Chapter 2
Intelligent Process Control Using Control Charts—I: Control Charts for Variables

Murat Gülbay and Cengiz Kahraman

Abstract Shewhart's control charts are used when you have enough and exact observed data. In case of incomplete and vague data, they can be still used by the help of the fuzzy set theory. In this chapter, we develop the fuzzy control charts for variables, which are namely $\overline{X}$ and R and $\overline{X}$ and S charts. Triangular fuzzy numbers have been used in the development of these charts. Unnatural patterns have been examined under fuzziness. Besides, fuzzy EWMA charts have been also developed in this chapter. For each fuzzy case, we present a numerical example.

Keywords Shewhart's control charts · EWMA control charts · Fuzzy sets · Triangular fuzzy numbers · Unnatural pattern

2.1 Introduction

Process control is an engineering discipline dealing with maintaining the output of a specific process, generally called a quality characteristic, within a desired range. Type of processes using the process control can be categorized into three main groups which are discrete, batch, and continuous processes. Applications having elements of both discrete, batch and continuous process control are often called hybrid applications.

A process may either be classified as "in control" or "out of control". The boundaries for these classifications are set by calculating mean, standard deviation, and range from a set of process data randomly collected when it is under stable operation. Based on the statistical methods, analytical decision-making tools which

M. Gülbay (✉)
Technical Sciences Vocational School, Gaziantep University, Gaziantep, Turkey
e-mail: gulbay@gantep.edu.tr

C. Kahraman
Department of Industrial Engineering, Istanbul Technical University,
34367 Maçka Istanbul, Turkey

C. Kahraman and S. Yanık (eds.), *Intelligent Decision Making in Quality Management*, Intelligent Systems Reference Library 97,
DOI 10.1007/978-3-319-24499-0_2

allow practitioners to measure, monitor, and control the process behavior working normally or not, are called "Statistical Process Control (SPC)". The most successful SPC tool is control charts, originally developed by Walter Shewhart in the early 1920s. Comparing with boundaries of a stable process with a graphical display, they enable online data tracing and abnormal conditions warning, which are an essential tool for continuous quality control. Basically, the control charts are the graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time to monitor and show how the process is performing and how the capabilities are affected by changes to the process. This information is then used to make quality improvements. The control charts attempt to distinguish between two types of process variation that impede peak performance. These variations are as follows:

- Common cause variation, which is intrinsic to the process and will always be present.
- Special cause variation, which stems from external sources indicating that the process has assignable situation(s).

Based on the monitored quality characteristics in numerical or in "conforming" or "nonconforming" measurements, the control charts are categorized into two main groups, variables and attributes. This chapter deals with the control charts for variables. The most commonly used control charts for variables use the mean ($\bar{x}$, μ), range (R), and standard deviations (σ, s) in terms of paired $\overline{X}$ and R charts, paired $\overline{X}$ and s charts, and moving average charts.

2.2 Classical Shewhart Control Charts for Variables

2.2.1 $\overline{X}$ and R Control Charts

Many quality characteristics can be expressed in terms of a precise numerical measurement. One of the efficient ways of determining whether the process is in control or not is checking the process mean and process variability. $\overline{X}$ Charts are used to control the process mean while R charts are used to control the process variability. In general, they are paired and interpreted by looking both of the control charts. When the sample size is constant and relatively small, say $n \leq 10$, the usage of $\overline{X}$ and R charts advantageous.

2.2.1.1 Control Limits for $\overline{X}$ and R Control Charts

Suppose that a quality characteristic "X" is normally distributed with the parameters of μ and σ both known. For a sample size of n (X_1, X_2, ..., X_n), the average of the sample is

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{2.1}$$

and it is known that $\bar{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. The probability is $1 - \alpha$ that any sample mean will fall between

$$\mu + z_{\alpha/2}\sigma_{\bar{x}} = \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \textit{and} \quad \mu - z_{\alpha/2}\sigma_{\bar{x}} = \mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \tag{2.2}$$

If μ and σ are known, Eq. 2.2 can be used as upper and lower control limits on a control chart for sample means. It is customary to replace $z_{\alpha/2}$ by 3, so that three-sigma limits are employed.

In practice, we do not know μ and σ and estimate them from preliminary samples or subgroups usually based on at least 20–25 samples taken when the process is thought to be in control. If m samples are available, each containing n observations on the quality characteristic. Let $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ be the average of each sample. Then, the best estimator of the process average μ is the grand average, and would be used as the center line of the $\overline{X}$ chart.

$$\overline{\overline{X}} = \frac{\overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_m}{m} \tag{2.3}$$

The range of a sample (R) is the difference between the largest and smallest observations and the average range $(\overline{R})$ can be written as given in Eqs. 2.4 and 2.5, respectively.

$$\mathrm{R_i} = \mathrm{X_{i,max}} - \mathrm{X_{i,min}} \quad \mathrm{i} = 1, 2, \ldots, \mathrm{m} \tag{2.4}$$

$$\overline{\mathrm{R}} = \frac{\mathrm{R_1} + \mathrm{R_2} + \cdots + \mathrm{R_m}}{\mathrm{m}} \tag{2.5}$$

The formulas for constructing the control limits on the $\overline{X}$ and R charts are tabulated in Table 2.1. Development of these equations can be found in Montgomery (2001).

The constants A_2, D_3, and D_4 depend on the sample (observation) size and are tabulated for various sample sizes in Appendix A.

These initial set of control limits is usually treated as trial limits and subject to subsequent revision. The past hypothesis that is the process is thought to be in

Table 2.1 Control limits for $\overline{X}$ and R charts

	$\overline{X}$ chart	R chart
Center Line (*CL*)	$\overline{\overline{X}}$	$\overline{R}$
Lower Control Limit (*LCL*)	$\overline{\overline{X}} - A_2\overline{R}$	$D_3\overline{R}$
Upper Control Limit (*UCL*)	$\overline{\overline{X}} + A_2\overline{R}$	$D_4\overline{R}$

control when samples are takes should be checked. If one or more of the samples plot out of control, the hypothesis is rejected and trial control limits should be revised. This can be done by examining the out of control points, and looking for assignable causes. If an assignable cause is found, the point is eliminated and control limits are recalculated based on the remaining samples. Recalculated control limits are called revised control limits. This revision process is continued until all points plot in control, and the final limits are adapted to the process as the control chart limits.

2.2.1.2 A Numerical Example

In a packaging process, 25 samples of size of 4 are taken in order to control the process mean and deviation. Data obtained from the packaging process is shown in Table 2.2. Let's construct the $\overline{X}$ and R chart.

Table 2.2 Data for the example

Sample number	Observations				$\overline{X}_i$	R_i
	I	II	III	IV		
1	51.98	49.21	49.73	50.16	50.27	2.77
2	50.94	50.28	50.77	51.40	50.85	1.13
3	50.87	51.67	49.89	52.68	51.28	2.79
4	47.15	46.25	48.05	49.91	47.84	3.66
5	48.97	52.20	49.86	52.46	50.87	3.49
6	50.43	51.08	52.99	50.41	51.23	2.58
7	48.51	51.18	52.02	51.09	50.70	3.51
8	50.65	52.73	51.65	52.86	51.97	2.22
9	51.70	50.93	50.80	48.43	50.46	3.27
10	52.77	52.70	48.01	52.93	51.60	4.93
11	48.36	52.59	49.70	51.55	50.55	4.23
12	49.15	51.07	48.33	49.94	49.62	2.73
13	52.07	48.51	48.90	51.15	50.16	3.56
14	52.24	51.01	51.15	52.74	51.79	1.73
15	52.19	48.85	52.28	49.34	50.67	3.43
16	52.53	49.63	51.25	51.15	51.14	2.90
17	48.16	52.89	52.84	50.86	51.19	4.73
18	52.36	48.84	52.88	48.22	50.58	4.66
19	49.00	51.83	49.48	51.67	50.49	2.83
20	52.69	49.86	51.27	52.28	51.52	2.84
21	51.88	48.09	50.64	49.61	50.05	3.79
22	48.33	49.81	51.88	48.23	49.56	3.65
23	48.81	50.90	48.84	52.12	50.17	3.32
24	50.68	49.19	51.66	50.71	50.56	2.47
25	51.21	51.25	50.83	52.34	51.41	1.50
				Average	50.661	3.148

For the sample size of 4, the constants of A_2, D_3, and D_4 are 0.729, 0, and 2.282, respectively. Mean and range of each subgroup are determined by using Eqs. 2.3 and 2.4, and also shown in Table 2.2.

Trial control limits for the given process to construct $\overline{X}$ and R charts are as follows.

For $\overline{X}$ chart

$$CL = \overline{\overline{X}} = 50.661$$
$$UCL = \overline{\overline{X}} + A_2\overline{R} = 50.661 + 0.729 \times 3.148 = 52.956$$
$$LCL = \overline{\overline{X}} - A_2\overline{R} = 50.661 - 0.729 \times 3.148 = 48.366$$

For R chart

$$CL = \overline{R} = 3.148$$
$$UCL = D_4\overline{R} = 2.282 \times 3.148 = 7.184$$
$$LCL = D_3\overline{R} = 0 \times 3.148 = 0$$

By looking the $\overline{X}_i$'s of the 25 samples, it can be clearly seen that sample 4 plot out of control which requires for calculation of the revised control limits. Eliminating sample 4, revised control limits can be calculated by taking remaining 24 samples into consideration as shown in Table 2.3.

Revised control limits for the given process to construct $\overline{X}$ and R charts are as follows.

For $\overline{X}$ chart

$$CL = \overline{\overline{X}} = 50.779$$
$$UCL = \overline{\overline{X}} + A_2\overline{R} = 50.779 + 0.729 \times 3.127 = 53.059$$
$$LCL = \overline{\overline{X}} - A_2\overline{R} = 50.779 - 0.729 \times 3.127 = 48.499$$

For R chart

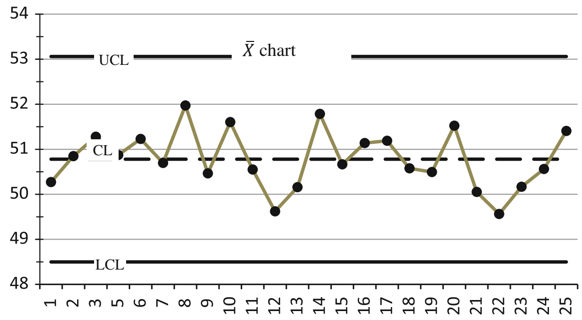
$$CL = \overline{R} = 3.127$$
$$UCL = D_4\overline{R} = 2.282 \times 3.127 = 7.184$$
$$LCL = D_3\overline{R} = 0 \times 3.127 = 0$$

Since all points plot in control, these limits can be set as the control limits to construct $\overline{X}$ and R charts as given in Figs. 2.1 and 2.2.

Table 2.3 Data for the calculation of the revised control limits

Sample number	Observations				$\overline{X}_i$	R_i
	I	II	III	IV		
1	51.98	49.21	49.73	50.16	50.27	2.77
2	50.94	50.28	50.77	51.40	50.85	1.13
3	50.87	51.67	49.89	52.68	51.28	2.79
5	48.97	52.20	49.86	52.46	50.87	3.49
6	50.43	51.08	52.99	50.41	51.23	2.58
7	48.51	51.18	52.02	51.09	50.70	3.51
8	50.65	52.73	51.65	52.86	51.97	2.22
9	51.70	50.93	50.80	48.43	50.46	3.27
10	52.77	52.70	48.01	52.93	51.60	4.93
11	48.36	52.59	49.70	51.55	50.55	4.23
12	49.15	51.07	48.33	49.94	49.62	2.73
13	52.07	48.51	48.90	51.15	50.16	3.56
14	52.24	51.01	51.15	52.74	51.79	1.73
15	52.19	48.85	52.28	49.34	50.67	3.43
16	52.53	49.63	51.25	51.15	51.14	2.90
17	48.16	52.89	52.84	50.86	51.19	4.73
18	52.36	48.84	52.88	48.22	50.58	4.66
19	49.00	51.83	49.48	51.67	50.49	2.83
20	52.69	49.86	51.27	52.28	51.52	2.84
21	51.88	48.09	50.64	49.61	50.05	3.79
22	48.33	49.81	51.88	48.23	49.56	3.65
23	48.81	50.90	48.84	52.12	50.17	3.32
24	50.68	49.19	51.66	50.71	50.56	2.47
25	51.21	51.25	50.83	52.34	51.41	1.50
				Average	50.779	3.127

Fig. 2.1 $\overline{X}$ chart



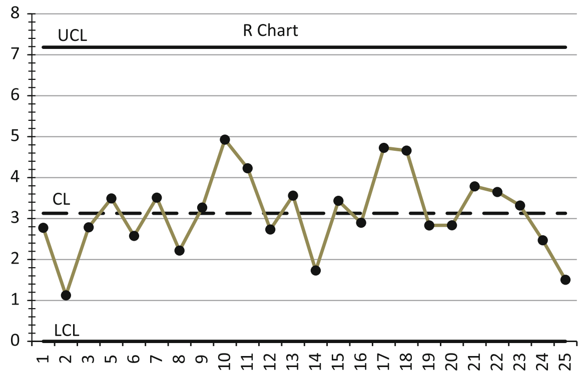


Fig. 2.2 *R* chart

2.2.2 $\overline{X}$ and *S* Control Charts

When the sample size is variable and relatively large, say $n > 10$, the usage of $\overline{X}$ and s charts is advantageous.

2.2.2.1 Control Limits for $\overline{X}$ and *S* Control Charts

For the cases where a standard value is known and given for σ, control limits for the S chart can be determined as follows:

$$\mathrm{CL} = c_4\sigma \tag{2.6}$$

$$\mathrm{UCL} = B_6\sigma \tag{2.7}$$

$$\mathrm{LCL} = B_5\sigma \tag{2.8}$$

where the constants c4, B5, and B6 depend on the sample (observation) size and are tabulated for various sample sizes in Appendix A.

When σ is unknown, we may write the parameters of the $\overline{X}$ and S control chart as given in the equations shown in Table 2.4.

where,

Table 2.4 Control limits for $\overline{X}$ and R charts

	$\overline{X}$ chart	s chart
CL	$\overline{\overline{X}}$	$\overline{s}$
LCL	$\overline{\overline{X}} - A_3\overline{s}$	$B_3\overline{s}$
UCL	$\overline{\overline{X}} + A_3\overline{s}$	$B_4\overline{s}$

$$s = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}} \tag{2.9}$$

$$\overline{s} = \frac{s_1 + s_2 + \cdots + s_n}{n} \tag{2.10}$$

The constants c_4, A_3, B_3, B_4, B_5 and B_6 depend on the sample (observation) size and are tabulated for various sample sizes in Appendix A.

If the sample size is variable, a weighted average approach is used in calculating $\overline{\overline{x}}$ and $\overline{s}$ as given below. In this case, since the sample size differs, the constants c4, A_3, B_3, B_4, B_5 and B_6 for each subgroup will be different. Hence, upper and lower control limits for each subgroup will also change.

$$\overline{\overline{X}} = \frac{\sum_{i=1}^{m} n_i \overline{X}_i}{\sum_{i=1}^{m} n_i} \tag{2.11}$$

$$\overline{s} = \left[\frac{\sum_{i=1}^{m}(n_i - 1)s_i^2}{\sum_{i=1}^{m}(n_i - m)}\right]^{1/2} \tag{2.12}$$

2.2.2.2 A Numerical Example

Consider the example given in Sect. 2.2.1.2. Calculation of the $\overline{x}_i$ and s_i for each subgroup are tabulated in Table 2.5.

For the sample size of 4, the constants of A_3, B_3, and B_4 are 1.628, 0, and 2.266, respectively. Trial control limits for the given process to construct $\overline{X}$ and S charts are as follows.

For $\overline{X}$ chart

$$CL = \overline{\overline{X}} = 50.661$$
$$UCL = \overline{\overline{X}} + A_3\overline{s} = 50.661 + 1.628 \times 1.452 = 53.025$$
$$LCL = \overline{\overline{X}} - A_3\overline{s} = 50.661 - 1.628 \times 1.452 = 48.297$$

For s chart

$$CL = \overline{s} = 1.452$$
$$CL = B_4\overline{s} = 2.266 \times 1.452 = 3.290$$
$$LCL = B_3\overline{s} = 0 \times 1.452 = 0$$

By looking the $\overline{X}_i$'s of the 25 samples, it can be clearly seen that sample 4 plot out of control which requires for calculation of the revised control limits.

Table 2.5 Data for the $\overline{X}$ and S chart

Sample number	Observations				$\overline{X}_i$	s_i
	I	II	III	IV		
1	51.98	49.21	49.73	50.16	50.27	1.21
2	50.94	50.28	50.77	51.40	50.85	0.46
3	50.87	51.67	49.89	52.68	51.28	1.18
4	47.15	46.25	48.05	49.91	47.84	1.56
5	48.97	52.20	49.86	52.46	50.87	1.73
6	50.43	51.08	52.99	50.41	51.23	1.21
7	48.51	51.18	52.02	51.09	50.70	1.52
8	50.65	52.73	51.65	52.86	51.97	1.04
9	51.70	50.93	50.80	48.43	50.46	1.41
10	52.77	52.70	48.01	52.93	51.60	2.40
11	48.36	52.59	49.70	51.55	50.55	1.89
12	49.15	51.07	48.33	49.94	49.62	1.16
13	52.07	48.51	48.90	51.15	50.16	1.73
14	52.24	51.01	51.15	52.74	51.79	0.84
15	52.19	48.85	52.28	49.34	50.67	1.83
16	52.53	49.63	51.25	51.15	51.14	1.19
17	48.16	52.89	52.84	50.86	51.19	2.23
18	52.36	48.84	52.88	48.22	50.58	2.38
19	49.00	51.83	49.48	51.67	50.49	1.46
20	52.69	49.86	51.27	52.28	51.52	1.26
21	51.88	48.09	50.64	49.61	50.05	1.60
22	48.33	49.81	51.88	48.23	49.56	1.71
23	48.81	50.90	48.84	52.12	50.17	1.63
24	50.68	49.19	51.66	50.71	50.56	1.02
25	51.21	51.25	50.83	52.34	51.41	0.65
				Average	50.661	1.452

Eliminating sample 4, revised control limits can be calculated by taking remaining 24 samples into consideration as shown in Table 2.6.

Revised control limits for the given process to construct $\overline{X}$ and S charts are as follows.

Table 2.6 Data for the calculation of the revised control limits

Sample number	Observations				$\overline{X}_i$	s_i
	I	II	III	IV		
1	51.98	49.21	49.73	50.16	50.27	1.21
2	50.94	50.28	50.77	51.40	50.85	0.46
3	50.87	51.67	49.89	52.68	51.28	1.18
5	48.97	52.20	49.86	52.46	50.87	1.73
6	50.43	51.08	52.99	50.41	51.23	1.21
7	48.51	51.18	52.02	51.09	50.70	1.52
8	50.65	52.73	51.65	52.86	51.97	1.04
9	51.70	50.93	50.80	48.43	50.46	1.41
10	52.77	52.70	48.01	52.93	51.60	2.40
11	48.36	52.59	49.70	51.55	50.55	1.89
12	49.15	51.07	48.33	49.94	49.62	1.16
13	52.07	48.51	48.90	51.15	50.16	1.73
14	52.24	51.01	51.15	52.74	51.79	0.84
15	52.19	48.85	52.28	49.34	50.67	1.83
16	52.53	49.63	51.25	51.15	51.14	1.19
17	48.16	52.89	52.84	50.86	51.19	2.23
18	52.36	48.84	52.88	48.22	50.58	2.38
19	49.00	51.83	49.48	51.67	50.49	1.46
20	52.69	49.86	51.27	52.28	51.52	1.26
21	51.88	48.09	50.64	49.61	50.05	1.60
22	48.33	49.81	51.88	48.23	49.56	1.71
23	48.81	50.90	48.84	52.12	50.17	1.63
24	50.68	49.19	51.66	50.71	50.56	1.02
25	51.21	51.25	50.83	52.34	51.41	0.65
				Average	50.779	1.448

For $\overline{X}$ chart

$$CL = \overline{\overline{X}} = 50.779$$
$$UCL = \overline{\overline{X}} + A_3\bar{s} = 50.779 + 1.628 \times 1.448 = 53.136$$
$$LCL = \overline{\overline{X}} - A_3\bar{s} = 50.779 - 1.628 \times 1.448 = 48.422$$

For s chart

$$CL = \bar{s} = 1.448$$
$$UCL = B_4\bar{s} = 2.266 \times 1.448 = 3.281$$
$$LCL = B_3\bar{s} = 0 \times 1.448 = 0$$

Fig. 2.3 $\overline{X}$ chart

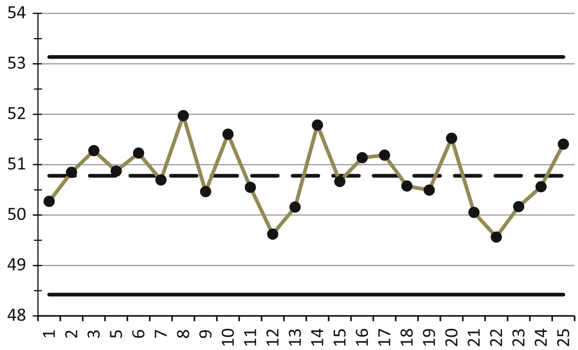
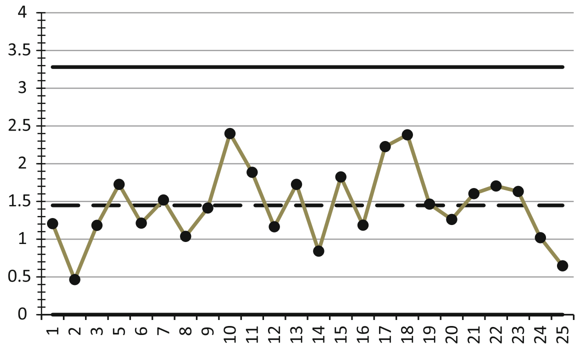


Fig. 2.4 S chart



Since all points plot in control, these limits can be set as the control limits to construct $\overline{X}$ and S charts as given in Figs. 2.3 and 2.4.

2.3 Moving Average (MA) Control Charts

In the cases where data are collected slowly over a period of time, or data are expensive to collect, moving average (MA) control charts are beneficial. The MA charts can help bringing trends to light more rapidly than conventional charts. However, run tests are not valid, since the adjacent points on the MA charts are not independent. As another disadvantage, there is a tendency to forget that individual observations have more variability than do the averages.

Moving average charts use the central limit theorem to make data approximately normal. There are two types of the moving average charts which are most commonly used: Exponentially weighted moving average charts (EWMA) and generally weighted moving average charts (GWMA).

2.3.1 Exponentially Weighted Moving Average (EWMA) Control Charts

The traditional EWMA control chart was introduced by Roberts in 1959 as below.
The statistic that is calculated is:

$$\mathrm{EWMA_t} = \lambda \mathrm{X_t} + (1 - \lambda)\mathrm{EWMA_{t-1}} \text{for } \mathrm{t} = 1, 2, \ldots, \mathrm{n} \tag{2.13}$$

where $EWMA_0$ is the mean of the historical data (target) and is equal to $\overline{\overline{X}}$, X_t refers to the observation at time t, n is the number of observations to be monitored, and $0 < \lambda \leq 1$ is a constant determining the depth of memory of the EWMA. The parameter λ determines the rate at which the older data enter into the calculation of the EWMA statistic where λ = 1 implies that only the most recent measurement from the observations influences the EWMA. In another words, a large value of λ that is closer to 1 gives more weight to recent data and a small value of λ that is closer to 0 gives more weight to the older data. The parameter λ is usually set between 0.2 and 0.3 although the choice is somewhat arbitrary (Montgomery 2001).

If X_t's are independent random variables with a known standard deviation of the population σ and a variance of σ^2/n, then the variance of the $EWMA_t$ becomes

$$\sigma^2_{\mathrm{EWMA_t}} = \frac{\sigma^2}{\mathrm{n}}\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2\mathrm{t}}\right] \tag{2.14}$$

As t increases, $\sigma^2_{EWMA_t}$ reaches to a limiting value of

$$\sigma^2_{\mathrm{EWMA}} = \frac{\sigma^2}{\mathrm{n}}\left(\frac{\lambda}{2-\lambda}\right) \tag{2.15}$$

For a moderately large number of sample size, the control limits for the traditional EWMA control charts can be expressed as follows:

$$\mathrm{CL_{EWMA}} = \overline{\overline{\mathrm{X}}} \tag{2.16}$$

$$\mathrm{UCL_{EWMA}} = \overline{\overline{\mathrm{X}}} + 3\frac{\sigma}{\sqrt{\mathrm{n}}}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.17}$$

$$\mathrm{LCL_{EWMA}} = \overline{\overline{\mathrm{X}}} - 3\frac{\sigma}{\sqrt{\mathrm{n}}}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.18}$$

If t is small, the control limits for the traditional EWMA control charts can be expressed as follows:

$$\mathrm{CL_{EWMA}} = \overline{\overline{\mathrm{X}}} \tag{2.19}$$

$$\mathrm{UCL_{EWMA}} = \overline{\overline{\mathrm{X}}} + 3\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]} \tag{2.20}$$

$$\mathrm{LCL_{EWMA}} = \overline{\overline{\mathrm{X}}} - 3\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]} \tag{2.21}$$

If σ is unknown and estimated from the samples, then $\overline{R}$ can be used for constructing traditional EWMA charts. In this case, the control limits are as follows:

$$\mathrm{CL_{EWMA}} = \overline{\overline{\mathrm{X}}} \tag{2.22}$$

$$\mathrm{UCL_{EWMA}} = \overline{\overline{\mathrm{X}}} + A_2\overline{\mathrm{R}}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.23}$$

$$\mathrm{LCL_{EWMA}} = \overline{\overline{\mathrm{X}}} - A_2\overline{\mathrm{R}}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.24}$$

where $\overline{R}$ is the mean of the ranges of the samples, and A_2 is a constant given in Appendix A.

2.3.1.1 A Numerical Example

Consider a process with the parameters of $EWMA_0 = 50.0$ and $s = 2.0539$ calculated from historical data. For the following 20 points observed, let us construct EWMA control charts.

51.9	47.1	53.1	49.4	50	47.6	50.2	50.1	51.4	50.6
49.8	47.5	9.9	50.9	47.6	51.5	52.8	52.3	54.9	50.1

With λ chosen to be 0.3 the parameter $\sqrt{\frac{\lambda}{2-\lambda}}$ is equal to 0.4201. CL, LCL and UCL for the EWMA chart can be calculated as follows.

$$CL_{EWMA} = \overline{\overline{X}} = 50.0$$

$$UCL_{EWMA} = \overline{\overline{X}} + 3\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} = 50.0 + 3(2.0539)(0.4201) = 52.5884$$

Table 2.7 EWMA statistics for 20 points

t	X	EWMA
0		50.00
1	51.9	50.57
2	47.1	49.53
3	53.1	50.60
4	49.4	50.24
5	50.0	50.17
6	47.6	49.40
7	50.2	49.64
8	50.1	49.78
9	51.4	50.26
10	50.6	50.36
11	49.8	50.20
12	47.5	49.39
13	49.9	49.54
14	50.9	49.95
15	47.6	49.24
16	51.5	49.92
17	52.8	50.78
18	52.3	51.24
19	54.9	52.34
20	50.1	51.67

$$LCL_{EWMA} = \overline{\overline{X}} + 3\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}} = 50.0 - 3(2.0539)(0.4201) = 47.4115$$

EWMA statistics of the 20 points are calculated by using Eq. 2.13 and summarized in Table 2.7. Constructed EWMA chart is illustrated in Fig. 2.5.

2.3.2 *Maximum Generally Weighted Moving Average (MaxGWMA) Control Charts*

The EWMA chart is widely used to detect small shifts in process mean and it has successfully become a source of inspiration to the many researchers as in the reviews by Xie (1999), Han and Tsung (2004), Eyvazian et al. (2008), Li and Wang (2010), Zhang et al. (2010), Sheu et al. (2012). On the basis of maximum statistic values, Chen and Cheng developed a Maxtype chart which effectively controls both process mean and variability on a single chart (Chen and Cheng 1998); Xie further examined numerous EWMA-type control charts and resulted that the MaxEWMA chart is superior to others in detecting small shifts of the process mean and

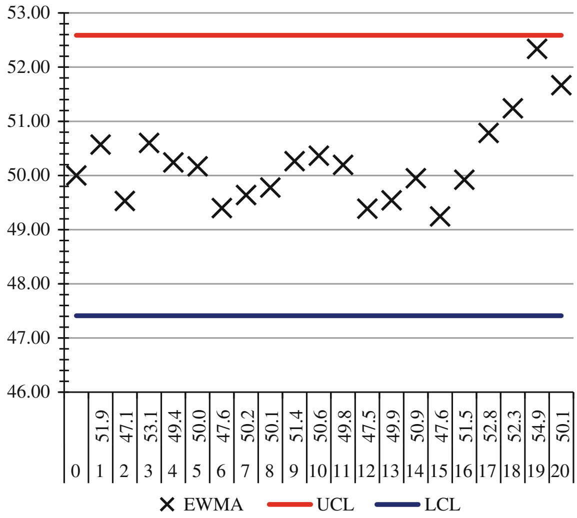


Fig. 2.5 EWMA Chart for the given data

variability as well as in identifying the source and the direction of an out-of-control signal (Xie 1999). Sheu and Lin created the generally weighted moving average (GWMA) chart which can detect small shifts much quicker than the EWMA can (Shu et al. 2014). By combing the advantages of the MaxEWMA chart and GWMA chart, Sheu et al. proposed a new chart called the maximum generally weighted moving average (MaxGWMA) chart which was found to be more sensitive under abnormal variations of on-line manufacturing processes than the MaxEWMA chart (Sheu et al. 2012).

Let X be the key quality characteristic with a normal distribution $N(\mu_0, \sigma_0^2)$, where μ_0 is the process mean and σ_0 is the process standard deviation. If the new mean is $\mu_{1=}\mu_0 \pm \delta\sigma_0$, then the process mean is said to have a shift of δ $(\delta \neq 0)$ standard deviation. Similarly, if the new standard deviation is $\sigma_1 = (1+\rho)\sigma_0$, then the process is said to have a shift of ρ standard deviation in variability. In real cases, μ_0 and σ_0 are usually unknown and can be estimated from the randomly collected sample data of which at least 20–25 in-control samples are recommended. Assume that m random subgroups and each subgroup containing n observation of x are collected. The sample average of the ith sample $(\overline{x}_i)$ and the grand sample average $\left(\overline{\overline{x}}\right)$ can be calculated by using formulas below.

$$\overline{\mathrm{x}}_{\mathrm{i}} = \frac{1}{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{x}_{\mathrm{ij}} \text{ for } \mathrm{i} = 1, 2, \ldots, \mathrm{m} \tag{2.25}$$

$$\overline{\overline{\mathrm{x}}} = \frac{1}{\mathrm{m}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{ij}} \text{ for } \mathrm{i} = 1, 2, \ldots, \mathrm{m} \tag{2.26}$$

In the same way, the standard deviation of the ith sample (s_i) and the average of the m standard deviations ($\bar{s}$) can be calculated by using the following formulas.

$$s_i = \sqrt{\frac{\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_i\right)^2}{n-1}} \tag{2.27}$$

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i \tag{2.28}$$

The unbiased estimators of the μ_0 and σ_0 are then given by

$$\mu_0 = E(\bar{x}) = \bar{\bar{x}} \tag{2.29}$$

$$\sigma_0 = E(\bar{s}) = \bar{s}/c_4 \tag{2.30}$$

where the value of the c_4 is a constant and can be found from Appendix A.

For the computation of the MaxGWMA statistic, two mutually independent statistics, M_i and S_i are defined as follows.

$$M_i = \frac{(\bar{x}_i - \mu_0)}{\sigma_0/\sqrt{n}} \tag{2.31}$$

$$S_i = \emptyset^{-1}\left\{F\left[\frac{(n-1)s_i^2}{\sigma_0^2}, n-1\right]\right\} \tag{2.32}$$

where F(a, b) refers to the chi-square distribution of a with b degrees of freedom, and $\emptyset^{-1}$ is the inverse of the standard normal distribution.

Let A be an event of interest and t be the counting number of samples between two adjacent occurrences of A. $p_j = P(t > j)$ is the probability that A does not occur in the first j samples. The probability of p_j of the occurrence of A at the jth sample can be calculated by Shu et al. (2014)

$$p_j = P(t > j-1) = P_{j-1} - P_j \tag{2.33}$$

Remember that for $\forall j > 1$ and $j < i$, we have $P_j > P_i$.

GWMA statistics for the ith subgroup are given by

$$U_i = \sum_{j=1}^{i} p_j M_{i+1-j} \tag{2.34}$$

$$V_i = \sum_{j=1}^{i} p_j S_{i+1-j} \tag{2.35}$$

For the ease of computation, We chose $P_j = q^{j^\alpha}$, where q is called a design parameter that is a constant with the value in [0,1] and α is called an adjustment parameter determined by the practitioner (Sheu and Lin 2003). Obviously, the traditional EWMA chart is a special case of GWMA chart when $\alpha = 1$ and $q = 1 - \lambda$. Now, the probability p_j of the occurrence of A at the jth sample can be rewritten as

$$p_j = q^{(j-1)^\alpha} - q^{j^\alpha} \quad (2.36)$$

If the process is not shifting, with respect to the independence of the M_i and S_i, then GWMA statistics U_i and V_i are also mutually independent and follow the same standard normal distribution. Thus, their variances can be determined by

$$\sigma^2_{U_i} = \sigma^2_{V_i} = \eta_i = \sum_{j=1}^{i} p_j \quad (2.37)$$

The statistic (MG) used to construct the MaxGWMA chart is defined as

$$MG_i = \max(|U_i|, |V_i|) \quad (2.38)$$

A small value of MG_i indicates that the process mean and process variability are close to their respective targets, while a large value of MG_i indicates that the process mean and process variability are away from their respective targets.

Since MG_i is nonnegative, only upper control limit for the ith subgroup formulated below is used to monitor MG_i (Sheu et al. 2012).

$$UCL_i = E(MG_i) + L\sqrt{\sigma^2(MG_i)} \quad (2.39)$$

where L is a constant.

Based on desired in control ARL_0, sample size n, and optimal values of parameters q, α and L for an initial state of the MaxGWMA chart, the approximate value of UCL_i can be given as (Sheu et al. 2012)

$$UCL_i = (1.12838 + 0.60281L)\sqrt{\eta_i} \quad (2.40)$$

In the MaxGWMA chart, each of the MG_i values is compared with the UCL_i and the following judgement can be performed about whether the process is in control or out of control.

$$\text{Process Control for MaxGWMA} = \left\{ \begin{array}{lll} MG_i \le UCL_i & ; & \text{in control} \\ MG_i > UCL_i & ; & \text{out of control} \end{array} \right\} \quad (2.41)$$

If there is a change in the process mean and/or process variability, Table 2.8 can be used to identify the situations (Sheu et al. 2012).

Table 2.8 Indications of the out of control points

Situation	Symbol	Indication
$MG_i = U_i$ and $\|V_i\| \leq UCL_i$	m^+	An increase in the process mean
$MG_i = -U_i$ and $\|V_i\| \leq UCL_i$	m_-	An decrease in the process mean
$MG_i = V_i$ and $\|U_i\| \leq UCL_i$	v^+	An increase in the process variability
$MG_i = -V_i$ and $\|U_i\| \leq UCL_i$	v_-	A decrease in the process variability
$U_i > UCL_i$ and $V_i > UCL_i$	$++$	An increase in both the process mean and the process variability
$-U_i > UCL_i$ and $-V_i > UCL_i$	$--$	A decrease in both the process mean and the process variability
$U_i > UCL_i$ and $-V_i > UCL_i$	$+-$	An increase in the process mean and a decrease in the process variability
$-U_i > UCL_i$ and $V_i > UCL_i$	$-+$	A decrease in the process mean and an increase in the process variability

2.4 Unnatural Patterns for Control Charts

The usual SPC control chart limit rules display at the 3-sigma level. In this case, a simple threshold test decides if a process is in or out of control. Once a process is brought under control using the simple 3-sigma level tests, quality professionals often want to increase the sensitivity of the control chart by detecting and correcting problems before the process excludes 3-sigma control limits. Based on the probability, more complex tests rely on more complicated decision-making criteria by examining the patterns of the points (sample characteristic) on the control chart and presenting a set of rules with respect to the very low probability of occurrence. These rules utilize historical data and look for a non-random (unnatural) pattern that can signify that the process is out of control, before reaching the normal ±3 sigma limits. In another words, a process may signal an out of control condition even its characteristic plots in control. The rules that characterize an out of control signal through the control chart limits are called “unnatural pattern rules” or “non-random pattern rules”.

The most popular of unnatural (non-random) pattern rules are the Western Electric Rules, also known as the WECO Rules, or WE Runtime Rules. First implemented by the Western Electric Co. in the 1920s, these quality control guidelines were codified in the 1950s and form the basis for the other entire rule sets (Western Electric Company: Statistical Quality Control Handbook, Indianapolis, Indiana 1956). Different industries have developed their own variants based on the WECO Rules. Other sets of rules which are common enough to recognize an identifying name, i.e. named rules, are “Nelson Rules (1984) ”, “Juran Rules (2010) ”, “Duncan Rules (1986) ”, “Automotive

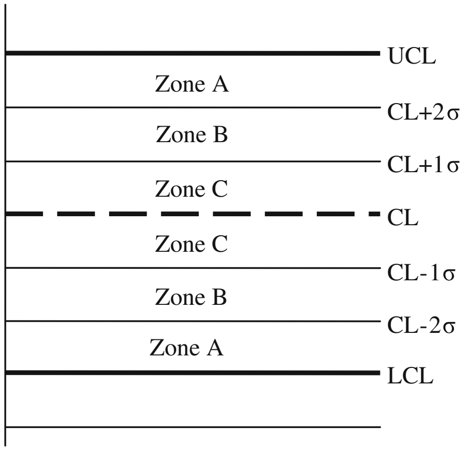


Fig. 2.6 Zones in a control chart

Industry Action Group (AIAG) Rules (Detroit 1995) ", "Gitlow Rules (1989) ", and "Westgard Rules (2014) ".

In general, when identifying these rules, the region between the usual ±3 sigma limits are divided into six region and the pattern is explained with respect to ±1, 2, and 3 sigma limits as shown in the Fig. 2.6.

Based on the zones illustrated by Fig. 2.6, some "Named Unnatural Pattern Rules" are explained in the following sections.

2.4.1 Western Electric Rules

In the Western Electric Rules, a process is accepted to signal an out of control if any of the following criteria are observed (Western Electric Company 1956):

1. One of the any point outside one of the 3-sigma control limits: If a point lies outside either of ±3 sigma limits, there is only a 0.27 % chance that this was caused by the normal process.
2. Two out of the three consecutive points outside of the 2-sigma control limits and on the same side of the center line: The probability that any point will fall outside the warning limit of 2-sigma is only 5 %. The chance that two out of three points in a row fall outside the warning limits is only about 1 %.
3. Four out of the five consecutive points outside of the 1-sigma control limits and on the same side of the center line: In normal processing, 68 % of points fall within 1-sigma of the mean. The probability that 4 of 5 points fall outside of one sigma is only about 3 %.
4. Eight consecutive points on the same side of the center line: The probability of getting eight points on the same side of the mean is only around 1 %.

Remember that these rules apply separately to both sides of the center line at a time. Therefore, in the WECO Rules there are eight actual alarm conditions. There are also additional WE Rules related with the trends of the points. These are often referred to as Western Electric Supplemental Rules.

5. Six points in a row increasing or decreasing: Sometimes this rule is changed to seven points rising or falling.
6. Fifteen points in a row within one sigma: In normal operation, 68 % of points will fall within one sigma of the mean. The probability that 15 points in a row will do so, is less than 1 %.
7. Fourteen points in a row alternating direction. The chances that the second point is always higher than (or always lower than) the preceding point, for all seven pairs is only about 1 %.
8. Eight points in a row outside one sigma. Since 68 % of points lie within one sigma of the mean, the probability that eight points in a row fall outside of the one-sigma line is less than 1 %.

2.4.2 Nelson Rules

The Nelson rules are almost identical to the combination of the WECO. The only difference is in Rule #4 where nine consecutive points on the same side of the center line is accepted as a signal (Nelson 1984).

2.4.3 Other Named Rules

In general, a given rule specifies two test conditions: Being a value of N points out of M consecutive points above and below of a specified sigma control limits. From this point of view, named rules mentioned in Sect. 2.4 are summarized and tabulated in Table 2.9.

2.5 Ranking Fuzzy Numbers and Direct Fuzzy Approach

Fuzzy numbers as they are used to represent uncertainties are an important issue in research in fuzzy set theory and their applications (Gülbay and Kahraman 2006). Because of the suitability for representing uncertain values, fuzzy numbers have been widely used in many applications. When quality characteristic and control limits are represented as fuzzy numbers, the main problem is to decide whether the quality characteristic lies within their respective fuzzy control limits or not in order to decide about the process: in-control or out of control. In such situations, a

Table 2.9 Summarization of some named rules in the form of N/M

Rule	Named rules						
	WECO	Nelson	Juran	Duncan	AIAG	Gitlow	Westgard
Outside ±3σ limits	1/1	1/1	1/1	1/1	1/1	1/1	1/1
Outside ±2σ limits	2/3	2/3	2/3	2/3		2/3	2/2 or 2/3
Outside ±1σ limits	4/5	4/5	4/5	4/5		4/5	4/4 or 3/4
On the same side of centerline	8/8	9/9	9/9		7/7	8/8	10/10
Increasing or decreasing in a row	6/6		6/6	7/7	7/7	8/8	7/7
Within ±1σ	15						
Outside ±1σ	8/8		8/8				
Outside ±2σ							1/1
Alternating	14/14						
Opposite sides of ±2σ							2/2

comparison of the fuzzy numbers is required. Various methods to manipulate fuzzy numbers have been developed to overcome the problem illustrated in Fig. 2.7 (Chen and Chen 2009; Chen and Sanguansat 2011; Deng et al. 2006; Wang and Lee 2008; Yager 1978; Zimmermann 1996).

The results of studies on ranking fuzzy numbers have been used in application areas especially where decision-making and data analysis have a vital importance. The ranking methods can be classified in three categories. The first category directly transforms each fuzzy number into a crisp real number and the second category compares a fuzzy number to all the other n − 1 fuzzy numbers to obtain its mapping into a positive real number. The third category differs substantially from the first two.

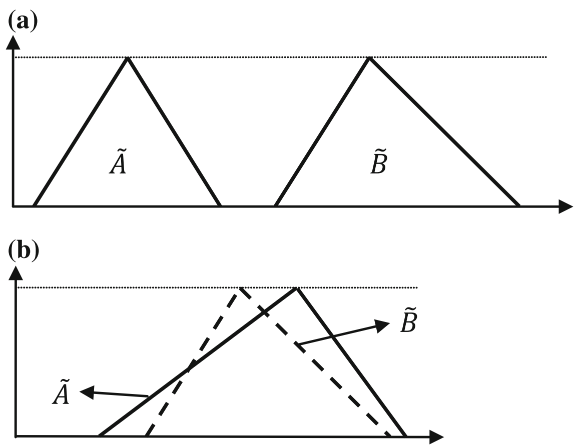


Fig. 2.7 Illustration of ranking two fuzzy numbers. **a** $\tilde{A} < \tilde{B}$. **b** Not clear

In this category, a method for pairwise ranking or preference for all pairs of fuzzy numbers is determined and then based on these pairwise orderings, a final order of the n fuzzy numbers is attempted (Shureshjani and Darehmiraki 2013). The significance of ranking fuzzy numbers for solving real world decision problems in a fuzzy environment has led to tremendous efforts being spent on the development of various ranking approaches (Bortolan and Degani 1985; Chen and Hwang 1992; Cheng 1998; Choobineh and Li 1993; Chu and Tsao 2002; Detyniecki and Yager 2001; Dias 1993; Dubois and Prade 1978, 1980; Fortemps and Roubens 1996; Jain 1976, 1978; Kim et al. 1998; Lee et al. 1994; Lee and Lee-Kwang 1999; Lee and Li 1998; Liu and Han 2005; Murakami 1983; Raj and Kumar 1999; Requena et al. 1994; Tran and Duckstein 2002; Wang et al. 2009; Zadeh 1965). To whom more interested to the ranking methods for fuzzy numbers, it is suggested to read (Brunelli and Mezeib 2013) for further knowledge.

For the fuzzy quality control chart studies we present a direct fuzzy comparison method to compare fuzzy numbers because the method enables the user to have a fuzzy decision about the comparison (Gülbay and Kahraman 2007).

Let $\tilde{X} = (X_a, X_b, X_c, X_d)$ be the fuzzy quality characteristic; $\widetilde{LCL} = (LCL_1, LCL_2, LCL_3, LCL_4)$ and $\widetilde{UCL} = (UCL_1, UCL_2, UCL_3, UCL_4,)$ be fuzzy lower control limit and fuzzy upper control limit, respectively, represented by trapezoidal fuzzy numbers. A decision about whether the process is in control can be made according to the percentage area of the sample which remains inside the $\widetilde{UCL}$ and/or $\widetilde{LCL}$. When the fuzzy sample is completely involved by the fuzzy control limits, the process is said to be "in-control". If a fuzzy sample is totally excluded by the fuzzy control limits, the process is said to be "out-of-control". Otherwise, a sample is partially included by the fuzzy control limits. In this case, if the percentage area which remains inside the fuzzy control limits (β_j) is equal or greater than a predefined acceptable percentage (β), then the process can be accepted as "rather in-control". Otherwise, it can be stated as "rather out of control". The possible decisions resulting from "Direct Fuzzy Approach (DFA) are illustrated in Fig. 2.8. The parameters to determine the sample's area outside the control limits for any a-level cut are LCL_1, LCL_2, UCL_3, UCL_4, a, b, c, d, and α. The shapes of the control limits and fuzzy samples are formed by the lines of $\overline{LCL_1 LCL_2}$, UCL_3UCL_4, $\overline{ab}$, and $\overline{cd}$,. A flowchart to calculate area of the fuzzy sample outside the control limits is given in Fig. 2.9. The sample's area above the upper control limits, A^U_{out}, and sample area falling below the lower control limits, A^L_{out}, can be calculated according to the flowchart given in Fig. 88. The equations to compute A^U_{out} and A^L_{out} are given in Appendix B. Then, the total area outside the fuzzy control limits, A_{out}, is the sum of the areas below the fuzzy lower control limit and above the fuzzy upper control limit. The percentage sample area within the fuzzy control limits is calculated as

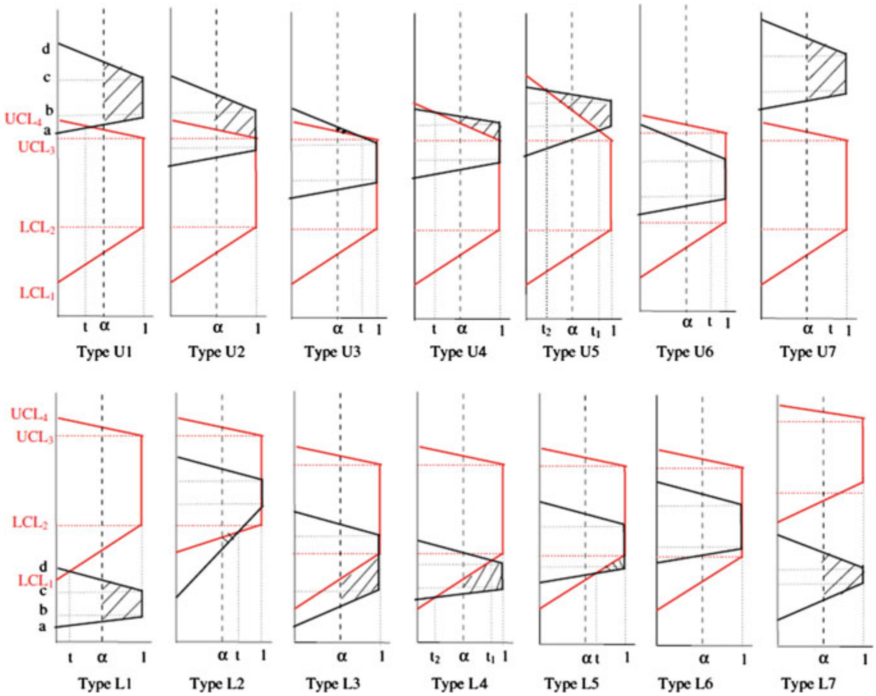


Fig. 2.8 Illustration of the possible areas outside the fuzzy control limits at α-level cut

$$\beta_j^\alpha = \frac{S_j^\alpha - A_{out}^\alpha}{S_j^\alpha} \tag{2.42}$$

where S_j^α is the sample's area at α-level cut. Remember that performing α-level cut is not a must but a preference if decided by a quality practitioner. Furthermore, the acceptable percentage (β) is set by the quality practitioner with respect to the tightness of the inspection.

2.6 Fuzzy Approaches for Control Charts for Variables

Many quality characteristics can be expressed in terms of a numerical measurement such as length, width, weight, temperature, volume etc. A process is either "in control" or "out of control" depending on numeric observation values. For many problems, control limits could not be so precise. Uncertainty comes from the measurement system including operators and gauges, and environmental conditions (Senturk and Erginel 2009). A research work incorporating uncertainty into decision analysis is basically done through the probability theory and/or the fuzzy set theory.

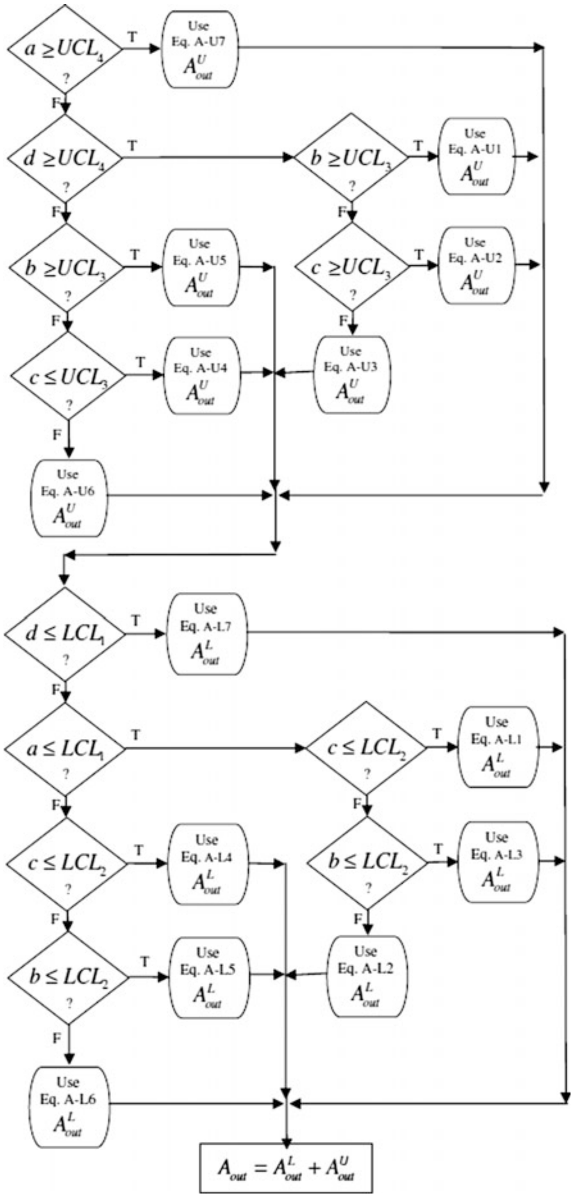


Fig. 2.9 Flowchart to compute the area of a fuzzy sample (a,b,c,d) falling outside the fuzzy control limits. (See Appendix B for the equations)

The former represents the stochastic nature of decision analysis while the latter captures the subjectivity of human behaviour. A rational approach toward decision-making should take human subjectivity into account, rather than employing only objective probability measures. The fuzzy set theory is a perfect means for modeling uncertainty (or imprecision) arising from mental phenomena which is

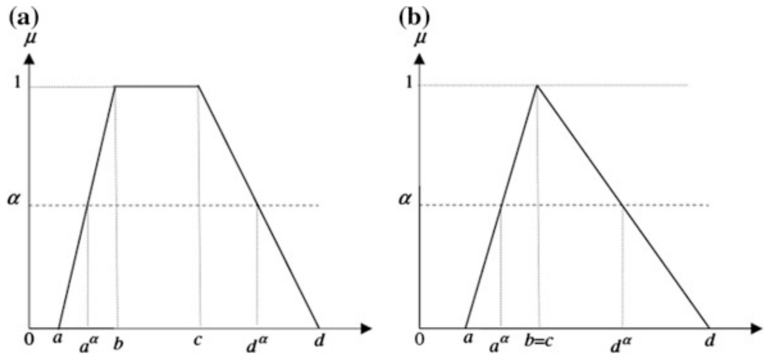


Fig. 2.10 Representation of a sample by trapezoidal and/or triangular fuzzy numbers: **a** Trapezoidal (a,b,c,d) and **b** triangular (a,b,b,d)

neither random nor stochastic. When human subjectivity plays an important role in defining the quality characteristics, the classical control charts may not be applicable since they require certain information. The judgment in classical process control results in a binary classification as "*in-control*" or "out-of-control" while fuzzy control charts may handle several intermediate decisions. Fuzzy control charts are inevitable to use when the statistical data in consideration are uncertain or vague; or available information about the process is incomplete or includes human subjectivity (Gülbay and Kahraman 2007). In the fuzzy case, each sample, or subgroup, is represented by a trapezoidal fuzzy number (a,b,c,d) or a triangular fuzzy number (a, b,d), or (a,c,d) with an α-cut (if necessary) as shown in Fig. 2.10.

X-R and X-s fuzzy control charts can be presented as given in Sects. 2.6.1 and 2.6.2 (Gülbay and Kahraman 2006).

2.6.1 Fuzzy $\overline{X}$ and R Control Charts

Let quality characteristic of a sample with a size of n be represented as fuzzy triangular numbers by $\overline{X}_i\left(X_{ija}, X_{ijb}, X_{ijc},\right)$ $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n.$ Using the fuzzy arithmetic the mean of the each subgroup and grand average of the samples can be calculated by Equations below.

$$\tilde{\overline{\mathrm{X}}}_{\mathrm{i}} = \left(\frac{\sum_{j=1}^{n} \mathrm{X_{ija}}}{\mathrm{n}}, \frac{\sum_{j=1}^{n} \mathrm{X_{ijb}}}{\mathrm{n}}, \frac{\sum_{j=1}^{n} \mathrm{X_{ijc}}}{\mathrm{n}}\right) = (\mathrm{X_{ia}}, \mathrm{X_{ib}}, \mathrm{X_{ic}}) \qquad (2.43)$$
$$\mathrm{i} = 1, 2, \ldots, \mathrm{m}; \quad \mathrm{j} = 1, 2, \ldots, \mathrm{n}$$

$$\tilde{\overline{\overline{\mathrm{X}}}} = \left(\frac{\sum_{i=1}^{m} \mathrm{X_{ia}}}{\mathrm{m}}, \frac{\sum_{i=1}^{m} \mathrm{X_{ib}}}{\mathrm{m}}, \frac{\sum_{i=1}^{m} \mathrm{X_{ic}}}{\mathrm{m}}\right) = \left(\overline{\overline{\mathrm{X}}}_{\mathrm{a}}, \overline{\overline{\mathrm{X}}}_{\mathrm{b}}, \overline{\overline{\mathrm{X}}}_{\mathrm{c}}\right) \qquad (2.44)$$

The fuzzy range of each subgroup can be represented by the equation below.

$$\tilde{R}_i = \tilde{X}_{ij,max} - \tilde{X}_{ij,min} \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \tag{2.45}$$

In crisp calculation, the maximum and minimum values of *R* can be easily determined. But it is not so easy to decide which fuzzy range observation is maximum and minimum. If represented fuzzy numbers are not intersecting, one can easily say that the fuzzy number with the most left support is smallest or minimum and the fuzzy number with the most right support is the greatest or maximum. In case where fuzzy observations have intersecting supports the problem about the ranking fuzzy numbers arises. Fuzzy numbers cannot be easily compared to each other. So, in decision analysis it is very difficult to distinguish the best possible course of action among alternatives defined by means of fuzzy numbers. Comparing and ranking fuzzy numbers in a given situation is complex and challenging (Yeh and Deng 2004; Sun and Wu 2006; Asady 2010). This is because fuzzy numbers usually represented by the possibility distribution (Zimmermann 2000; Dubois and Prade 1994) often overlap each other in many practical situations (Cheng 1998; Yeh and Deng 2004). It is difficult to clearly determine which fuzzy number is larger or smaller than another for a given situation, in particular when these two fuzzy numbers are similar (Kim and Park 1990; Deng 2007). Consequently, there are many fuzzy ranking methods, but an exhaustive review of ranking methods would be beyond the scope of this chapter. An attempt to list most of the ranking methods was made in Rao and Shankar (2011). DFA presented in Sect. 2.5 can also be used to find the greatest and smallest of the fuzzy numbers in any sample group.

Once the maximum and minimum fuzzy observation is decided, the fuzzy range can be determined by the following equations.

$$\tilde{R}_i = \tilde{X}_{ij,max} - \tilde{X}_{ij,min} = \tilde{X}_i\left(X_{ija}, X_{ijb}, X_{ijc},\right)_{max} - \left(X_{ija}, X_{ijb}, X_{ijc},\right)_{min} \tag{2.46}$$

$$\tilde{R}_i = (\tilde{X}_{ija,max} - \tilde{X}_{ijc,min}, \tilde{X}_{ijb,max} - \tilde{X}_{ijb,min}, \tilde{X}_{ijc,max} - \tilde{X}_{ija,min}) = (R_{ia}, R_{ib}, R_{ic}) \tag{2.47}$$

After calculating range of each subgroup, the fuzzy mean of the ranges can be defined as:

$$\tilde{\overline{R}} = \left(\frac{\sum_{i=1}^{m} R_{ia}}{m}, \frac{\sum_{i=1}^{m} R_{ib}}{m}, \frac{\sum_{i=1}^{m} R_{ic}}{m}\right) = \left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) \tag{2.48}$$

Control limits for the fuzzy $\tilde{\overline{X}}$ control charts, are then formulized as follows:

$$\widetilde{CL} = \tilde{\overline{\overline{X}}} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) = (CL_1, CL_2, CL_3) \tag{2.49}$$

Table 2.10 Summary of the fuzzy control limits for the $\overline{X} - R$ control chart

	$\overline{X}$ chart	R chart
Center Line $\widetilde{CL} = (CL_1, CL_2, CL_3)$	$\left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right)$	$\left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right)$
Lower Control Limit $\widetilde{LCL} = (LCL_1, LCL_2, LCL_3)$	$\left(\overline{\overline{X}}_a - A_2\overline{R}_c, \overline{\overline{X}}_b - A_2\overline{R}_b, \overline{\overline{X}}_c - A_2\overline{R}_a\right)$	$\left(D_3\overline{R}_a, D_3\overline{R}_b, D_3\overline{R}_c\right)$
Upper Control Limit $\widetilde{UCL} = (UCL_1, UCL_2, UCL_3)$	$\left(\overline{\overline{X}}_a + A_2\overline{R}_a, \overline{\overline{X}}_b + A_2\overline{R}_b, \overline{\overline{X}}_c + A_2\overline{R}_c\right)$	$\left(D_4\overline{R}_a, D_4\overline{R}_b, D_4\overline{R}_c\right)$

$$\begin{aligned}\widetilde{UCL} &= \tilde{\overline{\overline{X}}} + A_2\tilde{\overline{R}} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) + A_2\left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) \\ &= \left(\overline{\overline{X}}_a + A_2\overline{R}_a, \overline{\overline{X}}_b + A_2\overline{R}_b, \overline{\overline{X}}_c + A_2\overline{R}_c\right) \\ &= (UCL_1, UCL_2, UCL_3)\end{aligned} \tag{2.50}$$

$$\begin{aligned}\widetilde{LCL} &= \tilde{\overline{\overline{X}}} - A_2\tilde{\overline{R}} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) - A_2\left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) \\ &= \left(\overline{\overline{X}}_a - A_2\overline{R}_c, \overline{\overline{X}}_b - A_2\overline{R}_b, \overline{\overline{X}}_c - A_2\overline{R}_a\right) \\ &= (LCL_1, LCL_2, LCL_3)\end{aligned} \tag{2.51}$$

Remember that the constants A_2 as well as D_3 and D_4 depend on the sample (number of observation in each sample) size and are tabulated for various sample sizes in Appendix A.

Fuzzy control limits for the R charts can be derived in the same way.

$$\widetilde{CL} = \tilde{\overline{\overline{R}}} = \left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) = (CL_1, CL_2, CL_3) \tag{2.52}$$

$$\begin{aligned}\widetilde{UCL} &= D_4\tilde{\overline{R}} = D_4\left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) = \left(D_4\overline{R}_a, D_4\overline{R}_b, D_4\overline{R}_c\right) \\ &= (UCL_1, UCL_2, UCL_3)\end{aligned} \tag{2.53}$$

$$\widetilde{LCL} = D_3\tilde{\overline{R}} = \left(D_3\overline{R}_a, D_3\overline{R}_b, D_3\overline{R}_c\right) = (LCL_1, LCL_2, LCL_3) \tag{2.54}$$

Fuzzy control limits for the $\overline{X} - R$ control chart are summarized in Table 2.10.

2.6.2 *Fuzzy $\overline{X}$ and S Control Charts*

Determination of the control limits for paired $\overline{X}$ and s charts are based on the standard deviation as mentioned in Sect. 2.2.2.1. Hence, average standard deviation of the subgroups need to be firstly calculated.

Let quality characteristic of a sample with a size of n be represented as fuzzy triangular numbers by $\tilde{X}_i\left(X_{ija},\ X_{ijb},\ X_{ijc},\right) i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Using the fuzzy arithmetics, the fuzzy standard deviation of the each subgroup and fuzzy average standard deviation of the samples can be derived by the Equations below.

$$\begin{aligned}\tilde{s}_i &= \sqrt{\frac{\sum_{j=1}^{n}\left(\tilde{X}_{ij} - \tilde{\bar{X}}_i\right)^2}{n-1}} = \sqrt{\frac{\sum_{j=1}^{n}\left[\left(X_{ija}, X_{ijb}, X_{ijc}\right) - \left(\overline{X}_{ia}, \overline{X}_{ib}, \overline{X}_{ic}\right)\right]^2}{n-1}}\\ &= \left(s_{ia}, s_{ib}, s_{ic}\right)\end{aligned} \tag{2.55}$$

$$\tilde{\bar{s}} = \frac{\sum_{i=1}^{m}\tilde{s}_i}{m} = \left(\frac{\sum_{i=1}^{m}s_{ia}}{m}, \frac{\sum_{i=1}^{m}s_{ib}}{m}, \frac{\sum_{i=1}^{m}s_{ic}}{m}\right) = \left(\bar{s}_a, \bar{s}_b, \bar{s}_c\right) \tag{2.56}$$

The control limits of fuzzy $\overline{X}$ control chart based on standard deviation are obtained as follows:

$$\widetilde{CL} = \tilde{\overline{\overline{X}}} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) = (CL_1,\ CL_2,\ CL_3) \tag{2.57}$$

$$\begin{aligned}\widetilde{UCL} = \tilde{\overline{\overline{X}}} + A_3\tilde{\bar{s}} &= \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) + A_3(\bar{s}_a,\ \bar{s}_b,\ \bar{s}_c)\\ &= \left(\overline{\overline{X}}_a + A_3\bar{s}_a,\ \overline{\overline{X}}_b + A_3\bar{s}_b, \overline{\overline{X}}_c + A_3\bar{s}_c\right)\\ &= (UCL_1,\ UCL_2,\ UCL_3)\end{aligned} \tag{2.58}$$

$$\begin{aligned}\widetilde{LCL} = \tilde{\overline{\overline{X}}} - A_3\tilde{\bar{s}} &= \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) - A_3(\bar{s}_a,\ \bar{s}_b,\ s_c)\\ &= \left(\overline{\overline{X}}_a - A_3\bar{s}_c,\ \overline{\overline{X}}_b - A_3\bar{s}_b,\ \overline{\overline{X}}_c - A_3\bar{s}_a\right)\\ &= (LCL_1, LCL_2, LCL_3)\end{aligned} \tag{2.59}$$

Similarly, the control limits of fuzzy s control chart are derived as follows:

$$\widetilde{CL} = \tilde{\bar{s}} = (\bar{s}_a,\ \bar{s}_b,\ \bar{s}_c) = (CL_1,\ CL_2,\ CL_3) \tag{2.62}$$

$$\widetilde{UCL} = B_4\tilde{\bar{s}} = B_4(\bar{s}_a,\ \bar{s}_b,\ \bar{s}_c) = (B_4\bar{s}_a,\ B_4\bar{s}_b,\ B_4\bar{s}_c) = (UCL_1,\ UCL_2,\ UCL_3)$$

$$\widetilde{LCL} = D_3\bar{s} = (B_3\bar{s}_a,\ B_3\bar{s}_b,\ B_3\bar{s}_c) = (LCL_1,\ LCL_2,\ LCL_3) \tag{2.61}$$

2.6.3 Fuzzy Exponentially Weighted Moving Average (FEWMA) Control Charts

Depending whether fuzzy process mean and fuzzy process standard deviation is known or not, FEWMA charts can be constructed as explained in Sects. 2.6.3.1 and 2.6.3.2

2.6.3.1 Fuzzy EWMA Control Charts When $\tilde{\sigma}$ Are Known

Let $\tilde{X}_i = (X_a, X_b, X_c)_i$ and $\tilde{\overline{\overline{X}}} = \left(\overline{X}_a, \overline{X}_b, \overline{X}_c\right)$ be the fuzzy observations for the ith sample and fuzzy grand averages of the t randomly collected sample data represented by triangular fuzzy numbers, respectively. Assume that fuzzy standard deviation $\tilde{\sigma}$ is known and represented by triangular fuzzy number as $\tilde{\sigma} = (\sigma_a,\ \sigma_b,\ \sigma_c)$

If the sample number t is moderately large, the parameter $\left[1-(1-\lambda)^{2t}\right]$ reaches to a limiting value of 1 and can be omitted from the formula. Hence, the control limits for the fuzzy EWMA control chart is given as follows:

$$\widetilde{\mathrm{CL}}_{\mathrm{EWMA}} = \tilde{\overline{\overline{\mathrm{X}}}} = \left(\overline{\mathrm{X}}_{\mathrm{a}}, \overline{\mathrm{X}}_{\mathrm{b}}, \overline{\mathrm{X}}_{\mathrm{c}}\right) \tag{2.62}$$

$$\widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \tilde{\overline{\overline{\mathrm{X}}}} + \frac{3}{\sqrt{\mathrm{n}}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.63}$$

$$\widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \tilde{\overline{\overline{\mathrm{X}}}} - \frac{3}{\sqrt{\mathrm{n}}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.64}$$

Replacing the values of the $\tilde{\overline{\overline{X}}}$ and $\tilde{\sigma}$ to the equations above and performing simple fuzzy arithmetics, $\widetilde{UCL}_{EWMA}$ and $\widetilde{LCL}_{EWMA}$ for the moderately large number of samples can be rewritten as

$$\widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \tilde{\overline{\overline{\mathrm{X}}}} + \frac{3}{\sqrt{\mathrm{n}}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}} = \left(\overline{\mathrm{X}}_{\mathrm{a}}, \overline{\mathrm{X}}_{\mathrm{b}}, \overline{\mathrm{X}}_{\mathrm{c}}\right) + \frac{3}{\sqrt{\mathrm{n}}}(\sigma_{\mathrm{a}}, \sigma_{\mathrm{b}}, \sigma_{\mathrm{c}})\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.65}$$

$$\widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \left(\overline{\mathrm{X}}_{\mathrm{a}} + \frac{3\sigma_{\mathrm{a}}}{\sqrt{\mathrm{n}}}\sqrt{\frac{\lambda}{2-\lambda}}, \overline{\mathrm{X}}_{\mathrm{b}} + \frac{3\sigma_{\mathrm{b}}}{\sqrt{\mathrm{n}}}\sqrt{\frac{\lambda}{2-\lambda}}, \overline{\mathrm{X}}_{\mathrm{c}} + \frac{3\sigma_{\mathrm{c}}}{\sqrt{\mathrm{n}}}\sqrt{\frac{\lambda}{2-\lambda}}\right) \tag{2.66}$$

$$\widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \tilde{\bar{\bar{X}}} - \frac{3}{\sqrt{n}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}} = \left(\overline{X}_a, \overline{X}_b, \overline{X}_c\right) - \frac{3}{\sqrt{n}}(\sigma_a, \sigma_b, \sigma_c)\sqrt{\frac{\lambda}{2-\lambda}} \tag{2.67}$$

$$\widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \left(\overline{X_a} + \frac{3\sigma_c}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}, \overline{X_b} + \frac{3\sigma_b}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}, \overline{X_c} + \frac{3\sigma_a}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}\right) \tag{2.68}$$

Similarly, if the sample number t is small, control limits for the fuzzy EWMA control chart can be given as follows:

$$\widetilde{\mathrm{CL}}_{\mathrm{EWMA}} = \tilde{\bar{\bar{X}}} = \left(\overline{X}_a, \overline{X}_b, \overline{X}_c\right) \tag{2.69}$$

$$\widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \tilde{\bar{\bar{X}}} + \frac{3}{\sqrt{n}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]} \tag{2.70}$$

$$\widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \tilde{\bar{\bar{X}}} - \frac{3}{\sqrt{n}}\tilde{\sigma}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]} \tag{2.71}$$

By replacing the values of $\tilde{\bar{\bar{X}}}$ and $\tilde{\sigma}$, control limits for the fuzzy EWMA chart for small sample sizes can be given as

$$\begin{aligned}\widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \Bigg(&\overline{X_a} + \frac{3\sigma_a}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \overline{X_b} \\ &+ \frac{3\sigma_b}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \overline{X_c} \\ &+ \frac{3\sigma_c}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}\Bigg)\end{aligned} \tag{2.72}$$

$$\begin{aligned}\widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \Bigg(&\overline{X_a} + \frac{3\sigma_c}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \overline{X_b} + \frac{3\sigma_b}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}, \overline{X_c} \\ &+ \frac{3\sigma_a}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}\Bigg)\end{aligned} \tag{2.73}$$

Readers who want to apply α-level cuts to the control limits can refer to (Şentürk et al. 2014; Gülbay et al. 2004).

2.6.3.2 Fuzzy EWMA Control Charts When $\tilde{\sigma}$ Are Unknown

Let $\tilde{R}_i = (R_a, R_b, R_c)_i$ and $\tilde{\overline{R}} = (\overline{R}_a, \overline{R}_b, \overline{R}_c)$ be the fuzzy range of the ith sample and fuzzy average range of the t samples for i = 1, 2, …, t. If fuzzy standard deviation, $\tilde{\sigma}$, is unknown, an unbiased estimator of the $\tilde{\sigma}$ can be determined from the ranges. Control limits for the fuzzy EWMA charts for the small sample sizes of t become as follows

$$\widetilde{\mathrm{CL}}_{\mathrm{EWMA}} = \tilde{\overline{\overline{\mathrm{X}}}} = \left(\overline{\mathrm{X}}_{\mathrm{a}}, \overline{\mathrm{X}}_{\mathrm{b}}, \overline{\mathrm{X}}_{\mathrm{c}}\right) \tag{2.74}$$

$$\begin{aligned} \widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} &= \tilde{\overline{\overline{\mathrm{X}}}} + \mathrm{A}_2\tilde{\overline{\mathrm{R}}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]} \\ &= \left(\overline{\mathrm{X}}_{\mathrm{a}}, \overline{\mathrm{X}}_{\mathrm{b}}, \overline{\mathrm{X}}_{\mathrm{c}}\right) + \mathrm{A}_2\left(\overline{\mathrm{R}}_{\mathrm{a}}, \overline{\mathrm{R}}_{\mathrm{b}}, \overline{\mathrm{R}}_{\mathrm{c}}\right)\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]} \end{aligned} \tag{2.75}$$

$$\begin{aligned} \widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} &= \tilde{\overline{\overline{\mathrm{X}}}} - \mathrm{A}_2\tilde{\overline{\mathrm{R}}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]} \\ &= \left(\overline{\mathrm{X}}_{\mathrm{a}}, \overline{\mathrm{X}}_{\mathrm{b}}, \overline{\mathrm{X}}_{\mathrm{c}}\right) - \mathrm{A}_2\left(\overline{\mathrm{R}}_{\mathrm{a}}, \overline{\mathrm{R}}_{\mathrm{b}}, \overline{\mathrm{R}}_{\mathrm{c}}\right)\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]} \end{aligned} \tag{2.76}$$

Performing fuzzy arithmetic to the above equations, we obtain

$$\begin{aligned} \widetilde{\mathrm{UCL}}_{\mathrm{EWMA}} = \Bigg(&\overline{\mathrm{X}}_{\mathrm{a}} + \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{a}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}, \overline{\mathrm{X}}_{\mathrm{b}} \\ &+ \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{b}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}, \overline{\mathrm{X}}_{\mathrm{c}} \\ &+ \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{c}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}\Bigg) \end{aligned} \tag{2.77}$$

$$\begin{aligned} \widetilde{\mathrm{LCL}}_{\mathrm{EWMA}} = \Bigg(&\overline{\mathrm{X}}_{\mathrm{a}} - \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{c}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}, \overline{\mathrm{X}}_{\mathrm{b}} \\ &- \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{b}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}, \overline{\mathrm{X}}_{\mathrm{c}} \\ &- \mathrm{A}_2\overline{\mathrm{R}}_{\mathrm{a}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2\mathrm{t}}\right]}\Bigg) \end{aligned} \tag{2.78}$$

For moderately large sample size of t, the parameter $\left[1-(1-\lambda)^{2t}\right]$ tends to be 1 and can be ignored from the equations above.

2.6.4 Fuzzy Maximum Generally Weighted Moving Average (FMaxGWMA) Charts

Let $X_{ij}(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$ be fuzzy observations. For any given $0 \le \alpha \le 1$, the corresponding real-values lower and upper bounds can be obtained as $\left(X_{ij}\right)_{\alpha}^{L}$ and $\left(X_{ij}\right)_{\alpha}^{U}$, respectively. A real-valued data for the lower and upper bounds of the $\tilde{\bar{X}}_i$ and $\tilde{s}_i$ can be written as (Shu et al. 2014)

$$\tilde{\bar{X}}_{i_\alpha}^{U} = \frac{1}{n}\sum_{j=1}^{n}\left(X_{ij}\right)_{\alpha}^{U} \quad \text{and} \quad \tilde{\bar{X}}_{i\alpha}^{L} = \frac{1}{n}\sum_{j=1}^{n}\left(X_{ij}\right)_{\alpha}^{L} \tag{2.79}$$

$$\tilde{s}_{i\alpha}^{U} = \sqrt{\frac{\sum_{j=1}^{n}\left(\left(X_{ij}\right)_{\alpha}^{U} - \tilde{\bar{X}}_{i\alpha}^{U}\right)^2}{n-1}} \quad \text{and} \quad \tilde{s}_{i\alpha}^{L} = \sqrt{\frac{\sum_{j=1}^{n}\left(\left(X_{ij}\right)_{\alpha}^{L} - \tilde{\bar{X}}_{i\alpha}^{L}\right)^2}{n-1}} \tag{2.80}$$

Then, we obtain unbiased estimators of the $\tilde{\sigma}_{\alpha}^{U}$ and $\tilde{\sigma}_{\alpha}^{L}$ as follows

$$\mu_{\alpha}^{U} = \tilde{\bar{\bar{X}}}_{\alpha}^{U} = \frac{1}{m}\sum_{i=1}^{m}\tilde{\bar{X}}_{i\alpha}^{U} \quad \text{and} \quad \mu_{\alpha}^{L} = \tilde{\bar{\bar{X}}}_{\alpha}^{L} = \frac{1}{m}\sum_{i=1}^{m}\tilde{\bar{X}}_{i\alpha}^{L} \tag{2.81}$$

$$\tilde{\bar{s}}_{\alpha}^{U} = \frac{1}{m}\sum_{i=1}^{m}\tilde{s}_{i_\alpha}^{U} \quad \text{and} \quad \tilde{\bar{s}}_{\alpha}^{L} = \frac{1}{m}\sum_{i=1}^{m}\tilde{s}_{i_\alpha}^{L} \tag{2.82}$$

$$\tilde{\sigma}_{\alpha}^{U} = \tilde{\bar{s}}_{\alpha}^{U}\Big/c_4 \quad \text{and} \quad \tilde{\sigma}_{\alpha}^{L} = \tilde{\bar{s}}_{\alpha}^{L}\Big/c_4 \tag{2.83}$$

Mutually independent statistics, M_i and S_i can also be rewritten in terms of the real-valued upper and lower bounds as

$$M_{i_\alpha}^{U} = \frac{\tilde{\bar{X}}_{i_\alpha}^{U} - \mu_{\alpha}^{U}}{\tilde{\sigma}_{\alpha}^{U}/\sqrt{n}} \quad \text{and} \quad M_{i_\alpha}^{L} = \frac{\tilde{\bar{X}}_{i_\alpha}^{L} - \mu_{\alpha}^{L}}{\tilde{\sigma}_{\alpha}^{L}/\sqrt{n}} \tag{2.84}$$

$$\tilde{S}_{i_\alpha}^{U} = \emptyset^{-1}\left\{F\left[\frac{(n-1)\tilde{s}_{i_\alpha}^{U}}{\left(\tilde{\sigma}_{\alpha}^{U}\right)^2}, n-1\right]\right\} \quad \text{and} \quad \tilde{S}_{i_\alpha}^{L} = \emptyset^{-1}\left\{F\left[\frac{(n-1)\tilde{s}_{i_\alpha}^{L}}{\left(\tilde{\sigma}_{\alpha}^{L}\right)^2}, n-1\right]\right\} \tag{2.85}$$

Finally, fuzzy GWMA statistics for the ith subgroup can be given by

$$\tilde{U}_{i\alpha}^{U} = \sum_{j=1}^{m} p_j \tilde{M}_{i+1-j_\alpha{}^U} \quad \text{and} \quad \tilde{U}_{i\alpha}^{L} = \sum_{j=1}^{m} p_j \tilde{M}_{i+1-j_\alpha{}^L} \tag{2.86}$$

$$\tilde{V}^{U}_{i_\alpha} = \sum_{j=1}^{m} p_j \tilde{S}^{U}_{i+1-j_\alpha} \quad \text{and} \quad \tilde{V}^{L}_{i_\alpha} = \sum_{j=1}^{m} p_j \tilde{S}^{L}_{i+1-j_\alpha} \tag{2.87}$$

Then, the fuzzy control limits of the F-MaxGWMA chart can be obtained from

$$\widetilde{UCL}^{U}_{i_\alpha} = \alpha UCL_i + (1-\alpha) UCL_{i+1} \tag{2.88}$$

$$\widetilde{UCL}^{L}_{i_\alpha} = \alpha UCL_i + (1-\alpha) UCL_{i-1} \tag{2.89}$$

where $UCL_0 = 0$.

The membership functions of $\widetilde{MG}_i$ required to be constructed for further identifying the manufacturing condition. Consider the closed interval C_α which is defined as:

$$C_\alpha = \left[\min\left\{\max\left(\left|\tilde{U}^{U}_{i_\alpha}\right|, \left|\tilde{V}^{U}_{i_\alpha}\right|\right), \max\left(\left|\tilde{U}^{L}_{i_\alpha}\right|, \left|\tilde{V}^{L}_{i_\alpha}\right|\right)\right\}, \max\left\{\max\left(\left|\tilde{U}^{U}_{i_\alpha}\right|, \left|\tilde{V}^{U}_{i_\alpha}\right|\right), \max\left(\left|\tilde{U}^{L}_{i_\alpha}\right|, \left|\tilde{V}^{L}_{i_\alpha}\right|\right)\right\}\right] \tag{2.90}$$

The membership functions of $\widetilde{MG}_i$ can be obtained by using the following expression

$$\xi_{\widetilde{MG}_i}(C) = \underbrace{\sup \alpha}_{0 \le \alpha \le 1} 1_{C_\alpha}(C) \tag{2.91}$$

Endpoints of the α-level closed interval $\widetilde{MG}_{i_\alpha} = \left[\widetilde{MG}^{L}_{i_\alpha}, \widetilde{MG}^{U}_{i_\alpha}\right]$ become

$$\widetilde{MG}^{L}_{i_\alpha} = \underbrace{\min}_{\alpha \le \beta \le 1} \min\left\{\max\left(\left|\tilde{U}^{U}_{i_\beta}\right|, \left|\tilde{V}^{U}_{i_\beta}\right|\right), \max\left(\left|\tilde{U}^{L}_{i_\beta}\right|, \left|\tilde{V}^{L}_{i_\beta}\right|\right)\right\} \tag{2.92}$$

$$\widetilde{MG}^{U}_{i_\alpha} = \underbrace{\max}_{\alpha \le \beta \le 1} \max\left\{\max\left(\left|\tilde{U}^{U}_{i_\beta}\right|, \left|\tilde{V}^{U}_{i_\beta}\right|\right), \max\left(\left|\tilde{U}^{L}_{i_\beta}\right|, \left|\tilde{V}^{L}_{i_\beta}\right|\right)\right\} \tag{2.93}$$

Now, to realize if the $\widetilde{MG}_i$ lie within their respective fuzzy control limits, comparisons of fuzzy numbers can be applied as mentioned in Sect. 2.5.

2.6.5 A Numerical Example for $\overline{X}$-R Control Chart

A company produces a material and wants to monitor its hardness measured by hardness testing equipment. Quality assistant takes a subgroup size of three, each also having three materials. For each material, the measured hardness values vary because of the material properties and gauge variability. To overcome the uncertainties caused by the non-uniform material properties, 3 readings for each sample are explained as a triangular fuzzy number as shown in Table 2.11. Company wants to construct x-R control charts for the uncertain hardness measurements using fuzzy control charts. Fuzzy mean and the fuzzy range for each sample are calculated by the equations given in Sect. 2.4.1, and shown in Table 2.11.

Fuzzy control limits are calculated according to the equations given in the previous sections. For n = 3, $A_2 = 1.023$, $D_3 = 0$, and $D_4 = 2.574$ are read from the coefficients table for variable control charts given in Appendix A.

Fuzzy control limits for $\overline{X}$ charts:

$$\widetilde{CL} = \tilde{\overline{\overline{X}}} = (22.7, 24.7, 27.0)$$

$$\begin{aligned}\widetilde{UCL} &= \left(\overline{\overline{X}}_a + A_2\overline{R}_a, \overline{\overline{X}}_b + A_2\overline{R}_b, \overline{\overline{X}}_c + A_2\overline{R}_c\right)\\ &= (22.7 + 1.023(-1.5), 24.7 + 1.023(2.7), 27.0 + 1.023(6.9))\end{aligned}$$

$$\widetilde{UCL} = (21.17, 27.46, 34.06)$$

$$\begin{aligned}\widetilde{LCL} &= \left(\overline{\overline{X}}_a + A_2\overline{R}_c, \overline{\overline{X}}_b + A_2\overline{R}_b, \overline{\overline{X}}_c + A_2\overline{R}_a\right)\\ &= (22.7 - 1.023(6.9), 24.7 + 1.023(2.7), 27.0 - 1.023(-1.5))\end{aligned}$$

$$\widetilde{LCL} = (14.21, 21.94, 28.53)$$

Fuzzy control limits for R charts:

$$\widetilde{CL} = \tilde{\overline{R}} = (-1.5, 2.7, 6.9)$$

$$\begin{aligned}\widetilde{UCL} = \left(D_4\overline{R}_a, D_4\overline{R}_b, D_4\overline{R}_c\right) &= (2.574(-1.5), 2.574(2.7), 2.574(6.9))\\ &= (-3.86, 6.95, 17.76)\end{aligned}$$

$$\widetilde{LCL} = \left(D_3\overline{R}_a, D_3\overline{R}_b, D_3\overline{R}_c\right) = (0, 0, 0)$$

Construction of fuzzy $\overline{X}$ control limits are shown in Fig. 2.11.

Assume that 21st sample's fuzzy average is $\overline{X}_{21} = (25.4, 28.5, 32.3)$ as shown in Fig. 2.12.

In order to decide whether $\tilde{X}_{21}$ plots an out of control or not we need to check if one of the conditions are met.

$$\widetilde{\mathrm{LCL}} \leq \tilde{\mathrm{X}}_{21} \leq \widetilde{\mathrm{UCL}}$$

Table 2.11 Data for the testing hardness of a material

Sample number	Observations in each sample									Average			Fuzzy mean			Fuzzy range				
	I			II			III													
	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c	X_a	X_b	X_c	I	II	III	Max	Min	X_a	X_b	X_c
1	24.7	27.7	29.7	21.4	23.4	24.4	23.2	26.2	27.2	23.1	25.7	27.1	714.38	513.88	622.40	I	II	0.3	4.3	8.3
2	22.5	25.5	27.5	20.7	21.7	24.7	22.5	24.5	26.5	21.9	23.9	26.3	605.21	483.53	579.59	I	II	−2.2	3.8	6.8
3	23.3	26.3	29.3	22.4	25.4	28.4	21.2	24.2	27.2	22.3	25.3	28.3	657.81	613.07	555.26	I	III	−3.9	2.1	8.1
4	22.6	25.6	28.6	23.0	24.0	26.0	24.3	27.3	28.3	23.3	25.7	27.7	621.73	578.61	681.29	III	II	−1.7	3.3	5.3
5	23.2	24.2	27.2	22.4	23.4	26.4	21.0	22.0	25.0	22.2	23.2	26.2	599.84	559.61	496.17	I	III	−1.8	2.2	6.2
6	23.0	26.0	27.0	23.1	26.1	29.1	23.3	25.3	27.3	23.1	25.8	27.8	612.26	648.14	617.45	II	I	−3.8	0.2	6.2
7	24.5	26.5	27.5	23.9	24.9	27.9	24.8	26.8	29.8	24.4	26.1	28.4	666.18	634.09	706.86	III	II	−3.1	1.9	5.9
8	21.6	24.6	27.6	21.4	22.4	23.4	23.6	25.6	26.6	22.2	24.2	25.9	573.50	490.66	619.28	III	II	0.2	3.2	5.2
9	21.5	23.5	24.5	22.1	25.1	28.1	22.2	25.2	28.2	21.9	24.6	26.9	520.14	594.49	602.50	III	I	−2.3	1.7	6.7
10	24.3	25.3	27.3	20.8	21.8	23.8	24.0	26.0	27.0	23.0	24.4	26.0	642.39	475.21	639.93	I	II	0.5	3.5	6.5
11	25.3	28.3	31.3	21.9	23.9	26.9	23.6	25.6	27.6	23.6	26.0	28.6	763.41	564.10	632.05	I	II	−1.6	4.4	9.4
12	21.0	22.0	23.0	23.6	24.6	25.6	21.0	22.0	25.0	21.8	22.8	24.5	471.63	592.18	496.33	II	I	0.6	2.6	4.6
13	23.6	26.6	28.6	23.8	26.8	29.8	22.0	24.0	25.0	23.1	25.8	27.8	657.54	681.17	542.36	II	III	−1.2	2.6	7.8
14	23.8	26.8	29.8	21.7	22.7	23.7	23.4	24.4	27.4	22.9	24.6	26.9	680.73	503.43	607.53	I	II	0.1	4.1	8.1
15	23.9	24.9	25.9	21.6	24.6	25.6	24.2	25.2	28.2	23.2	24.9	26.6	608.38	546.77	648.90	III	II	−1.4	0.6	6.6
16	23.2	25.2	28.2	21.9	23.9	25.9	22.2	24.2	25.2	22.4	24.4	26.4	624.93	546.99	550.76	I	II	−2.7	1.3	6.3
17	24.8	26.8	29.8	21.6	24.6	25.6	22.4	24.4	27.4	22.9	25.3	27.6	708.47	548.11	584.45	I	II	−0.8	2.2	8.2
18	21.6	22.6	24.6	24.3	27.3	30.3	21.3	24.3	27.3	22.4	24.7	27.4	511.90	707.48	559.52	II	I	−0.3	4.7	8.7
19	20.2	22.2	25.2	22.2	23.2	26.2	22.6	23.6	25.6	21.7	23.0	25.7	484.02	549.68	559.43	III	I	−2.6	1.4	5.4
20	21.8	22.8	25.8	22.2	23.2	26.2	24.2	27.2	30.2	22.7	24.4	27.4	532.56	550.53	704.53	III	I	−1.6	4.4	8.4
							Grand average X			22.7	24.7	27.0			Average range			−1.5	2.7	6.9

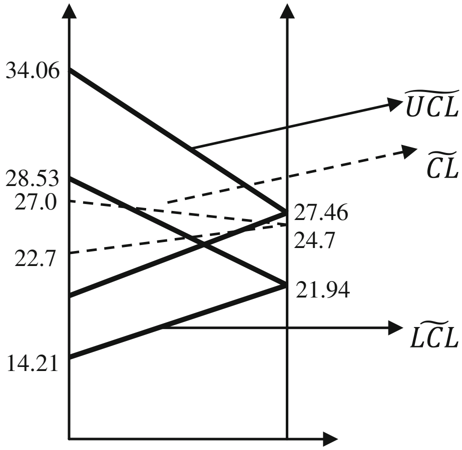


Fig. 2.11 Illustration of Fuzzy $\overline{X}$ control limits

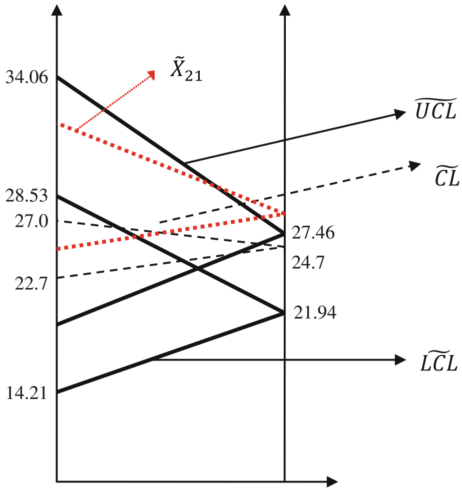


Fig. 2.12 Illustration of a new fuzzy observation: $\overline{X}_{21}$

If we use ranking methods explained in Sect. 2.5, we obtain a crisp decision that $\tilde{X}_{21}$ plots either out of control or in control condition. One of the best method is to use direct fuzzy approach control presented in Gülbay and Kahraman (2007) which allows quality professionals to decide and interpret the chart with the degree of membership that a point shows out of control or in control. Furthermore, by defining intermediate decisions between out of control and in control enables to the usage of various actions to correct the process.

2.7 Fuzzy Unnatural Pattern Analyses for Control Charts for Variables

2.7.1 Probability of Fuzzy Events

The formula for calculating the probability of a fuzzy event A is a generalization of the probability theory: in the case which a sample space X is a continuum or discrete, the probability of a fuzzy event P(A) is given by Yen and Langari (1999):

$$P(\tilde{A}) = \begin{cases} \int \mu_A(x) P_X(x) dx & \text{if X is continuous,} \\ \sum\limits_i \mu_A(x_i) P_X(x_i) dx & \text{if X is discrete.} \end{cases} \tag{2.94}$$

where P_X denotes a classical probability distribution function of X for continuous sample space and probability function for discrete sample space, and μ_A is a membership function of the event A.

The membership degree of a fuzzy sample that belongs to a region is directly related to its percentage area falling in that region, and therefore, it is continuous. For example, a fuzzy sample may be in zone B with a membership degree of 0.4 and in zone C with a membership degree of 0.6. While counting fuzzy samples in zone B, that sample is counted as 0.4.

2.7.2 Generation of Fuzzy Rules for Unnatural Patterns

The run rules are based on the premise that a specific run of data has a low probability of occurrence in a completely random stream of data. If a run occurs, then it is meant that something has changed in the process to produce such a nonrandom or unnatural pattern. Based on the expected percentages in each zone, sensitive run tests can be developed for analyzing the patterns of variation in the various zones. For the fuzzy control charts, based on the Western Electric rules (Western Electric Company 1956), the following fuzzy unnatural pattern rules can be defined. The probabilities of these fuzzy events are calculated using normal approach to binomial distribution (Gülbay and Kahraman 2006).

The probability of each fuzzy rule (event) below depends on the definition of the membership function which is subjectively defined so that the probability of each of the fuzzy rules is as close as possible to the corresponding classical rule for unnatural patterns. The idea behind this approach may justify the following rules (Gülbay and Kahraman 2006).

Rule 1: Any fuzzy data falling outside the three-sigma control limits with a ratio of more than predefined percentage (β) of sample area at desired α-level. The membership function for this rule can subjectively be defined as below:

$$\mu_1(x) = \begin{cases} 0 & ; \quad 0.85 \le x \le 1, \\ (x - 0.60)/0.25 & ; \quad 0.60 \le x \le 0.85, \\ (x - 0.10)/0.50 & ; \quad 0.10 \le x \le 0.60, \\ 1 & ; \quad 0 \le x \le 0.10, \end{cases} \tag{2.95}$$

Rule 2: A total membership degree around 2 from three consecutive points in zone A or beyond. Probability of a sample being in zone A (0.0214) or beyond (0.00135) is 0.02275. Let the membership function for this rule be defined as follows:

$$\mu_2(x) = \begin{cases} 0 & ; \quad 0 \le x \le 0.59, \\ (x - 0.59)/1.41 & ; \quad 0.59 \le x \le 2 \\ 1 & ; \quad 2 \le x \le 3 \end{cases} \tag{2.96}$$

The probability of the fuzzy event rule 2 is approximately 0.0015, which corresponds to the crisp case of this rule.

Rule 3: A total membership degree around 4 from five consecutive points in zone C or beyond:

$$\mu_3(x) = \begin{cases} 0 & ; \quad 0 \le x \le 2.42, \\ (x - 2.42)/1.58 & ; \quad 2.42 \le x \le 4, \\ 1 & ; \quad 4 \le x \le 5. \end{cases} \tag{2.97}$$

The probability of the fuzzy event rule 3 is approximately 0.0027

Rule 4: A total membership degree around 8 from eight consecutive points on the same side of the centerline with the membership function below and its probability is 0.0039:

$$\mu_4(x) = \begin{cases} 0 & ; \quad 0 \le x \le 2.54, \\ (x - 2.54)/5.46 & ; \quad 2.54 \le x \le 8. \end{cases} \tag{2.98}$$

Rule 5: A total membership degree around 7 from seven consecutive points on the same side of the center line. The fuzzy probability of this rule is 0.0079 when membership function is defined as below:

$$\mu_5(x) = \begin{cases} 0 & ; \quad 0 \le x \le 2.48, \\ (x - 2.48)/4.52 & ; \quad 2.48 \le x \le 7. \end{cases} \tag{2.99}$$

Rule 6: At least a total membership degree around 10 from 11 consecutive points on the same side of the center line. The fuzzy probability of this rule is 0.0058 when the membership function is defined as below:

$$\mu_6(x) = \begin{cases} 0 & ; \quad 0 \le x \le 9.33, \\ (x - 9.33)/0.67 & ; \quad 9.33 \le x \le 10, \\ 1 & ; \quad 10 \le x \le 11. \end{cases} \tag{2.100}$$

Rule 7: At least a total membership degree around 12 from 14 consecutive points on the same side of the center line. If the membership function is set as given below, then the fuzzy probability of the rule is equal to 0.0065.

$$\mu_7(x) = \begin{cases} 0 & ; \quad 0 \le x \le 11.33, \\ (x - 11.33)/0.67 & ; \quad 11.33 \le x \le 12, \\ 1 & ; \quad 12 \le x \le 14. \end{cases} \tag{2.101}$$

Rule 8: At least a total membership degree around 14 from 17 consecutive points on the same side of the center line. The probability of this fuzzy event with the membership function below is 0.0062.

$$\mu_7(x) = \begin{cases} 0 & ; \quad 0 \le x \le 13.34, \\ (x - 13.34)/0.66 & ; \quad 13.34 \le x \le 14, \\ 1 & ; \quad 14 \le x \le 17. \end{cases} \tag{2.102}$$

A framework for the application of the fuzzy unnatural pattern rules are as follows:

1. Determine $\pm 3\sigma$ fuzzy control limits.
2. Determine fuzzy regions of $\pm 1\sigma$ and $\pm 2\sigma$
3. For each sample, calculate the percentage of sample area that belongs to the regions of A, B, and C for both sides of the fuzzy center line.
4. For each fuzzy rule, check the last N points as defined in the rule and sum their percentage of sample area in the related region. Then, for that rule use its corresponding membership function to obtain the membership degree of the occurrence for the specified rule.
5. Repeat step 4 until all desired fuzzy rules are checked.

2.7.3 An Illustrative Example

Consider the case where a-three subgroup (n = 3) is taken for the construction of the fuzzy $\overline{X}$ control charts. For n = 3, the constant A_2 = 1.023. Grand average and average range for the 25 samples are calculated using the Eqs. 2.20–2.25 and given as

Table 2.12 Equations for calculation of fuzzy ±zσ control limits

z	Notation	Fuzzy $z\sigma$ limits
+3	$\widetilde{UCL}$	$\left(\overline{\overline{X}}_a + A_2\overline{R}_a, \overline{\overline{X}}_b + A_2\overline{R}_b, \overline{\overline{X}}_c + A_2\overline{R}_c\right)$
+2	$\widetilde{CL} + 2\sigma$	$\left(\overline{\overline{X}}_a + \left(\frac{2}{3}\right)A_2\overline{R}_a, \overline{\overline{X}}_b + \left(\frac{2}{3}\right)A_2\overline{R}_b, \overline{\overline{X}}_c + \left(\frac{2}{3}\right)A_2\overline{R}_c\right)$
+1	$\widetilde{CL} + 1\sigma$	$\left(\overline{\overline{X}}_a + \left(\frac{1}{3}\right)A_2\overline{R}_a, \overline{\overline{X}}_b + \left(\frac{1}{3}\right)A_2\overline{R}_b, \overline{\overline{X}}_c + \left(\frac{1}{3}\right)A_2\overline{R}_c\right)$
0	$\widetilde{CL}$	$\overline{\overline{X}} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right)$
−1	$\widetilde{CL} - 1\sigma$	$\left(\overline{\overline{X}}_a - \left(\frac{1}{3}\right)A_2\overline{R}_c, \overline{\overline{X}}_b - \left(\frac{1}{3}\right)A_2\overline{R}_b, \overline{\overline{X}}_c - \left(\frac{1}{3}\right)A_2\overline{R}_a\right)$
−2	$\widetilde{CL} - 2\sigma$	$\left(\overline{\overline{X}}_a - \left(\frac{2}{3}\right)A_2\overline{R}_c, \overline{\overline{X}}_b - \left(\frac{2}{3}\right)A_2\overline{R}_b, \overline{\overline{X}}_c - \left(\frac{2}{3}\right)A_2\overline{R}_a\right)$
−3	$\widetilde{LCL}$	$\left(\overline{\overline{X}}_a - A_2\overline{R}_c, \overline{\overline{X}}_b - A_2\overline{R}_b, \overline{\overline{X}}_c - A_2\overline{R}_a\right)$

Table 2.13 Fuzzy ±zσ control limits and their regions (see Fig. 2.6)

z	Notation	Fuzzy $z\sigma$ limits	Region
+3	$\widetilde{UCL}$	(50.2, 70.3, 90.5)	A
+2	$\widetilde{\mathrm{CL}} + 2\sigma$	(46.8, 65.2, 83.6)	B
+1	$\widetilde{\mathrm{CL}} + 1\sigma$	(43.4, 60.1, 76.8)	C
0	$\widetilde{\mathrm{CL}}$	(40.0, 55.0, 70.0)	C
−1	$\widetilde{\mathrm{CL}} - 1\sigma$	(36.6, 79.9, 63.2)	B
−2	$\widetilde{\mathrm{CL}} - 2\sigma$	(33.2, 44.8, 56.4)	A
−3	$\widetilde{\mathrm{LCL}}$	(29.8, 39.7, 49.5)	

$$\overline{\overline{X}} = (40, 55, 70)$$

$$\overline{R} = (10, 15, 20)$$

Fuzzy ±zσ limits can be calculated using the equations given in Table 2.12.

Replacing the values of the $\overline{\overline{X}}, \overline{R}$, and the constant A_2 into the equations above, we obtain fuzzy regions as given in Table 2.13.

Let the next sample be $\tilde{X}_{26} = (48.0,\ 60.0,\ 70.0)$. Let's construct the $\overline{X}$ control chart and see at what membership degree $\tilde{X}_{26}$ belongs to the regions of A, B, and C for the both sides of the fuzzy center line.

In Fig. 2.14, fuzzy control limits and fuzzy regions (see Fig. 2.13) are simplified in order to show the region A above the centerline and $\tilde{X}_{26}$.

As can be seen from the Fig. 2.14, only a little part of the sample area of $\tilde{\mathrm{X}}_{26}$ is out of the region A, namely most of its parts are in region A. The problem is to calculate the percentage area of $\tilde{\mathrm{X}}_{26}$ which is inside the region A. The percentage

Fig. 2.13 Illustration of $\pm z\sigma$ control limits and $\tilde{X}_{26}$

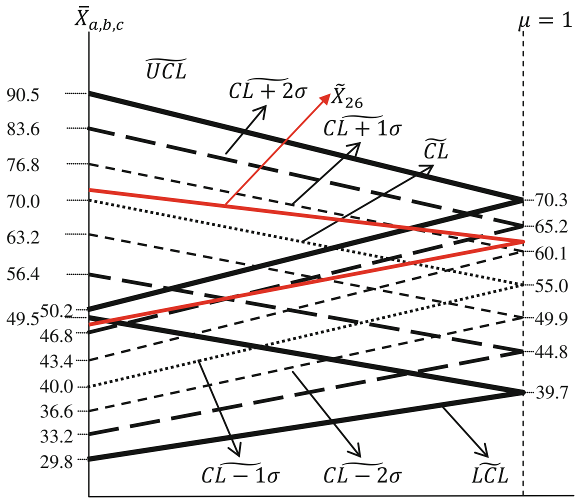
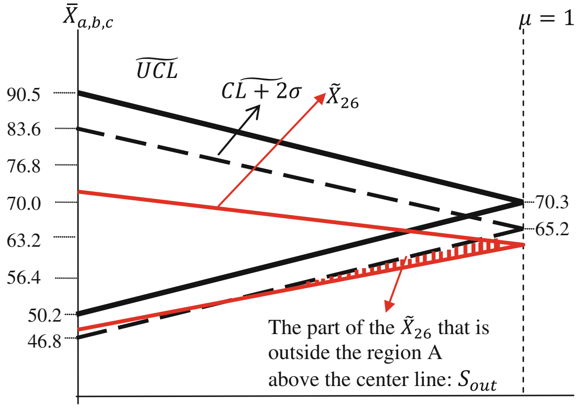


Fig. 2.14 Illustration of $+2\sigma$ and $+3\sigma$ control limits, and $\tilde{X}_{26}$



sample area within a specified region can be calculated using the formula given in Eq. 2.42.

$$\beta_j = \frac{S_j - S_{out}}{S_j} \tag{2.103}$$

where S_j is the sample area and S_{out} is the area of the sample outside the corresponding region. These calculations are a little hard, but by using simple software it can be easy to determine. Once control limits are specified a general formula can be derived for the area calculation and the percentage areas can be calculated using any spread sheets. The reader can refer to the Gülbay and Kahraman (2006) for the determination of the percentage areas. Suppose that β_{26}, β_{27}, and β_{28} are determined as 0.85, 0.50, 0,25. For these 3 consecutive samples the total degree of memberships is 0.85 + 0.50 + 0.25 = 1.60 for being in region A above the center line. Fuzzy rule 2 can be checked now to decide at what membership degree that rule is performed. Remember that membership degree of the rule 2 was subjectively defined as:

$$\mu_2(x) = \begin{cases} 0 & ; \quad 0 \le x \le 0.59, \\ (x - 0.59)/1.41 & ; \quad 0.59 \le x \le 2 \\ 1 & ; \quad 2 \le x \le 3 \end{cases}$$

Then,

$$\mu_2(1.60) = \{ (1.60 - 0.59)/1.41 = 0.716$$

The quality control professional can set a predefined value of μ to compare with the μ_i to accept or reject the occurrence of the rule and hence may justify a set of actions with respect to the calculated μ. The rest of the fuzzy rules are applied in the same way.

2.8 Conclusion

Control charts aim at detecting if any assignable cause exists in the considered process. If only random causes exist, no action is required. Otherwise, a corrective action is needed. We proposed fuzzy control charts to be used in case of incomplete and vague data for the process control. Fuzzy triangular fuzzy numbers have been preferred in the developed control charts because of their relative simplicity whereas the other types of fuzzy numbers can also be used. Trapezoidal fuzzy numbers or LR type fuzzy numbers can be replaced with triangular fuzzy numbers in these analyses. EWMA control charts are preferred when we need detecting small shifts. These charts have been also developed under fuzziness and a numerical example has been given. The new extensions of fuzzy sets such as type-2 fuzzy sets, Intuitionistic fuzzy sets, and hesitant fuzzy sets are the possible alternatives to extend our work. Each of these new extensions is also divided into a few types. For example, interval type Intuitionistic fuzzy sets for triangular Intuitionistic fuzzy sets are subalternatives for the new developments.

Appendix A

Table of coefficients for control charts for variables.

Observations in subgroup, n	A_2	A_3	c_4	B_3	B_4	B_5	B_6	d_2	d_3	D_1	D_2	D_3	D_4	E_2
2	1.880	2.659	0.798	0.000	3.267	0.000	2.606	1.128	0.853	0.000	3.686	0.000	3.267	2.660
3	1.023	1.954	0.886	0.000	2.568	0.000	2.276	1.693	0.888	0.000	4.358	0.000	2.574	1.772
4	0.729	1.628	0.921	0.000	2.266	0.000	2.088	2.059	0.880	0.000	4.698	0.000	2.282	1.457
5	0.577	1.427	0.940	0.000	2.089	0.000	1.964	2.326	0.864	0.000	4.918	0.000	2.114	1.290
6	0.483	1.287	0.952	0.030	1.970	0.029	1.874	2.534	0.848	0.000	5.078	0.000	2.004	1.184
7	0.419	1.182	0.959	0.118	1.882	0.113	1.806	2.704	0.833	0.204	5.204	0.076	1.924	1.109
8	0.373	1.099	0.965	0.185	1.815	0.179	1.751	2.847	0.820	0.388	5.306	0.136	1.864	1.054
9	0.337	1.032	0.969	0.239	1.761	0.232	1.707	2.970	0.808	0.547	5.393	0.184	1.816	1.010
10	0.308	0.975	0.973	0.284	1.716	0.276	1.669	3.078	0.797	0.687	5.469	0.223	1.777	0.975
11	0.285	0.927	0.975	0.321	1.679	0.313	1.637	3.173	0.787	0.811	5.535	0.256	1.744	0.945
12	0.266	0.886	0.978	0.354	1.646	0.346	1.610	3.258	0.778	0.922	5.594	0.283	1.717	0.921
13	0.249	0.850	0.979	0.382	1.618	0.374	1.585	3.336	0.770	1.025	5.647	0.307	1.693	0.899
14	0.235	0.817	0.981	0.406	1.594	0.399	1.563	3.407	0.763	1.118	5.696	0.328	1.672	0.881
15	0.223	0.789	0.982	0.428	1.572	0.421	1.544	3.472	0.756	1.203	5.741	0.347	1.653	0.864
16	0.212	0.763	0.984	0.448	1.552	0.440	1.526	3.532	0.750	1.282	5.782	0.363	1.637	0.849
17	0.203	0.739	0.985	0.466	1.534	0.458	1.511	3.588	0.744	1.356	5.820	0.378	1.622	0.836
18	0.194	0.718	0.985	0.482	1.518	0.475	1.496	3.640	0.739	1.424	5.856	0.391	1.608	0.824
19	0.187	0.698	0.986	0.497	1.503	0.490	1.483	3.689	0.734	1.487	5.891	0.403	1.597	0.813
20	0.180	0.680	0.987	0.510	1.490	0.504	1.470	3.735	0.729	1.549	5.921	0.415	1.585	0.803
21	0.173	0.663	0.988	0.523	1.477	0.516	1.459	3.778	0.724	1.605	5.951	0.425	1.575	0.794
22	0.167	0.647	0.988	0.534	1.466	0.528	1.448	3.819	0.720	1.659	5.979	0.434	1.566	0.786
23	0.162	0.633	0.989	0.545	1.455	0.539	1.438	3.858	0.716	1.710	6.006	0.443	1.557	0.778
24	0.157	0.619	0.989	0.555	1.445	0.549	1.429	3.895	0.712	1.759	6.031	0.451	1.548	0.770
25	0.153	0.606	0.990	0.565	1.435	0.559	1.420	3.931	0.708	1.806	6.056	0.459	1.541	0.763

Appendix B

The equations to compute sample area outside the control the limits.

$$A_{out}^{U} = \frac{1}{2}\left[\left(d^{\alpha} - UCL_4^{\alpha}\right) + \left(d^{t} - UCL_4^{t}\right)\right](\max(t-\alpha,0)) + \frac{1}{2}[(d^{z} - a^{z}) + (c-b)](\min(1-t,1-\alpha)) \tag{2.104}$$

where,

$$t = \frac{UCL_4 - a}{(b-a)+(c-b)} \text{ and } z = \max(t,\alpha)$$

$$A_{out}^{U} = \frac{1}{2}\left[\left(d^{\alpha} - UCL_4^{\alpha}\right) + (c - UCL_3)\right](1-\alpha) \tag{2.105}$$

$$A_{out}^{U} = \frac{1}{2}\left(d^{\alpha} - UCL_4^{\alpha}\right)(\max(t-\alpha,0)) \tag{2.106}$$

where

$$t = \frac{UCL_4 - d}{(UCL_4 - UCL_3) - (d-c)}$$

$$A_{out}^{U} = \frac{1}{2}\left[(c - UCL_3) + \left(d^{z} - UCL_4^{z}\right)\right](\min(1-t,1-\alpha)) \tag{2.107}$$

where

$$t = \frac{UCL_4 - d}{(UCL_4 - UCL_3) - (d-c)} \text{ and } z = \max(t,\alpha)$$

$$A_{out}^{U} = \frac{1}{2}\left[\left(d^{z_2} - UCL_4^{z_2}\right) + \left(d^{t_1} - UCL_4^{t_1}\right)\right](\min(\max(t_1-\alpha,0),t_1-t_2)) + \frac{1}{2}[(d^{z_1} - a^{z_1}) + (c-b)](\min(1-t_1,1-\alpha))$$

where

$$t_1 = \frac{UCL_4 - a}{(b-a)+(UCL_4 - UCL_3)}, \quad t_2 = \frac{UCL_4 - d}{(UCL_4 - UCL_3) - (d-c)}, \tag{2.108}$$

$$z_1 = \max(\alpha, t_1), \text{ and } z_2 = \max(\alpha, t_2)$$

$$A_{out}^{U} = 0 \tag{2.109}$$

$$A_{out}^{U} = \frac{1}{2}[(d^{\alpha} - a^{\alpha}) + (c - b)](1 - \alpha) \tag{2.110}$$

$$\begin{aligned} A_{out}^{L} = {} & \frac{1}{2}\left[\left(LCL_1^{\alpha} - a^{\alpha}\right) + \left(LCL_1^{t} - a^{t}\right)\right](\max(t - \alpha, 0)) \\ & + \frac{1}{2}[(d^{z} - a^{z}) + (c - b)](\min(1 - t, 1 - \alpha)) \end{aligned} \tag{2.111}$$

where

$$t = \frac{d - LCL_1}{(LCL_2 - LCL_1) + (d - c)} \text{ and } z = \max(\alpha, t)$$

$$A_{out}^{L} = \frac{1}{2}[(d^{\alpha} - a^{\alpha}) + (c - b)](1 - \alpha) \tag{2.112}$$

$$A_{out}^{L} = \frac{1}{2}\left[\left(LCL_1^{\alpha} - a^{\alpha}\right) + (LCL_2 - b)\right](1 - \alpha) \tag{2.113}$$

$$\begin{aligned} A_{out}^{L} = {} & \frac{1}{2}\left[\left(LCL_1^{z_2} - a^{z_2}\right) + \left(LCL_1^{t_1} - a^{t_1}\right)\right](\min(\max(t_1 - \alpha, 0), t_1 - t_2)) \\ & + \frac{1}{2}[(d^{z_1} - a^{z_1}) + (c - b)](\min(1 - t, 1 - \alpha)) \end{aligned} \tag{2.114}$$

where

$$t_1 = \frac{d - LCL_1}{(LCL_2 - LCL_1) + (d - c)},$$
$$t_2 = \frac{a - LCL_1}{(LCL_2 - LCL_1) - (b - a)}$$

$$z_1 = \max(\alpha, t_1), \text{ and } z_2 = \max(\alpha, t_2)$$

$$A_{out}^{L} = \frac{1}{2}\left[\left(LCL_1^{z} - a^{z}\right) + (LCL_2 - b)\right](\min(1 - t, 1 - \alpha)) \tag{2.115}$$

where

$$t = \frac{a - LCL_1}{(LCL_2 - LCL_1) - (b - a)}, \text{ and } z = \max(\alpha, t)$$

$$A_{out}^{L} = 0 \tag{2.116}$$

$$A_{out}^{L} = \frac{1}{2}[(d^{\alpha} - a^{\alpha}) + (c - b)](1 - \alpha) \tag{2.117}$$

References

Chen, G., Cheng, S.W.: Max-chart: combining X-bar and s-chart. Statistica Sinica **8**, 263–271 (1998)

Eyvazian, M., Naini, S.G.J., Vaghefi, A.: Monitoring process variability using exponentially weighted moving sample variance control charts. Int. J. Adv. Manuf. **39**, 261–270 (2008)

Han, D., Tsung, F.: A generalized EWMA control chart and its comparison with the optimal EWMA, CUSUM and GRL schemes. Ann Stat. **32**(1), 316–339 (2004)

Li, Z., Wang, Z.: An exponentially weighted moving average scheme with variable sampling intervals for monitoring linear profiles. Comput. Ind. Eng. **59**, 630–637 (2010)

Montgomery, D.C.: Introduction to Statistical Quality Control. Wiley, New York (2001)

Nelson, L.S.: The Shewhart control chart–tests for special causes. J. Qual. Technol. **16**(4), 238 (1984)

Sheu, S.H., Lin, T.C.: The generally weighted moving average control chart for detecting small shifts in the process mean. Qual. Eng. **16**, 209–231 (2003)

Sheu, S.H., Huang, C.J., Hsu, T.S.: Extended maximum generally weighted moving average control chart for monitoring process mean and variability. Comput. Ind. Eng. **62**, 216–225 (2012)

Shu, M.-H., Nguyen, T.-L., Hsu, B.-M.: Fuzzy MaxGWMA chart for identifying abnormal variations of on-line manufacturing processes with imprecise information. Expert Syst. Appl. **41**, 1342–1356 (2014)

Western Electric Company: Statistical Quality Control Handbook, Indianapolis, Indiana (1956)

Xie, H.: Contributions to qualimetry. Ph.D. thesis, Winnipeg, Canada: University of Manitoba (1999)

Zhang, J., Li, Z., Wang, Z.: A multivariate control chart for simultaneously monitoring process mean and variability. Comput. Stat. Data Anal. **54**, 2244–2252 (2010)

Juran's Quality Handbook: McGraw-Hill Professional; 6 edn. (2010). ISBN-10: 0071629734

Asady, B.: The revised method of ranking LR fuzzy numbers based on deviation degree. Expert Syst. Appl. **37**, 5056–5060 (2010)

Bortolan, G., Degani, R.: A review of some methods for ranking fuzzy numbers. Fuzzy Sets Syst. **15**, 1–19 (1985)

Brunelli, M., Mezeib, J.: How different are ranking methods for fuzzy numbers? A numerical study. Int. J. Approximate Reasoning **54**, 627–639 (2013)

Chen, S.-M., Chen, J.H.: Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. Expert Syst. Appl. **36**(3), 6833–6842 (2009)

Chen, S.-J., Hwang, C.-L.: Fuzzy Multiple Attribute Decision Making. Springer, New York (1992)

Chen, S.-M., Sanguansat, K.: Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers. Expert Syst. Appl. **38**(3), 2163–2171 (2011)

Cheng, C.H.: A new approach for ranking fuzzy numbers by distance method. Fuzzy Sets Syst. **95**, 307–317 (1998)

Choobineh, F., Li, H.: An index for ordering fuzzy numbers. Fuzzy Sets Syst. **54**, 287–294 (1993)

Chu, T.C., Tsao, C.T.: Ranking fuzzy numbers with an area between the centroid point and the original point. Comput. Math Appl. **43**, 111–117 (2002)

Deng, H.: A discriminative analysis of approaches to ranking fuzzy numbers in fuzzy decision making. In: Proceedings of the 4th IEEE International Conference on Fuzzy Systems and Knowledge Discovery, 26–29 Aug, Haikou, China (2007)

Deng, Y., Zhu, Z.F., Liu, Q.: Ranking fuzzy numbers with an area method using radius of gyration. Comput. Math Appl. **51**, 1127–1136 (2006)
Chrysler, Ford, GM: Measurement Systems Analysis Reference Manual. AIAG, Detroit, MI (1995)
Detyniecki, M., Yager, R.R.: Ranking fuzzy numbers using α-weighted valuations. Int. J. Uncertainty Fuzziness Knowl.-Based Syst. **8**(5), 573–592 (2001)
Dias, G.: Ranking alternatives using fuzzy numbers: a computational approach. Fuzzy Sets Syst. **56**, 247–252 (1993)
Dubois, D., Prade, H.: Operations on fuzzy numbers. Int. J. Syst. Sci. **9**, 613–626 (1978)
Dubois, D., Prade, H.: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York (1980)
Dubois, D., Prade, H.: Fuzzy sets—a convenient function for modeling vagueness and possibility. IEEE Trans. Fuzzy Syst. **2**, 16–21 (1994)
Duncan, A.J.: Quality Control and Industrial Statistics, 5th edn. Irwin, Homewood (1986)
Fortemps, P., Roubens, M.: Ranking and defuzzification methods based on area compensation. Fuzzy Sets Syst. **82**(3), 19–330 (1996)
Gitlow, H.S.: Tools and Methods for the Improvement of Quality (1989). ISBN-10: 0256056803
Gülbay, M., Kahraman, C.: Development of fuzzy process control charts and fuzzy unnatural pattern analyses. Comput. Stat. Data Anal. **51**, 434–451 (2006)
Gülbay, M., Kahraman, C.: An alternative approach to fuzzy control charts: direct fuzzy approach. Inf. Sci. **177**(6), 1463–1480 (2007)
Gülbay, M., Kahraman, C., Ruan, D.: α-cut fuzzy control charts for linguistic data. Int. J. Intell. Syst. **19**, 1173–1196 (2004)
Jain, R.: Decision-making in the presence of fuzzy variables. IEEE Trans. Syst. Man Cybern. **6**, 698–703 (1976)
Jain, R.: A procedure for multi-aspect decision making using fuzzy sets. Int. J. Syst. Sci. **8**, 1–7 (1978)
Kim, K., Park, K.S.: Ranking fuzzy numbers with index of optimism. Fuzzy Sets Syst. **35**, 143–150 (1990)
Kim, C.B., Seong, K.A., Lee-Kwang, H.: Design and implementation of fuzzy elevator group control system. IEEE Trans. Syst. Man Cybern. **28**, 277–287 (1998)
Lee, J.H., Lee-Kwang, H.: Distributed and cooperative fuzzy controller for traffic intersection group. IEEE Trans. Syst. Man Cybern. **29**, 263–271 (1999)
Lee, E.S., Li, R.J.: Comparison of fuzzy numbers based on the probability measure of fuzzy events. Comput. Math Appl. **15**, 887–896 (1998)
Lee, K.M., Cho, C.H., Lee-Kwang, H.: Ranking fuzzy values with satisfaction function. Fuzzy Sets Syst. **64**, 295–311 (1994)
Liu, X.-W., Han, S.-L.: Ranking fuzzy numbers with preference weighting function expectations. Comput. Math Appl. **49**, 1731–1753 (2005)
Murakami, S., Maeda, S., Imamura, S.: Fuzzy decision analysis on the development of centralized regional energy control system. In: Proceedings of the IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis, pp. 363–368 (1983)
Raj, A., Kumar, D.N.: Ranking alternatives with fuzzy weights using maximizing set and minimizing set. Fuzzy Sets Syst. **105**, 365–375 (1999)
Rao, P.P.B., Shankar, N.R.: Ranking fuzzy numbers with a distance method using circumcenter of centroids and an index of modality. Adv. Fuzzy Syst. **2011**, Article ID 178308 (2011). doi:10.1155/2011/178308
Requena, I., Delgado, M., Verdagay, J.I.: Automatic ranking of fuzzy numbers with the criterion of decision-maker learnt by an artificial neural network. Fuzzy Sets Syst. **64**, 1–19 (1994)
Senturk, S., Erginel, N.: Development of fuzzy and control charts using α-cuts. Inf. Sci. **179**(10), 1542–1551 (2009)
Şentürk, S., Erginel, N., Kaya, İ., Kahraman, C.: Fuzzy exponentially weighted moving average control chart for univariate data with a real case application. Appl. Soft Comput. **22**, 1–10 (2014)

Shureshjani, R.A., Darehmiraki, M.: A new parametric method for ranking fuzzy numbers. Indagationes Mathematicae **24**, 518–529 (2013)
Sun, H., Wu, J.: A new approach for ranking fuzzy numbers based on fuzzy simulation analysis method. Appl. Math. Comput. **174**, 755–767 (2006)
Tran, L., Duckstein, L.: Comparison of fuzzy numbers using a fuzzy distance measure. Fuzzy Sets Syst. **130**, 331–341 (2002)
Wang, Y.J., Lee, H.S.: The revised method of ranking fuzzy numbers with an area between the centroid and original points. Comput. Math Appl. **55**, 2033–2042 (2008)
Wang, Z.-X., Liu, Y.-J., Fan, Z.-P., Feng, B.: Ranking L-R fuzzy number based on deviation degree. Inf. Sci. **179**, 2070–2077 (2009)
Westgard, J.O., Westgard S.: Basic Quality Management Systems (2014). ISBN: 1-886958-28-9
Yager, R.R.: Ranking fuzzy subsets over the unit interval. In: Proceeding of the 17th IEEE International Conference on Decision and Control, San Diego, CA, pp. 1435–1437 (1978)
Yeh, C.H., Deng, H.: A practical approach to fuzzy utilities comparison in fuzzy multicriteria analysis. Int. J. Approximate Reasoning **35**(2), 179–194 (2004)
Yen, J., Langari, R.: Fuzzy Logic: Intelligence, Control, and Information. Prentice-Hall, Upper Saddle River (1999)
Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
Zimmermann, H.-J.: Fuzzy Set Theory and its Applications, 3rd edn. Kluwer Academic Publishers, Boston (1996)
Zimmermann, H.-J.: An application-oriented view of modeling uncertainty. Eur. J. Oper. Res. **122**, 190–198 (2000)