Category Theory and the Search for Universals: A Very Short Guide for Philosophers

Alberto Peruzzi

Abstract The aim of the paper is to present the categorical notion of an adjoint functor as a key to formally capturing the philosophical notion of "universal" especially as it figures in relation to semantics and epistemology. In the first part (first section to seventh section) the relevance of category theory for the main topics of analytic philosophy is suggested, in opposition to a widespread conservative attitude towards the entrenched conjunction of logic and ∈-based set theory. In the second part (8th section to 16th section) the concept of an adjunction is introduced and shown to provide the framework of some fundamental examples of universality. The paper is of an introductory character, because it is addressed to a broad philosophical audience, in particular to philosophically-oriented logicians and logically-educated philosophers with no previous knowledge of category theory.

Keywords Adjoint functor · Analytic philosophy · Category theory · Logic · Universality

Mathematics Subject Classification Primary 97E20 · Secondary 18C10 · 03G30

Motivations and Questions

Asked to characterise twentieth century philosophy, the list of features to be considered could be the subject of long debate. But among them there is undoubtedly an unprecedented attention to language and structure: an attention which also results in the mutual use of these notions within the study of each. As for the structure of language, it is widely acknowledged that its expressive power, its role in controlling the range of possible inferences, and finally the internal limitations related to self-reference have all been specified in logical terms. As for the language best suited to the description of structure in general, it was generally believed in the community of logicians and philosophers working in the analytic tradition that this was supplied by ∈-based set theory and that no structure could be defined without appealing to the notion of membership. Thus, logic + set theory in this view was taken to offer the main, and most comprehensive, framework for any philosophical investigation aiming at conceptual clarity and inferential rigour in the analysis of meaning and knowledge.

In fact, philosophy has always addressed issues which call for more than just thoughtful attention to meaning, rigour and the formal articulation of ontological inquiry. The twentieth century is no exception. But it is disputable whether the goals that philosophy

F.F. Abeles, M.E. Fuller (eds.), *Modern Logic 1850-1950, East and West*, Studies in Universal Logic, DOI 10.1007/978-3-319-24756-4_8

aims at beyond this can be pursued without such an underlying framework; and even the claim that the results of such investigations are untranslatable into such and such a framework already presupposes some competence in logic and set theory. Given the variety of formal languages, logical systems and axiomatic set theories we face today, any thesis as to the manner in which to pursue general philosophical inquiry on language and structure is at risk of vicious regress or being burdened with idiosyncratic technicalities.[1]

The search for underlying principles which can provide a unified view of the variety of languages, logics and structure-kinds (with their corresponding "signatures") brings philosophy back to the longstanding debate on "universals" in dealing with the structure of concepts. "Universals" have been taken in different senses, each deserving careful analysis. But such analysis can be expected to have consequences for our view of ontological architecture and the architecture of knowledge as well. If there is a mathematical clarification of the concept of "universality", it affects the search for "universal" principles on which to rely in order to supply a foundation of knowledge in any domain, once that knowledge is expressible in mathematical form. In the foundations of mathematics, such a clarification already exists and deeply affects the way we look at them.

Category theory identifies the principles that govern the architecture of mathematical structures in their mathematically significant relationships, and expresses these in the form of mutual constraints satisfying certain *universal* properties, which have consequences for the characterisation of each specific structure involved whilst at the same time providing cross-domain principles of characterisation.

The resulting conception of mathematical structure is independent, in general, of the internal composition of the objects in question—more specifically it does not depend on their being treated as *sets*—but this altered conception does not involve some vague appeal to the no less vague totality of relationships in which something stands to something else. Thus, as a conceptual framework, category theory is neither "substantialist" nor "relationist". What this theory aims at is rather the detection of a highly specific kind of relationship between structures, one that takes the form of special maps which we can define as "universal". Moreover, the explanatory power of such universality lies in its being sufficient to recover the known properties of the structures standing in such relationships, and in making it possible to discover new properties of those structures.

To what extent does the general notion of structure, as commonly used in mathematics, linguistics, physics and other sciences, fit with such a categorical framework? Can the principles governing such a framework lead to a satisfactory foundation of mathematics? How do they affect our understanding of logic? One principal achievement of category theory is that the basic notions of logic can be defined in terms of universal properties as expressed in the theory. But what is that specific kind of universality?

[1] For any given formalised philosophical thesis, by its "idiosyncratic" character I mean one *essentially* relative to a chosen formal "base", thus one that cannot remain stable under change of base.

Topics of Philosophical Interest

These cursory remarks, in much need of further clarification, and this list of questions suggest reasons why category theory deserves the attention both of philosophers and mathematicians interested in foundational issues. Further reasons for its study by philosophers are related to the way categorical universality affects the treatment of the following topics:

(a) MEANING. The concepts of category theory make possible a semantics that is more general, but no less rigorous, than the extensional semantics which forms the basis of classical model theory and the semantics for modal idioms that makes use of quantification over possible worlds. This gain in generality and flexibility allows a more abstract, but also more finely articulated, analysis of the traditional opposition between extensional and intensional contexts.

(b) THEORY AND TRUTH. Category theory strongly constrains and reformulates the concepts of “theory”, “interpretation” and “model” in an original and fruitful way, so that the resulting framework contains Tarski’s notion of truth as a special case, and at the same time also provides a sophisticated analysis of predication in natural languages.

(c) FORM OF PRINCIPLES. The special connection between these two features, (a) and (b), already reflects a striking and novel feature of categorical semantics. Of further philosophical relevance is the *equational* character of the principles expressing “universal” properties. This character, together with the emphasis on the search for what is “invariant”, suggests new ways of connecting the foundations of physics with the foundations of mathematics.

(d) LOGIC. As already mentioned, category theory raises the possibility of identifying the basic notions of logic (such as connectives and quantifiers, but also those of variables and constants) in terms of universal properties and hence offers a profound re-orientation in the understanding of those notions. For instance, the choice between classical logic and intuitionistic logic turns out to be neither a pragmatic matter (as selecting between two different conventions) nor due to any commitment to either a realistic or a constructive view of mathematical entities. Rather, it is determined by the algebra of sub-objects of any object in a given category (one equipped with a suitable structure to deal with logical notions). Just how far-reaching are the philosophical consequences of this re-orientation and the new meaning of the “universality” of the constructions involved will be a central theme in the following exposition.[2]

---

[2] There is already an extensive literature on each of the aspects (a)–(d), but it presupposes familiarity with categorical language. Those who wish to learn more about how the notions introduced here (along with others) have been put to work could begin by consulting the following papers: [3, 10, 13, 24]; in particular, for categorical logic see [5, 6, 31].

Which Organon for Philosophy?

Until now, the community of philosophers who studied and made use of category theory has been narrow and largely confined to those concerned with the philosophy of mathematics, see [21, 22, 30]. However, the substantial impact of the theory is now evident in various fields of research outside mathematics itself: the notions of category theory have proved to be relevant in theoretical computer science, linguistics, physics, systems theory, theoretical biology and even economics.

This is not the place to analyse the reasons for the delayed and circumscribed impact of these notions on topics of major importance for philosophers of language, science, logic and mathematics. Those who recognise the centrality of the analysis of language to philosophy should also recognise the scope of the re-orientation within that analysis which category theory produces—particularly for (a)–(d). The very idea that one has to formulate any philosophical question carefully and unambiguously, in order to check the relevant arguments, intrinsically poses the demand for appropriate tools. It seems that members of the philosophical community who follow the line of development initiated in the linguistic turn still think the tools devised a century ago for this aim cannot be superseded.

Ever since Aristotle, analysis of the structure of propositions and inferences has been an essential tool of philosophy. No-one lacking familiarity with this tool could expect to make progress in many central areas of philosophy.

Such lack of familiarity implies lack of awareness of the patterns of inference constitutive of rationality itself and hence the likelihood of errors in reasoning, given that most knowledge-claims are the result of inference and truth, is not transparent. While logic as the instrumental hygiene of all inquiry may not be sufficient to guarantee truth, it is a *sine qua non* of the *search* for truth. That lesson should never be forgotten. But the specific form the ideals of clarity and rigour have taken in history should not be transformed into a dogmatic specification of the form they must take.

If our intellectual spectacles distort or darken our vision, even our most modest cognitive claims may be compromised. Hence it is necessary to have the right equipment to ensure the spectacles are kept clean; and syllogistics was intended as part of that equipment. This was no longer so for early modern philosophers such as Bacon and Descartes, who made harsh criticisms of syllogistics. Descartes assigned the place previously held by logic to algebra. Bacon and the empiricists of the seventeenth and eighteenth centuries insisted that the warrant for knowledge-claims was unattainable by purely rational methods and the advance of knowledge was hindered by a logical attitude, which presumed knowledge to be expressed in subject-predicate form. Moreover, since Galileo the relational character of physical laws was recognised and emphasised as a characteristic feature of the new physics. Whereas, even at the period of its origin, the theory of the syllogism in the Organon was already ill-adapted to the relational principles and patterns of inference found in Euclidean geometry, the language of which was used by Galileo before the discovery of the Calculus and indeed by Newton afterwards.

The same limitation is far more apparent in relation to the principles and patterns of inference of post-Newtonian science, but already in ordinary discourse there are many arguments which stand in need of a richer formal framework. It is commonly thought that the lesson was learned and that the tools of formal logic fit much better with contemporary scientific theories, insofar as they are expressed in mathematical language. Mathematical logic was developed over a century ago precisely to identify and clarify the structure of thought, in particular as manifested in mathematics.

It was even held by some to have achieved such a final clarification, in the form of the claim that there is One True Universal Logic and that all mathematical notions are definable within that Logic. Such a claim seems very far-fetched today, when the manifold formal systems at the disposal of logicians are so numerous that the task of finding the one appropriate to a given universe of discourse (or of manufacturing the required system *ex novo* by means of suitable tinkering with logical axioms and rules) is the everyday activity of logicians. The presumption that a universal logic able to serve as an umbrella for all such systems, is the pre-ordained and timeless underlying format of thought has been of little help in guiding philosophical inquiry, except in pushing some philosophers towards the conclusion that there are contents of thought which escape the possibility of formalisation.

Ordinary language, in which most philosophical problems are still expressed and considered, is every bit as complex as mathematical language. Though modern formal logic has expanded into many other areas of philosophical inquiry, it has there faced further challenges posing such questions as to the limits of any axiomatic formal system.

That attention to language which has been so central in shaping "analytic philosophy" could not have existed without the formal treatment of connectives and quantifiers for a language with $n+1$-ary relations; in turn, that treatment is argued for in terms of some underlying mathematical structure, thought of as set-theoretic in nature. As already noted, not only are the specific resources of such a formalism assumed by arguments aimed at an ideal language in which the structure of thought is (allegedly) finally rendered transparent, but the very same resources are assumed in arguments supposed to demonstrate the inadequacy of formal methods to capture the expressive subtleties of natural language. Indeed, no serious account of such notions as truth, meaning, knowledge, explanation, objectivity or even free will can dispense with mastery of logical tools, though such mastery can by no means be a sufficient condition for such an account.

It is true that there have been philosophers, even amongst those inspired by Aristotle, whose work showed little familiarity or engagement with the theory of the syllogism. It is also true that those who do not care whether the lenses of their intellectual spectacles are clean or not, typically end up not being able to see clearly or very far. What is sometimes overlooked is that those who are so focused on the state of their spectacle lenses as to lose interest in the objects in their field of view also end up in the same way. Hence the ironic quality of Kant's remarks on "the gymnastics of the learned".

In the twentieth century, many analytic philosophers identified logical investigation of language as the core of First Philosophy. Differences as to which fragments of logic were of the deepest philosophical consequence were often at the heart of their arguments. For example, classical first order logic was customarily treated as the central tool and paradigmatic idiom of ontological inquiry—sometimes outfitted with modal or epistemic operators, sometimes modified to take account of other constraints and desiderata, such as constructivity. Irrespective of whether that choice of organon has always been fruitful and whether the categorical approach suggests a viable alternative and in reality a more comprehensive perspective, it is certainly a matter of historical record that this choice of logical tools regulated the way the scope and character of systematic ontological inquiry were conceived in analytic philosophy.

The Bias for Sets

The analytic "turn" has remained essentially tied, albeit often implicitly, to the assumptions embodied in semantics *à la* Tarski, which find expression in the language of set theory. The adoption of non-classical logical systems in treating certain problems or areas of inquiry did not fundamentally alter this situation. Set theory, conceived as the ontological hinterland of quantification theory, provided the fundamental framework. That some authors opted for an axiomatisation of set theory differing from classical ZF through a desire to reduce ontological commitments (e.g. to higher infinities) as a result of constructivist leanings did not essentially change matters.

It is true that more conceptually far-reaching alternative ontological frameworks have been proposed, such as mereology, prompted by philosophical motivations distinct from those underpinning the Tarskian option. But they have not established themselves as serious rivals, since there has been little convincing evidence either for (1) their relevance to mathematical practice, (2) their import for foundations, (3) their transferability to domains structured in terms of basic notions other than those they were originally intended to model, (4) their effectiveness in providing a general idiom of systematic scientific inquiry or (5) their help in clarifying our ontological commitments.

In all five of these respects, set theory continues to provide an inescapable point of reference: one that permits further and flexible specification tailored to the syntactic and semantic structure of a wide range of formal and natural languages and to the expression of the ontological commitments of a wide variety of bodies of knowledge.

No one should ignore the power of set theory, nor underestimate its adaptability. How else would its vocabulary have penetrated so many areas of mathematics? To this must be added the ease of grasping its basic notions—a feature which has made possible its introduction even into the primary school curriculum

Finally, regardless of the axiomatisation adopted, the theory of sets has allowed us to do an extraordinary thing, namely to give a unified foundation for the whole of mathematics: within specific branches specific axioms select certain sets rather than others, but the fact remains that all the subject matters of which mathematicians speak in different regions of their discipline can *au fond* always be regarded as variously structured sets.

In the early twentieth century, the efforts to axiomatise set theory were closely related to positions in the philosophy of mathematics. But later, as the landscape of alternative axiomatic systems for set theory developed into an intellectual mirror of the Amazonian rain forest, the technicalities involved in detecting and comparing the ontological commitments of one proposed system relative to another became more and more demanding (mainly in connection with issues about the size of large cardinals). As a result, philosophy of mathematics retreated into concerns about the nature of mathematical entities, and to a concern with the limitations on both provability and definability, stemming from Gödel's Second Incompleteness Theorem. This was paralleled by a concern with the consequences of Tarski's Theorem for the internally "ineffable" character of truth, with the Löwenheim-Skolem Theorem (leading to the Skolem Paradox), and with a special attention to the consequences deriving from the non-computable character of notions relevant for the philosophy of mind.

While mathematics has been an area of minority concern among linguists and philosophers of language, it is nonetheless striking that those philosophers for whom the analysis of language occupies a primary place in philosophy, should seem to think that such a strategy can stop short of trying to analyse the sources of the expressive power of the language of set theory in understanding the structure of any universe of discourse. In particular, analytic philosophers frequently discuss the notion of truth as if the problems posed for the notion of truth about the set-theoretic hierarchy could be by-passed.

The intuitive familiarity of the notion of membership seems tailor-made to illustrate a doubt, encountered in many areas of inquiry, about the status of definitions; a doubt which plays a crucial theoretical role—namely, whether the definiens is really less problematic than the definiendum. For, in set-theoretic terms, the truth (in a fixed valuation v) of a presumedly atomic statement such as "Mike is blond" is equivalent to the fact that the individual referred to by the name "Mike" (interpreted by v) belongs to the set of elements (of the given domain) which the valuation assigns to the predicate, i.e., v(Mike) $\in$ v(Blond). But notice: there are two far-reaching presuppositions involved in such a claim, namely, (1) that the reference of an individual constant could bypass the structure which makes the stability of individual entities possible and (2) that the use of any predicate could bypass the question of how a (purportedly timeless) collection is accessed.

Set theory, however, is not the last word, or at least was never proved to be so. Since the mid-twentieth century, category theory has developed to occupy a position in which it offers an alternative to $\in$-based set theory, since it provides a language no less expressive and, most importantly, aims to provide a more adequate framework for both the foundations and the architecture of mathematics (the two being no longer treated as separate issues). Moreover, it is one endowed with greater simplicity and ease of use, given suitable training. But above all it is one which provides a superior understanding of the relationship between different areas of mathematics and better reflects the evolving character of that relationship.[3]

To go back to the previous example, every philosopher is familiar with the distinction between type and token, but only a few care for investigating the structure of a collection of types. At its birth, the idea of a hierarchy of types with mutual relations (of logical significance) could be thought of simply as an antidote to paradoxes. But after the development of a typed lambda-calculus, type theory became a practical resource (think of "data-types"), proved to offer a general framework for computer science and, through the Curry-Howard correspondence (proofs-as-programs, propositions-as-types), for a new perspective on the whole field of logic itself. Over the last decades, various kinds of constructive type theory have been introduced, suitably reformulated and powered with respect to its original non-constructive version (whether the hierarchy is simple or ramified) and have been argued to provide as appropriate foundational systems, from Martin-Löf type theories to the recent "univalent program" focused on homotopy type theory. Such foundational projects have deep implications for the way philosophical questions about foundations of mathematics are formulated.[4]

Analytic philosophers have thus far given little attention to this new generation of type theories. However, they contain at least two aspects which should be of interest to them. The first is that, in view of Cantor's Theorem, the standard universe of sets contains no

[3]The phenomenology of these relationship is extensively described in [18].

[4]As for the relationships between intuitionistic type theory and topos theory, [4, 11] are essential references. At a more basic level, [1] is a clear presentation of the categorical approach to computability.

X (other than a singleton) such that $X^X \subset X$, whereas there are domains of theoretical relevance which call for such an X and these can be described in categorical terms. The second is that semantics for higher-order lambda-calculi makes an essential appeal to category theory. Conversely, there are type-theoretical translations of the language of categories, but such translations rely on assumptions which, however well motivated, are philosophically demanding: in fact, they involve a commitment to specific properties for the categories which support such translatability.

Passing to Categories

Many continue to believe that the language of sets has conceptual priority in some manner with respect to the primitive notions of category theory. Though this issue could be discussed endlessly, this introductory paper is not the place for such discussion. But to provide just one (biased) example of a parallel issue in intellectual history, the replacement in physics, of a language which centered around static qualities with one formulated in terms of dynamical quantities met with a similar resistance at the period of Galileo and his struggle with the Aristotelians of his time.

Further: (1) since the notion of collection is exploited in defining a category and (2) since the very notion of membership is itself defined within the categorical approach, rather than being taken as primitive, a certain puzzlement on the part of the mathematician, logician or philosopher trained to think in terms of set theory is to be expected.

For many years a wide-ranging debate has been conducted on whether and in what sense category theory can be claimed to provide a foundation which is really an alternative to the theory of sets, and as to the very sense—or multiple senses—which the expression “is a foundation for” can take with respect to mathematical notions and about the ultimate meaning of the methods of definition and construction of category theory. Beyond that foundational debate, a further controversy concerns the philosophical significance of category theory.

For example, there are those who have seen in category theory the revenge of Heraclitus over Parmenides in mathematics while others have seen the theory as the most precise formulation of the position which among philosophers goes under the label of structuralism in mathematics (and in some versions also in physics and in metaphysics). From the Erlangen Program to Bourbaki’s *structures mères*, there is indeed a “structuralist” tradition in mathematics which might support such a reading. But even setting aside the historical circumstance that Bourbaki mingled aspects of that tradition with a poorly designed set theory, the issue of “structuralism” deserves careful re-examination, especially on the part of dialectically-minded philosophers. Such examination is needed, on the one hand because previous attempts to develop a dialectical logic did not produce a system able to provide a foundation for the whole of mathematics; on the other hand such re-examination of the notion of structure is suggested by the fact that category theory does not aim to embody some exotic non-standard logic, but rather emphasises the intrinsic “emergence” of logical notions themselves out of mathematical structure.

The resulting changes in viewpoint on logic resulting from category-theoretic developments have been the subject of extensive discussion, although the task of putting these developments to work in the analysis of natural language has thus far been less explored, apart from some aspects associated with "categorial grammar".[5] Needless to say, such analysis can be of great importance in advancing research in this field. But thus far, recognition of the pertinence of category-theoretic notions for the logical analysis of natural language has been noticeably lacking in this area of philosophy.

Just as happened in the early years of analytic philosophy with respect to what was then the *novissimum organum*, i.e., mathematical logic, so with category theory: developments and changes in viewpoint in philosophy resting on the use of a new language and a new set of tools have met with widespread resistance and hostility.

Like it or not, the terms of the new debates which have opened about the foundations of mathematics, and to a lesser extent and in a more preliminary form, in semantics and the theory of meaning, with the advent of category theory are intelligible only to those with some knowledge of that theory. In this there is nothing strange or suspect, since to undertake philosophical reflection on a topic as it appears by means of a tool, one must first know something not only about the subject but also about the use and purpose of that tool. Just as to judge a musical instrument one must know how to play it, as well as knowing something about music.

After one century of analytic philosophy, it would be unthinkable that a graduate student wanting to do research on any topic, certainly on a "recognised" topic within the norms of the analytic community, and one which might lead to an academic career, could totally ignore logic. However, if that same student is told that, from a categorical standpoint, the investigation of logical notions involves some familiarity with topology and geometry, the reaction is often one of bewilderment or resistance. Yet the task of making explicit the knowledge which is presupposed or hidden in our use of language was at the core of the project initiated by the founding fathers of the "linguistic turn". The claim that some implicit knowledge of topological or geometric concepts is necessary for any understanding either of the way we talk about the common sense world or the realm of science should thus strike such a student as not only a profoundly consequential suggestion from the standpoint of general philosophical inquiry, but one firmly located within the agenda or syllabus which has shaped the methods and outlook of the analytic tradition.[6]

Universality Only Becomes Recognisable in an Expanded Universe

Whereas for the mathematically educated reader there are (at least in English) many good introductory texts to category theory (some of them freely accessible on the internet), the same is not true for those educated in philosophy.[7] As an aid for young philosophers

[5] Jim Lambek pioneered the investigation of categorial grammar in terms of category theory, see [9]. *En passant*, a purely lexical note: since the adjective "categorical" was already established in model theory as having a very specific meaning, one might opt for "categorial" in application to category theory. But unfortunately, "categorial" has a no less specific meaning within grammar. In the face of such an impasse, standard usage in the community in question is declared the winner.

[6] Readers who know Italian can find these aspects examined in [29].

[7] A starting point is [23]. For a book, see [16].

interested to learn what category theory has to offer, this essay aims to introduce, albeit very briefly, its basic notions and illustrate them with some examples. Therefore, in the following pages you will not find an account of the stages through which category theory has developed and which led to the program of the categorical foundation of mathematics; nor a discussion of the applications of these basic notions to other philosophical topics, such as those mentioned above.

Even cursory exposition of the categorical reformulation of logic, in particular of semantics, and of the motives underlying that reformulation would presuppose familiarity with those basic notions of category theory this work aims to provide. Still less will we be able to study specific applications of category theory to classical problems of philosophy, and analytic philosophy in particular. Such topics include the contrast between "intensional" and "extensional" aspects of meaning, the semantics of mass nouns, the non set-theoretic aspects of predication, whether the objectification of functions provides a good "meter-stick" for registering ontological commitments, and the study of recursion and diagonal arguments (in relation to self-reference)—to list only some relatively well-established uses of categorical methods in philosophy.[8]

But, just to illustrate the relevance of such methods for the analysis of language, we can point to the radical change of setting for the theory of definite descriptions, the oft-cited "paradigm of philosophy", that results from its being re-presented in a form drawing on the conceptual toolkit made available by category theory: the greater subtlety of analysis for singular terms made possible here by the use of these tools, [25], also suggests lines on which other central issues in philosophy may not just benefit from categorical treatment or reformulation, but may actually require it.

The following pages do not aspire to originality. There is, however, one aspect of the presentation here that differentiates it from other introductory material addressed to a philosophical audience. This is the emphasis on the concept of *universality*: one of the most recurrent and debated in philosophy. Not only does this concept receive a precise formulation in category theory: it can be considered its very leitmotif.

Among the tasks of philosophy, some presuppose that others have already been accomplished. The latter can thus be considered as basic. To this class belong the tasks of identifying the fundamental concepts involved in a given subject-matter, clarifying their exact sense (to be selected for theoretical aims) and distinguishing it from other senses manifested in common language. No less basic is the exploration of the precise ways in which such fundamental concepts are combined and in which their mode of combination contributes to the theoretical articulation of the subject in question. To understand the philosophical significance of a theory—in particular its epistemological basis—it is not enough to ask what kinds of things the theory speaks about, the ways in which they are related, how do we access them, and which grounds can support the theory's principles, once it is axiomatised. It is also necessary to provide an account of those patterns of construction that make it possible to use these concepts in unintended or unforeseen contexts and as a consequence of this cross-domain character, their role can be motivated in a non *ad hoc* manner.

In the case of category theory this additional requirement is decisive, (1) because these very patterns are the subject matter of the theory itself. Thus we find ourselves dealing

[8] These topics are covered by essays collected in [19].

with a situation involving an epistemically productive example of self-reference, and (2) because it is a theory that deals equally with objects and maps, entities and processes, sets and functions, data-types and algorithms, structures and their transformations: in sum, whenever it alludes to the existence of something, that existence condition is always characterised as the (unique)[9] result of a *universal* construction. Within category theory, the very notion of "for all *x*" which at first sight seems free of any reference to maps is shown actually to be the track of an implicit map, relatively to which the notion is defined by means of a universal property. The same applies to the notion of "for some *x*". Thus, both universal and existential quantification are instances of one and the same notion of universality, associated with the *adjunction* pattern (between functors), which is the core of the theory.

If, at the end of these philosophical considerations, I may be permitted a personal remark: I have written elsewhere that analytic philosophy, having provided the most important chapter of twentieth century philosophy, is dead. Indeed I thought I had contributed to the collection of arguments required for its epitaph. Now I want to add a rider. I should be willing to withdraw this obituary notice if it becomes clear it is possible to re-fashion the program of analytic philosophy in a consistent manner by expanding its underlying logical resources with the aid of category theory, rather than keeping them confined to the universe of sets (whilst not neglecting the role of sets and functions as a limiting instance of that of objects and maps).

To be brought to full fruition, such a re-conceived version of the analytic program likely requires a new generation of philosophers free from the burden of sacred texts and the evocation of standard authorities. The shift of focus over the last 20 years by many philosophers of language and those working on cognition, towards the new frontiers of computational science and neuroscience has, I believe, not so much undermined the neo-scholastic attitude inherent in much current work in analytic philosophy as spread it to further fields. But seminal work of a quite different sort, which cannot be indefinitely ignored, promises to bring about a shift in the overall picture and some of these developments lead me to believe such a re-conceived program is already taking shape. If the conceptual tools presented in the following sections were of help in its development, I would be glad to concede that my former epitaph was wrong.

Of Substance and Structure: Which Ontology?

It requires no appeal to *a priori* principles to recognise (1) that the objects which we refer to (be they individuals or species, particulars or universals, concrete or abstract entities) as well as the systems which they compose are more then just bags of dots; (2) that the features whereby objects and systems can be identified are not reducible to a fully bottom-up or fully top-down representation in terms of discrete quantities; (3) that, rather, the very existence of the objects and systems we refer to is the outcome of processes which are described in terms of continua and with respect to which objects and

[9]"Uniqueness" is always intended as *up to isomorphism*, in a sense that will be defined in what follows. The fact that a theory allows us to prove the existence only of those x's that are unique (in satisfying a given condition) marks one of the distinguishing features of the categorical approach.

systems preserve enough of their structure to be re-identified, to the extent needed for the relevant "qualitative" states to be discretely distinguished in language; (4) that, if objects and systems change, their variation must be subject to law-like regularities, otherwise they could not be reliably detected and stably referred to. This is a requirement quite independent of our success (or failure) to understand the principles governing the specific conditions of being and becoming.

The syntactic manipulation of symbols is no exception: if language is not extramundane, its patterns of construction are those of a hosted subsystem, and the existence of a range of interpretations for any set of linguistic expressions is made possible by the properties of the hosting system. In its turn, the structure of the whole system is constrained by the requirement that it jointly hosts the structure of language and the structure of models for any theory expressible in the language. Though the possibility of a common formalisation for both structures was one of the great achievements of set theory, the strict dependence of a theory on its syntactic presentations by means of axioms as well as the absence of constraints on interpretation (for it only has to be a function) left the relation between system and subsystem unprincipled.

The categorical approach, centered on a very general kind of algebra, provides a framework which takes account of (1)–(4) in a way which highlights the geometric structure of variable objects and systems, and better reflects the fact that relationships between language and world themselves have just this kind of principled genesis in the world.[10]

What matters most in this way of thinking is not the search for an ultimate ontology (the questions of what kinds of entities there are, which are entities of a given type and which of another, which possess a given property and which do not, in which relationships objects—and types—stand to one another, which are to be treated as fundamental—the basic "substances"—and which are to be seen as the derived/dependent/supervenient ones). But it does not at all follow from this that such a way of thinking is "structuralist" in nature (in spite of what many believe), nor does it follow that we should content ourselves with a pluralist ontology or a linguistic reformulation of the above substantialist perspective by means of even the most sophisticated type theory.[11]

Indeed, it should be noticed in passing that structuralism, in its most widespread current versions, is itself the limiting case of a particular ontological perspective: if entities are defined by the structure to which they belong, i.e., from the collection of mutual relations in which they stay, Structure itself takes on the role of Substance in former ontological traditions. For if Substance lies in a web of relations rather than in things, then, or so the structuralist seems to be claiming, it is Structure itself that now takes on the role of that-in-virtue-of-which any specific thing is identified.

One might object that structuralism is always confined to a specific language L and to a specific class of structures associated with L as the class of possible "universes-of-

[10]There are also "systemic" approaches to semantics, but so far they are inadequate for the management of logical syntax and call for notable modifications as we move from one semantic field to another. Thus they lack the uniformity found in the categorical approach.

[11]The objective conditions for the possibility of exercising any typing activity are usually set aside in the construction of a type theory. Consequently, the entire philosophical discussion on the choice of one type theory rather than another is essentially pre-Kantian.

discourse". But a consistent relationism based on the notion of role-in-a-structured-system lends itself naturally to recursive iteration. Therefore, any single structured system would in turn be defined by the whole set of relations it has with the others and with itself. As a whole, this recursive hierarchy of structured systems is absolutely indeterminate, since no one can have access to it in the limit and the totality of relations which link structured systems to each other is no less indeterminate.

But then: how to avoid a vicious circle or an infinite regress? In order to define each structured system S, some determinate (possibly finite) set R of relations between S and some determinate (possibly finite) set of systems ought to be sufficient. But which R? And why just those relations and those other systems, rather than others? Integral relationism cannot answer such questions without contradicting itself, because in order to answer it must appeal to constraints of a more than generically relational nature.[12] Category theory permits an answer. That is by no means its only claim on the attention of philosophers, but this feature of category theory alone should already be enough to compel such attention.

The Search for Universality at the Core of Category Theory

The old debate between substantialism and relationism took on many new guises in the twentieth century and the two opposing positions attracted new adherents whose articulation and exploration of the issues reflected their points of entry into the debate through concerns specific to disciplines as diverse as linguistics, physics, cognitive science, epistemology, as well as mathematics. For at least as long as the terms of the debate remain locked in the forms sketched here, it will presumably never end.

But whether objects are defined by the elements that compose them or by a network of mutual relations, it has to be the case that we can only identify the objects we are talking about within a given system (and furthermore, can only identify that system itself) if the reference of some expressions of our language to objects external to that system is kept fixed. This is to recognise what I have elsewhere labeled the Principle of Invariant Referential Potential [27]. Otherwise we could not even understand what we intend to uniquely characterise or argue why we are able (or not) to characterise them in one way rather than another.

Here it is crucial to recognise that the uniqueness in question does not refer to the identification of a thing-in-itself (be it a ur-element, an object, or a network of relations) but rather to the existence of universal properties, intended not so much as relative to an isolated system but rather as concerning a special kind of relation between coupled systems, with possibly different structure. It is precisely this requirement that is met by category theory and, more particularly, which is supplied by the concept of adjunction between functors.

In the definition of adjunction four concepts come into play: *naturalness*, *isomorphism*, *functor* and *category*. They are not mutually independent. Indeed, each concept assumes the following one. Only when all of these concepts, starting from that of category, have

[12]The escape route provided by saying that anything (hence any such added constraint too) can be relationally conceived, though not all at the same time, is still waiting for a consistently relational justification.

been defined is it possible to understand why adjunction allows us to capture a notion as philosophically important as that of "universality", one which spans the whole history of philosophy.

It is not the first time the concept of adjunction has been assigned such a central role. It may be objected that the concept of universality is itself more general than the concept of adjunction and that it retains this wider generality in category theory too. But, in all mathematically significant cases of universality, it is noteworthy in my view that we find ourselves dealing with adjoint functors. This circumstance brings to mind a saying of Saunders Mac Lane, that in mathematics it is correct and fruitful generality, as opposed to maximal generality, that is to be sought. As was noted for the first time by Bill Lawvere, the notion of adjunction allows us to make precise a sense of the notion of the foundations of mathematics which is distinct, in conception aim and method, from the main proposals advanced under that label in the first half of the twentieth century and formally refined in the second half.

To reiterate: this paper is not the first to place the concept of adjunction at the heart of category theory or to recognise it as a tool of great power in the genesis of particular mathematical discoveries and in the overall conceptual organisation of mathematics. Nor is it the first time that someone suggested its overall philosophical importance, [2]. Lawvere was the first, [12], to propose such a conception of foundations. He also argued that the Thesis of the Dialectical Unity of Opposites can be given exact expression in the form of adjoint functors, [15]. This paper tries to provide the minimal toolkit required for understanding such claims and for devising applications to further topics.

The Definition of an Adjunction and Its Prerequisites

The most common definition of an adjunction is as follows: a functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$, which is written $F\colon \mathbf{C} \to \mathbf{D}$, is a *left adjoint* of G (and G is a *right adjoint* of F) if and only if, for each object A in $\mathbf{C}$ and each object B in $\mathbf{D}$, there is a natural isomorphism between morphisms in $\mathbf{D}$ from FA to B and morphisms in $\mathbf{C}$ from A to GB. The existence of an adjunction thus defined between F and G is denoted by $F \dashv G$.

Three concepts are involved in this definition: *natural isomorphism*, *functor*, and *category*.

Definition of a Category and Some Examples

We shall begin by saying that a *category* $\mathbf{C}$ is given by a collection of objects $A, B, C, \ldots$ and morphisms (arrows, maps) $f, g, h, \ldots$ (between objects) such that each morphism has a unique object as its domain (source) and a unique object as its codomain (target). So, $f\colon A \longrightarrow B$ signifies the morphism f is from A to B, where A is the domain and B the codomain of f.[13]

[13]The uniqueness condition in this definition can be relaxed if one aims at an even more general framework.

In addition, two consecutive morphisms $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ compose, giving rise to a morphism $g \circ f\colon AcC$. It is assumed that this composition is associative, i.e., that if $h\colon C \longrightarrow D$ is a third morphism then $h \circ (g \circ f) = (h \circ g) \circ f$. Finally, it is assumed there is an identity morphism id_{-} (also noted as 1_{-}) for each object "−", i.e., for any objects A and B there are morphisms $id_A\colon A \longrightarrow A$ and $id_B\colon B \longrightarrow B$, such that for every $f\colon A \longrightarrow B$ and any $g\colon B \longrightarrow C$, $id_B \circ f = f = f \circ id_A$. This special morphism therefore behaves for composition of morphisms as the identity element of a monoid, except that instead of one there are many: one for any given object.

Examples of categories: topological spaces form a category **TOP**, where the morphisms between spaces are continuous functions. Partially ordered sets (posets) with order-preserving ("monotonic") maps form the category **POS**. The category of **GRP** has groups as objects and its morphisms are maps between one group $<G, \odot, e>$ and another $<G', \odot', e'>$ ("homomorphisms") $f\colon G \to G'$ such that $f(x \odot y) = f(x) \odot' f(y)$ and $f(e) = e'$, i.e., the morphisms in **GRP** coincide with group homomorphisms. Sets form another category, **SET**, which has sets as objects and functions as morphisms.

Each object of these three categories, **POS**, **GRP**, **SET** can in turn be described as a category. Every partially ordered set **P** (such as a tree or a set with linear order) is a category, whose objects are the elements and between an element p and another q there is a (unique) morphism iff $p \leq q$. Any group as usually defined in algebra, that is, as a set with a structure such that between any two elements there is defined a binary operation $\odot$ which is associative, has an identity element e, and is such that every element is invertible, forms a category: it is a one-object category G, in which the only morphisms are endomorphisms $G \longrightarrow G$, corresponding to the elements of the group as usually defined, $\odot$ is composition of morphisms, and the identity element is the identity morphism. Any set can be seen as a category in which the only morphisms are the identities on each element of the set considered as an object. Note, however, that this feature is not general: the category of graphs has graphs as objects and arc-preserving maps as morphisms but a graph, with nodes as objects and arcs as morphisms, is *not* a category.

Functors and Natural Transformations

A functor is a map between two categories that meets certain minimal properties of conservation, such as that of being a category. To be a functor, a map $F\colon \mathbf{C} \to \mathbf{D}$ must be such that, to each object A of **C**, F associates an object FA of **D** and, to each morphism $f\colon A \longrightarrow B$ in **C**, F associates a morphism $Ff\colon FA \longrightarrow FB$ in **D**; in addition, the map F must be such that $F(g \circ f) = (Fg) \circ (Ff)$ and $F(id_A) = id_{FA}$ for each object A of **C**. That is, a functor preserves both the composition of morphisms and identities. Thus, if the objects of **POS**, **GRP**, **SET** are described as categories, the morphisms in these three categories are functors.

A functor defined in this way is said to be "covariant", because it preserves the order in which morphisms in a category are composed. A functor that reverses the composition is said to be "contravariant" and is a functor from $\mathbf{C}^{op}$ to **D**, where $\mathbf{C}^{op}$ is the opposite category of **C**: $\mathbf{C}^{op}$ is obtained by simply reversing the direction of morphisms. So, for any f from A to B, f^{op} goes from B to A).[14] We can also define functors that, in addition to preserving the category-structure from **C** to **D**, also preserve specific properties of C. For example, a functor is said to be "faithful" if it never confuses two distinct morphisms having the same domain and codomain, i.e., for any f and g: $A \longrightarrow B$, with A and B any objects of **C**, if $F(f) = F(g)$ then $f = g$.

A *natural transformation* between two functors F and H, both from **C** to **D** is a map φ such that for any object A to **C**, the component φ_A is a morphism that goes from FA to HA in **D** and it is also the case that, for f: $A \longrightarrow B$ in **C**, $\varphi_B \circ F(f) = G(f) \circ \varphi_A$.

The functors from one category **C** to another **D** in turn form a ("functorial") category $\mathbf{D}^{\mathbf{C}}$: its objects are the functors from **C** to **D** and the morphisms between its objects are the natural transformations. In the particular case of $\mathbf{C}^{\mathbf{C}}$ it will also have the identity functor $1_{\mathbf{C}}$: $\mathbf{C} \longrightarrow \mathbf{C}$. Considering the categories as objects and functors as morphisms, one can easily verify that the functors can be composed and that their composition is associative. We have thus satisfied all the conditions required to define the category **CAT** of all categories.[15]

Isomorphisms and Monomorphisms

Among the morphisms of a category, some have distinguished properties with respect to composition. Recall that our aim here was to make explicit the meaning of the notions which go into the definition of the notion of adjunction. Here is how the last of these are defined.

A morphism f from A to B in **C** is an *isomorphism* (in **C**) if there is (in **C**) a morphism g from B to A such that $g \circ f = id_A$ and $f \circ g = id_B$, that is, if f has both a left inverse and a right inverse. Finally an isomorphism between functors in $\mathbf{D}^{\mathbf{C}}$ is said to be "natural" if it is a natural transformation each component of which is an isomorphism.

There will be different types of isomorphisms as we move from one category to another, but it is remarkable that with such a compact definition, *and without the slightest nod in the direction of the notion of membership* ($\in$), we can characterise the notion of isomorphism in general, regardless of which category we are in. In **SET**, the category of sets and functions, isomorphisms coincide with the bijective correspondences, while an isomorphism in **TOP** between two spaces is a homeomorphism.

---

[14] Passing from a category to its opposite (or "dualising") highlights the fact that many mathematical notions, including those of logical interest in particular, are the mirror images of others. This provides a theoretical economy: if we have proved a proposition which is supposed to hold in general, its dual holds too (by a dual proof).

[15] As in the case of the set of all sets, care is needed with CAT too in order to avoid paradoxes. Such care usually takes the form of principles of size limitation and the most common method consists of dealing with *locally small* categories, i.e., such that for any two objects the collection of morphisms from one to the other is *small* (is a set). One might object that, by such a move, category theory demonstrates its conceptual dependence on set theory in a manner which runs counter to the claim that it can be viewed as an alternative to set theory for foundational purposes. There is more than one response to this objection and both the objection and the various responses to it are the subject of much debate. Since this paper was intended to avoid an excursion into such foundational controversies, I shall here simply refer the reader to the works of Colin McLarty for a detailed treatment. See, e.g., [19].

Among the various kinds of morphisms, there is also another particularly significant one: the monomorphisms. Given a category **C** as reference, a **C**-map $f\colon A \longrightarrow B$ is a monomorphism (abbreviated as "f is monic") if $f \circ h = f \circ k$, for every h and k from an arbitrary X to A, then $h = k$. In this case we write $f\colon A \rightarrowtail B$. The notion of epimorphism is the dual of that of monomorphism, i.e., f is an epimorphism (abbr. "f is epic") in **C** if it is a monomorphism in $\mathbf{C}^{\mathrm{op}}$). If a category has a terminal object, i.e., an object, labelled 1, such that from any other A there is one and only one morphism $A \to 1$, it is unique up to isomorphism. All morphisms $1 \to A$ are monic.

Example. In **SET** monics are injective functions and surjective functions are epic. Note that in algebra and topology there are categories in which it is easy to find examples of a monomorphism which is not injective (as a set-function) and of an epimorphism which is not surjective. Furthermore, in **SET**, elements x of a set A correspond to morphisms from any singleton to A and any singleton acts as a terminal object, thus one can write a: $1 \to A$ rather than $a \in A$.

This example is useful for two reasons: (1) it illustrates how two fundamental concepts of set theory—surjections and injections—can be defined without reference to $\in$; (2) it underlines that the notions of monic and epic are more general than the manner in which they occur in **SET**. In fact, In **SET** for there to be an isomorphism between two objects it is enough to ensure that there is a monic map between them which also turns out to be epic. This is not true in other categories.

Subobjects

Given $f\colon A \rightarrowtail B$, we consider another monic $f'\colon A' \rightarrowtail B$. If f is factored through f', that is, if there is a $g\colon A \longrightarrow A'$ such that $f = f' \circ g$, and if vice versa f' is factored through f, then there is an isomorphism between A and A' which allows us to define an equivalence between f and f'. The corresponding equivalence class identifies a *sub-object* of B. Just as in **SET** every set A has a family of sub-sets (which is in turn a set, $\mathcal{P}A$, i.e., the power-set or set of all subsets of A), so we can associate with each object A of a given category **C** the family of its sub-objects and this association is *functorial*: it is expressed by the existence of a functor *Sub*: $\mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ which behaves in the expected way.

In philosophical terms, the categorical analysis of the relationship between objects and sub-objects allows us to address the relationship between parts and wholes in a manner intrinsic to the universe of discourse (as a category) within which Part-Whole relationships are to be considered. Thus, what can be a part in one case may not be so in another. Part-Whole relations have to respect the constraint of what may be termed ontological homogeneity. This constraint has non-negligible effects when interpreting the predicates of a language in a universe of discourse U; and by induction (in passing from atomic to compound formulae) it affects the semantic evaluation v of the whole language. For instance, take a binary predicate P: if P is interpreted by v on a U in **SET** then $v(P)$ can be any subset of the Cartesian product $U \times U$; but if U is also a group and the interpretation is confined to **GRP**, $v(P)$ will have to be a subgroup of the product group $U \times U$.[16]

The same applies to the notion of element. The classic principle of extensionality ensures that the functor *Sub* is determined by elements. Indeed, in **SET**, an object being a collection of elements in extension, each subset A of B is associated with a characteristic function χ_A: $B \rightarrow \{0,1\}$, with $\chi_A(x) = 1$ if and only if $x \in A$; so, by contraposition, two subsets A and A' of B will be different if and only if $\chi_A \neq \chi_{A'}$. This condition holds if and only if there exists a morphism x: $1 \rightarrow B$ such that $\chi_A \circ x \neq \chi_{A'} \circ x$.

Many categories in which one can interpret first order languages or languages of order n, for $n > 1$, do not satisfy the property just defined: the elements ("global" elements) are not sufficient. Since 1 has a minimal internal structure in **SET** (the only proper sub-object just being $\varnothing$), there is no way in that category to talk about "partial" elements. Many categories have terminal objects which are richer in internal structure and so they allow for such "partiality". At the same time morphisms from 1 to another object have to meet further conditions than in the case of **SET** and for this reason are typically more rare.

As an example, take the category $\mathbf{SET}^{\mathbf{T}}$ of sets linearly ordered with respect to a parameter, which may be here thought of as sets indexed by Time. At any given time t there is the set $A(t)$ of things belonging to A at that time. In this category, which can also be described as a "slice" category **SET/T** of sets over Time, the terminal object is the identity functor 1_{T} from **T** to **T**, so that for there to be a (global) element of such a variable set, it is necessary that there is never a time t when the set is empty. But even though this condition is not satisfied by A, the algebra of its parts can be rich in structure. In category theory the phenomenology of relations between the concepts of "being an element of " and that of "being part of " is extremely wide and provides an indispensable tool for anyone who wishes to analyse the concept of an *individual* entity, taking into account the way entities vary. So it is possible, within one and the same mathematical picture, to progressively approximate the way individuals are identified and referred to in the real world, rather than resigning ourselves to the disconnect between a semantics in the format provided by set theory and the highly constrained language-world interface to which real speakers in physical reality are subject.

Universality by Adjunction

We can now return to the concept of adjunction to pinpoint its most important characteristics. The natural isomorphism φ that must exist between morphisms f: $FA \longrightarrow B$ and morphisms g: $A \longrightarrow GB$ means that, for any B' in **D** and k: $B \longrightarrow B'$, $\varphi(k \circ f) = Gk \circ \varphi f$ and that, for any A' in **C** and h: $A' \longrightarrow A$, $\varphi^{-1}(g \circ h) = \varphi^{-1}(g) \circ Fh$. That is, the association of a unique morphism $\varphi(f)$ with f is stable with respect to the composition of f with other morphisms in **D**, and the same is true for g and $\varphi^{-1}(g)$. Therefore, the association is uniform, in the sense that it does not depend on the choice of the particular objects (A, B) and morphisms (f, g) considered.

[16] In order for the interpretation to be *functorial*, the syntax too must be categorically reformulated. This step introduces intrinsic constraints on semantics, but it requires further details than those strictly needed for the aim of this paper and therefore will not be examined here.

Now, if $F \dashv G$, there are two special morphisms, denoted as the *unit* (η) and *co-unit* (ε) of the adjunction, which allow us to grasp the sense of universality which has been repeatedly mentioned. Indeed, in the case where $B = FA$, among the possible morphisms $f\colon FA \longrightarrow FA$ there will necessarily be id_{FA}, matched by φ with a morphism $\varphi(id_{FA})\colon A \longrightarrow GFA$ and this morphism, labelled η_A, is the *unit* of the adjunction. Thus, the unit is characterised by the fact that, since φ is natural, each morphism $g\colon A \longrightarrow GB$ is factored through η_A in one and only one way, namely $g = Gf \circ \eta_A$ for a unique $f\colon FA \longrightarrow B$.

As A varies in **C**, each morphism η_A behaves as a component of a natural transformation from the identity functor $1_{\mathbf{C}}$ to the functor **GF**. Symmetrically: in the case where $A = GB$, among the possible morphisms $g\colon GB \longrightarrow GB$ there will necessarily be id_{GB}, which φ^{-1} will put in correspondence with a morphism $FGB \longrightarrow B$ and this morphism, labelled ε_B, is precisely the *co-unit* of the adjunction; and, similarly, this dual notion of the unit is characterised by the fact that, since φ is natural, each morphism $f\colon FA \longrightarrow B$ is factored in one and only one way through ε_B, i.e., $f = \varepsilon_B \circ F(g)$ for a unique $g\colon A \longrightarrow GB$.

Note that if a functor has a (left or right) adjoint, this is unique (up to isomorphism). Note also that the left (right) adjoint of a functor may in turn have a further left (right) adjoint. Indeed this type of situation is often iterated, signalling the existence of deep intrinsic bonds between the different kinds of structure of **C** and **D**, [14].

The traditional way of defining a mathematical structure, such as an algebraic system, or an ordering or a space, makes reference to a set (as the underlying domain of the structure) satisfying additional properties, which often have the form of closure under certain operations. This indeed facilitates the procedure of "forgetting" the structure. For example: given a group G, one can consider just the set of its elements. The procedure can be made precise as a "forgetful" functor U: **GRP** → **SET**, but in place of **GRP** one can consider any other category **D** of sets with structure.

Here, what is relevant for universality is that such a procedure admits an "inverse" F: **SET** → **D**, in a particular sense: namely, the construction of a *free* structure over a given set, the elements of which are tagged as "generators". In the case of **D** = **GRP**, it will be a *free group*. The point is that any free structure X^* of a given kind has a universal property, i.e., given any function f from its set of generators to another set $U(Y)$, f factorizes through the underlying set $U(X^*)$ of X^* along a unique map g of the same kind-of-structure. In the case of a free group over a set X, with $f\colon X \longrightarrow U(Y)$, $f = U(g) \circ i$, where i is the insertion of generators in the underlying set of X^*. Thus there is a functor F, such that $X^* = F(X)$ and $U \dashv F$.

Now, it may be the case that, given a functor $G\colon \mathbf{D} \to \mathbf{C}$, only for a fixed object A in **C** an object B is available in **D** for which there exists a map $e\colon A \to GB$ with the analogous property of unique factorisation, i.e., such that there is a unique k with $Gk \circ e = g'$, for all $g'\colon A \longrightarrow GB'$. The pair <$B$, e> is said to be a *universal morphism* from A to G, in the sense that it is *free* with respect to G, from which one can solve a problem relative to **C** in a canonical way.

In addition, it may be that such a pair also exists for other objects in **C**, rather than for A only, but with characteristics different with respect to either B or e. Finally, this may occur without there being a universal morphism from F to B, dual to that from A to G. These cases lack the uniform character seen in the case of an adjunct pair. If and only if a universal morphism exists for *every* A in **C** and the dual condition holds for *every* B in **D**, do we have that $F \dashv G$.

So there is a notion of universality that is more general, because "sensitive to data" and therefore *weaker* than that captured by the concept of adjunction. However, Mac Lane was right to propose the slogan "Adjoint functors arise everywhere in mathematics",[17] because the basic constructions which arise time and again across different fields of mathematics correspond to that uniformity which is guaranteed by the existence of adjunctions. The fact that a category **C** admits, for example, the formation of products or function spaces between any two objects, is due to the existence of an adjoint functor to a functor from **C** to another category or from another category to **C**, independently of specific objects and maps in **C**, but even independently of their nature: be they sets and functions, spaces and continuous maps, groups and homomorphisms, propositions and proofs, etc.

Examples of Adjunction

From the algebra of natural numbers to that of the complex numbers, numerical systems differ in being closed or not with respect to certain operations. For example, the natural numbers are closed under sums and products but not with respect to the inverse of those operations. The rationals are also closed with respect to both inverses but not with respect to the inverse of the operation of raising by a power. Similarly, there are many types of categories, differentiated by their closure with respect to certain constructions. Closure constructions are examples of a fundamental mathematical concept—that of limit (and, dually, of co-limit[18]), which in its own turn tracks an underlying adjunction.

To define the notion of limit one starts from the consideration of a diagram within a given category **C**. A (finite) diagram $\mathcal{D}$ in **C** is any collection of **C**-objects and **C**-morphisms between them. A cone for $\mathcal{D}$ is given by an object V (the "vertex" of the cone) and a family of morphisms f_i (having domain V) to each object A_i of $\mathcal{D}$ such that, if h: $A_i \longrightarrow A_j$ is a morphism in D, $f_j = h \circ f_i$.

A limit (cone) $\mathscr{L}$, with vertex L, for $\mathcal{D}$ is a *universal* cone for $\mathcal{D}$ in the sense that, in addition to being a cone on $\mathcal{D}$, with a family of morphisms such that $l_j = h \circ l_i$, it has the following property: every other cone $\mathcal{D}'$ factorizes through L, i.e., there exists a (unique) morphism u from L to the vertex V of each other cone, such that $\mathrm{f}_i = l_i \circ u$. A co-limit is nothing more than a limit for $\mathbf{C}^{\mathrm{op}}$.

For example, if **C** admits a limit for each diagram of the form $A_1 \leftarrow A_0 \longrightarrow A_2$, **C** has finite Cartesian products of each pair of objects $<A_1, A_2>$. In fact, if a category has binary products (for pairs of objects), then it has products for each n-tuple of finite objects. The notion of sum, or co-product, is simply the dual of this, reversing the direction of the two morphisms: $\mathrm{A}_1 \rightarrow \mathrm{A}_0 \leftarrow \mathrm{A}_2$. If **C** has a limit for the empty diagram, **C** has a terminal object denoted by 1. The dual notion is that of the initial object, denoted by 0.

In a category that is a partial order **P**, the existence of finite limits is equivalent to the fact that every finite set of nodes in **P** has a lowest bound ($\cap$) and the existence of colimits (even infinite ones) to the fact that every set (even infinite) of nodes in **P** has a supremum ($\cup$). Note that the usual definition of a topological space says that the category of open

[17]In the Preface to the first edition of [17], p. vii.

[18]Usually written "colimit".

subsets of a space X has finite limits and possibly infinite colimits. Indeed, in **TOP** a morphism *f*, that is a continuous function, from a space X to a space Y is characterised by the fact that the inverse image, $f^{-1}(-)$, of any open subset—of *Y* is an open subset of *X* and also $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ and $f^{-1}(\cup U_i) = \cup f^{-1}(U_i)$, for each *U, V* opens in *Y*, with $i \in I$, where *I* may be infinite. That is, f^{-1} preserves finite limits and arbitrary colimits.

In **SET**, the product as defined above coincides with the usual Cartesian product, the terminal object is any singleton, the co-product is a disjoint sum and the initial object is the empty set Ø. In the category of groups, **GRP**, these categorical notions have further properties. In particular, the initial and the terminal object coincide: the trivial group consisting only of the identity is both 0 and 1.

If we now consider as objects propositions *p, q, r* ... and deductions as morphisms, i.e., $p \longrightarrow q$ means that *q* is deducible from *p*, we have another category **PROP**, which, once endowed of finite limits and colimits, directly mirrors a fragment of propositional logic. The fact that **PROP** has (finite) products is equivalent to the possibility of forming the conjunction of any (finite) number of propositions. Similarly, co-products corresponds to disjunctions. The initial object is the absurd proposition (False, corresponding to the bottom element $\perp$ of a Heyting or Boolean algebra) and the terminal object is a logical truth (True, corresponding to the top element $\top$ of the given algebra). In **CAT**, the concepts will be those of the corresponding product category $\mathbf{C} \times \mathbf{D}$, the terminal category **1** (formed from a single object with its identity morphism) and so on.

Limits (and colimits) in many other categories can be defined and they turn out to be the outcome of a suitable adjunction. For simplicity, however, we shall confine attention to the examples given so far.

Denoting by Δ: $\mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}$ the diagonal functor which associates to each object *A* of **C** its duplication $<A, A>$ and every morphism $f: A \to B$, its duplication $<f, f>$, the existence of binary products in C is equivalent to the fact that Δ has an additional right adjoint (the "product" functor $\times$). For a fixed object *B*, the existence of a right-adjoint for the parametric product functor - $\times B$: $\mathbf{C} \to \mathbf{C}$ (which associates $A \times B$ to an arbitrary *A*) is equivalent to the fact that **C** admits "exponentials", i.e., objects C^B. **C** is said to be "Cartesian closed" if there is a natural isomorphism between morphisms $A \times B \to C$ and morphisms $A \longrightarrow C^B$. In **SET**, C^B becomes the set of functions from *B* to *C*. If the objects *B* and C are spaces, the exponential is the space of all continuous functions from *B* to *C*, with the usual topology.

As in the case of products, the existence of sums is equivalent to the fact that Δ has a left adjoint (+ as a functor). In addition, considering the functor !: $\mathbf{C} \to 1$ (of functors to 1, there exists necessarily only one), if ! has a left adjoint, then **C** has an initial object, and if ! has a right adjoint, **C** has a terminal object.

In the case of **PROP**, in particular, the basic structure corresponding to the logical properties of the conjunction $\wedge$ and the disjunction $\vee$ is definable purely by *equations* between morphisms and their composites and these equations are associated with the existence of adjoints: $\wedge \dashv \Delta \dashv \vee$ and $\perp \dashv\, ! \dashv \top$. The exponential adjunction in this case gives us the connective of implication as right adjoint to - $\times B$.

The extraordinary progress in this approach to logic was made possible by Lawvere's pioneering research, which showed that such an equational presentation of logic by means of adjoints can also be extended to the quantifiers. In fact, $\exists$ and $\forall$ turn out to be respectively the left and right adjoints of the functor *Sub*: $\mathbf{C}^{\mathrm{op}} \to \mathbf{SET}$ which associates with each object the set of its subobjects: A *topos* is a Cartesian closed category with a "subobject classifier" (which acts as an intrinsic truth-values-object). Consequently, all the notions required for logic acquire equational form.

It can be shown that a left adjoint preserves colimits and a right adjoint preserves limits. It follows that the existence of a functor which is both a left and a right adjoint tells us that a substantial part of the structure of a category is preserved by the functor. Further constraints due to the form of the axioms of a theory result in a tight control on the variation of its models. A *generic* model of a theory T, roughly speaking, is one with "no junk" and "no noise": formally, it has a property analogous to that seen in the case of free objects, but now holding for the category of models of T, thus endowing it with the corresponding universality.

Even apart from the new aspects of logic which are related to its internalisation in suitable categories (such as a topos),[19] the effects of all this on our understanding of semantics are profound. Not only is the notion of model refined by passing from the universe of sets to different categories (every bit as rich in structure), but also the notion of theory itself, once re-described in categorical terms, leads to a much richer semantic landscape (for example, to a far from trivial semantics for theories that had no models in **SET** and therefore have to be considered inconsistent by "standard" formal semantics).

Concluding Remarks

The concept of universality that has emerged with the notion of adjunction between functors allows us both to rethink and to refine a number of philosophical notions that have been the centre of many controversies, both in semantics and epistemology. Among the most significant are (1) the notion of *prototype* (in cognitive semantics), definable as the unit of an adjunction,[20] and (2) the notion of *transcendental subject* (in epistemology), definable by means of the notion of generic model.

As for (1), once the category **K** of Kinds and the category **I** of Individuals are specified, a subcategory **B** of **K** can be identified as that of *basic* kinds; if the restriction of the typing functor $\sigma: \mathbf{I} \to \mathbf{K}$ to **B** has a right adjoint $\pi: \mathbf{B} \to \mathbf{I}$, it means that any relation between an individual a of basic kind B and another a' of a possibly different kind can be uniquely recovered by factorisation along the unit of the adjunction, i.e., through $\pi\sigma(a)$, which then acts exactly as the prototype of B.

As for (2), once any theory T is described as a category and a suitable collection of theories is associated with each knowing subject S, then the access S can have to a theory T_U relative to a given empirical universe of discourse U takes the form of a functor $F: T_U \longrightarrow S$. In case where F is uniquely factorisable through a generic subject Σ, we are justified in considering Σ as a universal subject which, by coding the conditions of possibility for any knowledge about any domain by any subject, can be taken as "transcendental", i.e., as a generic model of the notion of knowing subject.[21]

[19]The most extensive treatise on topos theory is [8].

[20]An adjunction-based approach to the theory of concepts was first explored by Ellerman in [7].

[21]For the details relative to these two constructions, see respectively [26, 28].

The fact that so many different notions related to the search for "universals" are instances of one and same formal pattern, namely that of adjunction, should be of considerable interest to philosophers. Of no less interest should be that behind this unification is a theoretical perspective focusing on the analysis of change-of-structure (of systems emerging within and constrained by the physical world) and the idea that in general this change is not reducible to point-like components, but requires attention to the inner cohesion of any structure under variation, without such a recognition inevitably leading to some loose "structuralist" variant of holism. Previous knowledge of the "change of base" technique in algebraic geometry was indeed one of the sources of inspiration for such a unifying framework; conversely, this generalisation shed light on the relevance of algebraic geometry for logic—something most philosophers, and especially those in the analytic tradition, would consider an alien topic.

Admittedly, what is still lacking is a *systematic* treatment that would clarify the linkages between the form taken within a categorical setting by classical philosophical issues about (1) the nature of mathematical entities and the status of principles supposed to have a foundational role, (2) the nature of meaning, as a main concern for philosophy of language and (3) the architecture of knowledge, as it is investigated in epistemology and philosophy of science. Those starting research in philosophy lack the guidance that could provide a unified understanding of the various faces of universality. Category theory provides the tools for that understanding and for the treatment of those various aspects in a comprehensive and mathematically precise framework.

Only a collaboration between category theorists, logicians, and philosophers will overcome that lack and advance our understanding. But this assumes a motivation and a readiness to co-operate and, to date, there has been little evidence of that on the part of the philosophers. This reflects the fact that, with a few notable exceptions, there has been no recognition on their part of the potential offered by category theory to advance the understanding and resolution of the problems studied particularly in the various sub-disciplines of analytic philosophy. This state of affairs urgently needs to change. It may shortly do so. My aim in this compressed and incomplete presentation of categorical "universality" has been to help that to happen.

References

1. A. Asperti, G. Longo, *Categories, Types and Structures* (MIT Press, Cambridge, MA, 1991)
2. S. Awodey, *Category Theory* (Oxford University Press, Oxford, 2010)
3. J.L. Bell, From absolute to local mathematics. Synthese **69**, 409–426 (1986)
4. J.L. Bell, *Toposes and Local Set Theories: An Introduction* (Clarendon, Oxford, 1988)
5. J.L. Bell, The development of categorical logic, in *Handbook of Philosophical Logic*, ed. by D. Gabbay and F. Guenthner (Springer, New York), vol 12 (2005), pp. 279–361
6. V. De Paiva, A. Rodin, Elements of categorical logic. Log. Univers. **7**, 265–273 (2013)
7. D. Ellerman, Category theory and concrete universals. Erkentnnis **28**, 409–429 (1988)
8. Johnstone, P.: *Skteches of an Elephant: A Topos Theory Compendium*, vol. 1–2 (Oxford University Press, Oxford, 2002–2003)
9. J. Lambek, Categorial and categorical grammar, in *Categorial Structures and Natural Language Structure*, ed. by R.T. Oehrle et al. (Springer, Dordrecht, 1988), pp. 297–317
10. J. Lambek, Are the traditional philosophies of mathematics really incompatible? Math. Intell. **16**, 56–62 (1994)

11. J. Lambek, P. Scott, *Introduction to Higher-Order Categorical Logic* (Cambridge University Press, Cambridge, 1986)
12. F.W. Lawvere, Adjointness in foundations. Dialectica **23**, 281–296 (1969)
13. F.W. Lawvere, Taking categories seriously. Revista Colombiana de Matematicas **20**, 147–178 (1986)
14. F.W. Lawvere, Cohesive toposes and Cantor's "lauter Einsen". Philos. Math. **2**, 5–15 (1994)
15. F.W. Lawvere, Unity and identity of opposites in calculus and physics. Appl. Categ. Struct. **4**, 167–174 (1996)
16. F.W. Lawvere, S. Schanuel, *Conceptual Mathematics*, 2nd edn. (Cambridge University Press, Cambridge, 1997)
17. S. Mac Lane, *Categories for the Working Mathematician* (Springer, Berlin, 1971)
18. S. Mac Lane, *Mathematics: Form and Function* (Springer, New York, 1986)
19. J. Macnamara, G. Reyes (eds.), *Logical Foundations of Cognition* (Oxford University Press, Oxford, 1994)
20. C. McLarty, Defining sets as sets of points of spaces. J. Philos. Log. **17**, 75–90 (1988)
21. C. McLarty, Learning from questions on categorical foundations. Philos. Math. **3**, 44–60 (2005)
22. J.-P. Marquis, *From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory* (Springer, New York, 2008)
23. J.P. Marquis, Category theory, Stanford Encyclopedia of Philosophy (2013) http://plato.stanford.edu/entries/category-theory/
24. A. Peruzzi, Forms of extensionality in topos theory, in *Temi e prospettive della logica e della filosofia della scienza contemporanee*, ed. by M.L. Dalla Chiara, M.C. Galavotti, vol. 1 (CLUEB, Bologna, 1988), pp. 223–226
25. A. Peruzzi, The theory of descriptions revisited. Notre Dame J. Formal Log. **30**, 91–104 (1988)
26. A. Peruzzi, Towards a real phenomenology of logic. Husserl Stud. **6**, 1–24 (1989). errata corrige, 253
27. A. Peruzzi, From Kant to entwined naturalism. Annali del Dipartimento di Filosofia **9**, 225–334 (1993)
28. A. Peruzzi, The geometric roots of semantics, in *Meaning and Cognition*, ed. by L. Albertazzi (John Benjamins, Amsterdam, 2000), pp. 169–211
29. A. Peruzzi, Il lifting categoriale dalla topologia alla logica. Annali del Dipartimento di Filosofia **11**, 51–78 (2005)
30. A. Peruzzi, The meaning of category theory for 21st century's philosophy. Axiomathes **16**, 425–460 (2006)
31. A. Peruzzi, Logic in category theory, in *Logic, Mathematics, Philosophy: Essays in Honor of John Bell*, ed. by M. Hallett, D. Devidi, P. Clark (Springer, New York, 2011), pp. 287–326

A. Peruzzi (✉)
Department of Philosophy, University of Florence, Florence, Italy
e-mail: alberto.peruzzi@unifi.it