Chapter 3
Integration

Calculus has two components, and, thus far, we have been dealing with only one of them, namely differentiation. Differentiation is a systematic procedure for disassembling quantities at an infinitesimal level. Integration, which is the second component and is the topic here, is a systematic procedure for assembling the same sort of quantities. One of Newton's great discoveries is that these two components complement one another in a way that makes each of them more powerful.

3.1 Elements of Riemann Integration

Suppose that $f : [a, b] \longrightarrow [0, \infty)$ is a bounded function, and consider the region Ω in the plane bounded below by the portion of the horizontal axis between $(a, 0)$ and $(b, 0)$, the line segment between $(a, 0)$ and $\big(a, f(a)\big)$ on the left, the graph of f above, and the line segment between $\big(b, f(b)\big)$ and $(b, 0)$ on the right. In order to compute the area of this region, one might chop the interval $[a, b]$ into $n \geq 1$ equal parts and argue that, if f is sufficiently continuous and therefore does not vary much over small intervals, then, when n is large, the area of each slice

$$\Big\{(x, y) \in \Omega : x \in \big[a + \tfrac{m-1}{n}(b-a), a + \tfrac{m}{n}(b-a)\big] \;\&\; 0 \leq y \leq f(x)\Big\},$$

where $1 \leq m \leq n$, should be well approximated by $\frac{b-a}{n} f\big(a + \frac{m-1}{n}(b-a)\big)$, the area of the rectangle $\big[a + \frac{m-1}{n}(b-a), a + \frac{m}{n}(b-a)\big] \times \big[0, f\big(a + \frac{m-1}{n}(b-a)\big)\big]$. Continuing this line of reasoning, one would then say that the area of Ω is obtained by adding the areas of these slices and taking the limit

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{m=1}^{n} f\big(a + \tfrac{m-1}{n}(b-a)\big).$$

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Of course, there are two important questions that should be asked about this procedure. In the first place, does the limit exist and, secondly, if it does, is there a compelling reason for thinking that it represents the area of Ω? Before addressing these questions, we will reformulate the preceding in a more flexible form. Say that two closed intervals are *non-overlapping* if their interiors are disjoint. Next, given a finite collection $\mathcal{C}$ of non-overlapping closed intervals $I \neq \emptyset$ whose union is $[a, b]$, a *choice function* is a map $\Xi : \mathcal{C} \longrightarrow [a, b]$ such that $\Xi(I) \in I$ for each $I \in \mathcal{C}$. Finally given $\mathcal{C}$ and Ξ, define the corresponding *Riemann sum* of a function $f : [a, b] \longrightarrow \mathbb{R}$ to be

$$\mathcal{R}(f; \mathcal{C}, \Xi) = \sum_{I \in \mathcal{C}} f\big(\Xi(I)\big)|I|, \quad \text{where } |I| \text{ is the length of } I.$$

What we want to show is that, as the *mesh size* $\|\mathcal{C}\| \equiv \max\{|I| : I \in \mathcal{C}\}$ tends to 0, for a large class of functions f these Riemann sums converge in the sense that there is a number $\int_a^b f(x)\,dx \in \mathbb{R}$, which we will call the *Riemann integral*, or simply the *integral*, of f on $[a, b]$, such that for every $\epsilon > 0$ one can find a $\delta > 0$ for which

$$\left|\mathfrak{R}(f; \mathcal{C}, \Xi) - \int_a^b f(x)\,dx\right| \leq \epsilon$$

for all $\mathcal{C}$ with $\|\mathcal{C}\| \leq \delta$ and all associated choice functions Ξ. When such a limit exists, we will say that f is *Riemann integrable* on $[a, b]$.

In order to carry out this program, it is helpful to introduce the *upper Riemann sum*

$$\mathcal{U}(f; \mathcal{C}) = \sum_{I \in \mathcal{C}} \left(\sup_I f\right)|I|, \quad \text{where } \sup_I f = \sup\{f(x) : x \in I\},$$

and the *lower Riemann sum*

$$\mathcal{L}(f; \mathcal{C}) = \sum_{I \in \mathcal{C}} \left(\inf_I f\right)|I|, \quad \text{where } \inf_I f = \inf\{f(x) : x \in I\}.$$

Lemma 3.1.1 *Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. For every $\mathcal{C}$ and choice function Ξ, $\mathcal{L}(f; \mathcal{C}) \leq \mathcal{R}(f; \mathcal{C}, \Xi) \leq \mathcal{U}(f; \mathcal{C})$. In addition, for any pair $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{L}(f; \mathcal{C}) \leq \mathcal{U}(f; \mathcal{C}')$. Finally, for any $\mathcal{C}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\|\mathcal{C}'\| < \delta \implies \mathcal{L}(f; \mathcal{C}) \leq \mathcal{L}(f; \mathcal{C}') + \epsilon \text{ and } \mathcal{U}(f; \mathcal{C}') \leq \mathcal{U}(f; \mathcal{C}) + \epsilon,$$

and therefore

$$\lim_{\|\mathcal{C}\| \to 0} \mathcal{L}(f; \mathcal{C}) = \sup_{\mathcal{C}} \mathcal{L}(f; \mathcal{C}) \leq \inf_{\mathcal{C}} \mathcal{U}(f; \mathcal{C}) = \lim_{\|\mathcal{C}\| \to 0} \mathcal{U}(f; \mathcal{C}).$$

Proof The first assertion is obvious. To prove the second, begin by observing that there is nothing to do if $\mathcal{C}' = \mathcal{C}$. Next, suppose that every $I \in \mathcal{C}$ is contained in some $I' \in \mathcal{C}'$, in which case $\sup_I f \le \sup_{I'} f$. Therefore (cf. Lemma 5.1.1 for a detailed proof), since each $I' \in \mathcal{C}'$ is the union of the I's in $\mathcal{C}$ which it contains,

$$\mathcal{U}(f;\mathcal{C}') = \sum_{I'\in\mathcal{C}'} \left(\sum_{\substack{I\in\mathcal{C}\\ I\subseteq I'}} \left(\sup_{I'} f\right) |I| \right)$$

$$\ge \sum_{I'\in\mathcal{C}'} \left(\sum_{\substack{I\in\mathcal{C}\\ I\subseteq I'}} \left(\sup_{I} f\right) |I| \right) = \sum_{I\in\mathcal{C}} \left(\sup_{I} f\right) |I| = \mathcal{U}(f;\mathcal{C}),$$

and, similarly, $\mathcal{L}(f;\mathcal{C}') \le \mathcal{L}(f;\mathcal{C})$. Now suppose that $\mathcal{C}$ and $\mathcal{C}'$ are given, and set

$$\mathcal{C}'' \equiv \{I \cap I' : I \in \mathcal{C},\ I' \in \mathcal{C}' \ \&\ I \cap I' \ne \emptyset\}.$$

Then for each I'' there exist $I \in \mathcal{C}$ and $I' \in \mathcal{C}'$ such that $I'' \subseteq I$ and $I'' \subseteq I'$, and therefore

$$\mathcal{L}(f;\mathcal{C}) \le \mathcal{L}(f;\mathcal{C}'') \le \mathcal{U}(f;\mathcal{C}'') \le \mathcal{U}(f,\mathcal{C}').$$

Finally, let $\mathcal{C}$ and $\epsilon > 0$ be given, and choose $a = c_0 \le c_1 \le \cdots \le c_K = b$ so that $\mathcal{C} = \{[c_{k-1}, c_k] : 1 \le k \le K\}$. Given $\mathcal{C}'$, let $\mathcal{D}$ be the set of $I' \in \mathcal{C}'$ with the property that $c_k \in \operatorname{int}(I')$ for at least one $1 \le k < K$, and observe that, since the intervals are non-overlapping, $\mathcal{D}$ can contain at most $K-1$ elements. Then each $I' \in \mathcal{C}' \setminus \mathcal{D}$ is contained in some $I \in \mathcal{C}$ and therefore $\sup_{I'} f \le \sup_I f$. Hence

$$\mathcal{U}(f;\mathcal{C}') = \sum_{I'\in\mathcal{D}} \left(\sup_{I'} f\right) |I'| + \sum_{I\in\mathcal{C}} \sum_{\substack{I'\in\mathcal{C}'\setminus\mathcal{D}\\ I'\subseteq I}} \left(\sup_{I'} f\right) |I'|$$

$$\le \left(\sup_{[a,b]} |f|\right)(K-1)\|\mathcal{C}'\| + \sum_{I\in\mathcal{C}} \sup_I f \left(\sum_{\substack{I'\in\mathcal{C}'\\ I'\subseteq I}} |I'| \right)$$

$$\le \left(\sup_{[a,b]} |f|\right)(K-1)\|\mathcal{C}'\| + \mathcal{U}(f;\mathcal{C}).$$

Therefore, if $\delta > 0$ is chosen so that $\left(\sup_{[a,b]} |f|\right)(K-1)\delta < \epsilon$, then $\mathcal{U}(f;\mathcal{C}') \le \mathcal{U}(f;\mathcal{C}) + \epsilon$ whenever $\|\mathcal{C}'\| \le \delta$. Applying this to $-f$ and noting that $\mathcal{L}(f;\mathcal{C}'') = -\mathcal{U}(-f;\mathcal{C}'')$ for any $\mathcal{C}''$, one also has that $\mathcal{L}(f;\mathcal{C}) \le \mathcal{L}(f;\mathcal{C}') + \epsilon$ if $\|\mathcal{C}'\| \le \delta$. □

Theorem 3.1.2 *If $f : [a, b] \longrightarrow \mathbb{R}$ is a bounded function, then it is Riemann integrable if and only if for each $\epsilon > 0$ there is a $\mathcal{C}$ such that*

$$\sum_{\substack{I \in \mathcal{C} \\ \sup_I f - \inf_I f \geq \epsilon}} |I| < \epsilon.$$

In particular, a bounded f will be Riemann integrable if it is continuous at all but a finite number of points. In addition, if $f : [a, b] \longrightarrow [c, d]$ is Riemann integrable and $\varphi : [c, d] \longrightarrow \mathbb{R}$ is continuous, then $\varphi \circ f$ is Riemann integrable on $[a, b]$.

Proof First assume that f is Riemann integrable. Given $\epsilon > 0$, choose $\mathcal{C}$ so that

$$\left| \mathcal{R}(f; \mathcal{C}, \Xi) - \int_a^b f(x)\, dx \right| < \frac{\epsilon^2}{6}$$

for all choice functions Ξ. Next choose choice functions $\Xi^{\pm}$ so that

$$f\big(\Xi^+(I)\big) + \frac{\epsilon^2}{3(b-a)} \geq \sup_I f \quad \text{and} \quad f\big(\Xi^-(I)\big) - \frac{\epsilon^2}{3(b-a)} \leq \inf_I f$$

for each $I \in \mathcal{C}$. Then

$$\mathcal{U}(f; \mathcal{C}) \leq \mathcal{R}(f; \mathcal{C}, \Xi^+) + \frac{\epsilon^2}{3} \leq \mathcal{R}(f; \mathcal{C}, \Xi^-) + \frac{2\epsilon^2}{3} \leq \mathcal{L}(f; \mathcal{C}) + \epsilon^2,$$

and so

$$\epsilon^2 \geq \mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) = \sum_{I \in \mathcal{C}} \left(\sup_I f - \inf_I f \right) |I| \geq \epsilon \sum_{\substack{I \in \mathcal{C} \\ \sup_I f - \inf_I f \geq \epsilon}} |I|.$$

Next assume that for each $\epsilon > 0$ there is a $\mathcal{C}$ such that

$$\sum_{\substack{I \in \mathcal{C} \\ \sup_I f - \inf_I f \geq \epsilon}} |I| < \epsilon.$$

Given $\epsilon > 0$, set $\epsilon' = \epsilon \left(4 \sup_{[a,b]} |f| + 2(b-a)\right)^{-1}$, and choose $\mathcal{C}_\epsilon$ so that

$$\sum_{\substack{I \in \mathcal{C}_\epsilon \\ \sup_I f - \inf_I f \geq \epsilon'}} |I| < \epsilon'$$

and therefore

$$\mathcal{U}(f;\mathcal{C}_\epsilon) - \mathcal{L}(f;\mathcal{C}_\epsilon) \le \left(2\sup_{[a,b]}|f|\right) \sum_{\substack{I\in\mathcal{C}_\epsilon \\ \sup_I f - \inf_I f \ge \epsilon'}} |I| + \epsilon'(b-a) = \frac{\epsilon}{2}.$$

Now, using Lemma 3.1.1, choose $\delta_\epsilon > 0$ so that $\mathcal{U}(f;\mathcal{C}) \le \mathcal{U}(f;\mathcal{C}_\epsilon) + \frac{\epsilon}{2}$ and $\mathcal{L}(f;\mathcal{C}) \ge \mathcal{L}(f;\mathcal{C}_\epsilon) - \frac{\epsilon}{2}$ when $\|\mathcal{C}\| < \delta_\epsilon$. Then

$$\|\mathcal{C}\| < \delta_\epsilon \implies \mathcal{U}(f;\mathcal{C}) \le \mathcal{L}(f;\mathcal{C}) + \epsilon,$$

and so, in conjunction with Lemma 3.1.1, we know that

$$\lim_{\|\mathcal{C}\|\to 0} \mathcal{U}(f;\mathcal{C}) = M = \lim_{\|\mathcal{C}\|\to 0} \mathcal{L}(f;\mathcal{C}) \quad \text{where } M = \sup_{\mathcal{C}} \mathcal{L}(f;\mathcal{C}).$$

Since, for any $\mathcal{C}$ and associated Ξ, $\mathcal{L}(f;\mathcal{C}) \le \mathfrak{R}(f;\mathcal{C},\Xi) \le \mathcal{U}(f;\mathcal{C})$, it follows that f is Riemann integral and that M is its integral.

Turning to the next assertion, first suppose that f is continuous on $[a,b]$. Then, because it is uniformly continuous there, for each $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $x,\ y \in [a,b]$ and $|y-x| \le \delta_\epsilon$. Hence, if $\|\mathcal{C}\| < \delta_\epsilon$, then $\sup_I f - \inf_I f < \epsilon$ for all $I \in \mathcal{C}$, and so

$$\sum_{\substack{I\in\mathcal{C} \\ \sup_I f - \inf_I f \ge \epsilon}} |I| = 0.$$

Now suppose that f is continuous except at the points $a \le c_0 < \cdots < c_K \le b$. For each $r > 0$, f is uniformly continuous on $F_r \equiv [a,b] \setminus \bigcup_{k=0}^{K}(c_k - r, c_k + r)$. Given $\epsilon > 0$, choose $0 < r < \min\{c_k - c_{k-1} : 1 \le k \le K\}$ so that $2r(K+1) < \epsilon$, and then choose $\delta > 0$ so that $|f(y) - f(x)| < \epsilon$ for $x,\ y \in F_r$ with $|y-x| \le \delta$. Then one can easily construct a $\mathcal{C}$ such that $I_k \equiv [c_k - r, c_k + r] \cap [a,b] \in \mathcal{C}$ for each $0 \le k \le K$ and all the other I's in $\mathcal{C}$ have length less than δ, and for such a $\mathcal{C}$

$$\sum_{\substack{I\in\mathcal{C} \\ \sup_I f - \inf_I f \ge \epsilon}} |I| \le \sum_{k=0}^{K} |I_k| \le 2r(K+1) < \epsilon.$$

Finally, to prove the last assertion, let $\epsilon > 0$ be given and choose $0 < \epsilon' \le \epsilon$ so that $|\varphi(\eta) - \varphi(\xi)| < \epsilon$ if $\xi,\ \eta \in [c,d]$ with $|\eta - \xi| < \epsilon'$. Next choose $\mathcal{C}$ so that

$$\sum_{\substack{I\in\mathcal{C} \\ \sup_I f - \inf_I f \ge \epsilon'}} |I| < \epsilon',$$

and conclude that

$$\sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi\circ f-\inf_I \varphi\circ f\geq\epsilon}} |I| \;\leq \sum_{\substack{I\in\mathcal{C}\\ \sup_I f-\inf_I f\geq\epsilon'}} |I| \leq \epsilon' \leq \epsilon.$$

□

The fact that a bounded function is Riemann integrable if it is continuous at all but a finite number of points is important. For example, if f is a bounded, continuous function on (a, b), then its integral on $[a, b]$ can be unambiguously defined by extending f to $[a, b]$ in any convenient manner (e.g. taking $f(a) = f(b) = 0$), and then taking its integral to be the integral of the extension. The result will be the same no matter how the extension is made.

When applied to a non-negative, Riemann integrable function f on $[a, b]$, Theorem 3.1.2 should be convincing evidence that the procedure we suggested for computing the area of the region Ω gives the correct result. Indeed, given any $\mathcal{C}, \mathcal{U}(f;\mathcal{C})$ dominates the area of Ω and $\mathcal{L}(f;\mathcal{C})$ is dominated by the area of Ω. Hence, since by taking $\|\mathcal{C}\|$ small we can close the gap between $\mathcal{U}(f;\mathcal{C})$ and $\mathcal{L}(f;\mathcal{C})$, there can be little doubt that $\int_a^b f(x)\,dx$ is the area of Ω. More generally, when f takes both signs, one can interpret $\int_a^b f(x)\,dx$ as the difference between the area above the horizontal axis and the area below.

The following corollary deals with several important properties of Riemann integrals. In its statement and elsewhere, if $f : S \longrightarrow \mathbb{C}$,

$$\|f\|_S \equiv \sup\{|f(x)| : x \in S\}$$

is the *uniform norm* of f on S. Observe that

$$\|fg\|_S \leq \|f\|_S\|g\|_S \text{ and } \|\alpha f + \beta g\|_S \leq |\alpha|\|f\|_S + |\beta|\|g\|_S$$

for all $\mathbb{C}$-valued functions f and g on S and $\alpha, \beta \in \mathbb{C}$.

Corollary 3.1.3 *If $f : [a, b] \longrightarrow \mathbb{R}$ is a bounded, Riemann integrable function and $a < c < b$, then f is Riemann integrable on both $[a, c]$ and $[c, b]$, and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx \quad \textit{for all } c \in (a, b). \tag{3.1.1}$$

Further, if $\lambda > 0$ and f is a bounded, Riemann integrable function on $[\lambda a, \lambda b]$, then $x \in [a, b] \longmapsto f(\lambda x) \in \mathbb{R}$ is Riemann integrable and

$$\int_{\lambda a}^{\lambda b} f(y)\,dy = \lambda \int_a^b f(\lambda x)\,dx. \tag{3.1.2}$$

Next suppose that f and g are bounded, Riemann integrable functions on $[a,b]$. Then,

$$f \le g \implies \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \tag{3.1.3}$$

and so, if f is a bounded, Riemann integrable function on $[a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \left|\int_a^b |f(x)|\,dx\right| \le \|f\|_S |b-a|. \tag{3.1.4}$$

In addition, fg is Riemann integrable on $[a,b]$, and, for all $\alpha,\ \beta \in \mathbb{R}$, $\alpha f + \beta g$ is also Riemann integrable and

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx. \tag{3.1.5}$$

Proof To prove the first assertion, for a given $\epsilon > 0$ choose a non-overlapping cover $\mathcal{C}$ of $[a,b]$ so that

$$\sum_{\substack{I\in\mathcal{C}\\ \sup_I f - \inf_I f \ge \epsilon}} |I| < \epsilon,$$

and set $\mathcal{C}' = \{I \cap [a,c] : I \in \mathcal{C}\}$. Then, since

$$\sup\{f(y) - f(x) : x,\ y \in I \cap [a,c]\} \le \sup\{f(y) - f(x) : x,\ y \in I\}$$

and $|I \cap [a,c]| \le |I|$,

$$\sum_{\substack{I'\in\mathcal{C}'\\ \sup_{I'} f - \inf_{I'} f \ge \epsilon}} |I'| \le \sum_{\substack{I\in\mathcal{C}\\ \sup_I f - \inf_I f \ge \epsilon}} |I| < \epsilon.$$

Thus, f is Riemann integrable on $[a,c]$. The proof that f is also Riemann integrable on $[c,b]$ is the same. As for (3.1.1), choose $\{\mathcal{C}_n : n \ge 1\}$ and $\{\mathcal{C}'_n : n \ge 1\}$ and associated choice functions $\{\Xi_n : n \ge 1\}$ and $\{\Xi'_n : n \ge 1\}$ for $[a,c]$ and $[c,b]$ so that $\|\mathcal{C}_n\| \vee \|\mathcal{C}'_n\| \le \frac{1}{n}$, and set $\mathcal{C}''_n = \mathcal{C}_n \cup \mathcal{C}'_n$ and $\Xi''_n(I) = \Xi_n(I)$ if $I \in \mathcal{C}_n$ and $\Xi''_n(I) = \Xi'_n(I)$ if $I \in \mathcal{C}'_n \setminus \mathcal{C}_n$. Then

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty} \mathfrak{R}(f;\mathcal{C}''_n, \Xi''_n) = \lim_{n\to\infty} \mathfrak{R}(f;\mathcal{C}_n, \Xi_n) + \lim_{n\to\infty} \mathfrak{R}(f;\mathcal{C}'_n, \Xi'_n)\\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.\end{aligned}$$

Turning to the second assertion, set $f_\lambda(x) = f(\lambda x)$ for $x \in [a,b]$ and, given a cover $\mathcal{C}$ and a choice function Ξ, take $I_\lambda = \{\lambda x : x \in I\}$, and define

$\mathcal{C}_\lambda = \{I_\lambda : I \in \mathcal{C}\}$ and $\varXi_\lambda(I_\lambda) = \lambda\varXi(I)$ for $I \in \mathcal{C}$. Then $\mathcal{C}_\lambda$ is a non-overlapping cover for $[\lambda a, \lambda b]$, $\varXi_\lambda$ is an associated choice function, $\|\mathcal{C}_\lambda\| = \lambda\|\mathcal{C}\|$, and

$$\mathcal{R}(f_\lambda; \mathcal{C}, \varXi) = \sum_{I\in\mathcal{C}} f\big(\lambda\varXi(I)\big)|I| = \lambda^{-1}\mathcal{R}(f; \mathcal{C}_\lambda, \varXi_\lambda) \longrightarrow \lambda^{-1}\int_{\lambda a}^{\lambda b} f(y)\,dy$$

as $\|\mathcal{C}\| \to 0$.

Next suppose that f and g are bounded, Riemann integrable functions on $[a, b]$. Obviously, for all $\mathcal{C}$ and $\varXi$, $\mathcal{R}(f; \mathcal{C}, \varXi) \le \mathcal{R}(g; \mathcal{C}, \varXi)$ if $f \le g$ and

$$\mathcal{R}(\alpha f + \beta g; \mathcal{C}, \varXi) = \alpha\mathcal{R}(f; \mathcal{C}, \varXi) + \beta\mathcal{R}(g; \mathcal{C}, \varXi)$$

for all $\alpha, \beta \in \mathbb{R}$. Starting from these and passing to the limit as $\|\mathcal{C}\| \to 0$, one arrives at the conclusions in (3.1.3) and (3.1.5). Furthermore, (3.1.4) follows from (3.1.3), since, by the last part of Theorem 3.1.2, $|f|$ is Riemann integrable and $\pm f \le |f| \le \|f\|_{[a,b]}$. Finally, to see that fg is Riemann integrable, note that, by the preceding combined with the last part of Theorem 3.1.2, $(f+g)^2$ and $(f-g)^2$ are both Riemann integrable and therefore so is $fg = \frac{1}{4}\big((f+g)^2 - (f-g)^2\big)$. □

There is a useful notation convention connected to (3.1.1). Namely, if $a < b$, then one defines

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

With this convention, if $\{a_k : 0 \le k \le \ell\} \subseteq \mathbb{R}$ and f is function that is Riemann integrable on $[a_0, a_\ell]$ and $[a_k, a_{k+1}]$ for each $0 \le k < \ell$, one can use (3.1.1) to check that

$$\int_{a_0}^{a_\ell} f(x)\,dx = \sum_{k=0}^{\ell-1}\int_{a_k}^{a_{k+1}} f(x)\,dx. \tag{3.1.6}$$

Sometimes one wants to integrate functions that are complex-valued. Just as in the real-valued case, one says that a bounded function $f : [a, b] \longrightarrow \mathbb{C}$ is *Riemann integrable* and has *integral* $\int_a^b f(x)\,dx$ on $[a, b]$ if the Riemann sums

$$\mathcal{R}(f; \mathcal{C}, \varXi) = \sum_{I\in\mathcal{C}} f\big(\varXi(I)\big)|I|$$

converge to $\int_a^b f(x)\,dx$ as $\|\mathcal{C}\| \to 0$. Writing $f = u + iv$, where u and v are real-valued, one can easily check that f is Riemann integrable if and only if both u and v are, in which case

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

From this one sees that, except for (3.1.3), the obvious analogs of the assertions in Corollary 3.1.3 continue to hold for complex-valued functions. Of course, one can no longer use (3.1.3) to prove the first inequality in (3.1.4). Instead, one can use the triangle inequality to show that $|\mathfrak{R}(f; \mathcal{C}, \Xi)| \le \mathfrak{R}(|f|; \mathcal{C}, \Xi)$ and then take the limit as $\|\mathcal{C}\| \to 0$.

There are two closely related extensions of the Riemann integral. In the first place, one often has to deal with an interval $(a, b]$ on which there is a function f that is unbounded but is bounded and Riemann integrable on $[\alpha, b]$ for each $\alpha \in (a, b)$. Even though f is unbounded, it may be that the limit $\lim_{\alpha \searrow a} \int_\alpha^b f(x)\,dx$ exists in $\mathbb{C}$, in which case one uses $\int_{(a,b]} f(x)\,dx$ to denote the limit. Similarly, if f is bounded and Riemann integrable on $[a, \beta]$ for each $\beta \in (a, b)$ and $\lim_{\beta \nearrow b} \int_a^\beta f(x)\,dx$ exists or if f is bounded and Riemann integrable on $[\alpha, \beta]$ for all $a < \alpha < \beta < b$ and $\lim_{\substack{\alpha \searrow a \\ \beta \nearrow b}} \int_\alpha^\beta f(x)\,dx$ exists in $\mathbb{C}$, then one takes $\int_{[a,b)} f(x)\,dx$ or $\int_{(a,b)} f(x)\,dx$ to be the corresponding limit. The other extension is to situations when one or both of the endpoints are infinite. In this case one is dealing with a function f which is bounded and Riemann integrable on bounded, closed intervals of $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$, and one takes

$$\int_{(-\infty,b]} f(x)\,dx, \quad \int_{[a,\infty)} f(x)\,dx, \quad \text{or} \int_{(-\infty,\infty)} f(x)\,dx$$

to be

$$\lim_{a \searrow -\infty} \int_a^b f(x)\,dx, \ \lim_{b \nearrow \infty} \int_a^b f(x)\,dx, \ \text{or} \lim_{\substack{b \nearrow \infty \\ a \searrow -\infty}} \int_a^b f(x)\,dx$$

if the corresponding limit exists. Notice that in any of these situations, if f is non-negative then the corresponding limits exist in $[0, \infty)$ if and only if the quantities of which one is taking the limit stay bounded. More generally, if f is a $\mathbb{C}$-valued function on an interval J and if f is Riemann integrable on each bounded, closed interval $I \subseteq J$, then $\int_J f(x)\,dx$ will exist if $\sup_I \int_I |f(x)|\,dx < \infty$, in which is case f is said to be *absolutely Riemann integrable* on J. To check this, suppose that $J = (a, b]$. Then, for $a < \alpha_1 < \alpha_2 < b$,

$$\left|\int_{\alpha_2}^b f(x)\,dx - \int_{\alpha_1}^b f(x)\,dx\right| = \left|\int_{\alpha_1}^{\alpha_2} f(x)\,dx\right| \le \int_{\alpha_1}^{\alpha_2} |f(x)|\,dx,$$

and so the existence of the limit $\lim_{\alpha \searrow a} \int_\alpha^b f(x)\,dx \in \mathbb{C}$ follows from the existence of $\lim_{\alpha \searrow a} \int_\alpha^b |f(x)|\,dx \in [0, \infty)$. When J is $[a, b)$, (a, b), $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$, the argument is essentially the same.

The following theorem shows that integrals are continuous with respect to uniform convergence of their integrands.

Theorem 3.1.4 *If $\{f_n : n \geq 1\}$ is a sequence of bounded, Riemann integrable $\mathbb{C}$-valued functions on $[a, b]$ that converge uniformly to the function $f : [a, b] \longrightarrow \mathbb{C}$, then f is Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof Observe that

$$|\mathcal{R}(f_n; \mathcal{C}, \Xi) - \mathcal{R}(f; \mathcal{C}, \Xi)| \leq \mathcal{R}\big(|f_n - f|; \mathcal{C}, \Xi\big) \leq (b-a)\|f_n - f\|_{[a,b]},$$

and conclude from this first that $\left\{\int_a^b f_n(x)\,dx : n \geq 1\right\}$ satisfies Cauchy's convergence criterion and second that, for each n,

$$\overline{\lim_{\|\mathcal{C}\|\to 0}} \left|\mathcal{R}(f; \mathcal{C}, \Xi) - \int_a^b f_n(x)\,dx\right| \leq (b-a)\|f_n - f\|_{[a,b]}.$$

Hence, $\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \int_a^b f_n(x)\,dx$ exists, and $\mathcal{R}(f; \mathcal{C}, \Xi) \longrightarrow \int_a^b f(x)\,dx$ as $\|\mathcal{C}\| \to 0$. □

3.2 The Fundamental Theorem of Calculus

In the preceding section we developed a lot of theory for integrals but did not address the problem of actually evaluating them. To see that the theory we have developed thus far does little to make computations easy, consider the problem of computing $\int_a^b x^k\,dx$ for $k \in \mathbb{N}$. When $k = 0$, it is obvious that every Riemann sum will be equal to $(b - a)$, and therefore $\int_a^b x^0\,dx = b - a$. To handle $k \geq 1$, first note that $\int_a^b x^k\,dx = \int_0^b x^k\,dx - \int_0^a x^k\,dx$ and, for any $c \in \mathbb{R}$

$$\int_0^{-c} x^k\,dx = (-1)^{k+1} \int_0^c x^k\,dx.$$

Hence, it suffices to compute $\int_0^c x^k\,dx$ for $c > 0$. Further, by the scaling property in (3.1.2),

$$\int_0^c x^k\,dx = c^{k+1} \int_0^1 x^k\,dx.$$

Thus, everything comes down to computing $\int_0^1 x^k\,dx$. To this end, we look at Riemann sums of the form

$$\frac{1}{n}\sum_{m=1}^{n}\left(\frac{m}{n}\right)^k = \frac{S_n^{(k)}}{n^{k+1}} \quad \text{where } S_n^{(k)} \equiv \sum_{m=1}^{n} m^k.$$

When n is even, one can see that $S_n^{(1)} = \frac{n(n+1)}{2}$ by adding the 1 to n, 2 to $n-1$, etc. When n is odd, one gets the same conclusion by adding n to $S_{n-1}^{(1)}$. Hence, we have shown that

$$\int_0^1 x\,dx = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \frac{1}{2},$$

which is what one would hope is the area of the right triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$. When $k \geq 2$ one can proceed as follows. Write the difference $(n+1)^{k+1} - 1$ as the telescoping sum $\sum_{m=1}^{n}\bigl((m+1)^{k+1} - m^{k+1}\bigr)$. Next, expand $(m+1)^{k+1}$ using the binomial formula, and, after changing the order of summation, arrive at

$$(n+1)^{k+1} - 1 = \sum_{j=0}^{k}\binom{k+1}{j} S_n^{(j)}.$$

Starting from this and using induction on k, one sees that $\lim_{n\to\infty}\frac{S_n^{(k)}}{n^{k+1}} = \frac{1}{k+1}$. Hence, we now know that $\int_0^1 x^k\,dx = \frac{1}{k+1}$. Combining this with our earlier comments, we have

$$\int_a^b x^k\,dx = \frac{b^{k+1} - a^{k+1}}{k+1} \quad \text{for all } k \in \mathbb{N} \text{ and } a < b. \tag{3.2.1}$$

There is something that should be noticed about the result (3.2.1). Namely, if one looks at $\int_a^b x^k\,dx$ as a function of the upper limit b, then b^k is its derivative. That is, as a function of its upper limit, the derivative of this integral is the integrand. That this is true in general is one of Newton's great discoveries.[1]

Theorem 3.2.1 (Fundamental Theorem of Calculus) *Let* $f : [a, b] \longrightarrow \mathbb{C}$ *be a continuous function, and set* $F(x) = \int_a^x f(t)\,dt$ *for* $x \in [a, b]$. *Then* F *is continuous on* $[a, b]$, *continuously differentiable on* (a, b), *and* $F' = f$ *there. Conversely, if* $F : [a, b] \longrightarrow \mathbb{C}$ *is a continuous function that is continuously differentiable on* (a, b), *then*

$$F' = f \text{ on } (a, b) \implies F(b) - F(a) = \int_a^b f(x)\,dx.$$

[1] Although it was Newton who made this result famous, it had antecedents in the work of James Gregory and Newton's teacher Isaac Barrow. Mathematicians are not always reliable historians, and their attributions should be taken with a grain of salt.

Proof Without loss in generality, we will assume that f is $\mathbb{R}$-valued.

Let F be as in the first assertion. Then, by (3.1.1), for $x,\ y \in [a, b]$,

$$F(y) - F(x) = \int_x^y f(t)\,dt = f(x)(y-x) + \int_x^y \big(f(t) - f(x)\big)\,dt.$$

Given $\epsilon > 0$, choose $\delta > 0$ so that $|f(t) - f(x)| < \epsilon$ for $t \in [a, b]$ with $|t - x| < \delta$. Then, by (3.1.4),

$$\left|\int_x^y \big(f(t) - f(x)\big)\,dt\right| < \epsilon|y - x| \quad \text{if } |y - x| < \delta,$$

and so

$$\left|\frac{F(y) - F(x)}{y - x} - f(x)\right| < \epsilon \quad \text{for } x,\ y \in [a, b] \text{ with } 0 < |y - x| < \delta.$$

This proves that F is continuous on $[a, b]$, differentiable on (a, b), and $F' = f$ there.

If f and F are as in the second assertion, set $\Delta(x) = F(x) - \int_a^x f(t)\,dt$. Then, by the preceding, Δ is continuous of $[a, b]$, differentiable on (a, b), and $\Delta' = 0$ on (a, b). Hence, by (1.8.1), $\Delta(b) = \Delta(a)$, and so, since $\Delta(a) = F(a)$, the asserted result follows. □

It is hard to overstate the importance of Theorem 3.2.1. The hands-on method we used to integrate x^k is unable to handle more complicated functions. Instead, given a function f, one looks for a function F such that $F' = f$ and then applies Theorem 3.2.1 to do the calculation. Such a function F is called an *indefinite integral* of f. By Theorem 1.8.1, since the derivative of the difference between any two of its indefinite integrals is 0, two indefinite integrals of a function can differ by at most an additive constant. Once one has F, it is customary to write

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{b} \equiv F(b) - F(a).$$

Here are a couple of corollaries of Theorem 3.2.1.

Corollary 3.2.2 *Suppose that f and g are continuous, $\mathbb{C}$-valued functions on $[a, b]$ which are continuously differentiable on (a, b), and assume that their derivatives are bounded. Then*

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f(x)g'(x)\,dx$$

where $f(x)g(x)\Big|_{x=a}^{b} \equiv f(b)g(b) - f(a)g(a)$.

Proof By the product rule, $(fg)' = f'g + gf'$, and so, by Theorem 3.2.1 and (3.1.5),

$$f(\beta)g(\beta) - f(\alpha)g(\alpha) = \int_\alpha^\beta f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

for all $a < \alpha < \beta < b$. Now let $\alpha \searrow a$ and $\beta \nearrow b$. □

The equation in Corollary 3.2.2 is known as the *integration by parts formula*, and it is among the most useful tools available for computing integrals. For instance, it can be used to give another derivation of Taylor's theorem, this time with the remainder term expressed as an integral. To be precise, let $f : (a, b) \longrightarrow \mathbb{C}$ be an $(n+1)$ times continuously differentiable function. Then, for $x,\ y \in (a, b)$,

$$\begin{aligned} f(y) = \sum_{m=0}^{n} \frac{(y-x)^m}{m!} f^{(m)}(x) \\ + \frac{(y-x)^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n+1)}\big((1-t)x + ty\big)\,dt. \end{aligned} \tag{3.2.2}$$

To check this, set $u(t) = f\big((1-t)x + ty\big)$. Then

$$u^{(m)}(t) = (y-x)^m f^{(m)}\big((1-t)x + ty\big),$$

and, by Theorem 3.2.1, $u(1) - u(0) = \int_0^1 u'(t)\,dt$, which is (3.2.2) for $n = 1$. Next, assume that

$$(*) \qquad u(1) = \sum_{m=0}^{n-1} \frac{u^{(m)}(0)}{m!} + \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} u^{(n)}(t)\,dt$$

for some $n \geq 1$, and use integration by parts to see that

$$\int_0^1 (1-t)^{n-1} u^{(n)}(t)\,dt = -\frac{(1-t)^n u^{(n)}(t)}{n}\Big|_{t=0}^1 + \frac{1}{n}\int_0^1 (1-t)^n u^{(n+1)}(t)\,dt.$$

Hence, by induction, (3.2.2) holds for all $n \geq 1$.

A second application of integration by parts is to the derivation of *Wallis's formula*:

$$\frac{\pi}{2} = \lim_{n\to\infty} \prod_{m=1}^{n} \frac{2m}{2m-1}\frac{2m}{2m+1} = \lim_{n\to\infty} \frac{4^n (n!)^2}{(2n+1)\left(\prod_{m=1}^n (2m-1)\right)^2}. \tag{3.2.3}$$

To prove his formula, we begin by using integration by parts to see that

$$\int_0^{\frac{\pi}{2}} \cos^m t\,dt = \int_0^{\frac{\pi}{2}} \sin' t \cos^{m-1} t\,dt = (m-1)\int_0^{\frac{\pi}{2}} \sin^2 t \cos^{m-2} t\,dt$$
$$= (m-1)\int_0^{\frac{\pi}{2}} \cos^{m-2} t\,dt - (m-1)\int_0^{\frac{\pi}{2}} \cos^m t\,dt,$$

and therefore that

$$\int_0^{\frac{\pi}{2}} \cos^m t\,dt = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} \cos^{m-2} t\,dt \quad \text{for } m \ge 2.$$

Since $\int_0^{\frac{\pi}{2}} \cos t\,dt = 1$, this proves that

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,dt = \prod_{m=1}^{n} \frac{2m}{2m+1} \quad \text{for } n \ge 1.$$

At the same time, it shows that $\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \frac{\pi}{4}$ and therefore that

$$\int_0^{\frac{\pi}{2}} \cos^{2n} t\,dt = \frac{\pi}{2}\prod_{m=1}^{n} \frac{2m-1}{2m} \quad \text{for } n \ge 1.$$

Thus

$$\frac{\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,dt}{\int_0^{\frac{\pi}{2}} \cos^{2n} t\,dt} = \frac{2}{\pi}\,\frac{4^n (n!)^2}{(2n+1)\left(\prod_{m=1}^{n}(2m-1)\right)^2}.$$

Finally, since

$$1 \ge \frac{\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,dt}{\int_0^{\frac{\pi}{2}} \cos^{2n} t\,dt} = \frac{2n}{2n+1}\,\frac{\int_0^{\frac{\pi}{2}} \cos^{2n-1} t\,dt}{\int_0^{\frac{\pi}{2}} \cos^{2n} t\,dt} \ge \frac{2n}{2n+1},$$

(3.2.3) follows.

Aside from being a curiosity, as Stirling showed, Wallis's formula allows one to compute the constant e^{Δ} in (1.8.7). To understand what he did, observe that $\prod_{m=1}^{n}(2m-1) = \frac{(2n)!}{2^n n!}$ and therefore, by (1.8.7), that

$$\frac{4^n (n!)^2}{(2n+1)\left(\prod_{m=1}^{n+1}(2m-1)\right)^2} = \frac{1}{2n+1}\left(\frac{4^n (n!)^2}{(2n)!}\right)^2$$
$$\sim \frac{1}{2n+1}\left(\frac{4^n e^{\Delta} n \left(\frac{n}{e}\right)^{2n}}{\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}\right)^2 = \frac{e^{2\Delta} n}{4n+2}.$$

Hence, after letting $n \to \infty$ and applying (3.2.3), one sees that $e^{2\Delta} = 2\pi$ and therefore that (1.8.7) can be replaced by

$$\sqrt{2\pi}e^{-\frac{1}{n}} \le \frac{n!e^n}{n^{n+\frac{1}{2}}} \le \sqrt{2\pi}e^{\frac{1}{n}}, \tag{3.2.4}$$

or, more imprecisely, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Here is another powerful tool for computing integrals.

Corollary 3.2.3 *Let $\varphi : [a, b] \longrightarrow [c, d]$ be a continuous function, and assume that φ is continuously differentiable on (a, b) and that its derivative is bounded. If $f : [c, d] \longrightarrow \mathbb{C}$ is a continuous function, then*

$$\int_a^b \big(f \circ \varphi(t)\big)\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx.$$

In particular, if $\varphi' > 0$ on (a, b) and $\varphi(a) \le c < d \le \varphi(b)$, then for any continuous $f : [c, d] \longrightarrow \mathbb{C}$,

$$\int_c^d f(x)\,dx = \int_{\varphi^{-1}(c)}^{\varphi^{-1}(d)} \big(f \circ \varphi(t)\big)\varphi'(t)\,dt.$$

Proof Set $F(t) = \int_{\varphi(a)}^{\varphi(t)} f(x)\,dx$. Then, $F(a) = 0$ and, by the chain rule and Theorem 3.2.1, $F' = (f \circ \varphi)\varphi'$. Hence, again by Theorem 3.2.1,

$$F(b) = \int_a^b \big(f \circ \varphi(t)\big)\varphi'(t)\,dt.$$

□

The equation in Corollary 3.2.3 is called the *change of variables formula*. In applications one often uses the mnemonic device of writing $x = \varphi(t)$ and $dx = \varphi'(t)\,dt$. For example, consider the integral $\int_0^1 \sqrt{1 - x^2}\,dx$, and make the change of variables $x = \sin t$. Then $dx = \cos t\,dt$, $0 = \arcsin 0$, and $\frac{\pi}{2} = \arcsin 1$. Hence $\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt$, which, as we saw in connection with Wallis's formula, equals $\frac{\pi}{4}$. In that $\left\{\left(x, \sqrt{1 - x^2}\right) : x \in [0, 1]\right\}$ is the portion of the unit circle in the first quadrant, this is the answer that one should have expected.

Here is a slightly more theoretical application of Theorem 3.2.1.

Corollary 3.2.4 *Suppose that $\{\varphi_n : n \ge 1\}$ is a sequence of $\mathbb{C}$-valued continuous functions on $[a, b]$ that are continuously differentiable on (a, b). Further, assume that $\{\varphi_n : n \ge 1\}$ converges uniformly on $[a, b]$ to a function φ and that there is a function ψ on (a, b) such that $\varphi_n' \longrightarrow \psi$ uniformly on $[a + \delta, b - \delta]$ for each $0 < \delta < \frac{b-a}{2}$. Then φ is continuously differentiable on (a, b) and $\varphi' = \psi$.*

Proof Given $x \in (a,b)$, choose $0 < \delta < \frac{b-a}{2}$ so that $x \in (a+\delta, b-\delta)$. Then

$$\varphi(x) - \varphi(a+\delta) = \lim_{n\to\infty} \int_{a+\delta}^{x} \varphi_n'(y)\,dy = \int_{a+\delta}^{x} \psi(y)\,dy,$$

and so φ is differentiable at x and $\varphi'(x) = \psi(x)$. □

3.3 Rate of Convergence of Riemann Sums

In applications it is often very important to have an estimate on the rate at which Riemann sums are converging to the integral of a function. There are many results that deal with this question, and in this section we will show that there are circumstances in which the convergence is faster than one might have expected.

Given a continuous function $f : [0,1] \longrightarrow \mathbb{C}$, it is obvious that

$$\left| \mathcal{R}(f;\mathcal{C},\Xi) - \int_0^1 f(x)\,dx \right| \le \sup\bigl\{|f(y) - f(x)| : x,\ y \in [0,1] \text{ with } |y-x| \le \|\mathcal{C}\|\}$$

for any finite collection $\mathcal{C}$ of non-overlapping closed intervals whose union is $[0,1]$ and any choice function Ξ. Hence, if f is continuously differentiable on $(0,1)$, then

$$\left| \mathcal{R}(f;\mathcal{C},\Xi) - \int_0^1 f(x)\,dx \right| \le \|f'\|_{(0,1)}\|\mathcal{C}\|.$$

Moreover, even if f has more derivatives, this is the best that one can say in general. On the other hand, as we will now show, one can sometimes do far better.

For $n \ge 1$, take $\mathcal{C}_n = \{I_{m,n} : 1 \le m \le n\}$, where $I_{m,n} = \bigl[\frac{m-1}{n}, \frac{m}{n}\bigr]$, $\Xi_n(I_{m,n}) = \frac{m}{n}$, and, for continuous $f : [0,1] \longrightarrow \mathbb{C}$, set

$$\mathcal{R}_n(f) \equiv \mathcal{R}(f;\mathcal{C}_n,\Xi_n) = \frac{1}{n}\sum_{m=1}^{n} f\bigl(\tfrac{m}{n}\bigr).$$

Next, assume that f has a bounded, continuous derivative on $(0,1)$, and apply integration by parts to each term, to see that

$$\int_0^1 f(x)\,dx - \mathcal{R}_n(f) = \sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \bigl(f(x) - f\bigl(\tfrac{m}{n}\bigr)\bigr)\,dx$$

$$= \sum_{m=1}^{\infty} \int_{I_{m,n}} \bigl(x - \tfrac{m-1}{n}\bigr)'\Bigl(f(x) - f\bigl(\tfrac{m}{n}\bigr)\Bigr)\,dx = -\sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \bigl(x - \tfrac{m-1}{n}\bigr) f'(x)\,dx.$$

Now add the assumption that $f(1) = f(0)$. Then $\int_0^1 f'(x)\,dx = f(1) - f(0) = 0$, and so

$$\sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \tfrac{m-1}{n}\big) f'(x)\,dx = \sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \tfrac{m-1}{n} - c\big) f'(x)\,dx$$

for any constant c. In particular, by taking $c = \frac{1}{2n}$ to make each of the integrals $\int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \frac{m-1}{n} - c\big)\,dx = 0$, we can write

$$\int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \tfrac{m-1}{n} - \tfrac{1}{2n}\big) f'(x)\,dx = \int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \tfrac{m-1}{n} - \tfrac{1}{2n}\big)\big(f'(x) - f'(\tfrac{m}{n})\big)\,dx,$$

and thereby conclude that

$$\int_0^1 f(x)\,dx - \mathcal{R}_n(f) = -\sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \big(x - \tfrac{m-1}{n} - \tfrac{1}{2n}\big)\big(f'(x) - f'(\tfrac{m}{n})\big)\,dx.$$

This already represents progress. Indeed, because $\int_{\frac{m-1}{n}}^{\frac{m}{n}} \big|x - \frac{m-1}{n} - \frac{1}{2n}\big|\,dx = \frac{1}{4n^2}$ for each $1 \le m \le n$, we have shown that

$$\left|\int_0^1 f(x)\,dx - \mathcal{R}_n(f)\right| \le \frac{\sup\big\{|f'(y) - f'(x)| : |y - x| \le \frac{1}{n}\big\}}{4n}$$

if f is continuously differentiable and $f(1) = f(0)$. Hence, if in addition, f is twice differentiable, then $|f'(y) - f'(x)| \le \|f''\|_{(0,1)}|y - x|$ and the preceding leads to

$$(*) \qquad \left|\int_0^1 f(x)\,dx - \mathcal{R}_n(f)\right| \le \frac{\|f''\|_{(0,1)}}{4n^2}.$$

Before proceeding, it is important to realize how crucial a role the choice of both $\mathcal{C}_n$ and Ξ_n play. The role that $\mathcal{C}_n$ plays is reasonably clear, since it is what allowed us to choose the constant c independent of the intervals. The role of Ξ_n is more subtle. To see that it is essential, consider the function $f(x) = e^{i2\pi x}$, which is both smooth and satisfies $f(1) = f(0)$. Furthermore, $\|f'\|_{[0,1]} = 2\pi$ and $\int_0^1 f(x)\,dx = 0$. However, if $\tilde{\Xi}_n(I_{m,n}) = \frac{m(n-1)}{n^2}$, then

$$\mathcal{R}(f; \mathcal{C}_n, \tilde{\Xi}_n) = \frac{1}{n}\sum_{m=1}^{n} e^{i2\pi\frac{m(n-1)}{n^2}} = \frac{e^{i2\pi\frac{n-1}{n^2}}}{n}\,\frac{1 - e^{i2\pi\frac{n-1}{n}}}{1 - e^{i2\pi\frac{n-1}{n^2}}}.$$

Since $n\big(1-e^{i2\pi\frac{n-1}{n}}\big)=n\big(1-e^{-i2\pi\frac{1}{n}}\big)\longrightarrow i2\pi$ and $n\big(1-e^{i2\pi\frac{n-1}{n^2}}\big)\longrightarrow -i2\pi$, it follows that

$$\lim_{n\to\infty} n\left|\mathcal{R}(f;\mathcal{C}_n,\tilde{\Xi}_n)-\int_0^1 f(x)\,dx\right|=1.$$

Thus, the analog of (∗) does not hold if one replaces Ξ_n by $\tilde{\Xi}_n$.

To go further, we introduce the notation

$$\Delta_n^{(k)}(f)\equiv\frac{1}{k!}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\big(x-\tfrac{m-1}{n}\big)^k\big(f(x)-f(\tfrac{m}{n})\big)\,dx\quad\text{for } k\ge 0.$$

Obviously, $\Delta_n^{(0)}(f)=\int_0^1 f(x)\,dx-\mathcal{R}_n(f)$. Furthermore, what we showed when f is continuously differentiable and $f(1)=f(0)$ is that $\Delta_n^{(0)}(f)=\frac{1}{2n}\Delta_n^{(0)}(f')-\Delta_n^{(1)}(f')$. By essentially the same argument, what we will now show is that, under the same assumptions on f,

$$\Delta_n^{(k)}(f)=\frac{1}{(k+2)!n^{k+1}}\Delta_n^{(0)}(f')-\Delta_n^{(k+1)}(f')\quad\text{for } k\ge 0. \tag{3.3.1}$$

The first step is to integrate each term by parts and thereby get

$$\Delta_n^{(k)}(f)=-\frac{1}{(k+1)!}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\big(x-\tfrac{m-1}{n}\big)^{k+1}f'(x)\,dx. \tag{3.3.2}$$

Because $\int_0^1 f'(x)\,dx=0$, the right hand side does not change if one subtracts $\frac{1}{(k+2)n^{k+1}}$ from each $\big(x-\frac{m-1}{n}\big)^{k+1}$, and, once this subtraction is made, one can, with impunity, subtract $f'(\frac{m}{n})$ from $f'(x)$ in each term. In this way one arrives at

$$\Delta_n^{(k)}(f)=-\frac{1}{(k+1)!}\sum_{m=1}^{n}\int_{\frac{m-1}{n}}^{\frac{m}{n}}\Big(\big(x-\tfrac{m-1}{n}\big)^{k+1}-\tfrac{1}{(k+2)n^{k+1}}\Big)\big(f'(x)-f'(\tfrac{m}{n})\big)\,dx,$$

which, after rearrangement, is (3.3.1).

We will say that a function $\varphi:\mathbb{R}\longrightarrow\mathbb{C}$ is *periodic*[2] if $\varphi(x+1)=\varphi(x)$ for all $x\in\mathbb{R}$. Notice that if $\varphi:\mathbb{R}\longrightarrow\mathbb{C}$ is a bounded periodic function that is Riemann integrable on bounded intervals, then, by (3.1.6),

$$\int_a^{a+1}\varphi(\xi)\,d\xi=\int_0^1\varphi(\xi)\,d\xi\quad\text{for all } a\in\mathbb{R}. \tag{3.3.3}$$

[2]In general, a function $f:\mathbb{R}\longrightarrow\mathbb{C}$ is said to be periodic if there is some $\alpha>0$ such that $f(x+\alpha)=f(x)$ for all $x\in\mathbb{R}$, in which case α is said to be a period of f. Here, without further comment, we will always be dealing with the case when $\alpha=1$ unless some other period is specified.

To check this, suppose that $n \le a < n+1$. Then, by (3.1.6),

$$\int_a^{a+1} \varphi(\xi)\,d\xi = \int_a^{n+1} \varphi(\xi)\,d\xi + \int_{n+1}^{a+1} \varphi(\xi)\,d\xi$$
$$= \int_a^{n+1} \varphi(\xi)\,d\xi + \int_n^{a} \varphi(\xi)\,\xi = \int_n^{n+1} \varphi(\xi)\,d\xi = \int_0^{1} \varphi(\xi)\,d\xi.$$

Now assume that $f : \mathbb{R} \longrightarrow \mathbb{C}$ is periodic and has $\ell \ge 1$ continuous derivatives. Starting from (3.3.1) and working by induction on ℓ, we see that

$$\Delta_n^{(0)}(f) = \frac{1}{n^\ell}\sum_{k=0}^{\ell} a_{k,\ell} n^{k+1} \Delta_n^{(k)}(f^{(\ell)}),$$

where

$$a_{0,0} = 1, \quad a_{0,\ell+1} = \sum_{k=0}^{\ell} \frac{a_{k,\ell}}{(k+2)!}, \quad \text{and } a_{k,\ell+1} = -a_{k-1,\ell} \text{ for } 1 \le k \le \ell.$$

The preceding can be simplified by observing that $a_{k,\ell} = (-1)^k a_{0,\ell-k}$ for $0 \le k \le \ell$, which means that

$$\Delta_n^{(0)}(f) = \frac{1}{n^{\ell+1}}\sum_{k=0}^{\ell} (-1)^k b_{\ell-k} n^{k+1} \Delta_n^{(k)}(f^{(\ell)})$$
$$\text{where } b_0 = 1 \text{ and } b_{\ell+1} = \sum_{k=0}^{\ell} \frac{(-1)^k}{(k+2)!} b_{\ell-k}. \tag{3.3.4}$$

If we now assume that f is $(\ell+1)$ times continuously differentiable, then, by (3.3.2),

$$\left|\Delta_n^{(k)}(f^{(\ell)})\right| \le \frac{1}{(k+1)!}\sum_{m=1}^{n} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \left(x - \tfrac{m-1}{n}\right)^{k+1} |f^{(\ell+1)}(x)|\,dx \le \frac{\|f^{(\ell+1)}\|_{[0,1]}}{(k+2)!n^{k+1}},$$

and so (3.3.4) leads to the estimate (see Exercise 3.13 for a related estimate)

$$\left|\int_0^1 f(x)\,dx - \mathcal{R}_n(f)\right| \le \frac{K_\ell \|f^{(\ell+1)}\|_{[0,1]}}{n^{\ell+1}} \quad \text{where } K_\ell = \sum_{k=0}^{\ell} \frac{|b_{\ell-k}|}{(k+2)!} \tag{3.3.5}$$

for periodic functions f having $(\ell+1)$ continuous derivatives. In other words, if f is periodic and has $(\ell+1)$ continuous derivatives, then the Riemann sum $\mathcal{R}_n(f)$ differs from $\int_0^1 f(x)\,dx$ by at most the constant K_ℓ times $\|f^{(\ell+1)}\|_{[0,1]} n^{-\ell-1}$.

To get a feeling for how K_ℓ grows with ℓ, begin by taking $f(x) = e^{i2\pi x}$. Then $\Delta_1^{(0)}(f) = -1$, $\|f^{(\ell+1)}\|_{[0,1]} = (2\pi)^{\ell+1}$, and so (3.3.5) says that $K_\ell \ge (2\pi)^{-\ell-1}$. To get a crude upper bound, let α be the unique element of $(0, 1)$ for which $e^{\frac{1}{\alpha}} = 1 + \frac{2}{\alpha}$, and use induction on ℓ to see that $|b_\ell| \le \alpha^\ell$. Hence, we now know that

$$(2\pi)^{-\ell-1} \le K_\ell \le e^{\frac{1}{\alpha}} \alpha^{\ell+2}.$$

Below we will get a much more precise result (cf. (3.4.9)), but in the meantime it should be clear that (3.3.5) is saying that the convergence of $\mathcal{R}_n(f)$ to $\int_0^1 f(x)\,dx$ is very fast when f is periodic, smooth, and the successive derivatives of f are growing at a moderate rate.

3.4 Fourier Series

Taylor's Theorem provides a systematic method for finding polynomial approximations to a function by scrutinizing in great detail the behavior of the function at a point. Although the method has many applications, it also has flaws. In fact, as we saw in the discussion following Lemma 1.8.3, there are circumstances in which it yields no useful information. As that example shows, the problem is that the behavior of a function away from a point cannot always be predicted from the behavior of it and its derivatives at the point. Speaking metaphorically, Taylor's method is analogous to attempting to lift an entire plank from one end.

Fourier introduced a very different approximation procedure, one in which the approximation is in terms of waves rather than polynomials. He took the trigonometric functions $\{\cos(2\pi mx) : m \ge 0\}$ and $\{\sin(2\pi mx) : m \ge 1\}$ as the model waves into which he wanted to resolve other functions. That is, he wanted to write a general continuous function $f : [0, 1] \longrightarrow \mathbb{C}$ as a usually infinite linear combination of the form

$$f(x) = \sum_{m=0}^{\infty} a_m \cos(2\pi mx) + \sum_{m=1}^{\infty} b_m \sin(2\pi mx),$$

where the coefficients a_m and b_m are complex numbers. Using (2.2.1), one sees that by taking $\hat{f}_0 = a_0$, $\hat{f}_m = \frac{a_m - ib_m}{2}$ for $m \ge 1$, and $\hat{f}_m = \frac{a_{-m}+ib_{-m}}{2}$ for $m \le -1$, an equivalent and more convenient expression is

$$f(x) = \sum_{m=-\infty}^{\infty} \hat{f}_m \mathfrak{e}_m(x) \quad \text{where } \mathfrak{e}_m(x) \equiv e^{i2\pi mx}. \tag{3.4.1}$$

One of Fourier's key observations is that if one assumes that f can be represented in this way and that the series converges well enough, then the coefficients $\hat{f}_m$ are given by

$$\hat{f}_m \equiv \int_0^1 f(x)\mathfrak{e}_{-m}(x)\,dx. \tag{3.4.2}$$

To see this, observe that (cf. Exercise 3.7 for an alternative method)

$$\begin{aligned}
&\int_0^1 \mathfrak{e}_m(x)\mathfrak{e}_{-n}(x)\,dx \\
&\quad = \int_0^1 \big(\cos(2\pi mx)\cos(2\pi nx) + \sin(2\pi mx)\sin(2\pi nx)\big)\,dx \\
&\qquad\qquad + i\int_0^1 \big(-\cos(2\pi mx)\sin(2\pi nx) + \sin(2\pi mx)\cos(2\pi nx)\big)\,dx \\
&\quad = \int_0^1 \cos\big(2\pi(m-n)x\big)\,dx + i\int_0^1 \sin\big(2\pi(m-n)x\big)\,dx = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m \neq n, \end{cases}
\end{aligned}$$

where, in the passage to the last line, we used (1.5.1). Hence, if the exchange in the order of summation and integration is justified, (3.4.2) follows from (3.4.1).

We now turn to the problem of justifying Fourier's idea. That is, if $\hat{f}_m$ is given by (3.4.2), we want to examine to what extent it is true that f is represented by (3.4.1). Thus let a continuous $f : [0, 1] \longrightarrow \mathbb{C}$ be given, determine $\hat{f}_m$ accordingly by (3.4.2), and define

$$f_r(x) = \sum_{m=-\infty}^{\infty} r^{|m|}\hat{f}_m\mathfrak{e}_m(x) \quad \text{for } r \in [0,1) \text{ and } x \in \mathbb{R}. \tag{3.4.3}$$

Because $|\hat{f}_m| \leq \|f\|_{[0,1]}$, the series defining f_r is absolutely convergent. In order to understand what happens when $r \nearrow 1$, observe that

$$\begin{aligned}
\sum_{m=-n}^{n} r^{|m|}\hat{f}_m\mathfrak{e}_m(x) &= \int_0^1 \left(\sum_{m=0}^{n}\big(r\mathfrak{e}_1(x-y)\big)^m\right) f(y)\,dy \\
&\qquad + \int_0^1 \left(\sum_{m=1}^{n}\big(r\mathfrak{e}_1(y-x)\big)^m\right) f(y)\,dy \\
&= \int_0^1 \left(\frac{1 - r^{n+1}\mathfrak{e}_{n+1}(x-y)}{1 - r\mathfrak{e}_1(x-y)} + \frac{r\mathfrak{e}_1(y-x) - r^{n+1}\mathfrak{e}_{n+1}(y-x)}{1 - r\mathfrak{e}_1(y-x)}\right) f(y)dy \\
&= \int_0^1 \frac{1 - r^2 - r^{n+1}\cos\big(2\pi(n+1)(x-y)\big) + r^{n+2}\cos(2\pi n(x-y))}{|1 - r\mathfrak{e}_1(x-y)|^2} f(y)\,dy.
\end{aligned}$$

Hence, by Theorem 3.1.4,

$$f_r(x) = \sum_{m=-\infty}^{\infty} r^{|m|}\hat{f}_m\mathfrak{e}_m(x) = \int_0^1 p_r(x-y)f(y)\,dy$$

$$\text{where } p_r(\xi) \equiv \frac{1-r^2}{|1 - r\mathfrak{e}_1(\xi)|^2} \text{ for } \xi \in \mathbb{R}. \tag{3.4.4}$$

Obviously the function p_r is positive, periodic, and even: $p_r(-\xi) = p_r(\xi)$. Furthermore, by taking $f = \mathbf{1}$ and noting that then $\hat{f}_0 = 1$ and $\hat{f}_m = 0$ for $m \neq 0$, we see from (3.4.4) and evenness that $\int_0^1 p_r(x - y)\, dy = 1$. Finally, note that if $\delta \in \big(0, \frac{1}{2}\big)$ and $\xi \notin \bigcup_{n=-\infty}^{\infty}[n - \delta, n + \delta]$, then

$$(*) \qquad |1 - r\mathfrak{e}_1(\xi)|^2 \geq 2\big(1 - \cos(2\pi\xi)\big) \geq \omega(\delta) \equiv 2\big(1 - \cos(2\pi\delta)\big) > 0$$

and therefore $p_r(\xi) \leq \frac{1-r^2}{\omega(\delta)}$.

Given a function $f : [0, 1] \longrightarrow \mathbb{C}$, its *periodic extension* to $\mathbb{R}$ is the function $\tilde{f}$ given by $\tilde{f}(x) = f(x - n)$ for $n \leq x < n + 1$. Clearly, $\tilde{f}$ will be continuous if and only if f is continuous and $f(0) = f(1)$. In addition, if $\ell \geq 1$, then $\tilde{f}$ will be ℓ times continuously differentiable if and only if f is ℓ times continuously differentiable on $(0, 1)$ and the limits $\lim_{x\searrow 0} f^{(k)}(x)$ and $\lim_{x\nearrow 1} f^{(k)}(x)$ exist and are equal for each $0 \leq k \leq \ell$.

Theorem 3.4.1 *Let $f : [0, 1] \longrightarrow \mathbb{C}$ be a continuous function, and define f_r for $r \in [0, 1)$ as in (3.4.3). Then, for each $r \in [0, 1)$, f_r is a periodic function with continuous derivatives of all orders. In fact, for each $k \geq 0$, the series $\sum_{m=-\infty}^{\infty}(i2\pi|m|)^k r^{|m|} \hat{f}_m \mathfrak{e}_m(x)$ is absolutely and uniformly convergent to $f_r^{(k)}(x)$. Furthermore, for each $\delta \in \big(0, \frac{1}{2}\big)$, $f_r \longrightarrow f$ as $r \nearrow 1$ uniformly on $[\delta, 1 - \delta]$. Finally, if $f(0) = f(1)$, then $f_r \longrightarrow f$ uniformly on $[0, 1]$.*

Proof Since each term in the sum defining f_r is periodic, it is obvious that f_r is also. In addition, since the sum converges uniformly on $[0, 1]$, f_r is continuous. To prove that f_r has continuous derivatives of all orders, we begin by applying Theorem 2.3.2 to see that $\sum_{m=-\infty}^{\infty} |m|^k r^{|m|} \leq 2\sum_{m=0}^{\infty} m^k r^m < \infty$.

In view of the preceding, we know that $\sum_{m=-\infty}^{\infty}(i2\pi|m|)r^{|m|}\hat{f}_m\mathfrak{e}_m(x)$ converges uniformly for $x \in [0, 1]$ to a function ψ. At the same time, if $\varphi_n(x) \equiv \sum_{m=-n}^{n} r^{|m|}\hat{f}e_m(x)$, then φ_n converges uniformly to f_r and $\varphi_n' = \sum_{m=-n}^{n}(i2\pi m)r^{|m|}$ $\hat{f}_m\mathfrak{e}_m(x)$ converges uniformly to ψ. Hence, by Corollary 3.2.4, f_r is differentiable and $f_r' = \psi$. More generally, assume that f_r has k continuous derivatives and that $f_r^{(k)} = \sum_{m=-\infty}^{\infty} r^{|m|}(i2\pi|m|)^k \hat{f}_m\mathfrak{e}_m$. Then a repetition of the preceding argument shows that $f_r^{(k)}$ is differentiable and that its derivative is $\sum_{m=-\infty}^{\infty} r^{|m|}(i2\pi|m|)^{k+1}\hat{f}_m\mathfrak{e}_m$.

Now take $\tilde{f}$ to be the periodic extension of f to $\mathbb{R}$. Since $\tilde{f}$ is bounded and Riemann integrable on bounded intervals, (3.3.3) together with $\int_0^1 p_r(x - y)\, dy = 1$ show that

$$f_r(x) - f(x) = \int_0^1 p_r(x - y)\big(f(y) - f(x)\big)\, dy$$
$$= \int_{-x}^{1-x} p_r(\xi)\big(f(\xi + x) - f(x)\big)\, d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} p_r(\xi)\big(\tilde{f}(\xi + x) - \tilde{f}(x)\big)\, d\xi$$

$$= \int_{-\frac{1}{2}}^{-\delta} p_r(\xi)\big(\tilde{f}(\xi+x) - \tilde{f}(x)\big)\,d\xi + \int_{-\delta}^{\delta} p_r(\xi)\big(\tilde{f}(\xi+x) - \tilde{f}(x)\big)\,d\xi$$
$$+ \int_{\delta}^{\frac{1}{2}} p_r(\xi)\big(\tilde{f}(\xi+x) - \tilde{f}(x)\big)\,d\xi$$

for $x \in [0, 1]$ and $0 < \delta < \frac{1}{2}$. Using (∗), one sees that the first and third terms in the final expression are dominated by $2\|f\|_{[0,1]}\frac{1-r^2}{\omega(\delta)}$ and therefore tend to 0 as $r \nearrow 1$. As for the second term, it is dominated by $\sup\{|\tilde{f}(y) - \tilde{f}(x)| : |y-x| \le \delta\}$. Hence, if $\tilde{f}$ is continuous at x, as it will be if $x \in (0, 1)$, then

$$\lim_{r\nearrow 1}\big|f_r(x) - f(x)\big| = \lim_{r\nearrow 1}\left|\int_0^1 p_r(x-y)\big(f(y) - f(x)\big)\,dy\right| = 0.$$

Moreover, the convergence is uniform on the interval $[\delta, 1-\delta]$, and if $f(0) = f(1)$ and therefore $\tilde{f}$ is continuous everywhere, then the convergence is uniform on [0, 1]. □

Even though the preceding result is a weakened version of it, we now know that Fourier's idea is basically sound. One important fact to which this weak version leads is the identity

$$\int_0^1 f(x)\overline{g(x)}\,dx = \sum_{m=-\infty}^{\infty} \hat{f}_m\overline{\hat{g}_m}. \tag{3.4.5}$$

To prove this, first observe that, because $\int_0^1 \mathfrak{e}_{m_1}(x)\mathfrak{e}_{-m_2}(x)\,dx$ is 0 if $m_1 \ne m_2$ and 1 if $m_1 = m_2$, one can use Theorem 3.1.4 to justify

$$\int_0^1 f_r(x)\overline{g_r(x)}\,dx = \sum_{m_1,m_2=-\infty}^{\infty} r^{|m_1|+|m_2|}\hat{f}_{m_1}\overline{\hat{g}_{m_2}}\int_0^1\int_0^1 \mathfrak{e}_{m_1}(x)\mathfrak{e}_{-m_2}(x)\,dx$$
$$= \sum_{m=-\infty}^{\infty} r^{2|m|}\hat{f}_m\overline{\hat{g}_m}.$$

At the same time, for each $\delta \in \big(0, \frac{1}{2}\big)$,

$$\overline{\lim_{r\nearrow 1}}\int_0^1 \big|f_r(x)\overline{g_r(x)} - f(x)\overline{g(x)}\big|\,dx$$
$$\le \overline{\lim_{r\nearrow 1}}\int_0^{\delta} \big|f_r(x)\overline{g_r(x)} - f(x)\overline{g(x)}\big|\,dx + \overline{\lim_{r\nearrow 1}}\int_{1-\delta}^1 \big|f_r(x)\overline{g_r(x)} - f(x)\overline{g(x)}\big|\,dx$$
$$\le 4\|f\|_{[0,1]}\|g\|_{[0,1]}\delta,$$

and so $\lim_{r\nearrow 1}\int_0^1 f_r(x)\overline{g_r(x)}\,dx = \int_0^1 f(x)\overline{g(x)}\,dx$. Taking $g = f$, we conclude that

$$\int_0^1 |f(x)|^2\,dx = \lim_{r\nearrow 1}\sum_{m=-\infty}^{\infty} r^{2|m|}|\hat{f}_m|^2,$$

and therefore that $\sum_{m=-\infty}^{\infty}|\hat{f}_m|^2 < \infty$. Hence, by Schwarz's inequality (cf. Exercise 2.3),

$$\sum_{m=-\infty}^{\infty}|\hat{f}_m||\hat{g}_m| \le \sqrt{\sum_{m=-\infty}^{\infty}|\hat{f}_m|^2}\sqrt{\sum_{m=-\infty}^{\infty}|\hat{g}_m|^2} < \infty,$$

and so the series $\sum_{m=-\infty}^{\infty}\hat{f}_m\overline{\hat{g}_m}$ is absolutely convergent. Thus, when we apply (1.10.1), we find that

$$\sum_{m=-\infty}^{\infty}\hat{f}_m\overline{\hat{g}_m} = \lim_{r\nearrow 1}\sum_{m=-\infty}^{\infty} r^{2|m|}\hat{f}_m\overline{\hat{g}_m} = \int_0^1 f(x)\overline{g(x)}\,dx.$$

The identity in (3.4.5) is known as *Parseval's equality*, and it has many interesting applications of which the following is an example. Take $f(x) = x$. Obviously, $\hat{f}_0 = \frac{1}{2}$, and, using integration by parts, one sees that $\hat{f}_m = \frac{1}{i2\pi m}$ for $m \ne 0$. Hence, by (3.4.5),

$$\frac{1}{3} = \int_0^1 |f(x)|^2\,dx = \frac{1}{4} + \frac{1}{4\pi^2}\sum_{m\ne 0}\frac{1}{m^2} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2},$$

from which we see that

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}. \tag{3.4.6}$$

The function $\zeta(z)$ given by $\sum_{m=1}^{\infty}\frac{1}{m^z}$ when the real part of z is greater than 1 is the famous *Riemann zeta function* which plays an important role in number theory (cf. Sect. 6.3). We now know the value of ζ at $z = 2$, and, as we will show below, we can also find its value at all positive, even integers. However, in order to do that computation, we will need to discuss when the *Fourier series* $\sum_{m=-\infty}^{\infty}\hat{f}_m\mathfrak{e}_m(x)$ converges to f. This turns out to be a very delicate question, and we will not attempt to describe any of the refined answers that have been found. Instead, we will deal only with the most elementary case.

Theorem 3.4.2 *Let $f : [0,1] \longrightarrow \mathbb{C}$ be a continuous function. If $\sum_{m=-\infty}^{\infty}|\hat{f}_m| < \infty$, then $f(0) = f(1)$ and $\sum_{m=-\infty}^{\infty}\hat{f}_m\mathfrak{e}_m$ converges uniformly to f. In particular, if $f(1) = f(0)$ and f has a bounded, continuous derivative on* $(0,1)$, *then $\sum_{m=-\infty}^{\infty}|\hat{f}_m| < \infty$ and therefore $\sum_{m=-\infty}^{\infty}\hat{f}_m\mathfrak{e}_m$ converges uniformly to f.*

Proof We already know that $f_r(x) \longrightarrow f(x)$ for all $x \in (0, 1)$, and therefore, by (1.10.1), if $x \in (0, 1)$ and $\sum_{m=-\infty}^{\infty} \hat{f}_m \mathfrak{e}_m(x)$ converges, it must converge to $f(x)$. Thus, since $|\hat{f}_m \mathfrak{e}_m(x)| \le |\hat{f}_m|$ if $\sum_{m=-\infty}^{\infty} |\hat{f}_m| < \infty$, $\sum_{m=-\infty}^{\infty} \hat{f}_m \mathfrak{e}_m(x)$ converges absolutely and uniformly on [0, 1] to a continuous, periodic function, and, because that function coincides with f on (0, 1), it must be equal to f on [0, 1].

Now assume that $f(1) = f(0)$ is and that f has a bounded, continuous derivative f' on (0, 1). Using integration by parts and Exercise 3.7, we see that, for $m \ne 0$,

$$\begin{aligned}\hat{f}_m &= \lim_{\delta\searrow 0}\int_\delta^{1-\delta} f(x)\mathfrak{e}_{-m}(x)\,dx\\ &= \lim_{\delta\searrow 0}\frac{1}{i2\pi m}\left(f(\delta)\mathfrak{e}_{-m}(\delta) - f(1-\delta)\mathfrak{e}_{-m}(1-\delta) + \int_\delta^{1-\delta} f'(x)\mathfrak{e}_{-m}(x)\,dx\right)\\ &= \frac{\widehat{f'_m}}{i2\pi m}.\end{aligned}$$

Hence, by Schwarz's inequality, Parseval's equality, and (3.4.6),

$$\sum_{m\ne 1}|\hat{f}_m| \le \frac{1}{2\pi}\left(\sum_{m\ne 0}\frac{1}{m^2}\right)^{\frac{1}{2}}\left(\sum_{m\ne 0}|\widehat{f'_m}|^2\right)^{\frac{1}{2}} \le \sqrt{\frac{1}{24}}\left(\int_0^1 |f'(x)|^2\,dx\right)^{\frac{1}{2}}. \qquad \square$$

Notice that the integration by parts step in the preceding has the following easy extension. Namely, suppose that $f : [0, 1] \longrightarrow \mathbb{C}$ is continuous and that f has $\ell \ge 1$ bounded, continuous derivatives on (0, 1). Further, assume that $f^{(k)}(0) \equiv \lim_{x\searrow 0} f^{(k)}(x)$ and $f^{(k)}(1) \equiv \lim_{x\nearrow 1} f(x)$ exist and are equal for $0 \le k < \ell$. Then, by iterating the argument given above, one sees that

$$\hat{f}_m = (i2\pi m)^{-\ell}\widehat{(f^{(\ell)})}_m \quad \text{for } m \in \mathbb{Z}\setminus\{0\}.$$

Returning to the computation of $\zeta(2\ell)$, recall the numbers b_k introduced in (3.3.4), and set

$$P_\ell(x) = \sum_{k=0}^{\ell}\frac{(-1)^k b_{\ell-k}}{k!}x^k \quad \text{for } \ell \ge 0 \text{ and } x \in \mathbb{R}.$$

Then $P_0 \equiv 1$ and $P_\ell = -P'_{\ell+1}$. In addition, if $\ell \ge 2$, then

$$P_\ell(1) = b_\ell - b_{\ell-1} + \sum_{k=2}^{\ell}\frac{(-1)^k b_{\ell-k}}{k!}$$

and, by (3.3.4),

$$\sum_{k=2}^{\ell} \frac{(-1)^k b_{\ell-k}}{k!} = \sum_{k=0}^{\ell-2} \frac{(-1)^k b_{\ell-2-k}}{(k+2)!} = b_{\ell-1}.$$

Hence $P_\ell(1) = b_\ell = P_\ell(0)$ for all $\ell \geq 2$. In particular, these lead to

$$(\widehat{P_\ell})_0 = -\int_0^1 P'_{\ell+1}(x)\,dx = P_{\ell+1}(0) - P_{\ell+1}(1) = 0 \quad \text{for} \quad \ell \geq 1$$

and

$$(\widehat{P_\ell})_m = (-1)^{\ell-1}(i2\pi m)^{1-\ell}(\widehat{P_1})_m \quad \text{for} \quad \ell \geq 2 \quad \text{and} \quad m \neq 0.$$

Since $P_1(x) = b_1 - b_0 x = \frac{1}{2} - x$, we can use integration by parts to see that

$$(\widehat{P_1})_m = -\int_0^1 x e^{-i2\pi m x}\,dx = (i2\pi m)^{-1} \quad \text{for} \quad m \neq 0,$$

and therefore

$$P_\ell(x) = -\left(\frac{i}{2\pi}\right)^\ell \sum_{m\neq 0} \frac{\mathfrak{e}_m(x)}{m^\ell} \quad \text{for} \quad \ell \geq 2 \quad \text{and} \quad x \in [0,1]. \tag{3.4.7}$$

Taking $x = 0$ in (3.4.7), we have that

$$b_{2\ell+1} = 0 \quad \text{and} \quad b_{2\ell} = \frac{(-1)^{\ell+1} 2\zeta(2\ell)}{(2\pi)^{2\ell}} \tag{3.4.8}$$

for $\ell \geq 1$. Knowing that $b_\ell = 0$ for odd $\ell \geq 3$, the recursion relation in (3.3.4) for b_ℓ simplifies to

$$b_\ell = \begin{cases} 2^{-\ell} & \text{if } \ell \in \{0,1\} \\ \frac{1}{2\ell!} - \sum_{k=0}^{\frac{\ell-2}{2}} \frac{b_{2k}}{(\ell-2k+1)!} & \text{if } \ell \geq 2 \text{ is even} \\ 0 & \text{if } \ell \geq 2 \text{ is odd.} \end{cases} \tag{3.4.9}$$

Now we can go the other direction and use (3.4.8) and (3.4.9) to compute ζ at even integers:

$$\zeta(2\ell) = (-1)^{\ell+1} 2^{2\ell-1} \pi^{2\ell} b_{2\ell} \quad \text{for } \ell \geq 1. \tag{3.4.10}$$

Finally, starting from (3.4.10), one sees that the K_ℓ's in (3.3.5) satisfy $\lim_{\ell\to\infty}(K_\ell)^{\frac{1}{\ell}} = (2\pi)^{-1}$.

In the literature, the numbers $\ell! b_\ell$ are called the *Bernoulli numbers*, and they have an interesting history. Using (3.4.9) together with (3.4.10), one recovers (3.4.6) and sees that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$. Using these relations to compute ζ at larger even integers is elementary but tedious. Perhaps more interesting than such computations is the observation that, when $\ell \geq 2$, $P_\ell(x)$ is an ℓth order polynomial whose periodic extension from $[0, 1]$ is $(\ell - 2)$ times differentiable. That such polynomials exist is not obvious.

3.5 Riemann–Stieltjes Integration

The topic of this concluding section is an easy but important generalization, due to Stieltjes, of Riemann integration. Namely, given bounded, $\mathbb{R}$-valued functions φ and ψ on $[a, b]$, a finite cover $\mathcal{C}$ of $[a, b]$ by non-overlapping closed intervals I, and an associated choice function Ξ, set

$$\mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi) = \sum_{I \in \mathcal{C}} \varphi\big(\Xi(I)\big) \Delta_I \psi,$$

where $\Delta_I \psi$ denotes the difference between the value of ψ at the right hand end point of I and its value at the left hand end point. We will say that φ is *Riemann–Stieltjes integrable* on $[a, b]$ with respect to ψ if there exists a number $\int_a^b \varphi(x)\, d\psi(x)$, known as the *Riemann–Stieltjes* of φ with respect to ψ, such that for each $\epsilon > 0$ there is a $\delta > 0$ for which

$$\left| \mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi) - \int_a^b \varphi(x)\, d\psi(x) \right| < \epsilon$$

whenever $\|\mathcal{C}\| < \delta$ and Ξ is any associated choice function for $\mathcal{C}$. Obviously, when $\psi(x) = x$, this is just Riemann integration. In addition, it is clear that if φ_1 and φ_2 are Riemann–Stieltjes integrable with respect to ψ, then, for all $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \varphi_1 + \alpha_2 \varphi_2$ is also and

$$\int_a^b \big(\alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x)\big)\, d\psi(x) = \alpha_1 \int_a^b \varphi_1(x)\, d\psi(x) + \alpha_2 \int_a^b \varphi_2(x)\, d\psi(x).$$

Also, if $\check{\psi}(x) = \psi(a + b - x)$ and φ is Riemann–Stieltjes integrable with respect to ψ, then $x \rightsquigarrow \varphi(a + b - x)$ is Riemann–Stieltjes integrable with respect to $\check{\psi}$ and

$$\int_a^b \varphi(a + b - x)\, d\check{\psi}(x) = -\int_a^b \varphi(x)\, d\psi(x). \tag{3.5.1}$$

In general it is hard to determine which functions φ are Riemann–Stieltjes integrable with respect to a given function ψ. Nonetheless, the following simple lemma shows that there is an inherent symmetry between the roles of φ and ψ.

Lemma 3.5.1 *If φ is Riemann–Stieltjes integrable with respect to ψ, then ψ is Riemann–Stieltjes integrable with respect to φ and*

$$\int_a^b \varphi(x)\,d\psi(x) = \varphi(b)\psi(b) - \varphi(a)\psi(b) - \int_a^b \psi(x)\,d\varphi(x). \tag{3.5.2}$$

Proof Let $\mathcal{C} = \big\{[\alpha_{m-1}, \alpha_m] : 1 \le m \le n\big\}$, where $a = \alpha_0 \le \cdots \le \alpha_n = b$, and let Ξ be an associated choice function. Set $\beta_0 = a$, $\beta_m = \Xi([\alpha_{m-1}, \alpha_m])$ for $1 \le m \le n$, and $\beta_{n+1} = b$, and define $\mathcal{C}' = \{[\beta_{m-1}, \beta_m] : 1 \le m \le n+1\}$ and $\Xi'([\beta_{m-1}, \beta_m]) = \alpha_{m-1}$ for $1 \le m \le n+1$. Then

$$\begin{aligned}
\mathcal{R}(\psi|\varphi; \mathcal{C}, \Xi) &= \sum_{m=1}^{n} \psi(\beta_m)\big(\varphi(\alpha_m) - \varphi(\alpha_{m-1})\big) \\
&= \sum_{m=1}^{n} \psi(\beta_m)\varphi(\alpha_m) - \sum_{m=0}^{n-1} \psi(\beta_{m+1})\varphi(\alpha_m) \\
&= \psi(\beta_n)\varphi(\alpha_n) - \sum_{m=1}^{n-1} \varphi(\alpha_m)\big(\psi(\beta_{m+1}) - \psi(\beta_m)\big) - \psi(\beta_1)\varphi(\alpha_0) \\
&= \psi(b)\varphi(b) - \psi(a)\varphi(a) - \sum_{m=0}^{n} \varphi(\alpha_m)\big(\psi(\beta_{m+1}) - \psi(\beta_m)\big) \\
&= \psi(b)\varphi(b) - \psi(a)\varphi(a) - \mathcal{R}(\varphi|\psi; \mathcal{C}', \Xi').
\end{aligned}$$

Noting that $\|\mathcal{C}'\| \le 2\|\mathcal{C}\|$, one now sees that if φ is Riemann–Stieltjes integrable with respect to ψ, then ψ is Riemann–Stieltjes integrable with respect to φ and (3.5.2) holds. □

As we will see, Lemma 3.5.1 is an interesting generalization of the integration by parts, but it does little to advance us toward an understanding of the basic problem. In addressing that problem, the following analog of Theorem 3.1.2 will play a central role.

Lemma 3.5.2 *If ψ is non-decreasing on $[a, b]$, then φ is Riemann–Stieltjes integrable with respect to ψ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\|\mathcal{C}\| < \delta \implies \sum_{\substack{I \in \mathcal{C} \\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta_I \psi < \epsilon.$$

In particular, every continuous function φ is Riemann–Stieltjes integrable with respect to ψ. In addition, if φ is Riemann–Stieltjes integrable with respect to ψ and $c \in (a, b)$, then it is Riemann–Stieltjes integrable with respect of ψ on both $[a, c]$ and $[c, b]$ and

$$\int_a^b \varphi(x)\, d\psi(x) = \int_a^c \varphi(x)\, d\psi(x) + \int_c^b \varphi(x)\, d\psi(x).$$

Finally, if $\varphi : [a, b] \longrightarrow [c, d]$ is Riemann–Stieltjes integrable with respect to ψ and $f : [c, d] \longrightarrow \mathbb{R}$ is continuous, then $f \circ \varphi$ is again Riemann–Stieltjes integrable with respect to ψ.

Proof Once the first part is proved, the other assertions follow in exactly the same way as the analogous assertions followed from Theorem 3.1.2.

In proving the first part, we will assume, without loss in generality, that $\Delta \equiv \Delta_{[a,b]}\psi > 0$. Now suppose that φ is Riemann–Stieltjes integrable with respect to ψ. Given $\epsilon > 0$, choose $\delta > 0$ so that

$$\|\mathcal{C}\| \le \delta \implies \left|\mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi) - \int_a^b \varphi(x)\, d\psi(x)\right| \le \frac{\epsilon^2}{4}$$

for all associated choice functions Ξ. Next, given $\mathcal{C}$, choose, for each $I \in \mathcal{C}$, $\Xi_1(I),\ \Xi_2(I) \in I$ so that $\varphi\big(\Xi_1(I)\big) \ge \sup_I \varphi - \frac{\epsilon^2}{4\Delta}$ and $\varphi\big(\Xi_2(I)\big) \le \inf_I \varphi + \frac{\epsilon^2}{4\Delta}$. Then $\|\mathcal{C}\| < \delta$ implies that

$$\frac{\epsilon^2}{2} \ge \mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi_1) - \mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi_2) \ge \sum_{I \in \mathcal{C}} \left(\sup_I \varphi - \inf_I \varphi\right) \Delta_I \psi - \frac{\epsilon^2}{2},$$

and so

$$\epsilon^2 \ge \epsilon \sum_{\substack{I \in \mathcal{C} \\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta_I \psi.$$

To prove the converse, we introduce the upper and lower Riemann–Stieltjes sums

$$\mathcal{U}(\varphi|\psi; \mathcal{C}) = \sum_{I \in \mathcal{C}} \sup_I \varphi \Delta_I \psi \text{ and } \mathcal{L}(\varphi|\psi; \mathcal{C}) = \sum_{I \in \mathcal{C}} \inf_I \varphi \Delta_I \psi.$$

Just as in the Riemann case, one sees that

$$\mathcal{L}(\varphi|\psi; \mathcal{C}) \le \mathfrak{R}(\varphi|\psi; \mathcal{C}, \Xi) \le \mathcal{U}(\varphi|\psi; \mathcal{C})$$

for all $\mathcal{C}$ and associated rate functions Ξ, and $\mathcal{L}(\varphi|\psi; \mathcal{C}) \le \mathcal{U}(\varphi|\psi; \mathcal{C}')$ for all $\mathcal{C}$ and $\mathcal{C}'$. Further,

$$\mathcal{U}(\varphi|\psi;\mathcal{C}) - \mathcal{L}(\varphi|\psi;\mathcal{C}) = \sum_{I\in\mathcal{C}} \left(\sup_I \varphi - \inf_I \varphi\right) \Delta_I \psi$$
$$\leq \epsilon\Delta + 2\|\varphi\|_{[a,b]} \sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta_I \psi,$$

and so, under the stated condition, for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|\mathcal{C}\| < \delta \implies \mathcal{U}(\varphi|\psi;\mathcal{C}) - \mathcal{L}(\varphi|\psi;\mathcal{C}) < \epsilon.$$

As a consequence, we know that if $\|\mathcal{C}\| < \delta$ then, for any $\mathcal{C}'$

$$\mathcal{U}(\varphi|\psi;\mathcal{C}) \leq \mathcal{L}(\varphi|\psi;\mathcal{C}) + \epsilon \leq \mathcal{U}(\varphi|\psi;\mathcal{C}') + \epsilon,$$

and similarly, $\mathcal{L}(\varphi|\psi;\mathcal{C}) \geq \mathcal{L}(\varphi|\psi;\mathcal{C}') - \epsilon$. From these it follows that $M \equiv \inf_{\mathcal{C}} \mathcal{U}(\varphi|\psi;\mathcal{C}) = \sup_{\mathcal{C}} \mathcal{L}(\varphi|\psi;\mathcal{C})$ and

$$\lim_{\|\mathcal{C}\|\to 0} \mathcal{U}(\varphi|\psi;\mathcal{C}) = M = \lim_{\|\mathcal{C}\|\to 0} \mathcal{L}(\varphi|\psi;\mathcal{C}),$$

and at this point the rest of the argument is the same as the one in the proof of Theorem 3.1.2. □

The reader should take note of the distinction between the first assertion here and the analogous one in Theorem 3.1.2. Namely, in Theorem 3.1.2, the condition for Riemann integrability was that there exist some $\mathcal{C}$ for which

$$\sum_{\substack{I\in\mathcal{C}\\ \sup_I f - \inf_I f > \epsilon}} |I| < \epsilon,$$

whereas here we insist that

$$\|\mathcal{C}\| < \delta \implies \sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta_I \psi < \epsilon.$$

The reason for this is that the analog of the final assertion in Lemma 3.1.1 is true for ψ only if ψ is continuous. When ψ is continuous, then the condition that $\|\mathcal{C}\| < \delta$ can be removed from the first assertion in Lemma 3.5.2.

In order to deal with ψ's that are not monotone, we introduce the quantity

$$\operatorname{var}_{[a,b]}(\psi) \equiv \sup_{\mathcal{C}} \sum_{I\in\mathcal{C}} |\Delta_I \psi| < \infty,$$

where $\mathcal{C}$ denotes a generic finite cover of $[a, b]$ by non-overlapping closed intervals, and say that ψ has *finite variation* on $[a, b]$ if $\operatorname{var}_{[a,b]}(\psi) < \infty$. It is easy to check that

$\text{var}_{[a,b]}(\psi_1+\psi_2) \le \text{var}_{[a,b]}(\psi_1) + \text{var}_{[a,b]}(\psi_2)$ and that $\text{var}_{[a,b]}(\psi) = |\psi(b)-\psi(a)|$ if ψ is monotone (i.e., it is either non-decreasing or non-increasing). Thus, if ψ can be written as the difference between two non-increasing functions ψ^+ and ψ^-, then it has bounded variation and

$$\text{var}_{[a,b]}(\psi) \le \big(\psi^+(b) - \psi^+(a)\big) + \big(\psi^-(b) - \psi^-(a)\big).$$

We will now show that every function of bounded variation admits such a representation. To this end, define

$$\text{var}^{\pm}_{[a,b]}(\psi) = \sup_{\mathcal{C}} \sum_{I\in\mathcal{C}} (\Delta_I\psi)^{\pm}.$$

Lemma 3.5.3 *If ψ has bounded variation on $[a,b]$, then*

$$\Delta_{[a,b]}\psi = \text{var}^{+}_{[a,b]}(\psi) - \text{var}^{-}_{[a,b]}(\psi) \textit{ and } \text{var}_{[a,b]}(\psi) = \text{var}^{+}_{[a,b]}(\psi) + \text{var}^{-}_{[a,b]}(\psi).$$

Proof Obviously,

$$\sum_{I\in\mathcal{C}} |\Delta_I\psi| = \sum_{I\in\mathcal{C}} (\Delta_I\psi)^+ + \sum_{I\in\mathcal{C}} (\Delta_I\psi)^-$$

and

$$\Delta_{[a,b]}\psi = \sum_{I\in\mathcal{C}} (\Delta_I\psi)^+ - \sum_{I\in\mathcal{C}} (\Delta_I\psi)^-.$$

From the first of these, it is clear that $\text{var}_{[a,b]}(\psi) \le \text{var}^{+}_{[a,b]}(\psi) + \text{var}^{-}_{[a,b]}(\psi)$. From the second we see that $\text{var}^{\pm}_{[a,b]}(\psi) \le \text{var}^{\mp}_{[a,b]}(\psi) \pm \Delta_{[a,b]}\psi$, and therefore that $\Delta_{[a,b]}\psi = \text{var}^{+}_{[a,b]}(\psi) - \text{var}^{-}_{[a,b]}(\psi)$. Hence, if $\lim_{n\to\infty} \sum_{I\in\mathcal{C}_n} (\Delta_I\psi)^+ = \text{var}^{+}_{[a,b]}(\psi)$, then $\lim_{n\to\infty} \sum_{I\in\mathcal{C}_n} (\Delta_I\psi)^- = \text{var}^{-}_{[a,b]}(\psi)$, and so

$$\text{var}_{[a,b]}(\psi) \ge \lim_{n\to\infty} \left(\sum_{I\in\mathcal{C}_n} (\Delta_I\psi)^+ + \sum_{I\in\mathcal{C}_n} (\Delta_I\psi)^- \right) = \text{var}^{+}_{[a,b]}(\psi) + \text{var}^{-}_{[a.b]}(\psi).$$

□

Given a function ψ of bounded variation on $[a,b]$, define $V_\psi(x) = \text{var}_{[a,x]}(\psi)$ and $V^{\pm}_\psi(x) = \text{var}^{\pm}_{[a,x]}(\psi)$ for $x \in [a,b]$. Then V_ψ, V^+_ψ, and V^-_ψ are all non-decreasing functions that vanish at a, and, by Lemma 3.5.3, $\psi(x) = \psi(a) + V^+_\psi(x) - V^-_\psi(x)$ and $V_\psi(x) = V^+_\psi(x) + V^-_\psi(x)$ for $x \in [a,b]$.

Theorem 3.5.4 *Let ψ be a function of bounded variation on $[a,b]$, and refer to the preceding. If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is a bounded function, then φ is Riemann–Stieltjes* *integrable with respect to V_ψ if and only if it is Riemann–Stieltjes integrable with respect to both V_ψ^+ and V_ψ^-, in which case*

$$\int_a^b \varphi(x)\,dV_\psi(x) = \int_a^b \varphi(x)\,dV_\psi^+(x) + \int_a^b \varphi(x)\,dV_\psi^-(x).$$

Moreover, if φ is Riemann–Stieltjes integrable with respect to V_ψ, then it is Riemann–Stieltjes integrable with respect to ψ,

$$\int_a^b \varphi(x)\,d\psi(x) = \int_a^b \varphi(x)\,dV_\psi^+(x) - \int_a^b \varphi(x)\,dV_\psi^-(x),$$

and

$$\left|\int_a^b \varphi(x)\,d\psi(x)\right| \le \int_a^b |\varphi(x)|\,dV_\psi(x) \le \|\varphi\|_{[a,b]}\mathrm{var}_{[a,b]}(\psi).$$

Proof Since $V_\psi = V_\psi^+ + V_\psi^-$, it is clear that

$$\sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta V_\psi < \epsilon \iff \sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta V_\psi^+ + \sum_{\substack{I\in\mathcal{C}\\ \sup_I \varphi - \inf_I \varphi > \epsilon}} \Delta V_\psi^- < \epsilon,$$

and therefore, by Lemma 3.5.2, φ is Riemann–Stieltjes integrable with respect to V_ψ if and only if it is with respect to V_ψ^+ and V_ψ^-. Furthermore, because

$$\mathfrak{R}(\varphi|V_\psi;\mathcal{C},\Xi) = \mathfrak{R}(\varphi|V_\psi^+;\mathcal{C},\Xi) + \mathfrak{R}(\varphi|V_\psi^-;\mathcal{C},\Xi)$$

and

$$\mathfrak{R}(\varphi|\psi;\mathcal{C},\Xi) = \mathfrak{R}(\varphi|V_\psi^+;\mathcal{C},\Xi) - \mathfrak{R}(\varphi|V_\psi^-;\mathcal{C},\Xi),$$

the other assertions follow easily. □

Finally, there is an important case in which Riemann–Stieltjes integrals reduce to Riemann integrals. Namely, if ψ is continuous on $[a,b]$ and continuously differentiable on (a,b), then, by (1.8.1),

$$\sum_{I\in\mathcal{C}} |\Delta_I\psi| \le \|\psi'\|_{(a.b)}(b-a),$$

and so ψ will have bounded variation if ψ' is bounded. Furthermore, if ψ' is bounded and $\varphi : [a,b] \longrightarrow \mathbb{R}$ is a bounded function which is Riemann integrable on $[a,b]$, then φ is Riemann–Stieltjes integrable with respect to ψ and

$$\int_a^b \varphi(x)\,d\psi(x) = \int_a^b \varphi(x)\psi'(x)\,dx. \tag{3.5.3}$$

To prove this, note that if $I \in \mathcal{C}$, then one can apply (1.8.1) to find an $\eta(I) \in I$ such that $\Delta_I \psi = \psi'\big(\eta(I)\big)|I|$. Thus

$$\mathfrak{R}(\varphi|\psi;\mathcal{C},\Xi) = \sum_{I\in\mathcal{C}} \varphi\big(\eta(I)\big)\psi'\big(\eta(I)\big)|I| + \sum_{I\in\mathcal{C}}\Big(\varphi\big(\Xi(I)\big) - \varphi\big(\eta(I)\big)\Big)\psi'\big(\eta(I)\big)|I|$$

and

$$\left|\sum_{I\in\mathcal{C}}\Big(\varphi\big(\Xi(I)\big) - \varphi\big(\eta(I)\big)\Big)\psi'\big(\eta(I)\big)|I|\right| \le \|\psi'\|_{(a,b)}\big(\mathcal{U}(\varphi;\mathcal{C}) - \mathcal{L}(\varphi;\mathcal{C})\big),$$

which, since φ is Riemann integrable, tends to 0 as $\|\mathcal{C}\| \to 0$. Hence, since $\varphi\psi'$, as the product of two Riemann integrable functions, is Riemann integrable, we see that $\mathfrak{R}(\varphi|\psi;\mathcal{C},\Xi) \longrightarrow \int_a^b \varphi(x)\psi'(x)\,dx$ as $\|\mathcal{C}\| \to 0$. The content of (3.5.3) is often abbreviated by the equation $d\psi(x) = \psi'(x)\,dx$. Notice that when φ and ψ are both continuously differentiable on (a,b), then (3.5.2) is the integration by parts formula.

3.6 Exercises

Exercise 3.1 Most integrals defy computation. Here are a few that don't. In each case, compute the following integrals.

(i) $\int_{[1,\infty)} \frac{1}{x^2}\,dx$ (ii) $\int_{[0,\infty)} e^{-x}\,dx$
(iii) $\int_a^b \sin x\,dx$ (iv) $\int_a^b \cos x\,dx$
(v) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ (vi) $\int_{[0,\infty)} \frac{1}{1+x^2}\,dx$
(vii) $\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$ (viii) $\int_0^1 \frac{x}{x^4+1}\,dx$

Exercise 3.2 Here are some integrals that play a role in Fourier analysis. In evaluating them, it may be helpful to make use of (1.5.1). Compute

(i) $\int_0^{2\pi} \sin(mx)\cos(nx)\,dx$ (ii) $\int_0^{2\pi} \sin^2(mx)\,dx$
(iii) $\int_0^{2\pi} \cos^2(mx)\,dx$,

for $m,\ n \in \mathbb{N}$.

Exercise 3.3 For $t > 0$, define *Euler's Gamma function* $\Gamma(t)$ for $t > 0$ by

$$\Gamma(t) = \int_{(0,\infty)} x^{t-1}e^{-x}\,dx.$$

Show that $\Gamma(t+1) = t\Gamma(t)$, and conclude that $\Gamma(n+1) = n!$ for $n \geq 1$. See (5.4.4) for the evaluation of $\Gamma\left(\frac{1}{2}\right)$.

Exercise 3.4 Find indefinite integrals for the following functions:

(i) x^α for $\alpha \in \mathbb{R}$ & $x \in (0, \infty)$ (ii) $\log x$
(iii) $\frac{1}{x \log x}$ for $x \in (0, \infty) \setminus \{1\}$ (iv) $\frac{(\log x)^n}{x}$ for $n \in \mathbb{Z}^+$ & $x \in (0, \infty)$.

Exercise 3.5 Let $\alpha, \beta \in \mathbb{R}$, and assume that $(\alpha a) \vee (\alpha b) \vee (\beta a) \vee (\beta b) < 1$. Compute $\int_a^b \frac{1}{(1-\alpha x)(1-\beta x)}\,dx$. When $\alpha = \beta = 0$, there is nothing to do, and when $\alpha = \beta \neq 0$, it is obvious that $\big(\alpha(1-\alpha x)\big)^{-1}$ is an indefinite integral. When $\alpha \neq \beta$, one can use the method of *partial fractions* and write

$$\frac{1}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right).$$

See Theorem 6.3.2 for a general formulation of this procedure.

Exercise 3.6 Suppose that $f : (a, b) \longrightarrow [0, \infty)$ has continuous derivatives of all orders. Then f is said to be *absolutely monotone* if it and all its derivatives are non-negative. If f is absolutely monotone, show that for each $c \in (a, b)$

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(c)}{m!}(x-c)^m \quad \text{for } x \in [c, b),$$

an observation due to S. Bernstein. In doing this problem, reduce to the case when $c = 0 \in (a, b)$, and, using (3.2.2), observe that $f(y)$ dominates

$$\frac{y^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(ty)\,dt \geq \left(\frac{y}{x}\right)^n \frac{x^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(tx)\,dt$$

for $n \geq 1$ and $0 < x < y < b$.

Exercise 3.7 For $\alpha \in \mathbb{R} \setminus \{0\}$, show that

$$\int_a^b e^{i\alpha t}\,dt = \frac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}.$$

Next, write $\cos t = \frac{e^{it}+e^{-it}}{2}$, and apply the preceding and the binomial formula to show that

$$\int_0^{2\pi} \cos^n t\,dt = \begin{cases} 0 & \text{if } n \in \mathbb{N} \text{ is odd} \\ 2^{-n+1}\pi\binom{n}{\frac{n}{2}} & \text{if } n \in \mathbb{N} \text{ is even.} \end{cases}$$

By combining this with (3.2.4), show that $\lim_{n\to\infty} n^{\frac{1}{2}} \int_0^{2\pi} \cos^{2n} t\,dt = 2\pi^{\frac{1}{2}}$.

Exercise 3.8 A more conventional way to introduce the logarithm function is to define it by

$$(*) \qquad \log y = \int_1^y \frac{1}{t}\,dt \quad \text{for } y \in (0,\infty).$$

The purpose of this exercise is to show, without knowing our earlier definition, that this definition works.

(**i**) Suppose that $\ell : (0,\infty) \longrightarrow \mathbb{R}$ is a continuous function with the properties that $\ell(xy) = \ell(x) + \ell(y)$ for all $x,\ y \in (0,\infty)$ and $\ell(a) > 0$ for some $a > 1$. By applying Exercise 1.10 to the function $f(x) = \ell(a^x)$, show that $\ell\big(a^x\big) = x\ell(a)$.

(**ii**) Referring to (**i**), show that ℓ is strictly increasing, tends to ∞ as $y \to \infty$ and to $-\infty$ as $y \searrow 0$. Conclude that there is a unique $b \in (1,\infty)$ such that $\ell(b) = 1$.

(**iii**) Continuing (**i**) and (**ii**), and again using Exercise 1.10, conclude that $\ell(b^x) = x$ for $x \in \mathbb{R}$ and $b^{\ell(y)} = y$ for $y \in (0,\infty)$. That is, ℓ is the logarithm function with base b

(**iv**) Show that the function log given by $(*)$ satisfies the conditions in (**i**) and therefore that there exists a unique $e \in (1,\infty)$ for which it is the logarithm function with base e.

Exercise 3.9 Show that

$$\lim_{x\to\infty} \frac{\log x}{x} \int_e^x \frac{1}{\log t}\,dt = 1.$$

Exercise 3.10 Let $f : [0,1] \longrightarrow \mathbb{C}$ be a continuous function whose periodic extension is continuously differentiable. Show that

$$\int_0^1 \left| f(x) - \int_0^1 f(y)\,dy \right|^2 dx = \int_0^1 |f(x)|^2\,dx - \left| \int_0^1 f(y)\,dy \right|^2 = \sum_{m\neq 0} |\hat{f}_m|^2$$
$$\leq (2\pi)^{-2} \sum_{m\neq 0} (2\pi m)^2 |\hat{f}_m|^2 = (2\pi)^{-2} \int_0^1 |f'(x)|^2\,dx.$$

As a consequence, one has the *Poincaré inequality*

$$\int_0^1 \left| f(x) - \int_0^1 f(y)\,dy \right|^2 dx \leq (2\pi)^{-2} \int_0^1 |f'(x)|^2\,dx$$

for any function whose periodic extension is continuously differentiable.

Exercise 3.11 Let $f : [a,b] \longrightarrow \mathbb{C}$ be a continuous function, and set $L = b - a$. If $\delta \in \big(0, \frac{L}{2}\big)$, show that, as $r \nearrow 1$,

$$\frac{1}{L}\sum_{m=-\infty}^{\infty} r^{|m|}\left(\int_a^b f(y)\mathfrak{e}_{-m}\left(\tfrac{y}{L}\right)dy\right)\mathfrak{e}_m\left(\tfrac{x}{L}\right)$$

converges to $f(x)$ uniformly for $x \in [a+\delta, b-\delta]$ and that the convergence is uniform for $x \in [a,b]$ if $f(b) = f(a)$. Perhaps the easiest way to do this is to consider the function $g(x) = f(a+Lx)$ and apply Theorem 3.4.1 to it. Next, assume that $f(a) = f(b)$ and that f has a bounded, continuous derivative on (a,b). Show that

$$f(x) = \frac{1}{L}\sum_{m=-\infty}^{\infty}\left(\int_a^b f(y)\mathfrak{e}_{-m}\left(\tfrac{y}{L}\right)dy\right)\mathfrak{e}_m\left(\tfrac{x}{L}\right),$$

where the convergence is absolute and uniform on $[a,b]$.

Exercise 3.12 Let $f : [0,1] \longrightarrow \mathbb{C}$ be a continuous function, and show that, as $r \nearrow 1$,

$$f_r(x) \equiv 2\sum_{m=1}^{\infty} r^m\left(\int_0^1 f(y)\sin(m\pi y)\,dy\right)\sin \pi x$$

converges uniformly to f on $[\delta, 1-\delta]$ for each $\delta \in \left(0,\frac{1}{2}\right)$. One way to do this is to define $g : [-1,1] \longrightarrow \mathbb{C}$ so that $g = f$ on $[0,1]$ and $g(x) = -f(-x)$ when $x \in [-1,0)$, and observe that

$$\int_{-1}^1 g(y)\mathfrak{e}_{-m}\left(\tfrac{y}{2}\right)dy = -2i\int_0^1 f(y)\sin(m\pi y)\,dy.$$

If $f(0) = 0$ and therefore g is continuous, one need only apply Exercise 3.11 to g to see that $f_r \longrightarrow f$ uniformly on $[0, 1-\delta]$ for each $\delta \in \left(0,\frac{1}{2}\right)$. When $f(0) \neq$ and therefore g is discontinuous at 0, after examining the proof of Theorem 3.4.1, one can show that $f_r \longrightarrow f$ on $[\delta, 1-\delta]$ because g is uniformly continuous there.

Next show that if $f : [0,1] \longrightarrow \mathbb{C}$ is continuous, then

$$\int_0^1 |f(x)|^2\,dx = 2\sum_{m=1}^{\infty}\left|\int_0^1 f(x)\sin(m\pi x)\,dx\right|^2.$$

Finally, assuming that $f(0) = 0 = f(1)$ and that f has a bounded, continuous derivative on $(0,1)$, show that

$$f(x) = 2\sum_{m=1}^{\infty}\left(\int_0^1 f(y)\sin \pi y\,dy\right)\sin \pi x,$$

where the convergence of the series is absolute and uniform on $[0,1]$.

Exercise 3.13 Fourier series provide an alternative and more elegant approach to proving estimates like the one in (3.3.5). To see this, suppose that $f : [0, 1] \longrightarrow \mathbb{C}$ is a function whose periodic extension is $\ell \geq 1$ times continuously differentiable. Then, as we have shown,

$$f(x) = \int_0^1 f(y)\,dy + \sum_{m \neq 0} \frac{\widehat{(f^{(\ell)})}_m}{(i2\pi m)^\ell} \mathfrak{e}_m(x),$$

where the series converges uniformly and absolutely. After showing that

$$\frac{1}{n}\sum_{k=1}^{n} \mathfrak{e}_m\left(\tfrac{k}{n}\right) = \begin{cases} 1 & \text{if } m \text{ is divisible by } n \\ 0 & \text{otherwise,} \end{cases}$$

conclude that

$$\mathcal{R}_n(f) - \int_0^1 f(y)\,dy = \sum_{m \neq 0} \frac{\widehat{(f^{(\ell)})}_{mn}}{(i2\pi mn)^\ell},$$

and from this show that

$$\left|\mathcal{R}_n(f) - \int_0^1 f(y)\,dy\right| \leq \frac{2\|f^{(\ell)}\|_{[0,1]}\zeta(\ell)}{(2\pi n)^\ell}.$$

Finally, use Schwarz's inequality and (3.4.5) to derive the estimate

$$\left|\mathcal{R}_n(f) - \int_0^1 f(y)\,dy\right| \leq \frac{\sqrt{2\zeta(2\ell)}}{(2\pi n)^\ell}\sqrt{\int_0^1 |f(x)|^2\,dx}.$$

Exercise 3.14 Think of the unit circle $\mathbb{S}^1(0, 1)$ as a subset of $\mathbb{C}$, and let $\varphi : \mathbb{S}^1(0, 1) \longrightarrow \mathbb{R}$ be a continuous function. The goal of this exercise is to show that there exists an analytic function f on $D(0, 1)$ such that $\lim_{z \to \zeta} \mathfrak{R}\big(f(z)\big) = \varphi(\zeta)$ for all $\zeta \in \mathbb{S}^1(0, 1)$.

(**i**) Set

$$a_m = \int_0^1 \varphi\big(e^{i2\pi\theta}\big)\mathfrak{e}_{-m}(\theta)\,d\theta \quad \text{for } m \in \mathbb{Z},$$

and show that $\overline{a_m} = a_{-m}$. Next define the function u on $D(0, 1)$ by

$$u\big(re^{i2\pi}\big) = \sum_{m=-\infty}^{\infty} r^{|m|} a_m \mathfrak{e}_m(\theta) \quad \text{for } r \in [0, 1) \text{ and } \theta \in [0, 1).$$

Show that u is a continuous, $\mathbb{R}$-valued function and, using Theorem 3.4.1, that $\lim_{z \to \zeta} u(z) = \varphi(\zeta)$ for $\zeta \in \mathbb{S}^1(0, 1)$.

(**ii**) Define v on $D(0,1)$ by

$$v\big(re^{i2\pi\theta}\big) = -i\sum_{m=1}^{\infty} r^m\big(a_m\mathfrak{e}_m(\theta) - a_{-m}\mathfrak{e}_{-m}(\theta)\big).$$

and show that v is a continuous, $\mathbb{R}$-valued function. Next, set $f = u + iv$, and show that

$$f(z) = a_0 + 2\sum_{m=1}^{\infty} a_m z^m \quad \text{for } z \in D(0,1).$$

In particular, conclude that f is analytic and that $\lim_{z\to\zeta}\Re\big(f(z)\big) = \varphi(\zeta)$ for $\zeta \in \mathbb{S}^1(0,1)$.

(**iii**) Assume that $\sum_{m=-\infty}^{\infty}|a_m| < \infty$, and define $H\varphi$ on $\mathbb{S}^1(0,1)$ by

$$H\varphi\big(e^{i2\pi\theta}\big) = -i\sum_{m=1}^{\infty}\big(a_m\mathfrak{e}_m(\theta) - a_{-m}\mathfrak{e}_{-m}(\theta)\big).$$

If f is the function in (**ii**), show that $\lim_{z\to\zeta}\Im\big(f(z)\big) = H\varphi(\zeta)$ for $\zeta \in \mathbb{S}^1(0,1)$. The function $H\varphi$ is called the *Hilbert transform* of φ, and it plays an important role in harmonic analysis.

Exercise 3.15 Let $\{x_n : n \ge 1\}$ be a sequence of distinct elements of $(a,b]$, and set $S(x) = \{n \ge 1 : x_n \le x\}$ for $x \in [a,b]$. Given a sequence $\{c_n : n \ge 1\} \subseteq \mathbb{R}$ for which $\sum_{n=1}^{\infty}|c_n| < \infty$, define $\psi : [a,b] \longrightarrow \mathbb{R}$ so that $\psi(x) = \sum_{n\in S(x)} c_n$. Show that ψ has bounded variation and that $\|\psi\|_{\text{var}} = \sum_{n=1}^{\infty}|c_n|$. In addition, show that if $\varphi : [a,b] \longrightarrow \mathbb{R}$ is continuous, then

$$\int_a^b \varphi(x)\,d\psi(x) = \sum_{n=1}^{\infty}\varphi(x_n)c_n.$$

Exercise 3.16 Suppose that $F : [a,b] \longrightarrow \mathbb{R}$ is a function of bounded variation.

(**i**) Show that, for each $\epsilon > 0$,

$$\varlimsup_{h\searrow 0}\big|F(x+h) - F(x)\big| \vee |F(x-h) - F(x)| \ge \epsilon$$

for at most a finite number of $x \in (a,b)$, and use this to show that F is Riemann integrable on $[a,b]$.

(**ii**) Assume that $\varphi : [a,b] \longrightarrow \mathbb{R}$ is a continuous function that has a bounded, continuous derivative on (a,b). Prove the *integration by parts* formula

$$\int_a^b \varphi(t)\,dF(t) = \varphi(b)F(b) - \varphi(a)F(a) - \int_a^b \varphi'(t)F(t)\,dt.$$

(**iii**) Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a function whose restriction to $[0,T]$ has bounded variation for each $T > 0$. Assuming that $\psi(0) = 0$ and that $\sup_{t\geq 1} t^{-\alpha}|\psi(t)| < \infty$ for some $\alpha \geq 0$, use (**ii**) to show that

$$L(\lambda) \equiv \int_{[0,\infty)} e^{-\lambda t}\, d\psi(t) = \lim_{T\to\infty} \int_0^T e^{-\lambda t}\, d\psi(t) = \lambda \int_{[0,\infty)} e^{-\lambda t}\psi(t)\, dt$$

for all $\lambda > 0$. The function $\lambda \rightsquigarrow L(\lambda)$ is called the *Laplace transform* of ψ.

(**iv**) Refer to (**iii**), and assume that $a = \lim_{t\to\infty} t^{-\alpha}\psi(t) \in \mathbb{R}$ exists. Show that, for each $T > 0$, $\lambda^\alpha L(\lambda)$ equals

$$\lambda^{1+\alpha} \int_0^T e^{-\lambda t}\psi(t)\, dt + \lambda^{1+\alpha} \int_{[T,\infty)} t^\alpha e^{-\lambda t}\big(t^{-\alpha}\psi(t) - a\big) dt + a \int_{[\lambda T,\infty)}^{\infty} t^\alpha e^{-t}\, dt,$$

and use this to conclude that (cf. Exercise 3.3)

$$(*) \qquad a = \lim_{\lambda\searrow 0} \frac{\lambda^\alpha L(\lambda)}{\Gamma(1+\alpha)}.$$

This equation is an example of the general principle that the behavior of ψ near infinity reflects the behavior of its Laplace transform at 0.

(**v**) The equation in $(*)$ is an integral version of the *Abel limit* procedure in 1.10.1, and it generalizes 1.10.1. To see this, let $\{c_n : n \geq 1\} \subseteq \mathbb{R}$ be given, and define

$$\psi(t) = \sum_{1\leq n\leq t} c_n \quad \text{for } t \in [0,\infty).$$

Assuming that $\sup_{n\geq 1} n^{-\alpha}|\psi(n)| < \infty$, use Exercise 3.15 to show that

$$\int_{[0,\infty)} e^{-\lambda t}\, d\psi(t) = \sum_{n=1}^{\infty} e^{-\lambda n} c_n \quad \text{for } \lambda > 0.$$

Next, assume that $a = \lim_{n\to\infty} n^{-\alpha}\psi(n) \in \mathbb{R}$ exists, and use $(*)$ to conclude that

$$a = \lim_{\lambda\searrow 0} \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \sum_{n=1}^{\infty} e^{-\lambda n} c_n.$$

When $\alpha = 0$, this is equivalent to 1.10.1.