Chapter 5
Integration in Higher Dimensions

Integration of functions in higher dimensions is much more difficult than it is in one dimension. The basic reason is that in order to integrate a function, one has to know how to measure the volume of sets. In one dimension, most sets can be decomposed into intervals (cf. Exercise 1.21), and we took the length of an interval to be its volume. However, already in $\mathbb{R}^2$ there is a vastly more diverse menagerie of shapes. Thus knowing how to integrate over one shape does not immediately tell you how to integrate over others. A second reason is that, even if one knows how to define the integral of functions in $\mathbb{R}^N$, in higher dimensions there is no comparable *deus ex machina* to replace The Fundamental Theorem of Calculus.

A thoroughly satisfactory theory that addresses the first issue was developed by Lebesgue, but, because it takes too much time to explain, his is not the theory presented here. Instead, we will stay with Riemann's approach.

5.1 Integration Over Rectangles

The simplest analog in $\mathbb{R}^N$ of a closed interval is a closed rectangle R,[1] a set of the form

$$\prod_{j=1}^{N}[a_j,b_j] = [a_1,b_1]\times\cdots\times[a_N,b_N] = \{\mathbf{x}\in\mathbb{R}^N : a_j\le x_j\le b_j \text{ for } 1\le j\le N\},$$

where $a_j \le b_j$ for each j. Such rectangles have three great virtues. First, if one includes the empty set $\emptyset$ as a rectangle, then the intersection of any two rectangles is again a rectangle. Secondly, there is no question how to assign the volume $|R|$ of a rectangle, it's got to be $\prod_{j=1}^{N}(b_j-a_j)$, the product of the lengths of its sides. Finally, rectangles are easily subdivided into other rectangles. Indeed, every subdivision of

[1]From now on, every rectangle will be assumed to be closed unless it is explicitly stated that it is not.

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the intervals making up its sides leads to a subdivision of the rectangle into sub-rectangles. With this in mind, we will now mimic the procedure that we carried out in Sect. 3.1.

Much of what follows relies on the following, at first sight obvious, lemma. In its statement, and elsewhere, two sets are said to be *non-overlapping* if their interiors are disjoint.

Lemma 5.1.1 *If $\mathcal{C}$ is a finite collection of non-overlapping rectangles each of which is contained in the rectangle R, then $|R| \geq \sum_{S\in\mathcal{C}} |S|$. On the other hand, if $\mathcal{C}$ is any finite collection of rectangles whose union contains a rectangle R, then $|R| \leq \sum_{S\in\mathcal{C}} |S|$.*

Proof Since $|S \cap R| \leq |S|$, we may and will assume throughout that $R \supseteq \bigcup_{S\in\mathcal{C}} S$. Also, without loss in generality, we will assume that $\operatorname{int}(R) \neq \emptyset$.

The proof is by induction on N. Thus, suppose that $N = 1$. Given a closed interval I, use a_I and b_I to denote its left and right endpoints. Determine the points $a_R \leq c_0 < \cdots < c_\ell \leq b_R$ so that

$$\{c_k : 0 \leq k \leq \ell\} = \{a_I : I \in \mathcal{C}\} \cup \{b_I : I \in \mathcal{C}\},$$

and set $\mathcal{C}_k = \{I \in \mathcal{C} : [c_{k-1}, c_k] \subseteq I\}$. Clearly $|I| = \sum_{\{k:\, I\in\mathcal{C}_k\}} (c_k - c_{k-1})$ for each $I \in \mathcal{C}$.[2]

When the intervals in $\mathcal{C}$ are non-overlapping, no $\mathcal{C}_k$ contains more than one $I \in \mathcal{C}$, and so

$$\begin{aligned}\sum_{I\in\mathcal{C}} |I| &= \sum_{I\in\mathcal{C}} \sum_{\{k:I\in\mathcal{C}_k\}} (c_k - c_{k-1}) = \sum_{k=1}^{\ell} \operatorname{card}(\mathcal{C}_k)(c_k - c_{k-1}) \\ &\leq \sum_{k=1}^{\ell} (c_k - c_{k-1}) \leq (b_R - a_R) = |R|.\end{aligned}$$

If $R = \bigcup_{I\in\mathcal{C}} I$, then $c_0 = a_R$, $c_\ell = b_R$, and, for each $0 \leq k \leq \ell$, there is an $I \in \mathcal{C}$ for which $I \in \mathcal{C}_k$. To prove this last assertion, simply note that if $x \in (c_{k-1}, c_k)$ and $\mathcal{C} \ni I \ni x$, then $[c_{k-1}, c_k] \subseteq I$ and therefore $I \in \mathcal{C}_k$. Knowing this, we have

$$\begin{aligned}\sum_{I\in\mathcal{C}} |I| &= \sum_{I\in\mathcal{C}} \sum_{\{k:I\in\mathcal{C}_k\}} (c_k - c_{k-1}) = \sum_{k=1}^{\ell} \operatorname{card}(\mathcal{C}_k)(c_k - c_{k-1}) \\ &\geq \sum_{k=1}^{\ell} (c_k - c_{k-1}) = (b_R - a_R) = |R|.\end{aligned}$$

[2] Here, and elsewhere, the sum over the empty set is taken to be 0.

Now assume the result for N. Given a rectangle S in $\mathbb{R}^{N+1}$, determine $a_S,\ b_S \in \mathbb{R}$ and the rectangle Q_S in $\mathbb{R}^N$ so that $S = Q_S \times [a_S, b_S]$. As before, choose points $a_R \le c_0 < \cdots < c_\ell \le b_R$ for $\{[a_S, b_S] : S \in \mathcal{C}\}$, and define

$$\mathcal{C}_k = \{S \in \mathcal{C} : [c_{k-1}, c_k] \subseteq [a_S, b_S]\}.$$

Then, for each $S \in \mathcal{C}$,

$$|S| = |Q_S|(b_S - a_S) = |Q_S| \sum_{\{k:S\in\mathcal{C}_k\}} (c_k - c_{k-1}).$$

If the rectangles in $\mathcal{C}$ are non-overlapping, then, for each k, the rectangles in $\{Q_S : S \in \mathcal{C}_k\}$ are non-overlapping. Hence, since $\bigcup_{S\in\mathcal{C}_k} Q_S \subseteq Q_R$, the induction hypothesis implies $\sum_{S\in\mathcal{C}_k} |Q_S| \le |Q_R|$ for each $1 \le k \le \ell$, and therefore

$$\begin{aligned}\sum_{S\in\mathcal{C}} |S| &= \sum_{S\in\mathcal{C}} |Q_S| \sum_{\{k:\,S\in\mathcal{C}_k\}} (c_k - c_{k-1}) \\ &= \sum_{k=1}^{\ell} (c_k - c_{k-1}) \sum_{S\in\mathcal{C}_k} |Q_S| \le (b_R - a_R)|Q_R| = |R|.\end{aligned}$$

Finally, assume that $R = \bigcup_{S\in\mathcal{C}} S$. In this case, $c_0 = a_R$ and $c_\ell = b_R$. In addition, for each $1 \le k \le \ell$, $Q_R = \bigcup_{S\in\mathcal{C}_k} Q_S$. To see this, note that if $\mathbf{x} = (x_1, \dots, x_{N+1}) \in R$ and $x_{N+1} \in (c_{k-1}, c_k)$, then $S \ni \mathbf{x} \implies [c_{k-1}, c_k] \subseteq [a_S, b_S]$ and therefore that $S \in \mathcal{C}_k$. Hence, by the induction hypothesis, $|Q_R| \le \sum_{S\in\mathcal{C}_k} \mathrm{vol}(Q_S)$ for each $1 \le k \le \ell$, and therefore

$$\begin{aligned}\sum_{S\in\mathcal{C}} |S| &= \sum_{S\in\mathcal{C}} |Q_S| \sum_{\{k:S\in\mathcal{C}_k\}} (c_k - c_{k-1}) \\ &= \sum_{k=1}^{\ell} (c_k - c_{k-1}) \sum_{S\in\mathcal{C}_k} |Q_S| \ge (b_R - a_R)|Q_R| = |R|.\end{aligned}$$

□

Given a rectangle $\prod_{j=1}^N [a_j, b_j]$, throughout this section $\mathcal{C}$ will be a finite collection of non-overlapping, closed rectangles R whose union is $\prod_{j=1}^N [a_j, b_j]$, and the mesh size $\|\mathcal{C}\|$ will be $\max\{\mathrm{diam}(R) : R \in \mathcal{C}\}$, where the diameter $\mathrm{diam}(R)$ of $R = \prod_{j=1}^N [r_j, s_j]$ equals $\sqrt{\sum_{j=1}^N (s_j - r_j)^2}$. For instance, $\mathcal{C}$ might be obtained by subdividing each of the sides $[a_j, b_j]$ into n equal parts and taking $\mathcal{C}$ to be the set of n^N rectangles

$$\prod_{j=1}^{N}\left[a_j + \tfrac{m_j-1}{n}(b_j - a_j), a_j + \tfrac{m_j}{n}(b_j - a_j)\right] \quad \text{for } 1 \le m_1, \dots, m_N \le n.$$

Next, say that $\Xi : \mathcal{C} \longrightarrow \mathbb{R}^N$ is a *choice function* if $\boldsymbol{\Xi}(R) \in R$ for each $R \in \mathcal{C}$, and define the *Riemann sum*

$$\mathcal{R}(f; \mathcal{C}, \boldsymbol{\Xi}) = \sum_{R\in\mathcal{C}} f\big(\boldsymbol{\Xi}(R)\big)|R|$$

for bounded functions $f : \prod_{j=1}^N [a_j, b_j] \longrightarrow \mathbb{R}$. Again, we say that f is *Riemann integrable* if there exists a $\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, d\mathbf{x} \in \mathbb{R}$ to which the Riemann sums $\mathcal{R}(f; \mathcal{C}, \Xi)$ converge, in the same sense as before, as $\|\mathcal{C}\| \to 0$, in which case $\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, d\mathbf{x}$ is called the *Riemann integral* or just the *integral* of f on $\prod_{j=1}^N [a_j, b_j]$.

There are no essentially new ideas needed to analyze when a function is Riemann integrable. As we did in Sect. 3.1, one introduces the upper and lower Riemann sums

$$\mathcal{U}(f; \mathcal{C}) = \sum_{R\in\mathcal{C}} \left(\sup_R f\right)|R| \quad \text{and} \quad \mathcal{L}(f; \mathcal{C}) = \sum_{R\in\mathcal{C}} \left(\inf_R f\right)|R|,$$

and, using the same reasoning as we did in the proof of Lemma 3.1.1, checks that $\mathcal{L}(f; \mathcal{C}) \le \mathcal{R}(f; \mathcal{C}, \boldsymbol{\Xi}) \le \mathcal{U}(f; \mathcal{C})$ for any $\boldsymbol{\Xi}$ and $\mathcal{L}(f; \mathcal{C}) \le \mathcal{U}(f; \mathcal{C}')$ for any $\mathcal{C}'$. Further, one can show that for each $\mathcal{C}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathcal{C}'\| < \delta \implies \mathcal{U}(f; \mathcal{C}') \le \mathcal{U}(f; \mathcal{C}) + \epsilon \text{ and } \mathcal{L}(f; \mathcal{C}') \ge \mathcal{L}(f; \mathcal{C}) - \epsilon.$$

The proof that such a δ exists is basically the same as, but somewhat more involved than, the corresponding one in Lemma 3.1.1. Namely, given $\delta > 0$ and a rectangle $R = \prod_{j=1}^N [c_j, d_j] \in \mathcal{C}$, define $R_k^-(\delta)$ and $R_k^+(\delta)$ to be the rectangles

$$\left(\prod_{1\le j<k} [a_j, b_j]\right) \times \left[a_k \vee (c_k - \delta), b_k \wedge (c_k + \delta)\right] \times \left(\prod_{k<j\le N} [a_j, b_j]\right)$$

and

$$\left(\prod_{1\le j<k} [a_j, b_j]\right) \times \left[a_k \vee (d_k - \delta), b_k \wedge (d_k + \delta)\right] \times \left(\prod_{k<j\le N} [a_j, b_j]\right)$$

for $1 \le k \le N$, with the understanding that the first factor is absent if $k = 1$ and the last factor is absent if $k = N$. Now suppose that $\|\mathcal{C}'\| < \delta$ and $R' \in \mathcal{C}'$. Then either $R' \subseteq R$ for some $R \in \mathcal{C}$ or there is an $1 \le k \le N$ and an $R \in \mathcal{C}$ such that the interior of the kth side of R' contains one of the end points of kth side of R, in which case $R' \subseteq R_k^-(\delta) \cup R_k^+(\delta)$. Thus, if $\mathcal{D}$ is the set of $R' \in \mathcal{C}'$ that are not contained in any $R \in \mathcal{C}$, then, because $\sup_{R'} f \leq \sup_R f$ if $R' \subseteq R$, one can use Lemma 5.1.1 to see that

$$\begin{aligned}\mathcal{U}(f;\mathcal{C}') - \mathcal{U}(f;\mathcal{C}) &= \sum_{R'\in\mathcal{C}'}\sum_{R\in\mathcal{C}}\left(\sup_{R'} f - \sup_{R} f\right)|R'\cap R| \\ &\leq \sum_{R'\in\mathcal{D}}\sum_{R\in\mathcal{C}}\left(\sup_{R'} f - \sup_{R} f\right)|R'\cap R| \leq 2\|f\|_{\prod_1^N [a_j,b_j]}\sum_{R'\in\mathcal{D}}|R'| \\ &\leq 2\|f\|_{\prod_1^N [a_j,b_j]}\sum_{k=1}^{N}\sum_{R\in\mathcal{C}}\big(|R_k^-(\delta)| + |R_k^+(\delta)|\big).\end{aligned}$$

Since $|R_k^{\pm}(\delta)| \leq \delta\prod_{j\neq k}(b_j - a_j)$, it follows that there exists a constant $A < \infty$ such that $\mathcal{U}(f;\mathcal{C}') \leq \mathcal{U}(f;\mathcal{C}) + A\delta$ if $\|\mathcal{C}'\| < \delta$.

With these preparations, we now have the following analog of Theorem 3.1.2. However, before stating the result, we need to make another definition. Namely, we will say that a subset Γ of the rectangle $\prod_{j=1}^N [a_j, b_j]$ is *Riemann negligible* if, for each $\epsilon > 0$ there is a $\mathcal{C}$ such that

$$\sum_{\substack{R\in\mathcal{C}\\ \Gamma\cap R\neq\emptyset}} |R| < \epsilon.$$

Riemann negligible sets will play an important role in our considerations.

Theorem 5.1.2 *Let $f : \prod_{j=1}^N [a_j, b_j] \longrightarrow \mathbb{C}$ be a bounded function. Then f is Riemann integrable if and only if for each $\epsilon > 0$ there is a $\mathcal{C}$ such that*

$$\sum_{\substack{R\in\mathcal{C}\\ \sup_R f - \inf_R f \geq \epsilon}} |R| < \epsilon.$$

In particular, f is Riemann integrable if it is continuous off of a Riemann negligible set. Finally if f is Riemann integrable and takes all its values in a compact set $K \subseteq \mathbb{C}$ and $\varphi : K \longrightarrow \mathbb{C}$ is continuous, then $\varphi \circ f$ is Riemann integrable.

Proof Except for the one that says f is Riemann integrable if it is continuous off of a Riemann negligible set, all these assertions are proved in exactly the same way as the analogous statements in Theorem 3.1.2.

Now suppose that f is continuous off of the Riemann negligible set Γ. Given $\epsilon > 0$, choose $\mathcal{C}$ so that $\sum_{R\in\mathcal{D}}|R| < \epsilon$, where $\mathcal{D} = \{R \in \mathcal{C} : R \cap \Gamma \neq \emptyset\}$. Then $K = \bigcup_{R\in\mathcal{C}\setminus\mathcal{D}} R$ is a compact set on which f is continuous. Hence, we can find a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $x, y \in K$ with $|y - x| \leq \delta$. Finally, subdivide each $R \in \mathcal{C}\setminus\mathcal{D}$ into rectangles of diameter less than δ, and take $\mathcal{C}'$ to be the cover consisting of the elements of $\mathcal{D}$ and the sub-rectangles into which the elements of $\mathcal{C}\backslash\mathcal{D}$ were subdivided. Then

$$\sum_{\substack{R'\in\mathcal{C}' \\ \sup_{R'} f - \inf_{R'} f \geq \epsilon}} |R'| \leq \sum_{R\in\mathcal{D}} |R| < \epsilon. \qquad \square$$

We now have the basic facts about Riemann integration in $\mathbb{R}^N$, and from them follow the Riemann integrability of linear combinations and products of bounded Riemann integrable functions as well as the obvious analogs of (3.1.1), (3.1.5), (3.1.4), and Theorem 3.14. The replacement for (3.1.2) is

$$\int_{\prod_1^N [\lambda a_j, \lambda b_j]} f(\mathbf{x})\, d\mathbf{x} = \lambda^N \int_{\prod_1^N [a_j, b_j]} f(\lambda \mathbf{x})\, dx \tag{5.1.1}$$

for bounded, Riemann integrable functions on $\prod_{j=1}^N [\lambda a_j, \lambda b_j]$. It is also useful to note that Riemann integration is *translation invariant* in the sense that if f is a bounded, Riemann integrable function on $\prod_{j=1}^N [c_j + a_j, c_j + b_j]$ for some $\mathbf{c} = (c_1, \dots, c_N) \in \mathbb{R}^N$, then $x \rightsquigarrow f(\mathbf{c}+\mathbf{x})$ is Riemann integrable on $\prod_{j=1}^N [a_j, b_j]$ and

$$\int_{\prod_1^N [c_j + a_k, c_j + b_j]} f(\mathbf{x})\, d\mathbf{x} = \int_{\prod_1^N [a_j, b_j]} f(\mathbf{c}+\mathbf{x})\, d\mathbf{x}, \tag{5.1.2}$$

a property that follows immediately from the corresponding fact for Riemann sums. In addition, by the same procedure as we used in Sect. 3.1, we can extend the definition of the Riemann integral to cover situations in which either the integrand f or the region over which the integration is performed is unbounded. Thus, for example, if f is a function that is bounded and Riemann integrable on bounded rectangles, then one defines

$$\int_{\mathbb{R}^N} f(\mathbf{x})\, d\mathbf{x} = \lim_{\substack{a_1 \vee \cdots \vee a_N \to -\infty \\ b_1 \wedge \cdots \wedge b_N \to \infty}} \int_{\prod_1^N [a_j, b_j]} f(\mathbf{x})\, d\mathbf{x}$$

if the limit exists.

5.2 Iterated Integrals and Fubini's Theorem

Evaluating integrals in N variables is hard and usually possible only if one can reduce the computation to integrals in one variable. One way to make such a reduction is to write an integral in N variables as N iterated integrals in one variable, one for each dimension, and the following theorem, known as *Fubini's Theorem*, shows this can be done. In its statement, if $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $1 \leq M < N$, then $\mathbf{x}_1^{(M)} \equiv (x_1, \dots, x_M)$ and $\mathbf{x}_2^{(M)} \equiv (x_{M+1}, \dots, x_N)$.

Theorem 5.2.1 *Suppose that* $f : \prod_{j=1}^N [a_j, b_j] \longrightarrow \mathbb{C}$ *is a bounded, Riemann integrable function. Further, for some* $1 \le M < N$ *and each* $\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j]$, *assume that* $\mathbf{x}_1^{(M)} \in \prod_{j=1}^M [a_j, b_j] \longmapsto f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)}) \in \mathbb{C}$ *is Riemann integrable. Then*

$$\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j] \longmapsto f_1^{(M)}(\mathbf{x}_2^{(M)}) \equiv \int_{\prod_1^M [a_j,b_j]} f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)})\, d\mathbf{x}_1^{(M)}$$

is Riemann integrable and

$$\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, d\mathbf{x} = \int_{\prod_{M+1}^N [a_j,b_j]} f_1^{(M)}(\mathbf{x}_2^{(M)})\, d\mathbf{x}_2^{(M)}.$$

In particular, this result applies if f *is a bounded, Riemann integrable function with the property that, for each* $\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j]$, $\mathbf{x}_1^{(M)} \in \prod_{j=1}^M [a_j, b_j] \longmapsto f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)}) \in \mathbb{C}$ *is continuous at all but a Riemann negligible set of points.*

Proof Given $\epsilon > 0$, choose $\delta > 0$ so that

$$\|\mathcal{C}\| < \delta \implies \left| \int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, dx - \mathcal{R}(f;\, \mathcal{C}, \boldsymbol{\Xi}) \right| < \epsilon$$

for every choice function $\boldsymbol{\Xi}$. Next, let $\mathcal{C}_2^{(M)}$ be a cover of $\prod_{j=M+1}^N [a_j, b_j]$ with $\|\mathcal{C}_2^{(M)}\| < \frac{\delta}{2}$, and let $\boldsymbol{\Xi}_2^{(M)}$ be an associated choice function. Finally, because $\mathbf{x}_1^{(M)} \rightsquigarrow f\big(\mathbf{x}_1^{(M)}, \boldsymbol{\Xi}_2^{(M)}(R_2)\big)$ is Riemann integrable for each $R_2 \in \mathcal{C}_2^{(M)}$, we can choose a cover $\mathcal{C}_1^{(M)}$ of $\prod_{j=1}^M [a_j, b_j]$ with $\|\mathcal{C}_1^{(M)}\| < \frac{\delta}{2}$ and an associated choice function $\boldsymbol{\Xi}_1^{(M)}$ such that

$$\sum_{R_2 \in \mathcal{C}_2^{(M)}} \left| \sum_{R_1 \in \mathcal{C}_1^{(M)}} f\big(\boldsymbol{\Xi}_1^{(M)}(R_1), \boldsymbol{\Xi}_2^{(M)}(R_2)\big)\big)|R_1| - f_1^{(M)}(\boldsymbol{\Xi}_2^{(M)}(R_2)\big) \right| |R_2| < \epsilon.$$

If

$$\mathcal{C} = \{R_1 \times R_2 :\, R_1 \in \mathcal{C}_1^{(M)} \ \&\ R_2 \in \mathcal{C}_2^{(M)}\}$$

and $\boldsymbol{\Xi}\big(R_1 \times R_2) = \big(\boldsymbol{\Xi}_1^{(M)}(R_1), \boldsymbol{\Xi}_2^{(M)}(R_2)\big)$, then $\|\mathcal{C}\| < \delta$ and

$$\mathcal{R}(f; \mathcal{C}, \boldsymbol{\Xi}) = \sum_{R_1 \in \mathcal{C}_1^{(M)}} \sum_{R_2 \in \mathcal{C}_2^{(M)}} f\big(\boldsymbol{\Xi}_1^{(M)}(R_1), \boldsymbol{\Xi}_2^{(M)}(R_2)\big)|R_1||R_2|,$$

and so

$$\begin{aligned}&\left|\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} - \mathcal{R}(f_1^{(M)}; \mathcal{C}_2^{(M)}, \boldsymbol{\Xi}_2^{(M)})\right| \\ &\le \left|\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} - \mathcal{R}(f; \mathcal{C}, \boldsymbol{\Xi})\right| \\ &\quad + \sum_{R_2\in\mathcal{C}_2^{(M)}} \left| \sum_{R_1\in\mathcal{C}_1^{(M)}} f\big(\boldsymbol{\Xi}_1^{(M)}(R_1), \boldsymbol{\Xi}_2^{(M)}(R_2)\big)|R_1| - f_1^{(M)}(\boldsymbol{\Xi}_2^{(M)}(R_2))\right| |R_2|\end{aligned}$$

is less than 2ϵ. Hence, $\mathcal{R}(f_1^{(M)}; \mathcal{C}_2^{(M)}, \boldsymbol{\Xi}_2^{(M)}\big)$ converges to $\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x}$ as $\|\mathcal{C}_2^{(M)}\| \to 0$. □

It should be clear that the preceding result holds equally well when the roles of $\mathbf{x}_1^{(M)}$ and $\mathbf{x}_2^{(M)}$ are reversed. Thus, if $f : \prod_{j=1}^N [a_j, b_j] \longmapsto \mathbb{C}$ is a bounded, Riemann integrable function such that $\mathbf{x}_1^{(M)} \in \prod_{j=1}^M [a_j, b_j] \longmapsto f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)}) \in \mathbb{C}$ is Riemann integrable for each $\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j]$ and $\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j] \longmapsto f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)})$ is Riemann integrable for each $\mathbf{x}_1^{(M)} \in \prod_{j=1}^N [a_j, b_j]$, then

$$\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} = \begin{cases} \int_{\prod_{M+1}^N [a_j,b_j]} f_1^{(M)}(\mathbf{x}_2^{(M)})\,d\mathbf{x}_2^{(M)} \\ \int_{\prod_1^M [a_j,b_j]} f_2^{(M)}(\mathbf{x}_2^{(M)})\,d\mathbf{x}_1^{(M)}, \end{cases} \tag{5.2.1}$$

where

$$f_1^{(M)}(\mathbf{x}_2^{(M)}) = \int_{\prod_1^M [a_j,b_j]} f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)})\,d\mathbf{x}_1^{(M)}$$

and

$$f_2^{(M)}(\mathbf{x}_1^{(M)}) = \int_{\prod_{M+1}^N [a_j,b_j]} f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)})\,d\mathbf{x}_2^{(M)}.$$

Corollary 5.2.2 *Let f be a continuous function on $\prod_{j=1}^N [a_j, b_j]$. Then for each $1 \le M < N$,*

$$\mathbf{x}_2^{(M)} \in \prod_{j=M+1}^N [a_j, b_j] \longmapsto f_1^{(M)}(\mathbf{x}_2^{(M)}) \equiv \int_{\prod_1^M [a_j,b_j]} f(\mathbf{x}_1^{(M)}, \mathbf{x}_2^{(M)})\,d\mathbf{x}_1^{(M)} \in \mathbb{C}$$

is continuous. Furthermore,

$$f_1^{(M+1)}(\mathbf{x}_2^{(M+1)}) = \int_{[a_M,b_M]} f_1^{(M)}(x_M, \mathbf{x}_2^{M+1})\,dx_M \quad \textit{for } 1 \le M < N-1$$

and

$$\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} = \int_{[a_N,b_N]} f_1^{(N-1)}(x_N)\,dx_N.$$

Proof Once the first assertion is proved, the others follow immediately from Theorem 5.2.1. But, because f is uniformly continuous, the first assertion follows from the obvious higher dimensional analog of Theorem 3.1.4. □

By repeated applications of Corollary 5.2.2, one sees that

$$\int_{\prod_{j=1}^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} = \int_{a_N}^{b_N} \left(\cdots\left(\int_{a_1}^{b_1} f(x_1,\dots,x_{N-1},x_N)\,dx_1\right)\cdots\right)dx_N.$$

The expression on the right is called an *iterated integral*. Of course, there is nothing sacrosanct about the order in which one does the integrals. Thus

$$\int_{\prod_{j=1}^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} = \int_{a_{\pi(N)}}^{b_{\pi(N)}} \left(\cdots\left(\int_{a_{\pi(1)}}^{b_{\pi(1)}} f(x_1,\dots,x_{N-1},x_N)\,dx_{\pi(1)}\right)\cdots\right)dx_{\pi(N)} \tag{5.2.2}$$

for any permutation π of $\{1,\dots,N\}$. In that it shows integrals in N variables can be evaluated by doing N integrals in one variable, (5.2.2) makes it possible to bring Theorem 3.2.1 to bear on the problem. However, it is hard enough to find one indefinite integral on $\mathbb{R}$, much less a succession of N of them. Nonetheless, there is an important consequence of (5.2.2). Namely, if $f(\mathbf{x}) = \prod_{j=1}^N f_j(x_j)$, where, for each $1 \le j \le N$, f_j is a continuous function on $[a_j, b_j]$, then

$$\int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\,d\mathbf{x} = \prod_{j=1}^N \int_{[a_j,b_j]} f_j(x_j)\,dx_j. \tag{5.2.3}$$

In fact, starting from Theorem 5.2.1, it is easy to check that (5.2.3) holds when each f_j is bounded and Riemann integrable.

Looking at (5.2.2), one might be tempted to think that there is an analog of the Fundamental Theorem of Calculus for integrals in several variables. Namely, taking π to be the identity permutation that leaves the order unchanged and thinking of the expression on the right as a function F of $(b_1,\dots,b_N)$, it becomes clear that $\partial_{\mathbf{e}_1}\dots\partial_{\mathbf{e}_N} F = f$. However, what made this information valuable when $N = 1$ is the fact that a function on $\mathbb{R}$ can be recovered, up to an additive constant, from its derivative, and that is why we could say that $F(b) - F(a) = \int_a^b f(x)\,dx$ for any

F satisfying $F' = f$. When $N \geq 2$, the equality $\partial_{\mathbf{e}_1} \ldots \partial_{\mathbf{e}_N} F = f$ provides much less information. Indeed, even when $N = 2$, if F satisfies $\partial_{\mathbf{e}_1}\partial_{\mathbf{e}_2} F = f$, then so does $F(x_1, x_2) + F_1(x_1) + F_2(x_2)$ for any choice of differentiable functions F_1 and F_2, and the ambiguity gets worse as N increases. Thus finding an F that satisfies $\partial_{\mathbf{e}_1} \ldots \partial_{\mathbf{e}_N} F = f$ does little to advance one toward finding the integral of f.

To provide an interesting example of the way in which Fubini's Theorem plays an important role, define *Euler's Beta function* $B : (0, \infty)^2 \longrightarrow (0, \infty)$ by

$$B(\alpha, \beta) = \int_{(0,1)} x^{\alpha-1}(1-x)^{\beta-1}\, dx.$$

It turns out that his Beta function is intimately related to his (cf. Exercise 3.3) Gamma function. In fact,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{5.2.4}$$

which means that $\frac{1}{B(\alpha,\beta)}$ is closely related to the binomial coefficients in the same sense that $\Gamma(t)$ is related to factorials. Although (5.2.4) holds for all $(\alpha, \beta) \in (0, \infty)^2$, in order to avoid distracting technicalities, we will prove it only for $(\alpha, \beta) \in [1, \infty)^2$. Thus let $\alpha, \beta \geq 1$ be given. Then, by (5.2.3) and (5.2.1), $\Gamma(\alpha)\Gamma(\beta)$ equals

$$\begin{aligned} &\lim_{r\to\infty} \int_{[0,r]^2} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}\, d\mathbf{x} \\ &\qquad = \lim_{r\to\infty} \int_0^r x_2^{\beta-1} \left(\int_0^r x_1^{\alpha-1} e^{-(x_1+x_2)}\, dx_1 \right) dx_2. \end{aligned}$$

By (5.1.2),

$$\int_0^r x_1^{\alpha-1} e^{-(x_1+x_2)}\, dx_1 = \int_{x_2}^{r+x_2} (y_1 - x_2)^{\alpha-1} e^{-y_1}\, dy_1,$$

and so

$$\begin{aligned} &\int_0^r x_2^{\beta-1} \left(\int_0^r x_1^{\alpha-1} e^{-(x_1+x_2)}\, dx_1 \right) dx_2 \\ &\qquad = \int_0^r x_2^{\beta-1} \left(\int_{x_2}^{r+x_2} (y_1 - x_2)^{\alpha-1} e^{-y_1}\, dy_1 \right) dx_2. \end{aligned}$$

Now consider the function

$$f(y_1, x_2) = \begin{cases} (y_1 - x_2)^{\alpha-1} x_2^{\beta-1} e^{-y_1} & \text{if } x_2 \in [0, r]\ \&\ x_2 \leq y_1 \leq r + x_2 \\ 0 & \text{otherwise} \end{cases}$$

on $[0, 2r]\times[0, r]$. Because the only discontinuities of f lie in the Riemann negligible set $\{(r+x_2, x_2) : x_2 \in [0, r]\}$, it is Riemann integrable on $[0, 2r]\times[0, r]$. In addition, for each $y_1 \in [0, 2r]$, $x_2 \rightsquigarrow f(y_1, x_2)$ and, for each $x_2 \in [0, r]$, $y_1 \rightsquigarrow f(y_1, x_2)$ have at most two discontinuities. We can therefore apply (5.2.1) to justify

$$\int_0^r x_2^{\beta-1}\left(\int_{x_2}^{r+x_2}(y_1-x_2)^{\alpha-1}e^{-y_1}\,dy_1\right)dx_2 = \int_0^r\left(\int_0^{2r} f(y_1,x_2)\,dy_1\right)dx_2$$
$$= \int_0^{2r}\left(\int_0^r f(y_1,x_2)\,dx_2\right)dy_1 = \int_0^{2r} e^{-y_1}\left(\int_{(y_1-r)^+}^{r\wedge y_1}(y_1-x_2)^{\alpha-1}x_2^{\beta-1}\,dx_2\right)dy_1.$$

Further, by (3.1.2)

$$\int_{(y_1-r)^+}^{r\wedge y_1}(y_1-x_2)^{\alpha-1}x_2^{\beta-1}\,dx_2 = y_1^{\alpha+\beta-1}\int_{(1-y_1^{-1}r)^+}^{1\wedge(y_1^{-1}r)}(1-y_2)^{\alpha-1}y_2^{\beta-1}\,dy_2.$$

Collecting these together, we have

$$\Gamma(\alpha)\Gamma(\beta) = \lim_{r\to\infty}\int_0^{2r} y_1^{\alpha+\beta-1}e^{-y_1}\left(\int_{(1-y_1^{-1}r)^+}^{1\wedge(y_1^{-1}r)}(1-y_2)^{\alpha-1}y_2^{\beta-1}\,dy_2\right)dy_1.$$

Finally,

$$\int_0^{2r} y_1^{\alpha+\beta-1}e^{-y_1}\left(\int_{(1-y_1^{-1}r)^+}^{1\wedge(y_1^{-1}r)}(1-y_2)^{\alpha-1}y_2^{\beta-1}\,dy_2\right)dy_1$$
$$= \int_0^r y_1^{\alpha+\beta-1}e^{-y_1}\,dy_1 B(\alpha,\beta)$$
$$+ \int_r^{2r} y_1^{\alpha+\beta-1}e^{-y_1}\left(\int_{(1-y_1^{-1}r)}^{y_1^{-1}r}(1-y_2)^{\alpha-1}y_2^{\beta-1}\,dy_2\right)dy_1,$$

and, as $r \to \infty$, the first term on the right tends to $\Gamma(\alpha+\beta)$ whereas the second term is dominated by $\int_r^\infty y_1^{\alpha+\beta-1}e^{-y_1}\,dy_1$ and therefore tends to 0.

The preceding computation illustrates one of the trickier aspects of proper applications of Fubini's Theorem. When one reverses the order of integration, it is very important to figure out what are the resulting correct limits of integration. As in the application above, the correct limits can look very different after the order of integration is changed.

The Eq. (5.2.4) provides a proof of *Stirling's formula* for the Gamma function as a consequence of (1.8.7). Indeed, by (5.2.4), $\Gamma(n+1+\theta) = \frac{n!\Gamma(\theta)}{B(n+1,\theta)}$ for $n \in \mathbb{Z}^+$ and $\theta \in [1, 2)$, and, by (3.1.2),

$$B(n+1,\theta)=n^{-\theta}\int_0^n y^{\theta-1}\big(1-\tfrac{y}{n}\big)^n\,dy.$$

Further, because $1-x\le e^{-x}$ for all $x\in\mathbb{R}$,

$$\int_0^n y^{\theta-1}\big(1-\tfrac{y}{n}\big)^n\,dy\le\int_0^\infty y^{\theta-1}e^{-y}\,dy=\Gamma(\theta),$$

and, for all $r>0$,

$$\varliminf_{n\to\infty}\int_0^n y^{\theta-1}\big(1-\tfrac{y}{n}\big)^n\,dy\ge\varliminf_{n\to\infty}\int_0^r y^{\theta-1}\big(1-\tfrac{y}{n}\big)^n\,dy=\int_0^r y^{\theta-1}e^{-y}\,dy.$$

Since the final expression tends to $\Gamma(\theta)$ as $r\to\infty$ uniformly fast for $\theta\in[1,2]$, we now know that

$$\frac{\Gamma(\theta)}{n^\theta B(n+1,\theta)}\longrightarrow 1$$

uniformly fast for $\theta\in[1,2]$. Combining this with (1.8.7) we see that

$$\lim_{n\to\infty}\frac{\Gamma(n+\theta+1)}{\sqrt{2\pi n}\,\big(\frac{n}{e}\big)^n n^\theta}\longrightarrow 1$$

uniformly fast for $\theta\in[1,2]$. Given $t\ge 3$, determine $n_t\in\mathbb{Z}^+$ and $\theta_t\in[1,2)$ so that $t=n_t+\theta_t$. Then the preceding says that

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{\sqrt{2\pi t}\,\big(\frac{t}{e}\big)^t}\sqrt{\frac{t}{t-\theta_t}}\left(\frac{t}{t-\theta_t}\right)^t e^{-\theta_t}=1.$$

Finally, it is obvious that, as $t\to\infty$, $\sqrt{\frac{t}{t-\theta_t}}$ tends to 1 and, because, by (1.7.5),

$$\log\left(\left(\frac{t}{t-\theta_t}\right)^t e^{-\theta_t}\right)=-t\log\left(1-\tfrac{\theta_t}{t}\right)-\theta_t\longrightarrow 0,$$

so does $\left(\frac{t}{t-\theta_t}\right)^t e^{-\theta_t}$. Hence we have shown that

$$\Gamma(t+1)\sim\sqrt{2\pi t}\left(\frac{t}{e}\right)^t\quad\text{as } t\to\infty \tag{5.2.5}$$

in the sense that $\lim_{t\to\infty}\frac{\Gamma(t+1)}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t}=1$.

5.3 Volume of and Integration Over Sets

We motivated our initial discussion of integration by computing the area under the graph of a non-negative function, and as we will see in this section, integration provides a method for computing the volume of more general regions. However, before we begin, we must first be more precise about what we will mean by the volume of a region.

Although we do not know yet what the volume of a general set Γ is, we know a few properties that volume should possess. In particular, we know that the volume of a subset should be no larger than that of the set containing it. In addition, volume should be additive in the sense that the volume of the union of disjoint sets should be the sum of their volumes. Taking these comments into account, for a given bounded set $\Gamma \subseteq \mathbb{R}^N$, we define the *exterior volume* $|\Gamma|_e$ of Γ to be the infimum of the sums $\sum_{R\in\mathcal{C}} |R|$ as $\mathcal{C}$ runs over all finite collections of non-overlapping rectangles whose union contains Γ.[3] Similarly, define the *interior volume* $|\Gamma|_i$ to be the supremum of the sums $\sum_{R\in\mathcal{C}} |R|$ as $\mathcal{C}$ runs over finite collections of non-overlapping rectangles each of which is contained in Γ. Clearly the notion of exterior volume is consistent with the properties that we want volume to have. To see that the same is true of interior volume, note that an equivalent description would have been that $|\Gamma|_i$ is the supremum of $\sum_{R\in\mathcal{C}} |R|$ as $\mathcal{C}$ runs over finite collections of rectangles that are mutually disjoint and each of which is contained in Γ. Indeed, given a $\mathcal{C}$ of the sort in the definition of interior volume, shrink the sides of each $R \in \mathcal{C}$ with $|R| > 0$ by a factor $\theta \in (0, 1)$ and eliminate the ones with $|R| = 0$. The resulting rectangles will be mutually disjoint and the sum of their volumes will be θ^N times that of the original ones. Hence, by taking θ close enough to 1, we can get arbitrarily close to the original sum.

Obviously Γ is Riemann negligible if and only if $|\Gamma|_e = 0$. Next notice that $|\Gamma|_i \le |\Gamma|_e$ for all bounded Γ's. Indeed, suppose that $\mathcal{C}_1$ is a finite collection of non-overlapping rectangles contained in Γ and that $\mathcal{C}_2$ is a finite collection of rectangles whose union contains Γ. Then, by Lemma 5.1.1,

$$\sum_{R_2\in\mathcal{C}_2} |R_2| \ge \sum_{R_2\in\mathcal{C}_2}\sum_{R_1\in\mathcal{C}_1} |R_1 \cap R_2| = \sum_{R_1\in\mathcal{C}_1}\sum_{R_2\in\mathcal{C}_2} |R_1 \cap R_2| \ge \sum_{R_1\in\mathcal{C}_1} |R_1|.$$

In addition, it is easy to check that

$$\begin{aligned} |\Gamma_1|_e \le |\Gamma_2|_e \text{ and } |\Gamma_1|_i \le |\Gamma_2|_i \quad &\text{if } \Gamma_1 \subseteq \Gamma_2 \\ |\Gamma_1 \cup \Gamma_2|_e \le |\Gamma_1|_e + |\Gamma_2|_e \text{ for all } &\Gamma_1 \ \&\ \Gamma_2, \\ \text{and } |\Gamma_1 \cup \Gamma_2|_i \ge |\Gamma_1|_i + |\Gamma_2|_i \text{ if } &\Gamma_1 \cap \Gamma_2 = \emptyset. \end{aligned}$$

[3]It is reasonable easy to show that $|\Gamma|_e$ would be the same if the infimum were taken over covers by rectangles that are not necessarily non-overlapping.

We will say that Γ is *Riemann measurable* if $|\Gamma|_{\mathrm{i}} = |\Gamma|_{\mathrm{e}}$, in which case we will call $\mathrm{vol}(\Gamma) \equiv |\Gamma|_{\mathrm{e}}$ the *volume* of Γ. Clearly $|\Gamma|_{\mathrm{e}} = 0$ implies that Γ is Riemann measurable and $\mathrm{vol}(\Gamma) = 0$. In particular, if Γ is Riemann negligible and therefore $|\Gamma|_{\mathrm{e}} = 0$, then Γ is Riemann measurable and has volume 0. In addition, if R is a rectangle, $|R|_{\mathrm{e}} \le |R| \le |R|_{\mathrm{i}}$, and therefore R is Riemann measurable and $\mathrm{vol}(R) = |R|$.

One suspects that these considerations are intimately related to Riemann integration, and the following theorem justifies that suspicion. In its statement and elsewhere, $\mathbf{1}_\Gamma$ denotes the *indicator function* of a set Γ. That is, $\mathbf{1}_\Gamma(\mathbf{x})$ is 1 if $\mathbf{x} \in \Gamma$ and is 0 if $\mathbf{x} \notin \Gamma$.

Theorem 5.3.1 *Let Γ be a subset of $\prod_{j=1}^N [a_j, b_j]$. Then Γ is Riemann measurable if and only if $\mathbf{1}_\Gamma$ is Riemann integrable on $\prod_{j=1}^N [a_j, b_j]$, in which case*

$$\mathrm{vol}(\Gamma) = \int_{\prod_{j=1}^N [a_j, b_j]} \mathbf{1}_\Gamma(\mathbf{x})\, d\mathbf{x}.$$

Proof First observe that, without loss in generality, we may assume that all the collections $\mathcal{C}$ entering the definitions of outer and inner volume can be taken to be subsets of non-overlapping covers of $\prod_{j=1}^N [a_j, b_j]$.

Now suppose that Γ is Riemann measurable. Given $\epsilon > 0$, choose a non-overlapping cover $\mathcal{C}_1$ of $\prod_{j=1}^N [a_j, b_j]$ such that

$$\sum_{\substack{R \in \mathcal{C}_1 \\ R \cap \Gamma \ne \emptyset}} |R| \le \mathrm{vol}(\Gamma) + \tfrac{\epsilon}{2}.$$

Then

$$\mathcal{U}(\mathbf{1}_\Gamma; \mathcal{C}_1) = \sum_{\substack{R \in \mathcal{C}_1 \\ R \cap \Gamma \ne \emptyset}} |R| \le \mathrm{vol}(\Gamma) + \tfrac{\epsilon}{2}.$$

Next, choose $\mathcal{C}_2$ so that

$$\sum_{\substack{R \in \mathcal{C}_2 \\ R \subseteq \Gamma}} |R| \ge \mathrm{vol}(\Gamma) - \tfrac{\epsilon}{2},$$

and observe that then $\mathcal{L}(\mathbf{1}_\Gamma; \mathcal{C}_2) \ge \mathrm{vol}(\Gamma) - \frac{\epsilon}{2}$. Hence if

$$\mathcal{C} = \{R_1 \cap R_2 : \ R_1 \in \mathcal{C}_1 \ \&\ R_2 \in \mathcal{C}_2\},$$

then

$$\mathcal{U}(\mathbf{1}_\Gamma; \mathcal{C}) \le \mathcal{U}(\mathbf{1}_\Gamma; \mathcal{C}_1) \le \mathrm{vol}(\Gamma) + \tfrac{\epsilon}{2} \le \mathcal{L}(\mathbf{1}_\Gamma; \mathcal{C}_2) + \epsilon \le \mathcal{L}(\mathbf{1}_\Gamma; \mathcal{C}) + \epsilon,$$

and so not only is $\mathbf{1}_\Gamma$ Riemann integrable but also its integral is equal to $\mathrm{vol}(\Gamma)$.

Conversely, if $\mathbf{1}_\Gamma$ is Riemann integrable and $\epsilon > 0$, choose $\mathcal{C}$ so that $\mathcal{U}(\mathbf{1}_\Gamma; \mathcal{C}) \le \mathcal{L}(\mathbf{1}_\Gamma; \mathcal{C}) + \epsilon$. Define associated choice functions $\boldsymbol{\Xi}_1$ and $\boldsymbol{\Xi}_2$ so that $\boldsymbol{\Xi}_1(R) \in \Gamma$ if $R \cap \Gamma \ne \emptyset$ and $\boldsymbol{\Xi}_2(R) \notin \Gamma$ unless $R \subseteq \Gamma$. Then

$$|\Gamma|_{\mathrm{e}} \le \sum_{\substack{R\in\mathcal{C}\\ R\cap\Gamma\ne\emptyset}} |R| = \mathfrak{R}(\mathbf{1}_\Gamma; \mathcal{C}, \boldsymbol{\Xi}_1) \le \mathfrak{R}(\mathbf{1}_\Gamma; \mathcal{C}, \boldsymbol{\Xi}_2) + \epsilon = \sum_{\substack{R\in\mathcal{C}\\ R\subseteq\Gamma}} |R| + \epsilon \le |\Gamma|_{\mathrm{i}} + \epsilon,$$

and so Γ is Riemann measurable. □

Corollary 5.3.2 *If Γ_1 and Γ_2 are bounded, Riemann measurable sets, then so are $\Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2$, and $\Gamma_2 \setminus \Gamma_1$. In addition,*

$$\mathrm{vol}\big(\Gamma_1 \cup \Gamma_2) = \mathrm{vol}(\Gamma_1) + \mathrm{vol}(\Gamma_2) - \mathrm{vol}(\Gamma_1 \cap \Gamma_2)$$

and

$$\mathrm{vol}\big(\Gamma_2 \setminus \Gamma_1\big) = \mathrm{vol}\big(\Gamma_2\big) - \mathrm{vol}\big(\Gamma_1 \cap \Gamma_2\big).$$

In particular, if $\mathrm{vol}(\Gamma_1 \cap \Gamma_2) = 0$, *then* $\mathrm{vol}(\Gamma_1 \cup \Gamma_2) = \mathrm{vol}(\Gamma_1) + \mathrm{vol}(\Gamma_2)$. *Finally, $\Gamma \subseteq \prod_{j=1}^N [a_j, b_j]$ is Riemann measurable if and only if for each $\epsilon > 0$ there exist Riemann measurable subsets A and B of $\prod_{j=1}^N [a_j, b_j]$ such that $A \subseteq \Gamma \subseteq B$ and* $\mathrm{vol}(B \setminus A) < \epsilon$.

Proof By Theorem 5.3.1, $\mathbf{1}_{\Gamma_1}$ and $\mathbf{1}_{\Gamma_2}$ are Riemann integrable. Thus, since

$$\mathbf{1}_{\Gamma_1\cap\Gamma_2} = \mathbf{1}_{\Gamma_1}\mathbf{1}_{\Gamma_2} \text{ and } \mathbf{1}_{\Gamma_1\cup\Gamma_2} = \mathbf{1}_{\Gamma_1} + \mathbf{1}_{\Gamma_2} - \mathbf{1}_{\Gamma_1\cap\Gamma_2},$$

that same theorem implies that $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ are Riemann measurable. At the same time,

$$\mathbf{1}_{\Gamma_2\setminus\Gamma_1} = \mathbf{1}_{\Gamma_2} - \mathbf{1}_{\Gamma_1\cap\Gamma_2},$$

and so $\Gamma_2 \setminus \Gamma_1$ is also Riemann measurable. Also, by Theorem 5.3.1, the equations relating their volumes follow immediately for the equations relating their indicator functions.

Turning to the final assertion, there is nothing to do if Γ is Riemann measurable since we can then take $A = \Gamma = B$ for all $\epsilon > 0$. Now suppose that for each $\epsilon > 0$ there exist Riemann measurable sets A_ϵ and B_ϵ such that $A_\epsilon \subseteq \Gamma \subseteq B_\epsilon \subseteq \prod_{j=1}^N [a_j, b_j]$ and $\mathrm{vol}(B_\epsilon \setminus A_\epsilon) < \epsilon$. Then

$$|\Gamma|_{\mathrm{i}} \ge \mathrm{vol}(A_\epsilon) \ge \mathrm{vol}(B_\epsilon) - \epsilon \ge |\Gamma|_{\mathrm{e}} - \epsilon,$$

and so Γ is Riemann measurable. □

It is reassuring that the preceding result is consistent with our earlier computation of the area under a graph. In fact, we now have the following more general result.

Theorem 5.3.3 *Assume that* $f : \prod_{j=1}^{N}[a_j, b_j] \longrightarrow \mathbb{R}$ *is continuous. Then the graph*

$$G(f) = \left\{ \big(\mathbf{x}, f(\mathbf{x})\big) : \mathbf{x} \in \prod_{j=1}^{N}[a_j, b_j] \right\}$$

is a Riemann negligible subset of $\mathbb{R}^{N+1}$. *Moreover, if, in addition,* f *is non-negative and* $\Gamma = \big\{(\mathbf{x}, y) \in \mathbb{R}^{N+1} : 0 \le y \le f(\mathbf{x})\big\}$, *then* Γ *is Riemann measurable and*

$$\mathrm{vol}(\Gamma) = \int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, d\mathbf{x}.$$

Proof Set $r = \|f\|_{\prod_1^N [a_j,b_j]}$, and for each $\epsilon > 0$ choose $\delta_\epsilon > 0$ so that

$$|f(\mathbf{y}) - f(\mathbf{x})| < \epsilon \quad \text{if } |\mathbf{y} - \mathbf{x}| \le \delta_\epsilon.$$

Next let $\mathcal{C}$ with $\|\mathcal{C}\| < \delta_\epsilon$ be a cover of $\prod_1^N [a_j, b_j]$ by non-overlapping rectangles, and choose $K \in \mathbb{Z}^+$ so that $\frac{r}{K+1} < \epsilon \le \frac{r}{K}$. Then for each $R \in \mathcal{C}$ there is a $1 \le k_R \le 2(K-1)$ such that

$$\big\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in R\big\} \subseteq R \times \left[-r + \tfrac{(k_R-1)r}{K}, -r + \tfrac{(k_R+2)r}{K}\right],$$

and therefore

$$|G(f)|_{\mathrm{e}} \le \frac{3r}{K} \sum_{R \in \mathcal{C}} |R| \le 6 \left(\prod_{j=1}^{N} (b_j - a_j) \right) \epsilon,$$

which proves that $G(f)$ is Riemann negligible.

Turning to the second assertion, note that all the discontinuities of $\mathbf{1}_\Gamma$ on $\prod_{j=1}^{N}[a_j, b_j] \times [0, r]$ are contained in $G(f)$, and therefore $\mathbf{1}_\Gamma$ is Riemann measurable. In addition, for each $\mathbf{x} \in \prod_{j=1}^{N}[a_j, b_j]$, $y \in [0, r] \longmapsto \mathbf{1}_\Gamma(\mathbf{x}, y) \in \{0, 1\}$ has at most one discontinuity. Hence, by Theorem 5.3.1 and (5.2.1),

$$\mathrm{vol}(\Gamma) = \int_{\prod_1^N [a_j,b_j]} \left(\int_0^r \mathbf{1}_\Gamma(\mathbf{x}, y)\, dy \right) d\mathbf{x} = \int_{\prod_1^N [a_j,b_j]} f(\mathbf{x})\, d\mathbf{x}. \qquad \square$$

Theorem 5.3.3 allows us to confirm that the volume (i.e., the area) of the closed unit ball $\overline{B(\mathbf{0}, 1)}$ in $\mathbb{R}^2$ is π, the half period of the trigonometric sine function. Indeed, $\overline{B(\mathbf{0}, 1)} = H_+ \cup H_-$, where

$$H_\pm = \left\{ (x_1, x_2) : 0 \le \pm x_2 \le \sqrt{1 - x_1^2} \right\}.$$

By Theorem 5.3.3 both H_+ and H_- are Riemann measurable, and each has area

$$\int_{-1}^{1} \sqrt{(1-x^2)}\, dx = 2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \big(1-\cos 2\theta\big)\, d\theta = \frac{\pi}{2}.$$

Finally, $H_+ \cap H_- = [-1, 1] \times \{0\}$ is a rectangle with area 0. Hence, by Corollary 5.3.2, the desired conclusion follows. Moreover, because $\mathbf{1}_{\overline{B(\mathbf{0},r)}}(\mathbf{x}) = \mathbf{1}_{\overline{B(\mathbf{0},1)}}(r^{-1}\mathbf{x})$, we can use (5.1.1) and (5.1.2) to see that

$$\operatorname{vol}\big(\overline{B(\mathbf{c}, r)}\big) = \pi r^2 \tag{5.3.1}$$

for balls in $\mathbb{R}^2$.

Having defined what we mean by the volume of a set, we now define what we will mean by the integral of a function on a set. Given a bounded, Riemann measurable set Γ, we say that a bounded function $f : \Gamma \longrightarrow \mathbb{C}$ is *Riemann integrable on* Γ if the function

$$\mathbf{1}_\Gamma f \equiv \begin{cases} f & \text{on } \Gamma \\ 0 & \text{off } \Gamma \end{cases}$$

is Riemann integrable on some rectangle $\prod_{j=1}^N [a_j, b_j] \supseteq \Gamma$, in which case the *Riemann integral* of f on Γ is

$$\int_\Gamma f(\mathbf{x})\, d\mathbf{x} \equiv \int_{\prod_1^N [a_j,b_j]} \mathbf{1}_\Gamma(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}.$$

In particular, if Γ is a bounded, Riemann measurable set, then every bounded, Riemann integrable function on $\prod_{j=1}^N [a_j, b_j]$ will be is Riemann integrable on Γ. In particular, notice that if $\partial\Gamma$ is Riemann negligible and f is a bounded function of Γ that is continuous off of a Riemann negligible set, then f is Riemann integrable on Γ. Obviously, the choice of the rectangle $\prod_{j=1}^N [a_j, b_j]$ is irrelevant as long as it contains Γ.

The following simple result gives an integral version of the intermediate value theorem, Theorem 1.3.6.

Theorem 5.3.4 *Suppose that* $K \subseteq \prod_{j=1}^N [a_j, b_j]$ *is a compact, connected, Riemann measurable set. If* $f : K \longrightarrow \mathbb{R}$ *is continuous, then there exists a* $\boldsymbol{\xi} \in K$ *such that*

$$\int_K f(\mathbf{x})\, dx = f(\boldsymbol{\xi})\operatorname{vol}(K).$$

Proof If $\operatorname{vol}(K) = 0$ there is nothing to do. Now assume that $\operatorname{vol}(K) > 0$. Then $\frac{1}{\operatorname{vol}(K)} \int_K f(\mathbf{x})\, d\mathbf{x}$ lies between the minimum and maximum values that f takes on K, and therefore, by Exercise 4.5 and Lemma 4.1.2, there exists a $\boldsymbol{\xi} \in K$ such that $f(\boldsymbol{\xi}) = \frac{1}{\operatorname{vol}(K)} \int_K f(\mathbf{x})\, d\mathbf{x}$. □

5.4 Integration of Rotationally Invariant Functions

One of the reasons for our introducing the concepts in the preceding section is that they encourage us to get away from rectangles when computing integrals. Indeed, if $\Gamma = \bigcup_{m=0}^{n} \Gamma_m$ where the Γ_m's are bounded Riemann measurable sets, then, for any bounded, Riemann integrable function f,

$$\int_{\Gamma} f(\mathbf{x})\,d\mathbf{x} = \sum_{m=0}^{n} \int_{\Gamma_m} f(\mathbf{x})\,d\mathbf{x} \quad \text{if } \mathrm{vol}\big(\Gamma_m \cap \Gamma_{m'}\big) = 0 \text{ for } m' \neq m, \tag{5.4.1}$$

since

$$0 \leq \sum_{m=0}^{n} \mathbf{1}_{\Gamma_m} f - \mathbf{1}_{\Gamma} f \leq 2\|f\|_{\mathrm{u}} \sum_{0 \leq m < m' \leq n} \mathbf{1}_{\Gamma_m \cap \Gamma_{m'}}.$$

The advantage afforded by (5.4.1) is that a judicious choice of the Γ_m's can simplify computations. For example, suppose that f is a function on the closed ball $\overline{B(\mathbf{0}, r)}$ in $\mathbb{R}^2$, and assume that $f(\mathbf{x}) = \tilde{f}(|\mathbf{x}|)$ for some continuous function $\tilde{f} : [0, r] \longrightarrow \mathbb{C}$. For each $n \geq 1$, set $\Gamma_{0,n} = \{\mathbf{0}\}$ and $\Gamma_{m,n} = \overline{B\big(\mathbf{0}, \frac{mr}{n}\big)} \setminus \overline{B\big(\mathbf{0}, \frac{(m-1)r}{n}\big)}$ if $1 \leq m \leq n$. By Corollary 5.3.2 and the considerations leading up to (5.3.1), we know that the $\Gamma_{m,n}$'s are Riemann measurable, and, obviously, for each $n \geq 1$ they are a cover of $\overline{B(\mathbf{0}, r)}$ by mutually disjoint sets. If we define

$$f_n(\mathbf{x}) = \sum_{m=0}^{n} \tilde{f}\left(\frac{(2m-1)r}{2n}\right) \mathbf{1}_{\Gamma_{m,n}},$$

then f_n is Riemann measurable, $f_n \longrightarrow f$ uniformly, and therefore

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = \lim_{n\to\infty} \int_{\overline{B(\mathbf{0},r)}} f_n(\mathbf{x})\,d\mathbf{x} = \lim_{n\to\infty} \sum_{m=1}^{n} \tilde{f}\big(\tfrac{(2m-1)r}{2n}\big) \mathrm{vol}(\Gamma_{m,n}).$$

Finally, by Corollary 5.3.2 and (5.3.1), $\mathrm{vol}(\Gamma_{m,n}) = \frac{(2m-1)\pi r^2}{n^2}$, and so

$$\int_{\Gamma_m} f_n(\mathbf{x})\,d\mathbf{x} = \frac{2\pi r}{n} \sum_{m=1}^{n} \tilde{f}\big(\tfrac{(2m-1)r}{2n}\big) \tfrac{(2m-1)r}{2n} = 2\pi \mathcal{R}(g; \mathcal{C}_n, \Xi_n),$$

where $g(\rho) = \rho \tilde{f}(\rho)$, $\mathcal{C}_n = \big\{\big[\frac{(m-1)r}{n}, \frac{mr}{n}\big] : 1 \leq m \leq n\big\}$ and $\Xi_n\big(\big[\frac{(m-1)r}{n}, \frac{mr}{n}\big]\big) = \frac{(2m-1)r}{n}$. Hence, we have now proved that

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = 2\pi \int_0^r \tilde{f}(\rho)\rho\,d\rho \quad \text{if } f(\mathbf{x}) = \tilde{f}(|\mathbf{x}|) \tag{5.4.2}$$

when $\tilde{f} : [0, r] \longrightarrow \mathbb{C}$ is continuous. The preceding is an example of how, by taking advantage of symmetry properties, one can sometimes reduce the computation of an integral in higher dimensions to one in lower dimensions. In this example the symmetry was the rotational invariance of both the region of integration and the integrand.

Here is a beautiful application of (5.4.2) to a famous calculation. It is known that the function $x \rightsquigarrow e^{-\frac{x^2}{2}}$ does not admit an indefinite integral that can be written as a concatenation of polynomials, trigonometric functions, and exponentials. Nonetheless, by combining (5.2.1) with (5.4.2), we will now show that

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}. \tag{5.4.3}$$

Given $r > 0$, use (5.2.3) to write

$$\left(\int_{-r}^{r} e^{-\frac{x^2}{2}}\, dx\right)^2 = \int_{-r}^{r} \left(\int_{-r}^{r} e^{-\frac{x_1^2+x_2^2}{2}}\, dx_1\right) dx_2 = \int_{[-r,r]^2} e^{-\frac{|\mathbf{x}|^2}{2}}\, d\mathbf{x}.$$

Next observe that

$$\int_{\overline{B(\mathbf{0},\sqrt{2}r)}} e^{-\frac{|\mathbf{x}|^2}{2}}\, d\mathbf{x} \geq \int_{[-r,r]^2} e^{-\frac{|\mathbf{x}|^2}{2}}\, d\mathbf{x} \geq \int_{\overline{B(\mathbf{0},r)}} e^{-\frac{|\mathbf{x}|^2}{2}}\, d\mathbf{x},$$

and that, by (5.4.2),

$$\int_{\overline{B(\mathbf{0},R)}} e^{-\frac{|\mathbf{x}|^2}{2}}\, d\mathbf{x} = 2\pi \int_0^R e^{-\frac{\rho^2}{2}} \rho\, d\rho = 2\pi\big(1 - e^{-\frac{R^2}{2}}\big).$$

Thus, after letting $r \to \infty$, we arrive at (5.4.3). Once one has (5.4.3), there are lots of other computations which follow. For example, one can compute (cf. Exercise 3.3) $\Gamma\big(\frac{1}{2}\big) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$. To this end, make the change of variables $y = (2x)^{\frac{1}{2}}$ to see that

$$\begin{aligned}\Gamma\big(\tfrac{1}{2}\big) &= \lim_{r\to\infty} \int_{r^{-1}}^{r} x^{-\frac{1}{2}} e^{-x}\, dx = \lim_{r\to\infty} 2^{\frac{1}{2}} \int_{(2r)^{-\frac{1}{2}}}^{(2r)^{\frac{1}{2}}} e^{-\frac{y^2}{2}}\, dy \\ &= 2^{\frac{1}{2}} \int_{[0,\infty)} e^{-\frac{y^2}{2}}\, dy = 2^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dy,\end{aligned}$$

and conclude that

$$\Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}. \tag{5.4.4}$$

We will now develop the N-dimensional analog of (5.4.2) for other $N \geq 1$. Obviously, the 1-dimensional analog is simply the statement that

$$\int_{-r}^{r} f(x)\,dx = 2\int_{0}^{r} f(\rho)\,d\rho$$

for even functions f on $[-r, r]$. Thus, assume that $N \geq 3$, and begin by noting that the closed ball $\overline{B(\mathbf{0}, r)}$ of radius $r \geq 0$ centered at the origin is Riemann measurable. Indeed, $\overline{B(\mathbf{0}, r)}$ is the union of the hemispheres

$$H_{+} \equiv \left\{\mathbf{x} : 0 \leq x_N \leq \sqrt{\sum_{j=1}^{N-1} x_j^2}\right\} \text{ and } H_{-} \equiv \left\{\mathbf{x} : -\sqrt{\sum_{j=1}^{N-1} x_j^2} \leq x_N \leq 0\right\},$$

and so, by Theorem 5.3.3 and Corollary 5.3.2, $\overline{B(\mathbf{0}, r)}$ is Riemann measurable. Further, by (5.1.2) and (5.1.1), for any $\mathbf{c} \in \mathbb{R}^N$, $\overline{B(\mathbf{c}, r)}$ is Riemann measurable and $\mathrm{vol}\big(\overline{B(\mathbf{c}, r)}\big) = \mathrm{vol}\big(\overline{B(\mathbf{0}, r)}\big)\Omega_N r^N$, where Ω_N is the volume of the closed unit ball $\overline{B(\mathbf{0}, 1)}$ in $\mathbb{R}^N$.

Proceeding in precisely the same way as we did in the derivation of (5.4.2) and using the identity $b^N - a^N = (b-a)\sum_{k=0}^{N-1} a^k b^{N-1-k}$, we see that, for any continuous $\tilde{f} : [0, r] \longrightarrow \mathbb{C}$,

$$\int_{\overline{B(\mathbf{0},r)}} \tilde{f}(|\mathbf{x}|)\,d\mathbf{x} = \lim_{n\to\infty} \frac{N\Omega_N r}{n} \lim_{n\to\infty} \sum_{m=1}^{n} \tilde{f}\big(\xi_{m,n}\big)\xi_{m,n}^{N-1},$$

where

$$\xi_{m,n} = \frac{r}{n}\left(\frac{1}{N}\sum_{k=0}^{N-1} m^k (m-1)^{N-1-k}\right)^{\frac{1}{N-1}} \in \left[\tfrac{(m-1)r}{n}, \tfrac{mr}{n}\right],$$

and conclude from this that

$$\int_{\overline{B(\mathbf{0},r)}} \tilde{f}(|\mathbf{x}|)\,d\mathbf{x} = N\Omega_N \int_0^r \tilde{f}(\rho)\rho^{N-1}\,d\rho. \tag{5.4.5}$$

Combining (5.4.5) with (5.4.3), we get an expression for Ω_N. By the same reasoning as we used to derive (5.4.3), one finds that

$$(2\pi)^{\frac{N}{2}} = \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx\right)^N = \lim_{r\to\infty}\int_{B(\mathbf{0},r)} e^{-\frac{|\mathbf{x}|^2}{2}}\,d\mathbf{x} = N\Omega_N \int_0^\infty \rho^{N-1}e^{-\frac{\rho^2}{2}}\,d\rho.$$

Next make the change of variables $\rho = (2t)^{\frac{1}{2}}$ to see that

$$\int_{[0,\infty)} \rho^{N-1}e^{-\frac{\rho^2}{2}}\,d\rho = 2^{\frac{N}{2}-1}\int_0^\infty t^{\frac{N}{2}-1}e^{-t}\,dt = 2^{\frac{N}{2}-1}\Gamma\big(\tfrac{N}{2}\big).$$

Thus, we now know that

$$\Omega_N = \frac{2\pi^{\frac{N}{2}}}{N\Gamma\big(\frac{N}{2}\big)} = \frac{\pi^{\frac{N}{2}}}{\Gamma\big(\frac{N}{2}+1\big)}.$$

By (5.4.4) and induction on N,

$$\Gamma\big(\tfrac{2N+1}{2}\big) = \pi^{\frac{1}{2}} 2^{-N} \prod_{k=1}^{N} (2k-1) = \pi^{\frac{1}{2}} \frac{(2N)!}{4^N N!},$$

and therefore

$$\Omega_{2N} = \frac{\pi^N}{N!} \quad \text{and} \quad \Omega_{2N-1} = \frac{4^N \pi^{N-1} N!}{(2N)!} \quad \text{for } N \ge 1.$$

Applying (3.2.4), we find that

$$\Omega_{2N} \sim (2\pi N)^{-\frac{1}{2}} \left(\frac{\pi e}{N}\right)^N \quad \text{and} \quad \Omega_{2N-1} \sim (\sqrt{2}\pi)^{-1} \left(\frac{\pi e}{N}\right)^N.$$

Thus, as N gets large, Ω_N, the volume of the unit ball in $\mathbb{R}^N$, is tending to 0 at a very fast rate. Seeing as the volume of the cube $[-1,1]^N$ that circumscribes $\overline{B(\mathbf{0},1)}$ has volume 2^N, this means that the 2^N corners of $[-1,1]^N$ that are not in $\overline{B(\mathbf{0},1)}$ take up the lion's share of the available space. Hence, if we lived in a large dimensional universe, the biblical tradition that a farmer leave the corners of his field to be harvested by the poor would be very generous.

5.5 Rotation Invariance of Integrals

Because they fit together so nicely, thus far we have been dealing exclusively with rectangles whose sides are parallel to the standard coordinate axes. However, this restriction obscures a basic property of integrals, the property of *rotation invariance*. To formulate this property, recall that $(\mathbf{e}_1, \dots, \mathbf{e}_N) \in (\mathbb{R}^N)^N$ is called an *orthonormal basis* in $\mathbb{R}^N$ if $(\mathbf{e}_i, \mathbf{e}_j)_{\mathbb{R}^N} = \delta_{i,j}$. The standard orthonormal basis $(\mathbf{e}_1^0, \dots, \mathbf{e}_N^0)$ is the one for which $(\mathbf{e}_i^0)_j = \delta_{i,j}$, but there are many others. For example, in $\mathbb{R}^2$, for each $\theta \in [0, 2\pi)$, $\big((\cos\theta, \sin\theta), (\mp\sin\theta, \pm\cos\theta)\big)$ is an orthonormal basis, and every orthonormal basis in $\mathbb{R}^2$ is one of these.

A rotation[4] in $\mathbb{R}^N$ is a map $\mathfrak{R} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ of the form $\mathfrak{R}(\mathbf{x}) = \sum_{j=1}^N x_j \mathbf{e}_j$ where $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is an orthonormal basis. Obviously $\mathfrak{R}$ is linear in the sense that

[4]The terminology that I am using here is slightly inaccurate. The term *rotation* should be reserved for $\mathfrak{R}$'s for which the determinant of the matrix $\big(((\mathbf{e}_i, \mathbf{e}_j^0)_{\mathbb{R}^N})\big)$ is 1, and I have not made a distinction

$$\mathfrak{R}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathfrak{R}(\mathbf{x}) + \beta\mathfrak{R}(\mathbf{y}).$$

In addition, $\mathfrak{R}$ preserves inner products: $\big(\mathfrak{R}(\mathbf{x}), \mathfrak{R}(\mathbf{y})\big)_{\mathbb{R}^N} = (\mathbf{x}, \mathbf{y})_{\mathbb{R}^N}$. To check this, simply note that

$$\big(\mathfrak{R}(\mathbf{x}), \mathfrak{R}(\mathbf{y})\big)_{\mathbb{R}^N} = \sum_{i,j=1}^{N} x_i y_j (\mathbf{e}_i, \mathbf{e}_j)_{\mathbb{R}^N} = \sum_{i=1}^{N} x_i y_i = (\mathbf{x}, \mathbf{y})_{\mathbb{R}^N}.$$

In particular, $|\mathfrak{R}(\mathbf{y}) - \mathfrak{R}(\mathbf{x})| = |\mathbf{y} - \mathbf{x}|$, and so it is clear that $\mathfrak{R}$ is one-to-one and continuous. Further, if $\mathfrak{R}$ and $\mathfrak{R}'$ are rotations, then so is $\mathfrak{R}' \circ \mathfrak{R}$. Indeed, if $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is the orthonormal basis for $\mathfrak{R}$, then

$$\mathfrak{R}' \circ \mathfrak{R}(\mathbf{x}) = \sum_{j=1}^{N} x_j \mathfrak{R}'(\mathbf{e}_j),$$

and, since

$$\big(\mathfrak{R}'(\mathbf{e}_i), \mathfrak{R}'(\mathbf{e}_j)\big)_{\mathbb{R}^N} = (\mathbf{e}_i, \mathbf{e}_j)_{\mathbb{R}^N} = \delta_{i,j},$$

$\big(\mathfrak{R}'(\mathbf{e}_1), \dots, \mathfrak{R}'(\mathbf{e}_N)\big)$ is an orthonormal basis. Finally, if $\mathfrak{R}$ is a rotation, then there is a unique rotation $\mathfrak{R}^{-1}$ such that $\mathfrak{R} \circ \mathfrak{R}^{-1} = \mathbf{I} = \mathfrak{R}^{-1} \circ \mathfrak{R}$, where $\mathbf{I}$ is the identity map: $\mathbf{I}(\mathbf{x}) = \mathbf{x}$. To see this, let $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ be the orthonormal bases for $\mathfrak{R}$, and set $\tilde{\mathbf{e}}_i = \big((\mathbf{e}_1)_i, \dots, (\mathbf{e}_N)_i\big)$ for $1 \le i \le N$. Using $(\mathbf{e}_1^0, \dots, \mathbf{e}_N^0)$ to denote the standard orthonormal basis, we have that $(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j)_{\mathbb{R}^N}$ equals

$$\sum_{k=1}^{N} (\mathbf{e}_k)_i (\mathbf{e}_k)_j = \sum_{k=1}^{N} (\mathbf{e}_k, \mathbf{e}_i^0)_{\mathbb{R}^N} (\mathbf{e}_k, \mathbf{e}_j^0)_{\mathbb{R}^N} = \big(\mathfrak{R}(\mathbf{e}_i^0), \mathfrak{R}(\mathbf{e}_j^0)\big)_{\mathbb{R}^N} = \delta_{i,j},$$

and so $(\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_N)$ is an orthonormal basis. Moreover, if $\tilde{\mathfrak{R}}$ is the corresponding rotation, then

$$\begin{aligned}
\tilde{\mathfrak{R}} \circ \mathfrak{R}(\mathbf{x}) &= \sum_{i=1}^{N} x_i \tilde{\mathfrak{R}}(\mathbf{e}_i) = \sum_{i,j=1}^{N} x_i (\mathbf{e}_i, \mathbf{e}_j^0)_{\mathbb{R}^N} \tilde{\mathbf{e}}_j \\
&= \sum_{i,j,k=1}^{N} x_i (\mathbf{e}_i, \mathbf{e}_j^0)_{\mathbb{R}^N} (\mathbf{e}_k, \mathbf{e}_j^0)_{\mathbb{R}^N} \mathbf{e}_k^0 = \sum_{i,k=1}^{N} x_i (\mathbf{e}_i, \mathbf{e}_k)_{\mathbb{R}^N} \mathbf{e}_k^0 = \mathbf{x}.
\end{aligned}$$

A similar computation shows that $\mathfrak{R} \circ \tilde{\mathfrak{R}} = \mathbf{I}$, and so we can take $\mathfrak{R}^{-1} = \tilde{\mathfrak{R}}$.

(Footnote 4 continued)
between them and those for which it is -1. That is, I am calling all orthogonal transformations rotations.

Because $\mathfrak{R}$ preserves lengths, it is clear that $\mathfrak{R}\big(\overline{B(\mathbf{c}, r)}\big) = \overline{B\big(\mathfrak{R}(\mathbf{c}), r\big)}$. $\mathfrak{R}$ also takes a rectangle into a rectangle, but unfortunately the image rectangle may no longer have sides parallel to the standard coordinate axes. Instead, they are parallel to the axes for the corresponding orthonormal basis. That is,

$$(*) \qquad \mathfrak{R}\left(\prod_{j=1}^{N}[a_j, b_j]\right) = \left\{\sum_{j=1}^{N} x_j \mathbf{e}_j : \mathbf{x} \in \prod_{j=1}^{N}[a_j, b_j]\right\}.$$

Of course, we should expect that $\text{vol}\left(\mathfrak{R}\left(\prod_{j=1}^{N}[a_j, b_j]\right)\right) = \prod_{j=1}^{N}(b_j - a_j)$, but this has to be checked, and for that purpose we will need the following lemma.

Lemma 5.5.1 *Let G be a non-empty, bounded, open subset of $\mathbb{R}^N$, and assume that*

$$\lim_{r \searrow 0} \big|(\partial G)^{(r)}\big|_{\text{i}} = 0$$

where $(\partial G)^{(r)}$ is the set of $\mathbf{y}$ for which there exists an $\mathbf{x} \in \partial G$ such that $|\mathbf{y} - \mathbf{x}| < r$. Then $\bar{G}$ is Riemann measurable and, for each $\epsilon > 0$, there exists a finite set $\mathcal{B}$ of mutually disjoint closed balls $\bar{B} \subseteq G$ such that $\text{vol}(\bar{G}) \leq \sum_{B \in \mathcal{B}} \text{vol}(\bar{B}) + \epsilon$.

Proof First note that ∂G is Riemann negligible and therefore that $\bar{G}$ is Riemann measurable. Next, given a closed cube $Q = \prod_{j=1}^{N}[c_j - r, c_j + r]$, let $\bar{B}_Q$ be the closed ball $\overline{B\big(\mathbf{c}, \frac{r}{2}\big)}$.

For each $n \geq 1$, let $\mathcal{K}_n$ be the collection of closed cubes Q of the form $2^{-n}\mathbf{k} + [0, 2^{-n}]^N$, where $\mathbf{k} \in \mathbb{Z}^N$. Obviously, for each n, the cubes in $\mathcal{K}_n$ are non-overlapping and $\mathbb{R}^N = \bigcup_{Q \in \mathcal{K}_n} Q$.

Now choose n_1 so that $|(\partial G)^{(2^{\frac{N}{2}-n_1})}|_{\text{i}} \leq \frac{1}{2}\text{vol}(\bar{G})$, and set

$$\mathcal{C}_1 = \{Q \in \mathcal{K}_{n_1} : Q \subseteq G\} \text{ and } \mathcal{C}_1' = \{Q \in \mathcal{K}_{n_1} : Q \cap G \neq \emptyset\}.$$

Then $\bar{G} \subseteq \bigcup_{Q \in \mathcal{C}_1'} Q$, $\bigcup_{Q \in \mathcal{C}_1' \setminus \mathcal{C}_1} Q \subseteq (\partial G)^{(2^{\frac{N}{2}-n_1})}$, and therefore

$$\sum_{Q \in \mathcal{C}_1} |Q| = \sum_{Q \in \mathcal{C}_1'} |Q| - \sum_{Q \in \mathcal{C}_1' \setminus \mathcal{C}_1} |Q| \geq \text{vol}(\bar{G}) - \frac{\text{vol}(\bar{G})}{2} = \frac{\text{vol}(\bar{G})}{2}.$$

Clearly the $\bar{B}_Q$'s for $Q \in \mathcal{C}_1$ are mutually disjoint, closed balls contained in G. Furthermore, $\text{vol}(\bar{B}_Q) = \alpha|Q|$, where $\alpha \equiv 4^{-N}\Omega_N$, and therefore

$$\text{vol}\left(G \setminus \bigcup_{Q \in \mathcal{C}_1} \bar{B}_Q\right) = \text{vol}(G) - \sum_{Q \in \mathcal{C}_1} \text{vol}(\bar{B}_Q) = \text{vol}(G) - \alpha \sum_{Q \in \mathcal{C}_1} |Q| \leq \beta \text{vol}(G),$$

where $\beta \equiv 1 - \frac{\alpha}{2}$. Finally, set $\mathcal{B}_1 = \{\bar{B}_Q : Q \in \mathcal{C}_1\}$.

Set $G_1 = G \setminus \bigcup_{\bar{B} \in \mathcal{B}_1} \bar{B}$. Then G_1 is again a non-empty, bounded, open set. Furthermore, since (cf. Exercise 4.1) $\partial G_1 \subseteq \partial G \cup \bigcup_{\bar{B} \in \mathcal{B}_1} \partial \bar{B}$, it is easy to see that $\lim_{r \searrow 0} \big|(\partial G_1)^{(r)}\big|_{\mathrm{i}} = 0$. Hence we can apply the same argument to G_1 and thereby produce a set $\mathcal{B}_2 \supseteq \mathcal{B}_1$ of mutually disjoint, closed balls $\bar{B}$ such that $\bar{B} \subseteq G_1$ for $\bar{B} \in \mathcal{B}_2 \setminus \mathcal{B}_1$ and

$$\mathrm{vol}\left(G \setminus \bigcup_{\bar{B} \in \mathcal{B}_2} \bar{B}\right) = \mathrm{vol}\left(G_1 \setminus \bigcup_{\bar{B} \in \mathcal{B}_2 \setminus \mathcal{B}_1} B_k\right) \le \beta \mathrm{vol}(G_1) \le \beta^2 \mathrm{vol}(G).$$

After m iterations, we produce a collection $\mathcal{B}_m$ of mutually disjoint closed balls $\bar{B} \subseteq G$ such that $\mathrm{vol}\left(G \setminus \bigcup_{\bar{B} \in \mathcal{B}_m} \bar{B}\right) \le \beta^m \mathrm{vol}(G)$. Thus, all that remains is to choose m so that $\beta^m \mathrm{vol}(G) < \epsilon$ and then do m iterations. □

Lemma 5.5.2 *If R is a rectangle and $\mathfrak{R}$ is a rotation, then $\mathfrak{R}(R)$ is Riemann measurable and has the same volume as R.*

Proof It is obvious that $\mathrm{int}(R)$ satisfies the hypotheses of Lemma 5.5.1, and, by using $(*)$, it is easy to check that $\mathrm{int}\big(\mathfrak{R}(R)\big)$ does also.

Next assume that $G \equiv \mathrm{int}(R) \ne \emptyset$. Clearly G satisfies the hypotheses of Lemma 5.5.1, and therefore for each $\epsilon > 0$ we can find a collection $\mathcal{B}$ of mutually disjoint closed balls $\bar{B} \subseteq G$ such that $\sum_{\bar{B} \in \mathcal{B}} \mathrm{vol}(\bar{B}) + \epsilon \ge \mathrm{vol}(\bar{G}) = \mathrm{vol}(R)$. Thus, if $\mathcal{B}' = \{\mathfrak{R}(\bar{B}) : \bar{B} \in \mathcal{B}\}$, then $\mathcal{B}'$ is a collection of mutually disjoint closed balls $\bar{B}' \subseteq \mathfrak{R}(R)$ such that

$$\mathrm{vol}(R) \le \sum_{\bar{B} \in \mathcal{B}} \mathrm{vol}(\bar{B}) + \epsilon = \sum_{\bar{B}' \in \mathcal{B}'} \mathrm{vol}(\bar{B}') + \epsilon \le \mathrm{vol}\big(\mathfrak{R}(R)\big) + \epsilon,$$

and so $\mathrm{vol}\big(\mathfrak{R}(R)\big) \ge |R|$. To prove that this inequality is an equality, apply the same line of reasoning to $G' = \mathrm{int}\big(\mathfrak{R}(R)\big)$ and $\mathfrak{R}^{-1}$ acting on $\mathfrak{R}(R)$, and thereby obtain

$$\mathrm{vol}(R) = \mathrm{vol}\big(\mathfrak{R}^{-1} \circ \mathfrak{R}(R)\big) \ge \mathrm{vol}\big(\mathfrak{R}(R)\big).$$

Finally, if $R = \emptyset$ there is nothing to do. On the other hand, if $R \ne \emptyset$ but $\mathrm{int}(R) = \emptyset$, for each $\epsilon > 0$ let $R(\epsilon)$ be the set of points $\mathbf{y} \in \mathbb{R}^N$ such that $\max_{1 \le j \le N} |y_j - x_j| \le \epsilon$ for some $\mathbf{x} \in R$. Then $R(\epsilon)$ is a closed rectangle with non-empty interior containing R, and so $\mathrm{vol}\big(\mathfrak{R}(R)\big) \le \mathrm{vol}\big(\mathfrak{R}(R(\epsilon))\big) = |R(\epsilon)|$. Since $\mathrm{vol}(R) = 0 = \lim_{\epsilon \searrow 0} |R(\epsilon)|$, it follows that $\mathrm{vol}(R) = \mathrm{vol}\big(\mathfrak{R}(R)\big)$ in this case also. □

Theorem 5.5.3 *If Γ is a bounded, Riemann measurable subset and $\mathfrak{R}$ is a rotation, then $\mathfrak{R}(\Gamma)$ is Riemann measurable and* $\mathrm{vol}\big(\mathfrak{R}(\Gamma)\big) = \mathrm{vol}(\Gamma)$.

Proof Given $\epsilon > 0$, choose $\mathcal{C}_1$ to be a collection of non-overlapping rectangles contained in Γ such that $\mathrm{vol}(\Gamma) \le \sum_{R\in\mathcal{C}_1}|R| + \epsilon$, and choose $\mathcal{C}_2$ to be a cover of Γ by non-overlapping rectangles such that $\mathrm{vol}(\Gamma) \ge \sum_{R\in\mathcal{C}_2}|R| - \epsilon$. Then

$$\begin{aligned}|\mathfrak{R}(\Gamma)|_{\mathrm{i}} &\ge \sum_{R\in\mathcal{C}_1}\mathrm{vol}\big(\mathfrak{R}(R)\big) = \sum_{R\in\mathcal{C}_1}|R| \ge \mathrm{vol}(\Gamma) - \epsilon \ge \sum_{R\in\mathcal{C}_2}|R| - 2\epsilon \\ &= \sum_{R\in\mathcal{C}_2}\mathrm{vol}\big(\mathfrak{R}(R)\big) - 2\epsilon \ge |\mathfrak{R}(\Gamma)|_{\mathrm{e}} - 2\epsilon.\end{aligned}$$

Hence, $|\mathfrak{R}(\Gamma)|_{\mathrm{e}} \le \mathrm{vol}(\Gamma) + 2\epsilon$ and $|\mathfrak{R}(\Gamma)|_{\mathrm{i}} \ge \mathrm{vol}(\Gamma) - 2\epsilon$ for all $\epsilon > 0$. □

Corollary 5.5.4 *Let $f : \overline{B(0,r)} \longrightarrow \mathbb{C}$ be a bounded function that is continuous off of a Riemann negligible set. Then, for each rotation $\mathfrak{R}$, $f \circ \mathfrak{R}$ is continuous off of a Riemann negligible set and*

$$\int_{\overline{B(\mathbf{0},r)}} f\circ\mathfrak{R}(\mathbf{x})\,d\mathbf{x} = \int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x}.$$

Proof Without loss in generality, we will assume throughout that f is real-valued.

If D is a Riemann negligible set off of which f is continuous, then $\mathfrak{R}^{-1}(D)$ contains the set where $f\circ\mathfrak{R}$ is discontinuous. Hence, since $\mathrm{vol}\big(\mathfrak{R}^{-1}(D)\big) = \mathrm{vol}(D) = 0$, $f \circ \mathfrak{R}$ is continuous off of a Riemann negligible set.

Set $g = \mathbf{1}_{\overline{B(\mathbf{0},r)}} f$. Then, by the preceding, both g and $g \circ \mathfrak{R}$ are Riemann integrable. By (5.4.1), for any cover $\mathcal{C}$ of $[-r, r]^N$ by non-overlapping rectangles and any associated choice function $\boldsymbol{\Xi}$,

$$\begin{aligned}\int_{\overline{B(\mathbf{0},r)}} f\circ\mathfrak{R}(\mathbf{x})\,d\mathbf{x} &= \sum_{R\in\mathcal{C}}\int_{\mathfrak{R}^{-1}(R)} g\circ\mathfrak{R}(\mathbf{x})\,d\mathbf{x} \\ &= \mathcal{R}(g;\mathcal{C},\boldsymbol{\Xi}) + \sum_{R\in\mathcal{C}}\int_{\mathfrak{R}^{-1}(R)}\Delta_R(\mathbf{x})\,d\mathbf{x},\end{aligned}$$

where $\Delta_R(\mathbf{x}) = g(\mathbf{x}) - g\big(\boldsymbol{\Xi}(R)\big)$. Since $\mathcal{R}\big(g;\mathcal{C},\boldsymbol{\Xi}\big)$ tends to $\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x}$ as $\|\mathcal{C}\| \to 0$, what remains to be shown is that the final term tends to 0 as $\|\mathcal{C}\| \to 0$. But $|\Delta_R(\mathbf{x})| \le \sup_R g - \inf_R g$ and therefore

$$\left|\sum_{R\in\mathcal{C}}\int_{\mathfrak{R}^{-1}(R)}\Delta_R(\mathbf{x})\,d\mathbf{x}\right| \le \sum_{R\in\mathcal{C}}\left(\sup_R g - \inf_R g\right)|R| = \mathcal{U}(g;\mathcal{C}) - \mathcal{L}(g;\mathcal{C}),$$

which tends to 0 as $\|\mathcal{C}\| \to 0$. □

Here is an example of the way in which one can use rotation invariance to make computations.

Lemma 5.5.5 *Let* $0 \le r_1 < r_2$ *and* $0 \le \theta_1 < \theta_2 < 2\pi$ *be given. Then the region*

$$\big\{(r\cos\theta, r\sin\theta) : (r,\theta) \in [r_1, r_2] \times [\theta_1, \theta_2]\big\}$$

has a Riemann negligible boundary and volume $\frac{r_2^2 - r_1^2}{2}(\theta_2 - \theta_1)$.

Proof Because this region can be constructed by taking the intersection of differences of balls with half spaces, its boundary is Riemann negligible. Furthermore, to compute its volume, it suffices to treat the case when $r_1 = 0$ and $r_2 = 1$, since the general case can be reduced to this one by taking differences and scaling.

Now define $u(\theta) = \mathrm{vol}\big(W(\theta)\big)$ where

$$W(\theta) \equiv \big\{(r\cos\omega, r\sin\omega) : (r,\omega) \in [0,1] \times [0,\theta]\big\}.$$

Obviously, u is a non-decreasing function of $\theta \in [0, 2\pi]$ that is equal to 0 when $\theta = 0$ and π when $\theta = 2\pi$. In addition, $u(\theta_1 + \theta_2) = u(\theta_1) + u(\theta_2)$ if $\theta_1 + \theta_2 \le 2\pi$. To see this, let $\mathfrak{R}_{\theta_1}$ be the rotation corresponding to the orthonormal basis $\big((\cos\theta_1, \sin\theta_1), (-\sin\theta_1, \cos\theta_1)\big)$, and observe that

$$W(\theta_1 + \theta_2) = W(\theta_1) \cup \mathfrak{R}_{\theta_1}\big(W(\theta_2)\big)$$

and that $\mathrm{int}\big(W(\theta_1)\big) \cap \mathrm{int}\big(\mathfrak{R}_{\theta_1}\big(W(\theta_2)\big)\big) = \emptyset$. Hence, the equality follows from the facts that the boundaries of $W(\theta_1)$ and $\mathfrak{R}\big(W(\theta_2)\big)$ are Riemann negligible and that $\mathfrak{R}_{\theta_1}\big(W(\theta_2)\big)$ has the same volume as $W(\theta_2)$. After applying this repeatedly, we get $nu\big(\frac{2\pi}{n}\big) = \pi$ and then that $u\big(\frac{2\pi m}{n}\big) = mu\big(\frac{2\pi}{n}\big)$ for $n \ge 1$ and $0 \le m \le n$. Hence, $u\big(\frac{2\pi m}{n}\big) = \frac{\pi m}{n}$ for all $n \ge 1$ and $0 \le m \le n$. Now, given any $\theta \in (0, 2\pi)$, choose $\{m_n \in \mathbb{N} : n \ge 1\}$ so that $0 \le \theta - \frac{2\pi m_n}{n} < \frac{2\pi}{n}$. Then, for all $n \ge 1$,

$$\big|u(\theta) - \tfrac{\theta}{2}\big| \le \big|u(\theta) - u\big(\tfrac{2\pi m_n}{n}\big)\big| + \big|\tfrac{\pi m_n}{n} - \tfrac{\theta}{2}\big| \le u\big(\tfrac{2\pi}{n}\big) + \tfrac{\pi}{n} \le \tfrac{2\pi}{n},$$

and so $u(\theta) = \frac{\theta}{2}$.

Finally, given any $0 \le \theta_1 < \theta_2 \le 2\pi$, set $\theta = \theta_2 - \theta_1$, and observe that $W(\theta_2)\backslash \mathrm{int}(W(\theta_1) = \mathfrak{R}_{\theta_1}\big(W(\theta)\big)$ and therefore that $W(\theta_2)\backslash\mathrm{int}(W(\theta_1)$ has the same volume as $W(\theta)$. □

5.6 Polar and Cylindrical Coordinates

Changing variables in multi-dimensional integrals is more complicated than in one dimension. From the standpoint of the theory that we have developed, the primary reason is that, in general, even linear changes of coordinates take rectangles into parallelograms that are not in general rectangles with respect to any orthonormal basis. Starting from the formula in terms of determinants for the volume of parallelograms, Jacobi worked out a general formula that says how integrals transform under continuously differentiable changes that satisfy a suitable non-degeneracy condition, but his theory relies on a familiarity with quite a lot of linear algebra and matrix theory. Thus, we will restrict our attention to changes of variables for which his general theory is not required.

We will begin with *polar coordinates* for $\mathbb{R}^2$. To every point $\mathbf{x} \in \mathbb{R}^2\backslash\{\mathbf{0}\}$ there exists a unique point $(\rho, \varphi) \in (0, \infty) \times [0, 2\pi)$ such that $x_1 = \rho \cos\varphi$ and $x_2 = \rho \sin\varphi$. Indeed, if $\rho = |\mathbf{x}|$, then $\frac{\mathbf{x}}{\rho} \in \mathbb{S}^1(\mathbf{0}, 1)$, and so φ is the distance, measured counterclockwise, one travels along $\mathbb{S}^1(\mathbf{0}, 1)$ to get from $(1, 0)$ to $\frac{\mathbf{x}}{\rho}$. Thus we can use the variables $(\rho, \varphi) \in (0, \infty) \times [0, 2\pi)$ to parameterize $\mathbb{R}^2\backslash\{\mathbf{0}\}$. We have restricted our attention to $\mathbb{R}^2\backslash\{\mathbf{0}\}$ because this parameterization breaks down at $\mathbf{0}$. Namely, $\mathbf{0} = (0\cos\varphi, 0\sin\varphi)$ for *every* $\varphi \in [0, 2\pi)$. However, this flaw will not cause us problems here.

Given a continuous function $f : \overline{B(\mathbf{0}, r)} \longrightarrow \mathbb{C}$, it is reasonable to ask whether the integral of f over $\overline{B(\mathbf{0}, r)}$ can be written as an integral with respect to the variables (ρ, φ). In fact, we have already seen in (5.4.2) that this is possible when f depends only on $|\mathbf{x}|$, and we will now show that it is always possible. To this end, for $\theta \in \mathbb{R}$, let $\mathfrak{R}_\theta$ be the rotation in $\mathbb{R}^2$ corresponding to the basis $\big((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\big)$. That is,

$$\mathfrak{R}_\theta \mathbf{x} = \big(x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta\big).$$

Using (1.5.1), it is easy to check that $\mathfrak{R}_\theta \circ \mathfrak{R}_\varphi = \mathfrak{R}_{\theta+\varphi}$. In particular, $\mathfrak{R}_{2\pi+\varphi} = \mathfrak{R}_\varphi$.

Lemma 5.6.1 *Let $f : \overline{B(\mathbf{0}, r)} \longrightarrow \mathbb{C}$ be a continuous function, and define*

$$\tilde{f}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f\big(\rho\cos\varphi, \rho\sin\varphi\big)\, d\varphi \quad \text{for } \rho \in [0, r].$$

Then, for all $\mathbf{x} \in \overline{B(\mathbf{0}, r)}$,

$$\tilde{f}(|\mathbf{x}|) = \frac{1}{2\pi}\int_0^{2\pi} f\big(\mathfrak{R}_\varphi \mathbf{x}\big)\, d\varphi.$$

Proof Set $\rho = |\mathbf{x}|$ and choose $\theta \in [0, 1)$ so that $\mathbf{x} = \big(\rho\cos(2\pi\theta), \rho\sin(2\pi\theta)\big)$. Equivalently, $\mathbf{x} = \mathfrak{R}_{2\pi\theta}(\rho, 0)$. Then by the preceding remarks about rotations in $\mathbb{R}^2$ and (3.3.3) applied to the periodic function $\xi \rightsquigarrow f\big(\mathfrak{R}_{2\pi\xi}(\rho, 0)\big)$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f\big(\mathfrak{R}_\varphi \mathbf{x}\big)\, d\varphi &= \frac{1}{2\pi}\int_0^{2\pi} f\big(\mathfrak{R}_{2\pi\theta+\varphi}(\rho, 0)\big) \\ &= \int_0^1 f\big(\mathfrak{R}_{2\pi(\theta+\varphi)}(\rho, 0)\big)\, d\varphi = \int_0^1 f\big(\mathfrak{R}_{2\pi\varphi}(\rho, 0)\big)\, d\varphi = \tilde{f}(\rho).\end{aligned}$$

□

Theorem 5.6.2 *If f is a continuous function on the ball $\overline{B(\mathbf{0},r)}$ in $\mathbb{R}^2$, then*

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = \int_0^r \rho\left(\int_0^{2\pi} f\big(\rho\cos\varphi,\rho\sin\varphi\big)\,d\varphi\right)d\rho$$
$$= \int_0^{2\pi}\left(\int_0^r f\big(\rho\cos\varphi,\rho\sin\varphi\big)\rho\,d\rho\right)d\varphi.$$

Proof By (5.4.2),

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = \int_{\overline{B(\mathbf{0},r)}} f\big(\mathfrak{R}_\varphi\mathbf{x}\big)\,d\mathbf{x}$$

for all φ. Hence, by (5.2.1), Lemma 5.6.1, and (5.4.2),

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\overline{B(\mathbf{0},r)}} f\big(\mathfrak{R}_\varphi\mathbf{x}\big)\,d\mathbf{x}\right)d\varphi$$
$$= \int_{\overline{B(\mathbf{0},r)}} \tilde{f}(|\mathbf{x}|)\,d\mathbf{x} = \int_0^r \tilde{f}(\rho)\rho\,d\rho,$$

which is the first equality. The second equality follows from the first by another application of (5.2.1). □

As a preliminary application of this theorem, we will use it to compute integrals over a *star shaped region*, a region G for which there exists a $\mathbf{c} \in \mathbb{R}^2$, known as the *center*, and a continuous function, known as the *radial function*, $r : [0, 2\pi] \longrightarrow (0,\infty)$ such that $r(0) = r(2\pi)$ and

$$G = \big\{\mathbf{c} + r\mathbf{e}(\varphi) : \varphi \in [0, 2\pi)\ \&\ r \in \big[0, r(\varphi)\big)\big\}, \tag{5.6.1}$$

where $\mathbf{e}(\varphi) \equiv (\cos\varphi, \sin\varphi)$. For instance, if G is a non-empty, bounded, convex open set, then for any $\mathbf{c} \in G$, G is star shaped with center at $\mathbf{c}$ and

$$r(\varphi) = \max\{r > 0 : \mathbf{c} + r\mathbf{e}(\varphi) \in \bar{G}\}.$$

Observe that

$$\partial G = \big\{\mathbf{c} + r(\varphi)\mathbf{e}(\varphi) : \varphi \in [0, 2\pi)\big\}.$$

and, as a consequence, we can show that ∂G is Riemann negligible. Indeed, for a given $\epsilon \in (0, 1]$ choose $n \geq 1$ so that $|r(\varphi_2) - r(\varphi_1)| < \epsilon$ if $|\varphi_2 - \varphi_1| \leq \frac{2\pi}{n}$ and, for $1 \leq m \leq n$, set

$$A_m = \big\{\mathbf{c} + \rho\mathbf{e}(\varphi) : \tfrac{2\pi(m-1)}{n} \leq \varphi < \tfrac{2\pi m}{n}\ \&\ \big|\rho - r\big(\tfrac{2\pi m}{n}\big)\big| \leq \epsilon\big\}.$$

Then $\partial G \subseteq \bigcup_{m=1}^{n} A_{m,n}$ and, by Lemma 5.5.5,

$$\mathrm{vol}(A_m) = \frac{2\pi^2}{n}\Big(\big(r\big(\tfrac{2\pi m}{n}\big)+\epsilon\big)^2 - \big(r\big(\tfrac{2\pi m}{n}\big)-\epsilon\big)^2\Big) \le \frac{8\pi^2\|r\|_{[0,2\pi]}\epsilon}{n},$$

and therefore there is a constant $K < \infty$ such that $|\partial G|_e \le K\epsilon$ for all $\epsilon \in (0,1]$. Finally, notice that G is path connected and therefore, by Exercise 4.5 is connected.

The following is a significant extension of Theorem 5.6.2.

Corollary 5.6.3 *If G is the region in (5.6.1) and $f : \bar{G} \longrightarrow \mathbb{C}$ is continuous, then*

$$\int_{\bar{G}} f(\mathbf{x})\, d\mathbf{x} = \int_0^{2\pi}\left(\int_0^{r(\varphi)} f(\mathbf{c}+\rho\mathbf{e}(\varphi))\, \rho d\rho\right) d\varphi.$$

Proof Without loss in generality, we will assume that $\mathbf{c} = \mathbf{0}$. Set $r_- = \min\{r(\varphi) : \varphi \in [0,2\pi]\}$ and $r_+ = \max\{r(\varphi) : \varphi \in [0,2\pi]\}$. Given $n \ge 1$, define $\eta_n : \mathbb{R} \longrightarrow [0,1]$ by

$$\eta_n(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{nt}{r_-} & \text{if } 0 < t \le \frac{r_-}{n} \\ 1 & \text{if } t > \frac{r_-}{n}, \end{cases}$$

and define α_n and β_n on $\mathbb{R}^2$ by

$$\alpha_n(\rho\mathbf{e}(\varphi)) = \eta_n\big(r(\varphi)-\rho\big) \text{ and } \beta_n(\rho\mathbf{e}(\varphi)) = \eta_n\big(r(\varphi)+\tfrac{r_-}{n}-\rho\big)$$

Then both α_n and β_n are continuous functions, α_n vanishes off of G and β_n equals 1 on $\bar{G}$. Finally define

$$f_n(\mathbf{x}) \equiv \begin{cases} \alpha_n(\mathbf{x}) f(\mathbf{x}) & \text{if } \mathbf{x} \in G \\ 0 & \text{if } \mathbf{x} \notin G. \end{cases}$$

Then f_n is continuous and therefore, by Theorem 5.6.2,

$$\begin{aligned}\int_{\bar{G}} f_n(\mathbf{x})\, d\mathbf{x} &= \int_{\overline{B(\mathbf{0},r_+)}} f_n(\mathbf{x})\, d\mathbf{x} = \int_0^{2\pi}\left(\int_0^{r_+} f_n(\rho\mathbf{e}(\varphi))\rho\, d\rho\right) d\varphi \\ &= \int_0^{2\pi}\left(\int_0^{r(\varphi)} f_n(\rho\mathbf{e}(\varphi))\rho\, d\rho\right) d\varphi.\end{aligned}$$

Clearly, again by Theorem 5.6.2,

$$\begin{aligned}\left|\int_{\bar G} f(\mathbf{x})\,d\mathbf{x} - \int_{\bar G} f_n(\mathbf{x})\,d\mathbf{x}\right| &\le \|f\|_{\bar G}\int_{\bar G}\big(1-\alpha_n(\mathbf{x})\big)\,d\mathbf{x}\\
&\le \|f\|_{\bar G}\int_{\overline{B(\mathbf{0},2r_+)}}\beta_n(\mathbf{x})\big(1-\alpha_n(\mathbf{x})\big)\,d\mathbf{x}\\
&= \|f\|_{\bar G}\int_0^{2\pi}\left(\int_0^{2r_+}\eta_n\big(r(\varphi)+\tfrac{r_-}{n}-\rho\big)\Big(1-\eta_n\big(r(\varphi)-\rho\big)\Big)\rho\,d\rho\right)d\varphi\\
&= \|f\|_{\bar G}\int_0^{2\pi}\left(\int_{r(\varphi)-\frac{r_-}{n}}^{r(\varphi)+\frac{r_-}{n}}\eta_n\big(r(\varphi)+\tfrac{r_-}{n}-\rho\big)\Big(1-\eta_n\big(r(\varphi)-\rho\big)\Big)\rho\,d\rho\right)d\varphi\\
&\le \frac{8\pi\|f\|_{\bar G}r_+r_-}{n}.\end{aligned}$$

At the same time,

$$\begin{aligned}&\left|\int_0^{2\pi}\left(\int_0^{r(\varphi)} f(\rho\mathbf{e}(\varphi))\rho\,d\rho\right)d\varphi - \int_0^{2\pi}\left(\int_0^{r(\varphi)} f_n(\rho\mathbf{e}(\varphi))\rho\,d\rho\right)d\varphi\right|\\
&\le \|f\|_{\bar G}\int_0^{2\pi}\left(\int_{r(\varphi)-\frac{r_-}{n}}^{r(\varphi)}\rho\,d\rho\right)d\varphi \le \frac{4\pi\|f\|_{\bar G}r_+r_-}{n}.\end{aligned}$$

Thus, the asserted equality follows after one lets $n\to\infty$. □

We turn next to *cylindrical coordinates* in $\mathbb{R}^3$. That is, we represent points in $\mathbb{R}^3$ as $(\rho\mathbf{e}(\varphi),\xi)$, where $\rho\ge 0$, $\varphi\in[0,2\pi)$, and $\xi\in\mathbb{R}$. Again the correspondence fails to be one-to-one everywhere. Namely, φ is not uniquely determined for $\mathbf{x}\in\mathbb{R}^3$ with $x_1=x_2=0$, but, as before, this will not prevent us from representing integrals in terms of the variables (ρ,φ,ξ).

Theorem 5.6.4 *Let $\psi:[a,b]\longrightarrow[0,\infty)$ be a continuous function, and set*

$$\Gamma=\{\mathbf{x}\in\mathbb{R}^3 : x_3\in[a,b]\ \&\ x_1^2+x_2^2\le\psi(x_3)^2\}.$$

Then Γ is Riemann measurable and

$$\int_\Gamma f(\mathbf{x})\,d\mathbf{x}=\int_a^b\left(\int_0^{\psi(\xi)}\rho\left(\int_0^{2\pi} f\big(\rho\mathbf{e}(\varphi),\xi\big)\,d\varphi\right)d\rho\right)d\xi$$

for any continuous function $f:\Gamma\longrightarrow\mathbb{C}$.

Proof Given $n\ge 1$, define $c_{m,n}=\big(1-\frac{m}{n}\big)a+\frac{m}{n}b$ for $0\le m\le n$, and set $I_{m,n}=[c_{m-1,n},c_{m,n}]$ and $\Gamma_{m,n}=\{\mathbf{x}\in\Gamma : x_3\in I_{m,n}\}$ for $1\le m\le n$. Next, for each $1\le m\le n$, set $\kappa_{m,n}=\min_{I_{m,n}}\psi$, $K_{m,n}=\max_{I_{m,n}}\psi$, and

$$D_{m,n}=\{\mathbf{x} : \kappa_{m,n}^2\le x_1^2+x_2^2\le K_{m,n}^2\ \&\ x_3\in I_{m,n}\}.$$

To see that Γ is Riemann measurable, we will show that its boundary is Riemann negligible. Indeed, $\partial\Gamma_{m,n} \subseteq D_{m,n}$, and therefore, by Theorem 5.2.1 and Lemma 5.5.5, $|\partial\Gamma_{m,n}|_e \le \frac{\pi(K_{m,n}^2-\kappa_{m,n}^2)(b-a)}{n}$. Since

$$\lim_{n\to\infty}\max_{1\le m\le n}(K_{m,n}-\kappa_{m,n}) = 0 \text{ and } |\partial\Gamma|_e \le \sum_{m=1}^{n}|\partial\Gamma_{m,n}|_e,$$

it follows that $|\partial\Gamma|_e = 0$. Of course, since each $\Gamma_{m,n}$ is a set of the same form as Γ, each of them is also Riemann measurable.

Now let f be given. Then

$$\int_\Gamma f(\mathbf{x})\,d\mathbf{x} = \sum_{m=1}^{n}\int_{\Gamma_{m,n}} f(\mathbf{x})\,d\mathbf{x} = \sum_{m=1}^{n}\int_{C_{m,n}} f(\mathbf{x})\,d\mathbf{x} - \sum_{m=1}^{n}\int_{\Gamma_{m,n}\setminus C_{m,n}} f(\mathbf{x})\,d\mathbf{x},$$

where $C_{m,n} \equiv \{\mathbf{x} : x_1^2 + x_2^2 \le \kappa_{m,n} \ \&\ x_3 \in I_{m,n}\}$. Since $\Gamma_{m,n}\setminus C_{m,n} \subseteq D_{m,n}$, the computation in the preceding paragraph shows that

$$\left|\sum_{m=1}^{n}\int_{\Gamma_{m,n}\setminus C_{m,n}} f(\mathbf{x})\,d\mathbf{x}\right| \le \frac{\|f\|_\Gamma \pi(b-a)}{n}\sum_{m=1}^{n}\left(K_{m,n}^2 - \kappa_{m,n}^2\right) \longrightarrow 0$$

as $n\to\infty$. Next choose $\xi_{m,n} \in I_{m,n}$ so that $\psi(\xi_{m,n}) = \kappa_{m,n}$, and set

$$\epsilon_n = \max_{1\le m\le n}\sup_{\mathbf{x}\in C_{m,n}} |f(\mathbf{x}) - f(x_1, x_2, \xi_{m,n})|.$$

Then

$$\left|\sum_{m=1}^{n}\int_{C_{m,n}} f(\mathbf{x})\,d\mathbf{x} - \sum_{m=1}^{n}\int_{C_{m,n}} f(x_1, x_2, \xi_{m,n})\,d\mathbf{x}\right| \le \epsilon_n \mathrm{vol}(\Gamma) \longrightarrow 0.$$

Finally, observe that $\int_{C_{m,n}} f(x_1, x_2, \xi_{m,n})\,d\mathbf{x} = \frac{b-a}{n} g(\xi_{m,n})$ where g is the continuous function on $[a, b]$ given by

$$g(\xi) \equiv \int_0^{\psi(\xi)} \rho\left(\int_0^{2\pi} f\big(\rho\mathbf{e}(\varphi), \xi\big)\,d\varphi\right)d\rho.$$

Hence, $\sum_{m=1}^{n}\int_{C_{m,n}} f(\mathbf{x})\,d\mathbf{x} = \mathcal{R}(g; \mathcal{C}_n, \Xi_n)$ where $\mathcal{C}_n = \{I_{m,n} : 1 \le m \le n\}$ and $\Xi_n(I_{m,n}) = \xi_{m,n}$. Now let $n\to\infty$ to get the desired conclusion. □

Integration over balls in $\mathbb{R}^3$ is a particularly important example to which Theorem 5.6.4 applies. Namely, take $a = -r$, $b = r$, and $\psi(\xi) = \sqrt{r^2 - \xi^2}$ for $\xi \in [-r, r]$. Then Theorem 5.6.4 says that

$$\int_{\overline{B(\mathbf{0},r)}} f(\mathbf{x})\,d\mathbf{x} = \int_{-r}^{r}\left(\int_0^{\sqrt{r^2-\xi^2}} \rho\left(\int_0^{2\pi} f\big(\rho\cos\varphi, \rho\sin\varphi, \xi\big)\,d\varphi\right)d\rho\right)d\xi. \tag{5.6.2}$$

There is a beautiful application of (5.6.2) to a famous observation made by Newton about his *law of gravitation*. According to his law, the force exerted by a particle of mass m_1 at $\mathbf{y} \in \mathbb{R}^3$ on a particle of mass m_2 at $\mathbf{b} \in \mathbb{R}^3\setminus\{\mathbf{x}\}$ is equal to

$$\frac{Gm_1m_2}{|\mathbf{y}-\mathbf{b}|^3}(\mathbf{y}-\mathbf{b}),$$

where G is the gravitational constant. Next, suppose that Ω is a bounded, closed, Riemann measurable region on which mass is continuously distributed with density μ. Then the force that the mass in Ω exerts on a particle of mass m at $\mathbf{b} \notin \Omega$ is given by

$$\int_\Omega \frac{Gm\mu(\mathbf{y})}{|\mathbf{y}-\mathbf{b}|^3}(\mathbf{y}-\mathbf{b})\,d\mathbf{y}.$$

Newton's observation was that if Ω is a ball and the mass density depends only on the distance from the center of the ball, then the force felt by a particle outside the ball is the same as the force exerted on it by a particle at the center of the ball with mass equal to the total mass of the ball. That is, if $\Omega = \overline{B(\mathbf{c}, r)}$ and $\mu : [0, r] \longrightarrow [0, \infty)$ is continuous, then for $\mathbf{b} \notin \overline{B(\mathbf{c}, r)}$,

$$\begin{aligned} \int_{\overline{B(\mathbf{c},r)}} \frac{Gm\mu(|\mathbf{y}-\mathbf{c}|)}{|\mathbf{y}-\mathbf{b}|^3}(\mathbf{y}-\mathbf{b})\,d\mathbf{y} &= \frac{GMm}{|\mathbf{c}-\mathbf{b}|^3}(\mathbf{c}-\mathbf{b}) \\ \text{where } M = \int_{\overline{B(\mathbf{c},r)}} \mu\big(|\mathbf{y}-\mathbf{c}|\big)\,d\mathbf{y}. \end{aligned} \tag{5.6.3}$$

(See Exercise 5.8 for the case when $\mathbf{b}$ lies inside the ball).

Using translation and rotations, one can reduce the proof of (5.6.3) to the case when $\mathbf{c} = \mathbf{0}$ and $\mathbf{b} = (0, 0, -D)$ for some $D > r$. Further, without loss in generality, we will assume that $Gm = 1$. Next observe that, by rotation invariance applied to the rotations that take (y_1, y_2, y_3) to $(\mp y_1, \pm y_2, y_3)$,

$$\int_{\overline{B(\mathbf{0},r)}} \frac{\mu(|\mathbf{y}|)}{|\mathbf{y}-\mathbf{b}|^3} y_i\,d\mathbf{y} = -\int_{\overline{B(\mathbf{0},r)}} \frac{\mu(|\mathbf{y}|)}{|\mathbf{y}-\mathbf{b}|^3} y_i\,d\mathbf{y}$$

and therefore

$$\int_{\overline{B(\mathbf{0},r)}} \frac{\mu(|\mathbf{y}|)}{|\mathbf{y}-\mathbf{b}|^3} y_i\,d\mathbf{y} = 0 \quad \text{for } i \in \{1, 2\}.$$

Thus, it remains to show that

$$(*) \qquad \int_{B(\mathbf{0},r)} \frac{\mu(|\mathbf{y}|)}{|\mathbf{y}-\mathbf{b}|^3} y_3 \, d\mathbf{y} = D^{-2} \int_{B(\mathbf{0},r)} \mu(|\mathbf{y}|) \, d\mathbf{y}.$$

To prove $(*)$, we apply (5.6.2) to the function

$$f(\mathbf{x}) = \frac{\mu(|\mathbf{x}|)(x_3 + D)}{\big(x_1^2 + x_2^2 + (x_3 + D)^2\big)^{\frac{3}{2}}}$$

to write the left hand side as $2\pi J$ where

$$J \equiv \int_{-r}^{r} \left(\int_0^{\sqrt{r^2-\xi^2}} \frac{\rho\mu(\sqrt{\rho^2+\xi^2})(\xi + D)}{\big(\rho^2 + (\xi + D)^2\big)^{\frac{3}{2}}} \, d\rho \right) d\xi.$$

Now make the change of variables $\sigma = \sqrt{\rho^2 + \xi^2}$ in the inner integral to see that

$$J = \int_{-r}^{r} (\xi + D) \left(\int_{|\xi|}^{r} \frac{\sigma\mu(\sigma)}{(\sigma^2 + 2\xi D + D^2)^{\frac{3}{2}}} \, d\sigma \right) d\xi,$$

and then apply (5.2.1) to obtain

$$J = \int_0^r \sigma\mu(\sigma) \left(\int_{-\sigma}^{\sigma} \frac{D + \xi}{(\sigma^2 + 2\xi D + D^2)^{\frac{3}{2}}} \, d\xi \right) d\sigma.$$

Use the change of variables $\eta = \sigma^2 + 2\xi D + D^2$ in the inner integral to write it as

$$\frac{1}{4D^2} \int_{(D-\sigma)^2}^{(D+\sigma)^2} \big(\eta^{-\frac{1}{2}} + (D^2 - \sigma^2)\eta^{-\frac{3}{2}}\big) \, d\eta = \frac{2\sigma}{D^2}.$$

Hence,

$$2\pi J = \frac{4\pi}{D^2} \int_0^r \mu(\sigma)\sigma^2 \, d\sigma.$$

Finally, note that $3\Omega_3 = 4\pi$, and apply (5.4.5) with $N = 3$ to see that

$$4\pi \int_0^r \mu(\sigma)\sigma^2 \, d\sigma = \int_{B(\mathbf{0},r)} \mu(|\mathbf{x}|) \, d\mathbf{x}.$$

We conclude this section by using (5.6.2) to derive the analog of Theorem 5.6.2 for integrals over balls in $\mathbb{R}^3$. One way to introduce polar coordinates for $\mathbb{R}^3$ is to think about the use of latitude and longitude to locate points on a globe. To begin with, one has to choose a reference axis, which in the case of a globe is chosen to be the one passing through the north and south poles. Given a point $\mathbf{q}$ on the globe, consider a plane $P_{\mathbf{q}}$ containing the reference axis that passes through $\mathbf{q}$. (There will be only one unless $\mathbf{q}$ is a pole.) Thinking of points on the globe as vectors with base at the center of the earth, the latitude of a point is the angle that $\mathbf{q}$ makes in $P_{\mathbf{q}}$ with the north pole $\mathbf{N}$. Before describing the longitude of $\mathbf{q}$, one has to choose a reference point $\mathbf{q}_0$ that is not on the reference axis. In the case of a globe, the standard choice is Greenwich, England. Then the longitude of $\mathbf{q}$ is the angle between the projections of $\mathbf{q}$ and $\mathbf{q}_0$ in the equatorial plane, the plane that passes through the center of the earth and is perpendicular to the reference axis.

Now let $\mathbf{x} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$. With the preceding in mind, we say that the *polar angle* of $\mathbf{x} = (x_1, x_2, x_3)$ is the $\theta \in [0, \pi]$ such that $\cos\theta = \frac{(\mathbf{x},\mathbf{N})_{\mathbb{R}^3}}{|\mathbf{x}|}$, where $\mathbf{N} = (0, 0, 1)$. Assuming that $\sigma = \sqrt{x_1^2 + x_2^2} > 0$, the *azimuthal angle* of $\mathbf{x}$ is the $\varphi \in [0, 2\pi)$ such that $(x_1, x_2) = (\sigma\cos\varphi, \sigma\sin\varphi)$. In other words, in terms of the globe model, we have taken the center of the earth to lie at the origin, the north pole and south poles to be $(0, 0, 1)$ and $(0, 0, -1)$, and "Greenwich" to be located at $(1, 0, 0)$. Thus the polar angle gives the latitude and the azimuthal angle gives the longitude.

The preceding considerations lead to the parameterization

$$\begin{aligned}(\rho, \theta, \varphi) &\in [0, \infty) \times [0, \pi] \times [0, 2\pi) \\ &\longmapsto \mathbf{x}_{(\rho,\theta,\varphi)} \equiv \big(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta\big) \in \mathbb{R}^3\end{aligned}$$

of points in $\mathbb{R}^3$. Assuming that $\rho > 0$, θ is the polar angle of $\mathbf{x}_{(\rho,\theta,\varphi)}$, and, assuming that $\rho > 0$ and $\theta \notin \{0, \pi\}$, φ is its azimuthal angle. On the other hand, when $\rho = 0$, then $\mathbf{x}_{(\rho,\theta,\varphi)} = \mathbf{0}$ for all $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$, and when $\rho > 0$ but $\theta \in \{0, \pi\}$, θ is uniquely determined but φ is not. In spite of these ambiguities, if $\mathbf{x} = \mathbf{x}_{(\rho,\theta,\varphi)}$, then (ρ, θ, φ) are called the *polar coordinates* of $\mathbf{x}$, and as we are about to show, integrals of functions over balls in $\mathbb{R}^3$ can be written as integrals with respect to the variables (ρ, θ, φ).

Let $f : \overline{B(\mathbf{0}, r)} \longrightarrow \mathbb{C}$ be a continuous function. Then, by (5.6.2) and (5.2.2), the integral of f over $\overline{B(\mathbf{0}, r)}$ equals

$$\int_0^{2\pi}\left(\int_{-r}^{r}\left(\int_0^{\sqrt{r^2-\xi^2}} \sigma f_\varphi\big(\sigma, \xi\big)\, d\sigma\right) d\xi\right) d\varphi,$$

where $f_\varphi(\sigma,\xi)=f\big(\sigma\cos\varphi,\sigma\sin\varphi,\xi\big)$. Observe that

$$\begin{aligned}&\int_{-r}^{r}\left(\int_0^{\sqrt{r^2-\xi^2}}\sigma f_\varphi\big(\sigma,\xi\big)\,d\sigma\right)d\xi\\&\quad=\int_0^{r}\left(\int_0^{\sqrt{r^2-\xi^2}}\sigma f_\varphi\big(\sigma,\xi\big)\,d\sigma\right)d\xi+\int_{-r}^{0}\left(\int_0^{\sqrt{r^2-\xi^2}}\sigma f_\varphi\big(\sigma,\xi\big)\,d\sigma\right)d\xi\\&\quad=\int_0^{r}\sigma\left(\int_0^{\sqrt{r^2-\sigma^2}}f_\varphi\big(\sigma,\xi\big)\,d\xi\right)d\sigma+\int_0^{r}\sigma\left(\int_0^{\sqrt{r^2-\sigma^2}}f_\varphi\big(\sigma,-\xi\big)\,d\xi\right)d\sigma,\end{aligned}$$

and make the change of variables $\xi=\sqrt{\rho^2-\sigma^2}$ to write

$$\int_0^{\sqrt{r^2-\sigma^2}}f_\varphi\big(\sigma,\pm\xi\big)\,d\xi=\int_{(\sigma,r]}f_\varphi\big(\sigma,\pm\sqrt{\rho^2-\sigma^2}\big)\frac{\rho}{\sqrt{\rho^2-\sigma^2}}\,d\rho.$$

Hence, we now know that

$$\begin{aligned}&\int_{-r}^{r}\left(\int_0^{\sqrt{r^2-\xi^2}}\sigma f_\varphi\big(\sigma,\xi\big)\,d\sigma\right)d\xi\\&\quad=\int_0^{r}\sigma\left(\int_{(\sigma,r]}\Big(f_\varphi\big(\sigma,\sqrt{\rho^2-\sigma^2}\big)+f_\varphi\big(\sigma,-\sqrt{\rho^2-\sigma^2}\big)\Big)\frac{\rho}{\sqrt{\rho^2-\sigma^2}}\,d\rho\right)d\sigma\\&\quad=\int_0^{r}\rho\left(\int_{[0,\rho)}\Big(f_\varphi\big(\sigma,\sqrt{\rho^2-\sigma^2}\big)+f_\varphi\big(\sigma,-\sqrt{\rho^2-\sigma^2}\big)\Big)\frac{\sigma}{\sqrt{\rho^2-\sigma^2}}d\sigma\right)d\rho,\end{aligned}$$

where we have made use of the obvious extension of Fubini's Theorem to integrals that are limits of Riemann integrals. Finally, use the change of variables $\sigma=\rho\sin\theta$ in the inner integral to conclude that

$$\int_{-r}^{r}\left(\int_0^{\sqrt{r^2-\xi^2}}\sigma f_\varphi\big(\sigma,\xi\big)\,d\sigma\right)d\xi=\int_0^{r}\rho^2\left(\int_0^{\pi}f_\varphi\big(\rho\sin\theta,\rho\cos\theta\big)\,d\theta\right)d\rho$$

and therefore, after an application of (5.2.2), that

$$\begin{aligned}&\int_{\overline{B(\mathbf{0},r)}}f(\mathbf{x})\,d\mathbf{x}\\&\quad=\int_0^{r}\rho^2\left(\int_0^{\pi}\left(\int_0^{2\pi}f\big(\rho\sin\theta\cos\varphi,\rho\sin\theta\sin\varphi,\rho\cos\theta\big)\,d\varphi\right)d\theta\right)d\rho.\end{aligned}\tag{5.6.4}$$

5.7 The Divergence Theorem in $\mathbb{R}^2$

Integration by parts in more than one dimension takes many forms, and in order to even state these results in generality one needs more machinery than we have developed. Thus, we will deal with only a couple of examples and not attempt to derive a general statement.

The most basic result is a simple application of The Fundamental Theorem of Calculus, Theorem 3.2.1. Namely, consider a rectangle $R = \prod_{j=1}^{N}[a_j, b_j]$, where $N \geq 2$, and let φ be a $\mathbb{C}$-valued function which is continuously differentiable on $\text{int}(R)$ and has bounded derivatives there. Given $\boldsymbol{\xi} \in \mathbb{R}^N$, one has

$$\begin{aligned} \int_R \partial_{\boldsymbol{\xi}}\varphi(\mathbf{x})\, d\mathbf{x} &= \sum_{j=1}^{N} \xi_j \left(\int_{R_j(b_j)} \varphi(\mathbf{y})\, d\sigma_{\mathbf{y}} - \int_{R_j(a_j)} \varphi(\mathbf{y})\, d\sigma_{\mathbf{y}} \right) \\ \text{where } R_j(c) &\equiv \left(\prod_{i=1}^{j-1}[a_i, b_i] \right) \times \{c\} \times \left(\prod_{i=j+1}^{N} [a_j, b_j] \right), \end{aligned} \tag{5.7.1}$$

where the integral $\int_{R_j(c)} \psi(\mathbf{y})\, d\sigma_{\mathbf{y}}$ of a function ψ over $R_j(c)$ is interpreted as the $(N-1)$-dimensional integral

$$\int_{\prod_{i \neq j}[a_i, b_i]} \psi(y_1, \ldots, y_{j-1}, c, y_{j+1}, \ldots, y_N)\, dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_N.$$

Verification of (5.7.1) is easy. First write $\partial_{\boldsymbol{\xi}}\varphi$ as $\sum_{j=1}^{N} \xi_j \partial_{\mathbf{e}_j}\varphi$. Second, use (5.2.2) with the permutation that exchanges j and 1 but leaves the ordering of the other indices unchanged, and apply Theorem 3.2.1.

In many applications one is dealing with an $\mathbb{R}^N$-valued function $\mathbf{F}$ and is integrating its *divergence*

$$\text{div}\mathbf{F} \equiv \sum_{j=1}^{N} \partial_{\mathbf{e}_j} F_j$$

over R. By applying (5.7.1) to each coordinate, one arrives at

$$\int_R \text{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x} = \sum_{j=1}^{N} \int_{R_j(b_j)} F_j(\mathbf{y})\, d\sigma_{\mathbf{y}} - \sum_{j=1}^{N} \int_{R_j(a_j)} F_j(\mathbf{y})\, d\sigma_{\mathbf{y}}, \tag{5.7.2}$$

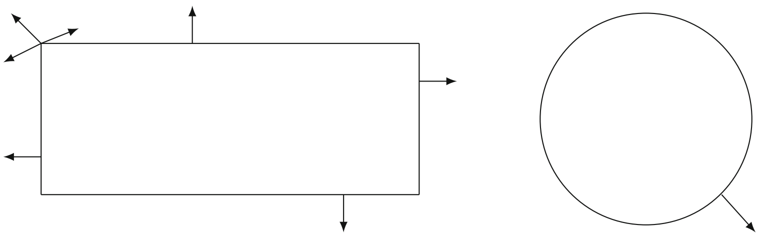
but there is a more revealing way to write (5.7.2). To explain this alternative version, let ∂G be the boundary of a bounded open subset G in $\mathbb{R}^N$. Given a point $\mathbf{x} \in \partial G$, say that $\boldsymbol{\xi} \in \mathbb{R}^N$ is a *tangent vector* to ∂G at $\mathbf{x}$ if there is a continuously differentiable path $\gamma : (-1, 1) \longrightarrow \partial G$ such that $\mathbf{x} = \gamma(0)$ and $\boldsymbol{\xi} = \dot{\gamma}(0) \equiv \frac{d\gamma}{dt}(0)$. That is, $\boldsymbol{\xi}$

is the velocity at the time when a path on ∂G passes through $\mathbf{x}$. For instance, when $G = \text{int}(R)$ and $\mathbf{x} \in R_j(a_j) \cup R_j(b_j)$ is not on one of the edges, then it is obvious that $\boldsymbol{\xi}$ is tangent to $\mathbf{x}$ if and only if $(\boldsymbol{\xi}, \mathbf{e}_j)_{\mathbb{R}^N} = 0$. If $\mathbf{x}$ is at an edge, $(\boldsymbol{\xi}, \mathbf{e}_j)_{\mathbb{R}^N}$ will be 0 for every tangent vector $\boldsymbol{\xi}$, but there will be $\boldsymbol{\xi}$'s for which $(\boldsymbol{\xi}, \mathbf{e}_j)_{\mathbb{R}^N} = 0$ and yet there is no continuously differentiable path that stays on ∂G, passes through $\mathbf{x}$, and has derivative $\boldsymbol{\xi}$ when it does. When $G = B(\mathbf{0}, r)$ and $\mathbf{x} \in \mathbb{S}^{N-1}(\mathbf{0}, r) \equiv \partial B(\mathbf{0}, r)$, then $\boldsymbol{\xi}$ is tangent to ∂G if and only if $(\boldsymbol{\xi}, \mathbf{x})_{\mathbb{R}^N} = 0$.[5] To see this, first suppose that $\boldsymbol{\xi}$ is tangent to ∂G at $\mathbf{x}$, and let γ be an associated path. Then

$$0 = \partial_t |\gamma(t)|^2 = 2\big(\gamma(t), \dot{\gamma}(t)\big)_{\mathbb{R}^N} = 2(\mathbf{x}, \boldsymbol{\xi})_{\mathbb{R}^N} \quad \text{at } t = 0.$$

Conversely, suppose that $(\mathbf{x}, \boldsymbol{\xi})_{\mathbb{R}^N} = 0$. If $\boldsymbol{\xi} = \mathbf{0}$, then we can take $\gamma(t) = \mathbf{x}$ for all t. If $\boldsymbol{\xi} \neq \mathbf{0}$, define $\gamma(t) = \big(\cos(r^{-1}|\boldsymbol{\xi}|t)\big)\mathbf{x} + \frac{r}{|\boldsymbol{\xi}|}\big(\sin(r^{-1}|\boldsymbol{\xi}|t)\big)\boldsymbol{\xi}$, and check that $\gamma(t) \in \mathbb{S}^{N-1}(\mathbf{0}, r)$ for all t, $\gamma(0) = \mathbf{x}$, and $\dot{\gamma}(0) = \boldsymbol{\xi}$.

Having defined what it means for a vector to be tangent to ∂G at $\mathbf{x}$, we now say that a vector $\boldsymbol{\eta}$ is a *normal vector* to ∂G at $\mathbf{x}$ if $(\boldsymbol{\eta}, \boldsymbol{\xi})_{\mathbb{R}^N} = 0$ for every tangent vector $\boldsymbol{\xi}$ at $\mathbf{x}$. For nice regions like balls, there is essentially only one normal vector at a point. Indeed, as we saw, $\boldsymbol{\xi}$ is tangent to $\mathbf{x} \in \partial B(\mathbf{0}, r)$ if and only if $(\boldsymbol{\xi}, \mathbf{x})_{\mathbb{R}^N} = 0$, and so every normal vector there will have the form $\alpha\mathbf{x}$ for some $\alpha \in \mathbb{R}$. In particular, there is a unique unit vector, known as the *outward pointing unit normal vector* $\mathbf{n}(\mathbf{x})$, that is normal to $\partial B(\mathbf{0}, r)$ at $\mathbf{x}$ and is pointing outward in the sense that $\mathbf{x} + t\mathbf{n}(\mathbf{x}) \notin B(\mathbf{0}, r)$ for $t > 0$. In fact, $\mathbf{n}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$. Similarly, when $\mathbf{x} \in R_j(a_j) \cup R_j(b_j)$ is not on an edge, every normal vector will be of the form $\alpha\mathbf{e}_j$, and the outward pointing normal unit normal at $\mathbf{x}$ will be $-\mathbf{e}_j$ or $\mathbf{e}_j$ depending on whether $\mathbf{x} \in R_j(a_j)$ or $\mathbf{x} \in R_j(b_j)$. However, when $\mathbf{x}$ is at an edge, there are too few tangent vectors to uniquely determine an outward pointing unit normal vector at $\mathbf{x}$.



normal to rectangle and circle

Fortunately, because this flaw is present only on a Riemann negligible set, it is not fatal for the application that we will make of these concepts to (5.7.2). To be precise, define $\mathbf{n}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \partial R$ that are on an edge, note that $\mathbf{n}$ is continuous off of a Riemann negligible subset of $\mathbb{R}^{N-1}$, and observe that (5.7.2) can be rewritten as

[5] This fact accounts for the notation $\mathbb{S}^{N-1}$ when referring to spheres in $\mathbb{R}^N$. Such surfaces are said to be $(N-1)$-dimensional because there are only $N-1$ linearly independent directions in which one can move without leaving them.

$$\int_R \operatorname{div}\mathbf{F}(\mathbf{x})\,d\mathbf{x} = \int_{\partial R} \big(\mathbf{F}(\mathbf{y}), \mathbf{n}(\mathbf{y})\big)_{\mathbb{R}^N}\, d\sigma_{\mathbf{y}}, \tag{5.7.3}$$

where

$$\int_{\partial R} \psi(\mathbf{y})\, d\sigma_{\mathbf{y}} \equiv \sum_{j=1}^{N} \left(\int_{R_j(a_j)} \psi(\mathbf{y})\, d\sigma_{\mathbf{y}} + \int_{R_j(b_j)} \psi(\mathbf{y})\, d\sigma_{\mathbf{y}} \right).$$

Besides being more aesthetically pleasing than (5.7.2), (5.7.3) has the advantage that it is in a form that generalizes and has a nice physical interpretation. In fact, once one knows how to interpret integrals over the boundary of more general regions, one can show that

$$\int_G \operatorname{div}\big(\mathbf{F}(\mathbf{x})\big)\, d\mathbf{x} = \int_{\partial G} \big(\mathbf{F}(\mathbf{y}), \mathbf{n}(\mathbf{y})\big)_{\mathbb{R}^N}\, d\sigma_{\mathbf{y}} \tag{5.7.4}$$

holds for quite general regions, and this generalization is known as the *divergence theorem*. Unfortunately, understanding of the physical interpretation requires one to know the relationship between $\operatorname{div}\mathbf{F}$ and the *flow* that $\mathbf{F}$ determines, and, although a rigorous explanation of this connection is beyond the scope of this book, here is the idea. In Sect. 4.5 we showed that if $\mathbf{F}$ satisfies (4.5.4), then it determines a map $\mathbf{X} : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by the equation

$$\dot{\mathbf{X}}(t, \mathbf{x}) = \mathbf{F}\big(\mathbf{X}(t, \mathbf{x})\big) \quad \text{with } \mathbf{X}(0, \mathbf{x}) = \mathbf{x}.$$

In other words, for each $\mathbf{x}$, $t \rightsquigarrow \mathbf{X}(t, \mathbf{x})$ is the path that passes through $\mathbf{x}$ at time $t = 0$ and has velocity $\mathbf{F}\big(\mathbf{X}(t, \mathbf{x})\big)$ for all t. Now think about mass that is initially uniformly distributed in a bounded region G and that is flowing along these paths. If one monitors the region to determine how much mass is lost or gained as a consequence of the flow, one can show that the rate at which change is taking place is given by the integral of $\operatorname{div}\mathbf{F}$ over G. If instead of monitoring the region, one monitors the boundary and measures how much mass is passing through it in each direction, then one finds that the rate of change is given by the integral of $\big(\mathbf{F}(\mathbf{x}), \mathbf{n}(\mathbf{x})\big)_{\mathbb{R}^N}$ over the boundary. Thus, (5.7.4) is simply stating that these two methods of measurement give the same answer.

We will now verify (5.7.4) for a special class of regions in $\mathbb{R}^2$. The main reason for working in $\mathbb{R}^2$ is that regions there are likely to have boundaries that are piecewise parameterized curves, which, by the results in Sect. 4.4, means that we know how to integrate over them. The regions G with which we will deal are *piecewise smooth star shaped regions in* $\mathbb{R}^2$ given by (5.6.1) with a continuous function $\varphi \in [0, 2\pi] \longmapsto r(\varphi) \in (0, \infty)$ that satisfies $r(0) = r(2\pi)$ and is piecewise smooth. Clearly the boundary of such a region is a piecewise parameterized curve. Indeed, consider the path $\mathbf{p}(\varphi) = \mathbf{c} + r(\varphi)\mathbf{e}(\varphi)$ where, as before, $\mathbf{e}(\varphi) = (\cos\varphi, \sin\varphi)$. Then the restriction $\mathbf{p}_1$ of $\mathbf{p}$ to $[0, \pi]$ and the restriction $\mathbf{p}_2$ of $\mathbf{p}$ to $[\pi, 2\pi]$ parameterize non-overlapping parameterized curves whose union is ∂G. Moreover, by (4.4.2), since

$$\dot{\mathbf{p}}(\varphi) = r'(\varphi)\mathbf{e}(\varphi) + r(\varphi)\dot{\mathbf{e}}(\varphi), \ \big(\mathbf{e}(\varphi), \dot{\mathbf{e}}(\varphi)\big)_{\mathbb{R}^2} = 0, \text{ and } |\mathbf{e}(\varphi)| = |\dot{\mathbf{e}}(\varphi)| = 1,$$

we have that

$$\int_{\partial G} f(\mathbf{y})\, d\sigma_{\mathbf{y}} = \int_0^{2\pi} f\big(\mathbf{c} + r(\varphi)\mathbf{e}(\varphi)\big)\sqrt{r(\varphi)^2 + r'(\varphi)^2}\, d\varphi.$$

Next observe that,

$$\mathbf{t}(\varphi) \equiv \big(r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi, r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi\big)$$

is tangent to ∂G at $\mathbf{p}(\varphi)$, and therefore that the outward pointing unit normal to ∂G at $\mathbf{p}(\varphi)$ is

$$\mathbf{n}\big(\mathbf{p}(\varphi)\big) = \pm\frac{\big(r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi, r(\varphi)\sin\varphi - r'(\varphi)\cos\varphi\big)}{\sqrt{r(\varphi)^2 + r'(\varphi)^2}} \tag{5.7.5}$$

for $\varphi \in [0, 2\pi]$ in intervals where r is continuously differentiable. Further, since

$$\big(\mathbf{n}\big(\mathbf{p}(\varphi)\big), \mathbf{p}(\varphi) - \mathbf{c}\big)_{\mathbb{R}^2} = \frac{r(\varphi)^2}{\sqrt{r(\varphi)^2 + r'(\varphi)^2}} > 0$$

and therefore

$$|\mathbf{p}(\varphi) + t\mathbf{n}(\varphi) - \mathbf{c}|^2 > r(\varphi)^2 \quad \text{for } t > 0,$$

we know that the plus sign is the correct one. Taking all these into account, we see that (5.7.4) for G is equivalent to

$$\begin{aligned}\int_G \operatorname{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x} = &\int_0^{2\pi} \big(r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi\big) F_1\big(\mathbf{c} + r(\varphi)\mathbf{e}(\varphi)\big)\, d\varphi \\ &+ \int_0^{2\pi} \big(r(\varphi)\sin\varphi - r'(\varphi)\cos\varphi\big) F_2\big(\mathbf{c} + r(\varphi)\mathbf{e}(\varphi)\big)\, d\varphi.\end{aligned} \tag{5.7.6}$$

In proving (5.7.6), we will assume, without loss in generality, that $\mathbf{c} = \mathbf{0}$. Hence, what we have to show is that

$$(*) \qquad \begin{aligned}\int_G \partial_{\mathbf{e}_1} f(\mathbf{x})\, d\mathbf{x} &= \int_0^{2\pi} \big(r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi\big) f\big(r(\varphi)\mathbf{e}(\varphi)\big)\, d\varphi \\ \int_G \partial_{\mathbf{e}_2} f(\mathbf{x})\, d\mathbf{x} &= \int_0^{2\pi} \big(r'(\varphi)\sin\varphi - r'(\varphi)\big) f\big(r(\varphi)\mathbf{e}(\varphi)\big)\, d\varphi\end{aligned}$$

To perform the required computation, it is important to write derivatives in terms of the variables ρ and φ. For this purpose, suppose that f is a continuously differentiable function on an open subset of $\mathbb{R}^2$, and set $g(\rho, \varphi) = f(\rho\cos\varphi, \rho\sin\varphi)$. Then

$$\partial_\rho g(\rho,\varphi) = \cos\varphi\,\partial_{\mathbf{e}_1} f(\rho\cos\varphi, \rho\sin\varphi) + \sin\varphi\,\partial_{\mathbf{e}_2} f(\rho\cos\varphi, \rho\sin\varphi)$$

and

$$\partial_\varphi g(\rho,\varphi) = -\rho\sin\varphi\,\partial_{\mathbf{e}_1} f(\rho\cos\varphi, \rho\sin\varphi) + \rho\cos\varphi\,\partial_{\mathbf{e}_2} f(\rho\cos\varphi, \rho\sin\varphi),$$

and therefore

$$\rho\partial_{\mathbf{e}_1} f(\rho\mathbf{e}(\varphi)) = \rho\cos\varphi\,\partial_\rho g(\rho,\varphi) - \sin\varphi\,\partial_\varphi g(\rho,\varphi)$$

and

$$\rho\partial_{\mathbf{e}_2} f(\rho\mathbf{e}(\varphi)) = \rho\sin\varphi\,\partial_\rho g(\rho,\varphi) + \cos\varphi\,\partial_\varphi g(\rho,\varphi).$$

Thus, if f is a continuous function on $\bar{G}$ that has bounded, continuous first order derivatives on G, then

$$\int_G \partial_{\mathbf{e}_1} f(\mathbf{x})\,d\mathbf{x} = I - J,$$

where

$$I = \int_0^{2\pi} \cos\varphi \left(\int_0^{r(\varphi)} \rho\partial_\rho g(\rho,\theta)\,d\rho \right) d\varphi$$

and

$$J = \int_0^{2\pi} \sin\varphi \left(\int_0^{r(\varphi)} \partial_\varphi g(\rho,\theta)\,d\rho \right) d\varphi.$$

Applying integration by parts to the inner integral in I, we see that

$$I = \int_0^{2\pi} \cos\varphi\, g\big(r(\varphi),\varphi\big) r(\varphi)\,d\varphi - \int_0^{2\pi} \cos\varphi \left(\int_0^{r(\varphi)} g(\rho,\varphi)\,d\rho \right) d\varphi.$$

Dealing with J is more challenging. The first step is to write it as $J_1 + J_2$, where J_1 and J_2 are, respectively,

$$\int_0^{2\pi} \sin\varphi \left(\int_0^{r_-} \partial_\varphi g(\rho,\varphi)\,d\rho \right) d\varphi \text{ and } \int_0^{2\pi} \sin\varphi \left(\int_{r_-}^{r(\varphi)} \partial_\varphi g(\rho,\varphi)\,d\rho \right) d\varphi,$$

and $r_- \equiv \min\{r(\varphi) : \varphi \in [0, 2\pi]\}$. By (5.2.1) and integration by parts,

$$J_1 = \int_0^{r_-} \left(\int_0^{2\pi} \sin\varphi\,\partial_\varphi g(\rho,\varphi)\,d\varphi \right) d\rho = -\int_0^{r_-} \left(\int_0^{2\pi} \cos\varphi\, g(\rho,\varphi)\,d\varphi \right) d\rho.$$

To handle J_2, choose $\theta_0 \in [0, 2\pi]$ so that $r(\theta_0) = r_-$, and choose $\theta_1, \dots, \theta_\ell \in [0, 2\pi]$ so that r' is continuous on each of the open intervals with end points θ_k and θ_{k+1}, where $\theta_{\ell+1} = \theta_0$. Now use (3.1.6) to write J_2 as $\sum_{k=0}^{\ell} J_{2,k}$, where

$$J_{2,k} = \int_0^{2\pi} \sin\varphi \left(\int_{r(\varphi\wedge\theta_k)}^{r(\varphi\wedge\theta_{k+1})} \partial_\varphi g(\rho, \varphi)\, d\rho \right) d\varphi,$$

and then make the change of variables $\rho = r(\theta)$ and apply (5.2.1) to obtain

$$\begin{aligned} J_{2,k} &= \int_0^{2\pi} \sin\varphi \left(\int_{\varphi\wedge\theta_k}^{\varphi\wedge\theta_{k+1}} \partial_\varphi g\big(r(\theta), \varphi\big) r'(\theta)\, d\theta \right) d\varphi \\ &= \int_{\theta_k}^{\theta_{k+1}} r'(\theta) \left(\int_\theta^{2\pi} \sin\varphi\, \partial_\varphi g\big(r(\theta), \varphi\big)\, d\varphi \right) d\theta. \end{aligned}$$

Hence,

$$J_2 = \int_0^{2\pi} r'(\theta) \left(\int_\theta^{2\pi} \sin\varphi\, \partial_\varphi g\big(r(\theta), \varphi\big)\, d\varphi \right) d\theta,$$

which, after integration by parts is applied to the inner integral, leads to

$$J_2 = -\int_0^{2\pi} \sin\theta\, g\big(r(\theta), \theta\big) r'(\theta)\, d\theta - \int_0^{2\pi} r'(\theta) \left(\int_\theta^{2\pi} \cos\varphi\, g\big(r(\theta), \varphi\big)\, d\varphi \right) d\theta.$$

After applying (5.2.1) and undoing the change of variables in the second integral on the right, we get

$$J_2 = -\int_0^{2\pi} \sin\theta\, g\big(r(\theta), \theta\big) r'(\theta)\, d\theta - \int_0^{2\pi} \cos\varphi \left(\int_{r_-}^{r(\varphi)} g(\rho, \varphi)\, d\rho \right) d\varphi$$

and therefore

$$J = -\int_0^{2\pi} \sin\varphi\, g\big(r(\varphi), \varphi\big) r'(\varphi)\, d\varphi - \int_0^{2\pi} \cos\varphi \left(\int_0^{r(\varphi)} g(\rho, \varphi)\, d\rho \right) d\varphi.$$

Finally, when we subtract J from I, we arrive at

$$\int_G \partial_{\mathbf{e}_1} f(\mathbf{x})\, d\mathbf{x} = \int_0^{2\pi} \big(r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi \big) g\big(r(\varphi), \varphi\big)\, d\varphi.$$

Proceeding in exactly the same way, one can derive the second equation in ($*$), and so we have proved the following theorem.

Theorem 5.7.1 *If $G \subseteq \mathbb{R}^2$ is a piecewise smooth star shaped region and if $\mathbf{F} : \bar{G} \longrightarrow \mathbb{R}^2$ is continuous on $\bar{G}$ and has bounded, continuous first order derivatives on G, then (5.7.5), and therefore (5.7.4) with $\mathbf{n} : \partial G \longrightarrow \mathbb{S}^1(0, 1)$ given by (5.7.5), hold.*

Corollary 5.7.2 *Let G be as in Theorem 5.7.1, and suppose that $\mathbf{a}_1, \dots, \mathbf{a}_\ell \in G$ and $r_1, \dots, r_\ell \in (0, \infty)$ have the properties that $\overline{B(\mathbf{a}_k, r_k)} \subseteq G$ for each $1 \le k \le \ell$ and that $\overline{B(\mathbf{a}_k, r_k)} \cap \overline{B(\mathbf{a}_{k'}, r_{k'})} = \emptyset$ for $1 \le k < k' \le \ell$. Set $H = G \setminus \bigcup_{k=1}^{\ell} \overline{B(\mathrm{a}_k, r_k)}$. If $\mathbf{F} : \bar{H} \longrightarrow \mathbb{R}^2$ is a continuous function that has bounded, continuous first order derivatives on H, then $\int_H \mathrm{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x}$ equals*

$$\int_{\partial G} \big(\mathbf{F}(\mathbf{y}), \mathbf{n}(\mathbf{y})\big)_{\mathbb{R}^2}\, d\sigma_{\mathbf{y}} \\ - \sum_{k=1}^{\ell} r_k \int_0^{2\pi} \Big(F_1\big(\mathbf{a}_k + r_k \mathbf{e}(\varphi)\big) \cos\varphi + F_2\big(\mathbf{a}_k + r_k \mathbf{e}(\varphi)\big) \sin\varphi\Big)\, d\varphi.$$

Proof First assume that $\mathbf{F}$ has bounded, continuous derivatives on the whole of G. Then Theorem 5.7.1 applies to $\mathbf{F}$ on $\bar{G}$ and its restriction to each ball $\overline{B(\mathbf{a}_k, r_k)}$, and so the result follows from Theorem 5.7.1 when one writes the integral of div$\mathbf{F}$ over $\bar{H}$ as

$$\int_{\bar{G}} \mathrm{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x} - \sum_{k=1}^{\ell} \int_{\overline{B(\mathbf{a}_k, r_k)}} \mathrm{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x}.$$

To handle the general case, define $\eta : \mathbb{R} \longrightarrow [0, 1]$ by

$$\eta(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1+\sin\left(\pi(t-\frac{1}{2})\right)}{2} & \text{if } 0 < t \le 1 \\ 1 & \text{if } t > 1. \end{cases}$$

Then η is continuously differentiable. For each $1 \le k \le \ell$, choose $R_k > r_k$ so that $\overline{B(\mathbf{a}_k, R_k)} \subseteq G$ and $\overline{B(\mathbf{a}_k, R_k)} \cap \overline{B(\mathbf{a}_{k'}, R_{k'})} = \emptyset$ for $1 \le k < k' \le \ell$. Define

$$\psi_k(\mathbf{x}) = \eta\left(\frac{|\mathbf{x} - \mathbf{a}_k|^2 - r_k^2}{R_k^2 - r_k^2}\right) \quad \text{for } \mathbf{x} \in \mathbb{R}^2$$

and

$$\tilde{\mathbf{F}}(\mathbf{x}) = \sum_{k=1}^{\ell} \psi_k(\mathbf{x}) \mathbf{F}(\mathbf{x})$$

if $\mathbf{x} \in \bar{H}$ and $\tilde{\mathbf{F}}(\mathbf{x}) = 0$ if $\mathbf{x} \in \bigcup_{k=1}^{\ell} B(\mathbf{a}_k, r_k)$. Then $\tilde{\mathbf{F}}$ is continuous on $\bar{G}$ and has bounded, continuous first order derivatives on G. In addition, $\tilde{\mathbf{F}} = \mathbf{F}$ on $G \setminus \bigcup_{k=1}^{\ell} \overline{B(\mathbf{a}_k, R_k)}$. Hence, if $H' = \bar{G} \setminus \bigcup_{k=1}^{\ell} \overline{B(\mathbf{a}_k, R_k)}$, then, by the preceding, $\int_{H'} \mathrm{div}\mathbf{F}(\mathbf{x})\, d\mathbf{x}$ equals

$$\int_{\partial G} \big(\mathbf{F}(\mathbf{y}), \mathbf{n}(\mathbf{y})\big)_{R^2} - \sum_{k=1}^{\ell} R_k \int_0^{2\pi} \Big(F_1\big(\mathbf{a}_k + R_k \mathbf{e}(\varphi)\big) \cos\varphi + F_2\big(\mathbf{a}_k + R_k \mathbf{e}(\varphi)\big) \sin\varphi\Big)\, d\varphi,$$

and so the asserted result follows after one lets each R_k decrease to r_k. □

We conclude this section with an application of Theorem 5.7.1 that plays a role in many places. One of the consequence of the Fundamental Theorem of Calculus is that every continuous function f on an interval (a, b) is the derivative of a continuously differentiable function F on (a, b). Indeed, simply set $c = \frac{a+b}{2}$ and take $F(x) = \int_c^x f(t)\, dt$. With this in mind, one should ask whether an analogous statement holds in $\mathbb{R}^2$. In particular, given a connected open set $G \subseteq \mathbb{R}^2$ and a continuous function $\mathbf{F} : G \longrightarrow \mathbb{R}^2$, is it true that there is a continuously differentiable function $f : G \longrightarrow \mathbb{R}$ such that $\mathbf{F}$ is the gradient of f? That the answer is *no* in general can be seen by assuming that $\mathbf{F}$ is continuously differentiable and noticing that if f exists then

$$\partial_{\mathbf{e}_2} F_1 = \partial_{\mathbf{e}_2} \partial_{\mathbf{e}_1} f = \partial_{\mathbf{e}_1} \partial_{\mathbf{e}_2} f = \partial_{\mathbf{e}_1} F_2.$$

Hence, a necessary condition for the existence of f is that $\partial_{\mathbf{e}_2} F_1 = \partial_{\mathbf{e}_1} F_2$, and when this condition holds $\mathbf{F}$ is said to be *exact*. It is known that exactness is sufficient as well as necessary for a continuously differentiable $\mathbf{F}$ on G to be the gradient of a function when G is what is called a simply connected region, but to avoid technical difficulties, we will restrict ourselves to star shaped regions.

Corollary 5.7.3 *Assume that G is a star shaped region in $\mathbb{R}^2$ and that $\mathbf{F} : G \longrightarrow \mathbb{R}^2$ is a continuously differentiable function. Then there exists a continuously differentiable function $f : G \longrightarrow \mathbb{R}$ such that $\mathbf{F} = \nabla f$ if and only if $\mathbf{F}$ is exact.*

Proof Without loss in generality, we will assume that $\mathbf{0}$ is the center of G. Further, since the necessity has already been shown, we will assume that $\mathbf{F}$ is exact.

Define $f : G \longrightarrow \mathbb{R}$ by $f(\mathbf{0}) = 0$ and

$$f\big(r\mathbf{e}(\varphi)\big) = \int_0^r \big(F_1(\rho\mathbf{e}(\varphi)) \cos\varphi + F_2(\rho\mathbf{e}(\varphi)) \sin\varphi\big)\, d\rho$$

for $\varphi \in [0, 2\pi)$ and $0 < r < r(\varphi)$. Clearly $F_1(\mathbf{0}) = \partial_{\mathbf{e}_1} f(\mathbf{0})$ and $F_2(\mathbf{0}) = \partial_{\mathbf{e}_2} f(\mathbf{0})$.

We will now show that $F_1 = \partial_{\mathbf{e}_1} f$ at any point $(\xi_0, \eta_0) \in G \setminus \{\mathbf{0}\}$. This is easy when $\eta_0 = 0$, since $f(\xi, 0) = \int_0^\xi F_1(t, 0)\, dt$. Thus assume that $\eta_0 \neq 0$, and consider points

(ξ, η_0) not equal to (ξ_0, η_0) but sufficiently close that $(t, \eta_0) \in G$ if $\xi \wedge \xi_0 \le t \le \xi \vee \xi_0$. What we need to show is that

$$(*) \qquad f(\xi, \eta_0) - f(\xi_0, \eta_0) = \int_{\xi_0}^{\xi} F_1(t, \eta_0)\, dt.$$

To this end, define $\tilde{\mathbf{F}} = (F_2, -F_1)$. Then, because $\mathbf{F}$ is exact, $\operatorname{div}\tilde{\mathbf{F}} = 0$ on G. Next consider the region H that is the interior of the triangle whose vertices are $\mathbf{0}$, (ξ_0, η_0), and (ξ, η_0). Then H is a piecewise smooth star shaped region and so, by Theorem 5.7.1, the integral of $(\tilde{\mathbf{F}}, \mathbf{n})_{\mathbb{R}^2}$ over ∂H is 0. Thus, if we write (ξ_0, η_0) and (ξ, η_0) as $r_0\mathbf{e}(\varphi_0)$ and $r\mathbf{e}(\varphi)$, then

$$\begin{aligned} 0 = \int_{\partial H} \big(\tilde{\mathbf{F}}(\mathbf{y}), \mathbf{n}(\mathbf{y})\big)_{\mathbb{R}^2}\, d\mathbf{y} = &\int_0^{r_0} \Big(\tilde{\mathbf{F}}\big(\rho\mathbf{e}(\varphi_0)\big), \mathbf{n}\big(\rho\mathbf{e}(\varphi_0)\big)\Big)_{\mathbb{R}^2} d\rho \\ &+ \int_0^{r} \Big(\tilde{\mathbf{F}}\big(\rho\mathbf{e}(\varphi)\big), \mathbf{n}\big(\rho\mathbf{e}(\varphi)\big)\Big)_{\mathbb{R}^2} d\rho \\ &+ \int_{\xi\wedge\xi_0}^{\xi\vee\xi_0} \big(\tilde{\mathbf{F}}(t, \eta_0), \mathbf{n}(t, \eta_0)\big)_{\mathbb{R}^2}\, dt. \end{aligned}$$

If $\eta_0 > 0$ and $\xi > \xi_0$, then

$$\mathbf{n}\big(\rho\mathbf{e}(\varphi)\big) = (\sin\varphi, -\cos\varphi), \ \ \mathbf{n}\big(\rho\mathbf{e}(\varphi_0)\big) = (-\sin\varphi_0, \cos\varphi_0),$$

and $\mathbf{n}(t, \eta_0) = (0, 1)$, and therefore

$$\begin{aligned} f(\xi, \eta_0) &= \int_0^{r} \Big(\tilde{\mathbf{F}}\big(\rho\mathbf{e}(\varphi)\big), \mathbf{n}\big(\rho\mathbf{e}(\varphi)\big)\Big)_{\mathbb{R}^2} d\rho, \\ f(\xi_0, \eta_0) &= -\int_0^{r_0} \Big(\tilde{\mathbf{F}}\big(\rho\mathbf{e}(\varphi_0)\big), \mathbf{n}\big(\rho\mathbf{e}(\varphi_0)\big)\Big)_{\mathbb{R}^2} d\rho, \\ \int_{\xi_0}^{\xi} F_1(t, \eta_0)\, dt &= -\int_{\xi\wedge\xi_0}^{\xi\vee\xi_0} \big(\tilde{\mathbf{F}}(t, \eta_0), \mathbf{n}(t, \eta_0)\big)_{\mathbb{R}^2}\, dt, \end{aligned}$$

and so $(*)$ holds. If $\eta_0 > 0$ and $\xi < \xi_0$, then the sign of $\mathbf{n}$ changes in each term, and therefore we again get $(*)$, and the cases when $\eta_0 < 0$ are handled similarly.

The proof that $F_2 = \partial_{\mathbf{e}_2} f$ follows the same line of reasoning and is left as an exercise. □

5.8 Exercises

Exercise 5.1 Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^2\backslash\{\mathbf{0}\}$. Then Schwarz's inequality says that the ratio

$$\rho \equiv \frac{(\mathbf{x}, \mathbf{y})_{\mathbb{R}^2}}{|\mathbf{x}||\mathbf{y}|}$$

is in the open interval $(-1, 1)$ unless $\mathbf{x}$ and $\mathbf{y}$ lie on the same line, in which case $\rho \in \{-1, 1\}$. Euclidean geometry provides a good explanation for this. Indeed, consider the triangle whose vertices are $\mathbf{0}$, $\mathbf{x}$, and $\mathbf{y}$. The sides of this triangle have lengths $|\mathbf{x}|$, $|\mathbf{y}|$, and $|\mathbf{y}-\mathbf{x}|$. Thus, by the law of the cosine, $|\mathbf{y}-\mathbf{x}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}|\cos\theta$, where θ is the angle in the triangle between $\mathbf{x}$ and $\mathbf{y}$. Use this to show that $\rho = \cos\theta$. The same explanation applies in higher dimensions since there is a plane in which $\mathbf{x}$ and $\mathbf{y}$ lie and the analysis can be carried out in that plane.

Exercise 5.2 Show that

$$\int_{\mathbb{R}} e^{\lambda x} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi} e^{\frac{\lambda^2}{2}} \quad \text{for all } \lambda \in \mathbb{R}.$$

One way to do this is to make the change of variables $y = x - \lambda$ to see that

$$\int_{\mathbb{R}} e^{\lambda x} e^{-\frac{x^2}{2}}\, dx = e^{\frac{\lambda^2}{2}} \int_{\mathbb{R}} e^{-\frac{(y-\lambda)^2}{2}}\, dy,$$

and then use (5.1.2) and (5.4.3).

Exercise 5.3 Show that $|\Gamma|_{\mathrm{e}} = |\mathfrak{R}(\Gamma)|_{\mathrm{e}}$ and $|\Gamma|_{\mathrm{i}} = |\mathfrak{R}(\Gamma)|_{\mathrm{i}}$ for all bounded sets $\Gamma \subseteq \mathbb{R}^N$ and all rotations $\mathfrak{R}$, and conclude that $\mathfrak{R}(\Gamma)$ is Riemann measurable if and only if Γ is. Use this to show that if Γ is a bounded subset of $\mathbb{R}^N$ for which there exists a $\mathbf{x}_0 \in \mathbb{R}^N$ and an $\mathbf{e} \in \mathbb{S}^{N-1}(\mathbf{0}, 1)$ such that $(\mathbf{x} - \mathbf{x}_0, \mathbf{e})_{\mathbb{R}^N} = 0$ for all $\mathbf{x} \in \Gamma$, then Γ is Riemann negligible.

Exercise 5.4 An integral that arises quite often is one of the form

$$I(a, b) \equiv \int_{(0,\infty)} t^{-\frac{1}{2}} e^{-a^2 t - \frac{b^2}{t}}\, dt,$$

where $a, b \in (0, \infty)$. To evaluate this integral, make the change of variables $\xi = at^{\frac{1}{2}} - bt^{-\frac{1}{2}}$. Then $\xi^2 = at - 2ab + t^{-1}$ and $t^{\frac{1}{2}} = \frac{\xi + \sqrt{\xi^2 + 4ab}}{2a}$, the plus being dictated by the requirement that $t \geq 0$. After making this substitution, arrive at

$$I(a, b) = \frac{e^{-2ab}}{a} \int_{\mathbb{R}} e^{-\xi^2} \big(1 + (\xi^2 + 4ab)^{-\frac{1}{2}} \xi\big)\, d\xi = \frac{e^{-2ab}}{a} \int_{\mathbb{R}} e^{-\xi^2}\, d\xi,$$

from which it follows that $I(a, b) = \frac{\pi^{\frac{1}{2}} e^{-2ab}}{a}$. Finally, use this to show that

$$\int_{(0,\infty)} t^{-\frac{3}{2}} e^{-a^2 t - \frac{b^2}{t}}\, dt = \frac{\pi^{\frac{1}{2}} e^{-2ab}}{b}.$$

Exercise 5.5 Recall the cardiod described in Exercise 2.2, and consider the region that it encloses. Using the expression $z(\theta) = 2R(1 - \cos\theta)e^{i\theta}$ for the boundary of this region after translating by $-R$, show that the area enclosed by the cardiod is $6\pi R^2$. Finally, show that the arc length of the cardiod is $16R$, a computation in which you may want to use (1.5.1) to write $1 - \cos\theta$ as a square.

Exercise 5.6 Given $a_1, a_2, a_3 \in (0, \infty)$, consider the region Ω in $\mathbb{R}^3$ enclosed by the ellipsoid $\sum_{i=1}^3 \frac{x_i^2}{a_i^2} = 1$. Show that the boundary of Ω is Riemann negligible and that the volume of Ω is $\frac{4\pi a_1 a_2 a_3}{3}$. When computing the volume V, first show that

$$V = 2a_3 \int_{\tilde{\Omega}} \sqrt{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}}\, dx_1 dx_2 \quad \text{where } \tilde{\Omega} = \left\{ \mathbf{x} \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \le 1 \right\}.$$

Next, use Fubini's Theorem and a change of variables in each of the coordinates to write the integral as $a_1 a_2 \int_{B(\mathbf{0},1)} \sqrt{1 - |\mathbf{x}|^2}\, d\mathbf{x}$, where $B(\mathbf{0}, 1)$ is the unit ball in $\mathbb{R}^2$. Finally, use (5.6.2) to complete the computation.

Exercise 5.7 Let Ω be a bounded, closed region in $\mathbb{R}^3$ with Riemann negligible boundary, and let $\mu : \Omega \longrightarrow [0, \infty)$ be a continuous function. Thinking of μ as a mass density, one says that the *center of gravity* of Ω with mass distribution μ is the point $\mathbf{c} \in \mathbb{R}^3$ such that $\int_\Omega \mu(\mathbf{y})(\mathbf{y} - \mathbf{c})\, d\mathbf{y} = \mathbf{0}$. The reason for the name is that if Ω is supported at this point $\mathbf{c}$, then the net effect of gravity will be 0 and so the region will be balanced there. Of course, $\mathbf{c}$ need not lie in Ω, in which case one should think of Ω being attached to $\mathbf{c}$ by weightless wires.

Obviously, $\mathbf{c} = \frac{\int_\Omega \mu(\mathbf{y})\mathbf{y}\, d\mathbf{y}}{M}$, where $M = \int_\Omega \mu(\mathbf{y})\, d\mathbf{y}$ is the total mass. Now suppose that $\Omega = \{\mathbf{y} \in \mathbb{R}^3 : x_1^2 + x_2^2 \le x_3 \le h\}$, where $h > 0$, has a constant mass density. Show that $\mathbf{c} = \left(0, 0, \frac{3h}{4}\right)$.

Exercise 5.8 We showed that for a ball $\overline{B(\mathbf{c}, r)}$ in $\mathbb{R}^3$ with a continuous mass distribution that depends only of the distance to $\mathbf{c}$, the gravitational force it exerts on a particle of mass m at a point $\mathbf{b} \notin \overline{B(\mathbf{c}, r)}$ is given by (5.6.3). Now suppose that $\mathbf{b} \in \overline{B(\mathbf{c}, r)}$, and set $D = |\mathbf{b} - \mathbf{c}|$. Show that

$$\int_{\overline{B(\mathbf{c},r)}} \frac{Gm\mu(|\mathbf{y} - \mathbf{c}|)}{|\mathbf{y} - \mathbf{b}|}\, d\mathbf{y} = \frac{GmM_D}{D^2}(\mathbf{c} - \mathbf{b})$$
$$\text{where } M_D = \int_{\overline{B(\mathbf{c},D)}} \mu(|\mathbf{y} - \mathbf{c}|)\, d\mathbf{y}.$$

In other words, the forces produced by the mass that lies further than $\mathbf{b}$ from $\mathbf{c}$ cancel out, and so the particle feels only the force coming from the mass between it and the center of the ball.

Exercise 5.9 Let $B(\mathbf{0}, r)$ be the ball of radius r in $\mathbb{R}^N$ centered at the origin. Using rotation invariance, show that

$$\int_{B(\mathbf{0},r)} x_i \, d\mathbf{x} = 0 \quad \text{and} \quad \int_{B(\mathbf{0},r)} x_i x_j \, d\mathbf{x} = \frac{\Omega_N r^{N+2}}{N+2} \delta_{i,j} \text{ for } 1 \le i,\, j \le N.$$

Next, suppose that $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is twice continuously differentiable, and let $\mathcal{A}(f, r) = (\Omega_N r^N)^{-1} \int_{B(\mathbf{0},r)} f(\mathbf{x})\, d\mathbf{x}$ be the average value of f on $B(\mathbf{0}, r)$. As an application of the preceding, show that $\frac{A(f,r)-f(\mathbf{0})}{r^2} \longrightarrow \frac{1}{2(N+2)} \sum_{i=1}^{N} \partial_{\mathbf{e}_i}^2 f(\mathbf{0})$ as $r \searrow 0$.

Exercise 5.10 Suppose that G is an open subset of $\mathbb{R}^2$ and that $\mathbf{F} : G \longrightarrow \mathbb{R}^2$ is continuous. If $\mathbf{F} = \nabla f$ for some continuously differentiable $f : G \longrightarrow \mathbb{R}$ and $\mathbf{p} : [a, b] \longrightarrow G$ is a piecewise smooth path, show that

$$f(b) - f(a) = \int_a^b \Big(\mathbf{F}\big(\mathbf{p}(t)\big), \dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N} dt$$

and therefore that $\int_a^b \Big(\mathbf{F}\big(\mathbf{p}(t)\big), \dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N} dt = 0$ if $\mathbf{p}$ is closed (i.e., $\mathbf{p}(b) = \mathbf{p}(a)$). Now assume that G is connected and that $\int_a^b \Big(\mathbf{F}\big(\mathbf{p}(t)\big), \dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N} dt = 0$ for all piecewise smooth, closed paths $\mathbf{p} : [a, b] \longrightarrow G$. Using Exercise 4.5, show that for each $\mathbf{x}, \mathbf{y} \in G$ there is a piecewise smooth path in G that starts at $\mathbf{x}$ and ends at $\mathbf{y}$. Given a reference point $\mathbf{x}_0 \in G$ and an $\mathbf{x} \in G$, show that

$$f(\mathbf{x}) \equiv \int_a^b \Big(\mathbf{F}\big(\mathbf{p}(t)\big), \dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N} dt$$

is the same for all piecewise smooth paths $\mathbf{p} : [a, b] \longrightarrow G$ such that $\mathbf{p}(a) = \mathbf{x}_0$ and $\mathbf{p}(b) = \mathbf{x}$. Finally, show that f is continuously differentiable and that $\mathbf{F} = \nabla f$.