Towards a Probabilistic Interpretation of Game Logic

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Abstract. Game logic is a modal logic the modalities of which model the interaction of two players, Angel and Demon. It is known that game logic is not adequately interpreted through relation based Kripke models. The basic mechanism behind neighborhood models, which are used instead, is given through effectivity functions. We give a brief introduction to effectivity functions based on sets, indicate some of their coalgebraic properties, and move on to a definition of stochastic effectivity functions over general measurable spaces. An interpretation of game logics in terms of these effectivity functions is sketched, and their relationship to probabilistic Kripke models and to the interpretation of the PDL fragment is indicated.

Modal Logics and Games. The formulas of a modal logics are given through the grammar

$$\varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle \varphi.$$

Here p is an atomic proposition, and a is a modality, which usually models actions. Thus $\langle a \rangle \varphi$ holds in a world $w \in W$ iff we can make a transition by executing action $a \in A$ into a world w' in which formula φ holds. This indicates the usual interpretation of the logic: we associate with each action a a relation $R_a \subseteq W \times W$ and define $w \models \langle a \rangle \varphi$ iff $w' \models \varphi$ for some $w' \in R_a(w) := \{w'' \mid \langle w, w'' \rangle \in R_a\}$; the Boolean connectives are interpreted as usual, and each atomic proposition p is associated with a set $V(p) \subseteq W$ such that $w \models \pi$ iff $w \in V(p)$. Collect these data into a relation based *Kripke model* $\big(W, (R_a)_{a \in A}, V\big)$.

If the modalities carry a structure of their own, one would expect that this is reflected in the interpreting relations. This is the case, e.g., with Propositional Dynamic Logic (PDL) or with Game Logic (GL), which are intended to model simple programs, and two person games, respectively. We assume for the latter that we have two adversaries, Angel and Demon, playing against each other, taking turns. The grammar for games is given through

$$g ::= \gamma \mid g_1 \cap g_1 \mid g_1 \cup g_2 \mid g_1; g_2 \mid g^* \mid g^d \mid g^\times \mid \varphi?$$

with $\gamma \in \Gamma$ a primitive game [6]. Here $g_1 \cup g_2$ denotes the nondeterministic choice between games g_1 and g_2, $g_1; g_2$ is the sequential play of g_1 and g_2 in that order, and g^* is iteration of game g a finite number of times (including zero). The game $\varphi?$ tests whether or not formula φ holds, where φ is a formula from the logic. $\varphi?$ serves as a guard: $(\varphi?; g_1) \cup (\neg\varphi?; g_2)$ tests whether φ holds, if it does g_1 is

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played, otherwise, g_2 is. This describes the moves of Angel. The moves of player Demon are given by $g_1 \cap g_2$, where Demon chooses between games g_1 and g_2; this is demonic choice (in contrast to angelic choice $g_1 \cup g_2$). With $g^\times$, Demon decides to play game g a finite number of times (including not at all), and g^d indicates that Angel and Demon change places.

The informal meaning of $\langle g\rangle\varphi$ is that formula φ holds after game g is played. Let us just indicate informally by $\langle g\rangle\varphi$ that Angel has a strategy in game g which makes sure that playing g results in a state which satisfies formula φ. We assume the game to be *determined*: if one player does not have a winning strategy, then the other one has. Thus if Angle does not have a φ-strategy, then Demon has a $\neg\varphi$-strategy, and vice versa. This means that we can derive the way Demon plays the game from the way Angel does, and vice versa. Thus we may express demonic choice $g_1 \cap g_2$ through $(g_1^d \cup g_2^d)^d$, and demonic iteration $g^\times$ through angelic iteration $\big((g^d)^*\big)^d$); clearly, g^{dd} should be the same as g. In contrast to Banach-Mazur games, we do not describe formally what a strategy is.

Neighborhood Models. Game logics are usually interpreted through *neighborhood models*, which associate with each primitive game $\gamma \in \Gamma$ and each world $w \in W$ a set $N_\gamma(w)$ of subsets of W, $A \in N_\gamma(w)$ indicating that Angel has a strategy for achieving a state in A upon playing γ in state w. Thus $N_\gamma(w)$ is an upper closed subset of the power set $\mathcal{P}(W)$ of W, hence $A \in N_\gamma(w)$ and $A \subseteq B$ implies $B \in N_\gamma(w)$; the elements of $N_\gamma(w)$ are perceived as neighborhoods of w under γ. These models are more general than Kripke models: given a relation $R \subseteq W \times W$, $w \mapsto \{A \subseteq W \mid R(w) \subseteq A\}$ yields for each $w \in W$ an upper closed set. Associating with N_γ a monotone map $N_\gamma^+ : \mathcal{P}(W) \to \mathcal{P}(W)$ through $N_\gamma^+(A) := \{w \in W \mid w \in N_\gamma(w)\}$, we may perform a syntax directed translation from games to maps $\mathcal{P}(W) \to \mathcal{P}(W)$, e.g., $N_{g_1;g_2}^+ := N_{g_1}^+ \circ N_{g_2}^+$, or $N_{g^d}^+(A) := W \setminus N_g^+(W \setminus A)$. In this way, each game g gets associated with such a monotone map N_g^+. We interpret the modal formula $\langle g\rangle\varphi$ by defining $[\![\langle g\rangle\varphi]\!] := N_g^+([\![\varphi]\!])$, where, as usual, $w \in [\![\varphi]\!]$ iff $w \models \varphi$.

A coalgebraic point of view notices that the assignment $\mathbb{V} : W \mapsto \{V \subseteq \mathcal{P}(W) \mid V \text{ is upper closed}\}$ is the functorial part of a monad, and that each N_γ is a Kleisli morphism for this monad, hence a coalgebra for $\mathbb{V}$. Composition of games is interpreted through Kleisli composition in the $\mathbb{V}$-monad; the actions of Demon may be obtained through *demonization* (the demonization of $f : W \to \mathbb{V}(W)$ is given by $\partial f : w \mapsto \{A \mid W \setminus A \notin f(w)\}$). The transformation $N_\gamma \mapsto N_\gamma^+$ is given by a natural transformation of the functors $\mathcal{P} \to \mathbb{V}$.

Neighborhood models are strictly more general than Kripke models, which turn out to be not adequate for interpreting general game logics. This is the reason why: The interpretation of games through Kripke models is disjunctive, which means that $\langle g_1; (g_2 \cup g_3)\rangle\varphi$ is semantically equivalent to $\langle g_1; g_2 \cup g_1; g_3\rangle\varphi$ for all games g_1, g_2, g_3. This, however, is evidently not desirable: Angle's decision after playing g_1 whether to play g_2 or g_3 should not be equivalent to decide whether to play $g_1; g_2$ or $g_1; g_3$. Neighborhood models in their greater generality do not display this equivalence [7].

A Stochastic Interpretation of Game Logic. We modify first the modal formulas $\langle g \rangle \varphi$ to the conditional modal formulas $\langle g \rangle_r \varphi$, indicating now that formula φ should hold after playing g with a probability not smaller than $r \in [0, 1]$. It replaces also sets of worlds by sets of probability distributions over these worlds. Playing game g in state w, $N_g(w)$ is an upper closed set, the elements of which are now probability distributions over W, $A \in N_g(w)$ indicating that Angel has a strategy for achieving a distribution of new states taken from A. So this sounds like simply replacing the set of states by the set of distributions over the states. But things are not that straightforward, unfortunately. The reason is that we need this new kind of neighborhood models be adaptable to the requirements provided by the algebraic structure of the games, in particular it should support the composition of games, and it should be closed under demonization.

This leads to the definition of stochastic effectivity functions, which model a particular kind of stochastic nondeterminism [4,2]. One first notes that the set of worlds W should carry a measurable structure, so that measures can be defined on it. The set $\mathbb{P}(W)$ of all probabilities on W then carries also a measurable structure, which is given in a fairly natural way by evaluating probabilities at events [3]. So an effectivity function P on world W should map W to the upper closed measurable subsets of $\mathbb{P}(W)$. This looks like an easy combination of two monads — the probability functor $\mathbb{P}$ is a well known monadic functor, and the upper closed functor is also monadic. Unfortunately, this does not work out well, because the composition of two monads is usually not a monad, bad luck.

The following technical construction helps to bypass this difficulty. Assume we have a measurable subset $H \subseteq \mathbb{P}(W) \times [0, 1]$, which may be thought of as a combination of measures with their numerical evaluations, e.g., $H = \{\langle \mu, q \rangle \mid \mu(A) \geq q\}$ for some measurable set A of worlds, then $H^q := \{\mu \mid \langle \mu, q \rangle \in H\}$ cuts H at q (imagine a set in the plane and look at its horizontal cuts). It can be shown that H^q is a measurable set of probabilities. We want the set $\{\langle w, q \rangle \in W \times [0, 1] \mid H^q \in P(w)\}$ be a measurable subset of $W \times [0, 1]$ for all such H; if this is the case, we call the effectivity function *t-measurable.*

Just to get the idea, assume that K is a stochastic transition kernel on W, hence $K(w)$ is a probability on W for each $w \in W$, then $w \mapsto \{A \subseteq \mathbb{P}(W) \text{ measurable} \mid K(w) \in A\}$ is such a t-measurable effectivity function (this is comparable to moving from a point to the ultrafilter generated by it). Another example comes from finite transition systems. Let the world W be finite and R a transition system on W with $R(w) \neq \emptyset$ for all $w \in W$, define the set of all weighted transitions from w through $\kappa(w) := \{\sum_{w' \in R(w)} \alpha_{w'} \cdot \delta_{w'} \mid \alpha_{w'} \geq 0 \text{ rational}, \sum_{w' \in R(w)} \alpha_{w'} = 1\}$, then $P(w) := \{A \subseteq \mathbb{P}(W) \text{ measurable} \mid \kappa(w) \subseteq A\}$ defines a t-measurable effectivity function on W. Also, if the effectivity function P is t-measurable, then $A \in \partial P(w)$ iff the complement of A is not in $P(w)$ defines a t-measurable effectivity function, the demonization of P.

As a whole, t-measurable effectivity functions have some fairly interesting algebraic properties [2], and they may be used for defining the semantics of game logics. This will be sketched now. The basic technical approach is to associate with each game g a set transformer, depending on a threshold value r, specifically, to define for $A \subseteq W$ the set $\Sigma(g \mid A, r)$ of states for which Angel has a strategy to achieve a member of A after playing g with a probability not smaller than r as the next state. For example, $\Sigma(\gamma \mid A, r)$ is defined for the primitive game $\gamma \in \Gamma$ as $\{w \in W \mid \{\mu \mid \mu(A) \geq r\} \in N_\gamma(w)\}$, so we look at all worlds for which Angel can achieve a distribution which evaluates A not smaller than r. Similarly, we define $\Sigma(g^d \mid A, r)$ as $W \setminus \Sigma(g \mid W \setminus A, r)$, thus Demon can reach a state in A with probability greater than r iff Angel cannot reach a state in $W \setminus A$ with probability greater than r. Finally — and here t-measurability kicks in — we define for the composition $\gamma; g$ with the primitive game $\gamma \in \Gamma$ and game g the transformation $\Sigma(\gamma; g \mid A, r) := \{w \in W \mid Q_g(A, r) \in N_\gamma(w)\}$, where $Q_g(A, r) := \{\mu \in \mathbb{P}(W) \mid \int_0^1 \mu(\Sigma(g \mid A, s))\ ds \geq r\}$. For an explanation, assume that $\Sigma(g \mid A, r)$ is already defined for each r as the set of states for which Angel has a strategy to achieve a state in A through playing g with probability not smaller than r. Given a distribution μ over the states, the integral $\int_0^1 \mu(\Sigma(g \mid A, s))\ ds$ is the expected value for entering a state in A through playing g for μ. The set $Q_g(A, q)$ collects all distributions, the expected value of which is not smaller than q. We collect all states such that Angel has this set in its portfolio when playing γ in this state. Selecting this set from the portfolio means that, when playing γ and subsequently g, a state in A may be reached with probability not smaller than q.

These are just some salient points in the definition of the transformation. Other cases have to be defined, depending on the games' syntax, in particular, $\Sigma(g^* \mid A, r)$ has to be determined; the details are outlined in [3, Section 4.9.4]. We have

Theorem: *If the measurable space W is complete, then $\Sigma(g \mid \cdot, r)$ transforms measurable sets into measurable sets.* $\dashv$

The reason why we need a complete measurable space here is that $\Sigma(g^* \mid A, r)$ involves some unpleasant uncountable Boolean operations, under which, however, this class of spaces is closed.

With this in mind, we can define an interpretation for modal formulas inductively through $[\![\langle g \rangle_r \varphi]\!] := \Sigma(g \mid [\![\varphi]\!], r)$, starting from some assignment of primitive propositions to measurable sets. It follows that each validity set is measurable, provided W is complete.

As in the set-valued case above, we have this property.

Proposition: *If the interpretation is Kripke generated, then it is disjunctive.* $\dashv$

Suppose that we consider only Angel's moves and forget about Demon. Then we have the PDL-fragment of game logic, which is somewhat easier to interpret. It turns out that the interpretation suggested here generalizes the known interpretations from [5,1].

Proposition: *A Kripke generated interpretation coincides on the PDL fragment with the one defined through Kleisli composition in the Giry monad.* $\dashv$

Thus the composition of programs can be described in an equivalent way through the convolution of Markov transition kernels.

Now, What? Well, it is interesting to investigate expressivity, i.e., the relationship of logical equivalence, bisimilarity and behavioral equivalence for these models. These properties have to be defined for stochastic effectivity functions (partial suggestions have been proposed in [4,2]). It would also be interesting to know whether simpler models of stochastic nondeterminism can be used for an interpretation, which would have to support the composition of games; a monad would be nice.

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