Chapter 1
Introduction

Abstract We introduce the core concepts and formalisms that are needed in our search for a noncommutative geometric description of supersymmetric theories. We start with a concise overview of supersymmetry and the minimal supersymmetric extension of the Standard Model (MSSM). We then provide a bird's eye view of noncommutative geometry, geared towards its applications in high-energy physics.

1.1 Supersymmetry

The past decades have witnessed the birth of a plethora of 'Beyond the Standard Model' theories, trying to remedy one or more of its shortcomings such as the absence of the gravitational force, the large quantum corrections to the Higgs mass and no account of dark matter. *Supersymmetry* (SUSY) is a particular example of such a theory. The purpose of this section is to very briefly discuss its basic notions, apply it to the Standard Model (SM) and review some relevant properties of the result. Good introductions to supersymmetry are [3, 19, 29, 30]. A more mathematical approach can be found in [20].

In the 1960s the question was raised whether there might be extensions of the Poincaré algebra, incorporating a symmetry that would prove to be valuable for physics. Coleman and Mandula [11] proved that—given certain conditions—the Poincaré algebra constitutes all the symmetries of the S-matrix.

Several years later however, Haag et al. [23] showed that extending the Poincaré algebra *can* possibly lead to new physics, if one extends the notion of a Lie algebra (as is the Poincaré algebra) to that of a graded Lie algebra. Elements of such an algebra have a specific degree which determines whether they satisfy commutator or anti-commutator relations. The Poincaré algebra (having only zero-degree elements) is then extended with a set of variables Q^i_a and their conjugates $\bar{Q}^i_a$ ($i = 1, \dots, N$,[1] $a = 1, 2$) of degree 1 (i.e. they satisfy anti-commutation relations), transforming in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group respectively. This extended algebra is called the *supersymmetry algebra*.

[1]The possible values for N, the number of supersymmetry generators, depend on the space-time dimension. For example, for $d = 4$, $N = 1, 2, 4$ or 8.

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Throughout this book we will be considering the case $N = 1$ only.

The nature of these 'fermionic' generators $Q, \bar{Q}$ is then that they relate bosons and fermions. Schematically:

$$Q|\text{boson}\rangle = |\text{fermion}\rangle, \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle.$$

To be a bit more precise:

Definition 1.1 (*Supersymmetry transformation*) For a constant, infinitesimal two-component spinor ε and its conjugate $\bar{\varepsilon}$, we define (cf. [36, p. 21]) a *supersymmetry transformation* on any representation ζ of the Poincaré algebra as

$$\delta_\varepsilon \zeta := [(\varepsilon Q) + (\bar{\varepsilon}\bar{Q})]\zeta. \tag{1.1}$$

Here εQ and $\bar{\varepsilon}\bar{Q}$ denote the usual Lorentz invariant products of two anti-commuting two-component spinors and conjugate spinors respectively.

If we define such a $\delta_\varepsilon \zeta_i(x)$ for each of the fields $\zeta_1, \ldots, \zeta_n$ appearing in a theory, we can talk about whether or not its action is invariant under supersymmetry. If

$$\delta S[\zeta_1, \ldots, \zeta_n] := \frac{\mathrm{d}}{\mathrm{d}t} S[\zeta_1 + t\delta_\varepsilon \zeta_1, \ldots, \zeta_n + t\delta_\varepsilon \zeta_n]\Big|_{t=0} \tag{1.2}$$

equals 0, we call the system *supersymmetric*. A particularly simple example of a supersymmetric system is the following.

Example 1.2 (*Wess-Zumino* [37]) The action of a system containing a free Weyl fermion ξ and complex scalar field ϕ, is (in the notation of [19]) given by

$$S[\phi, \xi, \bar{\xi}] = \int \Big(|\partial_\mu \phi|^2 + i\xi\sigma^\mu [\partial_\mu] \bar{\xi} \Big) \mathrm{d}^4 x, \tag{1.3}$$

where $\sigma^\mu = (I_2, \sigma^a)$ with σ^a, $a = 1, 2, 3$ the Pauli matrices, $\bar{\xi}$ is the Hermitian conjugate of ξ and $X[\partial_\mu]Y := \frac{1}{2} X \partial_\mu Y - \frac{1}{2}(\partial_\mu X)Y$. This action is seen to be invariant under the transformations

$$\delta_\varepsilon \phi := \sqrt{2}\varepsilon\xi, \qquad \delta_{\bar{\varepsilon}} \xi := -\sqrt{2} i \sigma^\mu \bar{\varepsilon} \partial_\mu \phi, \tag{1.4}$$

see [19, Sect. 4.2]. Fields such as ϕ and ξ are called each other's *superpartners*.

Actually, (1.3) is only supersymmetric *on shell*, i.e. to prove supersymmetry one has to invoke the equations of motion for ξ. This is caused by the fields having the same number of degrees of freedom on shell, but not off shell. We can make this work off shell as well by introducing a complex scalar (*auxiliary*) field F that appears in the Lagrangian through $\mathscr{L}_F = |F(x)|^2$. Modifying the transformations (1.4) slightly to contain F, supersymmetry is seen to hold both on shell and off shell.

The example above is a nice illustration of the necessary condition that the total number of fermionic and bosonic degrees of freedom has to be the same in order for a system to exhibit supersymmetry at all.

Example 1.3 (*Wess-Zumino* [37]) Another important example of a supersymmetric model is the *super Yang-Mills system*, whose action is given by

$$\int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda\sigma^\mu [\partial_\mu]\bar{\lambda} + \frac{1}{2} D^2 \right). \tag{1.5}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength (curvature) of a $u(1)$ gauge field A_μ, λ a Weyl spinor and D is a real $u(1)$ auxiliary field. The latter must again be added to ensure an equal number of bosonic and fermionic degrees of freedom both on and off shell. This action is seen to be invariant under the transformations

$$\begin{aligned} \delta A_\mu &= \varepsilon\sigma_\mu\bar{\lambda} + \lambda\sigma^\mu\bar{\varepsilon}, \\ \delta\lambda &= -\frac{i}{4}\sigma^\mu\sigma^\nu F_{\mu\nu}\varepsilon + D\varepsilon, \\ \delta D &= i\partial_\mu(\lambda\sigma^\mu\bar{\varepsilon} + \bar{\lambda}\bar{\sigma}^\mu\varepsilon), \end{aligned}$$

where $\bar{\sigma}^\mu = (I_2, -\sigma^a)$ (see [19], Chaps. 4.1 and 4.4).

In Table 1.1 the role of the auxiliary fields is explicated for the Wess-Zumino and the super Yang-Mills models. For both the bosonic degrees of freedom are seen to be equal to the fermionic ones.

In many of the more advanced treatments of supersymmetry (e.g. [36]), ordinary space is extended to a *superspace* $(x^\mu, \theta, \overline{\theta})$ (where θ and $\overline{\theta}$ are two-component Grassmann variables). The particle content of a certain model is then described in terms of *superfields* (fields depending on all coordinates of superspace and containing the particles that are each other's superpartners). Two key examples are the *chiral superfield* Φ, with the particle content of Example 1.2, and the *vector superfield* V, whose particle content is that of Example 1.3. The action is recovered by integrating certain combinations of the superfields Φ and V over superspace by means of a *Berezin integral*. In this way the actions (1.3) for the chiral superfield and (1.5) for the vector superfield can be recovered.

Table 1.1 The number of real degrees of freedom both on and off shell for the Wess-Zumino and Super Yang-Mills models

Wess-Zumino:	ϕ	F	ξ	Super Yang-Mills:	A_μ	D	λ
Off shell:	2	2	4	Off shell:	3	1	4
On shell:	2	0	2	On shell:	2	0	2

In all cases the bosonic and fermionic number of degrees of freedom coincide

Table 1.2 The particle content of the νMSSM, the minimal supersymmetric extension of the standard model featuring a right-handed neutrino

Superfield	Spin				Representation
		0	$\frac{1}{2}$	1	
Left-handed (s)quark	Q_L	$\tilde{q}_L$	q_L	–	(1/6, 2, 3)
Up-type (s)quark	U_R	$\tilde{u}_R$	u_R	–	(2/3, 1, 3)
Down-type (s)quark	D_R	$\tilde{d}_R$	d_R	–	(−1/3, 1, 3)
Left-handed (s)lepton	L_L	$\tilde{l}_L$	l_L	–	(−1/2, 2, 1)
Up-type (s)lepton	N_R	$\tilde{\nu}_R$	ν_R	–	(0, 1, 1)
Down-type (s)lepton	E_R	$\tilde{e}_R$	e_R	–	(−1, 1, 1)
Gluon, gluino	V	–	g	g_μ	(0, 1, 8)
$SU(2)$ gauge bosons, gauginos	W	–	$\boldsymbol{\lambda}$	$\mathbf{W}_\mu$	(0, 3, 1)
B-boson, bino	B	–	λ_0	B_μ	(0, 1, 1)
Up-type Higgs(ino)	H_u	h_u	$\tilde{h}_u$	–	(1/2, 2, 1)
Down-type Higgs(ino)	H_d	h_d	$\tilde{h}_d$	–	(−1/2, 2, 1)

Each line represents one superfield, with particle content as indicated. All superpartners are in the same representation of the gauge group. The last column gives the representation of the gauge group that the particles are in. The first number in that column denotes the hypercharge of the $U(1)$-representation. The second number denotes the dimension of the $SU(2)$-representation: 1 for trivial/singlet, 2 for fundamental/defining and 3 for adjoint. The third number is the dimension of the $SU(3)$-representation: 1,3 or 8

1.1.1 The Supersymmetric Version of the Standard Model

When considering gauge theories, superpartners need to be in the same representation of the gauge group. It is clear that the Standard Model by itself is not supersymmetric. We have to introduce its superpartners to *make* it supersymmetric however:

Example 1.4 (*MSSM*) The *Minimally Supersymmetric Standard Model* (MSSM) is the supersymmetric theory that is obtained by adding to the particle content a superpartner[2] for each type of SM particles. In addition an extra Higgs doublet and its superpartner are introduced with hypercharge opposite to that of the other pair. One of the two pairs will give mass to the up-type particles, the other to the down-type ones. The adjective ’minimally’ is justified by the fact that the MSSM is the smallest (i.e. with the least number of additional superpartners) viable supersymmetric extension of the SM. See Table 1.2 and e.g. [10, 19] for details.

The following nomenclature is used. The name of superpartners of the fermions get a prefix ‘s’ (i.e. selectron, stop, etc.). The superpartners of the bosons get the suffix ‘ino’ (i.e. gluino, higgsino, etc.).

[2]This makes it an example of $N = 1$ supersymmetry.

Having two higgsino doublets with opposite hypercharge is necessary because adding only one higgsino doublet to the fermionic content of the SM will generate a chiral anomaly. The second higgsino is needed to cancel this anomaly again [19, Sect. 8.2].

The various superpartners are not only distinguished by their spin, but also by their *R-parity*. This is a $\mathbb{Z}_2$-grading (or 'discrete gauge symmetry') that for the MSSM is equal to

$$R_p = (-1)^{2S+3B+L}, \tag{1.6}$$

where S is the spin of the particle, B is its baryon number and L its lepton number. It follows that all SM particles (including the extra Higgses) have R-parity $+1$, whereas all superpartners have R-parity -1.

The list of the MSSM's merits is quite impressive. See [10, ch. 1] for a short overview. Here we will pick out three:

1. The MSSM makes the Higgs mass more stable. Roughly speaking, for each of the loop-interactions contributing to the mass of the Higgs there is a second such interaction that features a superpartner. This second contribution compensates for the first one.
2. If R-parity is conserved in the MSSM, the lightest particle that has $R_p = -1$ cannot decay and thus provides a cold Dark Matter candidate.
3. The additional particle content of the MSSM makes it possible for the three coupling constants g_1, g_2 and g_3 to evolve via the Renormalization Group Equations in such a way that they exactly meet at one energy scale. This hints at the existence of a Grand Unified Theory, that is hoped for by many theorists. See also Sect. 1.2.3.

Despite the theoretical arguments in favour of the MSSM, so far no experimental hints for its existence have been detected [4].

1.2 Noncommutative Geometry

Although noncommutative geometry (NCG, [13]) is a branch of mathematics, there is a number of applications in physics. The aim of this section is to provide a bird's eye view of NCG in relation with its foremost such application. This is the interpretation of the Standard Model as a geometrical theory, a line of thought that started with the Connes-Lott model [16] and culminated in [5] with the full SM, including a prediction of the Higgs boson mass. As much as possible we will focus on ideas and concepts and avoid the use of rigorous but technical statements, referring to the literature instead. Good general introductions to the field are e.g. [22, 27, 35] and [33] focusing on the applications to particle physics.

1.2.1 Spectral Triples

The basic device in noncommutative geometry is a *spectral triple*, thought of describing a *noncommutative manifold*.

Definition 1.5 ([13]) A *spectral triple* is a triple $(\mathscr{A}, \mathscr{H}, D)$, where $\mathscr{A}$ is a unital, involutive algebra that is represented as bounded operators on a Hilbert space $\mathscr{H}$ on which also a *Dirac operator* D acts. The latter is an (unbounded) self-adjoint operator that has compact resolvent and in addition $[D, a]$ is bounded for all $a \in \mathscr{A}$.

We will write $\langle ., . \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ for the inner product in $\mathscr{H}$.

This is a rather abstract object. To make it a bit more tangible, we turn to the case of a compact Hausdorff space M. To make it more interesting for us, we require this space to be enriched with extra structures. We will restrict ourselves to *Riemannian spin manifolds*, spaces that *locally* look like the Euclidean space $\mathbb{R}^n$ (for some n) on which a Riemannian metric g (locally: $g_{\mu\nu}$) exists and that admit spinors.[3]

- The algebra $C^\infty(M, \mathbb{C})$ is the subalgebra of $C(M, \mathbb{C})$ containing only *smooth* (i.e. infinitely differentiable) functions. It can be made involutive (just as $C(M)$ itself) by defining $f^* : M \to \mathbb{C}$ through $(f^*)(x) := \overline{f(x)} \in \mathbb{C}$ for all $x \in M$.
- The Hilbert space that is compatible with this algebra is $L^2(M, S)$—or $L^2(S)$ for short. It consists of square-integrable , spinor-valued functions ψ (i.e. for each $x \in M$, $\psi(x) \in S_x$ is a spinor). The number of components of that spinor depends on the dimension m of the manifold M: $\dim S_x = 2^n$, with $m = 2n$ or $m = 2n + 1$, according to whether m is even or odd.
- The Levi-Civita connection—the unique connection on M that is compatible with the metric g—can be lifted to act on spinor-valued functions. This leads to the operator

$$\partial\!\!\!/_M := i\gamma^\mu(\partial_\mu + \omega_\mu), \tag{1.7}$$

where the term

$$\omega_\mu = -\frac{1}{4}\widetilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b$$

accounts for the manifold M being curved [22, Sect. 9.3]. Here the latin indices a, b indicate the use of a frame field h, diagonalising the metric $g^{\mu\nu} = h^\mu_a h^\nu_b \delta^{ab}$ and γ-matrices

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}, \qquad \gamma^\mu = h^\mu_a \gamma^a, \tag{1.8}$$

[3] One should keep in mind though that Minkowski space is not an example of a Riemannian manifold. Rather it is pseudo-Riemannian since its metric is diagonal with negative entries.

and $\widetilde{\Gamma}^b_{\mu a} := \Gamma^\lambda_{\mu\nu} h^\nu_a h^b_\lambda$, with $\Gamma^\lambda_{\mu\nu}$ the Christoffel symbols of the Levi-Civita connection. From the metric g thus a Dirac operator is derived and conversely [12] the metric is completely determined by the Dirac operator.

Together these three objects form the *canonical spectral triple*:

Example 1.6 (*Canonical spectral triple* [13] Chap. 6.1)] The triple

$$(\mathscr{A}, \mathscr{H}, D) = (C^\infty(M), L^2(M, S), \partial\!\!\!/_M = i\gamma^\mu(\partial_\mu + \omega_\mu))$$

is called the canonical spectral triple. Here M is a compact Riemannian spin-manifold and $L^2(M, S)$ denotes the square-integrable section of the corresponding spinor bundle. The Dirac operator $\partial\!\!\!/_M$ is associated to the unique spin connection, which in turn is derived from the Levi-Civita connection on M.

The canonical spectral triple may be said to have served as the motivating example of the field; NCG is more or less modelled to be a generalization of it.

In the physics parlance the canonical spectral triple roughly speaking determines a physical *system*: the algebra encodes space(-time), the Hilbert space contains spinors 'living' on that space(-time) and $\partial\!\!\!/_M$ determines how these spinors propagate.

A second important example is that of a *finite spectral triple*:

Example 1.7 (*Finite spectral triple*) For a finite-dimensional algebra $\mathscr{A}_F$, a finite-dimensional left module $\mathscr{H}_F$ of $\mathscr{A}_F$ and a Hermitian matrix $D_F : \mathscr{H}_F \to \mathscr{H}_F$, we call $(\mathscr{A}_F, \mathscr{H}_F, D_F)$ a *finite spectral triple*.

We will go into (much) more detail on finite spectral triples in Sect. 1.2.4.

Given a spectral triple one can enrich it with two operators. The first of these, indicated by J, has a role similar to that of charge conjugation, whereas the other, indicated by γ, allows one to make a distinction between positive ('left-handed') and negative ('right-handed') chirality elements of a (reducible) Hilbert space:

- We call a spectral triple *even* if there exists a grading $\gamma : \mathscr{H} \to \mathscr{H}$, with $[\gamma, a] = 0$ for all $a \in \mathscr{A}$ such that

$$\gamma D = -D\gamma. \tag{1.9}$$

- We call a spectral triple *real* if there exists an antiunitary operator (*real structure*) $J : \mathscr{H} \to \mathscr{H}$, satisfying

$$J^2 = \varepsilon \,\mathrm{id}_{\mathscr{H}}, \qquad JD = \varepsilon' DJ, \qquad \varepsilon, \varepsilon' \in \{\pm\}. \tag{1.10}$$

The real structure implements a right action a^o of $a \in \mathscr{A}$ on $\mathscr{H}$, via $a^o := Ja^*J^*$ that is required to be compatible with the left action:

$$[a, Jb^*J^*] = 0 \quad , \tag{1.11}$$

i.e. $(a\psi)b = a(\psi b)$ for all $a, b \in \mathscr{A}$, $\psi \in \mathscr{H}$. The Dirac operator and real structure are required to be compatible via the *first-order condition*:

$$[[D, a], Jb^*J^*] = 0 \quad \forall\, a, b \in \mathscr{A}. \tag{1.12}$$

- If a spectral triple is both real and even there is the additional compatibility relation

$$J\gamma = \varepsilon'' \gamma J, \qquad \varepsilon'' \in \{\pm\}. \tag{1.13}$$

We denote such an enriched spectral triple by $(\mathscr{A}, \mathscr{H}, D; J, \gamma)$ and call it a *real, even spectral triple* [14]. The eight different combinations for the three signs above determine the *KO-dimension* of the spectral triple, cf. Table 1.3. For more details we refer to [14, 17, 22] (Fig. 1.1).

Example 1.8 The canonical spectral triple (Example 1.5) can be extended by a real structure J_M ('charge conjugation'). When $\dim M$ is even it can also be extended by a grading $\gamma_M := (-i)^{\dim M/2}\gamma^1 \ldots \gamma^M$ ('chirality', often denoted as $\gamma^{\dim M+1}$). The KO-dimension of a canonical spectral triple always equals the dimension of the manifold M [14] (see also [22, Sect. 9.5]).

For $\dim M = 4$, the case we will be focussing on, we have

$$\gamma^5 := -\gamma^1\gamma^2\gamma^3\gamma^4,$$

which, using that $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ (cf. (1.8)), indeed satisfies $(\gamma^5)^2 = \mathrm{id}_{L^2(S)}$ and $(\gamma^5)^* = \gamma^5$. This enables us to reduce the space $L^2(M, S)$ into eigenspaces of γ^5:

Table 1.3 The various possible KO-dimensions and the corresponding values for the signs $J^2 = \varepsilon\, \mathrm{id}_{\mathscr{H}}$, $JD = \varepsilon' DJ$ and $J\gamma = \varepsilon''\gamma J$

KO-dimension:	0	1	2	3	4	5	6	7
$J^2 = \varepsilon\, \mathrm{id}_{\mathscr{H}}$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$JD = \varepsilon' DJ$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
$J\gamma = \varepsilon''\gamma J$	$+$		$-$		$+$		$-$	

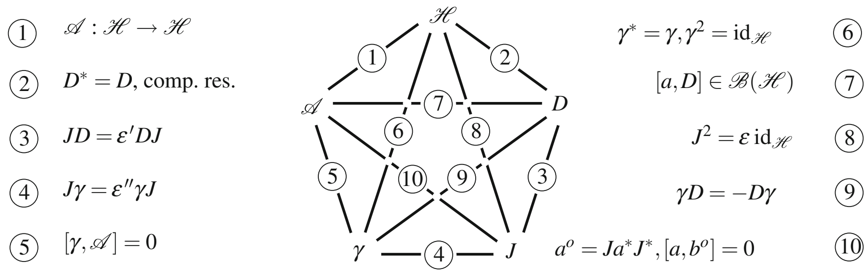

Fig. 1.1 A pictorial overview of the various relations that hold between the constituents of a real and even spectral triple. Not depicted here is the first order condition (1.12)

$$L^2(S) = L^2(S)_+ \oplus L^2(S)_-, \quad L^2(S)_\pm = \{\psi \in L^2(S), \gamma^5\psi = \pm\psi\}.$$

Also, γ^5 is seen to anticommute with $\partial\!\!\!/_M$. As for the real structure J, it is given (cf. [27, Sect. 5.7]) pointwise as $(J\psi)(x) := C(x)\bar{\psi}(x)$ with $C(x)$ a *charge conjugation matrix* and the bar denotes complex conjugation. One obtains [22, Sect. 9.4] a charge conjugation operator that satisfies

$$C^2 = -1, \qquad C\partial\!\!\!/_M = \partial\!\!\!/_M C, \qquad \gamma^5 C = C\gamma^5.$$

Table 1.3 shows that the KO-dimension indeed equals $\dim M$.

Example 1.9 As in the general case a finite spectral triple (Example 1.7) is called real if there exists a J_F (implementing a bimodule structure of $\mathscr{H}_F$) and even when there exists a grading γ_F on $\mathscr{H}_F$.

Given any two spectral triples $(\mathscr{A}_{1,2}, \mathscr{H}_{1,2}, D_{1,2}; J_{1,2}, \gamma_{1,2})$ their tensor product

$$(\mathscr{A}_1 \otimes \mathscr{A}_2, \mathscr{H}_1 \otimes \mathscr{H}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, J_\otimes, \gamma_1 \otimes \gamma_2),$$

is again a spectral triple. Here generally $J_\otimes = J_1 \otimes J_2$, but with the following exceptions: $J_\otimes = J_1\gamma_1 \otimes J_2$ when the sum of the respective KO-dimensions is 1 or 5 and $J_\otimes = J_1 \otimes J_2\gamma_2$ when the KO-dimension of the first spectral triple is 2 or 6 and that of the other one is even [18, 34]. The form of the Dirac operator of the tensor product is necessary to ensure that it anti-commutes with $\gamma_1 \otimes \gamma_2$ and that the resolvent remains compact. It follows that the KO-dimension of this tensor product is the sum of the KO-dimensions of the separate spectral triples. In the canonical spectral triple the algebra encodes space(-time), in a finite spectral triple it will seen to be intimately connected to the gauge group (see (1.37) ahead). In describing particle models we need both. We therefore take the tensor product of a canonical and a finite spectral triple. In the case that $\dim M = 4$ this reads

$$(C^\infty(M, \mathscr{A}_F), L^2(M, S \otimes \mathscr{H}_F), \partial\!\!\!/_M \otimes 1 + \gamma^5 \otimes D_F, J_M \otimes J_F, \gamma^5 \otimes \gamma_F), \tag{1.14}$$

with $C^\infty(M) \otimes \mathscr{A}_F \simeq C^\infty(M, \mathscr{A}_F)$. Spectral triples of this form are generally referred to as *almost-commutative geometries* [24]. Noncommutative geometry can thus be said to put the external and internal degrees of freedom of particles on similar footing. To obtain one's favourite particle physics model (in four dimensions) the key is to construct the right finite spectral triple that accounts for the gauge group and all internal degrees of freedom and interactions.

1.2.2 Gauge Fields and the Action Functional

Two more concepts need to be introduced, both arising from the question "what is the natural notion of equivalence for spectral triples and what is an invariant for this equivalence?". To this end we start by defining the notion of *unitarily equivalent* spectral triples:

Definition 1.10 (*Unitarily equivalent spin geometries*) Two (real and even) spectral triples $(\mathscr{A}, \mathscr{H}, D; J, \gamma)$ and $(\mathscr{A}, \mathscr{H}, D'; J', \gamma')$ are said to be *unitarily equivalent*, if there exists a unitary operator U on $\mathscr{H}$ such that

- $UaU^* = \sigma(a) \; \forall \, a \in \mathscr{A}$,
- $D' = UDU^*$,
- $J' = UJU^*$,
- $\gamma' = U\gamma U^*$.

Here σ denotes an *automorphism* of the algebra $\mathscr{A}$.

Given an algebra $\mathscr{A}$ we can form the group of unitary elements of $\mathscr{A}$:

$$U(\mathscr{A}) := \{u \in \mathscr{A}, uu^* = u^*u = 1\}$$

and construct unitary operators $U := uJuJ^*$:

$$U : \mathscr{H} \to \mathscr{H}, \quad \psi \to u\psi u^*. \tag{1.15}$$

Using this group we can construct a particular kind of unitary equivalence for spectral triples, where the automorphism σ is seen to be an *inner automorphism*, i.e. $UaU^* = uau^*$, where we have used (1.11) and that $J^2 = \varepsilon$ id. This leads to the following result [14].

Lemma 1.11 *For $U = uJuJ^*$ with $u \in U(\mathscr{A})$, the real and even spectral triples $(\mathscr{A}, \mathscr{H}, D; \gamma, J)$ and*

$$(\mathscr{A}, \mathscr{H}, D + A + \varepsilon' JAJ^*; J, \gamma) \quad \textit{with} \quad A = u[D, u^*], u \in U(\mathscr{A}), \tag{1.16}$$

are unitarily equivalent.

This result implies that the class of unitarily equivalent spectral triples for $U = uJuJ^*$, $u \in U(\mathscr{A})$ differ only by the *inner fluctuations* of the Dirac operator. A more general—but also a somewhat more involved—way to look at this is by using the notion of *Morita equivalence* of spectral triples [15]. In this way the inner fluctuations A of

$$D \to D_A := D + A + \varepsilon' JAJ^* \tag{1.17}$$

are seen to be the self-adjoint elements of

$$\Omega^1_D(\mathscr{A}) := \left\{ \sum_n a_n[D, b_n],\ a_n, b_n \in \mathscr{A} \right\}. \tag{1.18}$$

The action of U (Lemma 1.11) on D_A (i.e. $D_A \mapsto UD_AU^*$) induces one on the inner fluctuations:

$$A \mapsto A^u := uAu^* + u[D, u^*], \tag{1.19}$$

an expression that is reminiscent of the way gauge fields transform in quantum field theory. Note that the inner fluctuations that arise using the argument of unitary equivalence in fact only correspond to *pure gauges*.

In the case of a canonical spectral triple—for which the left and right actions coincide—that has $JD = DJ$, the inner fluctuations vanish [27, Sect. 8.3]. In the case of an almost-commutative geometry both components $\partial\!\!\!/_M$ and D_F of the Dirac operator generate inner fluctuations. For these we will write

$$D_A := \partial\!\!\!/_A + \gamma_M \otimes \Phi, \tag{1.20}$$

where $\partial\!\!\!/_A = i\gamma^\mu(\partial_\mu + \omega_\mu \otimes \mathrm{id}_{\mathscr{H}_F} + \mathbb{A}_\mu)$, with

$$\mathbb{A}_\mu = \sum_n \Big(a_n[\partial_\mu, b_n] - \varepsilon' J a_n[\partial_\mu, b_n]J^* \Big), \qquad a_n, b_n \in C^\infty(M, \mathscr{A}_F), \tag{1.21}$$

skew-Hermitian and

$$\Phi = D_F + \sum_n \Big(a_n[D_F, b_n] + \varepsilon' J a_n[D_F, b_n]J^* \Big), \qquad a_n, b_n \in C^\infty(M, \mathscr{A}_F).$$

The relative minus sign between the two terms in $\mathbb{A}_\mu$ comes from the identity $J_M\gamma^\mu J_M^* = -\gamma^\mu$ for even-dimensional $\dim M$. The terms will later be seen to contain all gauge fields of the theory [14]. The inner fluctuations of the finite Dirac operator D_F (see also (1.35)) are seen to parametrize all scalar fields, such as the Higgs field. Interestingly, this view places gauge and scalar fields on the same footing, something that is not the case in QFT. See Table 1.4 for an overview of the origin of the various fields.

The second and last ingredient that we will need here is a natural, gauge invariant, action functional. For that we want something which only depends on the data that are present in the spectral triple. The most natural choice [7] for that turns out to be

$$S[\zeta, A] := \frac{1}{2}\langle J\zeta, D_A\zeta\rangle + \mathrm{tr} f(D_A/\Lambda), \qquad \zeta \in \frac{1}{2}(1 + \gamma_M \otimes \gamma_F)\mathscr{H} \equiv \mathscr{H}^+, \tag{1.22}$$

Table 1.4 The various possible fields that are ingredients of physical theories and the NCG-objects they originate from in the case of an almost-commutative geometry

Type of field	NCG-object
Fermions	$L^2(M, S) \otimes \mathscr{H}_F$
Scalar bosons	$\Omega^1_{D_F}(\mathscr{A})$
Gauge bosons	$\Omega^1_{\partial\!\!\!/_M}(\mathscr{A})$

consisting of the *fermionic action* and the *spectral action* respectively. Here f is a positive, even function, Lambda is a (unknown) mass scale[4] and the trace of the second term is over the entire Hilbert space.

Using that $J^2 = \varepsilon$, $DJ = \varepsilon' JD$ the fermionic action is seen to satisfy

$$\langle J\xi, D_A\zeta\rangle = \varepsilon\varepsilon'\langle J\zeta, D_A\xi\rangle \qquad \forall\, \xi, \zeta \in \mathscr{H}, \tag{1.23}$$

i.e. it is either symmetric or antisymmetric. In its original [2, 16] form, the expression for the fermionic action did not feature the real structure (nor the factor $\frac{1}{2}$) and did not have elements of only $\mathscr{H}^+$ as input. It was shown [8] that for a suitable choice of a spectral triple it does yield the full fermionic part of the Standard Model Lagrangian (see Sect. 1.2.3), including the Yukawa interactions, but suffered from the fact that the fermionic degrees of freedom were twice what they should be, as pointed out in [28]. Furthermore it does not allow a theory with massive right-handed neutrinos. Adding J to the expression for the fermionic action and requiring $\{J, \gamma\} = 0$ allows restricting its input to $\mathscr{H}^+$ without vanishing altogether. This expression is seen to solve both problems at the same time [5] (see also [17]). We will not further go into details but refer to the mentioned literature instead.

Despite its deceivingly simple form, the second term of (1.22) is a rather complicated object and in practice one has to resort to approximations for calculating it explicitly. Most often this is done [8] via a *heat kernel expansion* [21]. In four dimensions and for a suitable Dirac operator this reads:

$$\operatorname{tr} f(D_A/\Lambda) \sim 2\Lambda^4 f_4 a_0(D_A^2) + 2\Lambda^2 f_2 a_2(D_A^2) + f(0)a_4(D_A^2) + \mathscr{O}(\Lambda^{-2}), \tag{1.24}$$

where f_2, f_4 are the second and fourth *moments* of f and the (*Seeley-DeWitt*) coefficients $a_{0,2,4}(D_A^2)$ only depend on the square of the Dirac operator. For a general almost-commutative geometry on a flat 4-dimensional Riemannian spin-manifold without boundary this reads:

[4]The parameter Λ more or less serves as a cut-off, and will in the derivation of the SM (Sect. 1.2.3 ahead) be interpreted as the GUT-scale.

$$\mathrm{tr} f\left(\frac{D_A}{\Lambda}\right) \sim \int_M \left[\frac{f(0)}{8\pi^2}\left(-\frac{1}{3}\,\mathrm{tr}_F\,\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} + \mathrm{tr}_F\,\Phi^4 + \mathrm{tr}_F[D_\mu, \Phi]^2\right) - \frac{\Lambda^2}{2\pi^2} f_2\,\mathrm{tr}_F\,\Phi^2 + \frac{\Lambda^4}{2\pi^2} f_4 \mathcal{N}(F)\right] + \mathcal{O}(\Lambda^{-2}), \tag{1.25}$$

where tr_F denotes the trace over the finite Hilbert space, $\mathcal{N}(F) = \dim(\mathcal{H}_F)$ and $\mathbb{F}_{\mu\nu}$ is the (skew-Hermitian) curvature (or field strength) of $\mathbb{A}_\mu$, i.e.

$$\mathbb{F}_\mu\nu = [\partial_\mu + \mathbb{A}_\mu, \partial_\nu + \mathbb{A}_\nu]. \tag{1.26}$$

Note that—in contrast to 'normal' high energy physics—there is no question of adding some terms to the action by hand in order to make something work. The action (1.22) is simply fixed by the spectral triple.

1.2.3 The Noncommutative Standard Model (NCSM)

We now have all the essential ingredients to obtain the Standard Model [5]. We take a compact, 4-dimensional Riemannian spin manifold M without boundary and the corresponding canonical spectral triple. We take the tensor product with a finite spectral triple whose algebra is

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

where with $\mathbb{H}$ we mean the quaternions and $M_3(\mathbb{C})$ the complex 3×3-matrices. Note that it is this finite algebra that makes the resulting spectral triple actually noncommutative. We denote the irreducible representations of its components with $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ respectively. In addition, we will need the anti-linear representation $\bar{\mathbf{1}}$, on which $\lambda \in \mathbb{C}$ acts as $\bar{\lambda}$. With $\mathbf{1}^o$, $\mathbf{2}^o$, etc. we denote the contragredient module. A natural bimodule of this algebra[5] (i.e. the finite Hilbert space),

$$(\mathbf{2} \otimes \mathbf{1}^o) \oplus (\mathbf{1} \otimes \mathbf{1}^o) \oplus (\bar{\mathbf{1}} \otimes \mathbf{1}^o) \oplus (\mathbf{2} \otimes \mathbf{3}^o) \oplus (\mathbf{1} \otimes \mathbf{3}^o) \oplus (\bar{\mathbf{1}} \otimes \mathbf{3}^o), \tag{1.27}$$

turns out to exactly describe the particle content of the Standard Model; l_L, ν_R, e_R, q_L, u_R and d_R respectively. From the noncommutative point of view having a right-handed neutrino is a desirable feature [5]. If we want to introduce a real structure J_F we also need $\mathbf{1} \otimes \mathbf{2}^o$, etc. (describing the antiparticles). We can construct a grading γ_F that distinguishes left- from right-handed particles and that anticommutes with the real structure. This makes the KO-dimension of the finite spectral triple equal to 6 and consequently that of the almost-commutative geometry equal to 2. This makes it possible to reduce the fermionic degrees of freedom [5, Sect. 4.4.1]. This Hilbert space describes only one generation of particles so we need to take three copies (or *generations*) of it.

[5] To be explicit, the element $(\lambda, q, m) \in \mathcal{A}_F$ acts on—say—$\mathbf{2} \otimes \mathbf{3}^o \ni v \otimes \bar{w}$ as $qv \otimes \bar{w}m = qv \otimes \overline{m^* w}$.

We can check that not only $SU(\mathscr{A}_F)$ (from (1.37)) equals the gauge group of the Standard Model $SU(3) \times SU(2) \times U(1)$ (modulo a finite group) but also that the resulting hypercharges of the representations match those of the particles of the Standard Model.

Then there is the Dirac operator D_F for the finite spectral triple. It is given by a hermitian matrix whose non-zero components are determined [5, Sect. 2.6] by 3×3-matrices Υ_ν, Υ_e, Υ_u, Υ_d and a symmetric 3×3-matrix Υ_R, that mix generations. The $\Upsilon_{\nu,e,u,d}$ map between the representations in $\mathscr{H}_F$ that describe the left- and right-handed (anti)leptons and (anti)quarks and are interpreted as the fermion mass mixing matrices. The component Υ_R maps between the representations that describe the right-handed neutrinos and their antiparticles and serves as a Majorana mass matrix.

A second step is to calculate the inner fluctuations of both Dirac operators. For $\partial\!\!\!/_M$, the inner fluctuations acting on $\mathbf{1}$ and $\overline{\mathbf{1}}$ are both seen to describe the same $U(1)$ gauge field. To also let the quarks interact with this field in the way they do in the SM, an additional constraint is imposed. This constraint asserts that the total inner fluctuations be traceless:

$$\mathrm{tr}_{\mathscr{H}_F} A_\mu = 0. \tag{1.28}$$

This is called the *unimodularity condition* [2, 13]. In addition it reduces the degrees of freedom of the gauge bosons to the right number. After applying this condition, the inner fluctuations of $\partial\!\!\!/_M$turn out to exactly describe the gauge bosons of the Standard Model; the hypercharge field B_μ, the weak-force bosons $\mathbf{W}_\mu$ and gluons g_μ. The inner fluctuations of D_F on the other hand are seen to describe a scalar field that—via the action—interacts with a left-handed and a right-handed lepton or quark: it is the famous Higgs field [5, Sect. 3.5]. Since the finite part of the right-handed neutrinos is in $\mathbf{1} \otimes \mathbf{1}^o \simeq \mathbb{C}$, the component Υ_R that describes their Majorana masses does not generate a field via the inner fluctuations (1.18).

If we calculate the spectral action for this spectral triple [5, Sect. 3.7], not only do we get the action of the full Standard Model but again the Einstein-Hilbert action of General Relativity too. Various coefficients of terms in the action are determined by variables that are characteristic for NCG (e.g. the moments f_n, Λ, etc.). This gives rise to relations between SM-parameters that are not present in the Standard Model. For example, if we normalize the kinetic terms of the gauge bosons we automatically find the relation

$$g_3^2 = g_2^2 = \frac{5}{3} g_1^2 \tag{1.29}$$

between the coupling constants of the strong, weak and hypercharge forces respectively [5, Sect. 4.2]. This relation suggests that the interpretation of the so far unknown value of Lambda is that of the energy scale at which our theory ‘lives’ and at which the three forces (hypercharge, weak and strong) are of the same strength. Looking at Fig. 1.2, this corresponds to the order of $10^{13} - 10^{17}$ GeV. There is also an additional relation

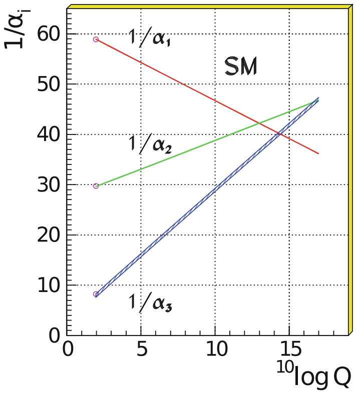

Fig. 1.2 The three (inverse) 'coupling constants' $\alpha_1 = \frac{5}{3}g_1^2/4\pi$, $\alpha_2 = g_2^2/4\pi$ and $\alpha_3 = g_3^2/4\pi$ as a function of the energy. At high energy they are seen to nearly meet in one point. The figure is taken from [25]

$$\lambda = 4g_2^2\frac{b}{a^2}, \quad b = \mathrm{tr}[(\Upsilon_\nu{}^*\Upsilon_\nu)^2 + (\Upsilon_e{}^*\Upsilon_e)^2 + 3(\Upsilon_u{}^*\Upsilon_u)^2 + 3(\Upsilon_d{}^*\Upsilon_d)^2],$$
$$a = \mathrm{tr}(\Upsilon_\nu{}^*\Upsilon_\nu + \Upsilon_e{}^*\Upsilon_e + 3\Upsilon_u{}^*\Upsilon_u + 3\Upsilon_d{}^*\Upsilon_d)$$

for the coefficient of the Higgs boson self-coupling. Using the value we find for g_2^2 from Fig. 1.2 and approximating the coefficients a, b we can infer [5, Sect. 5.2] that $\lambda(\Lambda) \approx 0.356$. Inserting this boundary condition into the renormalization group equation for λ we obtain a value for the Higgs boson mass at the electroweak scale in the order of 170 GeV (see [31] for a detailed analysis).

In addition, this scheme allows a retrodiction of the top quark mass. It is found to be $\lesssim$ 180 GeV [5, Sect. 5.4].

This would be a perfect end to the story, if it was not for two things. First of all, the observed Higgs mass (125.09 ± 0.24 GeV/c^2 [1]) is distinctly different from the above mass range. Second, though we pretended that the three forces are of equal strength at one specific energy-scale Λ, we know from experiment that—at least for the SM—they are in fact not completely, see Fig. 1.2. Nonetheless, the fact that NCG allows one to come up with a robust prediction of the Higgs mass in the first place (and that this prediction depends on the particle content, as illustrated by [9]) is a promising sign of NCG saying something about reality. Moreover, there is now evidence [6] that grand unification holds in the Pati-Salam models that have been derived previously from NCG.

1.2.4 Finite Spectral Triples and Krajewski Diagrams

Since we will be using real finite spectral triples (cf. Examples 1.7 and 1.9) extensively later on, we cover them in more detail. They are characterized by the following properties:

- The finite-dimensional algebra is (by Wedderburn's Theorem) a direct sum of matrix algebras:

$$\mathscr{A}_F = \bigoplus_i^K M_{N_i}(\mathbb{F}_i) \qquad \mathbb{F}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}. \tag{1.30}$$

- The finite Hilbert space is an $\mathscr{A}_F^{\mathbb{C}}$-bimodule, where $\mathscr{A}_F^{\mathbb{C}}$ is the *complexification* of $\mathscr{A}_F$. More specifically, it is a direct sum of tensor products of irreducible representations: $\mathbf{N}_i \equiv \mathbb{C}^{N_i}$ of $M_{N_i}(\mathbb{F}_i)$ for $\mathbb{F}_i = \mathbb{C}, \mathbb{R}$ and $\mathbf{N}_i \equiv \mathbb{C}^{2N_i}$ of $M_{N_i}(\mathbb{F}_i)$ for $\mathbb{F}_i = \mathbb{H}$, with the contragredient representation $\mathbf{N}_j^o$. The latter can be identified with the dual of $\mathbf{N}_j$. Thus $\mathscr{H}_F$ is generically of the form

$$\mathscr{H}_F = \bigoplus_{i \leq j \leq K} \left(\mathbf{N}_i \otimes \mathbf{N}_j^o\right)^{\oplus M_{N_i N_j}} \oplus \left(\mathbf{N}_j \otimes \mathbf{N}_i^o\right)^{\oplus M_{N_j N_i}}. \tag{1.31}$$

  The non-negative integers $M_{N_i N_j}$ denote the *multiplicity* of the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$. When various multiplicities all have one particular value M, we speak of (M) *generations* that are part of a *family*.
  In the rest of this book we will not consider representations such as the last part of (1.31), since these are incompatible with $J_F \gamma_F = -\gamma_F J_F$, necessary for avoiding the fermion doubling problem.
- The right $\mathscr{A}_F$-module structure is implemented by a real structure

$$J_F : \mathbf{N}_i \otimes \mathbf{N}_j^o \to \mathbf{N}_j \otimes \mathbf{N}_i^o \tag{1.32}$$

  that takes the adjoint: $J_F(\eta \otimes \bar{\zeta}) = \zeta \otimes \bar{\eta}$, for $\eta \in \mathbf{N}_i$ and $\zeta \in \mathbf{N}_j$. To be explicit: let $a := (a_1, \ldots, a_K) \in \mathscr{A}_F$ and $\eta \otimes \bar{\zeta} \in \mathbf{N}_i \otimes \mathbf{N}_j^o$, then

$$a^o := J_F a^* J_F^*(\eta \otimes \bar{\zeta}) = J_F a^* \zeta \otimes \bar{\eta} = J_F(a_j^* \zeta \otimes \bar{\eta}) = \eta \otimes \overline{a_j^* \zeta} \equiv \eta \otimes \bar{\zeta} a_j. \tag{1.33}$$

  From this it is clear that (1.11) entails the compatibility of the left and right action. For the Hilbert space the existence of a real structure (1.32) implies that $M_{N_i N_j} = M_{N_j N_i}$.
- For each component of the algebra for which $\mathbb{F}_i = \mathbb{C}$ we will a priori allow both the (complex) linear representation $\mathbf{N}_i$ and the anti-linear representation $\overline{\mathbf{N}}_i$, given by:

$$\pi(m)v := \overline{m}v, \qquad m \in M_{N_i}(\mathbb{C}), v \in \mathbb{C}^{N_i}.$$

- The finite Dirac operator D_F consists of components

$$D_{ij}^{kl} : \mathbf{N}_k \otimes \mathbf{N}_l^o \to \mathbf{N}_i \otimes \mathbf{N}_j^o. \tag{1.34}$$

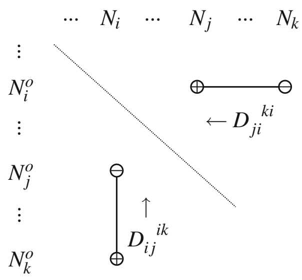

Fig. 1.3 An example of a Krajewski diagram. Each *circle* in the grid stands for a representation in $\mathscr{H}_F$. A *solid line* represents a component of the Dirac operator. As can be seen from the signs, $\{J_F, \gamma_F\} = 0$ here

The first order condition (1.12) implies that any component is either left- or right-linear with respect to the algebra [26]. This means that $i = k$ or $j = l$.[6] In both cases it is parametrized by a matrix; in the first case it constitutes of right multiplication with some $\eta_{lj} \in \mathbf{N}_l \otimes \mathbf{N}_j^o$, in the second case of left multiplication with some $\eta_{ik} \in \mathbf{N}_i \otimes \mathbf{N}_k^o$.

There exists a very useful graphical representation for finite spectral triples, called *Krajewski diagrams* [26]. Such a diagram consists of a two-dimensional grid, labeled by the various N_i and N_i^o, representing (the irreducible representations of) the algebra. Any representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ that occurs in $\mathscr{H}_F$ then can be represented as a *vertex* on the point (i, j) in this grid. If the finite spectral triple is even, each such representation has a value $\pm$ for the grading γ_F. We represent it by putting the sign in the corresponding vertex. For real spectral triples, a diagram has to be symmetric with respect to reflection across the diagonal from the upper left to the lower right corner. This is due to the role of J_F. The reflection of a particular vertex has the same or an opposite value for the grading, depending on whether J_F commutes or anticommutes with γ_F.

We can represent the component D_{ij}^{kl} of the Dirac operator in a Krajewski diagram by an *edge* from (k, l) to (i, j). Since the Dirac operator is self-adjoint, this means that there is also an edge from (i, j) to (k, l) and since it (anti)commutes with J_F, this means that there must also be an edge from (l, k) to (j, i). From the first order condition it follows [26] that these lines can only be horizontal or vertical. We provide a particularly simple example of a Krajewski diagram in Fig. 1.3, in which there are two vertices (and their conjugates) between which there is an edge.

Both as an example of the power of Krajewski diagrams and for future reference Fig. 1.4 shows the diagram that fully determines (the internal structure of) the Standard Model (c.f. Sect. 1.2.3). On each point there are in fact three vertices, corresponding to the three generations of particles. The finite Dirac operator was seen to be parametrized by the fermion mass mixing matrices $\Upsilon_{\nu,e,u,d} \in M_3(\mathbb{C})$. Their inner fluctuations generate scalars that are interpreted as the Higgs boson doublet (solid lines), connecting the left- and right-handed representations. Furthermore we have

[6] An exception to this rule is when one component of the algebra acts in the same way on more than one different representation in $\mathscr{H}_F$.

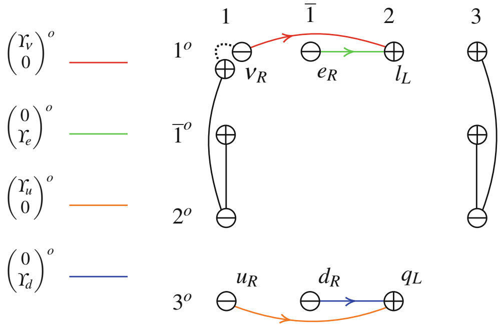

Fig. 1.4 The Krajewski diagram representing the Standard Model. The *color* of the edges denotes its parametrization

the possibility of adding a Majorana mass Υ_R for the right handed neutrino (dotted line).

The important result of [26] is that all properties of a finite spectral triple can be read off from a Krajewski diagram. Although Krajewski diagrams were thus developed as a tool to characterize or classify finite spectral triples (see also [33, Ch.3]), they have turned out to have an applicability beyond that, e.g. [32]. Here, we will use them also to determine the value of the trace of the second and fourth powers of the finite Dirac operator D_F (or Φ, including its fluctuations), appearing in the action functional (1.25). We notice [26, Sect. 5.4] that

- all contributions to the trace of the nth power of D_F are given by continuous, closed paths that are comprised of n edges in the Krajewski diagram.
- such paths can go back and forth along an edge.
- a step in the horizontal direction corresponds to a component D_{ij}^{kl} of D_F acting on the left of the bimodule $\mathscr{H}_F$, whereas a vertical step corresponds to a component D_{ij}^{kl} acting on the right via $J(D_{ij}^{kl})^* J^*$. Due to the tensor product structure, the trace that corresponds to a certain closed path is therefore the product of the horizontal and vertical contributions.
- if a closed path extends in only one direction, this means that the operator acts trivially on either the right or the left of the representation $\mathbf{N}_i \otimes \mathbf{N}_j^o$ at which the path started. The trace then yields an extra factor N_i or N_j, depending on the direction of the path.

As an example we have depicted in Fig. 1.5 all possible contributions to the trace of the fourth power of a D_F. This is the highest power that we shall encounter, as we are interested in the action (1.25). We introduce the notation $|X|^2 := \mathrm{tr}_N X^*X$, for $X^*X \in M_N(\mathbb{C})$. As an illustration of the factors appearing; in the second case a path can start at any of the three vertices, but when it starts in the middle one, it can either go first to the left or to the right. In addition, for a real spectral triple, each path appears in the same way in both directions, giving an extra factor 2. This last argument does not hold for the last case when $k = i$ and $l = j$, however.

A component D_{ij}^{kj} of the finite Dirac operator will develop inner fluctuations (1.18) that are of the form

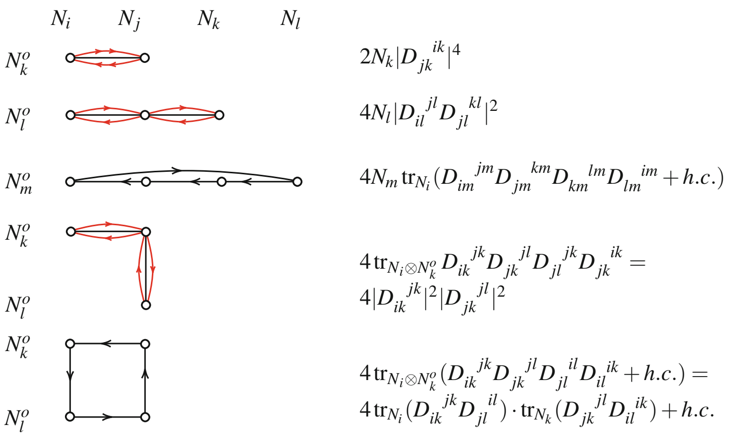

Fig. 1.5 All types of paths contributing to the fourth power of a finite Dirac operator. The last two only occur when it is part of a real spectral triple

$$\begin{aligned} D_{ij}^{kj} &\to D_{ij}^{kj} + \sum_n a_n[D_{ij}^{kj}, b_n] \\ &= D_{ij}^{kj} + \sum_n (a_n)_i(D_{ij}^{kj}(b_n)_k - (b_n)_i D_{ij}^{kj}), \qquad a_n, b_n \in \mathscr{A}, \end{aligned} \tag{1.35}$$

where $(a_n)_i$ denotes the ith component of the algebra element a_n. It describes a scalar Φ_{ik} in the representation $\mathbf{N}_i \otimes \mathbf{N}_k^o$. In the expansion (1.24) of the action for an almost commutative geometry the kinetic terms for the components of Φ appear via

$$\{\partial\!\!\!/_A, \gamma^5 \otimes \Phi\} = i\gamma^\mu\gamma^5[(\partial_A)_\mu, \mathrm{id}_{L^2(S)} \otimes \Phi].$$

We determine it for a component D_{ij}^{kj} of Φ in particular by applying it to an element $\zeta_{kj} \in L^2(M, S \otimes \mathbf{N}_k \otimes \mathbf{N}_j^o)$ and find that

$$\begin{aligned} [(\partial_A)_\mu, D_{ij}^{kj}]\zeta_{kj} &= (\partial_\mu + \omega_\mu)(\Phi_{ik}\zeta_{kj}) - ig_i A_{i\mu}\Phi_{ik}\zeta_{kj} + ig_j\Phi_{ik}\zeta_{kj}A_{j\mu} \\ &\qquad - \Phi_{ik}(\partial_\mu + \omega_\mu)(\zeta_{kj}) + ig_k\Phi_{ik}A_{k\mu}\zeta_{kj} - ig_j\Phi_{ik}\zeta_{kj}A_{j\mu} \\ &= \big(\partial_\mu(\Phi_{ik}) - ig_iA_{i\mu}\Phi_{ik} + ig_k\Phi_{ik}A_{k\mu}\big)\zeta_{kj} \\ &\equiv D_\mu(\Phi_{ik})\zeta_{kj}, \end{aligned} \tag{1.36}$$

where we have introduced the *covariant derivative* D_μ from which the operator ω_μ has dropped out completely. We have preliminarily introduced coupling constants $g_{i,k} \in \mathbb{R}$ and wrote $\mathbb{A}_\mu = -ig_iA_{i\mu} + ig_kA_{k\mu}^o$ (with $A_{i\mu}, A_{k\mu}$ Hermitian) to connect with the physics notation.

The *gauge group* that is associated to an algebra of the form (1.30) is given by

$$SU(\mathscr{A}_F) := \{u \equiv (u_1, \ldots, u_K) \in U(\mathscr{A}_F), \det{}_{\mathscr{H}_F}(u) = 1\}, \tag{1.37}$$

where $U(\mathscr{A}_F)$ was defined in (1.15) and with $\det_{\mathscr{H}_F}(u)$ we mean the determinant of the entire representation of u on $\mathscr{H}_F$. Applying $U = uJuJ^*$ to an element $\psi_{ij} \in \mathbf{N}_i \otimes \mathbf{N}_j^o \subset \mathscr{H}_F$ and typical component D_{ij}^{kj} of the finite Dirac operator yields

$$\psi_{ij} \to uJuJ^*\psi_{ij} = u_i\psi_{ij}u_j^* \tag{1.38a}$$

cf. (1.15) and

$$D_{ij}^{kj} \to uJuJ^*D_{ij}^{kj}u^*Ju^*J^* = u_iu_j^{*o}D_{ij}^{kj}u_k^*u_j^o = u_iD_{ij}^{kj}u_k^*, \tag{1.38b}$$

respectively.

We have now covered the most important ingredients for particle physics using almost-commutative geometries. In the next Chapter, we proceed by motivating the choice to search for supersymmetric theories that arise from noncommutative geometry.

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