thus the spectral action cannot be written off shell. If either $\mathbb{F}_i$ or $\mathbb{F}_j$ equals $\mathbb{R}$ or $\mathbb{H}$, then there is one $u(1)$-field, but the calculation for the action carries through as in Proposition 2.15 and there is no supersymmetric solution for M and $N_{i,j}$. Finally, if both $\mathbb{F}_{i,j}$ are equal to $\mathbb{C}$, there are two $u(1)$-fields and the cross term as in Remark 2.13 spoils supersymmetry.

Thus, all almost-commutative geometries for which $\mathscr{A}_F = M_{N_i}(\mathbb{F}_i) \oplus M_{N_j}(\mathbb{F}_j)$ and that have off-diagonal representations fail to be supersymmetric off shell.

The set-up described in this section has the same particle content as the supersymmetric version of a single ($R = +1$) particle–antiparticle pair and corresponds in that respect to a single chiral superfield in the superfield formalism [9, 4.3]. In constrast, its action is not fully supersymmetric. We stress however, that the scope of Proposition 2.15 is that of a *single* building block of the second type. As was mentioned before, the expressions for many of the coefficients typically vary with the contents of the finite spectral triple and they should only be assessed for the full model.

Another interesting difference with the superfield formalism is that a building block of the second type really requires two building blocks of the first type, describing gauginos and gauge bosons. In the superfield formalism a theory consisting of only a chiral multiplet, not having gauge interactions, is in many textbooks the first model to be considered. This underlines that noncommutative geometry inherently describes gauge theories.

There are ways to extend almost-commutative geometries by introducing new types of building blocks—giving new possibilities for supersymmetry—or by combining ones that we have already defined. In the next section we will cover an example of the latter situation, in which there arise interactions between two or more building blocks of the second type.

2.2.2.3 Interaction Between Building Blocks of the Second Type

In the previous section we have fully exploited the options that a finite algebra with two components over the complex numbers gave us. If we want to extend our theory, the finite algebra (2.3) needs to have a third summand—say $M_{N_k}(\mathbb{C})$. A building block of the first type (cf. Sect. 2.2.1) can easily be added, but then we already stumble upon severe problems:

Proposition 2.19 *The action (1.24) of an almost-commutative geometry whose finite algebra consists of three summands $M_{N_{i,j,k}}(\mathbb{C})$ over $\mathbb{C}$ and whose finite Hilbert space features building blocks $\mathscr{B}^{\pm}_{ij}$ and $\mathscr{B}^{\pm}_{ik}$ is not supersymmetric.*

Proof The inner fluctuations of the canonical Dirac operator on $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and $\mathbf{N}_i \otimes \mathbf{N}_k^o$ read:

$$\partial\!\!\!/_M + g_i A_i - g_j A_j^o + \frac{g_{B_i}}{N_i} B_i - \frac{g_{B_j}}{N_j} B_j, \quad \partial\!\!\!/_M + g_i A_i - g_k A_k^o + \frac{g_{B_i}}{N_i} B_i - \frac{g_{B_k}}{N_k} B_k,$$

where $A_{i,j,k} = \gamma_\mu A^\mu_{i,j,k}$, with $A^\mu_{i,j,k}(x) \in su(N_{i,j,k})$ and similarly $B^\mu_{i,j,k}(x) \in u(1)$. The unimodularity condition will, in the case that the representation of at least one of the two building blocks has $R = +1$, leave two of the three independent $u(1)$ fields—say—B_i and B_j. The kinetic terms of the gauge bosons on both representations will then feature a cross term (2.35) of different $u(1)$ field strengths, an obstruction for supersymmetry.

To resolve this, we allow—inspired by the NCSM—for one or more copies of the quaternions $\mathbb{H}$ in the finite algebra. If we define a building block of the first type over such a component (with the finite Hilbert space $M_2(\mathbb{C})$ as a bimodule of the complexification $M_1(\mathbb{H})^{\mathbb{C}} = M_2(\mathbb{C})$ of the algebra, instead of $\mathbb{H}$ itself, cf. [1, Sect. 4.1], [6]), the self-adjoint inner fluctuations of the canonical Dirac operator are already seen to be in $su(2)$ (e.g. traceless) prior to applying the unimodularity condition. On a representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ (from a building block $\mathscr{B}^{\pm}_{ij}$ of the second type), of which one of the indices comes from a component $\mathbb{H}$, only one $u(1)$ field will act.

From here on, using three or more components in the algebra, we will always assume at most two to be of the form $M_N(\mathbb{C})$ and all others to be equal to $\mathbb{H}$.

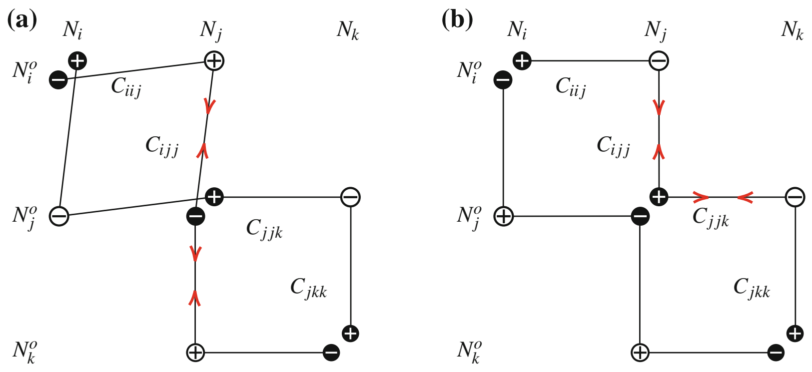

Fig. 2.4 In the case that there are two building blocks of the second type sharing one of their indices, there are extra interactions in the action. **a** Contributions when the gradings of the building blocks are different. **b** Contributions when the gradings of the building blocks are the same

The action of an almost commutative geometry whose finite spectral triple features two building blocks of the second type sharing one of their indices (i.e. that are in the same row or column in a Krajewski diagram) contains extra four-scalar contributions. The specific form of these terms depends on the value of the grading and of the indices appearing. When the first indices of two building blocks are the same, and they have the same grading (e.g. $\mathscr{B}_{ji}^{+}$ and $\mathscr{B}_{jk}^{+}$, cf. Fig. 2.4a) the resulting extra interactions are given by

$$S_{ij,jk}[\widetilde{\psi}_{ij}, \widetilde{\psi}_{jk}] = \frac{f(0)}{\pi^2} N_j \int_M |C_{ijj}\widetilde{\psi}_{ij} C_{jjk}\widetilde{\psi}_{jk}|^2 \sqrt{g}\mathrm{d}^4x. \tag{2.37}$$

In the other case (cf. Fig. 2.4b) it is given by

$$S_{ij,jk}[\widetilde{\psi}_{ij}, \widetilde{\psi}_{jk}] = \frac{f(0)}{\pi^2} \int_M |C_{ijj}\widetilde{\psi}_{ij}|^2 |C_{jjk}\widetilde{\psi}_{jk}|^2 \sqrt{g}\mathrm{d}^4x. \tag{2.38}$$

The paths corresponding to these contributions are depicted in Fig. 2.4.

However, to write all four-scalar interactions from the spectral action off shell in terms of the auxiliary fields $G_{i,j,k}$, one requires interactions of the form of both (2.37) and (2.38) to be present. The reason for this is the following. Upon writing the four-scalar part of the action of the building blocks $\mathscr{B}_{ij}$ and $\mathscr{B}_{jk}$ in terms of the auxiliary fields as in Lemma 2.10, we find for the terms with G_j in particular:

$$-\frac{1}{2n_j}\operatorname{tr}_{N_j} G_j^2 - \operatorname{tr}_{N_j} G_j\left(\overline{\widetilde{\psi}}_{ij}\mathscr{P}'_{j,i}\widetilde{\psi}_{ij}\right) - \operatorname{tr}_{N_j} G_j\left(\mathscr{P}'_{j,k}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\right).$$

On shell, the cross terms of this expression then give the additional four-scalar interaction

$$n_j|\mathscr{P}'^{1/2}_{j,i}\widetilde{\psi}_{ij}\mathscr{P}'^{1/2}_{j,k}\widetilde{\psi}_{jk}|^2 - \frac{n_j}{N_j}|\mathscr{P}'^{1/2}_{j,i}\widetilde{\psi}_{ij}|^2|\mathscr{P}'^{1/2}_{j,i}\widetilde{\psi}_{jk}|^2. \tag{2.39}$$

When the scaled counterparts (2.30) of $\mathscr{P}'_{j,i}$ and $\mathscr{P}'_{j,k}$ satisfy the constraints (2.33) for supersymmetry, this interaction reads

$$n_j g_j^2\Big(|\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2 - \frac{1}{N_j}|\widetilde{\psi}_{ij}|^2|\widetilde{\psi}_{jk}|^2\Big)$$

after scaling the fields. When having two or more building blocks of the second type that share one of their indices, we have either (2.37) or (2.38) in the spectral action, while we need (2.39) for a supersymmetric action. To possibly restore supersymmetry we need additional interactions, such as those of the next section.

2.2.3 *Third Building Block: Extra Interactions*

In a situation in which the finite algebra has three components and there are two adjacent building blocks of the second type, as depicted in Fig. 2.4b, there is allowed a component

$$D_{ij}{}^{kj} : \mathbf{N}_k \otimes \mathbf{N}_j^o \to \mathbf{N}_i \otimes \mathbf{N}_j^o \tag{2.40}$$

of the finite Dirac operator. We parametrize it with $\Upsilon_i{}^{k*}$, that acts (non-trivially) on family space. Such a component satisfies the first order condition and its inner fluctuations

$$\sum_n a_n[D_{ij}{}^{kj}, b_n] = \sum_n (a_i)_n\Big(\Upsilon_i{}^{k*}(b_k)_n - (b_i)_n\Upsilon_i{}^{k*}\Big)$$

generate a scalar $\widetilde{\psi}_{ik} \in \mathbf{N}_i \otimes \mathbf{N}_k^o$. Since there is no corresponding fermion ψ_{ik} present, a necessary condition for restoring supersymmetry is the existence of a building block $\mathscr{B}^{\pm}_{ik}$ of the second type. The component (2.40) then gives—amongst others—an extra fermionic contribution

$$\langle J_M\overline{\Psi}_{ij}, \gamma^5\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\Psi}_{jk}\rangle$$

to the action. Using the transformations (2.31) and (2.32), under which a building block of the second type is supersymmetric, we infer that this new term spoils supersymmetry. To overcome this, we need to add two extra components

$$D_{jk}{}^{ik} : \mathbf{N}_i \otimes \mathbf{N}_k^o \to \mathbf{N}_j \otimes \mathbf{N}_k^o, \qquad D_{ij}{}^{ik} : \mathbf{N}_i \otimes \mathbf{N}_k^o \to \mathbf{N}_i \otimes \mathbf{N}_j^o$$

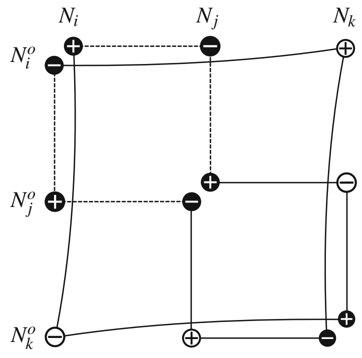

Fig. 2.5 A situation in which all three building blocks of the second type are present whose two indices are either i, j or k

to the finite Dirac operator, as well as their adjoints and the components that can be obtained by demanding that $[D_F, J_F] = 0$. We parametrize these two components with $\Upsilon_i{}^{j*}$ and $\Upsilon_j{}^{k*}$ respectively. They give extra contributions to the fermionic action that are of the form

$$\langle J_M \overline{\psi}_{jk}, \gamma^5 \overline{\widetilde{\psi}}_{ij} \Upsilon_i{}^{j*} \psi_{ik} \rangle + \langle J_M \overline{\psi}_{ij}, \gamma^5 \psi_{ik} \overline{\widetilde{\psi}}_{jk} \Upsilon_j{}^{k*} \rangle.$$

Both components require the representation $\mathbf{N}_i \otimes \mathbf{N}_k^o$ to have an eigenvalue of γ_F that is opposite to those of $\mathbf{N}_i \otimes \mathbf{N}_j^o$ and $\mathbf{N}_j \otimes \mathbf{N}_k^o$. This is the situation as is depicted in Fig. 2.5.

This brings us to the following definition:

Definition 2.20 For an almost-commutative geometry in which $\mathscr{B}_{ij}^{\pm}$, $\mathscr{B}_{ik}^{\mp}$ and $\mathscr{B}_{jk}^{\pm}$ are present, a *building block of the third type* $\mathscr{B}_{ijk}$ is the collection of all allowed components of the Dirac operator, mapping between the three representations $\mathbf{N}_i \otimes \mathbf{N}_j^o$, $\mathbf{N}_i \otimes \mathbf{N}_k^o$ and $\mathbf{N}_k \otimes \mathbf{N}_j^o$ and their conjugates. Symbolically it is denoted by

$$\mathscr{B}_{ijk} = (0, D_{ij}{}^{kj} + D_{jk}{}^{ik} + D_{ij}{}^{ik}) \in \mathscr{H}_F \oplus \mathrm{End}(\mathscr{H}_F). \tag{2.41}$$

The Krajewski diagram corresponding to $\mathscr{B}_{ijk}$ is depicted in Fig. 2.6.

The parameters of (2.41) are chosen such that the sfermions $\widetilde{\psi}_{ij}$ and $\widetilde{\psi}_{jk}$ are generated by the inner fluctuations of $\Upsilon_i{}^j$ and $\Upsilon_j{}^k$ respectively, whereas $\widetilde{\psi}_{ik}$ is generated by $\Upsilon_i{}^{k*}$. This is because $\widetilde{\psi}_{ik}$ crosses the particle/antiparticle-diagonal in the Krajewski diagram. Note that i, j, k are labels, not matrix indices.

There are several possible values of R that the vertices and edges can have. Requiring a grading that yields -1 on each of the diagonal vertices, all possibilities for an explicit construction of $R \in \mathscr{A}_F \otimes \mathscr{A}_F^o$ are given by $R = -P \otimes P^o$, $P = (\pm 1, \pm 1, \pm 1) \in \mathscr{A}_F$ where each of the three signs can vary independently. This yields 8 possibilities, but each of them appears in fact twice. Of the effectively four remaining combinations, three have one off-diagonal vertex that has $R = -1$ and in the other combination all three off-diagonal vertices have $R = -1$. These

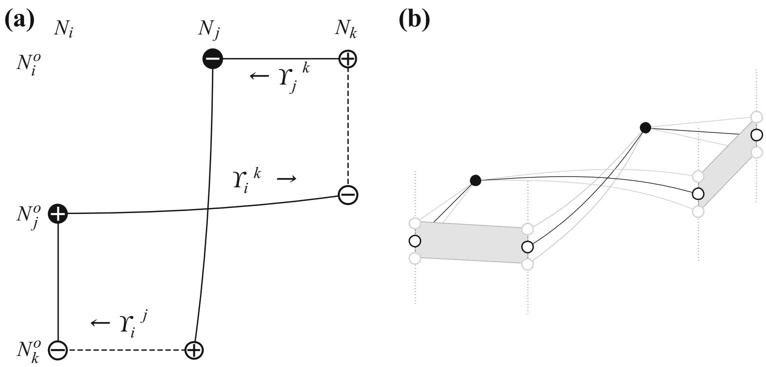

Fig. 2.6 A building block $\mathscr{B}_{ijk}$ of the third type in the language of Krajewski diagrams. **a** For clarity we have omitted here the edges and vertices that stem from the building blocks of the first and second type. **b** The same building block as shown on the *left side* but with the possible family structure of the two scalar fields with $R = 1$ being visualized

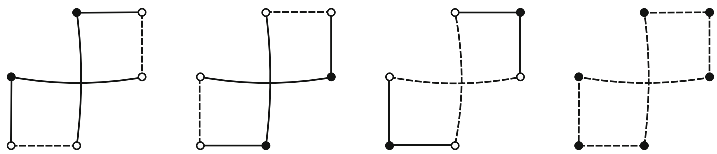

Fig. 2.7 All possible combinations of values for the R-parity operator in a building block of the third type. Three of those possibilities have one representation on which $R = -1$, in the other possibility all three of them have $R = -1$. This last option essentially entails having no family structure

four possibilities are depicted in Fig. 2.7. We will typically work in the case of the first image of Fig. 2.7, as is visualised in Fig. 2.6b, and will indicate where changes might occur when working in one of the other possibilities. If in this context the $R = 1$ representations in $\mathscr{H}_F$ come in M copies ('generations'), all components of the finite Dirac operator are in general acting non-trivially on these M copies, except C_{iij} and C_{ijj}, since they parametrize components of the finite Dirac operator mapping between $R = -1$ representations.

Note that in the action the expressions (2.18) for the pre-factors $\mathscr{N}_{ij}^2$, $\mathscr{N}_{ik}^2$ and $\mathscr{N}_{jk}^2$ of the sfermion kinetic terms all get an extra contribution from the new edges of the Krajewski diagram of Fig. 2.6. The first of these becomes

$$\mathscr{N}_{ij}^2 \to \frac{f(0)}{2\pi^2}\left(N_i C_{iij}^* C_{iij} + N_j C_{ijj}^* C_{ijj} + N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j\right). \tag{2.42}$$

The other two can be obtained replacing N_i, C_{iij}, C_{ijj} and $\Upsilon_i{}^j$ by their respective analogues.

The presence of a building block of the third type allows us to take a specific parametrization of the C_{iij} in terms of $\Upsilon_i{}^j$. To this end, we introduce the shorthand notations

$$q_i := \frac{f(0)}{\pi^2} g_i^2, \qquad r_i := q_i n_i, \qquad \omega_{ij} := 1 - r_i N_i - r_j N_j, \tag{2.43}$$

where we can infer from the normalization of the kinetic terms of the gauge bosons (i.e. setting $\mathscr{K}_i = 1$) that q_i must be rational. Then, similarly as in Proposition 2.15, we write out $C_{iij}^* C_{iij}$, with C_{iij} satisfying (2.33) from supersymmetry, and insert the pre-factor (2.42) of the kinetic term. This reads

$$C_{iij}^* C_{iij} = r_i \big(N_i C_{iij}^* C_{iij} + N_j C_{ijj}^* C_{ijj} + N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j \big).$$

Using $r_i C_{ijj}^* C_{ijj} = r_j C_{iij}^* C_{iij}$, which can be directly obtained from the result (2.33), we obtain

$$C_{iij}^* C_{iij} = \frac{r_i}{\omega_{ij}} N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j \tag{2.44}$$

for the parametrization of C_{iij} that satisfies (2.33). For future convenience we will take

$$C_{iij} = \varepsilon_{i,j} \sqrt{\frac{r_i}{\omega_{ij}}} \big(N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j \big)^{1/2}, \tag{2.45}$$

with $\varepsilon_{i,j} \in \{\pm\}$ the sign introduced in Theorem 2.12. The other parameter, C_{ijj}, can be obtained by $r_i \to r_j$, $\varepsilon_{i,j} \to \varepsilon_{j,i}$. This yields for the pre-factor (2.42) of the kinetic term of $\widetilde{\psi}_{ij}$:

$$\mathscr{N}_{ij}^2 = \frac{f(0)}{2\pi^2} \left(N_i \frac{r_i}{\omega_{ij}} + N_j \frac{r_j}{\omega_{ij}} + 1 \right) N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j = \frac{f(0)}{2\pi^2} \frac{1}{\omega_{ij}} N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j \tag{2.46}$$

prior to the scaling (2.28). When $\widetilde{\psi}_{ij}$ has $R = 1$ and therefore does not carry a family structure (as in Fig. 2.6b) then the trace over the representations where $\widetilde{\psi}_{ij} \overline{\widetilde{\psi}}_{ij}$ and $\overline{\widetilde{\psi}}_{ij} \widetilde{\psi}_{ij}$ are in, decouples from that over $M_M(\mathbb{C})$. Consequently, the third term in (2.42) and the right hand sides of the solutions (2.45) and (2.46) receive additional traces over family indices, i.e. $N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j \to N_k \operatorname{tr}_M \Upsilon_i{}^{j*} \Upsilon_i{}^j$. The strategy to write C_{iij} in terms of parameters of building blocks of the third type works equally well when the kinetic term of $\widetilde{\psi}_{ij}$ gets contributions from multiple building blocks of the third type. In that case $N_k \Upsilon_i{}^{j*} \Upsilon_i{}^j$ must be replaced by a sum of all such terms: $\sum_l N_l \Upsilon_{i,l}{}^{j*} \Upsilon_{i,l}{}^j$ (see e.g. Sect. 2.2.3.1), where the label l is used to distinguish the building blocks $\mathscr{B}_{ijl}$ that all give a contribution to the kinetic term of $\widetilde{\psi}_{ij}$.

There are several contributions to the action as a result of adding a building block of the third type. The action is given by

$$S_{ijk}[\zeta,\widetilde{\zeta}] = S_{f,ijk}[\zeta,\widetilde{\zeta}] + S_{b,ijk}[\widetilde{\zeta}], \tag{2.47}$$

with its fermionic part $S_{f,ijk}[\zeta,\widetilde{\zeta}]$ reading

$$\begin{aligned} S_{f,ijk}[\zeta,\widetilde{\zeta}] = {} & \langle J_M\overline{\psi}_{ij}, \gamma^5\psi_{ik}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\rangle + \langle J_M\overline{\psi}_{ij}, \gamma^5\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\psi}_{jk}\rangle \\ & + \langle J_M\overline{\psi}_{jk}, \gamma^5\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}\psi_{ik}\rangle + \langle J_M\overline{\psi}_{ik}, \gamma^5\Upsilon_i{}^{j}\widetilde{\psi}_{ij}\psi_{jk}\rangle \\ & + \langle J_M\overline{\psi}_{ik}, \gamma^5\psi_{ij}\Upsilon_j{}^{k}\widetilde{\psi}_{jk}\rangle + \langle J_M\psi_{jk}, \gamma^5\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k}\psi_{ij}\rangle. \end{aligned} \tag{2.48}$$

The bosonic part of the action is given by:

$$\begin{aligned} S_{b,ijk}[\widetilde{\zeta}] = {} & \frac{f(0)}{2\pi^2}\Big[N_i|\Upsilon_j{}^{k}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}|^2 + N_j|\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k}|^2 \\ & + N_k\,\mathrm{tr}_M(\Upsilon_i{}^{j*}\Upsilon_i{}^{j})^2|\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2\Big] \\ & + S_{b,ij,jk}[\widetilde{\zeta}] + S_{b,ik,jk}[\widetilde{\zeta}] + S_{b,ij,ik}[\widetilde{\zeta}], \end{aligned} \tag{2.49}$$

with

$$\begin{aligned} S_{b,ij,jk}[\widetilde{\zeta}] = {} & \frac{f(0)}{\pi^2}\Big[N_i|C_{iij}\widetilde{\psi}_{ij}\Upsilon_j{}^{k}\widetilde{\psi}_{jk}|^2 + N_k|\Upsilon_i{}^{j}\widetilde{\psi}_{ij}C_{jkk}\widetilde{\psi}_{jk}|^2 + |\widetilde{\psi}_{ij}|^2|\Upsilon_i{}^{j*}\Upsilon_j{}^{k}\widetilde{\psi}_{jk}|^2 \\ & + \Big(\mathrm{tr}\,\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(\overline{\widetilde{\psi}}_{ij}C_{iij}^*)^o(C_{iij}\widetilde{\psi}_{ij})^o\Upsilon_j{}^{k}\widetilde{\psi}_{jk} \\ & + \mathrm{tr}\,\overline{\widetilde{\psi}}_{jk}C_{jjk}^*(\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*})^o(\Upsilon_i{}^{j}\widetilde{\psi}_{ij})^oC_{jjk}\widetilde{\psi}_{jk} \\ & + \mathrm{tr}\,\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(\overline{\widetilde{\psi}}_{ij}C_{ijj}^*)^o(\Upsilon_i{}^{j}\widetilde{\psi}_{ij})^oC_{jjk}\widetilde{\psi}_{jk} + h.c.\Big)\Big], \end{aligned} \tag{2.50}$$

where the traces above are over $(\mathbf{N}_k \otimes \mathbf{N}_i^o)^{\oplus M}$. The fact that in this context $\widetilde{\psi}_{ij}$ has $R = 1$ makes it possible to separate the trace over the family-index in the last term of the first line of (2.49). A more detailed derivation of the four-scalar action that corresponds to a building block of the third type, including the expressions for $S_{b,ik,jk}[\widetilde{\zeta}]$ and $S_{b,ij,ik}[\widetilde{\zeta}]$, is given in Appendix 1.

The expression (2.49) contains interactions that in form we either have seen earlier (cf. (2.19), (2.37)) or that we needed but were lacking in a set-up consisting only of building blocks of the second type (cf. (2.38), see also the discussion in Sect. 2.2.2.3). In addition, it features terms that we need in order to have a supersymmetric action.

We can deduce from the transformations (2.31) that, for the expression (2.48) (i.e. the fermionic action that we have) to be part of a supersymmetric action, the bosonic action must involve terms with the auxiliary fields F_{ij}, F_{ik} and F_{jk} (that are available to us from the respective building blocks of the second type), coupled to two scalar fields. We will therefore formulate the most general action featuring these auxiliary fields and constrain its coefficients by demanding it to be supersymmetric in combination with (2.48). Subsequently, we will check if and when the spectral action (2.49) (after subtracting the terms that are needed for (2.38)) is of the correct form to be written off shell in such a general form. This will be done for the general case in Sect. 2.3.

The most general Lagrangian featuring the auxiliary fields F_{ij}, F_{ik}, F_{jk} that can yield four-scalar terms is

$$S_{b,ijk,\mathrm{off}}[F_{ij}, F_{ik}, F_{jk}, \widetilde{\zeta}] = \int_M \mathscr{L}_{b,ijk,\mathrm{off}}(F_{ij}, F_{ik}, F_{jk}, \widetilde{\zeta})\sqrt{g}\mathrm{d}^4x, \tag{2.51}$$

with

$$\begin{aligned}\mathscr{L}_{b,ijk,\mathrm{off}}(F_{ij}, F_{ik}, F_{jk}, \widetilde{\zeta}) = &-\mathrm{tr}\, F_{ij}^* F_{ij} + \big(\mathrm{tr}\, F_{ij}^*\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk} + h.c.\big)\\ &-\mathrm{tr}\, F_{ik}^* F_{ik} + \big(\mathrm{tr}\, F_{ik}^*\beta_{ik,j}^*\widetilde{\psi}_{ij}\widetilde{\psi}_{jk} + h.c.\big)\\ &-\mathrm{tr}\, F_{jk}^* F_{jk} + \big(\mathrm{tr}\, F_{jk}^*\beta_{jk,i}\overline{\widetilde{\psi}}_{ij}\widetilde{\psi}_{ik} + h.c.\big).\end{aligned}$$

Here $\beta_{ij,k}$, $\beta_{ik,j}$ and $\beta_{jk,i}$ are matrices acting on the generations and consequently the traces are performed over $\mathbf{N}_j^{\oplus M}$ (the first two terms) and $\mathbf{N}_k^{\oplus M}$ (the last four terms) respectively. Using the Euler-Lagrange equations the on shell counterpart of (2.51) is seen to be

$$S_{b,ijk,\mathrm{on}}[\widetilde{\zeta}] = \int_M \sqrt{g}\mathrm{d}^4x\Big(|\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}|^2 + |\beta_{ik,j}^*\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2 + |\beta_{jk,i}\overline{\widetilde{\psi}}_{ij}\widetilde{\psi}_{ik}|^2\Big)$$

cf. the second and third terms of (2.49). We have the following result:

Theorem 2.21 *The action consisting of the sum of* (2.48) *and* (2.51) *is supersymmetric under the transformations (2.31) and (2.32) if and only if the parameters of the finite Dirac operator are related via*

$$\Upsilon_j{}^k C_{jkk}^{-1} = -(C_{ikk}^*)^{-1}\Upsilon_i{}^k, \quad (C_{iik}^*)^{-1}\Upsilon_i{}^k = -\Upsilon_i{}^j C_{iij}^{-1}, \quad \Upsilon_i{}^j C_{ijj}^{-1} = -\Upsilon_j{}^k C_{jjk}^{-1} \tag{2.52}$$

and

$$\begin{aligned}&\beta_{ij,k}'^*\beta_{ij,k}' = \Upsilon_j'^k\Upsilon_j'^{k*} = \Upsilon_i'^k\Upsilon_i'^{k*}, \qquad \beta_{ik,j}'^*\beta_{ik,j}' = \Upsilon_i'^j\Upsilon_i'^{j*} = \Upsilon_j'^k\Upsilon_j'^{k*},\\ &\beta_{jk,i}'^*\beta_{jk,i}' = \Upsilon_i'^k\Upsilon_i'^{k*} = \Upsilon_i'^j\Upsilon_i'^{j*},\end{aligned} \tag{2.53}$$

where

$$\beta_{ij,k}' := \mathcal{N}_{jk}^{-1}\beta_{ij,k}\mathcal{N}_{ik}^{-1}, \quad \beta_{ik,j}' := \mathcal{N}_{jk}^{-1}\beta_{ik,j}\mathcal{N}_{ij}^{-1}, \quad \beta_{jk,i}' := \mathcal{N}_{ij}^{-1}\beta_{jk,i}\mathcal{N}_{ik}^{-1}$$

and

$$\Upsilon_i'^{j} := \Upsilon_i{}^{j}\mathcal{N}_{ij}^{-1}, \qquad \Upsilon_i'^{k} := \mathcal{N}_{ik}^{-1}\Upsilon_i{}^{k}, \qquad \Upsilon_j'^{k} := \Upsilon_j{}^{k}\mathcal{N}_{jk}^{-1}, \tag{2.54}$$

denote the scaled versions of the $\beta_{ij,k}$*'s and the* $\Upsilon_i{}^j$*'s respectively.*

Proof See Appendix section 'Third Building Block'.

For future use we rewrite (2.52) using the parametrization (2.45) for the C_{iij}, giving

$$\begin{aligned} &\varepsilon_{i,j}\sqrt{\omega_{ij}}\,\widetilde{\Upsilon}_i{}^j = -\varepsilon_{i,k}\sqrt{\omega_{ik}}\,\widetilde{\Upsilon}_i{}^k, \qquad \varepsilon_{j,i}\sqrt{\omega_{ij}}\,\widetilde{\Upsilon}_i{}^j = -\varepsilon_{j,k}\sqrt{\omega_{jk}}\,\widetilde{\Upsilon}_j{}^k, \\ &\varepsilon_{k,i}\sqrt{\omega_{ik}}\,\widetilde{\Upsilon}_i{}^k = -\varepsilon_{k,j}\sqrt{\omega_{jk}}\,\widetilde{\Upsilon}_j{}^k, \end{aligned} \tag{2.55}$$

where we have written

$$\begin{aligned} \widetilde{\Upsilon}_i{}^j &:= \Upsilon_i{}^j(N_k \operatorname{tr}\Upsilon_i{}^{j*}\Upsilon_i{}^j)^{-1/2}, \widetilde{\Upsilon}_i{}^k := (N_j\Upsilon_i{}^k\Upsilon_i{}^{k*})^{-1/2}\Upsilon_i{}^k, \\ \widetilde{\Upsilon}_j{}^k &:= \Upsilon_j{}^k(N_i\Upsilon_j{}^{k*}\Upsilon_j{}^k)^{-1/2}. \end{aligned} \tag{2.56}$$

There is a trace over the generations in the first term because the corresponding sfermion $\widetilde{\psi}_{ij}$ has $R = 1$ and consequently no family-index. Using these demands on the parameters, the (spectral) action from a building block of the third type becomes much more succinct. First of all it allows us to reduce all three parameters of the finite Dirac operator of Definition 2.20 to only one, e.g. $\Upsilon \equiv \Upsilon_i{}^j$. Second, upon using (2.52) the second and third lines of (2.50) are seen to cancel.[6] If the demands (2.52) and (2.53) are met, the on shell action (2.47) that arises from a building block $\mathscr{B}_{ijk}$ of the third type reads

$$\begin{aligned} S_{ijk}[\zeta, \widetilde{\zeta}, \mathbb{A}] = g_m\sqrt{\frac{2\omega_1}{q_m}}\Big[&\langle J_M\overline{\psi}_2, \gamma^5\widetilde{\Upsilon}\widetilde{\psi}_1\psi_3\rangle + \kappa_j\langle J_M\overline{\psi}_2, \gamma^5\psi_1\widetilde{\Upsilon}\widetilde{\psi}_3\rangle \\ &+ \kappa_i\langle J_M\psi_3, \gamma^5\overline{\widetilde{\psi}}_2\widetilde{\Upsilon}\psi_1\rangle + h.c.\Big] \\ + g_m^2\frac{4\omega_1}{q_m}\Big[&(1-\omega_2)|\widetilde{\Upsilon}\widetilde{\psi}_1\widetilde{\psi}_3|^2 + (1-\omega_1)|\widetilde{\Upsilon}\widetilde{\psi}_3\overline{\widetilde{\psi}}_2|^2 \\ + &(1-\omega_3)|\widetilde{\Upsilon}\overline{\widetilde{\psi}}_2\widetilde{\psi}_1|^2\Big]. \end{aligned} \tag{2.57}$$

Here we used the shorthand notations $ij \to 1$, $ik \to 2$, $jk \to 3$ and $\kappa_j = \varepsilon_{j,i}\varepsilon_{j,k}$, $\kappa_i = \varepsilon_{i,j}\varepsilon_{i,k}$ to avoid notational clutter as much as possible and where we have written everything in terms of $\widetilde{\Upsilon} \equiv \widetilde{\Upsilon}_i{}^j$ (as defined above), the parameter that corresponds

[6] More generally, this also happens for the other combinations: the four-scalar interactions of (2.99) are seen to cancel those of (2.102).

to the sfermion having $R = 1$ (and consequently also multiplicity 1). The index m in g_m and q_m can take any of the values that appear in the model, e.g. i, j or k. As with a building block of the second type there is a sign ambiguity that stems from those of the C_{iij}. In addition, the terms that are not listed here but are in (2.47) give contributions to terms that already appeared in the action from building blocks of the second type. See Sect. 2.3 for details on this.

For notational convenience we have used two different notations for scaled variables: $\widetilde{\Upsilon}_i{}^j$ from (2.2.3) and $\Upsilon_i'^j$ from (2.54). Using the expression (2.46) for $\mathscr{N}_{ij}$ in terms of $\Upsilon_i{}^j$ these are related via

$$\Upsilon_i'^k \equiv \mathscr{N}_{ik}^{-1}\Upsilon_i{}^k = \sqrt{\frac{2\pi^2}{f(0)}\omega_{ik}}\left(N_j\Upsilon_i{}^k\Upsilon_i{}^{k*}\right)^{-1/2}\Upsilon_i{}^k \equiv g_l\sqrt{\frac{2\omega_{ik}}{q_l}}\widetilde{\Upsilon}_i{}^k, \quad (2.58)$$

assuming that $\widetilde{\psi}_{ik}$ has $R = -1$. The other two scaled variables give analogous expressions but the order of Υ and Υ^* is reversed and the sfermion with $R = 1$ gets an additional trace over family indices.

Remark 2.22 Note that we can use this result to say something about the signs of the C_{iij} appearing in a building block of the third type. We first combine all three equations of (2.52) into one,

$$\Upsilon_j{}^k = (-1)^3(C_{iik}C_{ikk}^{-1})^*\Upsilon_j{}^k(C_{jjk}^{-1}C_{jkk})(C_{ijj}C_{iij}^{-1}),$$

when it is C_{iij} and C_{ijj} that do not have a family structure. All these parameters are only determined up to a sign. We will write

$$C_{iij}C_{ijj}^{-1} = s_{ij}\sqrt{\frac{n_i\mathscr{K}_j}{n_j\mathscr{K}_i}\frac{g_i}{g_j}}, \qquad \text{with } s_{ij} := \varepsilon_{i,j}\varepsilon_{j,i} = \pm 1,$$

cf. (2.33), etc. which gives $\Upsilon_j{}^k = -s_{ij}s_{jk}s_{ki}\Upsilon_j{}^k$ for the relation above. So for consistency either one, or all three combinations of C_{iij} and C_{ijj} associated to a building block $\mathscr{B}_{ij}$ that is part of a $\mathscr{B}_{ijk}$ must be of opposite sign.

Remark 2.23 If instead of $\widetilde{\psi}_{ij}$ it is $\widetilde{\psi}_{ik}$ or $\widetilde{\psi}_{jk}$ that has $R = 1$ (see Fig. 2.7) the demand on the parameters $\Upsilon_i{}^j$, $\Upsilon_i{}^k$ and $\Upsilon_j{}^k$ is a slightly modified version of (2.52):

$$(\Upsilon_j{}^kC_{jkk}^{-1})^t = -(C_{ikk}^*)^{-1}\Upsilon_i{}^k, \quad (C_{iik}^*)^{-1}\Upsilon_i{}^k = -\Upsilon_i{}^jC_{iij}^{-1}, \quad \Upsilon_i{}^jC_{ijj}^{-1} = -(\Upsilon_j{}^kC_{jjk}^{-1})^t, \quad (2.59)$$

where A^t denotes the transpose of the matrix A. This result can be verified by considering Lemma 2.43 for these cases.

By introducing a building block of the third type we generated the interactions that we lacked in a situation with multiple building blocks of the second type. The wish for supersymmetry thus forces us to extend any model given by Fig. 2.5 with a building block of the third type.

If we again seek the analogy with the superfield formalism, then a building block of the third type is a Euclidean analogy of an action on a Minkowskian background that comes from a superpotential term

$$\int \Big(\mathscr{W}(\{\Phi_m\})\Big|_F + h.c.\Big)\mathrm{d}^4x, \quad \text{with} \quad \mathscr{W}(\{\Phi_m\}) = f_{mnp}\Phi_m\Phi_n\Phi_p, \tag{2.60}$$

where $\Phi_{m,n,p}$ are chiral superfields, f_{mnp} is symmetric in its indices [9, Sect. 5.1] and with $|_F$ we mean multiplying by $\bar{\theta}\bar{\theta}$ and integrating over superspace $\int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$. To specify this statement, we write $\Phi_{ij} = \phi_{ij} + \sqrt{2}\theta\psi_{ij} + \theta\theta F_{ij}$ for a chiral superfield. Similarly, we introduce Φ_{jk} and Φ_{ki}. We then have that

$$\begin{aligned}\int_M \Big[\Phi_{ij}\Phi_{jk}\Phi_{ki}\Big]_F + h.c. = \int_M &-\psi_{ij}\phi_{jk}\psi_{ki} - \psi_{ij}\psi_{jk}\phi_{ki} - \phi_{ij}\psi_{jk}\psi_{ki}\\ &+ F_{ij}\phi_{jk}\phi_{ki} + \phi_{ij}\phi_{jk}F_{ki} + \phi_{ij}F_{jk}\phi_{ki} + h.c.\end{aligned}$$

This gives on shell the following contribution:

$$\begin{aligned}-\int_M \Big(&\psi_{ij}\phi_{jk}\psi_{ki} + \psi_{ij}\psi_{jk}\phi_{ki} + \phi_{ij}\psi_{jk}\psi_{ki}\\ &+ \frac{1}{2}|\phi_{jk}\phi_{ki}|^2 + \frac{1}{2}|\phi_{ij}\phi_{jk}|^2 + \frac{1}{2}|\phi_{ki}\phi_{ij}|^2 + h.c.\Big),\end{aligned}$$

to be compared with (2.57). In a set-up similar to that of Fig. 2.5, but with the chirality of one or two of the building blocks $\mathscr{B}_{ij}$, $\mathscr{B}_{jk}$ and $\mathscr{B}_{ik}$ being flipped, not all three components of D_F such as in Definition 2.20 can still be defined, see Fig. 2.8. Interestingly, one can check that in such a case the resulting action corresponds to a superpotential that is not holomorphic, but e.g. of the form $\Phi_{ij}\Phi_{ik}\Phi^{\dagger}_{jk}$ instead. To see this, we calculate the action (2.60) in this case, giving

$$\int_M \Big[\Phi_{ij}\Phi^{\dagger}_{jk}\Phi_{ki}\Big]_F + h.c. = \int_M -\psi_{ij}\phi^*_{jk}\psi_{ki} + F_{ij}\phi^*_{jk}\phi_{ki} + \phi_{ij}\phi^*_{jk}F_{ki} + h.c.,$$

which on shell equals

$$-\int_M \psi_{ij}\phi^*_{jk}\psi_{ki} + \frac{1}{2}|\phi^*_{jk}\phi_{ki}|^2 + \frac{1}{2}|\phi_{ij}\phi^*_{jk}|^2 + h.c.$$

This is indeed analogous to the interactions that the spectral triple depicted in Fig. 2.8 (still) gives rise to.

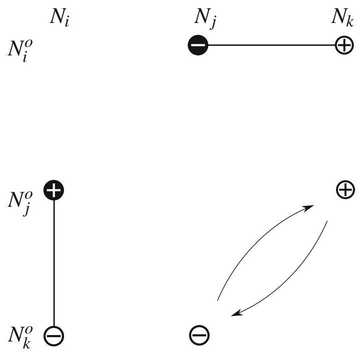

Fig. 2.8 A set-up similar to that of Fig. 2.6, but with the values of the grading reversed for $\mathbf{N}_j \otimes \mathbf{N}_k^o$ and its opposite. Consequently, only one of the three components that characterize a building block of the first type can now be defined

2.2.3.1 Interaction Between Building Blocks of the Third Type

Suppose we have two building blocks $\mathscr{B}_{ijk}$ and $\mathscr{B}_{ijl}$ of the third type that share two of their indices, as is depicted in Fig. 2.9. This situation gives rise to the following extra terms in the action:

$$\frac{f(0)}{\pi^2}\Big[N_j|\overline{\widetilde{\psi}}_{jk}C^*_{jjk}C_{jjl}\widetilde{\psi}_{jl}|^2 + N_i|\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\Upsilon_j{}^{l}\widetilde{\psi}_{jl}|^2 + |\Upsilon_j{}^{k}\widetilde{\psi}_{jk}|^2|\Upsilon_i{}^{l*}\widetilde{\psi}_{il}|^2\Big] + (i \leftrightarrow j)$$
$$+ \frac{f(0)}{\pi^2}\Big(N_i \operatorname{tr} C_{iik}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\Upsilon_j{}^{l}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il}C^*_{iil}$$
$$+ N_j \operatorname{tr} \Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}C^*_{jjk}C_{jjl}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il}\Upsilon_i{}^{l} + h.c.\Big), \tag{2.61}$$

where with '$(i \leftrightarrow j)$' we mean the expression preceding it, but everywhere with i and j interchanged. The first line of (2.61) corresponds to paths within the two building blocks $\mathscr{B}_{ijk}$ and $\mathscr{B}_{ijl}$ (such as the ones depicted in Fig. 2.9a) and the second line corresponds to paths of which two of the edges come from the building blocks of the second type that were needed in order to define the building blocks of the third type (Fig. 2.9b).

If we scale the fields appearing in this expression according to (2.28) and use the identity (2.52) for the parameters of a building block of the third type, we can write (2.61) more compactly as

$$4n_jr_jN_jg_j^2|\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}|^2 + 4\frac{g_m^2}{q_m}\omega_{ij}^2N_i|\overline{\widetilde{\psi}}_{jk}\widetilde{\Upsilon}_k{}^*\widetilde{\Upsilon}_l\,\widetilde{\psi}_{jl}|^2$$
$$+ 4\frac{g_m^2}{q_m}\omega_{ij}^2|\widetilde{\Upsilon}_k\,\widetilde{\psi}_{jk}|^2|\widetilde{\Upsilon}_l{}^*\widetilde{\psi}_{il}|^2 + (i \leftrightarrow j, \Upsilon \leftrightarrow \Upsilon^*)$$
$$+ \kappa_k\kappa_l\,4\frac{g_m^2}{q_m}(1-\omega_{ij})\omega_{ij} \operatorname{tr} \widetilde{\Upsilon}_l\,\widetilde{\Upsilon}_k{}^*\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il} + h.c., \tag{2.62}$$

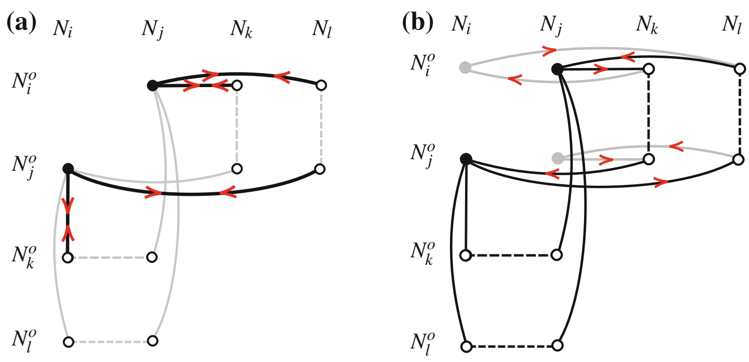

Fig. 2.9 In the case that there are two building blocks of the third type sharing two of their indices, there are extra four-scalar contributions to the action. They are given by (2.61). **a** Contributions corresponding to paths of which all four edges are from the building blocks $\mathscr{B}_{ijk}$ and $\mathscr{B}_{ijl}$ of the third type. **b** Contributions corresponding to paths of which two edges are from building blocks $\mathscr{B}_{ik}$ and $\mathscr{B}_{il}$ of the second type

where $\kappa_k = \varepsilon_{k,i}\varepsilon_{k,j}$, $\kappa_l = \varepsilon_{l,i}\varepsilon_{l,j} \in \{\pm 1\}$, $\widetilde{\Upsilon}_k \equiv \widetilde{\Upsilon}_{i,k}{}^j$ of $\mathscr{B}_{ijk}$ and $\widetilde{\Upsilon}_l \equiv \widetilde{\Upsilon}_{i,l}{}^j$ of $\mathscr{B}_{ijl}$, as defined in (2.2.3) but with contributions from two building blocks of the third type:

$$\widetilde{\Upsilon}_{i,k}{}^j = \Upsilon_{i,k}{}^j (N_k \operatorname{tr} \Upsilon_{i,k}{}^{j*}\Upsilon_{i,k}{}^j + N_l \operatorname{tr} \Upsilon_{i,l}{}^{j*}\Upsilon_{i,l}{}^j)^{-1/2}, \tag{2.63a}$$

$$\widetilde{\Upsilon}_{i,l}{}^j = \Upsilon_{i,l}{}^j (N_k \operatorname{tr} \Upsilon_{i,k}{}^{j*}\Upsilon_{i,k}{}^j + N_l \operatorname{tr} \Upsilon_{i,l}{}^{j*}\Upsilon_{i,l}{}^j)^{-1/2}. \tag{2.63b}$$

This expression can be generalized to any number of building blocks of the third type. In addition, we have assumed that $s_{ik}s_{il} = s_{jk}s_{jl}$ for the products of the relative signs between the parameters C_{iik} and C_{ikk} etc. (cf. Remark 2.22).

These new interactions must be accounted for by the auxiliary fields. The first and second terms are of the form (2.37) and should therefore be covered by the auxiliary fields $G_{i,j}$. The third term is of the form (2.38) and should consequently be described by the combination of $G_{i,j}$ and the $u(1)$-field H. The second line of (2.61) should be rewritten in terms of the auxiliary field F_{ij}. This can indeed be achieved via the off shell Lagrangian

$$-\operatorname{tr} F_{ij}^* F_{ij} + \left(\operatorname{tr} F_{ij}^*(\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk} + \beta_{ij,l}\widetilde{\psi}_{il}\overline{\widetilde{\psi}}_{jl}) + h.c.\right),$$

which on shell gives the following cross terms:

$$\operatorname{tr} \beta_{ij,l}^*\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il} + h.c. \tag{2.64}$$

In form, this indeed corresponds to the second line of (2.62). In Sect. 2.3 a more detailed version of this argument is presented.

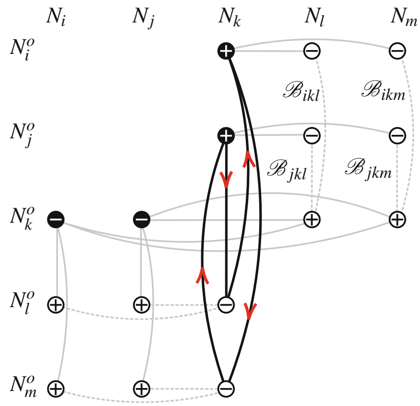

Fig. 2.10 When four building blocks of the third kind share one common index (in this case k) and each pair of building blocks shares one of its two remaining indices (i, j, l or m) with one other building block, there is an additional path that contributes to the trace of D_F^4 (including its inner fluctuations). The interaction is given by (2.65)

Furthermore, it can be that there are four different building blocks of the third type that all share one particular index—say $\mathscr{B}_{ikl}$, $\mathscr{B}_{ikm}$, $\mathscr{B}_{jkl}$ and $\mathscr{B}_{jkm}$, sharing index k—then there arises one extra interaction, that is of the form

$$N_k \frac{f(0)}{\pi^2}\big[\operatorname{tr} \Upsilon_i{}^{m*} \widetilde{\psi}_{im} \overline{\widetilde{\psi}}_{jm} \Upsilon_j{}^{m} \Upsilon_j{}^{l*} \widetilde{\psi}_{jl} \overline{\widetilde{\psi}}_{il} \Upsilon_i{}^{l} + h.c.\big].$$

Scaling the fields and rewriting the parameters using (2.55) gives

$$4\frac{g_n^2}{q_n}\omega_{ik}\omega_{jk} \operatorname{tr} \tilde{\Upsilon}_l \tilde{\Upsilon}_m^* \widetilde{\psi}_{im} \overline{\widetilde{\psi}}_{jm} (\tilde{\Upsilon}_l' \tilde{\Upsilon}_m'^*)^* \widetilde{\psi}_{jl} \overline{\widetilde{\psi}}_{il} + h.c., \tag{2.65}$$

where g_n can equal any of the coupling constants that appear in the theory and we have written

$$\tilde{\Upsilon}_m \equiv \Upsilon_{i,m}{}^{k} (N_m \Upsilon_{i,m}{}^{k*} \Upsilon_{i,m}{}^{k} + N_l \Upsilon_{i,l}{}^{k*} \Upsilon_{i,l}{}^{k})^{-1/2},$$
$$\tilde{\Upsilon}_m' \equiv \Upsilon_{j,m}{}^{k} (N_m \Upsilon_{j,m}{}^{k*} \Upsilon_{j,m}{}^{k} + N_l \Upsilon_{j,l}{}^{k*} \Upsilon_{j,l}{}^{k})^{-1/2},$$

and the same for $m \leftrightarrow l$. The path to which such an interaction corresponds, is given in Fig. 2.10. One can check that this interaction can only be described off shell by invoking either one or both of the auxiliary fields F_{ij} and F_{lm}. This means that in order to have a chance at supersymmetry, the finite spectral triple that corresponds to the Krajewski diagram of Fig. 2.10 requires in addition at least $\mathscr{B}_{ij}$ or $\mathscr{B}_{lm}$.

2.2.4 Higher Degree Building Blocks?

The first three building blocks that gave supersymmetric actions are characterized by one, two and three indices respectively. One might wonder whether there are building blocks of higher order, carrying four or more indices.

Each of the elements of a finite spectral triple is characterized by one (components of the algebra, adjoint representations in the Hilbert space), two (non-adjoint representations in the Hilbert space) or three (components of the finite Dirac operator that satisfy the order-one condition) indices. For each of these elements corresponding building blocks have been identified. Any object that carries four or more different indices (e.g. two or more off-diagonal representations, multiple components of a finite Dirac operator) must therefore be part of more than one building block of the first, second or third type. These blocks are, so to say, the irreducible ones.

This does not imply that there are no other building blocks left to be identified. However, as we will see in the next section, they are characterized by less than four indices.

2.2.5 Mass Terms

There is a possibility that we have not covered yet. The finite Hilbert space can contain two or more copies of one particular representation. This can happen in two slightly different ways. The first is when there is a building block $\mathscr{B}_{11'}$ of the second type, on which the same component $\mathbb{C}$ of the algebra acts both on the left and on the right in the same way. For the second way it is required that there are two copies of a particular building block $\mathscr{B}_{ij}$ of the second type. If the gradings of the representations are of opposite sign (in the first situation this is automatically the case for finite KO-dimension 6, in the second case by construction) there is allowed a component of the Dirac operator whose inner fluctuations will not generate a field, rather the resulting term will act as a mass term. In the first case such a term is called a Majorana mass term. We will cover both of them separately.

2.2.5.1 Fourth Building Block: Majorana Mass Terms

The finite Hilbert space can, for example due to some breaking procedure [6, 7], contain representations

$$\mathbf{1} \otimes \mathbf{1}'^o \oplus \mathbf{1}' \otimes \mathbf{1}^o \simeq \mathbb{C} \oplus \mathbb{C},$$

which are each other's antiparticles, e.g. these representations are not in the adjoint ('diagonal') representation, but the same component $\mathbb{C}$ of the algebra[7] acts on them. Then there is allowed a component ${D_{1'1}}^{11'}$ of the Dirac operator connecting the two. It satisfies the first order condition (1.12) and its inner fluctuations automatically vanish. Consequently, this component does not generate a scalar, unlike the typical component of a finite Dirac operator. Writing $(\xi, \xi') \in (\mathbb{C}\oplus\mathbb{C})^{\oplus M}$ (where M denotes

[7]For a component $\mathbb{R}$ in the finite algebra this would work as well, but such a component would not give rise to gauge interactions and is therefore unfavourable.

the multiplicity of the representation) for the finite part of the fermions, the demand of D_F to commute with J_F reads

$$(D_{11'}{}^{1'1}\bar{\xi}, D_{1'1}{}^{11'}\bar{\xi}') = \left(\overline{D_{1'1}{}^{11'}\xi}, \overline{D_{11'}{}^{1'1}\xi'}\right).$$

Using that $(D_{ij}{}^{ik})^* = D_{ik}{}^{ij}$ this teaches us that the component must be a symmetric matrix. It can be considered as a Majorana mass for the particle $\psi_{11'}$ whose finite part is in the representation $\mathbf{1} \otimes \mathbf{1}'^o$ (cf. the Majorana mass for the right handed neutrino in the Standard Model [7]). Then we have

Definition 2.24 For an almost-commutative geometry that contains a building block $\mathscr{B}_{11'}$ of the second type, a *building block of the fourth type* $\mathscr{B}_{maj}$ consists of a component

$$D_{1'1}{}^{11'} : \mathbf{1} \otimes \mathbf{1}'^o \to \mathbf{1}' \otimes \mathbf{1}^o$$

of the finite Dirac operator. Symbolically it is denoted by

$$\mathscr{B}_{maj} = (0, D_{1'1}{}^{11'}) \in \mathscr{H}_F \oplus \text{End}(\mathscr{H}_F),$$

where for the symmetric matrix that parametrizes this component we write Υ_{m}.

In the language of Krajewski diagrams such a Majorana mass is symbolized by a dotted line, cf. Fig. 2.11.

A $\mathscr{B}_{maj}$ adds the following to the action (1.24):

$$\begin{aligned}
&\frac{1}{2}\langle J_M \psi_{11'L}, \gamma^5 \Upsilon_{\text{m}}^* \psi_{11'L}\rangle + \frac{1}{2}\langle J_M \overline{\psi}_{11'R}, \gamma^5 \Upsilon_{\text{m}} \overline{\psi}_{11'R}\rangle \\
&\quad + \frac{f(0)}{\pi^2}\Big[|\Upsilon_{\text{m}} \overline{\widetilde{\psi}}_{11'} C^*_{111'}|^2 + |\Upsilon_{\text{m}} \overline{\widetilde{\psi}}_{11'} C^*_{1'1'1}|^2 \\
&\qquad\qquad + \sum_j \left(|\Upsilon_{\text{m}} \Upsilon_{1'}{}^{j} \widetilde{\psi}_{1'j}|^2 + |\Upsilon_{\text{m}}^* \Upsilon_1{}^{j*} \widetilde{\psi}_{1j}|^2\right)\Big]
\end{aligned}$$

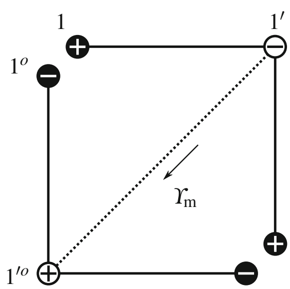

Fig. 2.11 A component of the finite Dirac operator that acts as a Majorana mass is represented by a *dotted line* in a Krajewski diagram

$$+ \frac{f(0)}{\pi^2} \sum_j \Big(\mathrm{tr}(\overline{\widetilde{\psi}}_{11'} C^*_{111'})^o \Upsilon_{\mathrm{m}} \Upsilon_{1'}{}^{j} \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1j} C^*_{11j}$$
$$+ \mathrm{tr}\, \Upsilon_{\mathrm{m}} (\overline{\widetilde{\psi}}_{11'} C^*_{1'1'1})^o C_{1'1'j} \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1j} \Upsilon_1{}^{j}$$
$$+ \mathrm{tr}\, \Upsilon_{\mathrm{m}} \Upsilon_{1'}{}^{j} \widetilde{\psi}_{1'j} (\overline{\widetilde{\psi}}_{11'} \Upsilon_1{}^{1'*})^o \overline{\widetilde{\psi}}_{1j} \Upsilon_1{}^{j} + h.c.\Big), \tag{2.66}$$

where the traces are over $(\mathbf{1} \otimes \mathbf{1}'^{o})^{\oplus M}$. In this expression, the first contribution comes from the inner product. The paths in the Krajewski diagram corresponding to the other contributions are depicted in Fig. 2.12. In this set-up it is $\widetilde{\psi}_{1'j}$ that does not have a family index. Consequently we can separate the traces over the family-index and that over $\mathbf{N}_j$ in the penultimate term of the second line of (2.66). We would like to rewrite the above action in terms of $\widetilde{\Upsilon} \equiv \Upsilon_{1'}{}^{j}$ by using the identity (2.59). For this we first need to rewrite the C_{iij} to the C_{ijj} by employing Remark 2.22. Writing out the family indices of the third and fourth line of (2.66) gives

$$\mathrm{tr}((\overline{\widetilde{\psi}}_{11'} C^*_{111'})^o \Upsilon_{\mathrm{m}})_a \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1jc} (C^*_{11j} (\Upsilon_{1'}{}^{j})^t)_{ca}$$
$$+ \mathrm{tr}(\Upsilon_{\mathrm{m}} (\overline{\widetilde{\psi}}_{11'} C^*_{1'1'1})^o)_a C_{1'1'j} \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1jc} (\Upsilon_1{}^{j})_{ca}$$
$$= \sqrt{\frac{n_1 \mathscr{K}_j}{n_j \mathscr{K}_1}} \frac{g_1}{g_j} (s_{1'j} - s_{1j} s_{11'}) \Big[\mathrm{tr}(\overline{\widetilde{\psi}}_{11'} C^*_{11'1'})^o \Upsilon_{\mathrm{m}} \Upsilon_{1'}{}^{j} \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1j} C^*_{1jj} \Big], \tag{2.67}$$

where a, b, c are family indices, s_{ij} is the product of the signs of C_{iij} and C_{ijj} (cf. the notation in Remark 2.22) and where we have used that Υ_{m} is a symmetric matrix.

Then to make things a bit more apparent, we scale the fields in (2.66) (with the third and fourth line replaced by (2.67)) according to (2.28) and put in the expressions for the C_{ijj} from (2.45), which gives

$$\frac{1}{2} \langle J_M \psi_{11'L}, \gamma^5 \Upsilon^*_{\mathrm{m}} \psi_{11'L} \rangle + \frac{1}{2} \langle J_M \overline{\psi}_{11'R}, \gamma^5 \Upsilon_{\mathrm{m}} \overline{\psi}_{11'R} \rangle$$
$$+ 4 r_1 |\Upsilon_{\mathrm{m}} \overline{\widetilde{\psi}}_{11'}|^2 + 2 \sum_j \omega_{1j} \Big(|\Upsilon_{\mathrm{m}} \widetilde{\Upsilon}_j|^2_M |\widetilde{\psi}_{1'j}|^2 + |\Upsilon^*_{\mathrm{m}} \widetilde{\Upsilon}_j{}^* \widetilde{\psi}_{1j}|^2 \Big)$$
$$+ \kappa_{1'} \kappa_j \sum_j 2 g_m \sqrt{\frac{2\omega_{1j}}{q_m}} \Big(\mathrm{tr}\, \overline{\widetilde{\psi}}_{11'} (r_1 + \omega_{1j} \widetilde{\Upsilon}_j \widetilde{\Upsilon}_j{}^*)^t \Upsilon_{\mathrm{m}} \widetilde{\Upsilon}_j \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1j} + h.c. \Big), \tag{2.68}$$

where we have written $|a|^2_M = \mathrm{tr}_M\, a^* a$ for the trace over the family-index, $\widetilde{\Upsilon}_j \equiv \widetilde{\Upsilon}_{1'}{}^{j}$, and where $\kappa_{1'} = \varepsilon_{1',j} \varepsilon_{1',1}$, $\kappa_j = \varepsilon_{j,1'} \varepsilon_{j,1} \in \{\pm 1\}$. We replaced $\overline{\widetilde{\psi}}{}^{o}_{11'}$ by $\overline{\widetilde{\psi}}_{11'}$ since these coincide when $\widetilde{\psi}_{11'}$ is a gauge singlet. Consequently, the traces are now over $\mathbf{1}^{\oplus M}$. In addition we used the relation (2.59) between $\Upsilon_1{}^{j}$, $\Upsilon_{1'}{}^{j}$ and $\Upsilon_1{}^{1'}$, the symmetry of Υ_{m} and that $g_1 \equiv g_{1'}$ (which follows from the set-up) and consequently

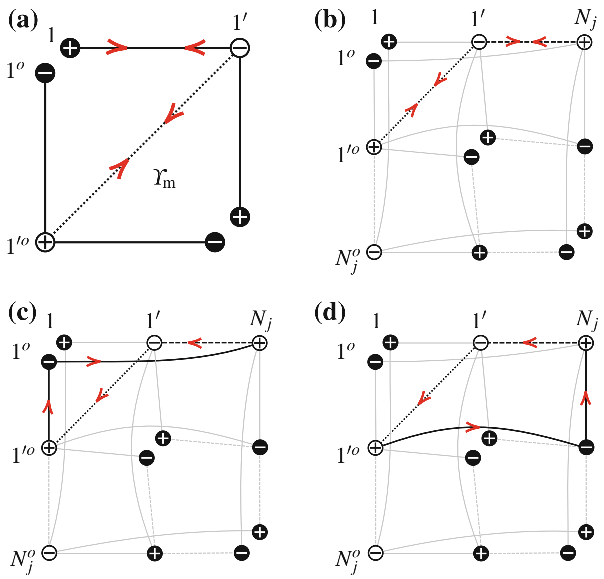

Fig. 2.12 In the case that there is a building block of the fourth type, there are extra interactions in the action. **a** A path featuring edges from a building block of the second type. **b** A path featuring edges from a building block of the third type. **c** A path featuring edges from building blocks of the second and third type. **d** A second path featuring edges from a building block of the third type

$r_1 = r_{1'}$ and $\omega_{1'j} = \omega_{1j}$. In contrast to the previous case, not all scalar interactions that appear here can be accounted for by auxiliary fields:

Lemma 2.25 *For a finite spectral triple that contains, in addition to building blocks of the first, second and third type, one building block of the fourth type, the only terms in the associated spectral action that can be written off shell using the available auxiliary fields are those featuring $\widetilde{\psi}_{11'}$ or its conjugate.*

Proof The bosonic terms in (2.66) must be the on shell expressions of an off shell Lagrangian that features the auxiliary fields available to us. Respecting gauge invariance, the latter must be

$$- \operatorname{tr} F_{11'}^* F_{11'} - \Big(\operatorname{tr} F_{11'}^* (\gamma_{11'} \overline{\widetilde{\psi}}_{11'} + \sum_j \beta_{11',j} \widetilde{\psi}_{1j} \overline{\widetilde{\psi}}_{1'j}) + h.c. \Big). \tag{2.69}$$

On shell this then gives the following contributions featuring $\widetilde{\psi}_{11'}$ and its conjugate:

$$|\gamma_{11'} \overline{\widetilde{\psi}}_{11'}|^2 + \sum_j \big(\operatorname{tr} \gamma_{11'} \overline{\widetilde{\psi}}_{11'} \widetilde{\psi}_{1'j} \overline{\widetilde{\psi}}_{1j} \beta_{11',j}^* + h.c. \big),$$

which corresponds at least in form to all bosonic terms of (2.68), except the second term of the second line.

We can use an argument similar to the one we used for building blocks of the third type:

Lemma 2.26 *The action consisting of the fermionic terms of* (2.68) *and the terms of* (2.69) *that do not feature* $\beta_{11',j}$ *or its conjugate is supersymmetric under the transformations (2.32) iff*

$$\gamma_{11'}^*\gamma_{11'} = \Upsilon_{\rm m}^*\Upsilon_{\rm m} \tag{2.70}$$

and the gauginos represented by the black vertices in Fig. 2.12a that have the same chirality are associated with each other.

Proof See Appendix section 'Fourth Building Block'.

Combining the above two Lemmas, then gives the following result.

Proposition 2.27 *The action* (2.68) *of a single building block of the fourth type breaks supersymmetry only softly via*

$$2\sum_j \omega_{1j}\Big(|\Upsilon_{\rm m}\widetilde{\Upsilon}_j|_M^2|\widetilde{\psi}_{1'j}|^2 + |\Upsilon_{\rm m}^*\widetilde{\Upsilon}_j{}^*\widetilde{\psi}_{1j}|^2\Big)$$

iff

$$r_1 = \frac{1}{4} \qquad \textit{and} \qquad \omega_{1j}\widetilde{\Upsilon}_j\,\widetilde{\Upsilon}_j{}^* = \Big(-\frac{1}{4} \pm \frac{\kappa_{1'}\kappa_j}{2}\Big)\,\mathrm{id}_M, \tag{2.71}$$

where the latter should hold for all j *appearing in the sum in* (2.66). *Here* $\kappa_{1'} = \varepsilon_{1',j}\varepsilon_{1',1}$, $\kappa_j = \varepsilon_{j,1'}\varepsilon_{j,1} \in \{\pm 1\}$.

Proof To prove this, we must match the coefficients of the contribution (2.68) to the spectral action from a building block $\mathscr{B}_{11'}$ to those of the auxiliary fields (2.69). This requires

$$\gamma_{11'} = 2\sqrt{r_1}e^{i\phi_\gamma}\Upsilon_{\rm m}, \quad \kappa_{1'}\kappa_j 2g_m\sqrt{\frac{2\omega_{1j}}{q_m}}(r_1\,\mathrm{id}_M + \omega_{1j}\widetilde{\Upsilon}_j\,\widetilde{\Upsilon}_j{}^*)^t\Upsilon_{\rm m}\widetilde{\Upsilon}_j \;= \gamma_{11'}(\beta_{11',j}^*)^t \tag{2.72}$$

for all j, where $e^{i\phi_\gamma}$ denotes the phase ambiguity left in $\Upsilon_{\rm m}$ from (2.70) and where we have used the symmetry of $\Upsilon_{\rm m}$. From supersymmetry $\gamma_{11'}$ is in addition constrained by (2.70), which requires the first relation of (2.71) to hold. For the building block $\mathscr{B}_{11'j}$ to have a supersymmetric action we demand

$$\beta_{11',j}^* = g_m\sqrt{\frac{2\omega_{1j}}{q_m}}e^{-i\phi_{\beta_j}}(\widetilde{\Upsilon}_j)^t,$$

which can be obtained by combining the demand (2.53) with the relation (2.58), but keeping Remark 2.23 in mind since it is $\widetilde{\psi}_{1'j}$ that does not have a family index. As is with $\Upsilon_{\rm m}$, the demand (2.53) determines $\beta_{11',j}$ only up to a phase ϕ_{β_j}. Comparing this with the second demand of (2.72), inserting (2.70) and using the symmetry of $\Upsilon_{\rm m}$, we must have

$$\phi_\gamma = \phi_{\beta_j} \mod \pi, \qquad 2(r_1 \,\mathrm{id}_M + \omega_{1j}\widetilde{\Upsilon}_j\widetilde{\Upsilon}_j^{\,*}) = \pm\kappa_{1'}\kappa_j 2\sqrt{r_1}\,\mathrm{id}_M\,.$$

Inserting the first relation of (2.71), its second relation follows. The second term of the second line of (2.68) cannot be accounted for by the auxiliary fields at hand, which establishes the result.

It is not per se impossible to write all of (2.68) off shell in terms of auxiliary fields, but to avoid the obstruction from Lemma 2.25 at least requires the presence of mass terms for the representation $\widetilde{\psi}_{1j}$ and $\widetilde{\psi}_{1'j}$ such as the ones that are discussed in the next section.

2.2.5.2 Fifth Building Block: 'mass' Terms

If there are two building blocks of the second type with the same indices—say i and j—but with different values for the grading, we are in the situation as depicted in Fig. 2.13. On the basis

$$\Big[(\mathbf{N}_i \otimes \mathbf{N}_j^o)_L \oplus (\mathbf{N}_j \otimes \mathbf{N}_i^o)_R \oplus (\mathbf{N}_i \otimes \mathbf{N}_j^o)_R \oplus (\mathbf{N}_j \otimes \mathbf{N}_i^o)_L\Big]^{\oplus M}, \tag{2.73}$$

the most general finite Dirac operator that satisfies the demand of self-adjointness, the first order condition (1.12) and that commutes with J_F is of the form

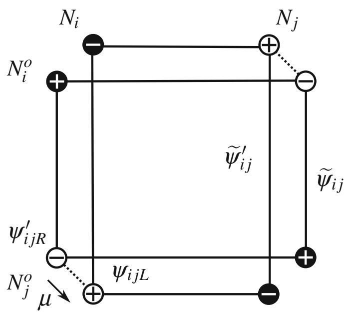

Fig. 2.13 The case with two building blocks of the second type that have the same indices but an opposite grading; a component of the finite Dirac operator mapping between the two copies will generate a mass-term, indicated by the *dotted line* with the 'μ'

$$D_F = \begin{pmatrix} 0 & 0 & \mu_i + \mu_j^o & 0 \\ 0 & 0 & 0 & (\mu_i^o)^* + \mu_j^* \\ \mu_i^* + (\mu_j^*)^o & 0 & 0 & 0 \\ 0 & \mu_i^o + \mu_j & 0 & 0 \end{pmatrix} \tag{2.74}$$

with $\mu_i \in M_{N_i M}(\mathbb{C})$ and $\mu_j \in M_{N_j M}(\mathbb{C})$. The inner fluctuations for general such matrices $\mu_{i,j}$ will generate scalar fields in the representations $M_{N_{i,j}}(\mathbb{C})$. If we want these components to result in mass terms in the action, we should restrict them both to only act non-trivially on possible generations, i.e. for a single generation the components are equal to a complex number. We will write $\mu := \mu_i + \mu_j^o \in M_M(\mathbb{C})$ for the restricted component.

This gives rise to the following definition.

Definition 2.28 For a finite spectral triple that contains building blocks $\mathscr{B}_{ij}^{\pm}$ and $\mathscr{B}_{ij}^{\mp}$ of the second type (both with multiplicity M), a *building block of the fifth type* is a component of D_F that runs between the representations of the two building blocks and acts only non-trivially on the M copies. Symbolically:

$$\mathscr{B}_{\text{mass},ij} = (0, {D_{ijL}}^{ijR}) \in \mathscr{H}_F \oplus \text{End}(\mathscr{H}_F).$$

We denote this component with $\mu \in M_M(\mathbb{C})$.

If for convenience we restrict to the upper signs for the chiralities of the building blocks and write

$$(\psi_{ijL}, \overline{\psi}_{ijR}, \psi'_{ijR}, \overline{\psi}'_{ijL})$$

for the elements of $L^2(M, S \otimes \mathscr{H}_F)$ on the basis (2.73) (where the first two fields are associated to $\mathscr{B}_{ij}^{+}$ and the last two to $\mathscr{B}_{ij}^{-}$), then the contribution of (2.74) to the fermionic action reads

$$\begin{aligned} S_{f,\text{mass}}[\zeta] &= \frac{1}{2}\langle J(\psi_{ijL}, \overline{\psi}_{ijR}, \psi'_{ijR}, \overline{\psi}'_{ijL}), \gamma^5 D_F(\psi_{ijL}, \psi'_{ijR}, \overline{\psi}_{ijR}, \overline{\psi}'_{ijL})\rangle \\ &= \langle J_M \overline{\psi}_{ijR}, \gamma^5 \mu \psi'_{ijR}\rangle + \langle J_M \overline{\psi}'_{ijL}, \gamma^5 \mu^* \psi_{ijL}\rangle. \end{aligned} \tag{2.75}$$

Let $\widetilde{\psi}$ and $\widetilde{\psi}'$ be the sfermions that are associated to $\mathscr{B}_{ij}^{+}$ and $\mathscr{B}_{ij}^{-}$ respectively, then the extra contributions to the spectral action as a result of adding this building block are given by

$$\begin{aligned} S_{b,\text{mass}}[\widetilde{\zeta}] = {}& \frac{f(0)}{\pi^2}(N_i|\mu^* C_{iij}\widetilde{\psi}_{ij}|^2 + N_j|\mu^* C_{ijj}\widetilde{\psi}_{ij}|^2 + N_i|\mu C'_{iij}\widetilde{\psi}'_{ij}|^2 + N_j|\mu C'_{ijj}\widetilde{\psi}'_{ij}|^2) \\ &+ \frac{f(0)}{\pi^2}\sum_k \Big[N_i \,\text{tr}\, \mu^* \overline{\widetilde{\psi}}'_{ij} C'^{*}_{iij} C_{iik} \widetilde{\psi}_{ik} \overline{\widetilde{\psi}}_{jk} {\Upsilon_j}^{k*} \\ &\qquad + N_j \,\text{tr}\, \overline{\widetilde{\psi}}'_{ij} C'^{*}_{ijj}\, \mu^* {\Upsilon_i}^{k*} \widetilde{\psi}_{ik} \overline{\widetilde{\psi}}_{jk} C^*_{jjk} + h.c. \\ &\qquad + \Big(N_j \,\text{tr}_M(\mu\mu^* {\Upsilon_i}^{k*} {\Upsilon_i}^{k})|\widetilde{\psi}_{ik}|^2 + N_i|\mu {\Upsilon_j}^{k} \widetilde{\psi}_{jk}|^2\Big)\Big], \end{aligned} \tag{2.76}$$

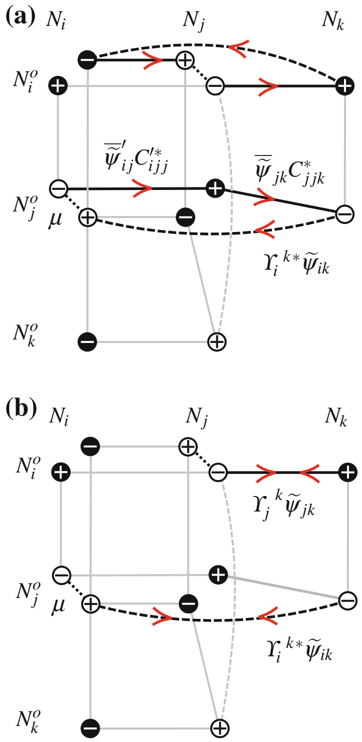

Fig. 2.14 In the case of a building block of the fifth type, there are various extra contributions to the action, depending on the content of the finite spectral triple. **a** A path with μ, featuring edges from a building block of the second and third type. **b** A path with μ, featuring only edges from building blocks of the third and fifth type

where the second and third lines arise in a situation where for some k, $\mathscr{B}_{ijk}$ is present. The paths corresponding to these expressions are depicted in Fig. 2.14. Here, the C_{iij} with a prime correspond to the components of the Dirac operator of $\mathscr{B}_{i\bar{j}}$. We assume that they also satisfy (2.33). In this context $\widetilde{\psi}_{ik}$ does not have a family-index and consequently we could separate the traces in the first term of the third line of (2.76).

In a similar way as with the building block of the fourth type we can rewrite the second line of (2.76) using Remarks 2.22 and 2.23, giving

$$\begin{aligned}\frac{f(0)}{\pi^2}\Big[& N_i \operatorname{tr}(\overline{\widetilde{\psi}}'_{ij} C'^*_{iij})_a C_{iik}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jkb}(\Upsilon_j{}^{k*}(\mu^*)^t)_{ba} \\ & + N_j \operatorname{tr}(\overline{\widetilde{\psi}}'_{ij} C'^*_{ijj})_a\, (\mu^*\Upsilon_i{}^{k*})_{ac}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jkb}(C^*_{jjk})_{bc} + h.c.\Big] \\ = s_{jk} & \left(\frac{N_i r_i + N_j r_j}{\sqrt{n_j n_k g_j g_k}}\right) \operatorname{tr} \overline{\widetilde{\psi}}'_{ij} C'^*_{ijj}\mu^*\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk} C^*_{jkk} + h.c. \qquad (2.77)\end{aligned}$$

Replacing the second line of (2.76) with (2.77) and then scaling the fields and rewriting $\Upsilon_i{}^j$ and $\Upsilon_j{}^k$ in terms of $\Upsilon_i{}^k \equiv \Upsilon$ using the identities (2.59), reduces the bosonic contribution (2.76) to

$$2(1-\omega_{ij})\big(|\mu^*\widetilde{\psi}_{ij}|^2 + |\mu\widetilde{\psi}'_{ij}|^2\big) + 2\sum_k \Bigg[\kappa_j g_l(1-\omega_{ij})\sqrt{\frac{2\omega_{ik}}{q_l}}\,\mathrm{tr}\,\overline{\widetilde{\psi}}'_{ij}\mu^*\widetilde{\Upsilon}^*\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk} + h.c. \\ + \omega_{ik}\Big(N_j|\Upsilon\mu|^2_M|\widetilde{\psi}_{ik}|^2 + N_i|\mu\widetilde{\Upsilon}\widetilde{\psi}_{jk}|^2\Big)\Bigg], \tag{2.78}$$

where we have again employed the notation $|a|^2_M = \mathrm{tr}_M\, a^*a$ for the trace over the family-index and used that $s_{jk}\varepsilon_{j,i}\varepsilon_{k,j} = \varepsilon_{j,i}\varepsilon_{j,k} \equiv \kappa_j \in \{\pm\}$. The index l can take any of the values that appear in the model.

Here we have a similar result as in the previous section:

Lemma 2.29 *For a finite spectral triple that contains, in addition to building blocks of the first, second and third type, one building block of the fifth type, the only terms in the associated spectral action that can be written off shell are those featuring* $\widetilde{\psi}_{ij}$, $\widetilde{\psi}'_{ij}$ *or their conjugates.*

Proof In order to rewrite the first terms of (2.78) in terms of auxiliary fields, we must introduce an interaction featuring one auxiliary field F and one sfermion. Since $\widetilde{\psi}_{ij}$ and $\widetilde{\psi}'_{ij}$ are in the same representation of the algebra, we can choose whether to couple $\widetilde{\psi}_{ij}$ to F_{ij} (corresponding to $\mathscr{B}^+_{ij}$) or to F'_{ij} (corresponding to $\mathscr{B}^-_{ij}$). The same holds for $\widetilde{\psi}'_{ij}$. Transforming the fermions in (2.75) according to (2.31) suggests that, in order to have a chance at supersymmetry, we must couple F'_{ij} to $\widetilde{\psi}_{ij}$ and F_{ij} to $\widetilde{\psi}'_{ij}$. We thus write

$$-\,\mathrm{tr}\,F^*_{ij}F_{ij} - \mathrm{tr}\,F'^*_{ij}F'_{ij} - \big(\mathrm{tr}\,F^*_{ij}\delta'_{ij}\widetilde{\psi}'_{ij} + \mathrm{tr}\,F'^*_{ij}\delta_{ij}\widetilde{\psi}_{ij} + h.c.\big), \tag{2.79}$$

with $\delta_{ij}, \delta'_{ij} \in M_M(\mathbb{C})$. This yields on shell $|\delta_{ij}\widetilde{\psi}_{ij}|^2 + |\delta'_{ij}\widetilde{\psi}'_{ij}|^2$, which is indeed of the same form as the first two terms in (2.78). In the case that there is a building block $\mathscr{B}_{ijk}$ of the third type present, the extra contributions to the action must come from the cross terms of

$$-\,\mathrm{tr}\,F^*_{ij}F_{ij} - \mathrm{tr}\,F'^*_{ij}F'_{ij} - \Big[\mathrm{tr}\,F^*_{ij}\big(\delta'_{ij}\widetilde{\psi}'_{ij} + \beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\big) + \mathrm{tr}\,F'^*_{ij}\delta_{ij}\widetilde{\psi}_{ij} + h.c.\Big],$$

where the interaction with $\beta_{ij,k}$ corresponds to the second term of (2.51). On shell this gives us the additional interaction

$$\mathrm{tr}\,\overline{\widetilde{\psi}}'_{ij}\delta'^*_{ij}\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk} + h.c. \tag{2.80}$$

In form, this indeed coincides with the second line of (2.78). The last two terms of (2.78) do not appear here and consequently they cannot be addressed using the auxiliary fields that are available to us when having only building blocks of the first, second and third type.

Similar as with the previous building blocks we can check what the demands for off shell supersymmetry are.

Lemma 2.30 *The action consisting of the fermionic action* (2.75) *and the off shell action* (2.79) *is supersymmetric under the transformations (2.32) if and only if*

$$\delta\delta^* = \mu^*\mu, \qquad \delta'\delta'^* = \mu\mu^*. \tag{2.81}$$

Proof See Appendix section 'Fifth Building Block'.

Combining the above lemmas gives the following result for a building block of the fifth type.

Proposition 2.31 *For a finite spectral triple that contains, in addition to building blocks of the first, second and third type, one building block of the fifth type, the action of a single building block of the fifth type breaks supersymmetry only softly via*

$$\omega_{ik}\Big(N_j|\widetilde{\Upsilon}\mu|^2_M|\widetilde{\psi}_{ik}|^2 + N_i|\mu\widetilde{\Upsilon}\widetilde{\psi}_{jk}|^2\Big)$$

iff

$$\omega_{ij} = \frac{1}{2}$$

and the product of the possible phases of δ'^* *and* $\beta_{ij,k}$ *(cf.* (2.81) *and* (2.53) *respectively) is equal to* $\varepsilon_{j,i}\varepsilon_{j,k}$.

Proof This follows from comparing the spectral action (2.78) with the off shell action (2.79) and using the demands (2.81) and (2.53).

The form of the soft breaking term suggests that, in order to let it be part of a truly supersymmetric action, we have the following necessary requirement. Each two building blocks of the second type that are connected to each other via an edge of a building block of the third type, both need to have a building block of the fifth type defined on them. In the case above this would have been $\widetilde{\psi}_{ik}$ and $\widetilde{\psi}_{jk}$.

2.3 Conditions for a Supersymmetric Spectral Action

Our aim is to determine whether the total action that corresponds to an almost-commutative geometry consisting of various of the five identified building blocks, is supersymmetric. More than once we used the following strategy for that. First, we identified the off shell counterparts for the contributions of $\mathrm{tr}_F\,\Phi^4$ to the (on shell) spectral action, using the available auxiliary fields and coefficients whose values were undetermined still. Second, we derived constraints for these coefficients based on the demand of having supersymmetry for the fermionic action and this off shell action. Finally, we should check if the off shell interactions correspond on shell to the spectral action again, when their coefficients satisfy the constraints that supersymmetry puts on them. If this is the case then the action from noncommutative geometry is an on shell counterpart of an off shell action that is supersymmetric.

In the previous sections we have experienced multiple times that the pre-factors of all bosonic interactions can get additional contributions when extending the almost-commutative geometry. As was stated before, we should therefore assess whether or not the demands from supersymmetry on the coefficients are satisfied for the final model only. In this section we will present an overview of all four-scalar interactions that have appeared previously, from which building blocks their pre-factors get what contributions and which demands hold for them. We identify several such demands, thus constructing a checklist for supersymmetry.

1. To have supersymmetry for a building block $\mathscr{B}_{ij}$ of the second type, the components of the finite Dirac operator should satisfy (2.33), after scaling them. For a single building block of the second type this demand can only be satisfied for $N_i = N_j$ and $M = 4$ (Proposition 2.15). When $\mathscr{B}_{ij}$ is part of a building block of the third type the demand is automatically satisfied via the solution (2.45).
2. A necessary requirement to have supersymmetry for any building block $\mathscr{B}_{ijk}$ of the third type (Sect. 2.2.3), is that the scaled parameters of the finite Dirac operator that make up such a building block satisfy

$$\omega_{jk}\widetilde{\Upsilon}_j{}^{k*}\widetilde{\Upsilon}_j{}^{k} = \omega_{ik}\widetilde{\Upsilon}_i{}^{k*}\widetilde{\Upsilon}_i{}^{k} = \omega_{ij}\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_i{}^{j} =: \Omega^*_{ijk}\Omega_{ijk}. \tag{2.82}$$

   This relation can be obtained from (2.55), multiplying each term with its conjugate. For notational convenience we have introduced the variable $\Omega^*_{ijk}\Omega_{ijk}$.
3. Terms $\propto |\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2$ appear for the first time with a building block of the second type ((2.19) in Sect. 2.2.2) but also get contributions from a building block $\mathscr{B}_{ijk}$ of the third type (first term of (2.49)). The total expression reads

$$\begin{aligned}
&\frac{f(0)}{2\pi^2}\Big[N_i|C^*_{iij}C_{iij}\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2 + N_j|C^*_{iij}C_{ijj}\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2 \\
&\qquad + \sum_k N_k|\Upsilon_{i,k}{}^{j*}\Upsilon_{i,k}{}^{j}\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2\Big] \\
&\to 2\frac{g_i^2}{q_i}\Big|\Big(N_i r_i^2 + \alpha_{ij}\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2\Big)^{1/2}\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}\Big|^2 \\
&\quad + 2\frac{g_j^2}{q_j}\Big|\Big(N_j r_j^2 + (1-\alpha_{ij})\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2\Big)^{1/2}\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}\Big|^2,
\end{aligned}$$

   upon scaling the fields. Here we have introduced a parameter $\alpha_{ij} \in \mathbb{R}$ that tells how any new contributions are divided over the initial two. Such terms can only be described off shell using the auxiliary fields G_i and G_j (cf. Lemma 2.10) via

$$-\frac{1}{2n_i}\operatorname{tr} G_i\big(G_i + 2n_i\mathscr{P}_i\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}\big) - \frac{1}{2n_j}\operatorname{tr} G_j\big(G_j + 2n_j\overline{\widetilde{\psi}}_{ij}\mathscr{P}_j\widetilde{\psi}_{ij}\big),$$

which on shell equals

$$\frac{n_i}{2}|\mathscr{P}_i\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2 + \frac{n_j}{2}|\overline{\widetilde{\psi}}_{ij}\mathscr{P}_j\widetilde{\psi}_{ij}|^2,$$

cf. (2.24). Comparing this with the above expression sets the coefficients $\mathscr{P}_i$ and $\mathscr{P}_j$:

$$\frac{n_i}{2}\mathscr{P}_i^2 = 2\frac{g_i^2}{q_i}\bigg(N_i r_i^2 + \alpha_{ij}\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2\bigg),$$
$$\frac{n_j}{2}\mathscr{P}_j^2 = 2\frac{g_j^2}{q_i}\bigg(N_j r_j^2 + (1-\alpha_{ij})\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2\bigg),$$

where there is an additional trace over the last terms if $\widetilde{\psi}_{ij}$ has no family index. If the action is supersymmetric then (2.33) can be used with $\mathscr{K}_i = \mathscr{K}_j = 1$ and the above relations read

$$\frac{r_i}{4} = N_i r_i^2 + \alpha_{ij}\sum_k N_k \operatorname{tr}[(\Omega^*_{ijk}\Omega_{ijk})^2],$$
$$\frac{r_j}{4} = N_j r_j^2 + (1-\alpha_{ij})\sum_k N_k \operatorname{tr}[(\Omega^*_{ijk}\Omega_{ijk})^2], \tag{2.83}$$

when $\widetilde{\psi}_{ij}$ has no family index and

$$\frac{r_i}{4}\,\mathrm{id}_M = N_i r_i^2\,\mathrm{id}_M + \alpha_{ij}\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2,$$
$$\frac{r_j}{4}\,\mathrm{id}_M = N_j r_j^2\,\mathrm{id}_M + (1-\alpha_{ij})\sum_k N_k(\Omega^*_{ijk}\Omega_{ijk})^2, \tag{2.84}$$

when it does. Here we have used that $r_i = q_i n_i$.

4. An interaction $\propto |\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2$ can receive contributions in two different ways; one comes from a building block $\mathscr{B}_{ijk}$ of the third type (2.57), the other comes from two adjacent building blocks $\mathscr{B}_{ijl}$ and $\mathscr{B}_{jkl}$ (first and second term of (2.62), but occurs only for particular values of the grading):

$$g_m^2\frac{4\omega_{ij}}{q_m}(1-\omega_{ik})|\widetilde{\Upsilon}_{i,k}{}^{j}\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2$$
$$+4\bigg(n_j r_j N_j g_j^2|\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2 + \frac{g_m^2}{q_m}\omega_{ij}\omega_{jk}N_l|\widetilde{\Upsilon}_{i,l}{}^{j}\widetilde{\psi}_{ij}\widetilde{\Upsilon}_{j,l}{}^{k}\widetilde{\psi}_{jk}|^2\bigg).$$

From this, however, we need to subtract the value $n_j g_j^2 |\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2$ that is expected from the cross term

$$- \operatorname{tr} G_j \big(\mathscr{P}_{j,i} \overline{\widetilde{\psi}}_{ij} \widetilde{\psi}_{ij} + \mathscr{P}_{j,k} \widetilde{\psi}_{jk} \overline{\widetilde{\psi}}_{jk}\big),$$

that should already be there when the almost-commutative geometry contains $\mathscr{B}_{ij}^{\pm}$ and $\mathscr{B}_{jk}^{\mp}$ but nevertheless does not appear in the spectral action (see Sect. 2.2.2.3 and the discussion above Theorem 2.47). The remaining terms must be accounted for by

$$- \operatorname{tr} F_{ik}^* F_{ik} + \big(\operatorname{tr} F_{ik}^* \beta'_{ik,j} \widetilde{\psi}_{ij} \widetilde{\psi}_{jk} + h.c.\big) \tag{2.85}$$

which equals

$$\operatorname{tr} \overline{\widetilde{\psi}}_{jk} \overline{\widetilde{\psi}}_{ij} \beta'_{ik,j} \beta'^*_{ik,j} \widetilde{\psi}_{ij} \widetilde{\psi}_{jk}$$

on shell. Since $\beta_{ik,j}\beta_{ik,j}^*$ is positive definite we can also write the above as

$$|(\beta'_{ik,j}\beta'^*_{ik,j})^{1/2} \widetilde{\psi}_{ij} \widetilde{\psi}_{jk}|^2.$$

Comparing the above relations, the off shell action (2.85) corresponds on shell to the spectral action, iff

$$\begin{aligned} \beta'_{ik,j}\beta'^*_{ik,j} = {} & g_m^2 \frac{4\omega_{ij}}{q_m}(1-\omega_{ik}) \widetilde{\Upsilon}_{i,k}{}^{j*} \widetilde{\Upsilon}_{i,k}{}^{j} - n_j g_j^2 \operatorname{id}_M \\ & + 4\bigg(n_j r_j N_j g_j^2 \operatorname{id}_M + \frac{g_m^2}{q_m} \omega_{ij}\omega_{jk} N_l (\widetilde{\Upsilon}_{i,l}{}^{j} \widetilde{\Upsilon}_{j,l}{}^{k})^* (\widetilde{\Upsilon}_{i,l}{}^{j} \widetilde{\Upsilon}_{j,l}{}^{k})\bigg), \end{aligned}$$

where we have assumed that it is $\widetilde{\psi}_{ij}$ not having a family structure. Furthermore, from the demand of supersymmetry $\beta'_{ik,j}$ must satisfy

$$\beta'_{ik,j}\beta'^*_{ik,j} = g_m^2 \frac{2\omega_{ij}}{q_m} \widetilde{\Upsilon}_{i}{}^{j*} \widetilde{\Upsilon}_{i}{}^{j} \equiv 2\frac{g_m^2}{q_m} \Omega_{ijk}^* \Omega_{ijk}$$

i.e. (2.53),[8] but with Υ' replaced by $\widetilde{\Upsilon}$ using (2.58). Combining the above two relations, we require that

$$\begin{aligned} \frac{2g_m^2}{q_m} \Omega_{ijk}^* \Omega_{ijk} = {} & 4\frac{g_m^2}{q_m}(1-\omega_{ik}) \Omega_{ijk}^* \Omega_{ijk} - n_j g_j^2 \operatorname{id}_M \\ & + 4\bigg(n_j r_j N_j g_j^2 \operatorname{id}_M + \frac{g_m^2}{q_m} \omega_{ij}\omega_{jk} N_l (\widetilde{\Upsilon}_{i,l}{}^{j} \widetilde{\Upsilon}_{j,l}{}^{k})^* (\widetilde{\Upsilon}_{i,l}{}^{j} \widetilde{\Upsilon}_{j,l}{}^{k})\bigg), \end{aligned}$$

[8] In fact, in (2.53) the variables are in reversed order compared to here but looking at (2.153)—from which the former is derived—one sees immediately that this also holds.

using the notation introduced in (2.82). Setting $m = j$ in particular, this reduces to

$$2(1-2\omega_{ik})\Omega^*_{ijk}\Omega_{ijk} - r_j\,\mathrm{id}_M + 4\Big(N_j r_j^2\,\mathrm{id}_M + \omega_{ij}\omega_{jk}N_l(\widetilde{\Upsilon}_{i,l}{}^j\widetilde{\Upsilon}_{j,l}{}^k)^*\left(\widetilde{\Upsilon}_{i,l}{}^j\widetilde{\Upsilon}_{j,l}{}^k\right)\Big) = 0. \quad (2.86)$$

5. The interaction $\propto \operatorname{tr}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il}$ only appears in the case of two adjacent building blocks $\mathscr{B}_{ijk}$ and $\mathscr{B}_{ijl}$ of the third type (cf. the Lagrangian (2.62)). Equating this term to (2.64) that appears from the auxiliary field F_{ij}, gives

$$\kappa_k\kappa_l 4\frac{g_m^2}{q_m}(1-\omega_{ij})\omega_{ij}\operatorname{tr}\widetilde{\Upsilon}_l\,\widetilde{\Upsilon}_k{}^*\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il} + h.c.$$
$$= \operatorname{tr}\beta'^*_{ij,l}\beta'_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}\widetilde{\psi}_{jl}\overline{\widetilde{\psi}}_{il} + h.c.,$$

with $\kappa_k = \varepsilon_{k,i}\varepsilon_{k,j}$, $\kappa_l = \varepsilon_{l,i}\varepsilon_{l,j}$. From the demand of supersymmetry $\beta'^*_{ij,l}$ and $\beta'_{ij,k}$ should satisfy (2.53). Their phases, if any, must be opposite modulo π for the action to be real. We write ϕ_{kl} for the remaining sign ambiguity. Inserting these demands above and using (2.58) requires that $\kappa_k\kappa_l 4\omega_{ij}(1-\omega_{ij}) = 2\phi_{kl}\omega_{ij}$ for this interaction to be covered by the auxiliary field F_{ij}. This has two solutions, the only acceptable of which is

$$\phi_{kl} = \kappa_k\kappa_l, \qquad \omega_{ij} = \frac{1}{2} \implies r_i N_i + r_j N_j = \frac{1}{2}, \quad (2.87)$$

where we have used (2.43).

6. From the spectral action interactions $\propto |\widetilde{\psi}_{ij}|^4$ only appear in the context of a building block of the second type as

$$\frac{f(0)}{\pi^2}|C_{iij}\widetilde{\psi}_{ij}|^2|C_{ijj}\widetilde{\psi}_{ij}|^2 \to 4\frac{g_l^2}{q_l}r_i r_j|\widetilde{\psi}_{ij}|^4,$$

see (2.15). Via the auxiliary fields on the other hand they appear in two ways; from the $G_{i,j}$ and via the $u(1)$-field H (see Lemma 2.10 for both). The latter give on shell the contributions

$$\left(\frac{\mathscr{Q}_{ij}^2}{2} - n_i\frac{\mathscr{P}_i^2}{2N_i} - n_j\frac{\mathscr{P}_j^2}{2N_j}\right)|\widetilde{\psi}_{ij}|^4,$$

where the minus-signs stem from the identity (2.26) between the generators $T^a_{i,j}$ of $su(N_{i,j})$. Demanding supersymmetry, $\mathscr{P}_i^2$ must equal g_i^2 and similarly $\mathscr{P}_j^2 = g_j^2$. In order for the interactions from the spectral action to equal the above equation, $\mathscr{Q}_{ij}^2$ is then set to be

$$\mathcal{Q}_{ij}^2 = \frac{g_l^2}{q_l}\left(8r_i r_j + \frac{r_i}{N_i} + \frac{r_j}{N_j}\right). \tag{2.88}$$

In the case that $\widetilde{\psi}_{ij}$ has family indices, the expressions for $\mathcal{P}_{i,j}^2$ and $\mathcal{Q}_{ij}^2$ must be multiplied with the $M \times M$ identity matrix id_M.

7. Interactions $\propto |\widetilde{\psi}_{ij}|^2|\widetilde{\psi}_{jk}|^2$ (having one common index j) appear via the spectral action in two different ways. First of all from two adjacent building blocks $\mathcal{B}_{ij}$ and $\mathcal{B}_{jk}$ of the second type (cf. (2.38)), and secondly from a building block of the third type (second line of (2.49)). This gives

$$\frac{f(0)}{\pi^2}\Big(|C_{ijj}\widetilde{\psi}_{ij}|^2|C_{jjk}\widetilde{\psi}_{jk}|^2 + |\widetilde{\psi}_{ij}|^2|\Upsilon_i{}^{j*}\Upsilon_j{}^{k}\widetilde{\psi}_{jk}|^2\Big)$$
$$\to 4\frac{g_l^2}{q_l}\Big(r_j^2|\widetilde{\psi}_{ij}|^2|\widetilde{\psi}_{jk}|^2 + \omega_{jk}\omega_{ij}|\widetilde{\psi}_{ij}|^2|\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_j{}^{k}\widetilde{\psi}_{jk}|^2\Big),$$

where we have assumed $\widetilde{\psi}_{ij}$ not to have a family-index. We can write this as

$$4\frac{g_l^2}{q_l}|\widetilde{\psi}_{ij}|^2\big|\big(r_j^2\,\mathrm{id}_M + \omega_{ij}\omega_{jk}(\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_j{}^{k})^*\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_j{}^{k}\big)^{1/2}\widetilde{\psi}_{jk}\big|^2.$$

From the auxiliary fields these terms can appear via G_j (with coefficients $\mathcal{P}_{j,i}$ and $\mathcal{P}_{j,k}$, i.e. as in (2.39)) and via the $u(1)$-field H with coefficients $\mathcal{Q}_{ij}$ and $\mathcal{Q}_{jk}$:

$$\left[\mathcal{Q}_{ij}\mathcal{Q}_{jk} - n_j\frac{\mathcal{P}_{j,i}\mathcal{P}_{j,k}}{N_j}\right]|\widetilde{\psi}_{ij}|^2|\widetilde{\psi}_{jk}|^2.$$

Equating the terms from the spectral action and those from the auxiliary fields, and inserting the values for the coefficients $\mathcal{P}_{j,i}$, $\mathcal{P}_{j,k}$ (from (2.33)), $\mathcal{Q}_{ij}$ and $\mathcal{Q}_{jk}$ (from (2.88)) that we obtain from supersymmetry, we require

$$\left(2r_i r_j + \frac{r_i}{4N_i} + \frac{r_j}{4N_j}\right)\left(2r_j r_k + \frac{r_j}{4N_j} + \frac{r_k}{4N_k}\right)\mathrm{id}_M$$
$$= \left[\left(r_j^2 + \frac{r_j}{4N_j}\right)\mathrm{id}_M + \omega_{ij}\omega_{jk}(\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_j{}^{k})^*\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_j{}^{k}\right]^2. \tag{2.89}$$

8. There are interactions $\propto |\widetilde{\psi}_{ik}|^2|\widetilde{\psi}_{jl}|^2$ and $\propto |\widetilde{\psi}_{jk}|^2|\widetilde{\psi}_{il}|^2$ that arise from two adjacent building blocks $\mathcal{B}_{ijk}$ and $\mathcal{B}_{ijl}$ of the third type. The first of these is given by

$$4\frac{g_m^2}{q_m}|(\omega_{ik}\widetilde{\Upsilon}_{i,j}{}^{k}\widetilde{\Upsilon}_{i,j}{}^{k*})^{1/2}\widetilde{\psi}_{ik}|^2|(\omega_{jl}\widetilde{\Upsilon}_{j,i}{}^{l*}\widetilde{\Upsilon}_{j,i}{}^{l})^{1/2}\widetilde{\psi}_{jl}|^2,$$

see (2.62). Since the interactions are characterized by four different indices, the auxiliary fields G_i cannot account for these and consequently they should be described by the $u(1)$-field H:

$$|\mathscr{Q}_{ik}^{1/2}\widetilde{\psi}_{ik}|^2|\mathscr{Q}_{jl}^{1/2}\widetilde{\psi}_{jl}|^2.$$

In order for the spectral action to be written off shell we thus require that

$$\mathscr{Q}_{ik}\mathscr{Q}_{jl} = 4\frac{g_m^2}{q_m}\Omega_{ijk}\Omega_{ijk}^*\Omega_{ijl}^*\Omega_{ijl}.$$

With $\mathscr{Q}_{ik}$ and $\mathscr{Q}_{jl}$ being determined by (2.88) from the demand of supersymmetry, we can infer from this that for the squares of these expressions we must have

$$\begin{aligned}\Big(2r_ir_k + \frac{r_i}{4N_i} + \frac{r_k}{4N_k}\Big)\,\mathrm{id}_M &= \Omega_{ijk}\Omega_{ijk}^*,\\ \Big(2r_jr_l + \frac{r_j}{4N_j} + \frac{r_l}{4N_l}\Big)\,\mathrm{id}_M &= \Omega_{ijl}^*\Omega_{ijl}.\end{aligned} \tag{2.90}$$

9. As was already covered in Sect. 2.2.5.1, a building block $\mathscr{B}_{\mathrm{maj}}$ of the fourth type only breaks supersymmetry softly iff

$$r_1 = \frac{1}{4} \qquad \text{and} \qquad \omega_{1j}\widetilde{\Upsilon}_j\widetilde{\Upsilon}_j^{\,*} = \Big(-\frac{1}{4} \pm \frac{\kappa_{1'}\kappa_j}{2}\Big)\,\mathrm{id}_M \tag{2.91}$$

(see Proposition 2.27), where the latter should hold for each building block $\mathscr{B}_{11'j}$ of the third type. Here $\kappa_{1'}, \kappa_j \in \{\pm 1\}$.

10. Covered in Sect. 2.2.5.2, a building block $\mathscr{B}_{\mathrm{mass},ij}$ of the fifth type also breaks supersymmertry only softly iff

$$\omega_{ij} = \frac{1}{2}, \tag{2.92}$$

see Proposition 2.31.

To be able to say whether an almost-commutative geometry that is built out of building blocks of the first to the fifth type has a supersymmetric action then entails checking whether all the relevant relations above are satisfied.

2.3.1 Applied to a Single Building Block of the Third Type

We apply a number of the demands above to the case of a single building block of the third type (and the building blocks of the second and first type that are needed to define it) to see whether this possibly exhibits supersymmetry. We will assume that ψ_{ij} has $R = -1$ (and consequently no family index), but of course we could equally well have taken one of the other two (see e.g. Remark 2.23). The generalization of Remark 2.14 for the expressions of the r_i that results from normalizing the gauge bosons' kinetic terms is

$$r_i = \frac{3}{2N_i + N_j + MN_k}, \quad r_j = \frac{3}{N_i + 2N_j + MN_k}, \quad r_k = \frac{3}{M(N_i + N_j) + 2N_k}.$$

For the first of the demands of the previous section, (2.82), one of the three terms that are equated to each other reads

$$\begin{aligned}\omega_{ik}\widetilde{\Upsilon}_i{}^k\widetilde{\Upsilon}_i{}^{k*} &\equiv \omega_{ik}(N_j\Upsilon_i{}^k\Upsilon_i{}^{k*})^{-1/2}\Upsilon_i{}^k\Upsilon_i{}^{k*}(N_j\Upsilon_i{}^k\Upsilon_i{}^{k*})^{-1/2}\\ &= \frac{\omega_{ik}}{N_j}\,\mathrm{id}_M = \omega_{ik}\widetilde{\Upsilon}_i{}^{k*}\widetilde{\Upsilon}_i{}^k,\end{aligned}$$

where we have used the definition (2.2.3) of $\widetilde{\Upsilon}_i{}^k$. Similarly,

$$\omega_{jk}\widetilde{\Upsilon}_j{}^{k*}\widetilde{\Upsilon}_j{}^k = \frac{\omega_{jk}}{N_i}\,\mathrm{id}_M \quad \text{and} \quad \omega_{ij}\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_i{}^j = \frac{\omega_{ij}}{N_k}\Upsilon_i{}^{j*}\Upsilon_i{}^j(\mathrm{tr}\,\Upsilon_i{}^{j*}\Upsilon_i{}^j)^{-1}$$

for the other two. Equating these, we obtain:

$$\frac{\omega_{ik}}{N_j}\,\mathrm{id}_M = \frac{\omega_{jk}}{N_i}\,\mathrm{id}_M = \frac{\omega_{ij}}{N_k}\Upsilon_i{}^{j*}\Upsilon_i{}^j(\mathrm{tr}\,\Upsilon_i{}^{j*}\Upsilon_i{}^j)^{-1}, \tag{2.93}$$

i.e. $\Upsilon_i{}^j$ is constrained to be proportional to a unitary matrix. Taking the trace gives the demand

$$M\frac{\omega_{ik}}{N_j} = M\frac{\omega_{jk}}{N_i} = \frac{\omega_{ij}}{N_k}. \tag{2.94}$$

Given the expressions for $r_{i,j,k}$ above, we can test whether this demand admits solutions. Indeed, we find

$$N_i = N_j = N_k \equiv N, \qquad M = 1 \vee 2. \tag{2.95}$$

In the first case we find that

$$r_iN_i = r_jN_j = r_kN_k = \frac{3}{4}, \qquad \omega_{ij} = \omega_{ik} = \omega_{jk} = -\frac{1}{2},$$

whereas in the second case we have

$$r_iN_i = r_jN_j = \frac{3}{5}, \quad r_kN_k = \frac{1}{2}, \quad \omega_{ij} = -\frac{1}{5}, \quad \omega_{ik} = \omega_{jk} = -\frac{1}{10}.$$

Next, we have the demand (2.83) to ensure that terms of the form $|\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}|^2$ can be written off shell in a supersymmetric manner. In this context it reads

$$\begin{aligned}\frac{r_i}{4} &= N_i r_i^2 + \alpha_{ij} N_k \omega_{ij}^2 \operatorname{tr}[(\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_i{}^{j})^2],\\ \frac{r_j}{4} &= N_j r_j^2 + \alpha_{ji} N_k \omega_{ij}^2 \operatorname{tr}[(\widetilde{\Upsilon}_i{}^{j*}\widetilde{\Upsilon}_i{}^{j})^2],\end{aligned}$$

for $\widetilde{\psi}_{ij}$ (where the trace in the last term comes from the fact that $\widetilde{\psi}_{ij}$ does not have family indices) and

$$\begin{aligned}\frac{r_k}{4}\,\mathrm{id}_M &= N_k r_k^2\,\mathrm{id}_M + \alpha_{kj} N_i \omega_{jk}^2 (\widetilde{\Upsilon}_j{}^{k*}\widetilde{\Upsilon}_j{}^{k})^2,\\ \frac{r_j}{4}\,\mathrm{id}_M &= N_j r_j^2\,\mathrm{id}_M + \alpha_{jk} N_i \omega_{jk}^2 (\widetilde{\Upsilon}_j{}^{k*}\widetilde{\Upsilon}_j{}^{k})^2,\\ \frac{r_k}{4}\,\mathrm{id}_M &= N_k r_k^2\,\mathrm{id}_M + \alpha_{ki} N_j \omega_{ik}^2 (\widetilde{\Upsilon}_i{}^{k}\widetilde{\Upsilon}_i{}^{k*})^2,\\ \frac{r_i}{4}\,\mathrm{id}_M &= N_i r_i^2\,\mathrm{id}_M + \alpha_{ik} N_j \omega_{ik}^2 (\widetilde{\Upsilon}_i{}^{k}\widetilde{\Upsilon}_i{}^{k*})^2,\end{aligned}$$

for $\widetilde{\psi}_{jk}$ and $\widetilde{\psi}_{ik}$ respectively. Here we have written $\alpha_{ji} = 1-\alpha_{ij}$, etc. We can remove all variables $\widetilde{\Upsilon}_i{}^{j}$, $\widetilde{\Upsilon}_i{}^{k}$ and $\widetilde{\Upsilon}_j{}^{k}$ by using the squares of the expressions in (2.93). This gives

$$\begin{aligned}\frac{N_i r_i}{4} &= (N_i r_i)^2 + \alpha_{ij} N_k \frac{\omega_{jk}^2}{N_i} M, & \frac{N_j r_j}{4} &= (N_j r_j)^2 + \alpha_{ji} N_k \frac{\omega_{ik}^2}{N_j} M,\\ \frac{N_k r_k}{4} &= (N_k r_k)^2 + \alpha_{kj} N_k \frac{\omega_{jk}^2}{N_i}, & \frac{N_j r_j}{4} &= (N_j r_j)^2 + \alpha_{jk} N_i \frac{\omega_{ik}^2}{N_j},\\ \frac{N_k r_k}{4} &= (N_k r_k)^2 + \alpha_{ki} N_k \frac{\omega_{ik}^2}{N_j}, & \frac{N_i r_i}{4} &= (N_i r_i)^2 + \alpha_{ik} N_j \frac{\omega_{jk}^2}{N_i},\end{aligned}$$

where the M in the first line above comes from taking the trace over id_M. Comparing the expressions featuring the same combinations $r_i N_i$, $r_j N_j$, $r_k N_k$ and using (2.94) we must have that

$$\alpha_{ij} N_k M = \alpha_{ik} N_j, \quad (1-\alpha_{jk}) N_i = (1-\alpha_{ik}) N_j, \quad (1-\alpha_{ij}) N_k M = \alpha_{jk} N_i.$$

Since both solutions (2.95) to the relation (2.94) have $N_i = N_j = N_k$, this solves

$$\alpha_{ij} = \frac{1}{2}, \qquad \alpha_{ik} = \frac{1}{2} M, \qquad \alpha_{jk} = \frac{1}{2} M$$

and the demands above reduce to

$$N_i r_i = 4(N_i r_i)^2 + 2\omega_{jk}^2 M, \qquad N_j r_j = 4(N_j r_j)^2 + 2\omega_{ik}^2 M,$$
$$N_k r_k = 4(N_k r_k)^2 + \omega_{ik}^2 (4 - 2M).$$

We can check that for neither of the two cases of (2.95) these are satisfied. As a cross check of this result we will employ one more demand.

In the context of a single building block of the third type the demand (2.86) that is necessary to write terms of the form $|\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}|^2$ off shell in a supersymmetric manner, reduces to

$$2(1 - 2\omega_{ik})\omega_{ik} = r_j N_j, \qquad 2(1 - 2\omega_{jk})\omega_{jk} = r_i N_i,$$
$$2(1 - 2\omega_{ij})\omega_{ij} \Upsilon_i^{\ j*} \Upsilon_i^{\ j} = r_k N_k \, \mathrm{id}_M \, \mathrm{tr}\, \Upsilon_i^{\ j*} \Upsilon_i^{\ j}.$$

We can use (2.94) to rewrite the last equation in terms of ω_{ik} or ω_{jk}. In any way, the LHS are seen to be negative for all values of ω_{ij}, ω_{ik} and ω_{jk} allowed by the solutions (2.95), whereas $r_i N_i$, $r_j N_j$ and $r_k N_k$ are necessarily positive. We thus get a contradiction.

A single building block of the third type (together with the building blocks needed to define it) is thus not supersymmetric.

2.4 Summary and Conclusions

The main subject of this chapter has been almost-commutative geometries of the form

$$(C^\infty(M, \mathscr{A}_F), L^2(M, S \otimes \mathscr{H}_F), \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F; \gamma_5 \otimes \gamma_F, J_M \otimes J_F)$$

of KO-dimension 2 on a flat, 4-dimensional background M. We have dressed these with a grading $R : \mathscr{H} \to \mathscr{H}$ called *R-parity*. We have shown that such almost-commutative geometries provide an arena suited for describing field theories that have a supersymmetric particle content. This was done by identifying five different *building blocks*; constituents of a finite spectral triple that yield an almost-commutative geometry whose particle content has an equal number of (off shell) fermionic and bosonic degrees of freedom. In addition they contain the right interactions to make them eligible for supersymmetric theories. These five building blocks are listed in Table 2.2.

Although we have not been using the notion of superspace and superfields, the building blocks themselves can thus be seen as an alternative. However, a significant difference between the two approaches is that if a certain superfield enters the action, then automatically all its component fields do too. For the components of our building blocks this need not be true; without *demanding* supersymmetry we are free to e.g. define a finite Hilbert space consisting of only the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ (and its conjugate), without its superpartner arising from a component of the finite Dirac operator. However, the philosophy to include each component of D_F that is not explicitly forbidden by the demands on a spectral triple turned out to be a fruitful

Table 2.2 The building blocks of a supersymmetric spectral triple

Building block	Required	Counterpart in superfield formalism
$\mathscr{B}_i$ (Sect. 2.2.1)	–	Vector multiplet
$\mathscr{B}_{ij}$ (Sect. 2.2.2)	$\mathscr{B}_i$, $\mathscr{B}_j$	Chiral multiplet
$\mathscr{B}_{ijk}$ (Sect. 2.2.3)	$\mathscr{B}_{ij}$, $\mathscr{B}_{ik}$, $\mathscr{B}_{jk}$	Superpotential with three chiral superfields
$\mathscr{B}_{\text{maj}}$ (Sect. 2.2.5.1)	$\mathscr{B}_{11'}$	Majorana mass for $\psi_{11'}$, $\widetilde{\psi}_{11'}$
$\mathscr{B}_{\text{mass},ij}$ (Sect. 2.2.5.2)	$\mathscr{B}_{ij}^{+}$, $\mathscr{B}_{ij}^{-}$	A mass(-like) term for ψ_{ij}, $\widetilde{\psi}_{ij}$

In the last column we have listed their counterparts in the superfield formalism

one in obtaining models that have a supersymmetric particle content, as long as we start by adding gauginos to the finite Hilbert space.

It is far from automatic, though, that when the field content is supersymmetric also the action is. First of all, there is a number of obstructions to a supersymmetric action:

1. A single building block $\mathscr{B}_i$ of the first type (i.e. without a building block $\mathscr{B}_{ij}$ of the second type, for some j) for which $N_i = 1$, has vanishing bosonic interactions (Remark 2.4).
2. A single building block $\mathscr{B}_{ij}$ of the second type that has $R = -1$, has two different $u(1)$ gauge fields that interact whereas the corresponding gauginos do not (Remark 2.13).
3. If the finite algebra contains more than two components $M_{N_i}(\mathbb{C})$, $M_{N_j}(\mathbb{C})$ and $M_{N_k}(\mathbb{C})$ over $\mathbb{C}$ and there is a set of two or more building blocks $\mathscr{B}_{ij}$, $\mathscr{B}_{ik}$ that share three different indices, then there are two different $u(1)$ gauge fields that interact, whereas the corresponding gauginos do not (Proposition 2.19).

Second, for a set-up that avoids these three obstructions, the question is whether the four-scalar interactions that are generated by the spectral action are rewritable as an off shell action in terms of the auxiliary fields that are available to us. On top of this, the pre-factors of the interactions with the auxiliary fields are dictated by supersymmetry. Both the form of the action functional used in noncommutative geometry and supersymmetry thus put demands on the pre-factors of interactions which together heavily constrain the number of possible solutions. Typical for almost-commutative geometries is that there are new contributions to various expressions when extending a model. The question whether for the 'full theory' the coefficients are such that these terms do have an off shell counterpart, is then phrased in terms of the demands listed in Sect. 2.3.

Despite all these technical calculations and detailed issues, we have a definite handle on which almost-commutative geometries exhibit a supersymmetric action and which do not. To obtain an exhaustive list of examples that do satisfy all demands requires an automated strategy, in which step by step models are extended with building blocks and it is checked whether they satisfy the aforementioned demands. Whatever the outcome of such a strategy will be, the examples of supersymmetric almost-commutative geometries will be sparse. This is markedly different from the more generic superfield formalism, but at the same time the models that do satisfy all demands will enjoy a very special status.

Appendix 1. The Action from a Building Block of the Third Type

In this section we derive in detail the action that comes from a building block $\mathscr{B}_{ijk}$ of the third type (cf. Sect. 2.2.3), such as that of Fig. 2.6. If we constrain ourselves for now to the off-diagonal part of the finite Hilbert space, then on the basis

$$\mathscr{H}_{F,\mathrm{off}} = (\mathbf{N}_i \otimes \mathbf{N}_j^o)_L \oplus (\mathbf{N}_i \otimes \mathbf{N}_k^o)_R \oplus (\mathbf{N}_j \otimes \mathbf{N}_k^o)_L \\ \oplus (\mathbf{N}_j \otimes \mathbf{N}_i^o)_R \oplus (\mathbf{N}_k \otimes \mathbf{N}_i^o)_L \oplus (\mathbf{N}_k \otimes \mathbf{N}_j^o)_R$$

the most general allowed finite Dirac operator is of the form

$$D_F = \begin{pmatrix} 0 & \Upsilon_j{}^{ko*} & 0 & 0 & 0 & \Upsilon_i{}^{k*} \\ \Upsilon_j{}^{ko} & 0 & \Upsilon_i{}^{j} & 0 & 0 & 0 \\ 0 & \Upsilon_i{}^{j*} & 0 & \Upsilon_i{}^{ko*} & 0 & 0 \\ 0 & 0 & \Upsilon_i{}^{ko} & 0 & \Upsilon_j{}^{k} & 0 \\ 0 & 0 & 0 & \Upsilon_j{}^{k*} & 0 & \Upsilon_i{}^{jo*} \\ \Upsilon_i{}^{k} & 0 & 0 & 0 & \Upsilon_i{}^{jo} & 0 \end{pmatrix} \tag{2.96}$$

We write for a generic element ζ of $\frac{1}{2}(1+\gamma)L^2(S \otimes \mathscr{H}_{F,\mathrm{off}})$

$$\zeta = (\psi_{ijL}, \psi_{ikR}, \psi_{jkL}, \overline{\psi}_{ijR}, \overline{\psi}_{ikL}, \overline{\psi}_{jkR})$$

where $\overline{\psi}_{ijR} \in L^2(S_- \otimes \mathbf{N}_j \otimes \mathbf{N}_i^o)$, etc. Applying the matrix (2.96) to this element yields

$$\begin{aligned}\gamma^5 D_F \zeta = \gamma^5 \Big(& \psi_{ikR}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*} + \Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\psi}_{jkR}, \psi_{ijL}\Upsilon_j{}^{k}\widetilde{\psi}_{jk} + \Upsilon_i{}^{j}\widetilde{\psi}_{ij}\psi_{jkL}, \\ & \overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}\psi_{ikR} + \overline{\psi}_{ijR}\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}, \psi_{jkL}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k} + \Upsilon_j{}^{k}\widetilde{\psi}_{jk}\overline{\psi}_{ikL}, \\ & \overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\overline{\psi}_{ijR} + \overline{\psi}_{jkR}\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}, \overline{\psi}_{ikL}\Upsilon_i{}^{j}\widetilde{\psi}_{ij} + \overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k}\psi_{ijL}\Big).\end{aligned}$$

Notice that for the pairs (i, j) and (j, k) we always encounter $\widetilde{\psi}_{ij}$ in combination with $\Upsilon_i{}^{j}$, whereas for (i, k) it is the combination $\widetilde{\psi}_{ik}$ and $\Upsilon_i{}^{k*}$. This has to do with the fact that the sfermion $\widetilde{\psi}_{ik}$ crosses the particle/antiparticle-diagonal in the Krajewski diagram. Since

$$\begin{aligned} J\zeta &= J(\psi_{ijL}, \psi_{ikR}, \psi_{jkL}, \overline{\psi}_{ijR}, \overline{\psi}_{ikL}, \overline{\psi}_{jkR}) \\ &= (J_M\overline{\psi}_{ijR}, J_M\overline{\psi}_{ikL}, J_M\overline{\psi}_{jkR}, J_M\psi_{ijL}, J_M\psi_{ikR}, J_M\psi_{jkL}), \end{aligned}$$

the extra contributions to the inner product $\frac{1}{2}\langle J\zeta, \gamma^5 D_F\zeta\rangle$ are written as

$$\begin{aligned} &\frac{1}{2}\langle J\zeta, \gamma^5 D_F\zeta\rangle \\ &= \frac{1}{2}\langle J_M\overline{\psi}_{ijR}, \gamma^5(\psi_{ikR}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*} + \widetilde{\psi}_{ik}\Upsilon_i{}^{k*}\overline{\psi}_{jkR})\rangle \\ &\quad + \frac{1}{2}\langle J_M\overline{\psi}_{ikL}, \gamma^5(\psi_{ijL}\Upsilon_j{}^{k}\widetilde{\psi}_{jk} + \Upsilon_i{}^{j}\widetilde{\psi}_{ij}\psi_{jkL})\rangle \\ &\quad + \frac{1}{2}\langle J_M\overline{\psi}_{jkR}, \gamma^5(\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}\psi_{ikR} + \overline{\psi}_{ijR}\Upsilon_i{}^{k*}\widetilde{\psi}_{ik})\rangle \\ &\quad + \frac{1}{2}\langle J_M\psi_{ijL}, \gamma^5(\psi_{jkL}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k} + \Upsilon_j{}^{k}\widetilde{\psi}_{jk}\overline{\psi}_{ikL})\rangle \\ &\quad + \frac{1}{2}\langle J_M\psi_{ikR}, \gamma^5(\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\overline{\psi}_{ijR} + \overline{\psi}_{jkR}\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*})\rangle \\ &\quad + \frac{1}{2}\langle J_M\psi_{jkL}, \gamma^5(\overline{\psi}_{ikL}\Upsilon_i{}^{j}\widetilde{\psi}_{ij} + \overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k}\psi_{ijL})\rangle. \end{aligned}$$

Using the symmetry properties (2.164) of the inner product, this equals

$$\begin{aligned} &\langle J_M\overline{\psi}_{ijR}, \gamma^5\psi_{ikR}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}\rangle + \langle J_M\overline{\psi}_{ijR}, \gamma^5\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\psi}_{jkR}\rangle \\ &\quad + \langle J_M\overline{\psi}_{ikL}, \gamma^5\psi_{ijL}\Upsilon_j{}^{k}\widetilde{\psi}_{jk}\rangle + \langle J_M\overline{\psi}_{ikL}, \gamma^5\Upsilon_i{}^{j}\widetilde{\psi}_{ij}\psi_{jkL}\rangle \\ &\quad + \langle J_M\overline{\psi}_{jkR}, \gamma^5\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}\psi_{ikR}\rangle + \langle J_M\psi_{jkL}, \gamma^5\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^{k}\psi_{ijL}\rangle. \end{aligned}$$

We drop the subscripts L and R, keeping in mind the chirality of each field, and for brevity we replace $ij \to 1$, $ik \to 2$, $jk \to 3$:

$$\begin{aligned} S_{123,F}[\zeta, \widetilde{\zeta}] = &\langle J_M\overline{\psi}_1, \gamma^5\psi_2\overline{\widetilde{\psi}}_3\Upsilon_3{}^*\rangle + \langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2{}^*\widetilde{\psi}_2\overline{\psi}_3\rangle + \langle J_M\overline{\psi}_2, \gamma^5\psi_1\Upsilon_3\,\widetilde{\psi}_3\rangle \\ &+ \langle J_M\overline{\psi}_2, \gamma^5\Upsilon_1\,\widetilde{\psi}_1\psi_3\rangle + \langle J_M\overline{\psi}_3, \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1{}^*\psi_2\rangle + \langle J_M\psi_3, \gamma^5\overline{\widetilde{\psi}}_2\Upsilon_2\,\psi_1\rangle. \end{aligned} \tag{2.97}$$

The spectral action gives rise to some new interactions compared to those coming from building blocks of the second type. They arise from the trace of the fourth power of the finite Dirac operator and are given by the following list.

- From paths of the type such as the one in the upper left corner of Fig. 2.15 the contribution is

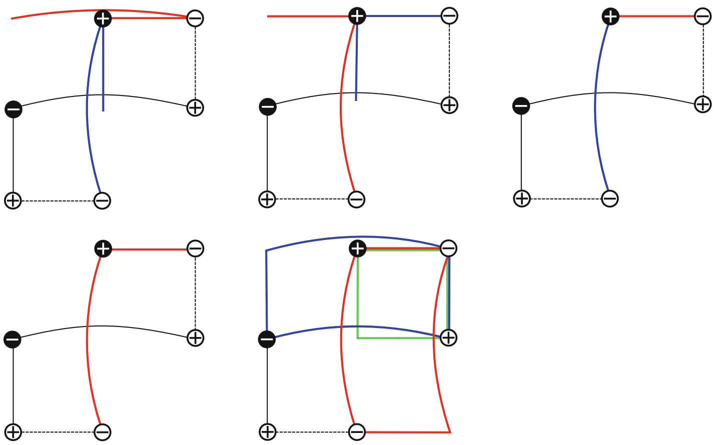

Fig. 2.15 The various contributions to tr D_F^4 in the language of Krajewski diagrams corresponding to a building block $\mathscr{B}_{ijk}$ of the third type

$$8\Big[N_i|C_{iij}\widetilde{\psi}_{ij}\Upsilon_j{}^k\widetilde{\psi}_{jk}|^2 + N_k|\Upsilon_i{}^j\widetilde{\psi}_{ij}C_{jkk}\widetilde{\psi}_{jk}|^2 + N_j|\overline{\widetilde{\psi}}_{ij}C^*_{ijj}\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}|^2 + N_k|\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}C_{ikk}\widetilde{\psi}_{ik}|^2 + N_i|\Upsilon_j{}^k\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{ik}C^*_{iik}|^2 + N_j|C_{jjk}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^k|^2\Big]. \tag{2.98}$$

Here the multiplicity $8 = 2(1 + 1 + 2)$ comes from the fact that there are three vertices involved in each path, on each of which the path can start. In the case of the 'middle' vertices the path can be traversed in two distinct orders. Furthermore a factor two comes from that each path occurs twice; also mirrored along the diagonal of the diagram.

- From paths such as the upper middle one in Fig. 2.15 the contribution is:

$$8\Big[\operatorname{tr}(C_{iij}\widetilde{\psi}_{ij})^o\Upsilon_j{}^k\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(\overline{\widetilde{\psi}}_{ij}C^*_{iij})^o + \operatorname{tr}(\Upsilon_i{}^j\widetilde{\psi}_{ij})^oC_{jjk}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}C^*_{jjk}(\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*})^o + \operatorname{tr}(\overline{\widetilde{\psi}}_{ij}C^*_{iij})^o\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^k(C_{iij}\widetilde{\psi}_{ij})^o + \operatorname{tr}(\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*})^oC_{iik}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{ik}C^*_{iik}(\Upsilon_i{}^j\widetilde{\psi}_{ij})^o + \operatorname{tr}(\overline{\widetilde{\psi}}_{ik}C^*_{ikk})^o\Upsilon_j{}^k\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(C_{ikk}\widetilde{\psi}_{ik})^o + \operatorname{tr}(\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^k)^oC_{jjk}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}C^*_{jkk}(\Upsilon_i{}^{k*}\widetilde{\psi}_{ik})^o\Big], \tag{2.99}$$

where the arguments for determining the multiplicity are the same as for the previous contribution.

- From paths such as the upper right one in Fig. 2.15, going back and forth along the same edge twice, the contribution is:

$$4\Big[N_i|\Upsilon_j{}^k\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}|^2 + N_j|\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^k|^2 + N_k|\Upsilon_i{}^j\widetilde{\psi}_{ij}\overline{\widetilde{\psi}}_{ij}\Upsilon_i{}^{j*}|^2\Big] \tag{2.100}$$

The multiplicity arises from 2 vertices on which the path can start and each such path occurs again reflected.

- From paths such as the lower left one in Fig. 2.15 the contribution is:

$$8\Big[|\widetilde{\psi}_{ij}|^2|\Upsilon_i{}^j\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}|^2+|\widetilde{\psi}_{ij}|^2|\Upsilon_i{}^{j*}\Upsilon_j{}^k\widetilde{\psi}_{jk}|^2+|\Upsilon_i{}^{k*}\widetilde{\psi}_{ik}|^2|\Upsilon_j{}^k\widetilde{\psi}_{jk}|^2\Big]. \tag{2.101}$$

- From paths such as the lower right one in Fig. 2.15 the contribution is:

$$\begin{aligned}8\Big[&\operatorname{tr}(\overline{\widetilde{\psi}}_{ik}C^*_{iik}(\Upsilon_i{}^j\widetilde{\psi}_{ij})^o(\overline{\widetilde{\psi}}_{ij}C^*_{iij})^o\Upsilon_i{}^{k*}\widetilde{\psi}_{ik})+\operatorname{tr}(\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(\overline{\widetilde{\psi}}_{ij}C^*_{ijj})^o(\Upsilon_i{}^j\widetilde{\psi}_{ij})^oC_{jjk}\widetilde{\psi}_{jk})\\&+\operatorname{tr}((\overline{\widetilde{\psi}}_{ik}\Upsilon_i{}^k)^oC_{jkk}\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{jk}\Upsilon_j{}^{k*}(C_{ikk}\widetilde{\psi}_{ik})^o)+h.c.\Big],\end{aligned} \tag{2.102}$$

corresponding with the blue, green and red paths respectively. The multiplicity arises from the fact that any such path has four vertices on which it can start and also occurs reflected around the diagonal. Besides, each path can also be traversed in the opposite direction, hence the 'h.c.'.

Adding (2.98)–(2.102) the total *extra* contribution to tr D_F^4 from adding a building block $\mathscr{B}_{ijk}$ of the third type, is given by (2.49).

Appendix 2. Supersymmetric Spectral Actions: Proofs

In this section we give the actual proofs and calculations of the Lemmas and Theorems presented in the text. First we introduce some notation. With $(.,.)_{\mathscr{S}}:\Gamma^\infty(\mathscr{S})\times\Gamma^\infty(\mathscr{S})\to C^\infty(M)$ we mean the $C^\infty(M)$-valued Hermitian structure on $\Gamma^\infty(\mathscr{S})$. The Hermitian form on $\Gamma^\infty(\mathscr{S})$ is to be distinguished from the $C^\infty(M)$-valued form on $\mathscr{H}\equiv L^2(M,S\otimes\mathscr{H}_F)$:

$$(.,.)_{\mathscr{H}}:\Gamma(\mathscr{S}\otimes\mathscr{H}_F)\times\Gamma(\mathscr{S}\otimes\mathscr{H}_F)\to C^\infty(M)$$

given by

$$(\psi_1,\psi_2)_{\mathscr{H}}:=(\zeta_1,\zeta_2)_{\mathscr{S}}\langle m_1,m_2\rangle_F,\qquad \psi_{1,2}=\zeta_{1,2}\otimes m_{1,2},$$

where $\langle\,.\,,\,.\,\rangle_F$ denotes the inner product on the finite Hilbert space $\mathscr{H}_F$. The inner product on the full Hilbert space $\mathscr{H}$ is then obtained by integrating over the manifold M:

$$\langle\psi_1,\psi_2\rangle_{\mathscr{H}}:=\int_M(\psi_1,\psi_2)_{\mathscr{H}}\sqrt{g}\mathrm{d}^4x.$$

If no confusion is likely to arise between $(.,.)_{\mathscr{S}}$ and $(.,.)_{\mathscr{H}}$, we omit the subscript.

In the proofs there appear a number of a priori unknown constants. To avoid confusion: capital letters always refer to parameters of the Dirac operator, lowercase letters always refer to proportionality constants for the superfield transformations. For the latter the number of indices determines what field they belong to: constants with one index belong to a gauge boson–gaugino pair, constants with two indices belong to a fermion–sfermion pair.

First Building Block

This section forms the proof of Theorem 2.35. In this case the action is given by (2.9). Its constituents are the—flat—metric g, the gauge field $A^j \in \mathrm{End}(\Gamma(\mathscr{S})\otimes su(N_j))$ and spinor $\lambda_j \in L^2(M, S\otimes su(N_j))$, both in the adjoint representation and the spinor after reducing its degrees of freedom (see Sect. 2.2.1.1).

Now for $\varepsilon \equiv (\varepsilon_L, \varepsilon_R) \in L^2(M, S)$, decomposed into Weyl spinors that vanish covariantly (i.e. $\nabla^S\varepsilon = 0$), we define

$$\begin{aligned}\delta A_j &= c_j\gamma^\mu\big[(J_M\varepsilon_R, \gamma_\mu\lambda_{jL})_{\mathscr{S}} + (J_M\varepsilon_L, \gamma_\mu\lambda_{jR})_{\mathscr{S}}\big]\\ &\equiv \gamma^\mu(\delta A_{\mu j+} + \delta A_{\mu j-}), \qquad (2.103a)\end{aligned}$$

$$\delta\lambda_{jL,R} = (c'_j F^j + c'_{G_j}G_j)\varepsilon_{L,R}, \qquad F^j \equiv \gamma^\mu\gamma^\nu F^j_{\mu\nu}, \qquad (2.103b)$$

$$\delta G_j = c_{G_j}\big[(J_M\varepsilon_L, \partial\!\!\!/_A\lambda_{jR})_{\mathscr{S}} + (J_M\varepsilon_R, \partial\!\!\!/_A\lambda_{jL})_{\mathscr{S}}\big], \qquad (2.103c)$$

where the coefficients $c_j, c'_j, c_{G_j}, c'_{G_j}$ are yet to be determined. In the rest of this section we will drop the index j for notational convenience and discard the factor n_j from the normalization of the gauge group generators, since it appears in the same way for each term.

- The fermionic part of the Lagrangian, upon transforming the fields, equals:

$$\begin{aligned}\langle J_M\lambda_L, \partial\!\!\!/_A\lambda_R\rangle \to \int_M &(J_M[c'F + c'_G G]\varepsilon_L, \partial\!\!\!/_A\lambda_R)_{\mathscr{H}} + (J_M\lambda_L, \partial\!\!\!/_A[c'F + c'_G G]\varepsilon_R)_{\mathscr{H}}\\ &+ gc(J_M\lambda_L, \gamma^\mu\,\mathrm{ad}[(J_M\varepsilon_L, \gamma_\mu\lambda_R)_{\mathscr{S}} + (J_M\varepsilon_R, \gamma_\mu\lambda_L)_{\mathscr{S}}]\lambda_R)_{\mathscr{H}}. \qquad (2.104)\end{aligned}$$

  Here we mean with $\mathrm{ad}(X)$ the adjoint: $\mathrm{ad}(X)Y := [X, Y]$.
- The kinetic terms for the gauge bosons transform to:

$$\begin{aligned}\frac{1}{4}\mathscr{K}\int_M \mathrm{tr}_N\, F^{\mu\nu}F_{\mu\nu} \to c\frac{\mathscr{K}}{2}\int_M \mathrm{tr}_N\, F^{\mu\nu}\Big(&\partial_{[\mu}\big[(J_M\varepsilon_R, \gamma_{\nu]}\lambda_L)_{\mathscr{S}} + (J_M\varepsilon_L, \gamma_{\nu]}\lambda_R)_{\mathscr{S}}\big]\\ &- ig[(J_M\varepsilon_R, \gamma_\mu\lambda_L)_{\mathscr{S}} + (J_M\varepsilon_L, \gamma_\mu\lambda_R)_{\mathscr{S}}, A_\nu]\\ &- ig[A_\mu, (J_M\varepsilon_R, \gamma_\nu\lambda_L)_{\mathscr{S}} + (J_M\varepsilon_L, \gamma_\nu\lambda_R)_{\mathscr{S}}]\Big)\sqrt{g}\mathrm{d}^4x, \qquad (2.105)\end{aligned}$$

  where $A_{[\mu}B_{\nu]} \equiv A_\mu B_\nu - A_\nu B_\mu$.

- And finally the term for the auxiliary fields transforms to

$$-\frac{1}{2}\int_M \mathrm{tr}_N\, G^2 \to -c_G \int_M \mathrm{tr}_N\, G\big[(J_M\varepsilon_R, \partial\!\!\!/_A\lambda_L)_{\mathscr{S}} + (J_M\varepsilon_L, \partial\!\!\!/_A\lambda_R)_{\mathscr{S}}\big]. \tag{2.106}$$

If we collect the terms of (2.104)–(2.106) containing the same field content, we get three groups of terms that separately need to vanish in order to have a supersymmetric theory. These groups are:

- one consisting of only one term with four fermionic fields (coming from the second line of (2.104)):

$$gc(J_M\lambda_L, \gamma^\mu\, \mathrm{ad}(J_M\varepsilon_L, \gamma_\mu\lambda_R)_{\mathscr{S}}\lambda_R)_{\mathscr{H}}. \tag{2.107}$$

  There is a second such term with $\varepsilon_L \to \varepsilon_R$ and $\lambda_R \to \lambda_L$ that is obtained via $(J_M\varepsilon_L, \gamma_\mu\lambda_R)_{\mathscr{S}} \to (J_M\varepsilon_R, \gamma_\mu\lambda_L)_{\mathscr{S}}$.
- one consisting of a gaugino and two or three gauge fields:

$$\int_M \Big[c'(J_M\lambda_L, \partial\!\!\!/_A F\varepsilon_R)_{\mathscr{H}} + c\frac{\mathscr{K}}{2}\,\mathrm{tr}_N\, F^{\mu\nu}\Big(\partial_{[\mu}(J_M\varepsilon_R, \gamma_{\nu]}\lambda_L)_{\mathscr{S}} \\ - ig\big[(J_M\varepsilon_R, \gamma_\mu\lambda_L)_{\mathscr{S}}, A_\nu\big] - ig\big[A_\mu, (J_M\varepsilon_R, \gamma_\nu\lambda_L)_{\mathscr{S}}\big]\Big)\Big] \tag{2.108}$$

  featuring the third term of (2.104) and the terms of (2.105) featuring λ_L. There is another such group with $\varepsilon_R \to \varepsilon_L$ and $\lambda_L \to \lambda_R$ consisting of the first term of (2.104) and the other terms of (2.105).
- one consisting of the auxiliary field G, a gauge field and a gaugino:

$$\int_M \Big[c'_G(J_M\lambda_L, \partial\!\!\!/_A G\varepsilon_R)_{\mathscr{H}} - c_G\,\mathrm{tr}_N\, G(J_M\varepsilon_R, \partial\!\!\!/_A\lambda_L)_{\mathscr{S}}\Big] \tag{2.109}$$

  featuring the second part of the third term of (2.104) and the first term of (2.106). There is another such group with $\varepsilon_R \to \varepsilon_L$ and $\lambda_L \to \lambda_R$.

We will tackle each of these groups separately in the following Lemmas.

Lemma 2.32 *The term* (2.107) *equals zero.*

Proof Evaluating (2.107) point-wise, applying the finite inner product and using the normalization for the generators of the gauge group, yields up to a constant factor

$$f^{abc}(J_M\lambda^a_L, \gamma^\mu\lambda^b_R)_{\mathscr{S}}(J_M\varepsilon_L, \gamma_\mu\lambda^c_R)_{\mathscr{S}}. \tag{2.110}$$

Here the f^{abc} are the structure constants of the Lie algebra $SU(N)$. We employ a Fierz transformation (See Appendix section 'Fierz Transformations'), using $C_{10} = -C_{14} = 4$, $C_{11} = C_{13} = -2$, $C_{12} = 0$, to rewrite (2.110) as

$$
\begin{aligned}
&f^{abc}(J_M\lambda_L^a, \gamma^\mu\lambda_R^b)_{\mathscr{S}}(J_M\varepsilon_L, \gamma_\mu\lambda_R^c)_{\mathscr{S}} \\
&\quad = -\frac{1}{4}f^{abc}\Big[4(J_M\varepsilon_L, \lambda_R^b)_{\mathscr{S}}(J_M\lambda_L^a, \lambda_R^c)_{\mathscr{S}} - 2(J_M\varepsilon_L, \gamma_\mu\lambda_R^b)_{\mathscr{S}}(J_M\lambda_L^a, \gamma^\mu\lambda_R^c)_{\mathscr{S}} \\
&\qquad\qquad - 2(J_M\varepsilon_L, \gamma_\mu\gamma^5\lambda_R^b)_{\mathscr{S}}(J_M\lambda_L^a, \gamma^\mu\gamma^5\lambda_R^c)_{\mathscr{S}} \\
&\qquad\qquad - 4(J_M\varepsilon_L, \gamma^5\lambda_R^b)_{\mathscr{S}}(J_M\lambda_L^a, \gamma^5\lambda_R^c)_{\mathscr{S}}\Big].
\end{aligned}
$$

The first and last terms on the right hand side of this expression are seen to cancel each other, whereas the second and third term add. We retain

$$
f^{abc}(J_M\lambda_L^a, \gamma^\mu\lambda_R^b)_{\mathscr{S}}(J_M\varepsilon_L, \gamma_\mu\lambda_R^c)_{\mathscr{S}} = f^{abc}(J_M\lambda_L^a, \gamma^\mu\lambda_R^c)_{\mathscr{S}}(J_M\varepsilon_L, \gamma_\mu\lambda_R^b)_{\mathscr{S}}.
$$

Since f^{abc} is fully antisymmetric in its indices, this expression equals zero.

Lemma 2.33 *The term* (2.108) *equals zero if and only if*

$$
2ic' = -c\mathscr{K}. \tag{2.111}
$$

Proof If we use that the spin connection is Hermitian and employ (2.163), this yields:

$$
\partial_\mu \delta A_{\nu+} = c(J_M\varepsilon_R, \gamma_\nu\nabla_\mu^S\lambda_L).
$$

Here we have used that $[\nabla_\mu^S, J_M] = 0$, that we have a flat metric and that $\nabla^S\varepsilon_{L,R} = 0$. Now using that $A_\mu(J_M\varepsilon_R, \gamma_\nu\lambda_L)_{\mathscr{S}} = (J_M\varepsilon_R, A_\mu\gamma_\nu\lambda_L)_{\mathscr{S}}$ and inserting these results into the second part of (2.108) gives

$$
c\frac{\mathscr{K}}{2}\int_M \mathrm{tr}_N\, F^{\mu\nu}(J_M\varepsilon_R, D_{[\mu}\gamma_{\nu]}\lambda_L)_{\mathscr{S}}, \quad D_\mu = \nabla_\mu^S - ig\,\mathrm{ad}(A_\mu).
$$

Using Lemma 2.53 and employing the antisymmetry of $F_{\mu\nu}$ we get

$$
c\mathscr{K}\int_M (J_M F^{\mu\nu}\varepsilon_R, D_\mu\gamma_\nu\lambda_L)_{\mathscr{H}}.
$$

We take the first term of (2.108) and write $\partial\!\!\!/_A F = i\gamma^\mu D_\mu\gamma^\nu\gamma^\lambda F_{\nu\lambda}$. We can commute the D_μ through the $\gamma^\nu\gamma^\lambda$-combination since the metric is flat. Employing the identity

$$
\gamma^\mu\gamma^\nu\gamma^\lambda = g^{\mu\nu}\gamma^\lambda + g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu + \varepsilon^{\sigma\mu\nu\lambda}\gamma^5\gamma_\sigma \tag{2.112}
$$

yields

$$
\partial\!\!\!/_A F = i\big(2g^{\mu\nu}\gamma^\lambda + \varepsilon^{\sigma\mu\nu\lambda}\gamma^5\gamma_\sigma\big)D_\mu F_{\nu\lambda},
$$

where the totally antisymmetric pseudotensor $\varepsilon^{\sigma\mu\nu\lambda}$ is defined such that $\varepsilon^{1234} = 1$. Applying this operator to ε_R gives

$$\partial\!\!\!/_A F\varepsilon_R = 2ig^{\mu\nu}\gamma^\lambda D_\mu F_{\nu\lambda}\varepsilon_R = 2i\gamma_\lambda D_\mu F^{\mu\lambda}\varepsilon_R,$$

for the other term cancels via the Bianchi identity and the fact that $\nabla^S\varepsilon_R = 0$. With the above results, (2.108) is seen to be equal to

$$2ic'\langle J\lambda_L, \gamma_\nu D_\mu F^{\mu\nu}\varepsilon_R\rangle + c\mathscr{K}\int_M (J_M F^{\mu\nu}\varepsilon_R, D_\mu\gamma_\nu\lambda_L)_{\mathscr{H}}. \tag{2.113}$$

Using the symmetry of the inner product, the result follows.

Lemma 2.34 *The term* (2.109) *equals zero iff*

$$c_G = -c'_G. \tag{2.114}$$

Proof Using the cyclicity of the trace, the symmetry property (2.165) of the inner product and Lemma 2.53, the second term of (2.109) can be rewritten to

$$c_G\int_M (J_M\lambda_L, \partial\!\!\!/_A G\varepsilon_R)_{\mathscr{H}}$$

from which the result immediately follows.

By combining the above three lemmas we can prove Theorem 2.5:

Proposition 2.35 *A spectral triple whose finite part consists of a building block of the first type (Definition 2.3) has a supersymmetric action* (2.9) *under the transformations (2.103) iff*

$$2ic' = -c\mathscr{K}, \quad c_G = -c'_G.$$

Second Building Block

We apply the transformations (2.10b), (2.31) and (2.32) to the terms in the action *that appear for the first time*[9] as a result of the new content of the spectral triple, i.e. (2.29). In the fermionic part of the action, the second and fourth terms transform under (2.31) to

[9]We need not investigate the terms originating from the Yang-Mills action, since together they were already supersymmetric.

$$\begin{aligned}\langle J_M\overline{\psi}_R, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle \to &\ \langle J_M c_{ij}^{\prime *}\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle + \langle J_M d_{ij}^{\prime *}F_{ij}^*\varepsilon_R, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle \\ &+ c_i'\langle J_M\overline{\psi}_R, \gamma^5 F_i\widetilde{C}_{i,j}\widetilde{\psi}\varepsilon_R\rangle + c_{G_i}'\langle J_M\overline{\psi}_R, \gamma^5 G_i\widetilde{C}_{i,j}\widetilde{\psi}\varepsilon_R\rangle \\ &+ \langle J_M\overline{\psi}_R, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)\rangle \end{aligned} \tag{2.115}$$

and

$$\begin{aligned}\langle J_M\psi_L, \gamma^5\overline{\widetilde{\psi}}\widetilde{C}_{i,j}^*\lambda_{iL}\rangle \to &\ c_{ij}'\langle J_M\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R, \gamma^5\overline{\widetilde{\psi}}\widetilde{C}_{i,j}^*\lambda_{iL}\rangle + d_{ij}'\langle J_M F_{ij}\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\widetilde{C}_{i,j}^*\lambda_{iL}\rangle \\ &+ c_i'\langle J_M\psi_L, \gamma^5\gamma^\mu\gamma^\nu\overline{\widetilde{\psi}}\widetilde{C}_{i,j}^*F_{i\mu\nu}\varepsilon_L\rangle + c_{G_i}'\langle J_M\psi_L, \gamma^5\overline{\widetilde{\psi}}\widetilde{C}_{i,j}^*G_i\varepsilon_L\rangle \\ &+ \langle J_M\psi_L, \gamma^5 c_{ij}^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)\widetilde{C}_{i,j}^*\lambda_{iL}\rangle \end{aligned} \tag{2.116}$$

respectively. We omit the terms with $\lambda_{jL,R}$ instead of $\lambda_{iL,R}$; transformation of these yield essentially the same terms. For the kinetic term of the $R = 1$ fermions (the first term of (2.16)) we have under the same transformations:

$$\begin{aligned}\langle J_M\overline{\psi}_R, \partial\!\!\!/_A\psi_L\rangle \to &\ \langle J_M c_{ij}^{\prime *}\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L, \partial\!\!\!/_A\psi_L\rangle + g_i c_i\langle J_M\overline{\psi}_R, \gamma^\mu[(J_M\varepsilon_L, \gamma_\mu\lambda_{iR}) \\ &+ (J_M\varepsilon_R, \gamma_\mu\lambda_{iL})]\psi_L\rangle + \langle J_M\overline{\psi}_R, \partial\!\!\!/_A c_{ij}'\gamma^5[\partial\!\!\!/_A\widetilde{\psi}]\varepsilon_R\rangle \\ &+ \langle J_M d_{ij}^{\prime *}F_{ij}^*\varepsilon_R, \partial\!\!\!/_A\psi_L\rangle + \langle J_M\overline{\psi}_R, \partial\!\!\!/_A d_{ij}'F_{ij}\varepsilon_L\rangle. \end{aligned} \tag{2.117}$$

As with the previous contributions to the action, we omit the terms δA_j (instead of δA_i) for brevity. In the bosonic action, we have the kinetic terms of the sfermions, transforming to

$$\begin{aligned}\mathrm{tr}_{N_j}\, D^\mu\overline{\widetilde{\psi}}D_\mu\widetilde{\psi} \to &+ig_i c_i\,\mathrm{tr}_{N_j}\left(\overline{\widetilde{\psi}}[(J_M\varepsilon_L, \gamma_\mu\lambda_{iR}) + (J_M\varepsilon_R, \gamma_\mu\lambda_{iL})]D^\mu\widetilde{\psi}\right) \\ &- ig_i c_i\,\mathrm{tr}_{N_j}\left(D_\mu\overline{\widetilde{\psi}}[(J_M\varepsilon_L, \gamma^\mu\lambda_{iR}) + (J_M\varepsilon_R, \gamma^\mu\lambda_{iL})]\widetilde{\psi}\right) \\ &+ \mathrm{tr}_{N_j}\left(D_\mu c_{ij}^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)D^\mu\widetilde{\psi}\right) + \mathrm{tr}_{N_j}\left(D_\mu\overline{\widetilde{\psi}}D^\mu c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)\right) \end{aligned} \tag{2.118}$$

(and terms with λ_j instead of λ_i) and from the terms with the auxiliary fields we have

$$\begin{aligned}\mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}\overline{\widetilde{\psi}}G_i \to &\ \mathrm{tr}_{N_i}\,\mathscr{P}_i c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)\overline{\widetilde{\psi}}G_i + \mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}c_{ij}^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)G_i \\ &+ c_{G_i}\,\mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}\overline{\widetilde{\psi}}[(J_M\varepsilon_L, \partial\!\!\!/_A\lambda_{iR}) + (J_M\varepsilon_R, \partial\!\!\!/_A\lambda_{iL})]. \end{aligned} \tag{2.119}$$

And finally we have the kinetic terms of the auxiliary fields F_{ij}, F_{ij}^* that transform to

$$\begin{aligned}\mathrm{tr}\, F_{ij}^*F_{ij} \to &\ \mathrm{tr}\, F_{ij}^*\Big[d_{ij}(J_M\varepsilon_R, \partial\!\!\!/_A\psi_L)_S + d_{ij,i}(J_M\varepsilon_R, \gamma^5\lambda_{iR}\widetilde{\psi})_{\mathscr{S}} - d_{ij,j}(J_M\varepsilon_R, \gamma^5\widetilde{\psi}\lambda_{jR})_{\mathscr{S}}\Big] \\ &+ \mathrm{tr}\Big[d_{ij}^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_R)_S + d_{ij,i}^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\lambda_{iL})_{\mathscr{S}} - d_{ij,j}^*(J_M\varepsilon_L, \gamma^5\lambda_{jL}\overline{\widetilde{\psi}})_{\mathscr{S}}\Big]F_{ij}, \end{aligned} \tag{2.120}$$

where the traces are over $\mathbf{N}_j^{\oplus M}$. Analyzing the result of this, we can put them in groups of terms featuring the very same fields. Each of these groups should separately give zero in order to have a supersymmetric action. We have:

- Terms with four fermionic fields; the fifth term of (2.115), and part of the second term of (2.117):

$$\langle J_M\overline{\psi}_R, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)\rangle + g_i c_i\langle J_M\overline{\psi}_R, \gamma^\mu(J_M\varepsilon_L, \gamma_\mu\lambda_{iR})\psi_L\rangle. \tag{2.121}$$

  The third term of (2.116) and the other part of the second term of (2.117) give a similar contribution but with $\varepsilon_L \to \varepsilon_R$, $\lambda_{iL} \to \lambda_{iR}$.
- Terms with one gaugino and two sfermions, consisting of the first term of (2.115), part of the first and second terms of (2.118), and part of the third term of (2.119):

$$\begin{aligned}&\langle J_M c_{ij}'^{*}\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L, \gamma^5\lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle + i g_i c_i\int \mathrm{tr}_{N_j}\left(\overline{\widetilde{\psi}}(J_M\varepsilon_L, \gamma_\mu\lambda_{iR})D^\mu\widetilde{\psi}\right)\\ &\quad - i g_i c_i\int \mathrm{tr}_{N_j}\left(D_\mu\overline{\widetilde{\psi}}(J_M\varepsilon_L, \gamma^\mu\lambda_{iR})\widetilde{\psi}\right) - c_{G_i}\int \mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}\overline{\widetilde{\psi}}(J_M\varepsilon_L, \partial\!\!\!/_A\lambda_{iR}).\end{aligned} \tag{2.122}$$

  The first term of (2.116), the other parts of the first and second terms of (2.118) and the other part of the third term of (2.119) give similar terms but with $\varepsilon_L \to \varepsilon_R$, $\lambda_{iR} \to \lambda_{iL}$.
- Terms with two gauge fields, a fermion and a sfermion, consisting of the third term of (2.115), the third term of (2.118) and the third term of (2.117):

$$\begin{aligned}&c_i'\langle J_M\overline{\psi}_R, \gamma^5 F_i\widetilde{C}_{i,j}\widetilde{\psi}\varepsilon_R\rangle + \int \mathrm{tr}_{N_j}\left(D_\mu c_{ij}^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)D^\mu\widetilde{\psi}\right)\\ &\quad + \langle J_M\overline{\psi}_R, \partial\!\!\!/_A\gamma^5 c_{ij}'[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R\rangle.\end{aligned} \tag{2.123}$$

  The fourth term of (2.116), the first term of (2.117) and the fourth term of (2.118) make up a similar group but with $\varepsilon_R \to \varepsilon_L$ and $\overline{\psi}_R \to \psi_L$.
- Terms with the auxiliary field G_i, consisting of the fourth term of (2.115) and the second term of (2.119):

$$c_{G_i}'\langle J_M\overline{\psi}_R, \gamma^5 G_i\widetilde{C}_{i,j}\widetilde{\psi}\varepsilon_R\rangle - \int \mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}c_{ij}^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)G_i. \tag{2.124}$$

  The fifth term of (2.116) and the first term of (2.119) make up another such group but with $\varepsilon_R \to \varepsilon_L$ and $\overline{\psi}_R \to \psi_L$.
- And finally all terms with either F_{ij} or F_{ij}^*, consisting of the second term of (2.115), the second term of (2.116), the fourth and fifth terms of (2.117) and the terms of (2.120) (of which we have omitted the terms with λ_j for now):

$$\langle J_M d'^*_{ij} F^*_{ij} \varepsilon_R, \gamma^5 \lambda_{iR} \widetilde{C}_{i,j} \widetilde{\psi} \rangle + \langle J_M d'^*_{ij} F^*_{ij} \varepsilon_R, \partial\!\!\!/_A \psi_L \rangle$$
$$- \int \mathrm{tr}_{N_j} F^*_{ij} \big[d_{ij} (J_M \varepsilon_R, \partial\!\!\!/_A \psi_L)_S + d_{ij,i} (J_M \varepsilon_R, \gamma^5 \lambda_{iR} \widetilde{\psi})_{\mathscr{S}} \big] \tag{2.125}$$

and

$$\langle J_M F_{ij} d'_{ij} \varepsilon_L, \gamma^5 \overline{\widetilde{\psi}} \widetilde{C}^*_{i,j} \lambda_{iL} \rangle + \langle J_M \overline{\psi}_R, \partial\!\!\!/_A d'_{ij} F_{ij} \varepsilon_L \rangle$$
$$- \int \mathrm{tr}_{N_j} \big[d^*_{ij} (J_M \varepsilon_L, \partial\!\!\!/_A \overline{\psi}_R)_S + d^*_{ij,i} (J_M \varepsilon_L, \gamma^5 \overline{\widetilde{\psi}} \lambda_{iL})_{\mathscr{S}} \big] F_{ij}.$$

We will tackle each of these five groups in the next five lemmas. For the first group we have:

Lemma 2.36 *The expression* (2.121) *vanishes, provided that*

$$\frac{1}{2} \widetilde{C}_{i,j} c_{ij} = -c_i g_i. \tag{2.126}$$

Proof Since the expression contains only fermionic terms, we need to prove this via a Fierz transformation, which is valid only point-wise. We will write

$$\lambda_i = \lambda^a \otimes T^a \in L^2(S_- \otimes su(N_i)_R),$$
$$\psi_L = \psi_{mn} \otimes e_{i,m} \otimes \bar{e}_{j,n} \in L^2(S_+ \otimes \mathbf{N}_i \otimes \mathbf{N}^o_j),$$
$$\overline{\psi}_R = \overline{\psi}_{rs} \otimes e_{j,r} \otimes \bar{e}_{i,s} \in L^2(S_- \otimes \mathbf{N}_j \otimes \mathbf{N}^o_i),$$

where a sum over a, m, n, r and s is implied, to avoid a clash of notation. Here the T^a are the generators of $su(N_i)$. Using this notation, (2.121) is point-wise seen to be equivalent to

$$(J_M \overline{\psi}_{jk}, \gamma^5 \lambda^a)(J_M \varepsilon_L, \gamma^5 \widetilde{C}_{i,j} c_{ij} \psi_{ij}) T^a_{ki} + g_i c_i (J_M \overline{\psi}_{jk}, \gamma^\mu \psi_{ij})(J_M \varepsilon_L, \gamma_\mu \lambda^a) T^a_{ki}.$$

Since it appears in both expressions, we may simply omit T^a_{ki} from our considerations. For brevity we will omit the subscripts of the fermions from here on. We then apply a Fierz transformation (see Appendix section 'Fierz Transformations') for the first term, giving:

$$\begin{aligned}
&(J_M \overline{\psi}, \gamma^5 \lambda^a)(J_M \varepsilon_L, \gamma^5 \psi) \\
&\quad = -\frac{C_{40}}{4} (J_M \overline{\psi}, \psi)(J_M \varepsilon_L, \lambda^a) - \frac{C_{41}}{4} (J_M \overline{\psi}, \gamma^\mu \psi)(J_M \varepsilon_L, \gamma_\mu \lambda^a) \\
&\quad - \frac{C_{42}}{4} (J_M \overline{\psi}, \gamma^\mu \gamma^\nu \psi)(J_M \varepsilon_L, \gamma_\mu \gamma_\nu \lambda^a) - \frac{C_{43}}{4} (J_M \overline{\psi}, \gamma^\mu \gamma^5 \psi)(J_M \varepsilon_L, \gamma_\mu \gamma^5 \lambda^a) \\
&\quad - \frac{C_{44}}{4} (J_M \overline{\psi}, \gamma^5 \psi)(J_M \varepsilon_L, \gamma^5 \lambda^a).
\end{aligned}$$

(Note that the sum in the third term on the RHS runs over $\mu < \nu$, see Example 2.56.) We calculate: $C_{40} = C_{43} = C_{44} = -C_{41} = -C_{42} = 1$ and use that ψ and $\overline{\psi}$ are of opposite parity, as are ψ and λ^a, to arrive at

$$\begin{aligned}(J_M\overline{\psi}, \gamma^5\lambda^a)(J_M\varepsilon_L, \gamma^5\psi) &= \frac{1}{4}(J_M\overline{\psi}, \gamma^\mu\psi)(J_M\varepsilon_L, \gamma_\mu\lambda^a)\\ &\quad - \frac{1}{4}(J_M\overline{\psi}, \gamma^\mu\gamma^5\psi)(J_M\varepsilon_L, \gamma_\mu\gamma^5\lambda^a)\\ &= \frac{1}{2}(J_M\overline{\psi}, \gamma^\mu\psi)(J_M\varepsilon_L, \gamma_\mu\lambda^a).\end{aligned}$$

Remark 2.37 From the action there in fact arises also a similar group of terms as (2.121), that reads

$$\langle J_M\overline{\psi}_R, \gamma^5\widetilde{C}_{j,i}c_{ij}(J_M\varepsilon_L, \gamma^5\psi_L)\lambda_{jR}\rangle - g_jc_j\langle J_M\overline{\psi}_R, \gamma^\mu\psi_L(J_M\varepsilon_L, \gamma_\mu\lambda_{jR})\rangle, \tag{2.127}$$

where the minus sign comes from the one in (1.19). Performing the same calculations, we find

$$\frac{1}{2}\widetilde{C}_{j,i}c_{ij} = c_jg_j \tag{2.128}$$

here.

Lemma 2.38 *The term* (2.122) *vanishes provided that*

$$\frac{1}{2}c_{ij}'^{*}\widetilde{C}_{i,j} = -g_ic_i = \mathscr{P}_ic_{G_i}. \tag{2.129}$$

Proof Using that $[J_M, \gamma^5] = 0$, $(\gamma^5)^* = \gamma^5$ and $(\gamma^5)^2 = 1$, the first term of (2.122) can be rewritten as

$$c_{ij}'^{*}\langle J_M[\not{\partial}_A, \overline{\widetilde{\psi}}]\varepsilon_L, \lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle = c_{ij}'^{*}\langle J_M\overline{\widetilde{\psi}}\varepsilon_L, \not{\partial}_A\lambda_{iR}\widetilde{C}_{i,j}\widetilde{\psi}\rangle,$$

where we have used the self-adjointness of $\not{\partial}_A$. The third term of (2.122) can be written as

$$g_ic_i\langle J_M\overline{\widetilde{\psi}}\varepsilon_L, \not{\partial}_A\lambda_{iR}\widetilde{\psi}\rangle, \tag{2.130}$$

where we have used that $\not{\partial}\varepsilon_L = 0$. On the other hand, the second and fourth terms of (2.122) can be rewritten to yield

$$\begin{aligned}&+ ig_ic_i\int \mathrm{tr}_{N_j}\left(\overline{\widetilde{\psi}}(J_M\varepsilon_L, \gamma_\mu\lambda_{iR})D^\mu\widetilde{\psi}\right) - c_{G_i}\,\mathrm{tr}_{N_i}\,\mathscr{P}_i\widetilde{\psi}\overline{\widetilde{\psi}}(J_M\varepsilon_L, \not{\partial}_A\lambda_{iR})_{\mathscr{S}}\\ &= g_ic_i\langle J_M\overline{\widetilde{\psi}}\varepsilon_L, \not{\partial}_A\lambda_{iR}\widetilde{\psi}\rangle\end{aligned} \tag{2.131}$$

provided that $g_i c_i = -\mathscr{P}_i c_{G_i}$. Then the two terms (2.130) and (2.131) cancel, provided that

$$c_{ij}^{\prime *}\widetilde{C}_{i,j} + 2g_i c_i = 0.$$

Lemma 2.39 *The expression* (2.123) *vanishes, provided that*

$$c_{ij}^{*} = c_{ij}^{\prime} = -2ic_i^{\prime}\widetilde{C}_{i,j}g_i^{-1} = 2ic_j^{\prime}\widetilde{C}_{j,i}g_j^{-1}. \tag{2.132}$$

Proof We start with (2.123):

$$c_i^{\prime}\langle J_M\overline{\psi}_R, \gamma^5 F_i\widetilde{C}_{i,j}\widetilde{\psi}\varepsilon_R\rangle + c_{ij}^{*}\int \mathrm{tr}_{N_j}\left(D_\mu(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)D^\mu\widetilde{\psi}\right)$$
$$-c_{ij}^{\prime}\langle J_M\overline{\psi}_R, \gamma^5\ \partial\!\!\!/_A[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R\rangle,$$

where we have used that $\{\gamma^5, \partial\!\!\!/_A\} = 0$. Note that the second term in this expression can be rewritten as

$$-c_{ij}^{*}\langle J_M\overline{\psi}_R, \gamma^5 D_\mu D^\mu\widetilde{\psi}\varepsilon_R\rangle$$

by using the cyclicity of the trace, the Leibniz rule for the partial derivative and Lemma 2.51. (We have discarded a boundary term here.) Together, the three terms can thus be written as

$$\langle J_M\overline{\psi}_R, \gamma^5\mathscr{O}\widetilde{\psi}\varepsilon_R\rangle, \quad \mathscr{O} = c_i^{\prime}\widetilde{C}_{i,j}F_i - c_{ij}^{*}D_\mu D^\mu - c_{ij}^{\prime}\ \partial\!\!\!/_A^2,$$

where we have used that $\partial\!\!\!/\varepsilon_R = 0$. We must show that the above expression can equal zero. Using Lemma 2.49 we have, on a flat background:

$$\partial\!\!\!/_A^2 + D_\mu D^\mu = -\frac{1}{2}\gamma^\mu\gamma^\nu\mathbb{F}_{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu(g_i F_{\mu\nu}^i - g_j F_{\mu\nu}^{j\,o})$$

since $\mathbb{A}_\mu = -ig_i\mathrm{ad}(A_\mu)$. Comparing the above equation with the expression for $\mathscr{O}$ we see that if $-c_{ij}^{*} = -c_{ij}^{\prime} = 2ic_i^{\prime}\widetilde{C}_{i,j}g_i^{-1}$, the operator $\mathscr{O}$—applied to $\widetilde{\psi}\varepsilon_R$—indeed equals zero. From transforming the fermionic action we also obtain the term

$$c_j^{\prime}\langle J_M\overline{\psi}_R, \gamma^5\widetilde{C}_{i,j}\widetilde{\psi}F_j\varepsilon_R\rangle$$

from which we infer the last equality of (2.132).

Lemma 2.40 *The expression* (2.124) *vanishes, provided that*

$$c_{ij}^{*}\mathscr{P}_i = c_{G_i}^{\prime}\widetilde{C}_{i,j}. \tag{2.133}$$

Proof The second term of (2.124) is rewritten using Lemmas 2.51, 2.53 and 2.54 to give

$$-c_{ij}^{*}\langle J_M\overline{\psi}_R, \gamma^5 G_i \mathscr{P}_i \widetilde{\psi}\varepsilon_R\rangle$$

establishing the result.

Then finally for the last group of terms we have:

Lemma 2.41 *The expression* (2.125) *vanishes, provided that*

$$d_{ij} = d_{ij}'^{*}, \qquad d_{ij,i} = d_{ij}'^{*}\widetilde{C}_{i,j}, \qquad d_{ij,j} = -d_{ij}'^{*}\widetilde{C}_{j,i}. \tag{2.134}$$

Proof The first two identities of (2.134) are immediate. The third follows from the term that we have omitted in (2.125), which is equal to the other term except that $\lambda_{iR}\widetilde{\psi} \to \widetilde{\psi}\lambda_{jR}$, $\widetilde{C}_{i,j} \to \widetilde{C}_{j,i}$ and $d_{ij,i} \to -d_{ij,j}$.

Combining the five lemmas above, we complete the proof of Theorem 2.12 with the following proposition:

Proposition 2.42 *A supersymmetric action remains supersymmetric $\mathscr{O}(\Lambda^0)$ after adding a 'building block of the second type' to the spectral triple if the scaled parameters in the finite Dirac operator are given by*

$$\widetilde{C}_{i,j} = \varepsilon_{i,j}\sqrt{\frac{2}{\mathscr{K}_i}}g_i\,\mathrm{id}_M, \qquad \widetilde{C}_{j,i} = \varepsilon_{j,i}\sqrt{\frac{2}{\mathscr{K}_j}}g_j\,\mathrm{id}_M \tag{2.135}$$

and if

$$c_{ij}' = c_{ij}^{*} = \varepsilon_{i,j}\sqrt{2\mathscr{K}_i}c_i = -\varepsilon_{j,i}\sqrt{2\mathscr{K}_j}c_j, \tag{2.136a}$$

$$d_{ij} = d_{ij}'^{*} = \varepsilon_{i,j}\sqrt{\frac{\mathscr{K}_i}{2}}\frac{d_{ij,i}}{g_i} = -\varepsilon_{j,i}\sqrt{\frac{\mathscr{K}_j}{2}}\frac{d_{ij,j}}{g_j}, \tag{2.136b}$$

$$\mathscr{P}_i^2 = g_i^2\mathscr{K}_i^{-1}, \tag{2.136c}$$

$$c_{G_i} = \varepsilon_i\sqrt{\mathscr{K}_i}c_i, \tag{2.136d}$$

with $\varepsilon_{ij}, \varepsilon_{ji}, \varepsilon_i \in \{\pm\}$.

Proof Using Lemmas 2.36, 2.38–2.41, the action is seen to be fully supersymmetric if the relations (2.126), (2.129), (2.132)–(2.134) can simultaneously be met. We can combine (2.126) and the second equality of (2.132) to yield

$$ic_i'\widetilde{C}_{i,j}^{*}\widetilde{C}_{i,j} = g_i^2c_i^{*} \qquad \Longrightarrow \qquad \widetilde{C}_{i,j}^{*}\widetilde{C}_{i,j}c_i = -\frac{2g_i^2}{\mathscr{K}_i}c_i^{*},$$

where in the last step we have used the relation (2.11) between c_i and c_i'. Inserting the expression for $\widetilde{C}_{i,j}$ from (2.30) and assuming that $c_i \in i\mathbb{R}$ to ensure the reality of $\widetilde{C}_{i,j}$, we find the first relation of (2.135). The other parameter, $\widetilde{C}_{j,i}$, can be obtained by invoking Remark 2.37 and using (2.132), leading to the second relation of (2.135). Plugging the former result into (2.132) and (2.134) (and invoking (2.11)) gives the second equality in (2.136a) and those of (2.136b) respectively. Combining (2.133), (2.135) and the second equality of (2.136a), we find

$$c_{G_i} = -g_i^{-1}\mathscr{K}_i\mathscr{P}_i c_i. \tag{2.137}$$

The combination of the second equality of (2.129) with (2.137) yields (2.136c). Finally, plugging this result back into (2.137) gives (2.136d).

Note that upon setting $\mathscr{K}_i \equiv 1$ (as should be done in the end) we recover the well known results for both the supersymmetry transformation constants and the parameters of the fermion–sfermion–gaugino interaction.

Third Building Block

The off shell counterparts of the *new interactions* that we get in the four-scalar action, are of the form (c.f. (2.98))

$$\begin{aligned} S_{123,B}[\zeta,\widetilde{\zeta},F] &= \int_M \Big[\operatorname{tr} F_{ij}^*(\beta_{ij,k}\widetilde{\psi}_{ik}\overline{\widetilde{\psi}}_{jk}) + \operatorname{tr}(\widetilde{\psi}_{jk}\overline{\widetilde{\psi}}_{ik}\beta_{ij,k}^*)F_{ij} + \operatorname{tr} F_{ik}^*(\beta_{ik,j}^*\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}) \\ &\quad + \operatorname{tr}(\overline{\widetilde{\psi}}_{jk}\overline{\widetilde{\psi}}_{ij}\beta_{ik,j})F_{ik} + \operatorname{tr}(\beta_{jk,i}\overline{\widetilde{\psi}}_{ij}\widetilde{\psi}_{ik})F_{jk}^* + \operatorname{tr}(\overline{\widetilde{\psi}}_{ik}\widetilde{\psi}_{ij}\beta_{jk,i}^*)F_{jk}\Big] \\ &\equiv \int_M \Big[\operatorname{tr} F_1^*(\beta_1\widetilde{\psi}_2\overline{\widetilde{\psi}}_3) + \operatorname{tr}(\overline{\widetilde{\psi}}_3\overline{\widetilde{\psi}}_1\beta_2)F_2 + \operatorname{tr}(\beta_3\overline{\widetilde{\psi}}_1\widetilde{\psi}_2)F_3^* + h.c.\Big] \\ &\to \int_M \Big[\operatorname{tr} F_1^*(\beta_1'\widetilde{\psi}_2\overline{\widetilde{\psi}}_3) + \operatorname{tr}(\overline{\widetilde{\psi}}_3\overline{\widetilde{\psi}}_1\beta_2')F_2 + \operatorname{tr}(\beta_3'\overline{\widetilde{\psi}}_1\widetilde{\psi}_2)F_3^* + h.c.\Big]. \end{aligned} \tag{2.138}$$

Here we have already scaled the fields according to (2.28) and have written

$$\beta_1' := \mathscr{N}_3^{-1}\beta_1\mathscr{N}_2^{-1}, \quad \beta_2' := \mathscr{N}_3^{-1}\beta_2\mathscr{N}_1^{-1}, \quad \beta_3' := \mathscr{N}_1^{-1}\beta_3\mathscr{N}_2^{-1}. \tag{2.139}$$

We apply the transformations (2.31) and (2.32) to the first term of (2.138) above, giving:

$$\begin{aligned} \operatorname{tr} F_1^*(\beta_1'\widetilde{\psi}_2\overline{\widetilde{\psi}}_3) \to \operatorname{tr}\Big[&\Big(d_1^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_1) + d_{1,i}^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}_1\lambda_{iL}) \\ &- d_{1,j}^*(J_M\varepsilon_L, \gamma^5\lambda_{jL}\overline{\widetilde{\psi}}_1)\Big)(\beta_1'\widetilde{\psi}_2\overline{\widetilde{\psi}}_3) \\ &+ \operatorname{tr} F_1^*\beta_1' c_2(J_M\varepsilon_R, \gamma^5\psi_2)\overline{\widetilde{\psi}}_3 + \operatorname{tr} F_1^*\beta_1'\widetilde{\psi}_2(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)c_3^*\Big], \end{aligned} \tag{2.140}$$

where $c_{1,2,3}$ should not be confused with the transformation parameter c_i of the building blocks of the first type. We have two more terms that can be obtained from the above ones by interchanging the indices 1, 2 and 3:

$$\begin{aligned}\operatorname{tr}(\overline{\widetilde{\psi}}_3\overline{\widetilde{\psi}}_1\beta_2')F_2 \to \operatorname{tr}\Big[(\overline{\widetilde{\psi}}_3\overline{\widetilde{\psi}}_1\beta_2')\Big(&d_2(J_M\varepsilon_L, \partial\!\!\!/_A\psi_2) + d_{2,i}(J_M\varepsilon_L, \gamma^5\lambda_{iL}\widetilde{\psi}_2)_{\mathscr{S}}\\ &- d_{2,k}(J_M\varepsilon_L, \gamma^5\widetilde{\psi}_2\lambda_{kL})\Big)\\ &+ \operatorname{tr} c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)\overline{\widetilde{\psi}}_1\beta_2'F_2 + \operatorname{tr}\overline{\widetilde{\psi}}_3 c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\beta_2'F_2\Big]\end{aligned} \tag{2.141}$$

and

$$\begin{aligned}\operatorname{tr} F_3^*(\beta_3'\overline{\widetilde{\psi}}_1\widetilde{\psi}_2) \to \operatorname{tr}\Big[\Big(&d_3^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_3)_S + d_{3,j}^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}_3\lambda_{jL})_{\mathscr{S}}\\ &- d_{3,k}^*(J_M\varepsilon_L, \gamma^5\lambda_{kL}\overline{\widetilde{\psi}}_3)\Big)(\beta_3'\overline{\widetilde{\psi}}_1\widetilde{\psi}_2)\\ &+ \operatorname{tr} F_3^*\beta_3'c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\widetilde{\psi}_2 + \operatorname{tr} F_3^*\beta_3'\overline{\widetilde{\psi}}_1 c_2(J_M\varepsilon_R, \gamma^5\psi_2)\Big].\end{aligned} \tag{2.142}$$

We can omit the other half of the terms in (2.138) from our considerations.

We introduce the notation

$$\Upsilon_1' := \Upsilon_1\,\mathscr{N}_1^{-1}, \qquad \Upsilon_2' := \mathscr{N}_2^{-1}\Upsilon_2\,, \qquad \Upsilon_3' := \Upsilon_3\,\mathscr{N}_3^{-1}, \tag{2.143}$$

for the scaled version of the parameters. Then for three of the fermionic terms of (2.97), after scaling the fields, we get:

$$\begin{aligned}\langle J_M\overline{\psi}_1, \gamma^5\psi_2\overline{\widetilde{\psi}}_3\Upsilon_3'^*\rangle \to\ &\langle J_M(c_1'^*\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}_1]\varepsilon_L + d_1'^*F_1^*\varepsilon_R), \gamma^5\psi_2\overline{\widetilde{\psi}}_3\Upsilon_3'^*\rangle\\ &+ \langle J_M\overline{\psi}_1, \gamma^5\psi_2 c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)\Upsilon_3'^*\rangle\\ &+ \langle J_M\overline{\psi}_1, \gamma^5(c_2'\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}_2]\varepsilon_L + d_2'F_2\varepsilon_R)\overline{\widetilde{\psi}}_3\Upsilon_3'^*\rangle,\end{aligned} \tag{2.144}$$

$$\begin{aligned}\langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2'^*\widetilde{\psi}_2\overline{\psi}_3\rangle \to\ &\langle J_M(c_1'^*\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}_1]\varepsilon_L + F_1^*d_1'^*\varepsilon_R), \gamma^5\Upsilon_2'^*\widetilde{\psi}_2\overline{\psi}_3\rangle\\ &+ \langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2'^*c_2(J_M\varepsilon_R, \gamma^5\psi_2)\overline{\psi}_3\rangle\\ &+ \langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2'^*\widetilde{\psi}_2(c_3'^*\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}_3]\varepsilon_L + d_3'^*F_3^*\varepsilon_R)\rangle,\end{aligned} \tag{2.145}$$

and

$$\begin{aligned}\langle J_M\overline{\psi}_3, \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1'^*\psi_2\rangle \to\ &\langle J_M(c_3'^*\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}_3]\varepsilon_L + d_3'^*F_3^*\varepsilon_R), \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1'^*\psi_2\rangle\\ &+ \langle J_M\overline{\psi}_3, \gamma^5 c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\Upsilon_1'^*\psi_2\rangle\\ &+ \langle J_M\overline{\psi}_3, \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1'^*(\gamma^5[\partial\!\!\!/_A, c_2'\widetilde{\psi}_2]\varepsilon_L + d_2'F_2\varepsilon_R)\rangle.\end{aligned} \tag{2.146}$$

We can safely omit the other terms of the fermionic action (2.97).

Collecting the terms from (2.140)–(2.146) containing the same variables, we obtain the following groups of terms:

- a group with three fermionic terms:

$$\begin{aligned}
&\langle J_M\overline{\psi}_1, \gamma^5\psi_2 c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)\Upsilon_3'^*\rangle + \langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2'^* c_2(J_M\varepsilon_R, \gamma^5\psi_2)\overline{\psi}_3\rangle \\
&\quad + \langle J_M\overline{\psi}_3, \gamma^5 c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\Upsilon_1'^*\psi_2\rangle \\
&= \langle J_M\overline{\psi}_1, \gamma^5\psi_{2a} c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_{3b})\rangle(\Upsilon_3'^*)_{ba} \\
&\quad + \langle J_M\overline{\psi}_1, \gamma^5 c_2(J_M\varepsilon_R, \gamma^5\psi_{2a})\overline{\psi}_{3b}\rangle(\Upsilon_2'^*)_{ba} \\
&\quad + \langle J_M\overline{\psi}_{3b}, \gamma^5 c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\psi_{2a}\rangle(\Upsilon_1'^*)_{ba},
\end{aligned} \tag{2.147}$$

  consisting of part of the second term of (2.144), the second term of (2.145) and the second term of (2.146). Here we have explicitly written possible family indices and have assumed that it is $\widetilde{\psi}_{ij}$ and ψ_{ij} that lack these.
- Three similar groups containing all terms with the auxiliary fields F_1^*, F_2 and F_3^* respectively:

$$\begin{aligned}
&\langle J_M d_1'^* F_1^*\varepsilon_R, \gamma^5\psi_2\overline{\widetilde{\psi}}_3\Upsilon_3'^*\rangle + \langle J_M d_1'^* F_1^*\varepsilon_R, \gamma^5\Upsilon_2'^*\widetilde{\psi}_2\overline{\psi}_3\rangle \\
&\quad + \int_M \operatorname{tr} F_1^*\beta_1' c_2(J_M\varepsilon_R, \gamma^5\psi_2)\overline{\widetilde{\psi}}_3 + \operatorname{tr} F_1^*\beta_1'\widetilde{\psi}_2 c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_3),
\end{aligned} \tag{2.148a}$$

$$\begin{aligned}
&\langle J_M\overline{\psi}_1, \gamma^5 d_2' F_2\overline{\widetilde{\psi}}_3\Upsilon_3'^*\varepsilon_R\rangle + \langle J_M\overline{\psi}_3, \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1'^* d_2' F_2\varepsilon_R)\rangle \\
&\quad + \int_M \operatorname{tr}\overline{\widetilde{\psi}}_3 c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\beta_2' F_2 + \operatorname{tr} c_3^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)\overline{\widetilde{\psi}}_1\beta_2' F_2
\end{aligned} \tag{2.148b}$$

  and

$$\begin{aligned}
&\langle J_M\overline{\psi}_1, \gamma^5\Upsilon_2'^*\widetilde{\psi}_2 d_3'^* F_3^*\varepsilon_R\rangle + \langle J_M d_3'^* F_3^*\varepsilon_R, \gamma^5\overline{\widetilde{\psi}}_1\Upsilon_1'^*\psi_2\rangle \\
&\quad + \int_M \operatorname{tr} F_3^*\beta_3' c_1^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\widetilde{\psi}_2 + \operatorname{tr} F_3^*\beta_3'\overline{\widetilde{\psi}}_1 c_2(J_M\varepsilon_R, \gamma^5\psi_2),
\end{aligned} \tag{2.148c}$$

  where, for example, the first group comes from parts of the first terms of (2.144) and of (2.145) and from the last two terms of (2.140).
- A group with the gauginos λ_{iL}, λ_{jL}:

$$\begin{aligned}
&\int_M \operatorname{tr}\big[d_{1,i}^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}_1\lambda_{iL}) - d_{1,j}^*(J_M\varepsilon_L, \gamma^5\lambda_{jL}\overline{\widetilde{\psi}}_1)\big](\beta_1'\widetilde{\psi}_2\overline{\widetilde{\psi}}_3) \\
&\qquad + \operatorname{tr}(\overline{\widetilde{\psi}}_3\overline{\widetilde{\psi}}_1\beta_2')\big[d_{2,i}(J_M\varepsilon_L, \gamma^5\lambda_{iL}\widetilde{\psi}_2) - d_{2,k}(J_M\varepsilon_L, \gamma^5\widetilde{\psi}_2\lambda_{kL})\big] \\
&\qquad + \operatorname{tr}\big[d_{3,j}^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}_3\lambda_{jL}) - d_{3,k}^*(J_M\varepsilon_L, \gamma^5\lambda_{kL}\overline{\widetilde{\psi}}_3)\big](\beta_3'\overline{\widetilde{\psi}}_1\widetilde{\psi}_2),
\end{aligned} \tag{2.149}$$

  coming from the second and third terms of (2.140)–(2.142) respectively.

- And finally three groups of terms containing the Dirac operator $\partial\!\!\!/_A$:

$$\langle J_M\overline{\psi}_1, c_2'[\partial\!\!\!/_A, \widetilde{\psi}_2]\overline{\overline{\psi}}_3\Upsilon_3'^*\varepsilon_L\rangle + \langle J_M\overline{\psi}_1, \Upsilon_2'^*\widetilde{\psi}_2 c_3'^*[\partial\!\!\!/_A, \overline{\overline{\psi}}_3]\varepsilon_L\rangle + \int_M \operatorname{tr} d_1^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_1)\beta_1'\widetilde{\psi}_2\overline{\overline{\psi}}_3, \tag{2.150a}$$

$$\langle J_M c_1'^*[\partial\!\!\!/_A, \overline{\overline{\psi}}_1]\varepsilon_L, \psi_2\overline{\overline{\psi}}_3\Upsilon_3'^*\rangle + \langle J_M c_3'^*[\partial\!\!\!/_A, \overline{\overline{\psi}}_3]\varepsilon_L, \overline{\overline{\psi}}_1\Upsilon_1'^*\psi_2\rangle + \int_M \operatorname{tr} \overline{\overline{\psi}}_3\overline{\overline{\psi}}_1\beta_2' d_2(J_M\varepsilon_L, \partial\!\!\!/_A\psi_2), \tag{2.150b}$$

and

$$\langle J_M c_1'^*[\partial\!\!\!/_A, \overline{\overline{\psi}}_1]\varepsilon_L, \Upsilon_2'^*\widetilde{\psi}_2\overline{\psi}_3\rangle + \langle J_M\overline{\psi}_3, \overline{\overline{\psi}}_1\Upsilon_1'^* c_2'[\partial\!\!\!/_A, \widetilde{\psi}_2]\varepsilon_L\rangle + \int_M \operatorname{tr} d_3^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_3)\beta_3'\overline{\overline{\psi}}_1\widetilde{\psi}_2, \tag{2.150c}$$

coming from parts of the first and third terms of (2.144)–(2.146) and from the first terms of (2.140)–(2.142).

Lemma 2.43 *The group* (2.147) *vanishes, provided that*

$$c_3^*\Upsilon_3'^* = c_2\Upsilon_2'^* = c_1^*\Upsilon_1'^* \tag{2.151}$$

Proof Since the terms contain four fermions, we must employ a Fierz transformation (Appendix section 'Fierz Transformations'). Point-wise, we have for the first term of (2.147) (omitting its pre-factor for now):

$$\begin{aligned}
&(J_M\overline{\psi}_1, \gamma^5\psi_2)(J_M\varepsilon_R, \gamma^5\overline{\psi}_3)\\
&= -\frac{C_{40}}{4}(J_M\overline{\psi}_1, \overline{\psi}_3)(J_M\varepsilon_R, \psi_2) - \frac{C_{41}}{4}(J_M\overline{\psi}_1, \gamma^\mu\overline{\psi}_3)(J_M\varepsilon_R, \gamma_\mu\psi_2)\\
&\quad - \frac{C_{42}}{4}(J_M\overline{\psi}_1, \gamma^\mu\gamma^\nu\overline{\psi}_3)(J_M\varepsilon_R, \gamma_\mu\gamma_\nu\psi_2) - \frac{C_{43}}{4}(J_M\overline{\psi}_1, \gamma^\mu\gamma^5\overline{\psi}_3)(J_M\varepsilon_R, \gamma_\mu\gamma^5\psi_2)\\
&\quad - \frac{C_{44}}{4}(J_M\overline{\psi}_1, \gamma^5\overline{\psi}_3)(J_M\varepsilon_R, \gamma^5\psi_2)\\
&= -\frac{1}{2}(J_M\overline{\psi}_1, \gamma^5\overline{\psi}_3)(J_M\varepsilon_R, \gamma^5\psi_2) + \frac{1}{4}(J_M\overline{\psi}_1, \gamma^\mu\gamma^\nu\overline{\psi}_3)(J_M\varepsilon_R, \gamma_\mu\gamma_\nu\psi_2),
\end{aligned}$$

where we have used that $C_{40} = C_{44} = -C_{42} = 1$ and that all fermions are of the same chirality. (Note that the sum in the last term runs over $\mu < \nu$, see Example 2.56.) Similarly, we can take the third term of (2.147), use the symmetries of the inner product for both terms, and apply the same transformation. This yields

$$
\begin{aligned}
&(J_M\overline{\psi}_3, \gamma^5(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)\psi_2)\\
&\quad= (J_M\psi_2, \gamma^5\overline{\psi}_3)(J_M\overline{\psi}_1, \gamma^5\varepsilon_R)\\
&\quad= -\frac{1}{2}(J_M\psi_2, \gamma^5\varepsilon_R)(J_M\overline{\psi}_1, \gamma^5\overline{\psi}_3) + \frac{1}{4}(J_M\psi_2, \gamma^\mu\gamma^\nu\varepsilon_R)(J_M\overline{\psi}_1, \gamma_\mu\gamma_\nu\overline{\psi}_3)\\
&\quad= -\frac{1}{2}(J_M\varepsilon_R, \gamma^5\psi_2)(J_M\overline{\psi}_1, \gamma^5\overline{\psi}_3) - \frac{1}{4}(J_M\varepsilon_R, \gamma^\mu\gamma^\nu\psi_2)(J_M\overline{\psi}_1, \gamma_\mu\gamma_\nu\overline{\psi}_3),
\end{aligned}
\tag{2.152}
$$

where we have used the symmetries (2.164) for the second inner product in each of the two terms of (2.152). We can add the two results, yielding

$$
\begin{aligned}
&(J_M\overline{\psi}_1, \gamma^5\psi_2)(J_M\varepsilon_R, \gamma^5\overline{\psi}_3 c_3^*\Upsilon_3'^*) + (J_M\overline{\psi}_3, \gamma^5(J_M\varepsilon_R, \gamma^5\overline{\psi}_1)c_1^*\Upsilon_1'^*\psi_2)\\
&\quad= -\frac{1}{2}(c_1^*\Upsilon_1'^* + c_3^*\Upsilon_3'^*)_{ba}(J_M\overline{\psi}_1, \gamma^5\overline{\psi}_{3b})(J_M\varepsilon_R, \gamma^5\psi_{2a})\\
&\qquad+ \frac{1}{4}(c_3^*\Upsilon_3'^* - c_1^*\Upsilon_1'^*)_{ba}(J_M\varepsilon_R, \gamma^\mu\gamma^\nu\psi_{2a})(J_M\overline{\psi}_1, \gamma^\mu\gamma^\nu\overline{\psi}_{3b}).
\end{aligned}
$$

When $c_3^*\Upsilon_3'^* = c_1^*\Upsilon_1'^* = c_2\Upsilon_2'^*$, this result is seen to cancel the remaining term in (2.147).

Lemma 2.44 *The groups of terms (2.148) vanish, provided that*

$$
\begin{aligned}
&c_2\beta_1' = -d_1'^*\Upsilon_3'^*, && c_3^*\beta_1' = -d_1'^*\Upsilon_2'^*, && c_3^*\beta_2' = -d_2'\Upsilon_1'^*,\\
&c_1^*\beta_2' = -d_2'\Upsilon_3'^*, && c_1^*\beta_3' = -d_3'^*\Upsilon_2'^*, && c_2\beta_3' = -d_3'^*\Upsilon_1'^*.
\end{aligned}
\tag{2.153}
$$

Proof This can readily be seen upon using Lemma 2.51, the cyclicity of the trace and Lemma 2.53.

Lemma 2.45 *The group of terms* (2.149) *vanishes, provided that*

$$
d_{1,i}^*\beta_1' = -d_{2,i}\beta_2', \qquad d_{1,j}^*\beta_1' = d_{3,j}^*\beta_3', \qquad d_{2,k}\beta_2' = -d_{3,k}^*\beta_3'. \tag{2.154}
$$

Proof This can readily be seen upon using the cyclicity of the trace and Lemma 2.53.

Lemma 2.46 *The three groups of terms (2.150) vanish, provided that*

$$
\Upsilon_3'^*c_2' = c_3'^*\Upsilon_2'^* = -d_1^*\beta_1', \quad \Upsilon_3'^*c_1'^* = c_3'^*\Upsilon_1'^* = -\beta_2'd_2, \quad c_1'^*\Upsilon_2'^* = \Upsilon_1'^*c_2' = -d_3^*\beta_3'. \tag{2.155}
$$

Proof This can be checked quite easily using the symmetry (2.164), the Leibniz rule for $\partial\!\!\!/_A$ and the fact that it is self-adjoint, that $\varepsilon_{L,R}$ vanish covariantly and Lemmas 2.53 and 2.54.

Combining the above lemmas, we get:

Proposition 2.47 *The extra action as a result of adding a building block $\mathscr{B}_{ijk}$ of the third type is supersymmetric if and only if the coefficients $\Upsilon_i{}^j$, $\Upsilon_i{}^k$ and $\Upsilon_j{}^k$ are related to each other via*

$$\Upsilon_i{}^j C_{iij}^{-1} = -(C_{iik}^*)^{-1}\Upsilon_i{}^k, \quad \Upsilon_i{}^j C_{ijj}^{-1} = -\Upsilon_j{}^k C_{jjk}^{-1}, \quad (C_{ikk}^*)^{-1}\Upsilon_i{}^k = -\Upsilon_j{}^k C_{jkk}^{-1}, \tag{2.156}$$

the constants of the transformations satisfy

$$|d_1|^2 = |d_2|^2 = |d_3|^2 = |c_1|^2 = |c_2|^2 = |c_3|^2 \tag{2.157}$$

and the coefficients β'_{ij} are given by

$$\beta_1'^*\beta_1' = \beta_2'^*\beta_2' = \beta_3'^*\beta_3' = \Upsilon_1'\Upsilon_1'^* = \Upsilon_2'\Upsilon_2'^* = \Upsilon_3'\Upsilon_3'^*. \tag{2.158}$$

Proof First of all, we plug the intermediate result (2.126) for $\widetilde{C}_{i,j}$ as given by (2.30) (but keeping in mind the results of Remark 2.37) into the Hermitian conjugate of the result (2.151) such that pairwise the same combination $c_i g_i$ appears on both sides. This yields

$$\Upsilon_i{}^j(-2c_i g_i)C_{iij}^{-1} = (-2c_i g_i C_{iik}^{-1})^*\Upsilon_i{}^k, \quad \Upsilon_i{}^j(2c_j g_j C_{ijj}^{-1}) = \Upsilon_j{}^k(-2c_j g_j)C_{jjk}^{-1},$$
$$(2c_k g_k C_{ikk}^{-1})^*\Upsilon_i{}^k = \Upsilon_j{}^k(2c_k g_k)C_{jkk}^{-1}.$$

Using that the $c_{i,j,k}$ are purely imaginary (cf. Theorem 2.42), we obtain (2.156). Secondly, comparing the relations (2.153) with (2.155) gives

$$d_1 d_1' = (c_2 c_2')^* = c_3 c_3', \quad (d_2 d_2')^* = c_1 c_1' = c_3 c_3', \quad d_3 d_3' = c_1 c_1' = (c_2 c_2')^*.$$

Using the relations (2.136a) and (2.136b) between the constraints, (2.157) follows. Plugging the relations from (2.157) into those of (2.153), we obtain

$$\beta_1'^*\beta_1' = \Upsilon_3'\Upsilon_3'^* = \Upsilon_2'\Upsilon_2'^*, \quad \beta_2'^*\beta_2' = \Upsilon_1'\Upsilon_1'^* = \Upsilon_3'\Upsilon_3'^*, \quad \beta_3'^*\beta_3' = \Upsilon_2'\Upsilon_2'^* = \Upsilon_1'\Upsilon_1'^*,$$

from which (2.158) directly follows.

N.B. Using (2.139) and (2.143) we can phrase the identities (2.158) in terms of the unscaled quantities $\beta_{1,2,3}$ and $\Upsilon_{1,2,3}$ as

$$\mathscr{N}_3^{-1}\beta_2 = \beta_3\mathscr{N}_2^{-1} = \Upsilon_1{}^*, \quad \mathscr{N}_3^{-1}\beta_1 = \beta_3\mathscr{N}_1^{-1} = \Upsilon_2{}^*, \quad \mathscr{N}_1^{-1}\beta_2 = \beta_1\mathscr{N}_2^{-1} = \Upsilon_3{}^*,$$

where we have used that $\mathscr{N}_1 \in \mathbb{R}$ since $\widetilde{\psi}_1$ has $R = 1$ (and consequently multiplicity 1).

Fourth Building Block

Phrased in terms of the auxiliary field $F_{11'} =: F$, a building block of the fourth type induces the following action:

$$\frac{1}{2}\langle J_M\psi, \gamma^5\Upsilon_{\mathrm{m}}^*\psi\rangle + \frac{1}{2}\langle J_M\overline{\psi}, \gamma^5\Upsilon_{\mathrm{m}}\overline{\psi}\rangle - \mathrm{tr}\left(F^*\gamma\overline{\widetilde{\psi}} + h.c.\right).$$

Here we have written $\psi := \psi_{11'L}$, $\overline{\psi} := \overline{\psi}_{11'R}$ and $\widetilde{\psi} := \widetilde{\psi}_{11'}$ for conciseness. Transforming the fields that appear in the above action, we have the following.

- From the first term:

$$\frac{1}{2}\langle J_M(c^*\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R + d^*F\varepsilon_L), \gamma^5\Upsilon_{\mathrm{m}}^*\psi\rangle + \frac{1}{2}\langle J_M\psi, \gamma^5\Upsilon_{\mathrm{m}}^*(c^*\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R + d^*F\varepsilon_L)\rangle.$$

- From the second term:

$$\frac{1}{2}\langle J_M(c\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L + dF^*\varepsilon_R), \gamma^5\Upsilon_{\mathrm{m}}\overline{\psi}\rangle + \frac{1}{2}\langle J_M\overline{\psi}, \gamma^5\Upsilon_{\mathrm{m}}(c\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L + dF^*\varepsilon_R)\rangle.$$

- From the terms with the auxiliary fields:

$$-\,\mathrm{tr}\left[d^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}) + d'^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\lambda_{1L}) - d''^*(J_M\varepsilon_L, \gamma^5\lambda_{1'L}\overline{\widetilde{\psi}})\right]\gamma\overline{\widetilde{\psi}}$$
$$-\, c^*\,\mathrm{tr}\, F^*\gamma(J_M\varepsilon_R, \gamma^5\overline{\psi})$$

  and

$$-\,\mathrm{tr}\,\widetilde{\psi}\gamma^*\left[d(J_M\varepsilon_R, \partial\!\!\!/_A\psi) + d'(J_M\varepsilon_R, \gamma^5\lambda_{1R}\widetilde{\psi}) - d''(J_M\varepsilon_R, \gamma^5\widetilde{\psi}\lambda_{1'R})\right]$$
$$-\, c\,\mathrm{tr}(J_M\varepsilon_L, \gamma^5\psi)\gamma^*F.$$

Here we have written $c := c_{ij}$, $d := d_{ij}$ (where we have expressed c'^*_{ij} as c_{ij} and d'^*_{ij} as d_{ij} using (2.136a) and (2.136b)) and $d' := d_{11',1}$, $d'' := d_{11',1'}$. We group all terms according to the fields that appear in them, leaving essentially the following three.

- The group consisting of all terms with F^* and $\overline{\psi}$:

$$\frac{1}{2}\langle J_M dF^*\varepsilon_R, \gamma^5\Upsilon_{\mathrm{m}}\overline{\psi}\rangle + \frac{1}{2}\langle J_M\overline{\psi}, \gamma^5\Upsilon_{\mathrm{m}}dF^*\varepsilon_R\rangle - c^*\int_M \mathrm{tr}\, F^*\gamma(J_M\varepsilon_R, \gamma^5\overline{\psi})$$
$$= \langle J_M F^*\varepsilon_R, \gamma^5(d\Upsilon_{\mathrm{m}} - c^*\gamma)\overline{\psi}\rangle,$$

  where we have used the symmetry of the inner product from Lemmas 2.51 and 2.53. This group thus only vanishes if

$$d\Upsilon_{\mathrm{m}} = c^*\gamma. \tag{2.159}$$

There is also a group of terms featuring F and ψ, but this is of the same form as the one above.

- A group of three terms with ψ and $\widetilde{\psi}$:

$$\frac{1}{2}\langle J_M c^*\gamma^5[\not{\partial}_A, \widetilde{\psi}]\varepsilon_R, \gamma^5\Upsilon_{\mathrm{m}}^*\psi\rangle + \frac{1}{2}\langle J_M\psi, \gamma^5\Upsilon_{\mathrm{m}}^* c^*\gamma^5[\not{\partial}_A, \widetilde{\psi}]\varepsilon_R\rangle$$
$$- \int_M \mathrm{tr}\,\widetilde{\psi}\gamma^* d(J_M\varepsilon_R, \not{\partial}_A\psi) = \langle J_M c^*\gamma^5[\not{\partial}_A, \widetilde{\psi}]\varepsilon_R, \gamma^5\Upsilon_{\mathrm{m}}^*\psi\rangle - \langle J_M\widetilde{\psi}\varepsilon_R, \not{\partial}_A\gamma^* d\psi\rangle,$$

where also here we have used Lemmas 2.51 and 2.53. Using the self-adjointness of $\not{\partial}_A$ this is only seen to vanish if

$$c^*\Upsilon_{\mathrm{m}}^* = \gamma^* d. \tag{2.160}$$

There is also a group of terms featuring $\overline{\psi}$ and $\overline{\widetilde{\psi}}$ but these are seen to be of the same form as the terms above.

- Finally, there are terms that feature gauginos:

$$- \int_M \Big[\mathrm{tr}\, d'^*(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\lambda_{1L}) - d''^*(J_M\varepsilon_L, \gamma^5\lambda_{1'L}\overline{\widetilde{\psi}})\Big]\gamma\overline{\widetilde{\psi}}$$
$$- \int_M \mathrm{tr}\,\widetilde{\psi}\gamma^*\Big[d'(J_M\varepsilon_R, \gamma^5\lambda_{1R}\widetilde{\psi}) - d''(J_M\varepsilon_R, \gamma^5\widetilde{\psi}\lambda_{1'R})\Big].$$

This expression is immediately seen to vanish when

$$d'^*\lambda_{1L} = d''^*\lambda_{1'L}, \qquad d'\lambda_{1R} = d''\lambda_{1'R}.$$

For this to happen we need that the gauginos are associated to each other and that $d' = d''$.

Combining the demands (2.159) and (2.160) we obtain

$$\Upsilon_{\mathrm{m}}^*\Upsilon_{\mathrm{m}} = \frac{|c|^2}{|d|^2}\gamma^*\gamma = \frac{|d|^2}{|c|^2}\gamma^*\gamma$$

i.e.

$$\Upsilon_{\mathrm{m}}^*\Upsilon_{\mathrm{m}} = \gamma^*\gamma, \qquad |d|^2 = |c|^2.$$

Fifth Building Block

We transform the fields that appear in the action according to (2.31) and (2.32). We suppress the indices i and j as much as possible, writing $c \equiv c_{ij}, d \equiv d_{ij}$ for the transformation coefficients (2.32) of the building block $\mathscr{B}^+_{ij}$ of the second type. We eliminate c'_{ij} and d'_{ij} in these transformations using the first relations of (2.136a) and (2.136b) so that we can write c', d' for those associated to $\mathscr{B}^-_{ij}$.

The first fermionic term of (2.78) transforms as

$$\begin{aligned}\langle J_M\overline{\psi}_R, \gamma^5\mu\psi'_R\rangle \to \langle J_M(\gamma^5 c[\partial\!\!\!/_A, \overline{\widetilde{\psi}}]\varepsilon_L + dF^*\varepsilon_R), \gamma^5\mu\psi'_R\rangle \\ + \langle J_M\overline{\psi}_R, \gamma^5\mu(c'^*\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}']\varepsilon_L + d'^*F'\varepsilon_R)\rangle.\end{aligned}$$

The second fermionic term of (2.78) transforms as

$$\begin{aligned}\langle J_M\overline{\psi}'_L, \gamma^5\mu^*\psi_L\rangle \to \langle J_M(c'\gamma^5[\partial\!\!\!/_A, \overline{\widetilde{\psi}}']\varepsilon_R + d'F'^*\varepsilon_L), \gamma^5\mu^*\psi_L\rangle \\ + \langle J_M\overline{\psi}'_L, \gamma^5\mu^*(c^*\gamma^5[\partial\!\!\!/_A, \widetilde{\psi}]\varepsilon_R + d^*F\varepsilon_L)\rangle.\end{aligned}$$

The four terms in (2.79) transform as

$$\begin{aligned}-\int_M \operatorname{tr} F'^*\delta\widetilde{\psi} \to -\int_M \Big(&\operatorname{tr}\big[d'^*(J_M\varepsilon_R, \partial\!\!\!/_A\overline{\psi}'_L) + d'^*_{ij,i}(J_M\varepsilon_R, \gamma^5\overline{\widetilde{\psi}}'\lambda_{iR}) \\ &- d'^*_{ij,j}(J_M\varepsilon_R, \gamma^5\lambda_{jR}\overline{\widetilde{\psi}}')\big]\delta\widetilde{\psi} + \operatorname{tr} F'^*\delta c(J_M\varepsilon_L, \gamma^5\psi_L)\Big), \\ -\int_M \operatorname{tr} F^*\delta'\widetilde{\psi}' \to -\int_M \Big(&\operatorname{tr}\big[d^*(J_M\varepsilon_L, \partial\!\!\!/_A\overline{\psi}_R) + d^*_{ij,i}(J_M\varepsilon_L, \gamma^5\overline{\widetilde{\psi}}\lambda_{iL}) \\ &- d^*_{ij,j}(J_M\varepsilon_L, \gamma^5\lambda_{jL}\overline{\widetilde{\psi}})\big]\delta'\widetilde{\psi}' + \operatorname{tr} F^*\delta' c'(J_M\varepsilon_R, \gamma^5\psi'_R)\Big), \\ -\int_M \operatorname{tr}\overline{\widetilde{\psi}}\delta^*F' \to -\int_M \Big(&\operatorname{tr} c^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)\delta^*F' + \operatorname{tr}\overline{\widetilde{\psi}}\delta^*\big[d'(J_M\varepsilon_L, \partial\!\!\!/_A\psi'_R) \\ &+ d'_{ij,i}(J_M\varepsilon_L, \gamma^5\lambda_{iL}\widetilde{\psi}') - d'_{ij,j}(J_M\varepsilon_L, \gamma^5\widetilde{\psi}'\lambda_{jL})\big]\Big)\end{aligned}$$

and

$$\begin{aligned}-\int_M \operatorname{tr}\overline{\widetilde{\psi}}'\delta'^*F \to -\int_M \Big(&\operatorname{tr} c'^*(J_M\varepsilon_L, \gamma^5\overline{\psi}'_L)\delta'^*F + \operatorname{tr}\overline{\widetilde{\psi}}'\delta'^*\big[d(J_M\varepsilon_R, \partial\!\!\!/_A\psi_L) \\ &+ d_{ij,i}(J_M\varepsilon_R, \gamma^5\lambda_{iR}\widetilde{\psi}) - d_{ij,j}(J_M\varepsilon_R, \gamma^5\widetilde{\psi}\lambda_{jR})\big]\Big).\end{aligned}$$

We group all terms that feature the same fields, which gives

- a group with F and F':

$$d'^*\langle J_M\overline{\psi}_R, \gamma^5\mu F'\varepsilon_R\rangle + d^*\langle J_M\overline{\psi}'_L, \gamma^5\mu^* F\varepsilon_L\rangle \\ - \int_M \Big(\operatorname{tr} c^*(J_M\varepsilon_R, \gamma^5\overline{\psi}_R)\delta^* F' + \operatorname{tr} c'^*(J_M\varepsilon_L, \gamma^5\overline{\psi}'_L)\delta'^* F\Big).$$

Using Lemmas 2.54 and 2.53 and employing the symmetries of the inner product (Lemma 2.51), this is seen to equal

$$d'^*\langle J_M\overline{\psi}_R, \gamma^5\mu F'\varepsilon_R\rangle + d^*\langle J_M\overline{\psi}'_L, \gamma^5\mu^* F\varepsilon_L\rangle \\ - c^*\langle J_M\overline{\psi}_R, \gamma^5\delta^* F'\varepsilon_R\rangle - c'^*\langle J_M\overline{\psi}'_L, \gamma^5\delta'^* F\varepsilon_L\rangle \\ = \langle J_M\overline{\psi}_R, \gamma^5\big[d'^*\mu - c^*\delta^*\big]F'\varepsilon_R\rangle + \langle J_M\overline{\psi}'_L, \gamma^5\big[d^*\mu^* - c'^*\delta'^*\big]F\varepsilon_L\rangle.$$

This only vanishes if

$$d'^*\mu = c^*\delta^*, \qquad d^*\mu^* = c'^*\delta'^*. \tag{2.161}$$

- a group with F^* and F'^*, that vanishes automatically if and only if (2.161) is satisfied.
- a group featuring ψ'_R and ψ_L:

$$\langle J_M c[\partial\!\!\!/_A, \overline{\overline{\psi}}]\varepsilon_L, \mu\psi'_R\rangle + c'\langle J_M[\partial\!\!\!/_A, \overline{\overline{\psi}}']\varepsilon_R, \mu^*\psi_L\rangle \\ - \int_M \Big(\operatorname{tr}\overline{\overline{\psi}}\delta^* d'(J_M\varepsilon_L, \partial\!\!\!/_A\psi'_R) + \operatorname{tr}\overline{\overline{\psi}}'\delta'^* d(J_M\varepsilon_R, \partial\!\!\!/_A\psi_L)\Big).$$

Employing Lemmas 2.53 and 2.54 this is seen to equal

$$\langle J_M c[\partial\!\!\!/_A, \overline{\overline{\psi}}]\varepsilon_L, \mu\psi'_R\rangle + c'\langle J_M[\partial\!\!\!/_A, \overline{\overline{\psi}}']\varepsilon_R, \mu^*\psi_L\rangle \\ - d'\langle J_M\overline{\overline{\psi}}\delta^*\varepsilon_L, \partial\!\!\!/_A\psi'_R\rangle - d\langle J_M\overline{\overline{\psi}}'\delta'^*\varepsilon_R, \partial\!\!\!/_A\psi_L\rangle.$$

Using the self-adjointness of $\partial\!\!\!/_A$, that $[\mu, \partial\!\!\!/_A] = 0$ and the symmetries of the inner product, this reads

$$\langle J_M\overline{\overline{\psi}}\varepsilon_L, \big[c\mu - d'\delta^*\big]\,\partial\!\!\!/_A\psi'_R\rangle + \langle J_M\overline{\overline{\psi}}'\varepsilon_R, \big[c'\mu^* - d\delta'^*\big]\,\partial\!\!\!/_A\psi_L\rangle.$$

We thus require that

$$c\mu = d'\delta^*, \qquad c'\mu^* = d\delta'^* \tag{2.162}$$

for this to vanish.
- a group with $\overline{\psi}_R$ and $\overline{\psi}'_L$ that vanishes if and only if (2.162) is satisfied.
- a group with the left-handed gauginos:

$$
\begin{aligned}
&- \int_M \Big(\mathrm{tr}\, \big[d^*_{ij,i} (J_M \varepsilon_L, \gamma^5 \overline{\widetilde{\psi}} \lambda_{iL}) - d^*_{ij,j} (J_M \varepsilon_L, \gamma^5 \lambda_{jL} \overline{\widetilde{\psi}}) \big] \delta' \widetilde{\psi}' \\
&\qquad + \mathrm{tr}\, \overline{\widetilde{\psi}} \delta^* \big[d'_{ij,i} (J_M \varepsilon_L, \gamma^5 \lambda_{iL} \widetilde{\psi}') - d'_{ij,j} (J_M \varepsilon_L, \gamma^5 \widetilde{\psi}' \lambda_{jL}) \big] \Big) \\
&= - \langle J_M \big(d^*_{ij,i} \delta' \widetilde{\psi}' \overline{\widetilde{\psi}} + d'_{ij,i} \widetilde{\psi}' \overline{\widetilde{\psi}} \delta^* \big) \varepsilon_L, \gamma^5 \lambda_{iL} \rangle \\
&\quad + \langle J_M \big(d^*_{ij,j} \overline{\widetilde{\psi}} \delta' \widetilde{\psi}' + d'_{ij,j} \overline{\widetilde{\psi}} \delta^* \widetilde{\psi}' \big) \varepsilon_L, \gamma^5 \lambda_{jL} \rangle,
\end{aligned}
$$

where we have used Lemmas 2.54 and 2.53. For this to vanish, we require that

$$
d^*_{ij,i} \delta' = - d'_{ij,i} \delta^*, \qquad d^*_{ij,j} \delta' = - d'_{ij,j} \delta^*.
$$

Inserting (2.162) above this is equivalent to

$$
d^*_{ij,i} \frac{c'^*}{d^*} = - d'_{ij,i} \frac{c}{d'}, \qquad d^*_{ij,j} \frac{c'^*}{d^*} = - d'_{ij,j} \frac{c}{d'}.
$$

- A group with the right-handed gauginos

$$
\begin{aligned}
&- \int_M \mathrm{tr}\, \big[d'^*_{ij,i} (J_M \varepsilon_R, \gamma^5 \overline{\widetilde{\psi}}' \lambda_{iR}) - d'^*_{ij,j} (J_M \varepsilon_R, \gamma^5 \lambda_{jR} \overline{\widetilde{\psi}}') \big] \delta \widetilde{\psi} \\
&\qquad - \int_M \mathrm{tr}\, \overline{\widetilde{\psi}}' \delta'^* \big[d_{ij,i} (J_M \varepsilon_R, \gamma^5 \lambda_{iR} \widetilde{\psi}) - d_{ij,j} (J_M \varepsilon_R, \gamma^5 \widetilde{\psi} \lambda_{jR}) \big] \\
&= - \langle J_M \big(d'^*_{ij,i} \delta \widetilde{\psi} \overline{\widetilde{\psi}}' + d_{ij,i} \widetilde{\psi} \overline{\widetilde{\psi}}' \delta'^* \big) \varepsilon_R, \gamma^5 \lambda_{iR} \rangle \\
&\quad + \langle J_M \big(d'^*_{ij,j} \overline{\widetilde{\psi}}' \delta \widetilde{\psi} + d_{ij,j} \overline{\widetilde{\psi}}' \delta'^* \widetilde{\psi} \big) \varepsilon_R, \gamma^5 \lambda_{jR} \rangle,
\end{aligned}
$$

which vanishes iff

$$
d'^*_{ij,i} \delta = - d_{ij,i} \delta'^*, \qquad d'^*_{ij,j} \delta = - d_{ij,j} \delta'^*.
$$

Combining all relations, above, we require that

$$
|c|^2 = |d'|^2, \qquad |c'|^2 = |d|^2, \qquad |d_{ij,i}|^2 = |d'_{ij,i}|^2, \qquad |d_{ij,j}|^2 = |d'_{ij,j}|^2,
$$

for the transformation constants and

$$
\delta \delta^* = \mu^* \mu, \qquad \delta' \delta'^* = \mu \mu^*
$$

for the parameters in the off shell action.

Appendix 3. Auxiliary Lemmas and Identities

In this section we provide some auxiliary lemmas and identities that are used in and throughout the previous proofs. The following two results can be found in any textbook on spin geometry, such as [12].

Lemma 2.48 *For the spin-connection* $\nabla^S : \Gamma(S) \to \mathscr{A}^1(M) \otimes_{C^\infty(M)} \Gamma(S)$ *on a flat manifold we have:*

$$[\nabla^S, \gamma^\mu] = 0. \tag{2.163}$$

Lemma 2.49 *Let* $\not{\partial}_A = -ic \circ (\nabla^S + \mathbb{A})$ *and* $D_\mu = (\nabla^S + \mathbb{A})_\mu$. *For a flat manifold, we have locally:*

$$\not{\partial}_A^2 + D_\mu D^\mu = -\frac{1}{2}\gamma^\mu\gamma^\nu\mathbb{F}_{\mu\nu}.$$

Corollary 2.50 *By applying the previous result, we have for* $\widetilde{\zeta}_{ik} \in C^\infty(M, \mathbf{N_i} \otimes \mathbf{N_k})$, $\varepsilon \in L^2(M, S)$

$$(\not{\partial}_A[\not{\partial}_A, \widetilde{\zeta}_{ik}]\varepsilon + D_\mu[D^\mu, \widetilde{\zeta}_{ik}])\varepsilon = \frac{1}{2}[\mathbb{F}, \widetilde{\zeta}_{ik}]\varepsilon + [D^\mu, \widetilde{\zeta}_{ik}]\nabla^S_\mu\varepsilon + [\not{\partial}_A, \widetilde{\zeta}_{ik}]\not{\partial}\varepsilon,$$

where the term with R vanished due to the commutator.

Lemma 2.51 *Let M be a four-dimensional Riemannian spin manifold and* $\langle\, .\, ,\, .\, \rangle : L^2(S) \times L^2(S) \to \mathbb{C}$ *the inner product on sections of the spinor bundle. For* $\mathscr{P}$ *a basis element of* $\Gamma(\mathbb{C}l(M))$, *we have the following identities:*

$$\langle J_M\zeta_1, \mathscr{P}\zeta_2\rangle = \pi_{\mathscr{P}}\langle J_M\zeta_2, \mathscr{P}\zeta_1\rangle, \qquad \pi_{\mathscr{P}} \in \{\pm\},$$

for any $\zeta_{1,2}$, *the Grassmann variables corresponding to* $\zeta'_{1,2} \in L^2(S)$. *The signs* $\pi_{\mathscr{P}}$ *are given by*

$$\begin{aligned} &\pi_{\mathrm{id}} = 1, && \pi_{\gamma^\mu} = -1, && \pi_{\gamma^\mu\gamma^\nu} = -1 \quad (\mu < \nu), \\ &\pi_{\gamma^\mu\gamma^5} = 1, && \pi_{\gamma^5} = 1. \end{aligned} \tag{2.164}$$

Proof Using that $J_M^2 = -1$ and $\langle J_M\zeta'_1, J_M\zeta'_2\rangle = \langle\zeta'_2, \zeta'_1\rangle$, we have

$$\langle J_M\zeta'_1, \mathscr{P}\zeta'_2\rangle = -\langle J_M\zeta'_1, J_M^2\mathscr{P}\zeta'_2\rangle = -\langle J_M\mathscr{P}\zeta'_2, \zeta'_1\rangle.$$

When considering Grassmann variables, we obtain an extra minus sign (see the discussion in [10, Sect. 4.2.6]). From $J_M\gamma^\mu = -\gamma^\mu J_M$, $(\gamma^\mu)^* = \gamma^\mu$ and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$, we obtain the result.

Corollary 2.52 *Similarly ([7, Sect. 4]) we find by using that* $\partial\!\!\!/_M^* = \partial\!\!\!/_M$ *and* $J_M \partial\!\!\!/_M = \partial\!\!\!/_M J_M$*, that*

$$\langle J_M \zeta_1, \partial\!\!\!/_M \zeta_2 \rangle = \langle J_M \zeta_2, \partial\!\!\!/_M \zeta_1 \rangle \tag{2.165}$$

for the Grassmann variables corresponding to any two $\zeta'_{1,2} \in L^2(S)$.

Lemma 2.53 *For any* $\widetilde{\psi} \in C^\infty(M, \mathbf{N}_i \otimes \mathbf{N}_j^o)$, $\psi \in L^2(S \otimes \mathbf{N}_j \otimes \mathbf{N}_i^o)$ *and* $\varepsilon \in L^2(S)$ *we have*

$$\mathrm{tr}_{N_i}\, \widetilde{\psi}(J_M \varepsilon, \psi)_{\mathscr{S}} = (J\widetilde{\psi}\varepsilon, \psi)_{\mathscr{H}}.$$

Proof This can be seen easily by writing out the elements in full detail:

$$\widetilde{\zeta} = f \otimes e \otimes \bar{e}', \qquad \psi = \zeta \otimes \eta \otimes \bar{\eta}', \qquad f \in C^\infty(M, \mathbb{C}), \zeta \in L^2(S).$$

Lemma 2.54 *Let* $\psi_1 \in L^2(S \otimes \mathbf{N}_i \otimes \mathbf{N}_j^o)$, $\psi_2 \in L^2(S \otimes \mathbf{N}_k \otimes \mathbf{N}_i^o)$, $\overline{\psi}_2 \in L^2(S \otimes \mathbf{N}_j \otimes \mathbf{N}_k^o)$, $\widetilde{\psi} \in C^\infty(M, \mathbf{N}_j \otimes \mathbf{N}_k^o)$ *and* $\widetilde{\psi}' \in C^\infty(M, \mathbf{N}_k \otimes \mathbf{N}_i^o)$*, then*

$$\langle J\psi_1\widetilde{\psi}, \psi_2 \rangle = \langle J\psi_1, \widetilde{\psi}\psi_2 \rangle \quad \textit{and} \quad \langle J\psi_1, \psi_2\widetilde{\psi}' \rangle = \langle J\widetilde{\psi}'\psi_1, \psi_2 \rangle. \tag{2.166}$$

Proof This can simply be proven by using that the right action is implemented via J and that J is an anti-isometry with $J^2 = \pm$.

Fierz Transformations

Details for the Fierz transformation in this context can be found in the Appendix of [2] but we list the main result here.

Definition 2.55 (*Orthonormal Clifford basis*) Let $Cl(V)$ be the Clifford algebra over a vector space V of dimension n. Then $\gamma_K := \gamma_{k_1} \ldots \gamma_{k_r}$ for all strictly ordered sets $K = \{k_1 < \cdots < k_r\} \subseteq \{1, \ldots, n\}$ form a basis for $Cl(V)$. If γ_K is as above, we denote with γ^K the element $\gamma^{k_1} \ldots \gamma^{k_r}$. The basis spanned by the γ_K is said to be *orthonormal* if $\mathrm{tr}\, \gamma_K \gamma_L = n n_K \delta_{KL}\ \forall\ K, L$. Here $n_K := (-1)^{r(r-1)/2}$, where r denotes the cardinality of the set K and with δ_{KL} we mean

$$\delta_{KL} = \begin{cases} 1 & \text{if } K = L \\ 0 & \text{else} \end{cases}. \tag{2.167}$$

Example 2.56 Take $V = \mathbb{R}^4$ and let $Cl(4, 0)$ be the Euclidean Clifford algebra [i.e. with signature $(+ + + +)$]. Its basis are the sixteen matrices

$$
\begin{array}{lr}
1 & \\
\gamma_\mu & \text{(4 elements)} \\
\gamma_\mu \gamma_\nu \quad (\mu < \nu) & \text{(6 elements)} \\
\gamma_\mu \gamma_\nu \gamma_\lambda \quad (\mu < \nu < \lambda) & \text{(4 elements)} \\
\gamma_1 \gamma_2 \gamma_3 \gamma_4 =: -\gamma_5. &
\end{array}
$$

We can identify

$$\gamma_1\gamma_2\gamma_3 = \gamma_4\gamma_5, \quad \gamma_1\gamma_3\gamma_4 = \gamma_2\gamma_5 \quad \gamma_1\gamma_2\gamma_4 = -\gamma_3\gamma_5, \quad \gamma_2\gamma_3\gamma_4 = -\gamma_1\gamma_5, \tag{2.168}$$

establishing a connection with the basis most commonly used by physicists.

We then have the following result:

Proposition 2.57 ((Generalized) Fierz identity) *If for any two strictly ordered sets K, L there exists a third strictly ordered set M and $c \in \mathbb{N}$ such that $\gamma_K \gamma_L = c\, \gamma_M$, we have for any $\psi_1, \dots, \psi_4$ in the n-dimensional spin representation of the Clifford algebra*

$$\langle \psi_1, \gamma^K \psi_2 \rangle \langle \psi_3, \gamma_K \psi_4 \rangle = -\frac{1}{n} \sum_L C_{KL} \langle \psi_3, \gamma^L \psi_2 \rangle \langle \psi_1, \gamma_L \psi_4 \rangle, \tag{2.169}$$

where the constants $C_{LK} \equiv n_L f_{LK}$, $f_{LK} \in \mathbb{N}$ are defined via $\gamma^K \gamma^L \gamma_K = f_{KL} \gamma^L$ (no sum over L). Here we have denoted by $\langle ., . \rangle$ the inner product on the spinor representation.

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