Simons Symposia

Paolo Cascini
James McKernan
Jorge Vitório Pereira *Editors*

Foliation Theory in Algebraic Geometry

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Preface

As part of the celebrations around the opening of the Simons Foundation offices in New York, the editors were invited to organise a conference on a topic of their choice. We chose birational geometry and foliation theory as there has been considerable activity in both areas in the last decade and there has also been increasing interaction between the two subjects. The conference “Foliation theory in algebraic geometry” took place September 3–7, 2013, at the recently opened Simons Foundation’s Gerald D. Fischbach Auditorium.

The conference attracted over seventy participants as well as locals from the New York area and was a great success. These are the proceedings of the conference. The talks included both survey talks on recent progress and original research and the articles are a reflection of these topics.

The articles in this proceedings should be of interest to people working in birational geometry and foliation theory and anyone wanting to learn about these subjects.

The editors would like to thank David Eisenbud for the initial invitation to organise a conference. They would also like to thank the Simons Foundation and Yuri Tschinkel for hosting the conference and to acknowledge the generous support of both the Simons Foundation and the NSF, under grant no. DMS 1339299. We would like to thank the speakers and all of the participants for making the conference a success. Finally they would like to recognise the invaluable support of Meghan Fazzi in the organisation of the conference.

London, UK — Paolo Cascini
La Jolla, CA, USA — James McKernan
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27 July 2015

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On Fano Foliations 2

Carolina Araujo and Stéphane Druel

Abstract In this paper we pursue the study of mildly singular del Pezzo foliations on complex projective manifolds started in [AD13].

Keywords Fano manifolds • Holomorphic foliations • Classification

Mathematical Subject Classification: 14M22, 37F75

1 Introduction

In recent years, techniques from higher dimensional algebraic geometry, specially from the minimal model program, have been successfully applied to the study of global properties of holomorphic foliations. This led, for instance, to the birational classification of foliations by curves on surfaces in [Bru04]. Motivated by these developments, we initiated in [AD13] a systematic study of *Fano foliations*. These are holomorphic foliations $\mathscr{F}$ on complex projective manifolds with ample anti-canonical class $-K_{\mathscr{F}}$. One special property of Fano foliations is that their leaves are always covered by rational curves, even when these leaves are not algebraic (see, for instance, [AD13, Proposition 7.5]).

The index $\iota_{\mathscr{F}}$ of a Fano foliation $\mathscr{F}$ on a complex projective manifold X is the largest integer dividing $-K_{\mathscr{F}}$ in $\mathrm{Pic}(X)$. In analogy with Kobayachi-Ochiai's theorem on the index of Fano manifolds (Theorem 2.2), we proved in [ADK08, Theorem 1.1] that the index of a Fano foliation $\mathscr{F}$ on a complex projective manifold is bounded above by its rank, $\iota_{\mathscr{F}} \leqslant r_{\mathscr{F}}$. Equality holds if and only if $X \cong \mathbb{P}^n$ and $\mathscr{F}$ is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r_{\mathscr{F}}}$. Our expectation is that Fano

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foliations with large index are the simplest ones. So we proceeded to investigate the next case, namely Fano foliation $\mathscr{F}$ of rank r and index $\iota_{\mathscr{F}} = r - 1$. We call such foliations *del Pezzo foliations*, in analogy with the case of Fano manifolds. In contrast to the case when $\iota_{\mathscr{F}} = r_{\mathscr{F}}$, there are examples of del Pezzo foliations with non-algebraic leaves. For instance, let $\mathscr{C}$ be a foliation by curves on $\mathbb{P}^k$ induced by a global vector field. If we take this vector field to be general, then the leaves of $\mathscr{C}$ are not algebraic. Now consider a linear projection $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$, with $n > k$, and let $\mathscr{F}$ be the foliation on $\mathbb{P}^n$ obtained as pullback of $\mathscr{C}$ via ψ. It is a del Pezzo foliation on $\mathbb{P}^n$, and its leaves are not algebraic (see Theorem 3.16(2) for the complete classification of del Pezzo foliations on $\mathbb{P}^n$). The first main result of [AD13] says that these are the only examples.

Theorem 1.1 ([AD13, Theorem 1.1]). *Let $\mathscr{F}$ be a del Pezzo foliation on a complex projective manifold $X \not\cong \mathbb{P}^n$. Then $\mathscr{F}$ is algebraically integrable, and its general leaves are rationally connected.*

One of the main ingredients in our study of Fano foliations is the notion of *log leaf* for an algebraically integrable foliation. Given an algebraically integrable foliation $\mathscr{F}$ on a complex projective manifold X, denote by $\tilde{e} : \tilde{F} \to X$ the normalization of the closure of a general leaf of $\mathscr{F}$. There is a naturally defined effective Weil divisor $\tilde{\Delta}$ on $\tilde{F}$ such that $K_{\tilde{F}} + \tilde{\Delta} = \tilde{e}^*K_{\mathscr{F}}$ (see Definition 3.6 for details). We call the pair $(\tilde{F}, \tilde{\Delta})$ a general *log leaf* of $\mathscr{F}$. In [AD13], we used the log leaf to define new notions of singularities for algebraically integrable foliations, following the theory of singularities of pairs from the minimal model program. Namely, we say that $\mathscr{F}$ has *log canonical singularities along a general leaf* if $(\tilde{F}, \tilde{\Delta})$ is log canonical. By Theorem 1.1, these notions apply to del Pezzo foliations on projective manifolds $X \not\cong \mathbb{P}^n$. In [AD13], we established the following classification of del Pezzo foliations with mild singularities.

Theorem 1.2 ([AD13, 9.1 and Theorems 1.3, 9.2, 9.6]). *Let $\mathscr{F}$ be a del Pezzo foliation of rank r on a complex projective manifold $X \not\cong \mathbb{P}^n$, and suppose that $\mathscr{F}$ has log canonical singularities and is locally free along a general leaf. Then*

- *either $\rho(X) = 1$;*
- *or $r \leqslant 3$ and X is a $\mathbb{P}^m$-bundle over $\mathbb{P}^k$.*

In the latter case, one of the following holds.

(1) *$X \cong \mathbb{P}^1 \times \mathbb{P}^k$, and $\mathscr{F}$ is the pullback via the second projection of a foliation $\mathscr{O}_{\mathbb{P}^k}(1)^{\oplus i} \subset T_{\mathbb{P}^k}$ for some $i \in \{1, 2\}$ ($r \in \{2, 3\}$).*

(2) *There exist*

- *an exact sequence of vector bundles $0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$ on $\mathbb{P}^k$; and*
- *a foliation by curves $\mathscr{C} \cong q^* \det(\mathscr{Q}) \subset T_{\mathbb{P}_{\mathbb{P}^k}(\mathscr{K})}$, where $q : \mathbb{P}_{\mathbb{P}^k}(\mathscr{K}) \to \mathbb{P}^k$ denotes the natural projection;*

such that $X \cong \mathbb{P}_{\mathbb{P}^k}(\mathscr{E})$, and $\mathscr{F}$ is the pullback of $\mathscr{C}$ via the relative linear projection $\mathbb{P}_{\mathbb{P}^k}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^k}(\mathscr{K})$. Moreover, one of the following holds.

(a) *$k = 1$, $\mathscr{Q} \cong \mathscr{O}_{\mathbb{P}^1}(1)$, $\mathscr{K}$ is an ample vector bundle such that $\mathscr{K} \not\cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for any integer a, and $\mathscr{E} \cong \mathscr{Q} \oplus \mathscr{K}$ ($r = 2$).*

(b) $k = 1$, $\mathscr{Q} \cong \mathscr{O}_{\mathbb{P}^1}(2)$, $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}$ *for some integer* $a \geqslant 1$, *and* $\mathscr{E} \cong \mathscr{Q} \oplus \mathscr{K}$ $(r = 2)$.

(c) $k = 1$, $\mathscr{Q} \cong \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus(m-1)}$ *for some integer* $a \geqslant 1$, *and* $\mathscr{E} \cong \mathscr{Q} \oplus \mathscr{K}$ $(r = 3)$.

(d) $k \geqslant 2$, $\mathscr{Q} \cong \mathscr{O}_{\mathbb{P}^k}(1)$, *and* $\mathscr{K}$ *is* V*-equivariant for some* $V \in H^0\big(\mathbb{P}^k, T_{\mathbb{P}^k} \otimes \mathscr{O}_{\mathbb{P}^k}(-1)\big) \setminus \{0\}$ $(r = 2)$.

Conversely, given $\mathscr{K}$, $\mathscr{E}$, *and* $\mathscr{Q}$ *satisfying any of the conditions above, there exists a del Pezzo foliation of that type.*

The goal of the present paper is to continue the classification of del Pezzo foliations on Fano manifolds $X \not\cong \mathbb{P}^n$ having log canonical singularities and being locally free along a general leaf. In view of Theorem 1.2, we need to understand del Pezzo foliations on Fano manifolds with Picard number 1. Our main result is the following.

Theorem 1.3. *Let* $\mathscr{F}$ *be a del Pezzo foliation of rank* $r \geqslant 3$ *on an* n*-dimensional Fano manifold* $X \not\cong \mathbb{P}^n$ *with* $\rho(X) = 1$, *and suppose that* $\mathscr{F}$ *has log canonical singularities and is locally free along a general leaf. Then* $X \cong Q^n$ *and* $\mathscr{F}$ *is induced by the restriction to* Q^n *of a linear projection* $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-r}$.

Remark 1.4. Codimension 1 del Pezzo foliations on Fano manifolds with Picard number 1 were classified in [LPT13, Proposition 3.7]. We extended this classification to mildly singular varieties, without restriction on the Picard number in [AD14, Theorem 1.3].

We also obtain a partial classification when $r = 2$ (Proposition 4.1).

In order to prove Theorem 1.3, we consider a general log leaf $(\tilde{F}, \tilde{\Delta})$ of $\mathscr{F}$. Under the assumptions of Theorem 1.3, $(\tilde{F}, \tilde{\Delta})$ is a *log del Pezzo pair*: it is a log canonical pair of dimension r satisfying $K_{\tilde{F}} + \tilde{\Delta} = (r-1)L$, where L is an ample divisor on $\tilde{F}$. The first step in the proof of Theorem 1.3 consists in classifying all log del Pezzo pairs. This is done in Section 2.4, using Fujita's theory of Δ-genus. Once we know the general log leaf $(\tilde{F}, \tilde{\Delta})$ of $\mathscr{F}$, we consider families of rational curves on X that restrict to special families of rational curves on $\tilde{F}$. The necessary results from the theory of rational curves are briefly reviewed in Section 2.1. The idea is to use these families of rational curves to bound the index of X from below. In order to obtain a good bound, we need to show that the dimension of these families of rational curves is big enough. Here enters a very special property of algebraically integrable Fano foliations having log canonical singularities along a general leaf: there is a common point contained in the closure of a general leaf [AD13, Proposition 5.3]. For our current purpose, we need the following strengthening of this result (see Definition 2.9 for the notion of log canonical center).

Proposition 1.5. *Let* $\mathscr{F}$ *be an algebraically integrable Fano foliation on a complex projective manifold* X *having log canonical singularities along a general leaf. Then there is a closed irreducible subset* $T \subset X$ *satisfying the following property. For a general log leaf* $(\tilde{F}, \tilde{\Delta})$, *there exists a log canonical center* S *of* $(\tilde{F}, \tilde{\Delta})$ *whose image in* X *is* T.

When $r \geqslant 3$, this allows us to show that $\iota_X \geqslant n$, and then use Kobayashi-Ochiai's theorem (Theorem 2.2) to conclude that $X \cong Q^n$. The classification of del Pezzo foliations on Q^n is established in Proposition 3.18.

Proposition 1.5 still holds in the more general setting of $\mathbb{Q}$-*Fano* foliations on possibly singular projective varieties. Since this may be useful in other situations, we present the theory of foliations on normal projective varieties in Section 3, and prove a more general version of Proposition 1.5 (Proposition 3.14).

Notation and Conventions We always work over the field $\mathbb{C}$ of complex numbers. Varieties are always assumed to be irreducible. We denote by $\mathrm{Sing}(X)$ the singular locus of a variety X.

Given a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on a variety X, we denote by $\mathscr{F}^*$ the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. If r is the generic rank of $\mathscr{F}$, then we denote by $\det(\mathscr{F})$ the sheaf $(\wedge^r \mathscr{F})^{**}$. For $m \in \mathbb{N}$, we denote by $\mathscr{F}^{[m]}$ the sheaf $(\mathscr{F}^{\otimes m})^{**}$. If $\mathscr{G}$ is another sheaf of $\mathscr{O}_X$-modules on X, then we denote by $\mathscr{F}[\otimes]\mathscr{G}$ the sheaf $(\mathscr{F} \otimes \mathscr{G})^{**}$.

If $\mathscr{E}$ is a locally free sheaf of $\mathscr{O}_X$-modules on a variety X, we denote by $\mathbb{P}_X(\mathscr{E})$ the Grothendieck projectivization $\mathrm{Proj}_X(\mathrm{Sym}(\mathscr{E}))$, and by $\mathscr{O}_{\mathbb{P}}(1)$ its tautological line bundle.

If X is a normal variety, we denote by T_X the sheaf $(\Omega^1_X)^*$.

We denote by Q^n a (possibly singular) quadric hypersurface in $\mathbb{P}^{n+1}$. Given an integer $d \geqslant 0$, we denote by $\mathbb{F}_d$ the surface $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-d))$. If moreover $d \geqslant 1$, we denote by $\mathbb{P}(1, 1, d)$ the cone in $\mathbb{P}^{d+1}$ over the rational normal curve of degree d.

2 Preliminaries

2.1 Fano Manifolds and Rational Curves

Definition 2.1. A *Fano manifold* X is a complex projective manifold whose anti-canonical class $-K_X$ is ample. The index ι_X of X is the largest integer dividing $-K_X$ in $\mathrm{Pic}(X)$.

Theorem 2.2 ([KO73]). *Let X be a Fano manifold of dimension $n \geqslant 2$ and index ι_X. Then $\iota_X \leqslant n+1$, and equality holds if and only if $X \cong \mathbb{P}^n$. Moreover, $\iota_X = n$ if and only if $X \cong Q^n \subset \mathbb{P}^{n+1}$.*

Families of rational curves provide a useful tool in the study of Fano manifolds. Next we gather some results from the theory of rational curves. In what follows, rational curves are always assumed to be proper. A *family of rational curves* on a complex projective manifold X is a closed irreducible subset of $\mathrm{RatCurves}^n(X)$. We refer to [Kol96] for details.

Definition 2.3. Let $\ell \subset X$ be a rational curve on a complex projective manifold, and consider its normalization $f : \mathbb{P}^1 \to X$. We say that ℓ is *free* if f^*T_X is globally generated.

2.4. Let X be a complex projective manifold, and $\ell \subset X$ a free rational curve. Let $x \in \ell$ be any point, and H_x an irreducible component of the scheme $\mathrm{RatCurves}^n(X, x)$ containing a point corresponding to ℓ. Then

$$\dim(H_x) = -K_X \cdot \ell - 2.$$

Notation 2.5. Let X be a Fano manifold with $\rho(X) = 1$, and $\mathscr{A}$ an ample line bundle on X such that $\mathrm{Pic}(X) = \mathbb{Z}[\mathscr{A}]$. For any proper curve $C \subset X$, we refer to $\mathscr{A} \cdot C$ as the *degree* of C. Rational curves of degree 1 are called *lines*. Note that if $C \subset X$ is a proper curve of degree d, then $\iota_X = \frac{-K_X \cdot C}{d}$.

One can use free rational curves on Fano manifolds with Picard number 1 to bound their index. The following is an immediate consequence of paragraph 2.4 above.

Lemma 2.6. *Let X be a Fano manifold with $\rho(X) = 1$. Suppose that there is an m-dimensional family V of rational curves of degree d on X such that:*

- *all curves from V pass though some fixed point $x \in X$; and*
- *some curve from V is free.*

Then $\iota_X \geqslant \frac{m+2}{d}$.

Remark 2.7. Let V be a family of rational curves on a complex projective manifold X. To guarantee that some member of V is a free curve, it is enough to show that some curve from V passes through a *general* point of X. More precisely, let H be an irreducible component of $\mathrm{RatCurves}^n(X)$ containing V. It comes with universal family morphisms

$$\begin{array}{ccc} U & \xrightarrow{\ e\ } & X\ , \\ {\scriptstyle \pi}\downarrow & & \\ H & & \end{array}$$

where $\pi : U \to H$ is a $\mathbb{P}^1$-bundle. Suppose that $e : U \to X$ is dominant. Then, by generic smoothness, there is a dense open subset $X^\circ \subset X$ over which $e : U \to X$ is smooth. On the other hand, by [Kol96, Proposition II.3.4], e is smooth at a point $u \in \pi^{-1}(t)$ if and only if the rational curve $\ell_t = e\big(\pi^{-1}(t)\big)$ is free.

2.2 Singularities of Pairs

We refer to [KM98, section 2.3] and [Kol13, sections 2 and 4] for details.

Definition 2.8. Let X be a normal projective variety, and $\Delta = \sum a_i \Delta_i$ an effective $\mathbb{Q}$-divisor on X, i.e., Δ is a nonnegative $\mathbb{Q}$-linear combination of distinct prime Weil divisors Δ_i's on X. Suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e., some nonzero multiple of it is a Cartier divisor.

Let $f : \tilde{X} \to X$ be a log resolution of the pair (X, Δ). There are uniquely defined rational numbers $a(E_i, X, \Delta)$'s such that

$$K_{\tilde{X}} + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_{E_i} a(E_i, X, \Delta)E_i.$$

The $a(E_i, X, \Delta)$'s do not depend on the log resolution f, but only on the valuations associated to the E_i's. The closed subvariety $f(E_i) \subset X$ is called the *center of E_i in X*. It also depends only on the valuation associated to E_i.

For a prime divisor D on X, we define $a(D, X, \Delta)$ to be the coefficient of D in $-\Delta$.

We say that the pair (X, Δ) is *log canonical* if, for some log resolution $f : \tilde{X} \to X$ of (X, Δ), $a(E_i, X, \Delta) \geqslant -1$ for every f-exceptional prime divisor E_i. If this condition holds for some log resolution of (X, Δ), then it holds for every log resolution of (X, Δ).

Definition 2.9. Let (X, Δ) be a log canonical pair. We say that a closed irreducible subvariety $S \subset X$ is a *log canonical center* of (X, Δ) if there is a divisor E over X with $a(E, X, \Delta) = -1$ whose center in X is S.

2.3 Polarized Varieties and Fujita's Δ-Genus

Definition 2.10. A *polarized variety* is a pair $(X, \mathscr{L})$ consisting of a normal projective variety X, and an ample line bundle $\mathscr{L}$ on X.

Definition 2.11 ([Fuj75]). The Δ-*genus* of an n-dimensional polarized variety $(X, \mathscr{L})$ is defined by the formula:

$$\Delta(X, \mathscr{L}) := n + c_1(\mathscr{L})^n - h^0(X, \mathscr{L}) \in \mathbb{Z}.$$

By [Fuj82, Corollary 2.12], $\Delta(X, \mathscr{L}) \geqslant 0$ for any polarized variety $(X, \mathscr{L})$. Next we recall the classification of polarized varieties with Δ-genus zero from [Fuj82].

Theorem 2.12 ([Fuj82]). *Let X be a normal projective variety of dimension $n \geqslant 1$, and $\mathscr{L}$ an ample line bundle on X. Suppose that $\Delta(X, \mathscr{L}) = 0$. Then one of the following holds.*

(1) $(X, \mathscr{L}) \cong (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$.
(2) $(X, \mathscr{L}) \cong (Q^n, \mathscr{O}_{Q^n}(1))$.
(3) $(X, \mathscr{L}) \cong (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$, *for some $d \geqslant 3$.*
(4) $(X, \mathscr{L}) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$.
(5) $(X, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1))$, *where $\mathscr{E}$ is an ample vector bundle on $\mathbb{P}^1$.*
(6) *$\mathscr{L}$ is very ample, and embeds X as a cone over a projective polarized variety of type (3–5) above.*

2.4 Classification of Log Del Pezzo Pairs

Definition 2.13. Let X be a normal projective variety of dimension $n \geqslant 1$, and Δ an effective $\mathbb{Q}$-divisor on X. We say that (X, Δ) is a *log del Pezzo pair* if (X, Δ) is log canonical, and $-(K_X + \Delta) \equiv (n-1)c_1(\mathscr{L})$ for some ample line bundle $\mathscr{L}$ on X.

Using Fujita's classification of polarized varieties with Δ-genus zero, we classify log del Pezzo pairs in Theorem 2.15 below.

Lemma 2.14. *Let $(X, \mathscr{L})$ be an n-dimensional polarized variety, with $n \geqslant 2$. Let (X, Δ) be a log del Pezzo pair such that $\Delta \neq 0$ and $-(K_X + \Delta) \equiv (n-1)c_1(\mathscr{L})$. Then $\Delta(X, \mathscr{L}) = 0$, and $\Delta \cdot c_1(\mathscr{L})^{n-1} = 2$.*

Proof. We follow the line of argumentation in the proof of [Fuj80, Lemma 1.10]. Since

$$c_1(\mathscr{L}^{\otimes t}) \equiv K_X + \Delta + c_1(\mathscr{L}^{\otimes n-1+t}),$$

we have that $h^i(X, \mathscr{L}^{\otimes t}) = 0$ for $i \geqslant 1$ and $t > 1-n$ by [Fuj11, Theorem 8.1]. Therefore $\chi(X, \mathscr{O}_X) = 1$ and $\chi(X, \mathscr{L}^{\otimes t}) = 0$ for $2-n \leqslant t \leqslant -1$. Hence, there are rational numbers a and b such that

$$\chi(X, \mathscr{L}^{\otimes t}) = \big(at^2 + bt + n(n-1)\big)\frac{\prod_{j=1}^{n-2}(t+j)}{n!}.$$

On the other hand, since X is normal, by Hirzebruch-Riemann-Roch,

$$\chi(X, \mathscr{L}^{\otimes t}) = \frac{c_1(\mathscr{L})^n}{n!}t^n - \frac{1}{2(n-1)!}K_X \cdot c_1(\mathscr{L})^{n-1}t^{n-1} + o(t^{n-1}).$$

Thus we have $a = c_1(\mathscr{L})^n$ and $b = \frac{n}{2}\Delta \cdot c_1(\mathscr{L})^{n-1} + (n-1)c_1(\mathscr{L})^n$. In particular,

$$h^0(X, \mathscr{L}) = \chi(X, \mathscr{L}) = n - 1 + c_1(\mathscr{L})^n + \frac{1}{2}\Delta \cdot c_1(\mathscr{L})^{n-1}.$$

One then computes that

$$\Delta(X, \mathscr{L}) = 1 - \frac{1}{2}\Delta \cdot c_1(\mathscr{L})^{n-1}.$$

Since $\Delta \neq 0$ and $\Delta(X, \mathscr{L}) \geqslant 0$, we must have $\Delta(X, \mathscr{L}) = 0$ and $\Delta \cdot c_1(\mathscr{L})^{n-1} = 2$. ◻

Theorem 2.15. *Let $(X, \mathscr{L})$ be an n-dimensional polarized variety, with $n \geqslant 1$. Let (X, Δ) be a log del Pezzo pair such that Δ is integral and nonzero, and $-(K_X + \Delta) \equiv (n-1)c_1(\mathscr{L})$. Then one of the following holds.*

(1) $(X, \mathscr{L}, \mathscr{O}_X(\Delta)) \cong (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1), \mathscr{O}_{\mathbb{P}^n}(2))$.
(2) $(X, \mathscr{L}, \mathscr{O}_X(\Delta)) \cong (Q^n, \mathscr{O}_{Q^n}(1), \mathscr{O}_{Q^n}(1))$.
(3) $(X, \mathscr{L}, \mathscr{O}_X(\Delta)) \cong (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d), \mathscr{O}_{\mathbb{P}^1}(2))$, *for some integer* $d \geqslant 3$.
(4) $(X, \mathscr{L}, \mathscr{O}_X(\Delta)) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2), \mathscr{O}_{\mathbb{P}^2}(1))$.
(5) $(X, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1))$ *for an ample vector bundle* $\mathscr{E}$ *on* $\mathbb{P}^1$. *Moreover, one of the following holds.*
   (a) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ *for some* $a \geqslant 2$, *and* $\Delta \sim_{\mathbb{Z}} \sigma + f$ *where* σ *is the minimal section and* f *is a fiber of* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}^1$.
   (b) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ *for some* $a \geqslant 2$, *and* Δ *is a minimal section.*
   (c) $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ *for some* $a \geqslant 1$, *and* $\Delta = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1))$.
(6) $\mathscr{L}$ *is very ample, and embeds* (X, Δ) *as a cone over* $\big((Z, \mathscr{M}), (\Delta_Z, \mathscr{M}_{|\Delta_Z})\big)$, *where* Z *is smooth and* $(Z, \mathscr{M}, \Delta_Z)$ *satisfies one of the conditions (3–5) above.*

Proof. By [CKP12, Theorem 0.1], we must have $-(K_X + \Delta) \sim_{\mathbb{Q}} (n-1)c_1(\mathscr{L})$.

If $n = 1$, then $-K_X \sim_{\mathbb{Q}} \Delta$ is ample, and hence $(X, \mathscr{L}, \mathscr{O}_X(\Delta))$ satisfies one of conditions (1–3) in the statement of Theorem 2.15.

Suppose from now on that $n \geqslant 2$. By Lemma 2.14, $\Delta(X, \mathscr{L}) = 0$, and so we can apply Theorem 2.12. Notice that if $(X, \mathscr{L})$ satisfies any of conditions (1–6) of Theorem 2.12, then $-(K_X + \Delta) \sim_{\mathbb{Z}} (n-1)c_1(\mathscr{L})$ since $X \setminus \mathrm{Sing}(X)$ is simply connected.

If $(X, \mathscr{L})$ satisfies any of conditions (1–4) of Theorem 2.12, one checks easily that $(X, \mathscr{L}, \Delta)$ satisfies one of conditions (1–4) in the statement of Theorem 2.15.

Suppose that $(X, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1))$ for an ample vector bundle $\mathscr{E}$ on $\mathbb{P}^1$, and write $\pi : X \to \mathbb{P}^1$ for the natural projection. Then

$$\Delta \in \big|\mathscr{O}_X(-K_X) \otimes \mathscr{L}^{\otimes 1-n}\big| = \big|\mathscr{L} \otimes \pi^*\big(\det(\mathscr{E}^*) \otimes \mathscr{O}_{\mathbb{P}^1}(2)\big)\big|.$$

Write $\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n)$, with $1 \leqslant a_1 \leqslant \cdots \leqslant a_n$. By the projection formula, $h^0\big(X, \mathscr{L} \otimes \pi^*\big(\det(\mathscr{E}^*) \otimes \mathscr{O}_{\mathbb{P}^1}(2)\big)\big) = h^0(\mathbb{P}^1, \mathscr{E} \otimes \det(\mathscr{E}^*) \otimes \mathscr{O}_{\mathbb{P}^1}(2))$, hence we must have $a_1 + \cdots + a_{n-1} \leqslant 2$. This implies that $(n, a_1, \ldots, a_{n-1}) \in \{(2,1), (2,2), (3,1,1)\}$. Thus either $\mathscr{E}$ satisfies condition (5a-c) in the statement of Theorem 2.15, or $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$, $\Delta \in |\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)|$, and hence X satisfies condition (2) with $n = 2$.

Finally, suppose that $\mathscr{L}$ is very ample, and embeds X as a cone with vertex V over a smooth polarized variety $(Z, \mathscr{M})$ satisfying one of conditions (3–5) in the statement of Theorem 2.12. Set $m := \dim(Z)$ and $s := n - m = \dim(V) + 1$. Let $e : Y \to X$ be the blow-up of X along V, with exceptional divisor E. We have $Y \cong \mathbb{P}_Z(\mathscr{M} \oplus \mathscr{O}_Z^{\oplus s})$, with natural projection $\pi : Y \to Z$, and tautological line bundle $\mathscr{O}_Y(1) \cong e^*\mathscr{L}$. The exceptional divisor E corresponds to the projection $\mathscr{M} \oplus \mathscr{O}_Z^{\oplus s} \twoheadrightarrow \mathscr{O}_Z^{\oplus s}$.

Let Δ_Y be the strict transform of Δ in Y. We are done if we prove that $\Delta_Y = \pi^*\Delta_Z$ for some divisor Δ_Z on Z.

Write $\Delta_Y \sim_{\mathbb{Z}} \pi^*\Delta_Z + kE$ for some integral divisor Δ_Z on Z, and some integer $k \geqslant 0$. Let $\sigma : Z \to Y$ be the section of π corresponding to a general surjection $\mathscr{M} \oplus \mathscr{O}_Z^{\oplus s} \twoheadrightarrow \mathscr{M}$. Then $\sigma(Z) \cap E = \emptyset$, and $\mathscr{N}_{\sigma(Z)/Y} \cong \mathscr{M}^{\oplus s}$. Moreover, $(\sigma(Z), \Delta_{Y|\sigma(Z)})$ is log canonical (see, for instance, [Kol97, Proposition 7.3.2]), and, by the adjunction formula, $-(K_{\sigma(Z)} + \Delta_{Y|\sigma(Z)}) \sim_{\mathbb{Z}} (m-1)c_1(\mathscr{O}_Y(1))_{|\sigma(Z)}$.

We have $h^0(Y, \mathscr{O}_Y(kE + \pi^*\Delta_Z)) = h^0(Y, \mathscr{O}_Y(\Delta_Y)) \geqslant 1$. On the other hand,

$$\begin{aligned} h^0(Y, \mathscr{O}_Y(kE + \pi^*\Delta_Z)) &= h^0(Y, \mathscr{O}_Y(k) \otimes \pi^*\mathscr{M}^{\otimes -k} \otimes \mathscr{O}_Y(\pi^*\Delta_Z)) \\ &= h^0(Z, S^k(\mathscr{M} \oplus \mathscr{O}_Z^{\oplus s}) \otimes \mathscr{M}^{\otimes -k} \otimes \mathscr{O}_Z(\Delta_Z)) \\ &= h^0(Z, S^k(\mathscr{O}_Z \oplus \mathscr{M}_Z^{\otimes -1 \oplus s}) \otimes \mathscr{O}_Z(\Delta_Z)). \end{aligned}$$

We claim that $h^0(Z, \mathscr{M}_Z^{\otimes -1} \otimes \mathscr{O}_Z(\Delta_Z)) = 0$. Indeed, suppose that $h^0(Z, \mathscr{M}_Z^{\otimes -1} \otimes \mathscr{O}_Z(\Delta_Z)) \neq 0$. Then $-K_Z \sim_{\mathbb{Z}} \Delta_Z + (m-1)c_1(\mathscr{M})$ since $\Delta_{Y|\sigma(Z)} \sim_{\mathbb{Z}} \left(\pi_{|\sigma(Z)}\right)^* \Delta_Z$, and hence $-K_Z \geqslant mc_1(\mathscr{M})$. Under these conditions, [AD14, Theorem 2.5] implies that $(Z, \mathscr{M}, \mathscr{O}_Z(\Delta_Z))$ is isomorphic to either $(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(1), \mathscr{O}_{\mathbb{P}^m}(2))$ or $(Q^m, \mathscr{O}_{Q^m}(1), \mathscr{O}_{Q^m}(1))$. This contradicts our current assumption that $(Z, \mathscr{M})$ satisfies one of conditions (3–5) in the statement of Theorem 2.12, and proves the claim.

Since $h^0(Z, \mathscr{M}_Z^{\otimes -1} \otimes \mathscr{O}_Z(\Delta_Z)) = 0$, we must have $h^0(Y, \mathscr{O}_Y(kE + \pi^*\Delta_Z)) = h^0(Z, \mathscr{O}_Z(\Delta_Z))$. Thus, replacing Δ_Z with a suitable member of its linear system if necessary, we may assume that $\Delta_Y = \pi^*\Delta_Z + kE$, and hence $k = 0$. Therefore $(X, \mathscr{L}, \Delta)$ satisfies condition (6) in the statement of Theorem 2.15. □

In dimension 2, we have the following classification, without the assumption that (X, Δ) is log canonical.

Theorem 2.16 ([Nak07, Theorem 4.8]). *Let (X, Δ) be a pair with* $\dim(X) = 2$ *and $\Delta \neq 0$. Suppose that $-(K_X + \Delta)$ is Cartier and ample. Then one of the following holds.*

(1) *$X \cong \mathbb{P}^2$ and* $\deg(\Delta) \in \{1, 2\}$.
(2) *$X \cong \mathbb{F}_d$ for some $d \geqslant 0$ and Δ is a minimal section.*
(3) *$X \cong \mathbb{F}_d$ for some $d \geqslant 0$ and $\Delta \sim_{\mathbb{Z}} \sigma + f$, where σ is a minimal section and f a fiber of $\mathbb{F}_d \to \mathbb{P}^1$.*
(4) *$X \cong \mathbb{P}(1,1,d)$ for some $d \geqslant 2$ and $\Delta \sim_{\mathbb{Z}} 2\ell$ where ℓ is a ruling of the cone $\mathbb{P}(1,1,d)$.*

3 Foliations

3.1 Foliations and Pfaff Fields

Definition 3.1. Let X be normal variety. A *foliation* on X is a nonzero coherent subsheaf $\mathscr{F} \subsetneq T_X$ such that

- $\mathscr{F}$ is closed under the Lie bracket, and
- $\mathscr{F}$ is saturated in T_X (i.e., $T_X/\mathscr{F}$ is torsion free).

The *rank* r of $\mathscr{F}$ is the generic rank of $\mathscr{F}$. The *codimension* of $\mathscr{F}$ is $q = \dim(X) - r$.

The *canonical class* $K_{\mathscr{F}}$ of $\mathscr{F}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{F}}) \cong \det(\mathscr{F})$.

Definition 3.2. A foliation $\mathscr{F}$ on a normal variety is said to be $\mathbb{Q}$-*Gorenstein* if its canonical class $K_{\mathscr{F}}$ is $\mathbb{Q}$-Cartier.

Definition 3.3. Let X be a variety, and r a positive integer. A *Pfaff field of rank* r on X is a nonzero map $\eta : \Omega_X^r \to \mathscr{L}$, where $\mathscr{L}$ is a reflexive sheaf of rank 1 on X such that $\mathscr{L}^{[m]}$ is invertible for some integer $m \geqslant 1$.

The *singular locus* S of η is the closed subscheme of X whose ideal sheaf $\mathscr{I}_S$ is the image of the induced map $\Omega_X^r[\otimes]\mathscr{L}^* \to \mathscr{O}_X$.

Notice that a $\mathbb{Q}$-Gorenstein foliation $\mathscr{F}$ of rank r on normal variety X naturaly gives rise to a Pfaff field of rank r on X:

$$\eta : \Omega_X^r = \wedge^r(\Omega_X^1) \to \wedge^r(T_X^*) \to \wedge^r(\mathscr{F}^*) \to \det(\mathscr{F}^*) \cong \det(\mathscr{F})^* \cong \mathscr{O}_X(K_{\mathscr{F}}).$$

Definition 3.4. Let $\mathscr{F}$ be a $\mathbb{Q}$-Gorenstein foliation on a normal variety X. The *singular locus* of $\mathscr{F}$ is defined to be the singular locus S of the associated Pfaff field. We say that $\mathscr{F}$ is *regular at a point* $x \in X$ if $x \notin S$. We say that $\mathscr{F}$ is *regular* if $S = \emptyset$.

Our definition of Pfaff field is more general than the one usually found in the literature, where $\mathscr{L}$ is required to be invertibile. This generalization is needed in order to treat $\mathbb{Q}$-Gorenstein foliations whose canonical classes are not Cartier.

3.5 (Foliations Defined by q-Forms). Let $\mathscr{F}$ be a codimension q foliation on an n-dimenional normal variety X. The *normal sheaf* of $\mathscr{F}$ is $N_{\mathscr{F}} := (T_X/\mathscr{F})^{**}$. The q-th wedge product of the inclusion $N_{\mathscr{F}}^* \hookrightarrow (\Omega_X^1)^{**}$ gives rise to a nonzero global section $\omega \in H^0\big(X, \Omega_X^q[\otimes]\det(N_{\mathscr{F}})\big)$ whose zero locus has codimension at least 2 in X. Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{F}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X, and $\omega \in H^0(X, \Omega_X^q[\otimes]\mathscr{L})$ a global section whose zero locus has codimension at least 2 in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1}[\otimes]\mathscr{L}$ given by the contraction with ω defines a foliation of codimension q on X. These constructions are inverse of each other.

3.2 Algebraically Integrable Foliations

Let X be a normal projective variety, and $\mathscr{F}$ a foliation on X. In this subsection we assume that $\mathscr{F}$ is *algebraically integrable*. This means that $\mathscr{F}$ is the relative tangent sheaf to a dominant rational map $\varphi : X \dashrightarrow Y$ with connected fibers. In this case, by a general leaf of $\mathscr{F}$ we mean the fiber of φ over a general point of Y. We start by defining the notion of *log leaf* when $\mathscr{F}$ is moreover $\mathbb{Q}$-Gorenstein. It plays a key role in our approach to $\mathbb{Q}$-Fano foliations.

Definition 3.6 (See [AD14, Definition 3.10] for details). Let X be a normal projective variety, $\mathscr{F}$ a $\mathbb{Q}$-Gorenstein algebraically integrable foliation of rank r on X, and $\eta : \Omega^r_X \to \mathscr{O}_X(K_{\mathscr{F}})$ its corresponding Pfaff field. Let $F \subset X$ be the closure of a general leaf of $\mathscr{F}$, and $\tilde{e} : \tilde{F} \to X$ the normalization of F. Let $m \geqslant 1$ be the Cartier index of $K_{\mathscr{F}}$, i.e., the smallest positive integer m such that $mK_{\mathscr{F}}$ is Cartier. Then η induces a generically surjective map $\otimes^m \Omega^r_{\tilde{F}} \to \tilde{e}^* \mathscr{O}_X(mK_{\mathscr{F}})$. Hence there is a canonically defined effective Weil $\mathbb{Q}$-divisor $\tilde{\Delta}$ on $\tilde{F}$ such that $mK_{\tilde{F}} + m\tilde{\Delta} \sim_{\mathbb{Z}} \tilde{e}^* mK_{\mathscr{F}}$.

We call the pair $(\tilde{F}, \tilde{\Delta})$ a *general log leaf* of $\mathscr{F}$.

The next lemma gives sufficient conditions under which the support of $\tilde{\Delta}$ is precisely the inverse image in $\tilde{F}$ of the singular locus of $\mathscr{F}$. It is an immediate consequence of [AD13, Lemma 5.6].

Lemma 3.7. *Let $\mathscr{F}$ be an algebraically integrable foliation on a complex projective manifold X. Suppose that $\mathscr{F}$ is locally free along the closure of a general leaf F. Let $\tilde{e} : \tilde{F} \to X$ be its normalization, and $(\tilde{F}, \tilde{\Delta})$ the corresponding log leaf. Then* $\mathrm{Supp}(\tilde{\Delta}) = \tilde{e}^{-1}(F \cap \mathrm{Sing}(\mathscr{F}))$.

Definition 3.8. Let X be normal projective variety, $\mathscr{F}$ a $\mathbb{Q}$-Gorenstein algebraically integrable foliation on X, and $(\tilde{F}, \tilde{\Delta})$ its general log leaf. We say that $\mathscr{F}$ has *log canonical singularities along a general leaf* if $(\tilde{F}, \tilde{\Delta})$ is log canonical.

Remark 3.9. In [McQ08, Definition I.1.2], McQuillan introduced a notion of log canonicity for foliations, without requiring algebraic integrability. If a $\mathbb{Q}$-Gorenstein algebraically integrable foliation $\mathscr{F}$ is log canonical in the sense of McQuillan, then $\mathscr{F}$ has log canonical singularities along a general leaf (see [AD13, Proposition 3.11] and its proof).

3.10 (The Family of Log Leaves). Let X be normal projective variety, and $\mathscr{F}$ an algebraically integrable foliation on X. We describe the *family of leaves* of $\mathscr{F}$ (see [AD13, Lemma 3.2 and Remark 3.8] for details). There is a unique irreducible closed subvariety W of $\mathrm{Chow}(X)$ whose general point parameterizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). It comes with a universal cycle $U \subset W \times X$ and morphisms:

$$\begin{array}{ccc} U & \xrightarrow{e} & X\ , \\ {\scriptstyle \pi}\downarrow & & \\ W & & \end{array}$$

where $e : U \to X$ is birational and, for a general point $w \in W$, $e\big(\pi^{-1}(w)\big) \subset X$ is the closure of a leaf of $\mathscr{F}$.

The variety W is called the *family of leaves* of $\mathscr{F}$.

Suppose moreover that $\mathscr{F}$ is $\mathbb{Q}$-Gorenstein, denote by $m \geqslant 1$ the Cartier index of $K_{\mathscr{F}}$, by r the rank of $\mathscr{F}$, and by $\eta : \Omega^r_X \to \mathscr{O}_X(K_{\mathscr{F}})$ the corresponding Pfaff field. Given a morphism $V \to W$ from a normal variety, let U_V be the normalization of $U \times_V W$, with induced morphisms:

$$\begin{array}{ccc} U_V & \xrightarrow{e_V} & X\ . \\ {\scriptstyle \pi_V}\downarrow & & \\ V & & \end{array}$$

Then η induces a generically surjective map $\otimes^m \Omega^r_{U_V/V} \to {e_V}^* \mathscr{O}_X(mK_{\mathscr{F}})$. Thus there is a canonically defined effective Weil $\mathbb{Q}$-divisor Δ_V on U_V such that $\det(\Omega^1_{U_V/V})^{\otimes m}[\otimes]\mathscr{O}_{U_V}(m\Delta_V) \cong {e_V}^* \mathscr{O}_X(mK_{\mathscr{F}})$. Suppose that $v \in V$ is mapped to a general point of W, set $U_v := (\pi_V)^{-1}(v)$, and $\Delta_v := (\Delta_V)_{|U_v}$. Then (U_v, Δ_v) coincides with the general log leaf $(\tilde{F}, \tilde{\Delta})$ defined above.

3.3 $\mathbb{Q}$-Fano Foliations

Definition 3.11. Let X be a normal projective variety, and $\mathscr{F}$ a $\mathbb{Q}$-Gorenstein foliation on X. We say that $\mathscr{F}$ is a $\mathbb{Q}$-*Fano foliation* if $-K_{\mathscr{F}}$ is ample. In this case, the *index* of $\mathscr{F}$ is the largest positive rational number $\iota_{\mathscr{F}}$ such that $-K_{\mathscr{F}} \sim_{\mathbb{Q}} \iota_{\mathscr{F}} H$ for a Cartier divisor H on X.

If $\mathscr{F}$ is a $\mathbb{Q}$-Fano foliation of rank r on a normal projective variety X, then, by [Hör14, Corollary 1.2], $\iota_{\mathscr{F}} \leqslant r$. Moreover, equality holds if and only if X is a generalized normal cone over a normal projective variety Z, and $\mathscr{F}$ is induced by the natural rational map $X \dashrightarrow Z$ (see also [ADK08, Theorem 1.1], and [AD14, Theorem 4.11]).

Definition 3.12. A $\mathbb{Q}$-Fano foliation $\mathscr{F}$ of rank $r \geqslant 2$ is called a *del Pezzo foliation* if $\iota_{\mathscr{F}} = r - 1$.

In [AD13, Proposition 5.3], we proved that algebraically integrable Fano foliations having log canonical singularities along a general leaf have a very special property: there is a common point contained in the closure of a general leaf. We strengthen this result in Proposition 3.14 below. It will be a consequence of the following theorem.

Theorem 3.13 ([ADK08, Theorem 3.1]). *Let X be a normal projective variety, $f : X \to C$ a surjective morphism onto a smooth curve, and Δ an effective Weil $\mathbb{Q}$-divisor on X such that (X, Δ) is log canonical over the generic point of C. Then $-(K_{X/C} + \Delta)$ is not ample.*

Proposition 3.14. *Let $\mathscr{F}$ be an algebraically integrable $\mathbb{Q}$-Fano foliation on a normal projective variety X, having log canonical singularities along a general leaf. Then there is a closed irreducible subset $T \subset X$ satisfying the following property. For a general log leaf $(\tilde{F}, \tilde{\Delta})$ of $\mathscr{F}$, there exists a log canonical center S of $(\tilde{F}, \tilde{\Delta})$ whose image in X is T.*

Proof. Let W be the normalization of the family of leaves of $\mathscr{F}$, U the normalization of the universal cycle over W, with universal family morphisms $\pi : U \to W$ and $e : U \to X$. As explained in 3.10, there is a canonically defined effective $\mathbb{Q}$-Weil divisor Δ on U such that $\det(\Omega^1_{U/W})^{\otimes m}[\otimes]\mathscr{O}_U(m\Delta) \cong e^*\mathscr{O}_X(mK_{\mathscr{F}})$, where $m \geqslant 1$ denotes the Cartier index of $K_{\mathscr{F}}$. Moreover, there is a smooth dense open subset $W_0 \subset W$ with the following properties. For any $w \in W_0$, denote by U_w the fiber of π over w, and set $\Delta_w := \Delta_{|U_w}$. Then

- U_w is integral and normal, and
- (U_w, Δ_w) has log canonical singularities.

To prove the proposition, suppose to the contrary that, for any two general points $w, w' \in W_0$, and any log canonical centers S_w and $S_{w'}$ of (U_w, Δ_w) and $(U_{w'}, \Delta_{w'})$ respectively, we have $e(S_w) \neq e(S_{w'})$.

Let $C \subset W$ be a (smooth) general complete intersection curve, and U_C the normalization of $\pi^{-1}(C)$, with induced morphisms $\pi_C : U_C \to C$ and $e_C : U_C \to X$. By [BLR95, Theorem 2.1'], after replacing C with a finite cover if necessary, we may assume that π_C has reduced fibers. As before, there is a canonically defined $\mathbb{Q}$-Weil divisor Δ_C on U_C such that $K_{U_C/C} + \Delta_C \sim_{\mathbb{Q}} e_C{}^*K_{\mathscr{F}}$. Therefore $K_{U_C} + \Delta_C \sim_{\mathbb{Q}} \pi_C^*K_C + e_C^*K_{\mathscr{F}}$ is a $\mathbb{Q}$-Cartier divisor. For a general point $w \in C$, we identify $\left(\pi_C^{-1}(w), \Delta_{C|\pi_C^{-1}(w)}\right)$ with (U_w, Δ_w), which is log canonical by assumption. Thus, by inversion of adjunction (see [Kaw07, Theorem]), the pair (U_C, Δ_C) has log canonical singularities over the generic point of C. Let $w \in C$ be a general point, and S_w any log canonical center of (U_w, Δ_w). Then there exists a reduced and irreducible closed subset $S_C \subset U_C$ such that:

- $S_w = S_C \cap U_w$, and
- S_C is a log canonical center of (U_C, Δ_C) over the generic point of C.

Moreover, our current assumption implies that

- $\dim(e_C(S_C)) = \dim(S_C)$.

Thus, by [Dem97, Proposition 7.2(ii)], there exist an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor E on U_C such that:

- $e_C^*(-K_{\mathscr{F}}) \sim_{\mathbb{Q}} A + E$, and
- for a general point $w \in C$, $\mathrm{Supp}(E)$ does not contain any log canonical center of (U_w, Δ_w).

Therefore $(U_C, \Delta_C + \epsilon E)$ is log canonical over the generic point of C for $0 < \epsilon \ll 1$. Notice that $e_C^*(-K_{\mathscr{F}}) - \epsilon E$ is ample since $e_C^*(-K_{\mathscr{F}})$ is nef and big, and hence

$$-(K_{U_C/C} + \Delta_C + \epsilon E) \sim_{\mathbb{Q}} e_C^*(-K_{\mathscr{F}}) - \epsilon E$$

is ample as well. But this contradicts Theorem 3.13, completing the proof of the proposition. □

Corollary 3.15. *Let $\mathscr{F}$ be an algebraically integrable Fano foliation on a complex projective manifold, and $(\tilde{F}, \tilde{\Delta})$ its general log leaf. Suppose that $\mathscr{F}$ is locally free along the closure of a general leaf. Then $\tilde{\Delta} \neq 0$.*

Proof. Denote by F the closure of a general leaf of $\mathscr{F}$. If $\tilde{\Delta} = 0$, then $\mathscr{F}$ is regular along F by Lemma 3.7. Hence $\mathscr{F}$ is induced by an almost proper map $X \dashrightarrow W$, and F is smooth. In particular $(\tilde{F}, \tilde{\Delta})$ is log canonical. But this contradicts Proposition 3.14. This proves that $\tilde{\Delta} \neq 0$. □

3.4 Foliations on $\mathbb{P}^n$

The *degree* $\deg(\mathscr{F})$ of a foliation $\mathscr{F}$ of rank r on $\mathbb{P}^n$ is defined as the degree of the locus of tangency of $\mathscr{F}$ with a general linear subspace $\mathbb{P}^{n-r} \subset \mathbb{P}^n$. By 3.5, a foliation on $\mathbb{P}^n$ of rank r and degree d is given by a twisted q-form $\omega \in H^0\big(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(q+d+1)\big)$, where $q = n - r$. Thus

$$d = \deg(K_{\mathscr{F}}) + r.$$

Jouanolou has classified codimension 1 foliations on $\mathbb{P}^n$ of degree 0 and 1. This has been generalized to arbitrary rank as follows.

Theorem 3.16.

(1) *[DC05, Théorème 3.8]. A codimension q foliation of degree* 0 *on $\mathbb{P}^n$ is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^q$.*

(2) *[LPT13, Theorem 6.2]. A codimension q foliation $\mathscr{F}$ of degree* 1 *on $\mathbb{P}^n$ satisfies one of the following conditions.*

   - *$\mathscr{F}$ is induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$, defined by q linear forms $L_1, \ldots, L_q$ and one quadratic form Q; or*
   - *$\mathscr{F}$ is the linear pullback of a foliation on $\mathbb{P}^{q+1}$ induced by a global holomorphic vector field.*

3.17. Let $\mathscr{F}$ be a codimension q foliation of degree 1 on $\mathbb{P}^n$.

In the first case described in Theorem 3.16(2), $\mathscr{F}$ is induced by the q-form on $\mathbb{C}^{n+1}$

$$\Omega = \sum_{i=1}^{q}(-1)^{i+1}L_i dL_1 \wedge \cdots \wedge \widehat{dL_i} \wedge \cdots \wedge dL_q \wedge dQ + (-1)^q 2Q dL_1 \wedge \cdots \wedge dL_q$$
$$= (-1)^q\Big(\sum_{i=q+1}^{n+1} L_j \frac{\partial Q}{\partial L_i}\Big) dL_1 \wedge \cdots \wedge dL_q$$
$$+ \sum_{i=1}^{q}\sum_{j=q+1}^{n+1}(-1)^{i+1}L_i \frac{\partial Q}{\partial L_j} dL_1 \wedge \cdots \wedge \widehat{dL_i} \wedge \cdots \wedge dL_q \wedge dL_j,$$

where $L_{q+1}, \dots, L_{n+1}$ are linear forms such that $L_1, \dots, L_{n+1}$ are linearly independent. Thus, the singular locus of $\mathscr{F}$ is the union of the quadric $\{L_1 = \cdots = L_q = Q = 0\} \cong Q^{n-q-1}$ and the linear subspace $\{\frac{\partial Q}{\partial L_{q+1}} = \cdots = \frac{\partial Q}{\partial L_{n+1}} = 0\}$.

In the second case described in Theorem 3.16(2), the singular locus of $\mathscr{F}$ is the union of linear subspaces of codimension at least 2 containing the center $\mathbb{P}^{n-q-2}$ of the projection.

3.5 Foliations on Q^n

In this subsection we classify del Pezzo foliations on smooth quadric hypersurfaces.

Proposition 3.18. *Let $\mathscr{F}$ be a codimension q del Pezzo foliation on a smooth quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$. Then $\mathscr{F}$ is induced by the restriction of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^q$.*

Proof. If $q = 1$, then the result follows from [AD14, Theorem 1.3]. So we assume from now on that $q \geqslant 2$.

By [AD13, Proposition 7.7], $\mathscr{F}$ is algebraically integrable, and its singular locus is nonempty by [AD14, Theorem 6.1].

Let $x \in Q^n$ be a point in the singular locus of $\mathscr{F}$, and consider the restriction $\varphi : Q^n \dashrightarrow \mathbb{P}^n$ to Q^n of the linear projection $\psi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ from x. Let $f : Y \to Q^n$ be the blow-up of Q^n at x with exceptional divisor $E \cong \mathbb{P}^{n-1}$, and $g : Y \to \mathbb{P}^n$ the induced morphism. Notice that g is the blow-up of $\mathbb{P}^n$ along the smooth codimension 2 quadric $Z = \varphi(\mathrm{Exc}(\varphi)) \cong Q^{n-2}$. Denote by H the hyperplane of $\mathbb{P}^n$ containing Z, and by F the exceptional divisor of g. Note that $g(E) = H$, and $f(F)$ is the hyperplane section of Q^n cut out by T_xQ^n. The codimension q del Pezzo foliation $\mathscr{F}$ is defined by a nonzero section $\omega \in H^0\big(Q^n, \Omega^q_{Q^n}(q+1)\big)$ vanishing at x. So it induces a twisted q-form $\alpha \in H^0\big(Y, \Omega^q_Y \otimes f^*\mathscr{O}_Q(q+1) \otimes \mathscr{O}_Y(-qE)\big) \cong H^0\big(Y, \Omega^q_Y \otimes g^*\mathscr{O}_{\mathbb{P}^n}(q+2) \otimes \mathscr{O}_Y(-F)\big)$. The restriction of α to $Y \setminus F$ induces a twisted q-form

$\tilde{\alpha} \in H^0\big(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(q+2)\big)$ such that $\tilde{\alpha}_z(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q) = 0$ for any $z \in Z$, $\mathbf{v}_1 \in T_z\mathbb{P}^n$, and $\mathbf{v}_i \in T_zZ$, $2 \leqslant i \leqslant q$. Denote by $\tilde{\mathscr{F}}$ the foliation on $\mathbb{P}^n$ induced by $\tilde{\alpha}$. There are two possibilities:

- Either $\tilde{\alpha}$ vanishes along the hyperplane H of $\mathbb{P}^n$ containing $Z \cong Q^{n-2}$, and hence $\tilde{\mathscr{F}}$ is a degree 0 foliation on $\mathbb{P}^n$; or
- $\tilde{\mathscr{F}}$ is a degree 1 foliation on $\mathbb{P}^n$, and either Z is contained in the singular locus of $\tilde{\mathscr{F}}$, or Z is invariant under $\tilde{\mathscr{F}}$.

We will show that only the first possibility occurs. In this case, it follows from Theorem 3.16(1) that $\mathscr{F}$ is induced by the restriction of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^q$.

Suppose to the contrary that $\tilde{\mathscr{F}}$ is a degree 1 foliation on $\mathbb{P}^n$, and either Z is contained in the singular locus of $\tilde{\mathscr{F}}$, or Z is invariant under $\tilde{\mathscr{F}}$. Recall the description of the two types of codimension q degree 1 on $\mathbb{P}^n$ from Theorem 3.16(2):

(1) Either the foliation is induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$, defined by q linear forms $L_1, \ldots, L_q$ and one quadratic form Q; or
(2) it is the linear pullback of a foliation on $\mathbb{P}^{q+1}$ induced by a global holomorphic vector field.

In case (2), the closure of the leaves and the singular locus are all cones with vertex $\mathbb{P}^{n-q-2}$. Since $Z \cong Q^{n-2}$ is a smooth quadric, we conclude that $\tilde{\mathscr{F}}$ must be of type (1), Z is invariant under $\tilde{\mathscr{F}}$, and $\tilde{\alpha}$ is as in the description of Ω in 3.17.

Since Z is invariant under $\tilde{\mathscr{F}}$, we must have $\{L_1 = \cdots = L_q = Q = 0\} \cong Q^{n-q-1} \subset Z$. We assume without loss of generality that $H = \{L_1 = 0\}$. Notice that $\{L_1 = \cdots = L_q = Q = 0\} \subsetneq Z$ since $q \geqslant 2$. Let $L_{q+1}, \ldots, L_{n+1} \in \mathbb{C}[t_1, \ldots, t_{n+1}]$ be linear forms such that $L_1, \ldots, L_{n+1}$ are linearly independent. Since Z is invariant under $\tilde{\mathscr{F}}$, $\varphi^*\tilde{\alpha}$ vanishes identically along $f(F) = \{\varphi^*L_1 = 0\}$. It follows from the description of the singular locus of $\tilde{\mathscr{F}}$ in 3.17 that we must have $\{\varphi^*\frac{\partial Q}{\partial L_{q+1}} = \cdots = \varphi^*\frac{\partial Q}{\partial L_{n+1}} = 0\} = \{\varphi^*L_1 = 0\}$. Hence, for $i \in \{q+1, \ldots, n+1\}$, $\varphi^*\frac{\partial Q}{\partial L_i} = a_i\varphi^*L_1$ for some complex number $a_i \in \mathbb{C}$. Then $\psi^*\tilde{\alpha} \in (\psi^*L_1)\cdot H^0\big(\mathbb{P}^{n+1}, \Omega^q_{\mathbb{P}^{n+1}}(q+1)\big) \subset H^0\big(\mathbb{P}^{n+1}, \Omega^q_{\mathbb{P}^{n+1}}(q+2)\big)$. Therefore, $\tilde{\mathscr{F}}$ is induced by a degree 0 foliation on $\mathbb{P}^{n+1}$. So $\tilde{\mathscr{F}}$ itself is a degree 0 foliation on $\mathbb{P}^n$, contrary to our assumption. This completes the proof of the proposition. □

4 Proof of Theorem 1.3

Let $X \ncong \mathbb{P}^n$ be an n-dimensional Fano manifold with $\rho(X) = 1$, and $\mathscr{F}$ a del Pezzo foliation of rank $r \geqslant 3$ on X. By Theorem 1.1, $\mathscr{F}$ is algebraically integrable. Let F be the closure of a general leaf of $\mathscr{F}$, $\tilde{e} : \tilde{F} \to X$ its normalization, and $(\tilde{F}, \tilde{\Delta})$ the corresponding log leaf. By assumption, $(\tilde{F}, \tilde{\Delta})$ is log canonical, and $\mathscr{F}$ is locally free along F.

Let $\mathscr{A}$ be an ample line bundle on X such that $\mathrm{Pic}(X) = \mathbb{Z}[\mathscr{A}]$, and set $\mathscr{L} := \tilde{e}^*\mathscr{A}$. Then $\det(\mathscr{F}) \cong \mathscr{A}^{r-1}$, and

$$-(K_{\tilde{F}} + \tilde{\Delta}) \sim_{\mathbb{Z}} -\tilde{e}^* K_{\mathscr{F}} \sim_{\mathbb{Z}} (r-1)c_1(\mathscr{L}).$$

By Corollary 3.15, $\tilde{\Delta} \neq 0$. So we can apply Theorem 2.15. Taking into account that if $\tilde{F}$ is singular, then its singular locus is contained in the support of $\tilde{\Delta}$ by Lemma 3.7, we get the following possibilities for the triple $(\tilde{F}, \mathscr{L}, \tilde{\Delta})$:

(1) $\big(\tilde{F}, \mathscr{L}, \mathscr{O}_{\tilde{F}}(\tilde{\Delta})\big) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1), \mathscr{O}_{\mathbb{P}^r}(2))$.
(2) $\big(\tilde{F}, \mathscr{L}, \mathscr{O}_{\tilde{F}}(\tilde{\Delta})\big) \cong (Q^r, \mathscr{O}_{Q^r}(1), \mathscr{O}_{Q^r}(1))$, where Q^r is a smooth quadric hypersurface in $\mathbb{P}^{r+1}$.
(3) $r = 1$ and $\big(\tilde{F}, \mathscr{L}, \mathscr{O}_{\tilde{F}}(\tilde{\Delta})\big) \cong (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d), \mathscr{O}_{\mathbb{P}^1}(2))$ for some integer $d \geqslant 3$.
(4) $r = 2$ and $\big(\tilde{F}, \mathscr{L}, \mathscr{O}_{\tilde{F}}(\tilde{\Delta})\big) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2), \mathscr{O}_{\mathbb{P}^2}(1))$.
(5) $(\tilde{F}, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1))$ for an ample vector bundle $\mathscr{E}$ of rank r on $\mathbb{P}^1$. Moreover, one of the following holds.
   (a) $r = 2$ and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ for some $a \geqslant 2$, and $\tilde{\Delta} \sim_{\mathbb{Z}} \sigma + f$ where σ is the minimal section and f a fiber of $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}^1$.
   (b) $r = 2$ and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ for some $a \geqslant 2$, and $\tilde{\Delta}$ is a minimal section.
   (c) $r = 3$ and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a)$ for some $a \geqslant 1$, and $\tilde{\Delta} = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1))$.
(6) $\mathscr{L}$ is very ample, and embeds $(\tilde{F}, \tilde{\Delta})$ as a cone over $\big((Z, \mathscr{M}), (\Delta_Z, \mathscr{M}_{|\Delta_Z})\big)$, where $(Z, \mathscr{M}, \Delta_Z)$ satisfies one of the conditions (2–5) above.

First we show that case (1) cannot occur. Suppose otherwise that $(\tilde{F}, \mathscr{L}, \mathscr{O}_{\tilde{F}}(\tilde{\Delta})) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1), \mathscr{O}_{\mathbb{P}^r}(2))$. By Proposition 3.14, there is a common point x contained in the closure of a general leaf. Since $(\tilde{F}, \mathscr{L}) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1))$, there is an irreducible $(n-1)$-dimensional family of lines on X through x sweeping out the whole X. Lemma 2.6 together with Theorem 2.2 imply that $X \cong \mathbb{P}^n$, contrary to our assumptions.

Next suppose that we are in case (2) or (5c). Note that $\tilde{\Delta}$ is irreducible in either case (in case (2), $\tilde{F}$ is a smooth quadric of dimension $r \geqslant 3$ and $\tilde{\Delta}$ is a hyperplane section). By Proposition 3.14, the image T of $\tilde{\Delta}$ in X is contained in the closure of a general leaf of $\mathscr{F}$. There is a family of lines on X, all contained in leaves of $\mathscr{F}$ and meeting T, that sweep out the whole X. In case (2), this corresponds to the family of lines on $\tilde{F} \cong Q^r$. In case (5c), it corresponds to the family of lines on fibers of $\tilde{F} \to \mathbb{P}^1$. Let $x \in T$ be a general point. Then there is an irreducible $(n-2)$-dimensional family of lines on X through x, and the general line in this family is free by Remark 2.7. By Lemma 2.6, $\iota_X \geqslant n$. Theorem 2.2 then implies that $X \cong Q^n$. By Proposition 3.18, $\mathscr{F}$ is induced by the restriction to Q^n of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-r}$.

Cases (3), (4), (5a), and (5b) do not occur since we are assuming $r \geqslant 3$.

Finally suppose that we are in case (6): $\mathscr{L}$ is very ample, and embeds $(\tilde{F}, \tilde{\Delta})$ as a cone over the pair $\big((Z, \mathscr{M}), (\Delta_Z, \mathscr{M}_{|\Delta_Z})\big)$, where $(Z, \mathscr{M}, \Delta_Z)$ satisfies one of the conditions (2–5) above. As in the proof of Theorem 2.15, set $m := \dim(Z)$, $s := r-m$, and let $e : Y \to \tilde{F}$ be the blow-up of $\tilde{F}$ along its vertex, with exceptional divisor E. Then $Y \cong \mathbb{P}_Z(\mathscr{M} \oplus \mathscr{O}_Z^{\oplus s})$, with natural projection $\pi : Y \to Z$. Moreover the strict transform Δ_Y of $\tilde{\Delta}$ in Y satisfies $\Delta_Y = \pi^*\Delta_Z$. A straightforward computation gives

$$K_Y + \pi^*\Delta_Z \sim_{\mathbb{Z}} e^*(K_{\tilde{F}} + \tilde{\Delta}) + (m-2)E.$$

On the other hand, by [AD13, 8.3], there exists an effective divisor B on Y such that

$$K_Y + E + B \sim_{\mathbb{Z}} e^*(K_{\tilde{F}} + \tilde{\Delta}).$$

Therefore

$$\Delta_Y = \pi^*\Delta_Z \sim_{\mathbb{Z}} (m-1)E + B.$$

We conclude that $m = 1$. Thus $\tilde{F}$ is isomorphic to a cone with vertex $V \cong \mathbb{P}^{r-2}$ over a rational normal curve of degree $d \geqslant 2$, and $\tilde{\Delta}$ is the union of two rulings Δ_1 and Δ_2, each isomorphic to $\mathbb{P}^{r-1}$.

By Proposition 3.14, there is a log canonical center S of $(\tilde{F}, \tilde{\Delta})$ whose image in X does not depend on the choice of the general log leaf. Either $S = V$, or $S = \Delta_i$ for some $i \in \{1, 2\}$. If $S = V$, then the lines through a general point of $\tilde{e}(V)$ sweep out the whole X. Lemma 2.6 together with Theorem 2.2 then imply that $X \cong \mathbb{P}^n$, contrary to our assumptions. We conclude that the image of V in X varies with $(\tilde{F}, \tilde{\Delta})$, and, for some $i \in \{1, 2\}$, $T = \tilde{e}(\Delta_i)$ is contained in the closure of a general leaf. There is a family of lines on X, all contained in leaves of $\mathscr{F}$ and meeting T, that sweep out the whole X. Let $x \in T$ be a general point. Since $V \subset \Delta_i$, and the image of V in X varies with $(\tilde{F}, \tilde{\Delta})$, there is an irreducible $(n-2)$-dimensional family of lines on X through x, and the general line in this family is free by Remark 2.7. By Lemma 2.6, $\iota_X \geqslant n$. Theorem 2.2 then implies that $X \cong Q^n$. By Proposition 3.18, $\mathscr{F}$ is induced by the restriction of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-r}$. □

Using Theorem 2.16 and the same arguments as in the proof of Theorem 1.3, one can get the following result for del Pezzo foliations of rank 2, without the assumption that $\mathscr{F}$ is log canonical along a general leaf.

Proposition 4.1. *Let $\mathscr{F}$ be a del Pezzo foliation of rank 2 on a complex projective manifold $X \not\cong \mathbb{P}^n$ with $\rho(X) = 1$, and suppose that $\mathscr{F}$ is locally free along a general leaf. Denote by $(\tilde{F}, \tilde{\Delta})$ the general log leaf of $\mathscr{F}$, and by $\mathscr{L}$ the pullback to $\tilde{F}$ of the ample generator of* Pic(X). *Then the triple $(\tilde{F}, \mathscr{O}_F(\tilde{\Delta}), \mathscr{L})$ is isomorphic to one of the following.*

(1) $\big(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2)\big)$;
(2) $\big(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2), \mathscr{O}_{\mathbb{P}^2}(1)\big)$;
(3) $\big(\mathbb{F}_d, \mathscr{O}_{\mathbb{F}_d}(\sigma + f), \mathscr{O}_{\mathbb{F}_d}(\sigma + (d+1)f)\big)$, *where $d \geqslant 0$, σ is a minimal section, and f is a fiber of $\mathbb{F}_d \to \mathbb{P}^1$;*
(4) $\big(\mathbb{P}(1,1,d), \mathscr{O}_{\mathbb{P}(1,1,d)}(2\ell), \mathscr{O}_{\mathbb{P}(1,1,d)}(d\ell)\big)$, *where $d \geqslant 2$, and ℓ is a ruling of the cone $\mathbb{P}(1, 1, d)$.*

Remark 4.2. There are examples of del Pezzo foliations of rank 2 on Grassmannians whose general log leaves are $(\tilde{F}, \tilde{\Delta}) = (\mathbb{P}^2, 2\ell)$ (see [AD13, 4.3]).

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Rational Curves on Foliated Varieties

Fedor Bogomolov and Michael McQuillan

Abstract The article refines and generalises the study of deformations of a morphism along a foliation begun by Y. Miyaoka, [Mi2]. The key ingredients are the algebrisation of the graphic neighbourhood, see Fact 3.3.1, which reduces the problem from the transcendental to the algebraic, and a p-adic variation of Mori's bend and break in order to overcome the "naive failure", see Remark 3.2.3, of the method in the required generality. Qualitatively the results are optimal for foliations of all ranks in all dimensions, and are quantitatively optimal for foliations by curves, for which the further precision of a cone theorem is provided.

Keywords Frobenius (theorem) • (foliated) (log) canonical singularities • Algorithmic resolution • Graphic neighbourhood • Ample (vector) bundle • Frobenius (map) • Bend and break • p-adic • Rationally connected • Cone of curves

Mathematical Subject Classification: 14E05 (32S65)

In a series of papers, notably [B1] and [B2], the first author showed, amongst other things, that a surprising interplay between the classical Frobenius theorem on the integrability of vector fields closed under Lie bracket and various algebro-geometric considerations gave rise to some rather strong restrictions on the "size" of sub-bundles of the cotangent bundle, with a particular corollary being inequalities for Chern numbers on algebraic surfaces. In refining this circle of ideas Y. Miyaoka, [Mi1], established that any quotient of the cotangent bundle of a surface of general type and positive index was big. Rather more remarkably, Miyaoka, [Mi2],

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subsequently considered the problem of sub-bundles of the tangent bundle with positive slope along a generic complete intersection of ample divisors, and by extending Mori's bend and break technique to "deformations along a foliation" showed that these hypothesis implied the existence of covering families of rationally chain connected varieties in the direction of the foliation. The surface case of this result is particularly clean, since it asserts that either the bundle of top weight forms $K_{\mathscr{F}}$ along the leaves is pseudo-effective, or the foliation is a fibration by rational curves. Continuing in these directions the second author in extending the results of the first author on boundedness of moduli of curves of a given genus on surfaces to curves with boundary employed Miyaoka's semi-positivity theorem in an essential way, cf. [M1]. The use of the said theorem therein was to force, under the hypothesis of a dense parabolic leaf, the existence of a global vector field defining the foliation on a crude version of what might be considered it's minimal model, i.e. a normal algebraic space on which the positive part of the Zariski decomposition of $K_{\mathscr{F}}$ coincides with the push forward of the latter. It was natural however to examine this question more carefully, not just in terms of a more delicate structure of the minimal model but to introduce the study of the birational geometry of foliations per se, cf. [M2, Br2, Br3]. As ever a prerequisite for such a study is the understanding of sub-varieties on which the cotangent bundle of our foliation is negative.

To fix ideas let us consider a foliation by curves, $\mathscr{F}$, on a smooth variety X. This is equivalent to giving a rank 1 torsion free quotient of Ω_X, whose Chern class we denote by $K_{\mathscr{F}}$. Miyaoka's theorem then asserts that if C is a "sufficiently movable" curve then either $K_{\mathscr{F}}.C \geq 0$ or $\mathscr{F}$ is a (possibly singular) fibration by rational curves. The main problem here is the hypothesis, "sufficiently movable" which more precisely means that C moves in a family $\{C_t \mid t \in T\}$ covering X such that generically C_t does not meet the singularities of $\mathscr{F}$, whereas one really wishes to understand the implications of the hypothesis $K_{\mathscr{F}}.C < 0$ for any curve C. The difficulties in extending Miyaoka's method, or its refinement by Shepherd-Barron, [SB], to this situation are formidable. Firstly one must establish that the hypothesis imply that the divided symmetric power algebra of $K_{\mathscr{F}}$ extends across a finitely generated extension of $\mathbb{Z}$, [Mi2], or rather more straightforwardly the foliation is defined by an inseparable scheme quotient in positive characteristic, [SB]. Even then there is the added complication that C may pass through the foliation singularities and so Mori's bend and break technique may not apply. The key to resolving this problem § 2 lies in finding a $\mathscr{F}$-invariant surface S containing C,[1] and so reduce the study to something more tractable. In finding our surface, however, we necessarily show that the leaves through C are algebraic curves, and whence reprove Miyaoka's theorem in this case without any appeal to reduction in positive characteristic, thanks to a theorem of Arakelov, [A], all be it that the most satisfying proof of Arakelov's theorem is to proceed via positive characteristic, cf. [S].

With this example in mind let us consider a more general situation. We denote by $(X, \mathscr{F})$ any variety equipped with an integrable (i.e. closed under Lie bracket)

[1] Concomitantly with the preparation of the original pre-print, c. may 2000, J.-B. Bost, [Bo], used what may be considered an arithmetic version of this trick which independently led him to discover the geometric variant, and its higher dimensional generalisations à la § 2.1.

foliation $\mathscr{F}$. The singularities of X are not our interest, but rather those of $\mathscr{F}$. What this latter should mean is a measure of how far away the foliation is from being given everywhere locally by a relatively smooth fibration. To understand this it is convenient to introduce an ambient smooth space M. The foliation is given by a sub-sheaf $\mathscr{T}_{\mathscr{F}}$ of $\mathscr{T}_X$ of rank r, say, and for any $x \in X$ we have the residue map,

$$\mathscr{T}_{\mathscr{F}} \otimes k(x) \longrightarrow T_M \otimes k(x) .$$

Should this map be an injection onto a subspace of dimension r, then Nakayama's lemma forces $\mathscr{F}$ to be a bundle in a neighbourhood of x, and better still the Frobenius theorem goes through verbatim to force $\mathscr{F}$ to be given locally by a relatively smooth fibration. Naturally then we introduce the notion of weak regularity, 1.1, which requires $\mathscr{T}_{\mathscr{F}}$ to be a bundle, and put:

$$\operatorname{sing}(\mathscr{F}) = \{x \in X \mid \dim(\operatorname{Im}\{T_{\mathscr{F}} \otimes k(x) \longrightarrow T_M \otimes k(x)\}) < r\} .$$

This variety stratifies naturally according to the rank of the image map, and for C any curve in X, we denote by $r(C)$ the generic rank. We may now state,

Main Theorem. (a) *Let $(X, \mathscr{F})$ be a weakly regular (integrable) foliated variety and C a curve in X with $T_{\mathscr{F}}|_C$ ample then for all $x \in C$ there is a $\mathscr{F}$-invariant rationally connected sub-variety $V_x \ni x$ of dimension $r(C)$.*
Here we use reduction modulo p, but only to resolve the problem for a foliation such that $X/\mathscr{F}$ exists as a scheme quotient over $\mathbb{C}$, so in fact,

(b) *Notations and hypothesis as above then the minimal degree of the rational curve connecting any two points in V_x is effectively computable. In particular there is a rational curve $L_x \ni x$ tangent to $\mathscr{F}$ such that for any nef. $\mathbb{R}$-divisor H,*

$$H.\, L_x \leq 2r(C)\, \frac{H.\, C}{-K_{\mathscr{F}_{r(C)}}\, C} .$$

The bound, *cf.* 3.3.1, however, on the degree of rational curves connecting any two points in item (a) may be much worse. One could reasonably expect, 3.2.2, that it should be as above with $-r(C)^{-1}K_{\mathscr{F}_{r(C)}}\, C$ replaced by the minimum slope of $T_{\mathscr{F}}|_C$, but this is not proved. In this notation, $\mathscr{F}_{r(C)}$ is the induced foliation on the sub-scheme of X where the generic rank is $r(C)$. It is not immediately clear that intersecting with the canonical class has any sense for $r(C) < r$, but well definedness will emerge in the course of the proof. One should also note that there is no need to suppose $T_{\mathscr{F}}$ is saturated in $\mathscr{T}_X$ provided there is closure under Lie bracket. Without closure under bracket one can of course find a foliation $\mathscr{G}$ corresponding to the minimal sub-sheaf $\mathscr{T}_{\mathscr{G}}$ of $\mathscr{T}_X$ closed under the same. It may happen, however, that this is not a bundle in a neighbourhood of our curve C, although it will certainly be as "ample" as one needs. This poses serious technical problems akin to the difficulty of doing deformation theory on singular varieties, and so one only obtains (a) and (b) under the weaker hypothesis that $\mathscr{G}$

is weakly regular in a neighbourhood of C. This is however perfectly sufficient to recover Miyaoka's semi-positivity theorem, where X is supposed normal and C moves in a large base point free family.

Our study however has so far revealed nothing if $r(C) = 0$. For foliations by curves this is equivalent to our curve being wholly contained in the singular locus of the foliation. To address this question we have recourse to the notion of foliated canonical singularities. This class of singularities may be understood as the maximal one for which birational geometry of foliations makes sense, equally the precise definition à la Kawamata-Mori et al. is given in § 1.1. To explain its implications observe that for a curve C with generic rank $r(C)$ contained in a component Y of the locus where the rank is the same there is an exact sequence of sheaves,

$$0 \longrightarrow \mathscr{N}_{r(C)} \longrightarrow T_{\mathscr{F}} \otimes \mathscr{O}_Y \longrightarrow T_{\mathscr{F}_{r(C)}}$$

where we take this as the definition of $\mathscr{N}$, and we have,

(c) *Let $(X, \mathscr{F})$ be a weakly regular (but not necessarily integrable) foliated variety with canonical singularities and C a curve with $r(C)$ the generic rank of the foliation along C then $\mathscr{N}_{r(C)} \otimes \mathscr{O}_C$ has a non-positive rank 1 quotient.*

Thus we have established that the rationally connected sub-varieties guaranteed by (a) exhaust the ampleness of $T_{\mathscr{F}}|_C$, and we deduce,

Corollary (4.2.1). *Let $(X, \mathscr{F})$ be a variety foliated by curves with foliated Gorenstein and canonical singularities (1.1.1) then there are countably many $\mathscr{F}$-invariant rational curves with $K_{\mathscr{F}}.L_i < 0$ such that if $\overline{\mathrm{NE}}(X)$ is the closed cone of effective curves and $\overline{\mathrm{NE}}(X)_{K_{\mathscr{F}}\geq 0}$ the sub-cone on which $K_{\mathscr{F}}$ is positive then,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathscr{F}}\geq 0} + \sum_i \mathbb{R}_+ [L_i] .$$

Better still,

(a) *The rays $\mathbb{R}_+ [L_i]$ are locally discrete in the upper half space $\mathrm{NS}(X)_{K_{\mathscr{F}}<0}$.*
(b) *If $(X, \mathscr{F})$ is not a ruling by rational curves $K_{\mathscr{F}}.L_i = -1$, $\forall i$, and $L_i \cap \mathrm{sing}(\mathscr{F}) \neq \emptyset$. Otherwise, $K_{\mathscr{F}}.L_i \in \{-1, -2\}$.*
(c) *Every extremal ray in the half space $\mathrm{NS}(X)_{K_{\mathscr{F}}<0}$ is of the form $\mathbb{R}_+ [L_i]$.*

Whence a smooth foliation either has $K_{\mathscr{F}}$ nef or it's a fibring by rational curves, and quite generally there are supporting Cartier divisors for extremal rays. Whence, we have the first step in a minimal model programme for foliations by curves.

For foliations of higher rank, our "p-adic" variation(s) on Mori's bend and break procedure, wherein the emphasis is switched from lifting rational curves in positive characteristic to lifting Frobenius is itself new, and, indeed, necessary since although bend and break produces, in positive characteristic, as many rational curves as one needs to establish the rational connectedness in the main theorem, only a limited number of these admit the characteristic independent degree bound of [MM]. To illustrate how "p-adic bend & break" does not proceed via such degree bounds we've added the following non-projective,

Little Theorem (III.4). *Let $X/\mathbb{C}$ be a proper algebraic space and $f : C \to X$ a map from a curve such that X is smooth in a neighbourhood of the image and f^*T_X is ample, then X is rationally chain connected. Consequently, [D, 4.9], a smooth proper algebraic space is rationally chain connected iff it admits such a curve.*

Whose proof is a lot simpler than the main theorem- twists of f by a sufficiently large power of Frobenius, and indeed positive characteristic deformations thereof, lift to characteristic zero by [SGA, Exposé III]. The principle in the main theorem is the same, but one has to add teeth to the curve in order to construct a suitable comb in the sense of [K, II.7.7] before the lifting can take place. Subsequent to acceptance, the second author while participating in the workshop on rationally connected varieties at the algebraic geometry laboratory named in honour of the first was amused (and not just because of the aforesaid context) to learn from Jason Starr's lectures that [dJS] employs a variation on this theme, *i.e.* creating enough deformations by the addition of teeth, to give a much simpler and characteristic free proof of the main theorem of [GHS], *i.e.* a rationally connected fibration over a curve has a section.

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1 Singularities

1.1 Definitions

Our objects of study are foliated varieties, i.e. a normal variety X equipped with a foliation $\mathscr{F}$, which we denote by $(X, \mathscr{F})$. On the other hand, we do not wish to think of this as two separate objects, but rather as a unified whole. As such the singularities which we wish to study have a priori nothing to do with the space X, although as suggested above we will make the technically convenient, and rather mild assumption that X is normal. To proceed further let us note that the precise definition of a foliation is simply a saturated sub-sheaf of the tangent sheaf, i.e.

$$\mathscr{T}_{\mathscr{F}} \hookrightarrow \mathscr{T}_X .$$

If in addition the sub-sheaf $\mathscr{T}_{\mathscr{F}}$ is closed under Lie bracket, then we say that $(X, \mathscr{F})$ is integrable. One has of course the classical theorem of Frobenius over the complex numbers, which asserts that if X is smooth and $\mathscr{T}_{\mathscr{F}}$ is a sub-bundle of X at some point, then closure under Lie bracket is equivalent to the foliation being given locally as a fibration, whence the appellation.

Now in the study of singularities of a variety per se, the main protagonists are the cotangent sheaf, and the canonical bundle. The former is the more classical and its relation with local algebra rather well understood. The latter is rather more recent, but its study is the essential prerequisite for birational geometry. Since our results will also encompass the case of the trivial foliation, i.e. simply the study of curves on varieties, it is not surprising that we will encounter a similar phenomenon. However rather than the cotangent sheaf of the foliation, we will work with the tangent sheaf, via which we introduce our first definition, viz:

Definition 1.1.1. A foliated variety $(X, \mathscr{F})$ is said to be weakly regular if it is given by a sub-bundle $T_{\mathscr{F}}$ sitting as a saturated sub-sheaf of the tangent sheaf $\mathscr{T}_X$.

This definition is immediately deserving of comment. In the case of the trivial foliation the definition asserts that the tangent sheaf of the variety is a bundle. This is a priori strictly weaker than the assertion that the variety is regular, i.e. that the cotangent sheaf is a bundle. It is however a conjecture of Zariski and Lipmann that the two are equivalent. At the other extreme if $(X, \mathscr{F})$ is a foliation by curves, given that X is normal, weak regularity is equivalent to being foliated Gorenstein as introduced in [M2], and amounts to the foliation being given everywhere by a vector field. Given that the said vector field is itself allowed to vanish in co-dimension 2, every foliation by curves on a non-singular variety is weakly regular. On the other hand, for a singular variety X, the condition is highly non-trivial, and can give a great deal of information about the foliation, cf. op. cit.

The role of normality of the underlying space in the above is fairly unimportant, where it really makes its appearance is in the consideration of the canonical sheaf of the foliation, i.e. the dual of the top exterior power of the tangent sheaf $\mathscr{T}_{\mathscr{F}}$. Naturally we denote this by $K_{\mathscr{F}}$, and follow Kawamata, Mori et al., by introducing a discrepancy function to measure the singularities. Specifically suppose $K_{\mathscr{F}}$ is "$\mathbb{Q}$-Cartier". For the moment let us be deliberately vague about what this may mean, and consider any proper birational map $p : (\tilde{X}, \tilde{\mathscr{F}}) \to (X, \mathscr{F})$ from a foliated variety to our original variety then for some divisors E_i contracted by p, and rational numbers a_i, we must have:

$$K_{\tilde{\mathscr{F}}} \equiv p^* K_{\mathscr{F}} + \sum_i a_i E_i$$

where "$\equiv$" is some suitable equivalence relation of divisors, such as rational or numerical. Given that $\tilde{X}$ is necessarily normal, whence non-singular in co-dimension 1, the numbers a_i depend only on E_i considered as prime valuations of the field of functions of X, and so we may define a map:

$$a(\ , X, \mathscr{F}) : \{\text{prime valuations } E \text{ of } k(X)^{\times}\} \longrightarrow \mathbb{Q}$$

where of course we implicitly assume that the valuations have non-empty centre, so that $a\,(\;, X, \mathscr{F})$ is even defined at the level of germs. Finally we are in a position to introduce the discrepancy of a foliated space, and to make another definition, i.e.

$$\operatorname{discrep}(X, \mathscr{F}) := \inf_E a\,(E, X, \mathscr{F})$$

Definition 1.1.2. A foliated space $(X, \mathscr{F})$ is said to have canonical singularities whenever $\operatorname{discrep}(X, \mathscr{F}) \geq 0$, and terminal singularities should this inequality be strict.

Necessarily in the case of the trivial foliation, $K_{\mathscr{F}}$ and K_X coincide, so that the definitions of canonical and terminal do like wise. However when the rank of the foliation differs from the dimension of X this is absolutely not so (Fig. 1). For example, consider the vector field $\partial = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}$ in a neighbourhood of the origin in $\mathbb{C}^2$. One may easily draw this, viz:

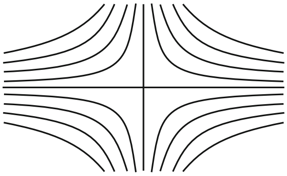


Fig. 1 A canonical but not terminal foliation singularity

The underlying space X is certainly smooth, and as such has terminal singularities in the usual sense (Fig. 2). However the pair $(X, \mathscr{F})$ is our object of study and this has discrepancy zero, i.e. it is properly canonical without being terminal. One can go further, and consider the vector field $\partial = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$. Again this is easily drawn, the underlying space is as before, but now the discrepancy is actually -1, for the foliated space. Actually these two examples are fairly representative of what can happen for an underlying smooth space X of dimension 2 together with a foliation by curves. Canonical singularities here are a slight generalisation, or more accurately the functorial correction, of so-called reduced singularities or those of Poincaré-Dulac type, cf. [M2], and every foliation by curves on a surface may be resolved to one with canonical singularities. Furthermore, as a moment's reflection on our initial example shows, this is best possible, i.e. unlike the trivial case one cannot resolve to a birational model with terminal singularities. Ultimately, however, one gains a rather better feeling for the definition and its interplay with the singularities of the underlying space by considering blow ups in foliation equivariant centres. This of course means that we specify a sub-variety Y of X, together with its sheaf of ideals $\mathscr{I}_Y$ and over an affine open subset U of X ask that for all derivations $\partial \in \Gamma(U, \mathscr{T}_{\mathscr{F}})$, $\partial(\mathscr{I}_Y) \subset \mathscr{I}_Y$. This definition behaves well with respect to localisation, so it easily globalises, and we have:

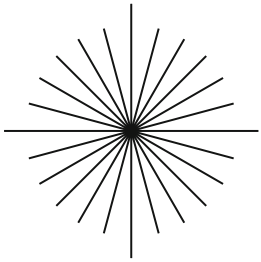


Fig. 2 A log-canonical but not canonical foliation singularity

Lemma 1.1.3. *Let $p : \tilde{X} = \mathrm{Bl}_Y(X) \to X$ be a blow up of a foliated variety $(X, \mathscr{F})$ in a $\mathscr{F}$-equivariant centre Y then for $\tilde{\mathscr{F}}$ the induced foliation we have a natural map,*

$$p^* \, \mathscr{T}_{\mathscr{F}} \longrightarrow \mathscr{T}_{\tilde{\mathscr{F}}} \, .$$

Proof. The question is local, so we may assume that X is affine, say Spec$A$, and $\mathscr{I}_Y$ is just the sheafication of an ideal I of A. The assertion is then simply that if $\partial \in \mathrm{Der}(A)$ lies in the tangent sheaf of the foliation then the a priori meromorphic vector field $p^* \partial$ is in fact holomorphic. This is easily verified, since locally a function f on $\tilde{X}$ is of the form $\frac{g}{h^d}$ where d is a non-negative integer, $g \in I^d$, and h being in I, where naturally we think of ourselves as looking at functions on the $h \neq 0$ part, then of course:

$$\partial(f) = \partial\left(\frac{g}{h^d}\right) = \frac{h\,\partial g - dg\,\partial h}{h^{d+1}}$$

and since I^d is also $\mathscr{F}$ equivariant, $\partial(f)$ is a function as required. □

As a simple illustration of the lemma consider a foliation by curves $(X, \mathscr{F})$ with underlying space X non-singular. Locally the foliation is given by a vector field ∂, and where ∂ vanishes is the singular locus of $\mathscr{F}$. Any point x in the singular locus is of course $\mathscr{F}$ equivariant, so in a neighbourhood of x the discrep is always less than or equal to zero. Consequently under such hypothesis if a "singularity" is "terminal" then the vector field is non-zero, i.e. the foliation is locally a smooth fibration. The essence of this conclusion extends, as we shall see, to arbitrary weakly regular integrable foliations. However it is not a classification of arbitrary terminal singularities, which can be found in [MP]. In any case to justify the above remark regarding weakly regular foliations we will need a further lemma, viz:

Lemma 1.1.4. *Let $p : (\tilde{X}, \tilde{\mathscr{F}}) \to (X, \mathscr{F})$ be the blow up of a foliated variety (in characteristic zero) in a $\mathscr{F}$-equivariant centre Y as in Lemma 1.1.3, and let $\rho : (X^\#, \mathscr{F}^\#) \to (\tilde{X}, \tilde{\mathscr{F}})$ be its normalisation then in fact ρ is $\mathscr{F}$-equivariant, i.e. we have a natural map,*

$$\rho^* \, p^* \, \mathscr{T}_{\mathscr{F}} \longrightarrow \mathscr{T}_{\mathscr{F}^\#} \, .$$

Proof. The assertion precisely regards the locus where $X^\#$ is not isomorphic to $\tilde{X}$. Better still since $X^\#$ is S_2 we only require to show the existence of the map in co-dimension 1, so let $D^\#$ in $X^\#$ be an irreducible divisor with generic point $\eta^\#$, and D, respectively η, the image of the same in $\tilde{X}$. Now if the foliation were generically transverse to D, then 1.2.1, on completion in D it would, around η, be a smooth projection onto a formal affinoid Y. The generic point of the latter is, however, the image of η, so Y would be smooth, and whence $\tilde{X}$ would already be normal at η. Thus, without loss of generality, the foliation leaves D invariant, so by [BM] the germ of ρ around η can be realised by a sequence of blow ups in $\tilde{\mathscr{F}}$ invariant centres, and we conclude by 1.1.3. □

Unlike the previous lemma, characteristic zero is essential here in order to deduce that the centres of the local blowing ups are equivariant under the foliation. Indeed in characteristic $p > 0$ the singular curve

$$x^p + y^l x^{p-1} + y^l = 0, \quad p \nmid l > 1$$

admits the smooth vector field $l(1 + x^{p-1})\partial_x + x^{p-2}y\partial_y$. Applying the lemma to arbitrary Gorenstein foliations by curves implies that either the foliation is locally integrable by the usual Frobenius type procedure or it has at best a canonical singularity. The difficulty in extending such considerations to arbitrary foliations relies on identifying suitable invariant centres. We will come back to this, and a more precise discussion of the above remarks on integrability in § 1.2. For the moment let us complete this introduction to singularities by way of some remarks on the condition $K_{\mathscr{F}}$ is "$\mathbb{Q}$-Cartier".

The most obvious sense of this is of course to consider the open embedding $j : (X_{sm}, \mathscr{F}_{sm}) \hookrightarrow (X, \mathscr{F})$ of the locus where say both X and $\mathscr{F}$ are smooth, and to demand that there is a positive integer m such that $j_* K_{\mathscr{F}_{sm}}^{\otimes m}$ is a Cartier divisor. This is rather strong, and so we term it $\mathbb{Q}$-foliated Gorenstein, and of course foliated Gorenstein if we can take $m = 1$. Equally for two-dimensional normal algebraic spaces, the definition of "$\mathbb{Q}$-Cartier" can be understood in a linguistically abusive, though not mathematically abusive, sense via Mumford's intersection theory. These remarks are, however, all rather parenthetical since we will be almost exclusively concerned with foliations which satisfy the Gorenstein condition.

1.2 Towards an Ideal Situation

We now wish to concentrate on how to ameliorate the singularities of a foliated variety in a neighbourhood of a curve. In this section we will concentrate on foliations by curves. This not only provides some calculations essential to the general case, but also illustrates the key features of what we are after without the technical complications that arise in the higher rank case. The ideal of course would be to find a neighbourhood of the curve, birationally, where the foliation is everywhere integrable. In absolute generality this is impossible, indeed it is even so on smooth surfaces. Nevertheless the impossibility only occurs for curves invariant by the foliation. Since we intend to allow arbitrary singularities on the underlying space X we will consider an embedding $X \hookrightarrow M$ of our variety into a smooth variety. In addition everything will be arbitrarily local, so let's just work in the analytic topology. Observe that for a vector field ∂ on X and $x \in X$ it is completely unclear from the definition whether $\partial \neq 0$ in $\mathscr{T}_X \otimes k(x)$ implies $\partial \neq 0$ in $T_M \otimes k(x)$. Indeed this difficulty is at the root of the Zariski-Lipmann conjecture. Whence although it may depend on the embedding let's call ∂ non-singular at x if ∂ is non-zero in $T_M \otimes k(x)$, and observe the following straightforward generalisation of the Frobenius theorem.

Lemma 1.2.1. *Let x be a non-singular point of a Gorenstein foliation by curves $(X, \mathscr{F})$ then there is a map $\pi : X \to Y$ of analytic spaces around x such that $\pi^* \Omega_Y$ is the conormal bundle of $\mathscr{F}$ in X, and π is relatively smooth.*

Proof. We proceed in the obvious way. Namely if I is the ideal of X in M we have the usual short exact sequence,

$$I/I^2 \longrightarrow \Omega_M|_X \longrightarrow \Omega_X \longrightarrow 0\,.$$

By the non-singularity hypothesis there is a vector field ∂ on M, non-vanishing at x, with $\partial(I) \subset I$ which induces our given foliation on X, where of course we permit as much localisation as we need. Now let $z_1, \ldots, z_n$ be coordinate functions on M, then without loss of generality $\partial(z_1) = 1$, as usual we may put,

$$y_i = \sum_{n=0}^{\infty} \frac{(-1)^n z_1^n \partial^n z_i}{n!}\,, \qquad 2 \le i \le n$$

and one easily checks $\partial y_i = 0$. Consequently we just put Y to be the image of X in $\mathbb{C}^{n-1}$ under the map, $(y_2, \ldots, y_n) : M \to \mathbb{C}^{n-1}$. □

Now we would like to obtain this situation around any point of a suitable curve C in X. The definition of suitable here is that C is not invariant by the foliation, so in particular the singular locus of the foliation meets C in a bunch of points. Let us concentrate on one of them, i.e. denote by X a sufficiently small neighbourhood of the point. It may of course happen that C is singular at such a point. The point is $\mathscr{F}$ equivariant so we are happy to blow up in it, so that whether self evidently or by a minor adaptation of Lemma 1.1.4 we note:

Fact 1.2.2. *Notations as above, there is a sequence of varieties,*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n = \tilde{X}$$

where $X_i \to X_{i-1}$ is obtained by blowing up in a foliation equivariant centre, or by normalisation, such that the proper transform of C in $\tilde{X}$ is non-singular at the induced foliation singularities.

Of course we like such sequences since they are "unramified" in the foliation direction by virtue of Lemmas 1.1.3 and 1.1.4, so now we want to use a similar sequence, starting with C smooth at $\mathrm{sing}(\mathscr{F})$, to get into the local integrability situation of Lemma 1.2.1.

Consequently let $x_1, \ldots, x_n$ be coordinates on our ambient smooth space M, and suppose C is given by $x_2 = \cdots = x_n = 0$. We of course consider a vector field ∂ on M which leaves the ideal I of X invariant, and restricts to our given foliation on X which is supposed Gorenstein. Necessarily we are supposing that ∂ is singular at the origin, we write $\partial = a_i \frac{\partial}{\partial x_i}$, index notation, and consider how ∂ transforms around C under the blow up $p : \tilde{M} \to M$ in the origin. We have a new coordinate system $x_1 = \xi_1$, $x_i = \xi_1 \xi_i$, with C transforming to $\xi_2 = \cdots = \xi_n = 0$. Putting ν_i to be the multiplicity of a_i at the origin, $a_i = \tilde{a}_i x_1^{\nu_i}$, and understanding this notation in the natural way when some $a_i = 0$, we obtain:

$$p^*\partial = \tilde{a}_1 \xi_1^{\nu_1} \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} (\tilde{a}_i \xi_1^{\nu_i - 1} - \tilde{a}_1 \xi_1^{\nu_1 - 1} \xi_i) \frac{\partial}{\partial \xi_i}\,.$$

Better still, we also have,

$$\mathrm{mult}_0(\tilde{a}_i|_C) = \mathrm{mult}_0(a_i|_C) - \nu_i\,.$$

Now we distinguish two cases. In the first $a_1 \neq 0$, then on replacing p by a sequence of blow ups invariant in the pull-back of ∂ (so even invariant by the foliation in the case of canonical singularities) we have without loss of generality,

$$p^*\partial = \xi_1^{\nu_1} \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} b_i \frac{\partial}{\partial \xi_i}\,.$$

Furthermore some b_i is not identically zero on C, so that possibly after some more blow ups of the same form, the pull-back of ∂ is a regular, and non-singular derivation along the proper transform of C. In the case that a_1 is identically zero, the same conclusion is even more immediate. Now let us summarise these reflections by way,

Definition 1.2.3. Call a foliation $\mathscr{F}$ on a variety X smoothly integrable at a point $x \in X$ if it arises as in Lemma 1.2.1.

Whence,

Proposition 1.2.4. *Let C be a non-invariant curve in a Gorenstein foliated variety $(X, \mathscr{F})$ then there is a neighbourhood $(X_0, \mathscr{F}_0)$ of C together with a proper birational map $p : (\tilde{X}_0, \tilde{\mathscr{F}}_0) \to (X_0, \mathscr{F}_0)$ such that $\tilde{\mathscr{F}}_0$ is smoothly integrable around* *every point of the proper transform of C, and better still there is a natural map by pulling back,* $p^* T_{\mathscr{F}_0} \to T_{\tilde{\mathscr{F}}_0}$.

1.3 General Case

We now wish to study arbitrary integrable and weakly regular foliations around curves. Our only restriction will be that the curve is not contained in the singular locus of the foliation. The study again being local this statement is to be understood in terms of the foliation not having full rank in the tangent space of some smooth variety M into which X is embedded. The tricky thing here is that a singular point x may no longer be foliation invariant. Indeed by definition they are invariant precisely when the map at the residue field level, $T_{\mathscr{F}} \otimes k(x) \to T_M \otimes k(x)$ is zero, so we have to do a little more work to identify invariant centres. To this end, and much as before, let $x_1, \ldots, x_n$ be coordinates on M and write,

$$\partial_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}, \qquad 1 \le i \le r, \quad 1 \le j \le n$$

where r is the rank of the foliation. It goes without saying that the ∂_i leave the ideal I_X of X in M invariant, and induce our given weakly regular foliations $\mathscr{F}$. Not surprisingly the matrix $A = [a_{ij}]$ of functions on X will play a key role, let us denote by s the dimension of the image of $T_{\mathscr{F}} \otimes k(x)$ in $T_M \otimes k(x)$ around our point of study x. Observe that by row and column reduction of matrices we may find $s \times (n-s)$ and $(r-s) \times (n-s)$ matrices B, D of functions in the maximal ideal at x such that without loss of generality A has the form,

$$\begin{bmatrix} I & B \\ 0 & D \end{bmatrix}$$

Equally, for the same reason, there is a $r \times (n-r)$ matrix of meromorphic functions A_0, such that,

$$A = \left[I \vdots A_0 \right].$$

Now with these notations let us pause to consider the case of $r = s$ and the Frobenius theorem in this context. This time let's start with the definition, viz:

Definition 1.3.1. A foliated variety $(X, \mathscr{F})$ is said to be smoothly integrable at $x \in X$, if there is a relatively smooth map $\pi : X \to Y$ in a neighbourhood of x such that $\pi^* \Omega_Y$ generates the conormal bundle of $\mathscr{F}$.

Then of course we have,

Lemma 1.3.2. *Notations as above if $s = r$ at $x \in X$, then $(X, \mathscr{F})$ is smoothly integrable at x.*

Proof. Proceeding as in Lemma 1.2.1 we can actually choose our coordinate functions on M, such that A_0 is not only a matrix of functions around x, but in fact the first row is identically zero. Now let us consider invariance under Lie bracket. By the invariance of I under the lifting of our various vector fields, this may simply be calculated in M and restricted to X. Consequently understanding A_0 as a $(r-1) \times (n-r)$ matrix of functions, we obtain a $(r-1) \times (r-1)$ matrix Λ of functions on X around x such that,

$$\left[0 \vdots \frac{\partial A_0}{\partial x_1}\right] = \Lambda \left[I \vdots A_0\right]$$

which of course forces $\frac{\partial}{\partial x_1} A_0$ to be identically zero on X. Consequently if we let X_1 be the image of X in $\mathbb{C}^{n-1}$ under the map $(x_2, \ldots, x_n) : M \to \mathbb{C}^{n-1}$, and π the induced map then π is relatively smooth and there is a foliation $\mathscr{F}_1$ on X_1, closed under Lie bracket of rank $(r-1)$ which induces our given foliation, so that we may conclude by induction. ☐

We can now turn to the situation of $s < r$ and see how far this discussion can be pushed. Allowing a meromorphic A_0, shows that at the first stage of the induction procedure (assuming of course $s \neq 0$) that there are $(s-1)$ meromorphic vector fields on X_1, which together with $\partial/\partial x_1$ generate our given foliation. Proceeding by induction gives a preferred coordinate system such that,

$$\partial_i = \frac{\partial}{\partial x_i}\ ,\ 1 \le i \le s \quad \partial_i = \sum_{j>s} a_{ij} \frac{\partial}{\partial x_j}\ ,\ i > s$$

with the a_{ij} holomorphic functions of $x_{s+1}, \ldots, x_n$ which all vanish at the origin. We have thus identified, locally, a suitable foliation invariant centre, viz: $x_{s+1} = \cdots = x_n = 0$. By way of our curve, C, passing through our singular point, we first consider its image under $(x_{s+1}, \ldots, x_n) : M \to \mathbb{C}^{n-s}$, and ask whether the curve itself is singular there or not. Arguing exactly as in 1.2.2 we may via a sequence of equivariant blow ups pass without loss of generality to the situation where the image of C is given by $x_{s+2} = \cdots = x_n = 0$.

Now for a curve C whose tangent space does not generically factor through that of the foliation, we may proceed more or less as in the case of foliations by curves, namely, blow up in centres of the type we have identified until such times as the rank increases. Once the rank increases change coordinates in the obvious way, resolve any singularities on the new projection of our curve, then blow up in some foliation equivariant centres until we increase the rank again. Consequently we obtain,

Proposition 1.3.3. *Let C be a curve in a weakly regular foliated variety $(X, \mathscr{F})$ whose tangent space does not generically factor through $\mathscr{F}$ and which is not contained in the singular locus of $\mathscr{F}$ then there is a neighbourhood $(X_0, \mathscr{F}_0)$ of*

C together with a proper birational map $p : (\tilde{X}_0, \tilde{\mathscr{F}}_0) \to (X_0, \mathscr{F}_0)$ such that $\tilde{\mathscr{F}}_0$ is smoothly integrable at every point in the proper transform of C, and pull-back of derivations yields a natural map $p^* T_{\mathscr{F}_0} \to T_{\tilde{\mathscr{F}}_0}$.

2 Algebraisation

2.1 The Graphic Neighbourhood

Let us concentrate our attention in this section on a curve C inside a foliated variety $(X, \mathscr{F})$ with the foliation of rank r, integrable, and with weakly regular singularities where additionally we will suppose that C is neither contained in the singularities nor does its tangent space factor through $T_{\mathscr{F}}$. Now by the considerations of § 1.2–3, we may find an open neighbourhood X_0 of C (either formally or in the analytic topology) together with a proper birational map (in the category of analytic spaces) $p : \tilde{X}_0 \to X_0$ such that the induced foliation $\tilde{\mathscr{F}}_0$ is smoothly integrable in a neighbourhood V of the proper transform $\tilde{C}$ of C. At this point we wish to consider the induced foliation $\mathscr{F}^{\#}$ on $V \times B$, where B is the normalisation of C, in a neighbourhood of the graph Γ of the natural map from B to $\tilde{C}$. The tangent bundle of $\mathscr{F}^{\#}$ is simply the pull-back of that of $\tilde{\mathscr{F}}_0$, and is of course smoothly integrable around Γ. Whence let us take a union of Δ_α, $\alpha \in A$, of small open analytic sets covering Γ, and denote by $\pi_\alpha : \Delta_\alpha \to Z_\alpha$ the relatively smooth map of analytic spaces which yields $\mathscr{F}^{\#}|_{\Delta_\alpha}$. Further for each α, we have an ideal I_α of functions on Δ_α, generated by functions on Z_α vanishing on Γ. Necessarily the I_α patch and define a smooth analytic sub-variety F of $\cup\Delta_\alpha$ of dimension $r + 1$, such that the normal bundle of Γ in F, $N_{\Gamma|F}$, is isomorphic to $T_{\tilde{\mathscr{F}}}|_\Gamma$ (Fig. 3). Rather more intuitively what we have done is create an analytic space F by adding to each point of B the germ of the locally smooth integrable sub-variety through each point of $\tilde{C}$ guaranteed by 1.3.3, while equipping F with a map ρ to B, and σ to X, i.e.

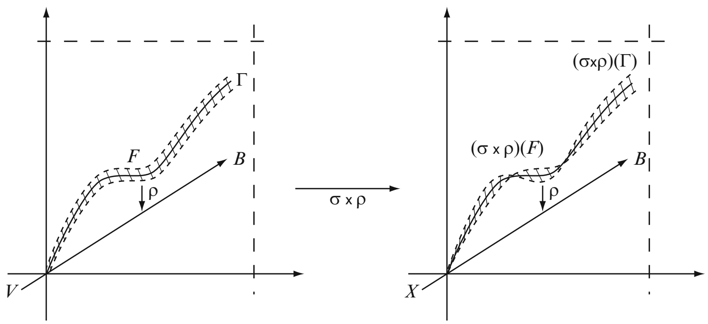


Fig. 3 The graphic neighbourhood

With this in mind we can deliver the coup de grâce to the transcendental nature of our problem by way of,

Fact 2.1.1. *If $T_{\mathscr{F}}|_C$ is ample then the Zariski closure of $\sigma \times \rho\,(F)$ is of dimension $(r+1)$, and as such every $\mathscr{F}$-integrable sub-variety through a point of $(\sigma \times \rho)(\Gamma)$ is algebraic.*

Proof. Since F comes equipped with a projection to B which pushes forward an integrable sub-variety to a point, it is wholly sufficient to prove the claim on the Zariski closure. Equally both X and B are algebraic, so all we need show is that for any line bundle L on F there is a constant $C(L)$ such that $h^0(F, L^{\otimes n}) \leq Cn^{r+1}$, for all positive integers. Better still if $\hat{F}$ is the completion of F along C, we have an injection $H^0(F, L^{\otimes n}) \hookrightarrow H^0(\hat{F}, L^{\otimes n})$, so we might as well just consider $\hat{F}$, and this is essentially a trivial exercise. Specifically for $m \in N$, let F_m be the m^{th} infinitesimal thickening, and observe by construction that there is a map $T_{\mathscr{F}}|_\Gamma \to N_{\Gamma|F}$ which is generically an isomorphism, so $N_{\Gamma|F}$ is ample. On the other hand, we have the usual exact sequence,

$$0 \longrightarrow H^0(\Gamma, \mathrm{Sym}^m N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \longrightarrow H^0(F_{m+1}, L^{\otimes n}) \longrightarrow H^0(F_m, L^{\otimes n})\,.$$

Necessarily the first group vanishes for $m \leq C(L)n$, where the constant $C(L)$ is of the form $0\,(|\deg_\Gamma\ L|)$, and whence,

$$h^0(\hat{F}, L^{\otimes n}) \leq \sum_{k=0}^{Cn} h^0(\Gamma, \mathrm{Sym}^k N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \leq Cn^{r+1}$$

where the last inequality may involve a slightly different constant, but nevertheless only depends on L as required. □

Lest there be any confusion let us make,

Remark 2.1.2. Evidently the role of the analytic topology is only for convenience of exposition, since the above is really a proposition about formal schemes.

2.2 Cleaning Up

We will continue to concentrate on the example of the previous section. Thanks to 2.1.1, we have obtained an algebraic variety W of dimension $r+1$, fibred over B by ρ, together with a section s of ρ, such that every fibre of W over B projects to a $\mathscr{F}$-invariant sub-variety of X through the corresponding point of C. Even better our bundle of derivations $T_{\mathscr{F}}$ on X, lifts naturally to a bundle of $\mathscr{O}_B$-derivations, which we will continue to denote by $T_{\mathscr{F}}$, on W since after all dualising coherent sheaves are compatible with flat pull-back. Now intuitively we'd like to think of W/B as relatively smooth in a neighbourhood of $s(B)$ with relative tangent bundle $T_{\mathscr{F}}$. However, for the reasons detailed in § 1.2, this is potentially rather false around points where $\mathscr{F}$ is not smoothly integrable. Whence we seek to resolve W to $\tilde{W}$ in such a way that $T_{\mathscr{F}}$ will admit a map to $T_{\tilde{W}/B}$ around the section, and such that in a neighbourhood of the section the fibration will be smooth. In light of Lemma 1.1.3, what is therefore required is equivariant desingularisation with respect to $T_{\mathscr{F}}$ viewed as a bundle of derivations. Considered with respect to a fixed smooth embedding $W \hookrightarrow M$ we have as ever a stratification of the singularities by closed sub-schemes W_s defined as,

$$W_s = \{w \in W \mid \dim\left(\mathrm{Im}\left\{T_{\mathscr{F}} \otimes k(w) \longrightarrow T_M \otimes k(w)\right\}\right) \leq s\}.$$

Now let $Y \hookrightarrow W$ be a smooth centre in which we may wish to blow up in order to carry out the desingularisation algorithm of [BM]. For $y \in Y$, there exists a unique s such that $y \in W_s \backslash W_{s-1}$, where by convention W_{-1} is empty. We may apply the discussion pre-Proposition 1.3.3, to find a coordinate system $x_1, \ldots, x_n$ on M in the analytic topology with respect to which a basis of $T_{\mathscr{F}}$ around y is given by,

$$\partial_i = \frac{\partial}{\partial x_i}\,,\ 1 \leq i \leq s, \quad \partial_i = \sum_{j=s+1}^{n} a_{ij} \frac{\partial}{\partial x_j}\,,\ s < i \leq r$$

where $a_{ij} = a_{ij}(x_{s+1}, \ldots, x_n)$. Furthermore we have a locally smooth map, $\pi : W \rightarrow Z$ around y, given by $(x_{s+1}, \ldots, x_n)$. By construction the algorithmic desingularisation procedure respects π, so we may find functions $f_1, \ldots, f_m$ of $x_{s+1}, \ldots, x_n$ which generate $I_Y \otimes \mathscr{O}_{W,y}$, where the local ring is understood formally or analytically. With these preliminaries in mind for any derivation ∂ of $T_{\mathscr{F}}$ over an open subset U containing y, our explicit choice of coordinates imply,

$$\partial\,(I_Y) \subset \mathfrak{m}\,(y)\,.$$

Since this holds for all ∂ and all $y \in Y$, we see that Y is in fact $\mathscr{F}$ equivariant. Blowing up in Y therefore leaves $T_{\mathscr{F}}$ as a bundle of derivations on the blow up, and so we may continue with the algorithmic desingularisation procedure to obtain a $\mathscr{F}$-equivariant resolution $\tilde{W}$ of W. Our section s necessarily lifts to a section $\tilde{s}$ of $\tilde{W}$ which immediately forces $\tilde{W}$ to be relatively smooth in a neighbourhood of $\tilde{s}(B)$. Changing notations slightly, we have therefore established,

Better Fact 2.2.1. *Let $(X, \mathscr{F})$ be a weakly regular foliated variety of rank r and $f : B \rightarrow X$ a map from a smooth curve such that $f^* T_{\mathscr{F}}$ is ample, and $f(B)$ is not contained in the singularities of $\mathscr{F}$ nor generically tangent to $\mathscr{F}$, then there is a smooth algebraic variety W of dimension $(r+1)$, equipped with projections σ, ρ to X and B, respectively, together with a section s of ρ such that if W_b denotes the smooth sub-variety passing through $s(b)$ then $\sigma(W_b)$ is an $\mathscr{F}$-invariant sub-variety through $f(b)$. In addition there is a natural map, $\sigma^* T_{\mathscr{F}} \rightarrow T_{W/B}$ in a neighbourhood of the section.*

2.3 Complements

Firstly we consider the case where our curve C is not contained in the singularities but may be generically tangent to $\mathscr{F}$. Again let B be the normalisation of C, and continue to denote by $\mathscr{F}$ the induced foliation on the product $X \times B$, with Γ the graph of B. We observe that Γ is now a curve which is NOT generically tangent to $\mathscr{F}$, but $T_{\mathscr{F}}|_{\Gamma}$ is ample if it were already so for $T_{\mathscr{F}}|_C$. Passing to a modification $X \tilde{\times} B$ of our product around Γ, as in 1.3.3 we obtain a neighbourhood V of the proper transform $\tilde{\Gamma}$ on which the induced foliation $\tilde{\mathscr{F}}$ is smoothly integrable. Whence by 2.1.1 there is a $\mathscr{F}$-invariant sub-variety W_x through every point x of Γ of dimension the rank of $\mathscr{F}$. Projecting forward to X gives a $\mathscr{F}$-invariant sub-variety $W \supset C$ of the appropriate dimension. The equivariant desingularisation arguments of II.2 go through verbatim to yield a resolution $\tilde{W}$ such that $T_{\tilde{W}}$ is ample on the proper transform $\tilde{C}$ of C, and indeed we may even take the later isomorphic to B should we wish.

Let us finally consider the case where our curve C is contained in the singular locus, and let s be the generic rank of $\mathscr{F}$ along C. Denote therefore by Y a component of X_s containing C. The coordinate system pre-Proposition 1.3.3 imply that Y is $\mathscr{F}$-invariant so let $\mathscr{G}$ be the induced foliation of rank s. The immediate thing to note is that $\mathscr{G}$ is not necessarily weakly regular. However at a point y of C where the rank drops to $t < s$ we are guaranteed a space of derivations of Y of dimension $s-t$ if C is not tangent to $\mathscr{G}$, respectively $s-t-1$ otherwise, which are not identically zero. The resolution procedure of § 1.3 therefore goes through verbatim, and we make neighbourhoods of the normalisation B of C exactly as before which are equivariant under $T_{\mathscr{F}}$. Since Y is also $T_{\mathscr{F}}$-equivariant, and the new foliation is generically a quotient of $T_{\mathscr{F}}$, we therefore have:

Final Fact 2.3.1. *Let s be the generic rank of $\mathscr{F}$ along C, then for every point $x \in C$ there is a smooth algebraic variety W_x of dimension s together with a map $\sigma : W_x \to X$ such that $\sigma(W_x) \ni x$ is a $\mathscr{F}$-equivariant sub-variety. Better still if $f : B \to C$ is the normalisation then either,*

(a) *$T_{\mathscr{F}_s}$ is not generically tangent to C, and the W_x are the fibres of a smooth variety $\rho : W \to B$ through a section s, and there is a natural map, $\sigma^* T_{\mathscr{F}} \to T_{W/B}$, generically an isomorphism around B.*

(b) *$T_{\mathscr{F}_s}$ is generically tangent to C, W_x is independent of x and contains a copy of B such that there is a map of pairs $(W, B) \to (X, C)$ and a natural map $\sigma^* T_{\mathscr{F}} \to T_W$, generically an isomorphism around B.*

3 Deformation Theory

3.1 The Set Up

We have thus reduced our initial highly transcendental problem to an algebraic one, either to produce rational curves in a smooth variety in which there is a curve on which the ambient tangent space is ample, or to find rational curves in a family of varieties over a curve in which there is a section on which the relative tangent space is ample. The latter is less standard and implies the former by the graph construction à la 2.3 so we will concentrate upon it. Nevertheless, the former, *i.e.* the little theorem of the introduction, presents such noteworthy simplification that we devote § 3.4 to it. We will of course be following Mori's method of reduction modulo p, so in this section we make our set up with the necessary precision, and summarise the appropriate minor variations that we require from [K], specifically § 1.2 and 2.3.

To this end let B be a smooth projective curve over a field k of arbitrary characteristic, let $q : C \to B$ be a finite morphism from another curve, $\pi : X \to B$ a projective flat family and $f : C/B \to X/B$ a morphism of B-schemes (with respect to the structure maps q and π) where X/B is supposed relatively smooth in a neighbourhood of $f(C)$. We wish to study the scheme of B-morphisms, $\mathrm{Hom}_B(C,X)$ in a neighbourhood of f. The next proposition contains all that we will need,

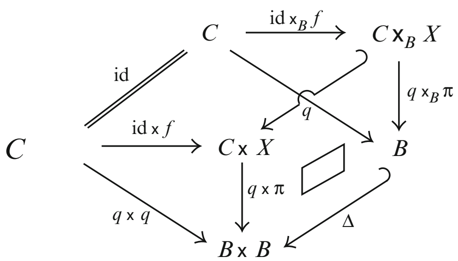
Proposition 3.1.1. *Notations as above then,*

(a) *The tangent space of* $\mathrm{Hom}_B(C,X)$ *at* f *is isomorphic to,* $H^0(C, f^*\,T_{X/B})$.
(b) *The dimension of every irreducible component of* $\mathrm{Hom}_B(C,X)$ *at* f *is at least,*

$$h^0(C, f^*\,T_{X/B}) - h^1(C, f^*\,T_{X/B})\,.$$

(c) *The deformations of* f *are unobstructed if* $H^1(C, f^*\,T_{X/B}) = 0$. *In particular should this occur* $\mathrm{Hom}_B(C,X)$ *is smooth at* f.
(d) *If* $Z \subset X$ *has co-dimension at least* 2, *and* $H^1(C, f^*\,T_{X/B}(-c)) = 0\ \forall$ *geometric points* c *of* C *then a generic* $g \in \mathrm{Hom}_B(C,X)$ *has image disjoint from* Z.

Proof. For part (a), we wish to consider the graph Γ of f in the relative product $C \times_B X$ and calculate its conormal bundle. To this end consider the commutative diagram,



where by definition the vertical square is Cartesian, and Γ is the image of id $\times_B f$. This in turn leads to a commutative diagram of exact sequences of sheaves on Γ, viz:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}} \otimes \mathscr{O}_\Gamma & = & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}} \otimes \mathscr{O}_\Gamma & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \frac{I_{\Gamma,C\times X}}{I^2_{\Gamma,C\times X}} & \longrightarrow & \Omega_{C\times X} \otimes \mathscr{O}_\Gamma & \longrightarrow & \Omega_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \frac{I_{\Gamma,C\times_B X}}{I^2_{\Gamma,C\times_B X}} & \longrightarrow & \Omega_{C\times_B X} \otimes \mathscr{O}_\Gamma & \longrightarrow & \Omega_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the notation $I_{\cdot,\cdot}$ means the ideal of whatever sub-scheme inside the other. Now the middle row is exact because of the natural splitting of $\Omega_{C\times X}$, while the middle column is exact for general nonsense. Specifically X is by hypothesis smooth around Γ, so $\Omega_{C\times X}$ is a bundle, the conormal sheaf $I_{\Delta,B\times B}/I^2_{\Delta,B\times B} \xrightarrow{\sim} \Omega_B$ is also a bundle, and the map from Ω_B to $\Omega_{C\times X}$ is not zero, so it must be an injection of sheaves. Whence the left-hand column is in fact exact, and so also is the bottom row by the 9 lemma, and/or a trivial diagram chase. From which we conclude an isomorphism via the splitting of the middle row,

$$I_{\Gamma,C\times_B X}/I^2_{\Gamma,C\times_B X} \xrightarrow{\sim} \Omega_\Gamma \backslash \Omega_{C\times_B X} \otimes \mathscr{O}_\Gamma \xrightarrow{\sim} \Omega_{X/B} \otimes \mathscr{O}_\Gamma$$

which proves (a). Better still $C \times_B X$ is non-singular around Γ by hypothesis, Γ is itself smooth, so the deformations are generically unobstructed which proves (b) and (c). To prove (d), we start by taking a smooth open neighbourhood U of f in $\mathrm{Hom}_B(C, X)$. For any geometric point $c \in C$, and for g in a possibly smaller U, we have by hypothesis, $H^1(C, g^* T_{X/B}(-c)) = 0$. Consequently if $\mathrm{Hom}_B(C, X, g|_c)$ is the space of morphisms taking c to $g(c)$, then a minor variation of the previous argument forces $U \cap \mathrm{Hom}_B(C, X, g|_c)$ to be smooth around g of dimension $h^0(C, g^* T_{X/B}) - \dim (X/B)$. On the other hand,

$$\{g \in U \mid g(C) \cap Z \neq \varphi\} = \bigcup_{c\in C} \bigcup_{x\in Z} \{g \in U \mid g(x) = z\} .$$

So that this space has dimension at most,

$$h^0(C, g^* T_{X/B}) - \dim (X/B) + \dim C \times_B Z \leq \dim U - 1$$

which completes the proof of (d). □

3.2 Bend and Break

We begin with a simple lemma,

Lemma 3.2.1. *Let E be an ample vector bundle on a smooth curve C over a field of positive characteristic, then there is a sufficiently high power $F_n : C \to C$, of the absolute Frobenius F such that for all geometric points c of C,*

$$H^1(C, F_n^* E(-c)) = 0 .$$

Proof. Fix an ample divisor D on C, and using F_n to denote as large a power of Frobenius as we require, we obtain by Riemann-Roch global sections of $F_n^* E\,(-D)$. Consequently for E of rank more than 1 we obtain a dévissage of bundles,

$$0 \longrightarrow E' \longrightarrow F_n^* E \longrightarrow E'' \longrightarrow 0$$

where E' is ample of rank 1. Moreover E'' is ample, and the proposition is clear for line bundles via Serre-duality, so for some high power of Frobenius F_m and $c \in C$ any geometric point we obtain a short exact sequence,

$$0 \longrightarrow F_m^* E'(-c) \longrightarrow F_{m+n}^* E(-c) \longrightarrow F_m^* E''(-c) \longrightarrow 0$$

where by induction the kernel and cokernel are acyclic. □

We return to the notation of the previous section, viz: B, C smooth projective curves, $q : C \to B$, $\pi : X \to B$ the structure maps, with the latter family flat, and $f : C/B \to X/B$ a map of B-schemes such that X is smooth in a neighbourhood of $f(C)$. Moreover let everything be over a field, k, of positive characteristic p, $f^* T_{X/B}$ ample, and f not contained in a fibre of B. Now consider a rational map—so, implicitly, well defined in co-dimension 2—$\nu : X \dashrightarrow X_1$ of B-schemes such that

(a) X_1 is smooth and geometrically irreducible over k.
(b) ν is an honest morphism over a Zariski open $U_1 \subset X_1$.
(c) ν is generically smooth.

Consequently there is a (unique) torsion free quotient $\mathscr{T}_1$ of $T_{X/B}$ which coincides with $\nu^* T_{X_1/B}$ over U_1, so, in particular, $\mathscr{T}_1$ is a vector bundle in co-dimension 2. As such, if $g : C_m \to X$ is a generic deformation of a sufficiently large power of the geometric Frobenius $C_m \to C$ of our given curve composed with f then by

Proposition 3.1.1.(d) and Lemma 3.2.1 we may suppose that its image is contained in the locus where $\mathscr{T}_1$ is a bundle, and we write

$$0 \to T^1 \to g^* T_{X/B} \to T_1 = g^* \mathscr{T}_1 \to 0$$

for the resulting exact sequence of bundles on C_m.

Next, let $\mathscr{O}(1)$ be the tautological bundle on $\mathbb{P}(T_{X/B})$, and $\mathbb{Q} \ni \varepsilon > 0$ such that $f^*\mathscr{O}(1)(-\varepsilon[c])$ is ample for $[c]$ the fibre over a closed point $c \in C$. Furthermore, observe, that for any bundle, E, in positive characteristic the pull-back F^*E by the absolute Frobenius is canonically a direct summand of the p-th symmetric power, so by the commutativity of Proj with base change (and provided the power F_m of Frobenius in the definition of g is sufficiently large) for any line bundle L on C_m of degree, $\ell < p^m \varepsilon$,

(d) $g^* T_{X/B} \otimes L^\vee$ (and whence $T_1 \otimes L^\vee$) is generated by global sections.
(e) $\mathrm{H}^1(C_m, g^* T_{X/B} \otimes L^\vee) = 0$.

In particular, therefore, the space of deformations (relative to B) of g which fix the intersection with ℓ generic fibres of ν is unobstructed and projects to a positive dimensional space of deformations of νg in X_1 fixing ℓ generic points. To these deformations, we can apply (uniformly in the moduli of g) Mori's bend and break, *cf.* [K, II.5]. In doing so (and bearing in mind that our interest is to obtain rational curves in the fibres of B) we can, conveniently, eschew some technicalities by observing that everything we've said so far is stable under any base change $B' \to B$, so, without loss of generality B (whence also C) isn't rational, and we obtain:

Fact 3.2.2. *There is a covering family $h_p : \mathbb{P}^1 \times M_p \to X$ of B-trivial rational curves, whose projection νh_p covers X_1 such that for H_1 ample on X_1 and $m \in M_p$*

$$\nu^* H_1 \cdot_{h_p(m)} \mathbb{P}^1 \leq \frac{2p^m}{\ell}\, \nu^* H_1 \cdot_f C$$

By way of concluding this section, let us make the following clarifying

Remark 3.2.3. It seems more than reasonable that the degree bound in 3.2.2 should hold for any ample bundle on X rather than just those that come from X_1. Nevertheless in trying to extend the proof, [MM], of the bound to this more general context one gets stuck by the (highly unlikely under the above hypothesis) possibility that a priori it may require many more blow ups to resolve certain rational maps to X rather than X_1. In consequence, we only get the weaker statement 3.2.2, and whence a certain amount of technical complication in the next section.

3.3 Finding Rationally Connected Varieties

We now consider, in characteristic zero, our data of maps $q : C \to B$, $\pi : X \to B$, $\pi_1 : X_1 \to B$, $\nu : X - \to X_1$ and $f : C/B \to X/B$ with all the previous hypothesis on smoothness ampleness, etc., and proceed to show that the fibres of X/B are rationally chain connected. None of our hypothesis are changed by supposing X non-singular, so let's throw that in for good measure. As usual we take our given data over $\mathbb{C}$ and find an integral affine $\mathbb{Z}$-scheme, S, of finite type over which everything is defined and our initial characteristic zero situation corresponds to the generic fibre. So in fact we'll actually use C, B, X, q, π, f, etc., for S-schemes and maps thereof, and rather abusively denote the generic fibre by the sub-script $\mathbb{C}$. In any case for $\mu_{\min}$ the minimal slope of the H-N filtration of $f^*T_{X/B}$, we can, for $p \gg 0$, take the $\varepsilon > 0$ prior to item (d) of § 3.2 to be any rational number less than $\mu_{\min}$. Consequently, given $\delta > 0$, for a closed point s of S with residue field of sufficiently large positive characteristic, we may apply 3.2.2 to conclude the existence of a component W_δ of the Chow scheme of X_1 parametrising 1-cycles with B-trivial rational components of degree at most $2\frac{H_1 \cdot_{\nu f} C}{\mu_{\min}} + \delta$ such that the map from the universal family $\mathscr{C}_\delta$ over W_δ to $X_{1,s}$ is dominant. Since this holds for all s in an open set, there is in fact such a component over the generic fibre. Better still since this must be true for all $\delta > 0$, and there are at most finitely many such components, there is a component W of the Chow scheme parametrising 1-cycles with rational components of degree at most $2\,\frac{H_1 \cdot_{\nu f} C}{\mu_{\min}}$ such that the universal family $\mathscr{C}$ dominates X_1.

We now apply these considerations inductively beginning with $\nu = \mathrm{id}_X$, so we have, *cf.* [K, IV.4.13], an equivalence relation, $\sim$, between points of X defined by $x \sim y$ iff they are joined by a chain of rational curves in W. In positive characteristic, there might be separability issues about the quotient map defined by such a relation, but, by the above, we are in characteristic zero, so we can find a map well defined in co-dimension 1, $\nu : X_{\mathbb{C}} \dashrightarrow X_{1,\mathbb{C}}$, where $X_{1,\mathbb{C}}$ is a smooth, with any two points of a generic fibre connected by a chain of curves in $W_{\mathbb{C}}$. In particular the generic fibre of ν is rationally chain connected, and everything extends to a diagram of B-schemes,

$$\begin{array}{ccc} X & \overset{\nu}{\dashrightarrow} & X_1 \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & B & \end{array}$$

which satisfy the hypothesis § 3.2.(a)–(c) when specialised to any closed point $s \in S$ of sufficiently large characteristic. Consequently, X_1 is covered by rational curves of degree at most $d_1 := 2\,\frac{H_1 \cdot_{\nu f} C}{\mu_{\min}}$, and we again have a quotient $\nu_1 : X_1 - \to X_2$ given by the equivalence of points joined by a rational curve of degree at most d_1.

Now by 3.2.2 these curves have some further good properties which are best exploited by working p-adically. Again bearing in mind that our goal is the rational connectedness of the fibres of $X_{\mathbb{C}} \to B_{\mathbb{C}}$ (or, better, in the first instance the fibres of $X - \to X_2$) we can, without loss of generality, suppose the following

(a) S is the spectrum of a complete D.V.R., V, with quotient field K of characteristic 0, and closed point $s \in S$ enjoying an algebraically closed residue field, k, of characteristic p.
(b) Replace $\nu : X - \to X_1$ by an (everywhere defined, smooth over an open subscheme of X_1 surjecting onto S) map of smooth projective S-varieties whose specialisation to k satisfies all of § 3.2.(a)–(e).
(c) Every smooth closed fibre of ν admits a very free (i.e. pull-back of the tangent bundle relative to ν is ample) rational curve.
(d) The generic rational curve, $\nu h_p(m) : \mathbb{P}^1_k \to X_1 \otimes k$, of 3.2.2 (which by the degree bound is necessarily separable) lifts to a rational curve, $\theta : \mathbb{P}^1_S \to X_1$.

Of which, (d) merits a little precision, to wit: prior to applying (a),(b) or (c) we can assume that the images of the M_p (albeit $M_{s'}$, as s' ranges over closed points would be better notation) of 3.2.2 are Zariski dense in a sub-variety, M', of S-rational curves of degree at most d_1. Throwing away closed subsets of S as necessary, we can therefore suppose that for generic s', $M'_{s'}$ meets the locus where M'/S is smooth, so that on subsequently applying step (a) for $k(s') \hookrightarrow k$ and suitable V, the generic k-point of M_s is also a smooth point of M'_s so it lifts to a S-point of M' by [SGA, Exposé III, 5.2].

In the same vein, for M' as above, and prior to applying step (a), consider the diagram in which the left- and right-hand squares are fibred

$$\begin{array}{ccccccc} X & \longleftarrow & X'_{\rm big} & \longleftarrow & X'_{\rm res} & \longleftarrow & X'_{m'} \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu'_{\rm big}} & & \downarrow{\scriptstyle \nu'_{\rm res}} & & \downarrow{\scriptstyle \nu'_{m'}} \\ X_1 & \longleftarrow & \mathbb{P}^1{}_S \times M' & = & \mathbb{P}^1{}_S \times M' & \longleftarrow & \mathbb{P}^1{}_S \times m' \end{array}$$

and $\nu'_{\rm res}$ is the algorithmic resolution of [BM] applied to $X'_{\rm big}$ over the function field of M' extended over S (which here is still finite type over $\mathbb{Z}$), so that by the generic smoothness of ν, $\nu'_{\rm res} = \nu'_{\rm big}$ over the generic point of X_1. As such, the locus of $m' \in M'$ where the right-hand vertical is not smooth over $k(m')$ is closed, non-dense, in M'. Consequently, by the same reasoning that we applied above to get item (d), we can assume all of (a)–(d) together with the existence of a diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu'} \\ X_1 & \xleftarrow{\ \theta\ } & \mathbb{P}^1{}_S \end{array}$$

which is a fibre square not just generically, but even over the generic point of $\mathbb{P}^1_k$; is a modification over a closed set not dominating the generic point of $\mathbb{P}^1_k$ otherwise; and all varieties therein are smooth S-schemes. Now by 3.2.2 ν'_k has a section, σ_k, and, indeed, a covering family of such, and we choose some very free (relative to ν') rational curves, $\gamma_i : C_i \to X'$, in some large number n, to be specified, of generic fibres of ν'_k, through points $\sigma_k(x_i) = \gamma_i(c_i)$ with the aim of constructing a smoothing, [K, II.7.1], over k, of the comb [K, II.7.7]

$$\gamma := \sigma_k \vee_i \gamma_i : C := \mathbb{P}^1_k \coprod_i C_i/x_i \sim c_i \to X'_k$$

to a rational curve $\gamma' : \mathbb{P}^1_k \to X'$ such that $(\gamma')^* T_{X'/k}$ is ample. Essentially this is [K, IV.6.10], but our needs are a little different so we spell it out. To begin with ν' may be supposed smooth in a neighbourhood of C, so by [K, II.7.9] a smoothing $\Gamma : Y/T \to X'_k$ over some smooth k-curve T with special fibre γ exists. Further provided $n \gg 0$ by [K, II.7.10.1]—applied to $\Gamma^* T_{X'_k/\mathbb{P}^1_k}(-D)$ for D an effective Cartier divisor on Y which misses the C_i—the restriction $(\Gamma_t)^* T_{X'_k/\mathbb{P}^1_k}$ to the generic fibre Y_t is ample, while $\nu'|_{Y_t}$ is finite, thus $(\nu'\Gamma_t)^* T_{\mathbb{P}^1/k}$ is ample too, so by the obvious exact sequence we have what we want with $\gamma' = \Gamma_t$, and whence

Fact 3.3.1. *There is a map $\gamma'_S : \mathbb{P}^1_S \to X'$ to the smooth locus of X'/S such that $(\gamma'_S)^* T_{X'/S}$ is ample relative to S.*

Proof. Denote by $\hat{}$ completion in fibres over s, and apply [SGA, Exposé III.5.8] to γ' to obtain $\hat{\gamma}' : \mathbb{P}^1_{\hat{S}} \to \hat{X}'$, specialising to γ', in order to conclude from the existence of the Hilbert-scheme/S and the fact that being ample is open. □

Exactly what the bound, d_2, on the degree, with respect to an ample divisor H on X, of the resulting rational curves in X'_K may be is not so clear, but it's manifestly a linear function (depending only on ν/B) of the H-degree of the curves $h_p(m)$ of 3.2.2—m generic, and again, an unfortunate notational confusion of s with p—which although, Remark 3.2.3, we don't know how this varies with p, it doesn't matter since p is just some fixed large prime affording (a)–(d). As such, in principle, it's effectively computable, and more importantly it has no dependence on the fibre over B which we specialised to in item (b), so that on returning to the situation over B the fibres of $X/B - \to X_2/B$ are connected by chains of rational curves of degree at most d_2, and we conclude by the obvious induction that the fibres of X/B are rationally chain connected. Alternatively, if one wants to be less constructive about it, then one could just start with $X/B - \to X_1/B$ the MRC-fibration/B of [K, IV.5.9] and argue as above.

3.4 Scholion: Algebraic Spaces

In this section we prove the little theorem of the introduction. By way of a much simpler variant of the arguments of the previous section we can, without loss of generality, start with a D.V.R. S of mixed characteristic as (even notationally) encountered in item (a) of § 3.3 (albeit the previous technically convenient conditions of completeness and algebraically closed residue field no longer serve any purpose); a smooth irreducible curve C/S, and a S-map $f : C \to X$ to a proper algebraic space such that X is S-smooth in a neighbourhood of the image and $f^* T_{X/S}$ is ample relative to S. Denoting by the sub-script $_k$ specialisation to the closed fibre, consider the problem of trying to lift the composition of f with a large, to be specified, power $F^m : C_k^{(m)} \to C_k$ of the geometric Frobenius- where, for obvious reasons, we've changed from a sub-script m to a super-script. Now, as it happens, [SGA, Exposé III.7.4], all curves over k can be lifted to S, but since $C_k^{(m)}$ is just a conjugate variety, this is trivial. Next we require the items (d) and (e) (over k) of § 3.2, but the change is purely notation, i.e. $\mathscr{O}(1)$ the tautological bundle on $\mathbb{P}(T_{X/S})$, and otherwise everything as per op. cit., with of course $m \gg 0$. Consequently, just as in 3.3.1, [SGA, Exposé III.5.8] applies again to yield a morphism $f^{(m)} : C^{(m)}/S \to X/S$ lifting $f_k F^{(m)}$ with (on replacing g by $f^{(m)}$ and S by B) the pleasing features (d) & (e) of § 3.2, but now over all of S. In particular, if we now specialise to the generic fibre, K, then we not only have many deformations of $f_K^{(m)}$, but Proposition 3.1.1.(d) is valid too, i.e. a generic deformation, g of f_m misses any co-dimension 2 sub-set we care to specify, and, of course, § 3.2.(d)–(e) hold (over K) for g up to the notational issue of replacing B by K. As such if $\nu : X_K - \to X_1$ is a K-map satisfying § 3.2.(a)–(b) (with item (a) over K, so (c) is automatic) then we have everything (i.e. § 3.2.(a)–(e)) we need to run bend and break (relative to ν) over K, and whence

Fact 3.4.1. *There is a covering family $h_\nu : \mathbb{P}^1_K \times M_\nu \to X_K$ of rational curves whose projections $\nu(h_\nu)$ cover X_1.*

To which we could also add the degree bound of 3.2.2 for H_1 a nef. divisor on X_1, but since things needn't be projective this may well be an empty statement. Nevertheless, we can conclude that X_K is rationally chain connected by a similar, and much easier, induction to that of § 3.3. Specifically, start with $\nu = \mathrm{id}_X$, apply 3.4.1 to get a covering family, define $\sim$ to be the equivalence of points joined by chains in this family, then take $\nu : X - \to X_1$ to be the quotient by $\sim$, etc. Given that this absolute case is so much easier than the previous relative case, let us again offer a clarifying

Remark 3.4.2. The obstruction to doing much that same thing relative to B instead of the trickier reduction to 3.2.2 via 3.3.1 is that B need not be rational, and should this occur it's simply impossible to lift the composition

$$C_k^{(m)} \xrightarrow{F^{(m)}} C_k \xrightarrow{q_k} B_k$$

as B/S-schemes. Conversely, if B were rational then this could be done, and everything would be as above, but to take B rational may well cause the loss of B-smoothness about the image of f and Proposition 3.1.1 would collapse. As such projectivity is being employed in the main theorem not just in the algebraisation lemma, 2.1.1, but also the proof § 3.2–3.3 of rational connectedness.

4 Principal Results and Corollaries

4.1 The Main Theorem

Consider a weakly regular integrable foliated variety $(X, \mathscr{F})$. The discussion of the previous chapters shows that if C is a curve in X, and $T_{\mathscr{F}}|_C$ is ample then through every point $x \in C$ there is a rationally chain connected variety whose dimension is the generic rank along C and whence by [K&] rationally connected, $\mathscr{F}$-invariant sub-variety V_x with appropriate bounds on the degree of the connecting curves. It therefore remains to discuss the case of curves contained in the singular locus of $\mathscr{F}$ in more detail which is where the hypothesis of canonical singularities will intervene. To illustrate the principle idea let's suppose for the moment X is non-singular, and $\mathscr{F}$ is a foliation by curves. Denote by Z any sub-scheme contained in the singular locus of $\mathscr{F}$ considered scheme theoretically (i.e. if locally $\partial = a_i \frac{\partial}{\partial x_i}$, then the ideal of the singular locus is that generated by the a_i). Observe that we have maps,

$$\begin{array}{ccccc} I_Z & \subset & \mathcal{O}_X & & \\ \big\downarrow & & \big\downarrow d & & \\ & & \Omega_X & \longrightarrow & K_{\mathcal{F}}\otimes I_{\mathrm{sing}(F)} \subset K_{\mathcal{F}}\otimes I_Z \\ & & & \searrow^{D} & \big\downarrow \otimes\mathcal{O}_Z \\ I_Z/I_Z^2 & & \dashrightarrow & & K_{\mathcal{F}}\otimes I_Z/I_Z^2 \end{array}$$

We claim that the total composite of the maps on the right factors through I_Z/I_Z^2, and better still defines an $\mathscr{O}_Z$-linear map, which we will denote by D. The linearity is automatic by the Leibniz rule, and this in turn automatically forces the said factorisation. Let's observe some simple facts about this map. Firstly suppose $W \subset Z$ then we get maps for either sub-scheme, and a natural commutative diagram,

$$\begin{array}{ccc} I_Z/I_Z^2 & \longrightarrow & I_W/I_W^2 \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ K_{\mathscr{F}} \otimes I_Z/I_Z^2 & \longrightarrow & K_{\mathscr{F}} \otimes I_W/I_W^2 \end{array}$$

Now suppose we're considering a singular point x, with coordinate functions $x_1, \ldots, x_n$, and say ∂ a derivation defining the foliation at the point, then of course, $\partial = \sum_i a_i \frac{\partial}{\partial x_i}$, but $a_i - a_{ij} x_j \in \mathfrak{m}^2$, the square of the ideal at the point, with a_{ij} constants. In this case D is just the linear transform of the residue of the cotangent bundle given by,

$$D : \Omega_X \otimes k(x) \longrightarrow \Omega_X \otimes k(x) : dx_i \mapsto a_{ij}\, dx_j\,.$$

This map may of course be zero, but a moments thought shows that this cannot happen if the singularities are canonical. Indeed for general Z, we can just use the maps in the middle row together with the natural map from I_Z/I_Z^2 to the cotangent bundle to define a map, again denoted D, from $\Omega_X \otimes \mathscr{O}_Z$ to $\Omega_X \otimes \mathscr{O}_Z(K_{\mathscr{F}})$ which at every point is as above. Whence in turn for each $1 \le n \le \dim X$ we obtain via symmetric functions a global section, $S_n \in \Gamma(Z, \mathscr{O}_Z(nK_{\mathscr{F}}))$. The issue is therefore whether S_n is zero or not. If not this contradicts the ampleness of $T_{\mathscr{F}}$ on taking Z to be our curve, so what we'll show is that if S_n is zero for all n, the singularity is not canonical. It is wholly sufficient to prove this at a singular point, so say notations as above with x the origin. If all the symmetric functions vanish then the matrix $[a_{ij}]$ is nilpotent. Linear changes of coordinates conjugate the matrix, so we can suppose:

$$\partial = \sum_{i=k}^{n-1} x_i \frac{\partial}{\partial x_{i+1}} + \delta$$

where $\delta \in \mathfrak{m}^2\, T_{X,x}$. We blow up in the origin, and look on the $x_n \neq 0$ patch, i.e. change coordinates to $x_n = \xi_n$, $x_i = \xi_i\, \xi_n$, $i < n$. Denote the blow up by $p : \tilde{X} \to X$, then we have:

$$p^* \partial = \sum_{i=k}^{n-2} \xi_i \frac{\partial}{\partial \xi_{i+1}} + p^* \delta (\operatorname{mod} \mathfrak{m}^2\, T_{X,0})$$

where $\mathfrak{m}$ continues to denote the maximal ideal at the origin. Superficially it may appear that we have reduced the complexity of the matrix, this may not however be the case since we can only guarantee,

$$p^* \delta = \sum_{i=1}^{n-1} a_i\, \xi_n \frac{\partial}{\partial \xi_i} \; (\operatorname{mod} \mathfrak{m}^2\, T_{X,0})$$

and so the dimension of the eigenspace may not increase. Now writing things rather more invariantly we have a new linear map,

$$\tilde{D} : \Omega_{\tilde{X}} \otimes k(0) \longrightarrow \Omega_{\tilde{X}} \otimes k(0)$$

with $\tilde{D}\, d\xi_n = 0$, and all the terms, $\tilde{\delta}$ say, that we have lumped under the appellation $\bmod \mathfrak{m}^2$ enjoy the additional property of being divisible by ξ_n, with the exception of a term of the form $-\sum_{i \neq n} \xi_{n-1}\, \xi_i \frac{\partial}{\partial \xi_i}$. So let's blow up in the origin again, but this time look at the $\xi_{n-1} \neq 0$ patch on the blow up, i.e. put $\xi_{n-1} = \zeta_{n-1}$, $\xi_i = \zeta_i\, \zeta_{n-1}$, so that denoting the blow up morphism again by p, and the maximal ideal of the new origin still by $\mathfrak{m}$, we have,

$$p^* \partial = \sum_{i=k}^{n-3} \zeta_i \frac{\partial}{\partial \zeta_{i+1}} + \sum_{i=1}^{n-2} a_i\, \zeta_n \frac{\partial}{\partial \zeta_i} \; (\bmod \mathfrak{m}^2\, T_{X,0})$$

where the sums are understood to be zero if $k < n-3$ or $1 < n-2$, respectively. The new linear map $\tilde{\tilde{D}}$, say, still has characteristic polynomial zero, but has at least one more eigenvector than D, so by induction, we conclude that our singularity could not have been canonical. Evidently to conclude the main theorem in this case it is sufficient that the singularities are canonical in dimension 1, i.e. canonical outside of a bunch of points. Furthermore by embedding in a smooth manifold, and using Lemma 1.1.4 to control any non-normality that the blow up procedure may introduce, we see that the hypothesis that X is smooth is not really essential, and whence arrive to our theorem in the case of foliations by curves.

The general situation is rather more delicate. We start as ever with a weakly regular foliated variety $(X, \mathscr{F})$ of rank r, and a curve C in X with s the generic rank of $\mathscr{F}$ along C. We have for Y a component of X_s containing C an exact sequence of the form,

$$0 \longrightarrow \mathscr{N} \longrightarrow T_{\mathscr{F}} \otimes \mathscr{O}_Y \longrightarrow \mathscr{T}_Y$$

where the image has rank s, and all the maps are generically of the same rank after tensoring with $\mathscr{O}_C$. Consequently if $f : B \to C$ is the normalisation, then there is a rank $r-s$ sub-bundle N of $f^*\, T_{\mathscr{F}}$ saturating the map on the left. This bundle admits a rather clean geometric description as follows, viz: we can find a neighbourhood V of the proper transform $\tilde{C}$ of C in some $\mathscr{F}$-equivariant modification of Y such that the induced foliation $\mathscr{G}$ is smoothly integrable with tangent bundle $T_{\mathscr{G}}$, say, and so N must be the kernel of the map of bundles $f^*\, T_{\mathscr{F}} \twoheadrightarrow f^*\, T_{\mathscr{G}}$. Now let $\mathscr{I}$ be the ideal sheaf of $\tilde{C}$ in the ambient modification $\tilde{X}$ of X, then we have an $\mathscr{O}_{\tilde{C}}$ linear map,

$$D : \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{I}/\mathscr{I}^2 \otimes \operatorname{Ker}\{T_{\mathscr{F}} \to T_{\mathscr{G}}\}^\vee .$$

This map is not in general extendable to Ω_V, but no matter, since $T_{\mathscr{F}} \to T_{\mathscr{G}}$ is a surjection of bundles around $\tilde{C}$, should E be the double dual of $f^* \mathscr{I}/\mathscr{I}^2$ we in fact have an induced map of bundles,

$$D : E \longrightarrow E \otimes N^{\vee} .$$

Finally letting L be the tautological bundle on $\mathbb{P}(N^{\vee})$ we have a map, $D : E \to E \otimes L$, and global sections given by symmetric functions $S_d \in H^0(\mathbb{P}(N^{\vee}), L^d)$. On the other hand, these functions are non-zero if and only if, in our usual coordinate system, the matrix a_{ij} of functions in $x_{s+1}, \ldots, x_n$ is non-nilpotent for each $s < i \leq r$. The identical analysis to before shows this is not possible for canonical singularities, and whence we get our desired global section over B of $\mathrm{Sym}^d N^{\vee}$ for some d.

4.2 Foliations by Curves

In the case of foliations by curves there are some particularly beautiful corollaries of the main theorem, since the canonical bundle and cotangent bundle of the foliation now coincide. In particular let $(X, \mathscr{F})$ be a variety foliated by curves, then à la Mori we introduce the closed cone $\overline{\mathrm{NE}}(X)$ inside $\mathrm{NS}_1(X)_{\mathbb{R}}$ generated by effective curves and consider the sub-cone,

$$\overline{\mathrm{NE}}(X)_{K_{\mathscr{F}} \geq 0} := \{\alpha \in \overline{\mathrm{NE}}(X) \mid K_{\mathscr{F}} \cdot \alpha \geq 0\} .$$

Then we obtain,

Theorem 4.2.1. *Let $(X, \mathscr{F})$ be a variety foliated by curves with foliated Gorenstein and foliated canonical singularities then there are countably many $\mathscr{F}$ invariant rational curves L_i with, $K_{\mathscr{F}} \cdot L_i < 0$ such that,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathscr{F}} \geq 0} + \sum_i \mathbb{R}_+[L_i] .$$

Better still,

(a) *The rays $\mathbb{R}_+[L_i]$ are locally discrete in the upper half space,* $\mathrm{NS}_1(X)_{K_{\mathscr{F}} < 0}$.
(b) *If $(X, \mathscr{F})$ is not a ruling by rational curves $K_{\mathscr{F}} \cdot L_i = -1$, $\forall i$.*
(c) *Otherwise, $-2 \leq K_{\mathscr{F}} \cdot L_i \leq -1$.*
(d) *Every extremal ray R of $\overline{\mathrm{NE}}(X)$ (i.e. $\alpha + \beta \in R$, $\alpha, \beta \in \overline{\mathrm{NE}}(X) \Rightarrow \alpha, \beta \in R$) lying in the half space $\mathrm{NS}_1(X)_{K_{\mathscr{F}} < 0}$ is of the form $\mathbb{R}_+[L_i]$.*

Proof. Everything in the corollary except (b) and (c) follows verbatim from the demonstration of the corresponding theorem for K_X, as found in [K], Theorem III.1.2. To prove (b) and (c) consider an embedding of X in a smooth ambient manifold M. Observe firstly that the foliation defines a quasi-section (i.e. a section in co-dimension 2) of $\mathbb{P}(\Omega^1_X) \to X$. Let $\tilde{X}$ be the closure of this section, and $\mathscr{M}$ the tautological restricted to it. Further let L be the normalisation of any one of our L_i, then L maps to $\tilde{X}$, by $\tilde{f}$ say, and:

$$\mathscr{M}\cdot_{\tilde{f}} L = -2 - \mathrm{Ram}_f$$

where $f : L \to L_i \hookrightarrow X$ is the corresponding map. Moreover we put,

$$s_{\mathrm{sing}(\mathscr{F})}(L) = K_{\mathscr{F}}\cdot_f L - \mathscr{M}\cdot_{\tilde{f}} L \geq 0$$

which is precisely the contribution of the locus where $\mathscr{F}$ is singular in the sense of § 1.2. Now if L_i is singular at $x \in X$, then in fact x is a singular point of f, so by a sequence of $\mathscr{F}$-equivariant blow ups we can assume that $L_i = L$ (remember $K_{\mathscr{F}}$ cannot change). Whence,

$$-1 \geq K_{\mathscr{F}}\cdot_f L = s_{\mathrm{sing}(\mathscr{F})}(L) - 2 \geq -2$$

and -2 is obtained only if L does not pass through the singularities of $\mathscr{F}$. Consequently $\mathscr{F}$ must be smoothly integrable in a neighbourhood of L, and we can apply the algorithmic decomposition procedure once more to obtain a desingularisation $X^{\#}$ of X which is $\mathscr{F}$-equivariant around L. However the classical Frobenius theorem now forces L to have flat, whence trivial normal bundle in $X^{\#}$. Consequently L moves in a $\dim X - 1$ family which covers $X^{\#}$. Should C be any curve in this family, $K_{\mathscr{F}}\cdot C = K_{\mathscr{F}}\cdot L < 0$, so there is an $\mathscr{F}$-invariant rational sub-curve of bounded degree through the generic point of X as required. □

We may also observe from the proof that to have an extremal ray on a foliated variety which is not a fibration in rational curves requires singularities, and so we even have,

Corollary 4.2.2. *Let $(X, \mathscr{F})$ be an everywhere smooth foliation by curves which is not a fibration in rational curves then $K_{\mathscr{F}}$ is nef.*

There is yet another case where usual Mori theory considerations, cf. [K], yield a theorem of some interest. Call $H_R \in \mathrm{NS}^1(X)$ a supporting function of an extremal ray R if H_R is nef., and $H_R\cdot \alpha = 0$ iff $\alpha \in R$, then:

Theorem 4.2.3. *Hypothesis as in IV.2.1, and $R \subset \overline{\mathrm{NE}}(X)$ an extremal ray in the half space $\mathrm{NS}_1(X)_{K_{\mathscr{F}}} < 0$ then there is a $\mathbb{Q}$-Cartier divisor H_R which is a supporting function for R, and moreover $n H_R - K_{\mathscr{F}}$ is ample for $n \in \mathbb{N}$ sufficiently large.*

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Local Structure of Closed Symmetric 2-Differentials

Fedor Bogomolov and Bruno De Oliveira

Abstract In this article we provide a description of the local structure of closed symmetric 2-differentials on a complex surface. The main technical result of the article is that a sum of two local singular analytic functions which both are constant along two different smooth holomorphic foliations can be locally holomorphic only if both functions have at most meromorphic singularities. As a corollary we proof that locally a product of two singular closed differentials on a surface is holomorphic only if the singularities of the differentials are at most exponents of local meromorphic functions.

Keywords Symmetric differential • Essential singularity • Foliation

Mathematical Subject Classification: 14F10, 32B30

1 Introduction

In the authors' previous work on symmetric differentials and their connection to the topological properties of the ambient manifold, a class of symmetric differentials was introduced: closed symmetric differentials [BoDeO11, BoDeO13]. Closed symmetric differentials are characterized by the possibility to locally decompose the differential as a product of closed holomorphic 1-differentials in a neighborhood of a point of the manifold. The property of being closed is conjecturally described

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by a nonlinear differential operator (in the case of dimension 2 and degree 2 this differential operator comes from the Gaussian curvature, see section 3.1).

In this article we give a description of the local structure of closed symmetric 2-differentials on complex surfaces, with an emphasis towards the local decompositions as products of 1-differentials. Recall that there is a general obstruction for a symmetric 2-differential to have a decomposition as a product of 1-differentials around a point x in the complex surface X, it might be impossible to order the two foliations defined by w near x (we then say that w is not locally split at x). This obstruction can be removed via a ramified covering of X, hence the results will be given for symmetric differentials that are locally split.

We show that a closed symmetric 2-differential w of rank 2 (i.e., defines two distinct foliations at the general point) has a subvariety $B_w \subset X$ outside of which w is locally the product of closed holomorphic 1-differentials. The main result, Theorem 3.6, gives a complete description of a (locally split) closed symmetric 2-differential in a neighborhood of a general point of B_w. A consequence of the main result is that the differential w still has a local decomposition into a product of closed 1-differentials (in a generalized sense) at the points of B_w. The closed 1-differentials involved in the local decompositions might have to be multi-valued and acquire singularities along B_w. Note that if we were considering local decompositions of a locally split holomorphic symmetric 2-differential into a product of 1-differentials (not necessarily closed), then the 1-differentials involved can be chosen to be holomorphic, i.e., no singularities need to occur. On the other hand, it is also true that by multiplying one of the 1-differentials by an arbitrary function and the other by its inverse, that arbitrary singularities can occur in the decomposition. An important feature of decompositions of symmetric 2-differentials of rank 2 as products of closed 1-differentials is that they are unique up to multiplicative constants, hence there is no ambiguity on the singularities that occur.

The singularities that occur in the decomposition of a closed holomorphic symmetric 2-differential w when we require that the 1-differentials are closed can be essential singularities along the locus B_w. A key feature of Theorem 3.6 giving the local structure of w around points in B_w is that we have a control on these essential singularities, they come from exponentials of meromorphic functions acquiring poles of a bounded order along B_w. Before describing the nature of the bound, we need to describe our result characterizing the locus B_w. In the case w is locally split (always the case after a ramified cover), we show that any irreducible component of B_w must be simultaneously a leaf of both foliations defined by w. The bound on the order of the poles along an irreducible component of B_w is the order of contact of both foliations along that irreducible component.

This article addresses the case of closed symmetric 2-differentials, we expect that a straightforward generalization of our methods will provide similar results on the local structure of closed symmetric differentials of arbitrary degree and give control of the singularities that occur on the decompositions as product of closed 1-differentials.

2 General Set Up

A symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ on a complex manifold X defines at each point where $w(x) \neq 0$ a cone in tangent space T_xX with vertex the origin and defined at infinity ($\mathbb{P}^{n-1}$) by a variety of degree m. If X is a complex surface then one gets a distribution of d ($d \leq m$) lines, which will be integrable, defining a non-singular d-web at the general point. In higher dimensions the cones will not be necessarily union of hyperplanes and even if they are hyperplanes their distributions need not be integrable. Here, we should note that the class of symmetric differentials that is studied in this work, closed symmetric differential (see below), will in all dimensions be connected to webs on the manifold.

Definition 2.1. A symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ is split if it has a decomposition:

$$w = \psi_1 \dots \psi_m$$

where the ψ_i are meromorphic 1-forms or equivalently if $w = \mu_1 \dots \mu_m$, with $\mu_i \in H^0(X, \Omega^1_X \otimes L_i)$, where L_i are line bundles on X.

Geometrically being split means that the symmetric differential defines hyperplane distributions and moreover they can be numbered consistently globally.

Definition 2.2. A symmetric differential w on X has rank r if at a general point $x \in X$ $w(x)$ defines r distinct hyperplanes in T_xX.

Definition 2.3. A symmetric differential w on X is said to have a holomorphic closed decomposition if:

$$w = \mu_1 \dots \mu_m\ , \qquad \mu_i \text{ closed holomorphic 1-forms} \tag{1}$$

and a holomorphic closed decomposition at x if x has an analytic neighborhood where (1) holds.

Definition 2.4. A symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ is said to be:

1) closed, if w has an holomorphic closed decomposition at a general point $x \in X$.
2) of the 1st kind, if w has holomorphic closed decompositions at all $x \in X$.

Remarks. 1) The class of closed symmetric differentials of the 1st kind plays a special role in the motivation for considering closed symmetric differentials as a class of symmetric differentials having a stronger connection to the topology of the ambient manifold. We expand on this point below.

2) Our definitions of closed and 1st kind coincide with the usual definitions when m=1, i.e., holomorphic 1-forms. Our definition of closed asks for a holomorphic 1-form to be locally exact somewhere which by the identity principle implies it is locally exact everywhere and hence closed in the usual sense. Hence, for $m = 1$ our notions of closed and 1st kind coincide.

3) If the degree $m > 1$, then closed no longer implies of the 1st kind. This has far reaching geometric consequences and the main results of this work concern the locus where this failure comes from and the structure of the closed symmetric differentials near this locus.

Definition 2.5. The locus of X where a closed symmetric differential w fails to be of the 1st kind at, $B_w = \{x \in X|\ w$ has no holomorphic closed decomposition at x$\}$, will be called the breakdown locus of w.

A key feature of holomorphic closed decompositions is that they have rigidity properties. The level of rigidity has to do with a familiar notion in the theory of webs, the Abelian rank of a web.

Definition 2.6. Given the germ $w_x \in S^m\Omega^1_{X,x}$ with the holomorphic closed decomposition

$$w_x = \mu_1 \dots \mu_m \tag{2}$$

where $\mu_i \in \Omega^1_{X,cl,x}$ ($\Omega^1_{X,cl}$ is the sheaf of closed holomorphic 1-forms on X), we call an m-tuple $(f_1, \dots, f_m) \in \mathcal{M}^m_x$ satisfying:

$$\sum_{i=1}^{m} f_i\mu_i = 0 \quad \textit{with } \ df_i \wedge \mu_i \equiv 0.$$

an Abelian relation of the decomposition (2). The Abelian rank of the decomposition (2) is the dimension of the $\mathbb{C}$-vector space consisting of all Abelian relations of (2). The Abelian rank of a closed symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ is the Abelian rank of any holomorphic closed decomposition at the general point of x.

Remarks. 1) The definition of Abelian rank of a closed symmetric differential w is well defined, since there is an analytic subvariety of $R \subset X$ such that all holomorphic closed decompositions of w_x, $\forall x \in X \setminus R$, have the same Abelian rank.

2) It is a classical result of web theory that the Abelian rank of a decomposition (2) is finite if rank(w)=m, with upper bounds depending on the dimension and degree (for dimension 2 this is a result of G.Bol and also W.Blaschke see, for example, [ChGr78, He01] and [He04] for information on webs).

3) A general germ of a closed symmetric differential has trivial Abelian rank.

We concentrate our attention to the case of trivial Abelian rank which is the generic case and holds trivially for all closed symmetric 2-differentials (and rank 2).

Proposition 2.1. *Let X be a connected manifold and $w \in H^0(X, S^m\Omega^1_X)$ be a closed symmetric differential with an holomorphic closed decomposition $w = \mu_1 \dots \mu_m$. If the Abelian rank of w is trivial, then all holomorphic closed decomposition $w = \eta_1 \dots \eta_m$ of w on X have the closed 1-forms $\eta_i = c_i\mu_i$ with $c_i \in \mathbb{C}^*$ and $\prod_{i=1}^m c_i = 1$.*

Proof. Suppose $w = \eta_1 \dots \eta_m$ is an holomorphic closed decomposition of w and assume the η_i are ordered such that $\eta_i \wedge \mu_i = 0$. The condition $\eta_i \wedge \mu_i = 0$ in conjunction with η_i and μ_i being closed implies that $\eta_i = f_i\mu_i$ with $f_i \in \mathcal{M}(X)$ and $df_i \wedge \mu_i = 0$. Moreover $\mu_1 \dots \mu_m = \eta_1 \dots \eta_m$ gives:

(3) $$\prod_{i=1}^{m} f_i = 1$$

Pick a simply connected open set $U \subset X$ where $f_i|_U \in \mathcal{O}^*(U)$. Taking the logarithm and differentiating (3) restricted to U and using the identity principle we obtain:

(4) $$\sum_{i=1}^{m} \frac{df_i}{f_i} = 0$$

but $df_i \wedge \mu_i = 0$, hence $df_i = g_i \mu_i$ with $g_i \in \mathcal{M}(X)$ and $dg_i \wedge \mu_i = 0$. It follows that (4) gives rise to the Abelian relation at a general point $x \in X$:

$$\sum_{i=1}^{m} (\frac{g_i}{f_i})_x (\mu_i)_x = 0$$

The Abelian rank of w being trivial implies that the $(g_i)_x = 0$ and hence $df_i = 0$ on X, i.e., $f_i = c_i \in \mathbb{C}^*$.♦

A consequence of this proposition, see below, is the decomposition of symmetric differentials of the 1st kind with Abelian rank 0 into a product of twisted closed holomorphic 1-forms $\phi_i \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_i})$. In [BoDeO13] we use such decompositions to characterize the origins and the geometric implications of symmetric 2-differentials of the 1st kind.

Corollary 2.2. *Let X be a complex manifold and $w \in H^0(X, S^m\Omega^1_X)$ be of the 1st kind with trivial Abelian rank. Then there is a finite unramified cover $f : X' \to X$ (unnecessary if w is split) for which f^*w has a decomposition:*

$$f^*w = \phi_1 \ldots \phi_m$$

where $\phi_i \in H^0(X', \Omega^1_{X',cl} \otimes \mathbb{C}_{\rho_i})$, where the $\mathbb{C}_{\rho_i}$ are local systems of rank 1 on X' such that $\mathbb{C}_{\rho_1} \otimes \ldots \otimes \mathbb{C}_{\rho_m} \simeq \mathbb{C}$.

Proof. The differential w being of first kind implies that locally w is split, but w might fail to be globally split. This failure is measured by the monodromy coming from the local ordering of the foliations, i.e., we obtain a representation $\sigma : \pi_1(X, x) \to S_m$. Associated to this representation we get an unramified cover $f : X' \to X$ with degree a factor of $m!$ such that f^*w is split.

From now on we assume that w is split on X. The differential w being of the 1st kind gives that there is a Leray covering $\mathcal{U} = \{U_i\}$ of X where

$$w|_{U_i} = \phi_{1i} \ldots \phi_{mi}$$

with $\phi_{ki} \in H^0(U_i, \Omega^1_{X,cl})$. Since the differential w is split, we can order the $\{\phi_{ki}\}$ such that $\phi_{ki} \wedge \phi_{kj} = 0$ on the intersections $U_i \cap U_j$. Proposition 2.1 implies that on $U_i \cap U_j$

(5) $$\phi_{ki} = c_{k,ij} \phi_{kj}$$

with $c_{k,ij} \in \mathbb{C}^*$ and $\prod_{k=1}^{m} c_{k,ij} = 1$. The m collections $\{c_{k,ij}\}$ for $k = 1, \ldots, m$ are elements in $Z^1(X, \mathbb{C}^*)$ and give rise to the rank 1 local systems which we denote by $\mathbb{C}_{\rho_k}$ and satisfy $\mathbb{C}_{\rho_1} \otimes \ldots \otimes \mathbb{C}_{\rho_m} \simeq \mathbb{C}$ (we remark that the isomorphism classes of these local systems are completely determined by w). It follows from (5) that each collection for a fixed k, $\{\phi_{ik}\}$, gives a section $\phi_k \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_k})$ and the result holds.♦

The presence of twisted closed holomorphic 1-forms $\phi_i \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_i})$ has implications on both the topology and geometry of the manifold X. On the topological side one observes that the cohomology exact sequence associated to the short exact sequence $0 \to \mathbb{C}_\rho \to \mathcal{O} \otimes \mathbb{C}_\rho \to \Omega^1_{X,cl} \otimes \mathbb{C}_\rho \to 0$ implies that $H^1(X, \mathbb{C}_\rho) \geq h^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_\rho)$ (hence in particular $\pi_1(X)$ must infinite). On the geometric side, if X is compact Kähler the presence of non-torsion, i.e., $L_\rho = \mathcal{O} \otimes \mathbb{C}_\rho$ non-torsion, twisted closed holomorphic 1-forms implies that X is fibered over curves of genus $g \geq 1$ as follows from the work of Beauville, Lazarsfeld-Green, and Simpson (see [GrLa87], [Be92, Si93], and [Ar92]).

3 Local Structure of Closed 2-Differentials on Surfaces

3.1 Differential Operator for Closed Symmetric 2-Differentials

A symmetric differential of degree 2 on a complex surface can be viewed as a generalized complex counterpart of a Riemannian metric on a real surface. We are going to use this fact to motivate the differential operator characterizing closed symmetric 2-differentials.

Let $w \in H^0(X, S^2\Omega^1_X)$ on a complex surface X. There is an open cover of X, $\mathcal{U} = \{U_i\}$ by local holomorphic charts, where

$$w|_{U_i} = a_i(z)(dz_{1i})^2 + b_i(z)dz_{1i}dz_{2i} + c_i(z)(dz_{2i})^2$$

On each of these open sets we get the holomorphic functions $detw|_{U_i} = a_i(z)c_i(z) - b_i(z)^2/4$ which together form an element of $H^0(X, \mathcal{O}(2K_X))$, called the discriminant of w.

Definition 3.1. The discriminant divisor Disc_w of $w \in H^0(X, S^2\Omega^1_X)$ is the divisor of zeros of the section $\{detw|_{U_i}\}_{i\in I}$ of $\mathcal{O}(2K_X)$. The core discriminant divisor of w is $\mathrm{Disc}^0_w = \mathrm{Disc}_w - 2(w)_0$.

Geometrically, the support of Disc_w corresponds to the set of points where $w(x)$ either vanishes or defines one single line in T_xX. To better understand the support of Disc^0_w we give a new characterization of the divisor Disc^0_w. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X such that $w|_{U_i} = h_i\hat{w}_i$ with $h_i \in \mathcal{O}(U_i)$ and $\hat{w}_i$ vanishes only in codimension 2. The divisor Disc^0_w can described via the local information $\mathrm{Disc}^0_w \cap U_i = \mathrm{Disc}_{\hat{w}_i}$. By definition $\mathrm{Supp}(\mathrm{Disc}^0_w) \subset \mathrm{Supp}(\mathrm{Disc}_w)$, moreover a point

$x \in \mathrm{Supp}(\mathrm{Disc}_w) \setminus \mathrm{Supp}(\mathrm{Disc}_w^0)$ is such that $w(x) = 0$ but $\hat{w}_i(x)$ defines two distinct lines in $T_x X$. A more detailed characterization of the irreducible components of the divisors Disc_w and Disc_w^0 is given in section 3.2.

Let $x \in X \setminus \mathrm{Disc}_w^0$, then the germ w_x of w at x splits, $w_x = \mu_1 \mu_2$ with $\mu_i \in \Omega^1_{X,x}$. Say $x \in U_i$ with U_i as above, $w|_{U_i} = h_i \hat{w}_i$, then $x \in X \setminus \mathrm{Disc}_w^0$ implies that the discriminant of $\hat{w}_i$ does not vanish at x and therefore $\hat{w}_i$ (and hence w) splits at x, $\hat{w}_{ix} = \hat{\mu}_1 \hat{\mu}_2$. Moreover, since the discriminant of $\hat{w}_i$ is nonzero at x, then $\hat{\mu}_1 \wedge \hat{\mu}_2(x) \neq 0$, which implies that x has a neighborhood U_x with holomorphic a chart (z_1, z_2) such that:

$$w|_{U_x} = g(z) dz_1 dz_2 \tag{1}$$

with $g \in \mathcal{O}(U_x)$. The condition that $w|_{U_x}$ has a closed holomorphic decomposition, $w|_{U_x} = g(z) dz_1 dz_2 = \mu_1 \mu_2$ with μ_i closed holomorphic 1-forms, is equivalent to

$$g(z) = f_1(z_1) f_2(z_2) \tag{2}$$

($\mu_i \wedge dz_i \equiv 0$ implies that $\mu_i = f_i(z_i) dz_i$). The condition (2) can be characterized via the nonlinear differential equation $\frac{\partial^2 \ln g(z)}{\partial z_1 \partial z_2} = 0$.

It follows from the Brioschi's formula for the Gaussian curvature in terms of the 1st fundamental form [Sp99], that the differential operator giving the Gaussian curvature for a metric in the form $ds^2 = f(x) dx_1 dx_2$ is $K(ds^2) = -\frac{2}{f} \frac{\partial^2 \ln f(z)}{\partial x_1 \partial x_2}$. Hence the symmetric differential in (1) is closed if and only if $K_{\mathbb{C}}(w|_{U_x}) = 0$, where $K_{\mathbb{C}}$ is the operator obtained from K by replacing x_1, x_2 by z_1, z_2.

A symmetric 2-differential cannot be put locally in the form (1) everywhere, but this is not a problem since the differential operator K for the Gaussian curvature works for metrics whose 1st fundamental form is arbitrary (works formally if the 1st fundamental form is degenerate, i.e., with discriminant zero at some points). Hence $K_{\mathbb{C}}$ works for any symmetric 2-differential (for a general symmetric 2-differential w, $K_{\mathbb{C}}(w)$ will be a meromorphic function with poles along the discriminant locus). If w is a symmetric 2-differential satisfying $K_{\mathbb{C}}(w) = 0$, then $K_{\mathbb{C}}(w|_{U_x}) = 0$ with $U_x \subset X \setminus \mathrm{Disc}_w^0$ as in 2.1 and hence it by our previous paragraph w is closed. Hence we obtain:

Proposition 3.1. *Let $w \in H^0(X, S^2\Omega^1_X)$ be of rank 2 on a X a connected complex surface, then w being closed is equivalent to:*

$$K_{\mathbb{C}}(w) = 0$$

Moreover, $K_{\mathbb{C}}(w) = 0$ is equivalent to w is of the 1st kind on $X \setminus Disc_w^0$.

3.2 Characterization of the Breakdown Locus B_w

In this section w is a closed symmetric 2-differential of rank 2. We start by showing that the breakdown locus B_w has no isolated points and then proceed to show that B_w is an analytic subvariety of codimension 1 and to characterize geometrically its components.

Lemma 3.2. *B_w has no isolated points.*

Proof. It is enough to show that if w is of the 1st kind in a punctured ball $\mathbb{B}^*$, then it is of the 1st kind on the whole ball $\mathbb{B}$. Let $w \in H^0(\mathbb{B}, S^2\Omega^1_{\mathbb{B}})$ be of the 1st kind on the punctured ball $\mathbb{B}^*$. According to Corollary 2.2,

$$w|_{\mathbb{B}^*} = \phi_1\phi_2$$

with $(\phi_1, \phi_2) \in H^0(\mathbb{B}^*, \Omega^1_{X,cl} \otimes (\mathbb{C}_\rho \oplus \mathbb{C}^*_\rho))$. The triviality of the fundamental group of $\mathbb{B}^*$ implies that $\mathbb{C}_\rho \simeq \mathbb{C}^*_\rho \simeq \mathbb{C}$ and hence the ϕ_i can be chosen to be in $H^0(\mathbb{B}^*, \Omega^1_{X,cl})$. Again using $\pi_1(\mathbb{B}^*) = \{e\}$, it follows by integration that $\phi = df_i$ with $f_i \in \mathcal{O}(\mathbb{B}^*)$. Hartog's extension theorem implies that exist $\hat{f}_i \in \mathcal{O}(\mathbb{B})$ extending the f_i and hence w is of the 1st kind on $\mathbb{B}$ with $w = d\hat{f}_1 d\hat{f}_2$.♦

Proposition 3.1 tell us that that:

Corollary 3.2. *$B_w \subset Supp(Disc^0_w)$.*

To proceed we need to give a geometric description of the irreducible components of both discriminant loci. The support of the discriminant divisor decomposes into:

$$\mathrm{Supp}(\mathrm{Disc}_w) = N_w \cup S_w$$

where N_w and S_w are the union of all irreducible components of Disc_w of respectively odd and even multiplicities. The locus N_w corresponds to the points where w fails to split at. For the support of the core discriminant divisor we have:

$$\mathrm{Supp}(\mathrm{Disc}^0_w) = N_w \cup C_w \cup R_w$$

It follows from the characterization of Disc^0_w given after Definition 3.1 that any $x \in \mathrm{Supp}(\mathrm{Disc}^0_w)$ is in the closure of the locus of points $y \in X$ such that $\hat{w}_i(y)$ defines a single line in T_yX (where $w|_{U_i} = h_i\hat{w}_i$ with $y \in U_i$, $h_i \in \mathcal{O}(U_i)$ and $\hat{w}_i$ is a symmetric 2-differential vanishing at most in codimension 2). Note that Definition 3.1 gives directly that the divisor N_w is fully contained in $\mathrm{Supp}(\mathrm{Disc}^0_w)$, since only even multiples of irreducible components are subtracted from (Disc_w) to obtain (Disc^0_w).

The divisor C_w consists of the union of all irreducible components of $\mathrm{Supp}(\mathrm{Disc}_w)$ which are leaves simultaneously of the two foliations defined by the $\{\hat{w}_i\}_{i\in I}$ (a 2-differential defines two foliations where it splits). We will call the irreducible components of C_w the common leaves of w. The divisor R_w consists of all the irreducible components of $\mathrm{Supp}(\mathrm{Disc}^0_w)$ that are not in N_w or C_w. These will be the components for which at their general point x the two different foliations given by $\{\hat{w}_i\}_{i\in I}$ define leaves that are tangent at x but that do not coincide).

Theorem 3.4. *$B_w = N_w \cup C''_w$, where C''_w is a union of curves contained in C_w.*

Proof. The locus N_w is contained in B_w since the differential w splits on any $x \notin B_w$. Set $X' = X \setminus N_w$, $C'_w = C_w \cap X'$ and $R'_w = R_w \cap X'$ and get:

$$\mathrm{Supp}(\mathrm{Disc}^0_w) \cap X' = C'_w \cup R'_w$$

The desired result then follows if we show that the breakdown locus $B_{w|_{X'}}$ is an union of irreducible components of C'_w (with C''_w being the closure of this union).

By construction X' is the open subset of X where w is locally split. Hence given any $x \in X'$, there exists an open neighborhood U_x of x where $w|_{U_x} = \mu_1\mu_2$, $\mu_i \in H^0(U_x, \Omega^1_X)$. We can shrink U_x so that we can decompose $\mu_i = h_i\hat{\mu}_i$ with $h_i \in \mathcal{O}(U_x)$ and the $\hat{\mu}_i \in H^0(U_x, \Omega^1_X)$ are either nowhere vanishing or vanish only at x. Frobenius' theorem (if $\hat{\mu}_i(y) \neq 0$, then $\exists U_y$ open neighborhood of y where $\hat{\mu}_i = f_i du_i, f_i, u_i \in \mathcal{O}(U_y)$), then implies that the set $S \subset X'$ consisting of the points x where w fails to have a neighborhood U_x where $w|_{U_x} = g dz_1 dw_1$ with $g \in \mathcal{O}(U_x)$ and dz_1, dw_1 nowhere vanishing is discrete.

Consider the irreducible decomposition

$$C'_w \cup R'_w = \bigcup_{i=I} C'_{w,i} \cup \bigcup_{j=J} R'_{w,j}$$

where I, J are countable and $C'_{w,i}$ and $R'_{w,j}$ are the irreducible components of C'_w and R'_w, respectively. Below, we will first show that the irreducible components $R'_{w,j}$ intersect $B_{w|_{X'}}$ only inside S, i.e., $R'_{w,j} \cap B_{w|_{X'}} \subset S \cup \bigcup_{i=I} C'_{w,i}$. Second, we will show that the irreducible components $C'_{w,i}$ are such that $C'_{w,i} \subset B_{w|_{X'}}$ or $C'_{w,i} \cap B_{w|_{X'}} \subset S$. These two results (and Corollary 3.2) imply that $B_{w|_{X'}} = \bigcup_{i=I'} C'_{w,i} \cup S'$, with $S' \subset S$ and $I' \subset I$. The result then follows since $S' \subset \bigcup_{i=I'} C'_{w,i}$. The discreteness of S and $\bigcup_{i=I'} C'_{w,i}$ being an analytic subvariety of X' implies that if $x \in S'$ is not contained in $\bigcup_{i=I'} C'_{w,i}$, then x has a neighborhood U_x such that $U_x \cap B_w = x$, but by Lemma 3.2 B_w has no isolated points.

Claim: $R'_{w,j} \cap B_{w|_{X'}} \subset S \cup \bigcup_{i=I} C'_{w,i}$

Before proceeding, note that by the definition of the set S it follows that any $x \in X' \setminus S$ has a neighborhood U_x with $g, z_1, z_2, w_1 \in \mathcal{O}(U_x)$ such that

$$w|_{U_x} = g dz_1 dw_1$$

and $\phi = (z_1, z_2) : U_x \to \Delta \times \Delta$, Δ a disc centered at 0, is a biholomorphism with $\phi(x) = (0, 0)$.

We will show that any $x \in R'_{w,j} \cap [X' \setminus (S \cup \bigcup_{i=I} C'_{w,i})]$ cannot lie in B_w.

Let U_x be a neighborhood of x as in the previous paragraph. Consider the leaf $L = \{z_1 = 0\}$ of w on U_x passing through x. By hypothesis x is not in a common leaf of w, hence L cannot be a common leaf of w which implies that $L \not\subset \text{Supp}(\text{Disc}^0_w)$. If $L \subset \text{Supp}(\text{Disc}^0_w)$ then $dz_1 \wedge dw_1 = 0$ on L making L a leaf of dw_1 also, hence a common leaf for w. Hence $L \setminus [\text{Supp}(\text{Disc}^0_w) \cap L)] \neq \emptyset$

Pick $y \in L$ but not in $\text{Supp}(\text{Disc}^0_w)$, then by Proposition 3.1 there is a (connected) neighborhood U_y of y where $w|_{U_y} = f(z_1)g(w_1)dz_1dw_1$ with $f, h \in \mathcal{O}(U_y)$ ($f(z_1)$ denotes a function $f(z_1, z_2)$ depending only on z_1). Let Δ' be a disc centered at 0 such that $\Delta' \times z_2(y) \subset \phi(U_y)$ and $W_x = z_1^{-1}(\Delta')(= \phi^{-1}(\Delta' \times \Delta))$. The function f has a clear holomorphic extension $\hat{f} \in \mathcal{O}(U_y \cup W_x)$, with $\hat{f}|_{W_x}(z_1, z_2) = f(z_1, z_2(y))$.

The same reasoning applied to h will not give an extension of h to $W_x \cup U_y$, so instead we use the extension of f and consider the function $\hat{h} = \frac{g}{\hat{f}}$. Clearly, $\hat{h}|_{U_y} = h$ hence $\hat{h}$ is a function of w_1 alone. The function $\hat{h}|_{W_x}$ is holomorphic since the irreducible components of the polar divisor $(\hat{h}|_{W_x})_\infty$ if they exist must be some of the irreducible components of the divisor of zeros of $\hat{f}$ which will be a union of curves $\{z_1 = c\}$ and hence intersect non-trivially U_y but this intersection must be empty since $h|_{U_y} = h$ is holomorphic.

It follows from the previous two paragraphs that the closed holomorphic decomposition of w at U_y, $w|_{U_y} = f(z_1)h(w_1)dz_1dw_1$, propagates to give the closed holomorphic decomposition on the neighborhood W_x of x, $w|_{W_x} = \hat{f}(z_1)\hat{h}(w_1)dz_1dw_1$, making $x \notin B_w$.

Claim: $C'_{w,i} \subset B_{w|_{X'}}$ or $C'_{w,i} \cap B_{w|_{X'}} \subset S$.

In addition to the properties, described two paragraphs above, that we can guarantee for an open neighborhood U_x of $x \in X' \setminus S$, we can equally guarantee the existence of an open neighborhood $U'_x \subset U_x$ and $w_2 \in \mathcal{O}(U'_x)$ such that $\phi' = (w_1, w_2) : U'_x \to \Delta' \times \Delta'$ is a biholomorphism.

Consider the subsets $C^*_{w,i} = C'_{w,i} \cap (X' \setminus S)$ and $V_i = C^*_{w,i} \cap (X \setminus B_w)$. The set $C^*_{w,i}$ is connected since by the local parametrization theorem [De12] an irreducible component of an analytic variety punctured by a discrete set is connected. The subset V_i consisting of all points of $C^*_{w,i}$ where w has a local holomorphic decomposition is clearly open in $C^*_{w,i}$. We proceed to show that V_i is also closed in $C^*_{w,i}$. Since $C^*_{w,i}$ is connected, V_i being both open and closed implies the desired result that the irreducible components $C'_{w,i}$ are such that $C'_{w,i} \subset B_{w|_{X'}}$ (when $V_i = \emptyset$ and use B_w closed) or $C'_{w,i} \cap B_{w|_{X'}} \subset S$ (when $V_i = C^*_{w,i}$).

Let $x \in C^*_{w,i}$ be an accumulation point of V_i. Pick $y \in V_i \cap U'_x$, with U'_x as in two paragraphs above. Since $C^*_{w,i}$ is a common leaf of w, $y \in L_x = \{z_1 = 0\} = \{w_1 = 0\}$. Hence y has a neighborhood U_y such that $\phi(U_y) \supset \Delta'' \times z_2(y)$ and $\phi'(U_y) \supset \Delta'' \times w_2(y)$, Δ'' a disc centered at 0, where $w|_{U_y} = g dz_1 dw_1 = f(z_1)h(w_1)dz_1dw_1$ with $f, h \in \mathcal{O}(U_y)$. The functions f and h are clearly extendable to $\hat{f} \in \mathcal{O}(z_1^{-1}(\Delta'') \cup U_y)$ and $\hat{h} \in \mathcal{O}(w_1^{-1}(\Delta'') \cup U_y)$. By construction $W_x = z_1^{-1}(\Delta'') \cap w_1^{-1}(\Delta'')$ is a connected open set containing x and y and $g|_{W_x \cap U_y} = \hat{f}\hat{h}|_{W_x \cap U_y}$, hence $g|_{W_x} = \hat{f}\hat{h}|_{W_x}$ giving a holomorphic decomposition of w on the neighborhood W_x of x, i.e., $x \in V_i$.

3.3 Monodromy at B_w

Let $w \in H^0(X, S^2\Omega^1_X)$ be closed of rank 2 and $B_w = \sum_{j\in J}$, J countable, be the irreducible decomposition of the breakdown locus B_w. Let $\mathcal{U} = \{U_i\}$ be a Leray covering of $X \setminus B_w$ where

$$w|_{U_i} = \phi_{1i}\phi_{2i}$$

with $\phi_{ki} \in H^0(U_i, \Omega^1_{X,cl})$, $k = 1, 2$. The Abelian rank of a closed symmetric 2-differential of rank 2 is trivial, it follows then from Proposition 2.1 that if $U_i \cap U_j \neq \emptyset$, then

$$\begin{bmatrix}\phi_{1i}\\ \phi_{2i}\end{bmatrix} = g_{ij}\begin{bmatrix}\phi_{1j}\\ \phi_{2j}\end{bmatrix}$$

with $g_{ij} \in G = \left\{\begin{bmatrix} c & 0\\ 0 & c^{-1}\end{bmatrix}, \begin{bmatrix} 0 & c\\ c^{-1} & 0\end{bmatrix} \mid c \in \mathbb{C}^*\right\}$. The collection $\{g_{ij}\}$ gives a 1-cocycle with values in the group G, i.e., $\{g_{ij}\} \in Z^1(\mathcal{U}, G)$. Hence given $x_0 \in X\setminus B_w$, we obtain a representation $\rho : \pi_1(X \setminus B_w, x_0) \to G$.

If w is split, then Im$\rho \subset G'$, with $G' = \{\begin{bmatrix} c & 0\\ 0 & c^{-1}\end{bmatrix}, \forall c \in \mathbb{C}^*\} \subset G$ (this follows from being able to get a consistent ordering of the foliations on all the U_i). Since G' is Abelian we get a representation, $\rho_w : \pi_1(X \setminus B_w) \to G'$, that is independent of the base point and factors through $H_1(X \setminus B_w, \mathbb{Z})$, and gives:

$$\bar{\rho}_w : H_1(X \setminus B_w, \mathbb{Z}) \to G'$$

Associated with each irreducible component B_j, let $\gamma_i j \in H_1(X \setminus B_w, \mathbb{Z})$ be the class of a simple loop around $B_i j$ (boundary to a disc transversal to B_j centered at a general point of B_j) which can have either orientation.

Definition 3.2. Let $w \in H^0(X, S^2\Omega^1_X)$ be split, closed of rank 2 and $B_w = \sum_{j\in J} B_j$, J countable, be the irreducible decomposition of the breakdown locus B_w. To each irreducible component B_j we associate the monodromy index $M(B_j, w) = \{c, c^{-1}\}$, if $\bar{\rho}_w(\gamma_j) = \begin{bmatrix} c & 0\\ 0 & c^{-1}\end{bmatrix}$, with $\bar{\rho}_w$ and γ_j as above.

3.4 Local Form at B_w

The goal of this section and the main result of this article is to give the general form of a split closed symmetric 2-differential of rank 2 w at the general point of an irreducible component of the breakdown locus B_w. We will see that the closed decompositions of w can acquire essential singularities and have non-trivial monodromy at the breakdown locus B_w. We will show that the essential singularities have an algebraic feature, they come from exponential functions with meromorphic functions with poles along B_w as exponents. Moreover, we will give a bound on the order of the poles of the meromorphic functions appearing as exponents. The bounds come from the order of contact of the two foliations of w along the irreducible components of B_w.

We start with some examples of closed symmetric 2-differentials for which B_w is non-empty.

Example (Non-split). Let z_1 be a holomorphic coordinate of $\mathbb{C}^n$ and $f \in \mathcal{O}(\mathbb{C}^n)$, set $w = z_1(dz_1)^2 - (df)^2$. The differential is non split at all points in $\{z_1 = 0\}$ but it is closed since any point $y \in X \setminus \{z_1 = 0\}$ has a neighborhood U_y where $\sqrt{z_1}$ exists and hence w has a holomorphic exact decomposition $w|_{U_y} = d(\frac{2}{3}z_1^{\frac{3}{2}} + f)d(\frac{2}{3}z_1^{\frac{3}{2}} - f)$.

If the differential is locally split at x, then a 2nd layer of the failure of w to have a holomorphic closed decomposition at x is due to the monodromy in the factors of the closed decompositions (not the monodromy of the foliations) around B_w.

Example (Monodromy of the Closed Decompositions). Let $B \subset \mathbb{C}^2$ be a sufficiently small open ball about the origin where $1 + z_2$ is invertible. Consider $w = (1 + z_2)^\alpha dz_1 d[z_1(1 + z_2)]$. Recall that the differential w has a holomorphic closed decomposition at a point $x \in B$ if and only if we can decompose $(1 + z_2)^\alpha$ as a product of holomorphic functions of z_1 and $z_1(1 + z_2)$ near x. At points in the complement of $\{z_1 = 0\}$ we have the decomposition $(1 + z_2)^\alpha = z_1^{-\alpha}[z_1(1 + z_2)]^\alpha$, but at points in $\{z_1 = 0\}$ the functions involved are multi-valued, hence no holomorphic closed decomposition of w is possible at $x \in \{z_1 = 0\}$. In fact this monodromy is infinite if $\alpha \notin \mathbb{Q}$, meaning that even after finite ramified coverings the symmetric differential would not have an exact decomposition along the pre-image of $\{z_1 = 0\}$.

If the differential is both locally split at x and no monodromy occurs, then w a 3rd level of failure to have a holomorphic closed decomposition is due to the singularities of the 1-differentials on the decomposition.

Example (Meromorphic Singularities). $w = (dz_1)^2 + z_1 z_2 dz_1 dz_2 = dz_1(dz_1 + z_1 z_2 dz_2)$ has the common leaf $L = \{z_1 = 0\}$. The differential w is closed because the 1-form $dz_1 + z_1 z_2 dz_2$ has an integrating factor, $\frac{1}{z_1}$, which is a function of z_1. This integrating factor produces the closed meromorphic decomposition $w = d(\frac{z_1^2}{2})(\frac{1}{z_1}dz_1 + z_2 dz_2)$. Note that since the Abelian rank of w is trivial any other closed decomposition of w would differ just by multiplicative constants hence meromorphic singularities would be always present in the closed decompositions of w.

Example (Essential Singularities). This example shows that even essential singularities can occur, $w = e^{\frac{z_2}{1+z_1 z_2}} dz_1 d[z_1(1 + z_1 z_2)]$. The 1-differentials in the split closed decomposition are unique up to multiplicative constants, as it was shown in

Proposition 2.1, and the constants will cancel each other so in fact the decomposition is unique and has the form

$$w = e^{\frac{z_2}{1+z_1z_2}} dz_1 d[z_1(1+z_1z_2)] = e^{\frac{1}{z_1}} dz_1 e^{-\frac{1}{z_1(1+z_1z_2)}} d[z_1(1+z_1z_2)]$$

with essential singularities occurring on the closed 1-forms at $\{z_1 = 0\}$.

Lemma 3.5. *Let X be a complex 2-manifold, $w \in H^0(X, S^2\Omega^1_X)$ be split of rank 2 and L be an irreducible component of a common leaf of w. Then there is an $m \in \mathbb{N}$ such that the general point x of L has a neighborhood U_x with a holomorphic chart (z_1, z_2) where*

$$w|_{U_x} = f(z_1, z_2) dz_1 d[z_1(1 + z_1^m z_2)]$$

Proof. The differential w being split implies that every point $x \in X$ has an open neighborhood U_x such that $w|_{U_x} = \mu_1\mu_2$, $\mu_i \in H^0(X, \Omega^1_X)$. By shrinking U_x we can factor the 1-forms μ_i in the form $\mu_i = f_i\hat{\mu}_i$ with $f_i \in \mathcal{O}(U_x)$ and $\hat{\mu}_i$ either non-vanishing or vanishing only at x. Since by Frobenius theorem a non-vanishing 1-form in dimension 2 is integrable, it follows that there is a discrete set $S \subset X$ such that all $x \in X \setminus S$ have a neighborhood U_x where

$$w|_{U_x} = h\,dv\,dr \tag{3}$$

with $h, v, r \in \mathcal{O}(U_x)$, $v(x) = r(x) = 0$, dv and dr nowhere zero on U_x.

Let x be a general point of L, using the notation of (3) we have $L \cap U_x = \{v = 0\} = \{w = 0\}$ with:

$$r = vu$$

with u a unit on U_x. After shrinking U_x we can assume there is a holomorphic local chart on U_x, (v_1, v_2) such that $v_1 = v$. Consider the series expansion $u(v_1, v_2) = \sum_{i=0,j=0}^{\infty} c_{ij} v_1^i v_2^j$ and let $m = \min\{i | \exists j > 0 \text{ s.t. } c_{ij} \neq 0\}$ (the Taylor series of u must involve v_2 since $dv \wedge dr \not\equiv 0$). Then decompose u as

$$u(v_1, v_2) = s(v_1) + v_1^m(t(v_2) + v_1 g(v_1, v_2))$$

where $s(v_1) = \sum_{i=0}^{\infty} c_{i0} v_1^i$ is a holomorphic function in v_1 with $s(0) \neq 0$ and hence a unit in a neighborhood of 0. Note $t(0) = 0$ and more importantly $t(v_2)$ is not constant. Hence $dt(v_2)$ is non-vanishing at the general point of $L \cap U_x$. If $dt(v_2)(x) = 0$, then change x to make $dt(v_2)(x) \neq 0$.

Set $z_1 = v_1 s(v_1)$ and $z_2 = \frac{t(v_2) + v_1 g(v_1, v_2)}{s(v_1)^{m+1}}$. By construction $dz_1(x) \neq 0$, $dz_2(x) \neq 0$ and $dz_1 \wedge dz_2(x) \neq 0$ and

$$r = z_1(1 + z_1^m z_2)$$

giving the desired $w|_{U_x} = f(z_1, z_2) dz_1 d[z_1(1 + z_1^m z_2)]$ with $f = \frac{h}{s(v_1) + v_1 s'(v_1)}$.

Observing that $m = \text{ord}_{\{v_1=0\}}(\frac{\partial r}{\partial v_2}) - 1$, it follows that m is independent of the choice of v and r with dv and dr non-vanishing such that $w = hdvdr$ and the choice of holomorphic chart (v_1, v_2) with $v_1 = v$. The independence of m on the above choices plus the connectedness of L minus a discrete set of points implies that any other general point of L would give the same m and hence m is naturally associated to the irreducible component L.♦

Definition 3.3. An irreducible component L of a common leaf of $w \in H^0(X, S^2\Omega^1_X)$ of rank 2 is said to have order of contact m, $O(L, w) = m$, if in a neighborhood U_x of the general point $x \in L$ w is of the form as in Lemma 3.5, i.e., $w|_{U_x} = f(z_1, z_2)dz_1 d[z_1(1 + z_1^m z_2)]$.

Theorem 3.6. *Let X be a complex 2-manifold, $w \in H^0(X, S^2\Omega^1_X)$ be split, closed of rank 2 and L an irreducible component of a common leaf of w. Then the general point x in L has a neighborhood U_x where $w|_{U_x}$ has a decomposition of the form:*

$$w|_{U_x} = z_1^k(1 + z_1^m z_2)^\alpha e^{f(z_1)} e^{g(z_1(1+z_1^m z_2))} dz_1 d[z_1(1 + z_1^m z_2)]$$

where:

i) $m = O(L, w)$, $k = ord_L(w)_0$ ($(w)_0$ is the divisorial zero of w) and $\alpha = \frac{\log c}{2\pi i} + k$ for some $k \in \mathbb{Z}$ and $c \in M(L, w)$.
ii) f and g are meromorphic functions on Δ^* with poles of order at most m at 0.

Remark. The local form of $w|_{U_x}$ in the theorem can be rewritten as the following decomposition of $w|_{U_x}$ as the product of two closed 1-differentials (in a generalized sense since they might be multi-valued) with singularities along L:

$$w|_{U_x} = (z_1^\beta e^{f(z_1)} dz_1)([z_1(1 + z_1^m z_2)]^\alpha e^{g(z_1(1+z_1^m z_2))} d[z_1(1 + z_1^m z_2)])$$

with $\alpha + \beta = \text{ord}_L(w)_0$ and α, f and g as in the theorem.

Proof. According to the Lemma 3.5 the general point $x \in L$ has a neighborhood U_x with a holomorphic coordinate chart (z_1, z_2) such that $x = (0, 0)$, $L \cap U_x = \{z_1 = 0\}$ and $w|_{U_x} = v(z_1, z_2)dz_1 d[z_1(1 + z_1^m z_2)]$ with $v \in \mathcal{O}(U_x)$.

We claim that if we shrink U_x the divisor of zeros of $w|_{U_x}$ is $(v)_0 = kL$ and hence

$$w|_{U_x} = z_1^k \tilde{w}$$

with $\tilde{w} \in H^0(U_x, S^2\Omega^1_X)$ a nowhere vanishing closed symmetric differential of the form:

$$\tilde{w} = \tilde{v}(z_1, z_2)dz_1 d[z_1(1 + z_1^m z_2)] \quad (4)$$

with $\tilde{v}(z_1, z_2) \in \mathcal{O}^*(U_x)$.

By shrinking U_x we can make $(v)_0$ a finite union of irreducible components all passing through x. The differential w being closed implies (Theorem 3.4) that all $y \in U_x \setminus L$ have a neighborhood U_y such that $v|_{U_y} = f(z_1)g(z_1(1 + z_1^m z_2))$. This implies that if an irreducible component of $(v)_0$ is not L, then it must be a level set of z_1 or $z_1(1 + z_1^m z_2)$ not passing through x, a contradiction. It follows then that $(v)_0 = kL$ for some $k \in \mathbb{N}$ and (4) holds.

Note that we have the equality $M(L, \tilde{w}) = M(L, w)$, this can be seen, for example, by noting that the factor on the local holomorphic decompositions of w and $\tilde{w}$ corresponding to the foliation $d[z_1(1 + z_1^m z_2)]$ does not change (the 1-cocycle with values in $\mathbb{C}^*$ corresponding to this foliation remains unchanged) hence the index remains unchanged.

The neighborhood U_x can be chosen to be the bi-disc $U_x = \Delta_{\epsilon_1} \times \Delta_{\epsilon_2}$, $\epsilon_i > 0$, relative to the coordinate chart (z_1, z_2). On U_x we have two maps $\pi_1 : U_x \to \mathbb{C}$ given by $\pi_1(z_1, z_2) = z_1$ and $\pi_2 : U_x \to \mathbb{C}$ given by $\pi_2(z_1, z_2) = z_1(1 + z_1^m z_2)$.

Let $\mathcal{U} = \{U_i\}_{i=1,\ldots,k}$, $k \in \mathbb{N}$, be a Leray open covering of the punctured disc $\Delta^*_{\epsilon_1}$. The Leray covering $\{U_i \times \Delta_{\epsilon_2}\}_{i=1,\ldots,k}$ of $U_x \setminus \{z_1 = 0\}$ is such that one has the holomorphic closed decompositions on its open sets:

$$\tilde{w}|_{U_i \times \Delta_{\epsilon_2}} = \breve{f}_i(z_1)\breve{g}_i(z_1(1 + z_1^m z_2))dz_1 d[z_1(1 + z_1^m z_2)] \tag{5}$$

where $\breve{f}_i = \pi_1^* f_i$ with $f_i \in \mathcal{O}(U_i)$ and $\breve{g}_i = \pi_2^* g_i$ with $g_i \in \mathcal{O}(U'_i)$ ($U_i \subset U'_i = \pi_2(U_i \times \Delta_{\epsilon_2})$). The existence of such closed decomposition on the whole open sets $U_i \times \Delta_{\epsilon_2}$ is guaranteed since the open sets are simply connected and the fibers of both $\pi_1|_{U_i \times \Delta_{\epsilon_2}}$ and $\pi_2|_{U_i \times \Delta_{\epsilon_2}}$ are connected (assuming ϵ_1 and ϵ_2 are sufficiently small).

Since w is a symmetric differential of degree 2 and rank 2, the Abelian rank of w is trivial which due to Proposition 2.1 implies that on the intersections $U_i \cap U_j$:

$$f_i = c_{ij} f_j \qquad g_i = c_{ij}^{-1} g_j \tag{6}$$

The collection $\{c_{ij}\}$ defines a 1-cocycle in $Z^1(\mathcal{U}, \mathbb{C}^*)$ defining a representation of $\rho : \pi_1(\Delta^*_{\epsilon_1}) \to \mathbb{C}^*$. There is a natural homomorphism $\phi_L : \pi_1(\Delta^*_{\epsilon_1}) \to H_1(X \setminus B_w, \mathbb{Z})$ sending the class of a simple loop γ around the origin oriented positively to the class of a simple loop γ_L around the irreducible component L (as in Definition 3.2). By construction, $\rho(\gamma)$ is one of the diagonal entries of $\bar{\rho}_w(\gamma_L)$, i.e., $\rho(\gamma) \in$ M(L,w) $= \{c, c^{-1}\}$.

To simplify notation rescale the coordinates so that $U_x = \Delta \times \Delta$, Δ the unit disc centered at 0 and set the covering $\mathcal{U}$ of Δ^* to be $\{U_{-1}, U_0, U_1\}$ with $U_i = (0, 1) \times (\frac{(2i-1)}{3}\pi - \epsilon, \frac{(2i+1)}{3} + \epsilon)$, $\epsilon > 0$ sufficiently small, if expressed in polar coordinates.

Consider the universal covering map $e : \mathcal{H}^- \to \Delta^*$, $z \to e^z$, with $\mathcal{H}^- = \{z \in \mathbb{C} | \text{Re} z < 0\}$, and the open covering of $\mathcal{H}^-$, $\tilde{\mathcal{U}} = \{\tilde{U}_j\}_{j \in \mathbb{Z}}$ where the $\tilde{U}_j = (-\infty, 0) \times (\frac{(2j-1)}{3}\pi - \epsilon, \frac{(2j+1)}{3} + \epsilon)$. Note that $e : \tilde{U}_j \to U_{[j]}$, with $[j] \in \{-1, 0, 1\}$ and $j \equiv [j]$ mod 3, is a biholomorphism.

Let $\{\tilde{f}_j\}_{j\in\mathbb{Z}} \in C^0(\tilde{\mathcal{U}}, \mathcal{O}^*)$ be the 0-cochain defined by $\tilde{f}_j = f_{[j]} \circ e \in \mathcal{O}^*(\tilde{U}_j)$. The co-boundary $\delta\{\tilde{f}_j\}$ gives a 1-cocycle with values in $\mathbb{C}^*$, $\{\tilde{c}_{jj'}\} \in Z^1(\tilde{\mathcal{U}}, \mathbb{C}^*)$, since $\tilde{f}_j = \tilde{c}_{jj'}\tilde{f}_{j'}$ on $\tilde{U}_j \cap \tilde{U}_{j'}$. The space $\mathcal{H}^-$ being simply connected implies that $\{\tilde{c}_{jj'}\} \in B^1(\tilde{\mathcal{U}}, \mathbb{C}^*)$. Hence there is a collection $\{\tilde{c}_j\} \in C^0(\tilde{\mathcal{U}}, \mathbb{C}^*)$ such that $\tilde{c}_{j'}\tilde{f}_j = \tilde{c}_j\tilde{f}_j$ on $\tilde{U}_j \cap \tilde{U}_{j'}$ giving:

$$\{\tilde{c}_j\tilde{f}_j\} =: F \in \mathcal{O}^*(\mathcal{H}^-) \tag{7}$$

The function F, due to $\tilde{c}_{jj'} = c_{[j][j']}$ and the discussion following (6), satisfies the special transformation law

$$F(z + 2\pi i) = \rho(\gamma)F(z)$$

Since $e^{(\frac{\log\rho(\gamma)}{2\pi i})z}$ is function with the same transformation law as F, it follows that:

$$F = e^{(\frac{\log\rho(\gamma)}{2\pi i})z}\hat{f}(e^z)$$

with $\hat{f} \in \mathcal{O}^*(\Delta^*)$.

The above implies that if we set $c_i = \tilde{c}_i$, $i = -1, 0, 1$, then

$$c_i f_i|_{\hat{U}_i} = (z^{\frac{\log\rho(\gamma)}{2\pi i}}\hat{f})|_{\hat{U}_i}$$

with $z^{\frac{\log\rho(\gamma)}{2\pi i}}$ representing the principal branch of the power function and $\hat{U}_i = (0, 1) \times (\frac{(2i-1)}{3}\pi, \frac{(2i+1)}{3}\pi)$ (if expressed in polar coordinates). The same reasoning can be done with respect to the collection $\{g_i\}_{i=-1,0,1}$ using the collection $\{c_i^{-1}\}_{i=-1,0,1} \in C^0(\mathcal{U}, \mathbb{C}^*)$ to obtain

$$c_i^{-1}\, g_i|_{\hat{U}_i} = (z^{-\frac{\log\rho(\gamma)}{2\pi i}}\hat{g})|_{\hat{U}_i}$$

with $\hat{g} \in \mathcal{O}^*(\Delta^*)$ and $z^{-\frac{\log\rho(\gamma)}{2\pi i}}$ representing the principal branch of power function.

Finally, using the above descriptions of the collections $\{c_i\, f_i\}$ and $\{c_i^{-1}g_i\}$ it follows that we can rewrite the local holomorphic closed decompositions described in (5) by changing $\breve{f}_i$ and $\breve{g}_i$ to respectively $c_i\breve{f}_i$ and $c_i^{-1}\breve{g}_i$ and obtain the global decomposition of $\tilde{w}$ on $\Delta^* \times \Delta$:

$$\tilde{w}|_{\Delta^*\times\Delta} = (1 + z_1^m z_2)^{-\frac{\log\rho(\gamma)}{2\pi i}}\hat{f}(z_1)\hat{g}(z_1(1 + z_1 z_2))dz_1 d[z_1(1 + z_1^m z_2)] \tag{8}$$

recall that by construction $\rho(\gamma) \in M(L, w)$. Note that behind the global decomposition (8) there is a closed decomposition of $\tilde{w}$ on U_x but it involves the multi-valued functions and functions with singularities along L, $\tilde{w} = (z_1^{\frac{\log\rho(\gamma)}{2\pi i}}\hat{f}(z_1)dz_1)([z_1(1 + z_1^m z_2)]^{-\frac{\log\rho(\gamma)}{2\pi i}}\hat{g}(z_1(1 + z_1 z_2))d[z_1(1 + z_1^m z_2)])$.

The next goal is to understand the singularities that are possible for the functions $\hat{f}, \hat{g} \in \mathcal{O}^*(\Delta^*)$. To achieve this goal, we use the fact that the product of $\hat{f}(z_1)$ with $\hat{g}(z_1(1 + z_1^m z_2))$ extends to a holomorphic function on $\Delta \times \Delta$ since it satisfies:

$$(9) \qquad \hat{f}(z_1)\hat{g}(z_1(1 + z_1^m z_2))|_{\Delta^* \times \Delta} = \hat{v}(z_1, z_2)|_{\Delta^* \times \Delta}$$

where $\hat{v}(z_1, z_2) = \tilde{v}(z_1, z_2)(1 + z_1^m z_2)^{\frac{\log \rho(\gamma)}{2\pi i}} \in \mathcal{O}^*(\Delta \times \Delta)$.

The functions $\hat{f}$ and $\hat{g}$ do not necessarily have a well- defined logarithm on Δ^*, since $f_*, g_* : \pi_1(\Delta^*) \to \pi_1(\mathbb{C}^*)$ are not necessarily trivial. However, if we set $k_1 = f_*(\gamma) \in \pi_1(\mathbb{C}^*) = \mathbb{Z}$ and $k_2 = g_*(\gamma) \in \pi_1(\mathbb{C}^*) = \mathbb{Z}$ with γ a simple loop around 0 positively oriented, then $\hat{f}(z) = z^{k_1}\breve{f}(z)$ and $\hat{g}(z) = z^{k_2}\breve{g}(z)$ are such that the functions $\breve{f}, \breve{g} \in \mathcal{O}^*(\Delta^*)$ have well- defined logarithmic functions, $f = \log \breve{f}, g = \log \breve{g} \in \mathcal{O}(\Delta^*)$.

It follows from (9) that $\hat{f}(z)\hat{g}(z) = \hat{v}(z, 0)$ and hence $\hat{f}(z)\hat{g}(z) \in \mathcal{O}(\Delta^*)$ extends to a holomorphic function on Δ which forces $k_2 = -k_1$ ($(\hat{f}\hat{g})_* : \pi_1(\Delta^*) \to \pi_1(\mathbb{C}^*)$ is trivial since it factors through $\hat{v}(z, 0)_* : \pi_1(\Delta) \to \pi_1(\mathbb{C}^*)$ and $(\hat{f}\hat{g})_*(\gamma)) = f_*(\gamma) + g_*(\gamma)$). This implies that decomposition (8) can be rewritten as:

$$(10) \qquad \tilde{w} = (1 + z_1^m z_2)^{-\frac{\log \rho(\gamma)}{2\pi i} - k_1} e^{f(z_1)} e^{g(z_1(1+z_1 z_2))} dz_1 d[z_1(1 + z_1^m z_2)]$$

We are now interested in the singularities of $f, g \in \mathcal{O}(\Delta^*)$. It follows from (9) that:

$$(11) \qquad f(z_1) + g(z_1(1 + z_1^m z_2)) = \log \hat{v}(z_1, z_2)$$

where $\log \hat{v}(z_1, z_2) \in \mathcal{O}(U_x)$. To derive conditions on $f, g \in \mathcal{O}(\Delta^*)$ from (11) consider the Laurent series expansions:

$$f(z_1) = \sum_{i=-\infty}^{\infty} a_i z_1^i$$

$$g(z_1(1 + z_1^m z_2)) = \sum_{i=-\infty}^{\infty} b_i [z_1(1 + z_1^m z_2)]^i$$

for the sum $f(z_1) + g(z_1(1 + z_1^m z_2))$ to be holomorphic we must have

$$(12) \qquad \sum_{i=-\infty}^{-1} a_i z_1^i + \sum_{i=-\infty}^{-1} b_i [z_1(1 + z_1^m z_2)]^i =: r(z_1, z_2)$$

with $r(z_1, z_2)$ holomorphic. To simplify our notation, we quickly note that for $r(z_1, 0)$ to be holomorphic we must have $b_i = -a_i \ \forall i < 0$ from which it follows that

$$r(z_1, z_2) = \sum_{i=-\infty}^{-1} a_i z_1^i [1 - (1 + z_1^m z_2)^i]$$

Consider the expansion $1 - (1 + z_1^m z_2)^i = \sum_{k=1}^{\infty} c_k^{(i)} z_1^{km} z_2^k$. Of the coefficients $c_k^{(i)}$ we will only use the fact that they are all non-vanishing and $r(z_1, z_2) = \sum_{i=-\infty}^{-1} \sum_{k=1}^{\infty} a_i c_k^{(i)} z_1^{i+km} z_2^k$.

Reorganizing the terms of the last expansion of $r(z_1, z_2)$, one obtains:

$$r(z_1, z_2) = \sum_{j=-\infty}^{\infty} (\sum_{k=\min\{1, \ulcorner \frac{j}{m} \urcorner\}}^{\infty} a_{j-km} c_k^{(j-km)} z_2^k) z_1^j$$

The holomorphicity of $r(z_1, z_2)$ implies that $\forall j \leq -1$ the functions

$$s_j(z_2) = \sum_{k=1}^{\infty} a_{j-km} c_k^{(j-km)} z_2^k$$

must vanish, which using the non-vanishing of the $c_k^{(i)}$ implies that

$$a_i = 0 \qquad \forall i < -m$$

this jointly with the equality $b_i = -a_i \ \forall i < 0$ gives the desired result ii) stating that f and g are meromorphic functions with poles of order at most m at the origin.♦

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Aspects of the Geometry of Varieties with Canonical Singularities

Stefan Kebekus and Thomas Peternell

Abstract This survey reports on recent developments regarding the global structure of complex varieties which occur in the minimal model program.

Keywords Log-canonical space • klt singularity • Reflexive differential • Flat bundle • Étale fundamental group

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1 Introduction

This article is an extended version of two overview talks given by the authors in September 2013 at the Simons conference on "Foliation theory in algebraic geometry". We survey some recent developments regarding the global geometry of complex algebraic varieties with singularities occurring in the theory of minimal models. In other words, we are primarily interested in varieties with canonical or Kawamata log terminal (klt) singularities. More generally, we are interested in varieties X carrying a $\mathbb{Q}$-divisor such that (X, D) is klt, or perhaps log-canonical.

A typical problem is to understand the structure of complex projective varieties X with numerically trivial canonical class K_X. These are the minimal models of

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manifolds Y with Kodaira dimension $\kappa(Y) = 0$. If X happens to be smooth, which is rare in minimal model theory, then powerful methods from analysis, such as existence results for Kähler-Einstein metrics, can be applied to study the structure of X. In the singular case, however, new methods have to be developed, and we describe some steps along this way. For this, a good understanding of differential forms on varieties with canonical or klt singularities is required. Such an understanding is essential also in other circumstances, including moduli problems. Arguing along these lines, we show how the classical Decomposition Theorem for Kähler manifolds with vanishing first Chern class generalises to the singular case, to give a set of canonically defined foliations whose global geometry still needs to be explored.

In a similar vein, we recall a famous theorem of Yau, which asserts that any compact Kähler manifold with vanishing first and second Chern class is an étale quotient of a torus. Again, the proof of this result relies on the existence of a Kähler-Einstein metric. In case where X is projective and has canonical singularities we will prove an analogous statement, where the quotient is étale in codimension one. In a certain sense, this statement can be seen as saying that the foliations constructed above are trivial, and that the second Chern class might be understood as an obstruction against their triviality. The proof requires a good understanding of the difference of the algebraic fundamental group of X and that of its smooth locus. In particular, we need to understand the geometric meaning of the flatness of the smooth locus of X.

1.1 Outline of the Paper

We give a short description of the content of the paper. As it might have become clear, there are two main technical tools: the theory of good (=reflexive) differentials and the study of fundamental groups. Part I is devoted to the study of differential forms, with the technical core, the Extension Theorem, described in Section 2 and the applications being given in the subsequent Sections 3–7. Part II first discusses algebraic fundamental groups of varieties with klt singularities in Section 8, followed in Sections 9 and 10 by applications to varieties whose regular part is flat and to varieties with trivial canonical and trivial second Chern class. Finally, we mention how the topological main result completes the structure theory of Nakayama-Zhang on polarised endomorphisms.

1.2 Notation and Global Assumptions

Throughout the paper, we work over the complex number field. We use standard notation and follow the conventions of minimal model theory, as introduced in [Har77, KM98]. We will frequently consider *quasi-étale* morphisms, a concept which might be non completely standard: A finite, surjective morphism of normal varieties $\gamma : X \to Y$ is called *quasi-étale* if γ is étale in codimension one.

Part I: Extension of Differential Forms and Applications

2 Reflexive Differentials and the Extension Theorem

2.1 Statement of Result

We define the notion of reflexive differentials and state the main results of the paper [GKKP11] in this section. As there are other surveys available, we restrict ourselves to the minimal amount of material needed in later chapters and refer the reader to [Keb13a] for a more detailed introduction.

Definition 2.1. Let X be a normal variety or normal complex space. The sheaf of reflexive differentials on X is defined to be

$$\Omega_X^{[p]} := \left(\wedge^p \Omega_X^1\right)^{**},$$

where Ω_X^1 is the sheaf of Kähler differentials. If D is a reduced Weil divisor on X and if $\Omega_X^1(\log D)$ denotes the sheaf of Kähler differentials with logarithmic poles along D, then

$$\Omega_X^{[p]}(\log D) := \left(\wedge^p \Omega_X^1(\log D)\right)^{**}.$$

Notation 2.2. Let X be a normal variety or normal complex space. Given a coherent sheaf $\mathscr{A}$ on X and a positive number m, set $\mathscr{A}^{[m]} := (\mathscr{A}^{\otimes m})^{**}$. If $f : X' \to X$ is any morphism, then $f^{[*]}(\mathscr{A}) := (f^*(\mathscr{A}))^{**}$.

Notation 2.3. Let X be a normal variety or normal complex space and D a $\mathbb{Q}$-Weil divisor on X. A *log resolution of the pair* (X, D) is a birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, the exceptional locus E has pure codimension one and the set $\pi^{-1}(\operatorname{supp} D) \cup E$ is a divisor with simple normal crossing support. By Hironaka's theorem, log resolutions always exist.

In a simplified form, the main result of [GKKP11] can be stated as follows.

Theorem 2.4 ([GKKP11, Theorem 1.4]). *Let X be a quasi-projective variety such that $(X, 0)$ is klt. Let π be a log resolution of $(X, 0)$ and let p be any number. Then $\pi_*\Omega^p_{\tilde{X}} = \Omega^{[p]}_X$. Equivalently, $\pi_*\Omega^p_{\tilde{X}}$ is reflexive.* □

In the most general version, we consider a log-canonical pair (X, D). Then there exists a smallest closed algebraic set N such that (X, D) is klt outside N. The set N is called the non-klt locus of (X, D).

Theorem 2.5 ([GKKP11, Theorem 1.5]). *Let X be a quasi-projective variety carrying a $\mathbb{Q}$-Weil-divisor D such that (X, D) is log-canonical, with non-klt locus $N \subset X$. Let $\pi : \tilde{X} \to X$ be a log resolution with exceptional set E, and let $\tilde{D} \subset \tilde{X}$ be the largest reduced divisor contained in $\pi^{-1}(N)$. Then $\pi_*\Omega^p_{\tilde{X}}(\log \tilde{D}) = \Omega^{[p]}_X(\log D)$, for all numbers p.* □

We refer to both theorems as "extension theorems". In fact, Theorem 2.5 can be restated as follows. Given any open set $U \subseteq X$ with preimage $\tilde{U} = \pi^{-1}(U)$, Theorem 2.5 asserts that the restriction

$$\underbrace{H^0\big(\tilde{U},\ \Omega^p_{\tilde{X}}(\log \tilde{D})\big)}_{=H^0\left(U,\, \pi_* \Omega^p_{\tilde{X}}(\log \tilde{D})\right)} \to \underbrace{H^0\big(\tilde{U} \setminus E,\ \Omega^p_{X}(\log \tilde{D})\big)}_{=H^0\left(U,\, \Omega^{[p]}_X(\log D)\right)}$$

is surjective, and hence isomorphic.

2.2 Related Results

Building on results of Steenbrink-van Straten, [SvS85], Flenner proved in [Fle88] a version of Theorem 2.4 for $p \leq \operatorname{codim} X_{\mathrm{sing}}$, for all normal varieties and without any assumption on the nature of the singularities. Namikawa showed Theorem 2.4 for $p \in \{1, 2\}$, provided that X has canonical Gorenstein singularities, [Nam01, Section 1]. For further discussions we refer to [GKKP11].

We would like to emphasise that Theorems 2.4 and 2.5 are optimal if we want to have extension for all p. For examples and details, see [GKKP11, Section 3]. Relations to the notion of *Du Bois* singularities and pairs are discussed in [GK13, GK14]. Relations to h-differentials, the sheafification of Kähler differentials in Voevodsky's h-topology, are discussed in [JH13].

2.3 Extension in the Analytic Category

The Extension Theorems 2.4 and 2.5 are stated and proved in the algebraic category. In fact, the proof heavily uses parts of the minimal model program and certain vanishing theorems which are presently unavailable in the analytic category. However, there seems no reason why the Extension Theorem should not hold analytically. There is no problem to define notions as klt, log-canonical in the analytic category. In [GKP13a] a holomorphic version of Theorem 2.5 is established, provided the pair (X, D) is locally algebraic. This is to say that every point $p \in X$ admits an open Euclidean neighbourhood U which is open in a quasi-projective variety Y such that $D|_U$ is the restriction of a divisor on Y. Following a famous algebraisation result of M. Artin, [Art69, Theorem 3.8], examples for locally algebraic spaces are provided by complex spaces with isolated singularities. Other examples are given by Moishezon spaces. The best known result in the analytic category reads as follows.

Theorem 2.6 ([GKP13a, Section 2]). *Let X be a normal complex locally algebraic variety carrying a $\mathbb{Q}$-divisor D such that (X, D) is log-canonical, with non-klt locus*

$N \subset X$. Let $\pi : \tilde{X} \to X$ be a log resolution with exceptional set E. Let $\tilde{D}$ be the largest reduced divisor contained in $\pi^{-1}(N)$. Then $\pi_* \Omega^p_{\tilde{X}}(\log \tilde{D})$ is reflexive for all p. □

As pointed out, we expect that the Extension Theorems hold in full generality in the analytic category.

Conjecture 2.7. Theorem 2.6 holds without the assumption that X is locally algebraic.

3 Kodaira-Akizuki-Nakano Vanishing and the Poincaré Lemma

3.1 KAN Type Vanishing Results for Reflexive Differentials

Recall the statement of the classical Kodaira-Akizuki-Nakano Vanishing Theorem.

Theorem 3.1 ([AN54]). *Let X be a smooth projective variety and let $\mathscr{L}$ be an ample line bundle on X. Then*

(3.1.1) $$H^q\big(X,\, \Omega^p_X \otimes \mathscr{L}\big) = 0 \quad \text{for } p+q > n, \text{ and}$$

(3.1.2) $$H^q\big(X,\, \Omega^p_X \otimes \mathscr{L}^{-1}\big) = 0 \quad \text{for } p+q < n.$$

□

Assertions (3.1.1) and (3.1.2) are equivalent via Serre duality. Ramanujam [Ram72] gave a simplified proof of Theorem 3.1 and showed that it does not hold if one only requires $\mathscr{L}$ to be semi-ample and big. Esnault and Viehweg generalised Theorem 3.1 to logarithmic differentials, [EV86].

We ask for generalisations of Kodaira-Akizuki-Nakano vanishing to singular varieties, using reflexive differentials. In full generality, Kodaira-Akizuki-Nakano vanishing has been established for sheaves of reflexive differentials on varieties with quotient singularities, see [Ara88], as well as on toric varieties, see [CLS11, Theorem 9.3.1].

For varieties with more general types of singularities, vanishing results of KAN type are restricted to special values of p and q. It turns out that even for spaces with isolated terminal Gorenstein singularities, Theorem 3.1 does not hold for arbitrary $p+q > n$, respectively $p+q < n$. We begin the discussion with one generalisation of Assertion (3.1.2).

Theorem 3.2 ([GKP13a, Proposition 4.3]). *Let X be a normal projective variety of dimension n, let D be an effective $\mathbb{Q}$-divisor on X such that (X, D) is log-canonical, and let $\mathscr{L} \in \operatorname{Pic}(X)$ be an ample line bundle. Then*

$$H^0\big(X,\ \Omega_X^{[p]}(\log\lfloor D\rfloor)\otimes\mathscr{L}^{-1}\big)=0 \qquad \textit{for all } p<n, \ \textit{and} \tag{3.2.1}$$

$$H^1\big(X,\ \Omega_X^{[p]}(\log\lfloor D\rfloor)\otimes\mathscr{L}^{-1}\big)=0 \qquad \textit{for all } p<n-1. \tag{3.2.2}$$

If (X, D) *is additionally assumed to be dlt, then* $H^q\big(X,\ \mathscr{L}^{-1}\big)=0$ *for all* $q<n$. □

There are analogous generalisations of Assertion (3.1.1).

Theorem 3.3 ([GKP13a, Proposition 4.5]). *Let* X *be a normal projective variety of dimension* n*, let* D *be an effective* $\mathbb{Q}$*-divisor on* X *such that* (X, D) *is klt, and let* $\mathscr{L}\in \operatorname{Pic}(X)$ *be an ample line bundle. Then*

$$H^q\big(X,\ \omega_X\otimes\mathscr{L}\big)=0 \qquad \textit{for all } q>0, \textit{ and} \tag{3.3.1}$$

$$H^n\big(X,\ \Omega_X^{[p]}\otimes\mathscr{L}\big)=0 \qquad \textit{for all } p>0. \tag{3.3.2}$$

□

Example 3.4. The paper [GKP13a] exhibits a klt space X of dimension 4 and an ample line bundle $\mathscr{L}$ such that

$$H^2\big(X,\ \Omega_X^{[1]}\otimes\mathscr{L}^{-1}\big)\neq 0 \quad \text{and} \quad H^2\big(X,\ \Omega_X^{[3]}\otimes\mathscr{L}\big)\neq 0.$$

It follows that Kodaira-Akizuki-Nakano does not hold in full generality on a klt space, even when the space has only Gorenstein, terminal singularities. The example given in [GKP13a] starts with the threefold $Y := \mathbb{P}(T_{\mathbb{P}_2})$. Set $\tilde{X}=\mathbb{P}(\mathscr{O}_Y\oplus\mathscr{O}_Y(1))$ and let $\pi : \tilde{X}\to X$ be the contraction of the divisor $E=\mathbb{P}(\mathscr{O}_Y))$. Then $p=\pi(E)$ is a terminal Gorenstein singularity. The calculations for the cohomology groups are lengthy. We refer the reader to [GKP13a] for details.

3.2 Relation to the Poincaré Lemma for Reflexive Differential Forms

Needless to say, the Poincaré lemma is fundamental in the theory of complex manifolds. It is therefore natural to ask to which extent it holds also for reflexive differentials on singular. Results in this direction have been obtained by several authors, including Campana-Flenner, Greuel and Reiffen. The singularities discussed in their work are often isolated, rational or holomorphically contractible. A rather complete list of references is found in Jörder's paper [Jö14]. For locally algebraic klt spaces, the Poincaré lemma holds in degree one.

Theorem 3.5 ([GKP13a, Theorem 5.4]). *Let* X *be a normal complex space and* D *an effective* $\mathbb{Q}$*-divisor on* X *such that* (X, D) *is analytically klt and locally algebraic. Let* $\sigma\in H^0\big(X,\ \Omega_X^{[1]}\big)$ *be a closed holomorphic reflexive one-form on* X*. Then every*

$p \in X$ has an open neighbourhood U (in the Euclidean topology) and a holomorphic function $f \in \mathscr{O}_X(U)$ such that $\sigma|_U = df_U$. □

The notion of *analytic klt spaces*, which is rather self-explaining, is properly introduced in [GKP13a]. Notice that it is not difficult to construct counterexamples to Theorem 3.5 if (X, D) is only assumed to be log-canonical.

In his Freiburg Ph.D. thesis [Jö14] Jörder found a topological condition[1] which guarantees the validity of the Poincaré lemma in degree one, for normal, locally algebraic, complex spaces. Besides various other results, he showed that for projective varieties of dimension at least four with only one isolated rational singularity p, any failure of the Poincaré lemma in degree three yields

$$H^2\big(X,\, \Omega_X^{[q]} \otimes \mathscr{L}^{-1}\big) \neq 0, \quad \text{for any ample line bundle } \mathscr{L} \text{ over } X.$$

He also shows that the local divisor class groups of the singular points are obstructions to KAN type vanishing. In Example 3.4, it is the latter that give the non-vanishing, whereas the Poincaré lemma does hold everywhere. We refer the reader to [Jö14] for details. Poincaré lemmas in the context of h-differentials are also discussed there.

4 Varieties with Trivial Canonical Classes

4.1 Decomposition of Kähler Manifolds with Vanishing First Chern Class

We recall the famous structure theorem for Kähler manifolds with vanishing first Chern class.

Theorem 4.1 ([Bea83] and References There). *Let X be a compact Kähler manifold whose canonical divisor is numerically trivial. Then there exists a finite étale cover $X' \to X$ such that X' decomposes as a product*

$$X' = T \times \prod_{\nu} X_\nu$$

where T is a compact complex torus, and where the X_ν are irreducible and simply connected Calabi-Yau- or holomorphic-symplectic manifolds. □

Remark 4.2. Let X be a compact, simply connected Kähler manifold. We call X "Calabi-Yau" if $\omega_X \cong \mathscr{O}_X$ and $h^0\big(X,\, \Omega_X^p\big) = 0$ for all $p \notin \{0, \dim X\}$. We call

[1] Vanishing of a local intersection cohomology group.

X "irreducible holomorphic-symplectic", if $\omega_X \cong \mathscr{O}_X$ and if there exists a non-degenerate two-form whose wedge powers generate the ring of differential forms.

4.2 Decomposition of the Tangent Sheaf

Important as it is, the class of manifolds with vanishing first Chern class is too small from the point of view of birational classification of projective (or compact Kähler) manifolds. There, we are generally more interested in the structure of manifolds X with Kodaira dimension zero, $\kappa(X) = 0$. Conjecturally, any such X possesses a good minimal model X', which is $\mathbb{Q}$-factorial, has terminal singularities and a numerically trivial canonical divisor $K_{X'} \equiv_{\text{num}} 0$. Given one such X', a theorem of Kawamata, [Kaw85b, Theorem 1.1], asserts that there exists a positive number m such that $\mathscr{O}_X(mK_{X'}) \cong \mathscr{O}_{X'}$. We aim to prove a structure theorem for these varieties. Building on the Extension Theorem 2.4, the following infinitesimal analogue of Theorem 4.1 has been established in [GKP11].

Theorem 4.3 ([GKP11, Theorem 1.3]). *Let X be a normal, projective variety with at worst canonical singularities. Assume that the canonical divisor of X is numerically trivial, $K_X \equiv 0$. Then, there exists an Abelian variety A, a projective variety $\tilde{X}$ with at worst canonical singularities, a quasi-étale cover $f : A \times \tilde{X} \to X$, and a decomposition*

$$\mathscr{T}_{\tilde{X}} \cong \bigoplus \mathscr{E}_i$$

such that the following holds.

(4.3.1) The $\mathscr{E}_i$ are integrable saturated subsheaves of $\mathscr{T}_{\tilde{X}}$, with trivial determinants.

Further, if $g : \hat{X} \to \tilde{X}$ is any quasi-étale cover, then the following properties hold in addition.

(4.3.2) The sheaves $(g^*\mathscr{E}_i)^{**}$ are slope-stable with respect to any ample polarisation on $\hat{X}$.

(4.3.3) The irregularity of $\hat{X}$ is zero, $h^1\big(\hat{X}, \mathscr{O}_{\hat{X}}\big) = 0$.

Idea of Proof. Parts of the proof follow ideas of Bogomolov, [Bog74]. Consider a normal projective variety X as in Theorem 4.3. Set $n := \dim X$. From the work of Kawamata [Kaw85a] we obtain a quasi-étale cover $f : A \times X' \to X$ where A is an Abelian variety, where $\omega_{X'} \cong \mathscr{O}_{X'}$ and $q(X') = 0$, even after further quasi-étale covers of X'. Thus we will assume from now on that $q(X) = 0$, and we are allowed to pass to quasi-étale covers if we wish to do so.

Instead of decomposing $\mathscr{T}_X$ directly, we first show that there exists a decomposition in the ring of reflexive forms: given any number p and any reflexive form $\sigma \in H^0\big(X, \Omega_X^{[p]}\big)$, we show that there exists a complementary form $\tau \in H^0\big(X, \Omega_X^{[n-p]}\big)$ such that $\sigma|_{X_{\text{reg}}} \wedge \tau|_{X_{\text{reg}}}$ is a nowhere-vanishing top-form, defined on the smooth locus X_{reg}. In other words, we show that the natural pairing given by the wedge product,

$$\bigwedge : H^0\big(X,\, \Omega_X^{[p]}\big) \times H^0\big(X,\, \Omega_X^{[n-p]}\big) \longrightarrow H^0\big(X,\, \omega_X\big) \cong \mathbb{C}, \tag{4.3.4}$$

is non-degenerate. For this, we express the Pairing (4.3.4) in terms of Dolbeault cohomology. The Extension Theorem 2.4 and the fact that canonical singularities are rational allows to compare the relevant cohomology groups with those that exist on a resolution $\tilde{X}$ of singularities. Non-degeneracy of (4.3.4) comes out of non-degeneracy of the Serre duality pairings on $\tilde{X}$.

In order to construct a decomposition of the tangent sheaf, recall from Miyaoka's work [Miy87a, Miy87b] that the tangent sheaf $\mathscr{T}_X$ is slope-semistable with respect to any polarisation. Assuming that there exists a polarisation h where $\mathscr{T}_X$ is not stable, consider a destabilising subsheaf $\mathscr{E} \subsetneq \mathscr{T}_X$. It follows that the slope of $\mathscr{E}$ vanishes, $\mu_h(\mathscr{E}) = 0$, and it is easy to deduce from there that $c_1(\mathscr{E}) = 0$.

Passing to the *minimal dlt model*, we can assume that X is $\mathbb{Q}$-factorial, [BCHM10]. Using $\mathbb{Q}$-factoriality and $q(X) = 0$, we conclude that $\det \mathscr{E} \cong \mathscr{O}_X$, perhaps after passing to another étale cover. If $r := \operatorname{rank} \mathscr{E}$, we obtain a subsheaf

$$\det \mathscr{E} \cong \mathscr{O}_X \subset \Omega_X^{[r]}.$$

In other words, we have constructed a reflexive differential form $\sigma \in H^0\big(X,\, \Omega_X^{[r]}\big)$. Using the existence of a complementary form $\tau \in H^0\big(X,\, \Omega_X^{[r]}\big)$, one can show by linear algebra that $\mathscr{E}$ is a direct summand of $\mathscr{T}_X$. □

The proof of Theorem 4.1 uses the existence of a Ricci-flat Kähler-Einstein metric quite heavily. In the singular setting, the necessary differential-geometric tools, namely a Kähler-Einstein metric on the smooth locus of X with good boundary behaviour near the singularities, are not available so far—see [EGZ09] for recent developments in this direction. In order to pass from the infinitesimal decomposition of Theorem 4.3 to a physical decomposition of the variety as in Theorem 4.1, we would therefore propose to use different, more algebraic methods. The main problem is to show that the leaves of the foliation are algebraic, and then to analyse the structure of the closure of the leaves.

The following would be a conjectural analogue of Theorem 4.1. Together with the (conjectural) existence of good minimal models, a positive answer to this conjecture would give a rather satisfying structure theory for projective manifolds with vanishing Kodaira dimension.

Problem 4.4. Let X be a normal, $\mathbb{Q}$-factorial, projective variety with canonical singularities and trivial canonical class K_X. Suppose that $q(\hat{X}) = 0$ for all quasi-étale covers $\hat{X} \to X$. Then, there exists a quasi-étale cover $X' \to X$, such that X' is birational to a product

$$X' \sim_{\mathrm{biratl}} \prod_{\nu} X'_{\nu},$$

where the varieties X'_ν are $\mathbb{Q}$-factorial, with only canonical singularities, trivial canonical classes and the additional property that the tangent sheaf is *strongly stable*, that is, stable for any ample polarisation, even after passing to further quasi-étale covers.

4.3 Strongly Stable Varieties

Whether or not Problem 4.4 has a positive solution, canonical varieties with linearly trivial canonical divisor and strongly stable tangent bundle will be important building blocks in any structure theory for varieties of Kodaira dimension zero. In analogy to the distinction between *irreducible complex-symplectic* and *Calabi-Yau* manifolds, one can distinguish the following two basic types.

Definition 4.5. Let X be a normal projective variety with $K_X \cong \mathscr{O}_X$, having at worst canonical singularities.

(4.5.1) We call X *Calabi-Yau* if $H^0\big(\tilde{X}, \Omega_X^{[q]}\big) = 0$ for all numbers $0 < q < \dim X$ and all quasi-étale covers $\tilde{X} \to X$.

(4.5.2) We call X *irreducible holomorphic-symplectic* if there exists a reflexive 2-form $\sigma \in H^0\big(X, \Omega_X^{[2]}\big)$ such that σ is everywhere non-degenerate on X_{reg}, and such that for all quasi-étale covers $f : \tilde{X} \to X$, the exterior algebra of global reflexive forms is generated by $f^*(\sigma)$.

We expect that the dichotomy known from the smooth case will also hold for singular varieties.

Conjecture 4.6. Let X be a projective variety with canonical singularities. If $\omega_X \cong \mathscr{O}_X$ and if $\mathscr{T}_X$ is strongly stable, then X is either Calabi-Yau or irreducible holomorphic-symplectic, in the sense of Definition 4.5.

Remark 4.7. The converse of Conjecture 4.6 is known to hold: The tangent sheaf of any Calabi-Yau or irreducible symplectic variety is strongly stable, [GKP11, Proposition 8.20].

In the smooth case, Calabi-Yau- and irreducible complex-symplectic manifolds are distinguished by their holonomy representation. As this depends again on the Ricci-flat Kähler metric, we cannot use holonomy in the singular setting. However, the following theorem does provide some evidence that the conjecture might in fact still be true.

Theorem 4.8 ([GKP11, Propositions 8.15 and 8.21]). *Conjecture 4.6 holds if the dimension of X is no more than five.* □

Theorem 4.8 has been shown using stability properties of wedge powers of $\mathscr{T}_X$. In fact, one way to attack Conjecture 4.6 is to observe that in the smooth case, the classes of Calabi-Yau and irreducible holomorphic-symplectic manifolds are distinguished by stability properties of $\bigwedge^2 \mathscr{T}_X$.

Proposition 4.9. *Let X be a simply connected compact Kähler manifold with $c_1(X) = 0$. Fix an ample polarisation h. Then the following holds.*

(4.9.1) The manifold X is Calabi-Yau if and only if $\mathscr{T}_X$ and $\bigwedge^2 \mathscr{T}_X$ are both h-stable.
(4.9.2) The manifold X is irreducible symplectic if and only if $\mathscr{T}_X$ is h-stable and $\wedge^2 \mathscr{T}_X$ is h-semistable but not h-stable.

Idea of Proof. Using the Decomposition Theorem 4.1 and the smooth version of Theorem 4.3, we need only to show that the wedge power $\bigwedge^2 \mathscr{T}_X$ of a Calabi-Yau manifold X is h-stable. If not, consider a destabilising subsheaf $\mathscr{S} \subset \bigwedge^2 \mathscr{T}_X$, say of rank r. Since $\det \mathscr{S} = \mathscr{O}_X$, we obtain the non-vanishing

$$H^0\big(X, \bigwedge^r \bigwedge^2 \mathscr{T}_X\big) \neq 0.$$

However—using holonomy and representation theory—it is a standard fact, although possibly never stated explicitly in the literature, that, with $n = \dim X$,

$$H^0\big(X, \mathscr{T}_X^{\otimes m}\big) = \begin{cases} 1 & \text{if } m \text{ is a multiple of } n \\ 0 & \text{otherwise.} \end{cases}$$

If m is a multiple of n, the section comes from the direct summand $\mathscr{O}_X = (-aK_X) \subset \mathscr{T}_X^{\otimes m}$. This contradicts the above non-vanishing, since $\wedge^r \wedge^2 \mathscr{T}_X$ is a direct summand of some $\mathscr{T}_X^{\otimes m}$. □

To prove Conjecture 4.6 along these lines, a solution to the following problem would be needed.

Problem 4.10 ([GKP11, Problem 8.11]). Let X be a normal projective variety of dimension $n > 1$ with $K_X \cong \mathscr{O}_X$, having at worst canonical singularities. Assume that the tangent sheaf $\mathscr{T}_X$ is strongly stable. Then show that the following holds.

(4.10.1) For any odd numbers $q \neq n$ and any quasi-étale cover $\tilde{X} \to X$, we have $H^0\big(\tilde{X}, \Omega^{[q]}_{\tilde{X}}\big) = 0$.
(4.10.2) If there exists a quasi-étale cover $g : X' \to X$ and an even number $0 < q < n$ such that $H^0\big(X', \Omega^{[q]}_{X'}\big) \neq 0$, then there exists a reflexive 2-form $\sigma' \in H^0\big(X', \Omega^{[2]}_{X'}\big)$, symplectic on the smooth locus X'_{reg}, such that for any quasi-étale cover $f : \tilde{X} \to X'$, the exterior algebra of global reflexive forms on $\tilde{X}$ is generated by $f^*(\sigma')$. In other words,

$$\bigoplus_p H^0\Big(\tilde{X}, \Omega^{[p]}_{\tilde{X}}\Big) = \mathbb{C}\big[f^*(\sigma)\big].$$

It is shown in [GKP11, Proposition 8.21] that Problem 4.10 implies Conjecture 4.6. As indicated above, Problem 4.10 has been solved if $\dim X$ is at most five, [GKP11, Proposition 8.15]. It is certainly true if X is smooth, [GKP11, Proposition 8.13]. We expect that in (4.10.2), it will be unnecessary to pass to the cover $\tilde{X}$.

4.4 The Fundamental Group

The fundamental group $\pi_1(X)$ of a compact Kähler manifold X with $c_1(X) = 0$ is almost Abelian. In other words, there exists an Abelian subgroup in $\pi_1(X)$ of finite index. The proof of this result does not require the Structure and Decomposition Theorem 4.1, but nevertheless uses the existence of a Ricci-flat metric. A long-standing problem asks whether the same is true if only $\kappa(X) = 0$.

Conjecture 4.11. Let X be a projective (compact Kähler) manifold with $\kappa(X) = 0$. Then $\pi_1(X)$ is almost Abelian.

If X' is a minimal model of X, a result of Takayama [Tak03, Theorem 1.1], asserts that $\pi_1(X) \cong \pi_1(X')$. This leads us to conjecture the following.

Conjecture 4.12. Let X be a normal projective variety with at most terminal (canonical) singularities. If $K_X \equiv_{\mathrm{num}} 0$, then $\pi_1(X')$ is almost Abelian. If additionally $q(\tilde{X}) = 0$ for any quasi-étale cover $\tilde{X} \to X$, then $\pi_1(X)$ is finite.

The following result in this direction has been established. The proof relies on Campana's work, [Cam95], and on the methods introduced in Section 4.2.

Theorem 4.13 ([GKP11, Proposition 8.20]). *Let X be a normal, n-dimensional, projective variety with at worst canonical singularities. If K_X is torsion and if $\chi(X, \mathscr{O}_X) \neq 0$, then $\pi_1(X)$ is finite, of cardinality*

$$|\pi_1(X)| \leq \frac{2^{n-1}}{|\chi(X, \mathscr{O}_X)|}.$$

□

Theorem 4.14 ([GKP11, Corollary 8.25]). *Let X be a normal projective variety with at worst canonical singularities. Assume that $\dim X \leq 4$, and that the canonical divisor K_X is numerically trivial. Then $\pi_1(X)$ is almost Abelian, that is, $\pi_1(X)$ contains an Abelian subgroup of finite index.* □

The case $n = 3$ has been shown previously in [Kol95, 4.17.3].

5 Rationally Connected Varieties

5.1 Pluriforms on Rationally Connected Varieties

Rationally connected and rationally chain connected varieties play a prominent role in the structure theory of algebraic varieties. It is a basic fact that a rationally connected projective manifold X does not carry any pluriform, that is

(5.0.1) $$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \quad \forall m \in \mathbb{N}^+.$$

We refer the reader to [Kol96, IV.3.8] for a thorough discussion of this result. The key of the proof is the existence of many rational curves $C \subset X$ such that the restricted tangent bundle $T_X|_C$ is ample.

A well-known conjecture of Mumford asserts that (5.0.1) actually characterises rationally connected manifolds. This has been proven in dimension three by Kollár–Miyaoka–Mori, [KMM92, Thm. 3.2]. For an asymptotic version in any dimension, see [Pet06, CDP12]. As an immediate consequence of the Extension Theorem 2.4, the vanishing result (5.0.1) generalises to reflexive p-forms on spaces which support klt pairs.

Theorem 5.1 ([GKKP11, Theorem 5.1]). *Let X be a normal, rationally chain-connected projective variety. If there exists a $\mathbb{Q}$-divisor D on X such that (X, D) is klt, then $H^0\big(X, \Omega_X^{[p]}\big) = 0$ for all $1 \le p \le \dim X$.* □

Remark 5.2. At this point, the following remark might be useful. Let X be a normal, rationally chain-connected projective variety. If there exists a $\mathbb{Q}$-divisor D on X such that (X, D) is klt, then X is in fact rationally connected, cf. [HM07, Cor. 1.5].

It is natural to suspect that the vanishing (5.0.1) should also hold for pluriforms, that is, for section in reflexive tensor powers, $H^0\big(X, (\Omega^1_X)^{[m]}\big)$. Somewhat surprisingly, this is not always the case. This emphasises the fact that the statement of the Extension Theorem is not true for pluriforms. On the positive side, the following is known to hold.

Theorem 5.3 ([GKP13a, Theorem 1.3]). *Let X be a normal, rationally connected, projective variety. If X is factorial and has canonical singularities, then*

$$H^0\left(X, (\Omega^1_X)^{[m]}\right) = 0 \quad \textit{for all } m \in \mathbb{N}^+, \textit{ where } (\Omega^1_X)^{[m]} := \left((\Omega^1_X)^{\otimes m}\right)^{**}. \quad \square$$

Remark 5.4 (Relation Between Theorems 5.1 and 5.3). Let X be a normal space. Assume that there exists a $\mathbb{Q}$-divisor D on X such that (X, D) is klt. If X is factorial, then X has canonical singularities, cf. [KM98, Cor. 2.35].

Remark 5.5 (Necessity of the Assumption that X is Canonical). There are examples of rational surfaces X with log terminal singularities whose canonical bundle is torsion or even ample, cf. [Tot12, Example 10] or [Kol08, Example 43]. Since $H^0\big(X,\ \mathscr{O}_X(mK_X)\big) \subset H^0\big(X,\ (\Omega^1_X)^{[m\cdot \dim X]}\big)$, these examples show that the assumption that X has *canonical* singularities cannot be omitted in Theorem 5.3.

The proof of Theorem 5.3 uses the notion of semistable sheaves on singular spaces, where semistability is defined respect to a movable curve class α. We take the opportunity to correct an error in the proof given in [GKP13a]. An essential point in the proof is the fact that the reflexive tensor product of α-semistable sheaves is again α-semistable. In [GKP13a, Fact A.13], we referred to [CP11] for a proof, where however only the case that α is in the interior of the movable cone is established. The gap has been closed in [GKP14, Sect. 1.1.2 and Thm. 4.2].

Now, arguing by contradiction, one proceeds by analysing the maximal destabilising subsheaf $\mathscr{S}$ of a reflexive tensor power $(\Omega^1_X)^{[m]}$. The factoriality is used to conclude that $\det \mathscr{S}$ is a line bundle. This is important for calculations involving the restriction $\det \mathscr{S}|_C$: without the factoriality assumption, $\det \mathscr{S}|_C$ might contain torsion, which kills the argument. In fact, if $\mathscr{S}$ is a coherent sheaf on a smooth curve, then the positivity of $c_1(\mathscr{S})$ does not imply ampleness of $\mathscr{S}$. Instead, one might have $\mathscr{S} = A \oplus T$ with A a negative line bundle and T a (large) torsion sheaf.

Remark 5.6 (Theorem 5.3 in the $\mathbb{Q}$-Factorial Setting). If X is not factorial in Theorem 5.3, but still $\mathbb{Q}$-factorial, then not only the proof of Theorem 5.3 fails, but also the statement itself is false. A counterexample is given in [GKP13a, Example 3.7], by exhibiting a rationally connected surface S such that $H^0\big(S,\ (\Omega^1_S)^{[2]}\big) \neq 0$.

Two recent preprints of Wenhao Ou, [Ou13, Ou14], describe the structure of rationally connected surfaces and threefolds with canonical singularities carrying a non-zero pluriform.

Following [Cam95] in the smooth case, a refined Kodaira dimension can be defined also in the singular case.

Definition 5.7. Let X be a normal, $\mathbb{Q}$-factorial, projective variety. Set

$$\kappa^+(X) := \max\{\kappa(\det \mathscr{F}) \mid \mathscr{F} \subset \Omega^{[p]}_X \text{ a coherent subsheaf and } 1 \leq p \leq n\}.$$

Obviously, $\kappa^+(X) \geq \kappa(X)$. Unfortunately, $\kappa^+(X)$ does not behave well birationally, even when X has canonical singularities. In fact, [GKP13a, Example 3.7] exhibits a rational surface X supporting a rank-one, reflexive subsheaf $\mathscr{L} \subset \Omega^{[1]}_X$ such that $\mathscr{L}^{[2]}) = \mathscr{O}_X \subset (\Omega^1_X)^{[2]}$. Thus $\kappa^+(X) \geq 0$, whereas $\kappa^+(\hat{X}) = -\infty$ for any desingularisation $\hat{X}$ of X.

5.2 *The Tangent Bundle of Rationally Connected Varieties*

As already mentioned, a rationally connected manifold X carries many rational curves C such that $\mathscr{T}_X|_C$ is ample. It is natural to ask whether this generalises to klt varieties: assume that (X, Δ) is klt or that X has only canonical singularities. If X is rationally (chain) connected, can one find rational curves C through the general point of X such that $\mathscr{T}_X|_C$ is ample?[2] The answer is negative in general.

Proposition 5.8. *Let (X, Δ) be klt and rationally connected. Suppose that $H^0(X, (\Omega_X^1)^{[m]}) \neq 0$ for some m. Then there is no irreducible curve C through the general point of X, such that $\mathscr{T}_X|_C$ is ample. In particular, there does not exist a rational curve C not meeting the singular locus of X such that $\mathscr{T}_X|_C$ is ample.*

Proof. Fix a non-zero form $\omega \in H^0\big(X, (\Omega_X^1)^{[m]}\big) \neq 0$. Suppose to the contrary and assume that there is an irreducible curve C through the general point p of X, such that $\mathscr{T}_X|_C$ is ample. The form ω defines a morphism

$$\lambda : (\mathscr{T}_X^{\otimes m})^{**} =: \mathscr{T}_X^{[m]} \to \mathscr{O}_X.$$

Restricting to C and observing that C passes through a general point of X, we obtain a non-zero morphism

$$\lambda_C : \mathscr{T}_X^{[m]}|_C \to \mathscr{O}_C.$$

On the other hand, since $\mathscr{T}_X|_C$ is ample, so is $\mathscr{T}_X^{\otimes m}|_C = (\mathscr{T}_X^{\otimes m})|_C$. Using the generically injective map $\mathscr{T}_X^{\otimes m}|_C \to \mathscr{T}_X^{[m]}|_C$, we conclude that $\mathscr{T}_X^{[m]}|_C$ is ample. Hence $\lambda_C = 0$, a contradiction. □

5.3 Related and Complementary Results

In contrast to the non-existence of differential forms on rationally chain connected spaces non-existence of Kähler-differentials modulo torsion holds without any assumption as to the nature of the singularities.

Theorem 5.9 ([Keb13b, Theorem 4.1]). *Let X be a reduced, projective scheme. Assume that X is rationally chain connected. Then $H^0\big(X, \Omega_X^p/\mathrm{tor}\big) = 0$, for all p.* □

We do not assume that X is irreducible. The statement of Theorem 5.9 becomes wrong if one replaces Ω_X^p/tor with Kähler differentials. Examples are given in [Keb13b, Section 4]. There are related results for h-differentials, [JH13].

[2] Observe that the sheaf $\mathscr{T}_X$ need not be locally free. We refer the reader to [Anc82, Section 2] for the definition of ampleness for arbitrary coherent sheaves.

6 The Lipman-Zariski Conjecture

The Lipman-Zariski Conjecture [Lip65, page 874], originally stated as a question, asserts that a normal variety X whose tangent sheaf $\mathscr{T}_X$ is locally free, is smooth. Besides work of Lipman, the first results in this direction concern hypersurfaces and homogeneous complete intersections, and are due to Scheja-Storch [SS72, Chapter 9] and Hochster [Hoc75]. Generalising previous results by Steenbrink and van Straten, [SvS85], Flenner [Fle88] proved the Lipman-Zariski Conjecture if the singular locus of X has codimension at least 3. Källström established the conjecture for complete intersections, [Kä11].

As a consequence of the Extension Theorem 2.4, we obtain the conjecture in the klt case, where the singular locus is of codimension two in general.

Theorem 6.1 ([GKKP11, Theorem 6.1]). *Let X be a normal, projective klt variety. In other words, assume that $(X, 0)$ is klt. If the tangent sheaf $\mathscr{T}_X$ is locally free, X is smooth.*

Idea of Proof. Like most other proofs of special cases of the Lipman-Zariski Conjecture, Theorem 6.1 is shown by lifting differential forms to a resolution of singularities. In our case, the Extension Theorem 2.4 allows to do that. We argue by contradiction and assume that X is singular while $\mathscr{T}_X$ is locally free. Choose the so-called *functorial* or *canonical* resolution $\pi : \tilde{X} \to X$, which is a log resolution that commutes with smooth morphisms, see [Kol07]. By possibly shrinking X, we may assume that $\mathscr{T}_X$ is locally free; choose a basis $\theta_1, \dots, \theta_n$. These vector fields lift by [GKK10, Corollary 4.7] to logarithmic vector fields

$$\tilde{\theta}_j \in H^0\big(\tilde{X},\ \mathscr{T}_{\tilde{X}}(-\log E)\big).$$

Choose the dual basis outside E to obtain differential forms

$$\omega_j \in H^0\big(\tilde{X} \setminus E,\ \Omega^1_{\tilde{X}}\big), \quad \text{for all } 1 \le j \le n.$$

By the Extension Theorem 2.4, the ω_j are actually holomorphic forms on all of $\tilde{X}$. The identity $\omega_i(\tilde{\theta}_j) = \delta_{i,j}$ therefore holds everywhere on $\tilde{X}$. However, the $\tilde{\theta}_j$ are tangent to the exceptional divisor, providing a contradiction. □

6.1 Further Generalisations

Recently, Graf, Graf-Kovács [GK13] and Druel [Dru13] generalised Theorem 6.1 to the log-canonical case. Druel's proof is independent of the Extension Theorem and instead uses foliation theory, while Graf-Kovács use an Extension Theorem for Du Bois pairs. Finally, we mention that Jörder proved the Lipman-Zariski Conjecture in case where $\mathscr{T}_X$ has a local basis of *commuting* vector fields [Jö13], and in case where there exists holomorphic $\mathbb{C}^*$ action with non-negative weights whose fixed point locus is not contained in the singular locus of X, [Jö14].

7 Bogomolov-Sommese Vanishing and Hyperbolicity of Moduli Spaces

The Extension Theorem 2.5 has been applied to prove hyperbolicity properties of moduli spaces. One of the further key ingredients is a generalisation of the Bogomolov-Sommese Vanishing Theorem to singular varieties. Since these matters are explained in quite some detail in the survey paper [Keb13a], we only recall the most important results here.

The most general version of the Bogomolov-Sommese vanishing is due to Graf [Gra13], generalising [GKKP11, Theorem 7.2]. We refrain from stating the most general form, which works in the context of "Campana orbifolds" or "$\mathcal{C}$-pairs", but just cite the following, more intuitive version.

Theorem 7.1 ([Gra13, Theorem 1.3]). *Let* (X, D) *be a normal, projective, log-canonical pair. Assume that* $\mathscr{A} \subset \Omega_X^{[p]}(\log\lfloor D \rfloor)$ *is a reflexive sheaf of rank* 1*. Then* $\kappa(\mathscr{A}) \leq p$*.* □

It applies to moduli problems in the following way.

Theorem 7.2 ([KK10, Corollary 1.3]). *Let* $f^\circ : X^\circ \to Y^\circ$ *be a smooth projective family of canonically polarised varieties, over a quasi-projective manifold* Y° *of dimension* $\dim Y^\circ \leq 3$*. Then either*

(7.2.1) $\kappa(Y^\circ) = -\infty$ *and* $\operatorname{Var}(f^\circ) < \dim Y^\circ$*, or*
(7.2.2) $\kappa(Y^\circ) \geq 0$ *and* $\operatorname{Var}(f^\circ) \leq \kappa(Y^\circ)$*.*

Remark 7.3. Recall that by definition, $\kappa(Y^\circ) = \kappa(K_Y + D)$, where Y is a smooth projective compactification and $D = Y \setminus Y^\circ$.

Idea of Proof. Consider the case where $Y := Y^\circ$ is projective and K_Y linearly trivial. By Miyaoka's work [Miy87a, Miy87b], the sheaf of differential forms, Ω_Y^1 will then be semistable with respect to any polarisation. Now, if there was a family $f^\circ : X^\circ \to Y^\circ$ of positive variation, it has been shown by Viehweg-Zuo that a suitable symmetric product of Ω_Y^1 contains a positive subsheaf, violating semistability.

If Y° is not projective, then it can be compactified to Y by adding a boundary divisor D with simple normal crossings. Assume for simplicity that $K_Y + D \equiv_{\text{num}} 0$ and that the Picard-Number of $Y_{\min}$ is one, so that any line bundle is either numerically trivial, ample or anti-ample. In this setting, Bogomolov-Sommese vanishing can be used to replace Miyaoka's semistability argument, which is not available in the presence of boundary divisors: by Viehweg-Zuo, the existence of a non-trivial family would imply that $\Omega^1_Y(\log D)$ is not semistable. However, any maximally destabilising subsheaf would automatically be ample, violating Bogomolov-Sommese.

If the simplifying assumptions are not satisfied and if $\dim Y^\circ \le 3$, then one can apply the minimal model program to come to a singular space $Y_{\min}$ with numerically trivial log-canonical class. With sufficient technical work, the Extension Theorem allows to work on these spaces, and to adopt the ideas sketched above. □

Theorem 7.2 is in fact a consequence of the following more general result.

Theorem 7.4 ([JK11, Theorem 1.5]). *Let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarised varieties over a smooth quasi-projective base. If Y° is* special in the sense of Campana*, then the family f is isotrivial.* □

Remark 7.5. In case where Y° is compact, a somewhat weaker version of Theorem 7.2 has been shown in all dimensions by Patakfalvi, [Pat12]. Generalisations of Theorems 7.2 and 7.4 to all dimensions are contained in a preprint by Campana-Paun, [CP13], and in the upcoming PhD thesis of Behrouz Taji.

Part II: Local Fundamental Groups and Étale Covers

8 Étale Covers of a klt Space and Its Smooth Locus

8.1 Finiteness of Obstructions to Extending Finite Étale Covers from the Smooth Locus

Working with a singular complex algebraic variety X, one is often interested in comparing the set of finite étale covers of X with that of its smooth locus X_{reg}. More precisely, one may ask the following.

Question 8.1. What are the obstructions to extending finite étale covers of X_{reg} to X? How do the étale fundamental groups of X and of its smooth locus differ?

Our motivation to consider this question came from the study of varieties with canonical singularities and numerical trivial canonical classes and vanishing second Chern class in a suitable sense. This will be discussed in the subsequent Section 9.

Remark 8.2. If X is normal, then it is a basic fact that the natural push-forward map between étale fundamental groups,

$$\hat{\iota}_* : \hat{\pi}_1(\tilde{X}_{\text{reg}}) \to \hat{\pi}_1(\tilde{X}), \tag{8.4.1}$$

is surjective. Question 8.1 therefore asks for conditions to guarantee injectivity.

Building on recent boundedness theorem of Hacon-M^c^Kernan-Xu for $\mathbb{Q}$-Fano klt pairs, [HMX12, Corollary 1.8], Chenyang Xu recently gave a complete answer for klt spaces with isolated singularities.

Theorem 8.3 ([Xu12, Theorem 1]). *Let $0 \in (X, \Delta)$ be an analytic germ of an algebraic klt singularity. Then the algebraic local fundamental group $\hat{\pi}_1^{\mathrm{loc}}(X, 0)$ is finite.* □

In the setting of Theorem 8.3, recall that $0 \in X$ admits a basis of neighbourhoods U which are homeomorphic to the topological cone over the link $\mathrm{Link}(X, 0)$ The local fundamental group of $0 \in X$ is defined as the usual topological fundamental group of the link, that is, $\hat{\pi}_1^{\mathrm{loc}}(X, 0) := \pi_1\big(\mathrm{Link}(X, s)\big)$. The algebraic local fundamental group is its profinite completion.

Problem 8.4. It is an open question whether an analogue of Theorem 8.3 holds for the local fundamental group, [Kol11, Question 26] and [Xu12, Conjecture 1].

Building on Xu's result, the paper [GKP13b] establishes the following answer to Question 8.1. Recall from the Section 1.2 that a finite surjective morphism $f : X \to Y$ between normal varieties is *quasi-étale*, if it is étale outside a set of codimension two. Equivalently, if f is étale over the smooth locus of Y.

Theorem 8.5 ([GKP13b, Theorem 1.4]). *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Then, there exists a normal variety $\tilde{X}$ and a quasi-étale, Galois morphism $\gamma : \tilde{X} \to X$, such that the following, equivalent conditions hold.*

(8.5.1) Any finite, étale cover of $\tilde{X}_{\mathrm{reg}}$ extends to a finite, étale cover of $\tilde{X}$.
(8.5.2) The natural map $\hat{\iota}_* : \hat{\pi}_1(\tilde{X}_{\mathrm{reg}}) \to \hat{\pi}_1(\tilde{X})$ of étale fundamental groups induced by the inclusion of the smooth locus, $\iota : \tilde{X}_{\mathrm{reg}} \to \tilde{X}$, is an isomorphism. □

A few remarks and comments are perhaps in order. First of all, in Theorem 8.5 and throughout this paper, Galois morphisms are assumed to be finite and surjective, but need not be étale. Second, despite appearance to the contrary, Theorem 8.5 does *not* imply that the kernel of the push-forward morphism (8.4.1) is finite for all klt spaces. A counterexample is discussed in [GKP13b, Section 14.2]. Third, we point out that the variety $\tilde{X}$ of Theorem 8.5 is not unique. In fact, it is shown in [GKP13b, Section 14.3] by way of example that a unique, minimal choice of $\tilde{X}$ cannot exist in general.

8.2 Generalisations

Theorem 8.5 is in fact a corollary of the following, more general and much more involved result. In essence, Theorem 8.6 asserts that in any infinite tower of quasi-étale Galois morphisms over any sequence of increasingly smaller and smaller subsets of X, all but finitely many of the morphisms must in fact be étale.

Theorem 8.6 ([GKP13b, Theorem 2.1]). *Let X be a normal, complex, quasi-projective variety of dimension $\dim X \geq 2$. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Suppose further that we are given a descending chain of dense open subsets $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$, a closed reduced subscheme $S \subset X$ of codimension $\operatorname{codim}_X S \geq 2$, and a commutative diagram of morphisms between normal varieties,*

(8.6.1)
$$\begin{array}{ccccccccccc} & & Y_0 & \xleftarrow{\gamma_1} & Y_1 & \xleftarrow{\gamma_2} & Y_2 & \xleftarrow{\gamma_3} & Y_3 & \xleftarrow{\gamma_4} & \cdots \\ & & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\ X & \xleftarrow[\iota_0]{} & X_0 & \xleftarrow[\iota_1]{} & X_1 & \xleftarrow[\iota_2]{} & X_2 & \xleftarrow[\iota_3]{} & X_3 & \xleftarrow[\iota_4]{} & \cdots, \end{array}$$

where the following holds for all indices $i \in \mathbb{N}$.

(8.6.2) The morphisms ι_i are the inclusion maps.
(8.6.3) The morphisms γ_i are quasi-finite, dominant and étale away from the reduced preimage set $S_i := \eta_i^{-1}(S)_{\mathrm{red}}$.
(8.6.4) The morphisms η_i are finite, surjective, Galois, and étale away from S_i.

Then, all but finitely many of the morphisms γ_i are étale. Further, if S is not empty, then there exists an open subset $S^\circ \subseteq S$ and a number $N_S \in \mathbb{N}^+$, both depending only on X and S, such that the following holds.

(8.6.5) Setting $S' := S \setminus S^\circ$, we have $\dim S' < \dim S$.
(8.6.6) Given any index $i \in \mathbb{N}$ and any point $y \in \eta_i^{-1}(S^\circ)$, the ramification index of η_i^{an} at y is bounded by N_S, that is, $r\big(\eta_i^{an}, y\big) < N_S$. □

The setup of Theorem 8.6 is illustrated in Fig. 1. To better understand its meaning and its relation to Theorem 8.5, it is useful to consider Theorem 8.6 in the special case where $X = X_0 = X_1 = X_2 = \cdots = Y_0$, where the morphisms γ_i are finite and surjective, and where the morphisms η_i are of the form

$$\eta_i = \begin{cases} \mathrm{Id}_X & \text{if } i = 0 \\ \gamma_1 \circ \cdots \circ \gamma_{i-1} \circ \gamma_i & \text{if } i > 0 \end{cases}$$

Under these assumptions, Theorem 8.6 reduces to the following.

Theorem 8.7 ([GKP13b, Theorem 1.1]). *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Assume we are given a sequence of quasi-étale morphisms,*

(8.7.1)
$$X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} Y_3 \xleftarrow{\gamma_4} \cdots.$$

If the composed morphisms $\gamma_1 \circ \cdots \circ \gamma_i : Y_i \to X$ are Galois for every $i \in \mathbb{N}^+$, then all but finitely many of the morphisms γ_i are étale. □

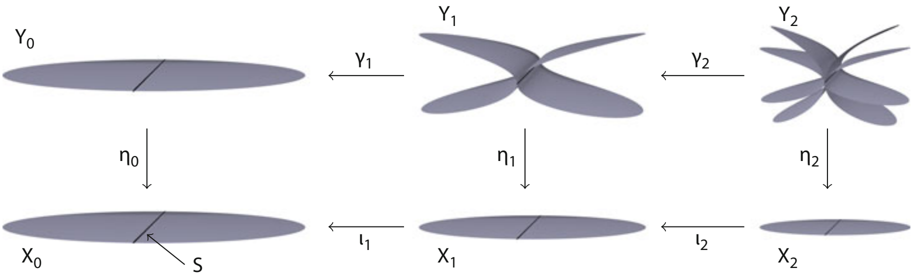

Fig. 1 Setup of Theorem 8.6. The figure shows the setup for the main result, Theorem 8.6, schematically. The morphisms η_i are Galois covers over a sequence $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$ is increasingly small open subsets of X. The morphisms γ_i between these covering spaces are étale away from the preimages of S. In Theorem 8.6, the set S is of codimension two or more. This aspect is difficult to illustrate and therefore not properly shown in the figure

Remark 8.8. By purity of the branch locus, the assumption that all morphisms γ_i of Theorem 8.7 are quasi-étale can also be formulated in one of the following, equivalent ways.

(8.8.1) All morphisms $\gamma_1 \circ \cdots \circ \gamma_i$ are étale over the smooth locus of Y_0.
(8.8.2) All morphisms γ_i are étale over the smooth locus of Y_{i-1}.

Theorem 8.5 quickly follows from Theorem 8.7 by assuming to the contrary: if no cover of X satisfied the conclusion of Theorem 8.5, we could inductively construct a tower of morphisms that are étale over the smooth loci, but not everywhere étale. Passing to the appropriate Galois closures, we can always achieve that the morphisms are Galois over X.

8.3 Idea of Proof

The main idea for the proof of Theorem 8.6 is roughly formulated as follows. If X has isolated singularities, then Theorem 8.6 can be easily deduced from Xu's work and nothing new needs to be done. If the singularities of X are not isolated, we employ Verdier's topological triviality of algebraic morphisms, [Ver76], in order to construct a suitable Whitney stratification. Then argue inductively, stratum-by-stratum, using cutting-down-arguments to reduce to the case of isolated singularities.

8.4 Immediate Applications

An array of morphisms as in Theorem 8.6 can inductively be constructed by fixing a point p of a klt space X, by choosing Weil divisors $D_i \subset Y_i$ that are $\mathbb{Q}$-Cartier near the preimages of p and taking the associated index-one-covers for the morphisms γ_i. The assertion that almost all morphisms γ_i are étale then implies that the divisors in question were Cartier near the preimages of p. This way, one constructs a "simultaneous index-one cover" for all divisors that are $\mathbb{Q}$-Cartier in a neighbourhood of p.

Theorem 8.9 ([GKP13b, Theorem 1.9]). *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Let $p \in X$ be any closed point. Then, there exists a Zariski-open neighbourhood X° of $p \in X$, a normal variety $\tilde{X}^\circ$ and a quasi-étale, Galois morphism $\gamma : \tilde{X}^\circ \to X^\circ$, such that the following holds for any Zariski-open neighbourhood $U = U(p) \subseteq X^\circ$ with preimage $\tilde{U} = \gamma^{-1}(U)$.*

(8.9.1) If $\tilde{D}$ is any $\mathbb{Q}$-Cartier divisor on $\tilde{U}$, then $\tilde{D}$ is Cartier.
(8.9.2) If D is any $\mathbb{Q}$-Cartier divisor on U, then $(\#\,\mathrm{Gal}(\gamma)) \cdot D$ is Cartier. □

Remark 8.10 ([GKP13b, Remark 1.10]). Under the assumptions of Theorem 8.9, there exists a number $N \in \mathbb{N}^+$ such that $N \cdot D$ is Cartier, whenever D is a $\mathbb{Q}$-Cartier divisor on X.

For applications on the global structure of Kähler space, as given below in the algebraic setting, it is highly desirable to extend the results presented in this section to the analytic category.

9 Flatness Criteria and Characterisation of Torus Quotients

9.1 Extension Results for Flat Sheaves

We aim to apply Theorem 8.5 to the study of flat sheaves on klt spaces. Since we are dealing with singular spaces, we do not attempt to define flat sheaves via connections. Instead, a flat sheaf $\mathscr{F}$ will always be an analytic, locally free sheaf, given by a representation of the fundamental group. More precisely, we will use the following definition.

Definition 9.1. If Y is any complex space, and $\mathscr{G}$ is any locally free sheaf on Y, we call $\mathscr{G}$ *flat* if it is defined by a representation of the topological fundamental group $\rho : \pi_1(Y) \to \mathrm{GL}_{\mathrm{rank}\,\mathscr{G}}(\mathbb{C})$. A locally free, algebraic sheaf on a complex algebraic variety Y is called flat if and only if the associated analytic sheaf on the underlying complex space Y^{an} is flat.

Now consider a normal variety X and a flat, locally free, analytic sheaf $\mathscr{F}^\circ$, defined on the complex manifold X^{an}_{reg}. We aim to extend $\mathscr{F}^\circ$ across the singularities, to a coherent sheaf that is defined on all of X. Unlike in the algebraic case, where extension over subsets of codimension two is easy, the extension problem for coherent analytic sheaves is generally hard. For flat sheaves, however, a fundamental theorem of Deligne, [Del70, II.5, Corollary 5.8 and Theorem 5.9], asserts that $\mathscr{F}^\circ$ is algebraic, and thus extends to a coherent, algebraic sheaf $\mathscr{F}$ on X. If the algebraic fundamental groups of X and X_{reg} agree, the following theorem shows that Deligne's extended sheaf $\mathscr{F}$ is again locally free and flat.

Theorem 9.2 ([GKP13b, Section 11.1]). *Let X be a normal, complex, quasi-projective variety, and assume that the natural inclusion map between étale fundamental groups, $\hat{\imath}_* : \hat{\pi}_1(\tilde{X}_{\text{reg}}) \to \hat{\pi}_1(\tilde{X})$, is isomorphic. If $\mathscr{F}^\circ$ is any flat, locally free, analytic sheaf defined on the complex manifold X^{an}_{reg}, then there exists a flat, locally free, analytic sheaf $\mathscr{F}$ on X^{an} such that $\mathscr{F}^\circ = \mathscr{F}|_{X^{an}_{\text{reg}}}$.*

Sketch of Proof. Set $Y := X^{an}$ and $Y^\circ := X^{an}_{\text{reg}}$. The sheaf $\mathscr{F}^\circ$ then corresponds to a representation $\rho^\circ : \pi_1(Y^\circ) \to \mathrm{GL}\big(\operatorname{rank}\mathscr{F}, \mathbb{C}\big)$. We need to show that this representation is induced by a representation of $\pi_1(Y)$. This is trivially true if the natural, surjective push-forward map of fundamental groups, $\iota_* : \pi_1(Y^\circ) \to \pi_1(Y)$ was known to be isomorphic. Our assumptions, however, guarantee only that the induced map $\hat{\imath}_*$ between profinite completions is an isomorphism.

Write $G := \operatorname{img}(\rho^\circ)$. As a finitely generated subgroup of the general linear group, G is residually finite by Malcev's theorem. Consequently, the profinite completion morphism $a : G \to \hat{G}$ is injective. The remaining proof is now purely group-theoretic. □

A combination of Theorems 8.5 and 9.2 immediately gives the following consequence.

Theorem 9.3 ([GKP13b, Theorem 1.13]). *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Then, there exists a normal variety $\tilde{X}$ and a quasi-étale, Galois morphism $\gamma : \tilde{X} \to X$, such that the following holds. If $\mathscr{G}^\circ$ is any flat, locally free, analytic sheaf on the complex space $\tilde{X}^{an}_{\text{reg}}$, there exists a flat, locally free, algebraic sheaf $\mathscr{G}$ on $\tilde{X}$ such that $\mathscr{G}^\circ$ is isomorphic to the analytification of $\mathscr{G}|_{\tilde{X}_{\text{reg}}}$.* □

9.2 Flatness Criteria

Theorem 9.3 can be used to show that many classical flatness criteria for semistable vector bundles, cf. [UY86, Kob87, Sim92, BS94], generalise to spaces with klt singularities, at least after passing to a suitable quasi-étale cover whose étale fundamental group coincides with that of its smooth locus.

9.2.1 Chern Classes on Singular Spaces

In view of the applications, we are mostly interested in flatness criteria for semistable sheaves with vanishing first and second Chern classes. The literature discusses several competing notions of Chern classes on singular spaces, all of which are technically challenging, cf. [Mac74, Alu06]. We will restrict ourselves to the following elementary definition, which suffices in our case. We refer the reader to [GKP13b, Section 4] for more details.

Definition 9.4. Let X be a normal variety and $\mathscr{E}$ a coherent sheaf of $\mathscr{O}_X$-modules. A *resolution of* $(X, \mathscr{E})$ is a proper, birational and surjective morphism $\pi : \tilde{X} \to X$ such that the space $\tilde{X}$ is smooth, and such that the sheaf $\pi^*(\mathscr{E})/_{\text{tor}}$ is locally free. If π is isomorphic over the open set where X is smooth and $\mathscr{E}/_{\text{tor}}$ is locally free, we call π a *strong resolution of* $(X, \mathscr{E})$.

The existence of a resolution of singularities combined with a classical result of Rossi, [Ros68, Thm. 3.5], shows that resolutions and strong resolutions of $(X, \mathscr{E})$ exist.

Definition 9.5. Let X be a normal, n-dimensional, quasi-projective variety and $\mathscr{E}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Assume we are given a number $i \in \mathbb{N}^+$ such that X is smooth in codimension i and such that $\mathscr{E}$ is locally free in codimension i. Given any resolution morphism $\pi : \tilde{X} \to X$ of $(X, \mathscr{E})$ and any set of Cartier divisors $L_1, \ldots L_{n-i}$ on X, we use the following shorthand notation

$$c_i(\mathscr{E}) \cdot L_1 \cdots L_{n-i} := c_i(\mathscr{F}) \cdot (\pi^* L_1) \cdots (\pi^* L_{n-i}) \in \mathbb{Z}.$$

where $\mathscr{F} := \pi^*\mathscr{E}/_{\text{tor}}$, and where $c_i(\mathscr{F})$ denote the classical Chern classes of the locally free sheaf $\mathscr{F}$ on the smooth variety $\tilde{X}$.

9.2.2 Flatness Criteria

Using the above definitions, we generalise a famous flatness criterion of Simpson, [Sim92], to the klt setting.

Theorem 9.6 ([GKP13b, Theorem 1.19]). *Let X be an n-dimensional, normal, complex, projective variety, smooth in codimension two. Assume that there exists a $\mathbb{Q}$-Weil divisor D such that (X, D) is klt. Let H be an ample Cartier divisor on X, and $\mathscr{E}$ be a reflexive, H-semistable sheaf. Assume that the following intersection numbers vanish*

(9.6.1) $$c_1(\mathscr{E}) \cdot H^{n-1} = 0, \quad c_1(\mathscr{E})^2 \cdot H^{n-2} = 0, \quad \textit{and} \quad c_2(\mathscr{E}) \cdot H^{n-2} = 0.$$

Then, there exists a normal variety $\tilde{X}$ and a quasi-étale, Galois morphism $\gamma : \tilde{X} \to X$, such that $(\gamma^*\mathscr{E})^{**}$ is locally free and flat, that is, $(\gamma^*\mathscr{E})^{**}$ is given by a linear representation of $\pi_1(\tilde{X})$.

Idea of Proof. The proof of Theorem 9.6 uses cutting-down arguments to reduce to the case of a smooth surface $S \subset X_{\text{reg}}$, where Simpson's flatness criterion [Sim92] can be applied. Hamm and Goreski-MacPherson's version of the Lefschetz theorem [GM88, II.1.2] implies that the sheaf $\mathscr{E}|_S$ extends to a flat sheaf that is defined on all of X_{reg}. Boundedness and some vanishing results for singular spaces identify this sheaf with $\mathscr{E}|_{X_{\text{reg}}}$. An application of Theorem 9.3 finishes the proof. □

9.3 Characterisation of Torus Quotients

As a classical consequence of Yau's theorem [Yau78] on the existence of Kähler-Einstein metrics, any Ricci-flat, compact Kähler manifold X with vanishing second Chern class is covered by a complex torus, cf. [LB70, Thm. 12.4.3] and [Kob87, Ch. IV, Cor. 4.15]. Using the flatness criteria discussed above, we generalise this result to the singular case, when X has terminal or canonical singularities.

To this end, recall from Theorem 6.1 that a klt space is smooth if and only if its tangent sheaf is locally free. Theorem 9.3 therefore implies the following criterion to guarantee that a given variety has quotient singularities and is a quotient of an Abelian variety.

Theorem 9.7 ([GKP13b, Corollary 1.15]). *Let X be a normal, complex, quasi-projective variety. Assume that (X, D) is klt for some $\mathbb{Q}$-divisor D. If $\mathscr{T}_{X_{\text{reg}}}$ is flat, then $\tilde{X}$ is smooth and X has only quotient singularities. If X is additionally assumed to be projective, then there exists an Abelian variety A and a quasi-étale Galois morphism $A \to \tilde{X}$.* □

With sufficient amount of technical work, the flatness criterion of semistable sheaves with vanishing first and second Chern classes, Theorem 9.6, will then imply the following.

Theorem 9.8 ([GKP13b, Theorem 1.16]). *Let X be a normal, complex, projective variety of dimension n with at worst canonical singularities. Assume that X is smooth in codimension two and that the canonical divisor is numerically trivial, $K_X \equiv 0$. Further, assume that there exist ample divisors $H_1, \dots, H_{n-2}$ on X and a desingularisation $\pi : \tilde{X} \to X$ such that $c_2(\mathscr{T}_{\tilde{X}}) \cdot \pi^*(H_1) \cdots \pi^*(H_{n-2}) = 0$. Then, there exists an Abelian variety A and a quasi-étale, Galois morphism $A \to X$.* □

There are, in fact, necessary and sufficient conditions for a variety to be a torus quotient, cf. [GKP13b, Section 12]. In dimension three, Theorem 9.8 has been established by Shepherd-Barron and Wilson in [SBW94], and our proof of Theorem 9.8 follows their line of reasoning. The article [SBW94] also asserts an variant of Theorem 9.8 for threefolds with canonical singularities.

10 Applications to Endomorphisms of Algebraic Varieties

In this final section we discuss an application of Theorem 8.7 to polarised endomorphisms of algebraic varieties. First we provide the relevant definition.

Definition 10.1. Let X be a normal, complex, projective variety. An endomorphism $f : X \to X$ is called *polarised* if there exists an ample Cartier divisor H and a positive number $q \in \mathbb{N}^+$ such that $f^*(H) \sim q \cdot H$.

In [NZ10], Nakayama and Zhang study the structure of varieties admitting polarised endomorphisms. They conjecture in [NZ10, Conjecture 1.2] that any variety of this kind is either uniruled or covered by an Abelian variety, with a quasi-étale covering map. They prove the conjecture in [NZ10, Theorem 3.3] under an additional assumption concerning fundamental groups of smooth loci of Euclidean-open subsets of X, which turns out to be an immediate consequence of Theorem 8.7. The following result is thus established.

Theorem 10.2 ([NZ10, Conjecture 1.2] and [GKP13b, Theorem 1.20]). *Let X be a normal, complex, projective variety admitting a non-trivial polarised endomorphism. Assume that X is not uniruled. Then, there exists an Abelian variety A and quasi-étale morphism $A \to X$.* □

Theorem 10.2 has consequences for the structure theory of varieties with endomorphisms. The following results have been shown in [NZ10], conditional to the assumption that [NZ10, Conjecture 1.2] = Theorem 10.2 holds true. The definition of the invariant $q^\natural$ is recalled below.

Theorem 10.3 ([NZ10, Theorem 1.3] and [GKP13b, Theorem 13.1]). *Let $f : X \to X$ be a non-isomorphic, polarised endomorphism of a normal, complex, projective variety X of dimension n. Then $\kappa(X) \le 0$ and $q^\natural(X, f) \le n$. Furthermore, there exists an Abelian variety A of dimension* $\dim A = q^\natural(X, f)$ *and a commutative diagram of normal, projective varieties,*

$$\begin{array}{ccccccc}
A & \xleftarrow{\ \omega\ } & Z & \xrightarrow{\ \rho\ } & V & \xrightarrow{\ \tau\ } & X \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_Z} & & \downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f} \\
A & \underset{\text{flat, surjective}}{\xleftarrow{\ \omega\ }} & Z & \underset{\text{biratl.}}{\xrightarrow{\ \rho\ }} & V & \underset{\text{finite, surjective, quasi-étale}}{\xrightarrow{\ \tau\ }} & X,
\end{array}$$

where all vertical arrows are polarised endomorphism, and every fibre of ω is irreducible, normal and rationally connected. In particular, X is rationally connected if $q^\natural(X, f) = 0$.

Moreover, the fundamental group $\pi_1(X)$ contains a finitely generated, Abelian subgroup of finite index whose rank is at most $2 \cdot q^\natural(X, f)$. □

Remark 10.4 ([NZ10, page 992f]). In the setting of Theorem 10.3, the number $q^\sharp(X,f)$ is defined as the supremum of irregularities $q(\tilde{X}') = h^1\big(\tilde{X}', \mathscr{O}_{\tilde{X}'}\big)$ of a smooth model $\tilde{X}'$ of X' for all quasi-étale morphism $\tau : X' \to X$ admitting an endomorphism $f' : X' \to X'$ with $\tau \circ f' = f \circ \tau$.

Even for general ramified endomorphisms $f : X \to X$ it is known that X is uniruled. For this fact and further information we refer to [AKP08]. It would definitely be interesting to establish both theorems for polarised endomorphisms of Kähler varieties.

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This overview article summarises the content of the research articles [GKKP11, GKP11, GKP13a, GKP13b] as well as some recent developments by Graf, Jörder and others, and aims to put them into perspective. The results presented here are therefore not new. The exposition frequently follows the original articles. Proper references will be given throughout.

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Geometric Structures and Substructures on Uniruled Projective Manifolds

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Abstract In a series of works on uniruled projective manifolds starting in the late 1990's, Jun-Muk Hwang and the author have developed the basics of a geometric theory of uniruled projective manifolds arising from the study of varieties of minimal rational tangents (VMRTs), i.e., the collection at a general point of tangents to minimal rational curves passing through the point. From its onset, our theory is a cross-over between algebraic geometry and differential geometry. While we deal with problems in algebraic geometry, the heart of our perspective is differential-geometric in nature, revolving around foliations, G-structures, differential systems, etc. and dealing with various issues relating to connections, curvature and integrability.

The current article is written with the aim of highlighting certain aspects in the geometric theory of VMRTs revolving around the theme of analytic continuation of geometric structures and substructures. For the parts of the article where adequate exposition already exists, we recall fundamental elements and results in the theory essential for the understanding of more recent development and provide occasional examples for illustration. The presentation will be more systematic on sub-VMRT structures since the latter topic is relatively new. We will discuss various perspectives concerning sub-VMRT structures, and indicate how the subject has intimate links with other areas of mathematics including several complex variables, local differential geometry and Kähler geometry.

Keywords Uniruled projective manifold • Minimal rational curve • Variety of minimal rational tangents • VMRT structure • Sub-VMRT structure • Analytic continuation • Parallel transport

Mathematical Subject Classification: 14J45, 32M15, 53B20

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Uniruled projective manifolds play an important role in algebraic geometry. By the seminal work of Mori [Mr79], rational curves always exist on a projective manifold whenever the canonical line bundle fails to be numerically effective, and by Miyaoka-Mori [MM86] any Fano manifold is uniruled. While much knowledge is gained from Mori theory in the case of higher Picard numbers, the structure of uniruled projective manifolds of Picard number 1 is hard to grasp from a purely algebro-geometric perspective. In a series of works on uniruled projective manifolds starting with Hwang-Mok [HM98], Jun-Muk Hwang and the author have developed the basics of a geometric theory of uniruled projective manifolds arising from the study of varieties of minimal rational tangents (VMRTs), i.e., the collection at a general point of the variety of tangents to minimal rational curves passing through the point. The theory was from its onset a cross-over between algebraic geometry and differential geometry. While we dealt with classical problems in algebraic geometry and axiomatics were derived from basics in the deformation theory of rational curves, the heart of our perspective was differential-geometric in nature, revolving around tautological foliations, G-structures, differential systems, etc., and dealing with various issues relating to connections, curvature, integrability, etc., while techniques from several complex variables on analytic continuation were brought in to allow for a passage from transcendental objects defined on open sets in the Euclidean topology to algebraic objects in the Zariski topology.

Given any uniruled projective manifold X, fixing a polarization and minimizing degrees of free rational curves we obtain a minimal rational component $\mathcal{K}$. Basic to $(X, \mathcal{K})$ is the double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, where $\rho : \mathcal{U} \to \mathcal{K}$ is the universal family whose fibers are (unparametrized) minimal rational curves, and $\mu : \mathcal{U} \to X$ is the evaluation map. At a general point $x \in X$ a point on the fiber $\mathcal{U}_x$ corresponds to a minimal rational curve with a marking at x, and the VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ is the image of $\mathcal{U}_x$ under the tangent map. The double fibration and the VMRT structure $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T(X)$, endowed with a tautological foliation, set the stage on which the basics of our geometric theory have been developed.

In this article, by a geometric structure we mean a VMRT structure $\pi : \mathcal{C}(X) \to X$ or its restriction to a connected open set U on X, and by a geometric substructure we mean a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) \subset \mathbb{P}T(S)$, on a complex submanifold S of some open subset of X, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, which among other things is assumed to dominate S. For a VMRT structure $\pi : \mathcal{C}(X) \to X$, of principal importance here is the tautological foliation on $\mathcal{C}(X)$ transported from the fibration $\rho : \mathcal{U} \to \mathcal{K}$ by means of the tangent map, and solutions to various questions concerning the tautological foliation have strong implications leading to rigidity phenomena or characterization results on uniruled projective manifolds. As to sub-VMRT structures, a basic question is whether the tautological foliation on $\mathcal{C}(X)$ is tangent to $\mathcal{C}(S)$, and an affirmative answer to the question leads to rational saturation for germs of submanifolds inheriting certain types of sub-VMRT structures and characterization of various classes of special uniruled projective subvarieties.

There is a wide scope of phenomena and problems concerning geometric structures and substructures in complex geometry, and those on uniruled projective manifolds arising from the consideration of minimal rational curves in particular, and the current article is an exposition on selected aspects of such phenomena and problems arising from VMRTs. Concerning geometric structures we will be exclusively concerned with those arising from or directly related to known uniruled projective manifolds, especially rational homogeneous spaces of Picard number 1, leaving aside the topic of general VMRT structures, for which the reader is referred to two expository articles of Hwang [Hw12, Hw15] (and references therein) on VMRTs from the perspective of Cartanian geometry. For geometric substructures our focus will be on sub-VMRT structures on rational homogeneous spaces of Picard number 1 modeled on certain admissible pairs (X_0, X) of such manifolds, while results will also be formulated for sub-VMRT structures on uniruled projective manifolds in general satisfying various notions of nondegeneracy related to the second fundamental form (cf. Mok [Mk08a], Hong-Mok [HoM10, HoM13], Hong-Park [HoP11], Hwang [Hw14b], Zhang [Zh14], Mok-Zhang [Mz15]). An overview on the topic of germs of complex submanifolds on uniruled projective manifolds will be given in the article.

The current article is written with the aim of highlighting certain aspects in an area of research arising from the study of geometric structures modeled on varieties of minimal rational tangents. As a number of surveys and expository articles are available at different stages on various aspects in the development of the subject (Hwang-Mok [HM99a], Hwang [Hw01], Kebekus-Sola Conde [KS06], Mok [Mk08b], Hwang [Hw12, Hw15]), for the parts of the article where adequate exposition already exists, we are contented with recalling fundamental elements and results in the theory which are essential for the understanding of more recent development and with providing examples occasionally for the purpose of illustration. The presentation will be more systematic in the last section on sub-VMRT structures since the latter topic is relatively new. At the end of the section, we will discuss various perspectives concerning sub-VMRT structures, and indicate how the subject, a priori arising from the study of uniruled projective manifolds and their subvarieties, has intimate links with other areas of mathematics including several complex variables, local differential geometry, and Kähler geometry. Already on this topic there is the prospect of exciting cross-fertilization of ideas and methodology, and the subject will thrive with further investigation on problems intrinsic to the study of VMRTs and also with applications to be explored on these and other related areas of mathematics.

1 Minimal Rational Curves on Uniruled Projective Manifolds

1.1 Minimal Rational Components and the Universal Family

For a projective variety $W \subset \mathbb{P}^N$ we denote by $\mathrm{Chow}(W)$ the Chow space of all cycles C on W, and by $[C] \in \mathrm{Chow}(W)$ the member corresponding to the cycle C. Each irreducible component of $\mathrm{Chow}(W)$ is projective. For two projective varieties Y and Z we denote by $\mathrm{Hom}(Y, Z)$ the set of all morphisms from Y to Z. Through the use of Hilbert schemes, $\mathrm{Hom}(Y, Z)$ is endowed the structure of a complex space such that each of its irreducible components is projective (cf. Kollár [Ko96, Chapter 1]).

By a rational curve on a projective manifold X we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$, which will be denoted by $[f]$ when regarded as an element of $\mathrm{Hom}(\mathbb{P}^1, X)$. A rational curve $[f]$ is said to be free if and only if the vector bundle f^*T_X on $\mathbb{P}^1$ is semipositive, i.e., isomorphic to a direct sum of holomorphic line bundles $\mathcal{O}(a_k)$ of degree $a_k \geq 0$. The basic objects of our study are the uniruled projective manifolds, i.e., projective manifolds that are "filled up" by rational curves. Equivalently a projective manifold X is uniruled if and only if there exists a free rational curve on X. A smooth hypersurface $X \subset \mathbb{P}^n$ of degree $\leq n-1$ is uniruled by projective lines, and those of degree n are uniruled by rational curves of degree 2. By Mori-Miyaoka [MM86] any Fano manifold is uniruled. For the basics on rational curves in algebraic geometry we refer the reader to Kollár [Ko96].

Let X be a uniruled projective manifold. Fixing an ample line bundle L on X, let $f_0 : \mathbb{P}^1 \to X$ be a free rational curve realizing the minimum of $\deg(h^*L)$ among all *free* rational curves $[h] \in \mathrm{Hom}(\mathbb{P}^1, X)$. Let $\check{\mathcal{H}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be an irreducible component containing $[f_0]$ and $\mathcal{H} \subset \check{\mathcal{H}}$ be the subset consisting of free rational curves. $\check{\mathcal{H}}$ is quasi-projective and $\mathcal{H} \subset \check{\mathcal{H}}$ is a dense Zariski open subset. Since each member $[f] \in \mathcal{H}$ is a free rational curve, there is no obstruction in deforming $f : \mathbb{P}^1 \to X$, and, passing to normalization if necessary, $\mathcal{H}$ will be endowed the structure of a quasi-projective manifold. Any member $f : \mathbb{P}^1 \to X$ of $\mathcal{H}$ must be generically injective (i.e., f must be birational onto its image) by the freeness of f and by the minimality of $\deg(f^*L)$ among free rational curves. Thus, $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$ by the assignment $(\gamma, [f]) \mapsto [f \circ \gamma]$ for $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ and $[f] \in \mathcal{H}$. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$, the quotient $\mathcal{K} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$ is a complex manifold. There is a canonical morphism $\alpha : \mathcal{H} \to \mathrm{Chow}(X)$ defined by $\alpha([f]) = [f(\mathbb{P}^1)]$ mapping $\mathcal{K}$ onto a Zariski open subset $\mathcal{Q}$ of some irreducible subvariety $\mathcal{Z}$ of $\mathrm{Chow}(X)$. The mapping α is invariant under the action of $\mathrm{Aut}(\mathbb{P}^1)$ and it descends to a bijective holomorphic map $\nu : \mathcal{K} \to \mathcal{Q}$. Hence, ν is a normalization, and $\mathcal{K}$ is a quasi-projective manifold. We call $\mathcal{K}$ a minimal rational component on X. There is a smallest subvariety $B \subset X$ such that every member of K passing through any point $x \varepsilon X - B$ is a free rational curve. We call $B \subset X$ the bad locus of (X, K).

On a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component we have a universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ called the universal family of $\mathcal{K}$, where $\mathcal{U} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1; 0)$, and $\rho : \mathcal{U} \to \mathcal{K}$ is the canonical projection which realizes $\mathcal{U}$ as the total space of a holomorphic fiber bundle with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1;0) \cong \mathbb{P}^1$. We have canonically the evaluation map $\mu : \mathcal{U} \to X$, and we write $\mathcal{U}_x := \mu^{-1}(x)$. From the Bend-and-Break Lemma of Mori [Mr79] it follows that a general member κ of a minimal rational component $\mathcal{K}$ corresponds to a standard rational curve, i.e., κ is the equivalence class modulo the action of $\mathrm{Aut}(\mathbb{P}^1)$ of some $[f] \in \mathrm{Hom}(\mathbb{P}^1, X)$ such that $f^*T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$, $1 + p + q = n := \dim(X)$. Note that any standard rational curve $f : \mathbb{P}^1 \to X$ is immersive and generically injective. In the sequel, to avoid clumsy language the term "minimal rational curve" will sometimes also be used to describe the image of a minimal rational curve belonging to $\mathcal{H}$ under the canonical map $\beta : \mathcal{H} \to \mathcal{K}$.

1.2 Varieties of Minimal Rational Tangents and the Tautological Foliation

Let $(X, \mathcal{K})$ be a uniruled projective manifold X equipped with a minimal rational component. Denote by $\rho : \mathcal{U} \to \mathcal{K}$ the universal family over $\mathcal{K}$, and by $\mu : \mathcal{U} \to X$ the accompanying evaluation map. By definition $\rho : \mathcal{U} \to \mathcal{K}$, as a holomorphic fiber bundle with fibers isomorphic to $\mathbb{P}^1$, is equipped with a tautological foliation whose leaves are the fibers of ρ. Let $x \in X$ be a general point and $u \in \mathcal{U}_x$, corresponding to a minimal rational curve with a marking at x. Let $f : \mathbb{P}^1 \to X$ be a parametrization of $\rho(u) \in \mathcal{K}$, $\widehat{f} : \mathbb{P}^1 \to \mathcal{U}$ its tautological lifting such that $\widehat{f}(0) = u$ (hence $f(0) = x$). If f is an immersion at 0 we define $\tau_x(u) = [df(T_0(\mathbb{P}^1))] \in \mathbb{P}T_0(X)$. For a general point $x \in X$ this defines the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathbb{P}T_x(X)$, which is a holomorphic immersion at a general point of $\mathcal{U}_x$ corresponding to a standard rational curve with a marking at x, and denote by $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ the strict transform of τ_x, so that $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ is a priori a generically finite dominant rational map, and $\pi : \mathcal{C}(X) \to X$ is equipped at general points with a multi-foliation $\mathcal{F}$ transported from the tautological foliation on $\mathcal{U}$ by means of the tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$.

Standard rational curves play a special role with regard to the tangent map. To simplify the notation we will state the following result for embedded standard rational curves ℓ. The general case, in which standard rational curves are known only to be immersed, can be stated with a slight modification. For an embedded minimal rational curve ℓ we have $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, and we denote by $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p$ the positive part of $T(X)|_\ell$. For the normal bundle $N_{\ell|X}$ for $\ell \subset X$ we have $N_{\ell|X} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^q$. From the deformation theory of rational curves we have (cf. Mok [Mk08b, (2.4), Lemma 2]).

Lemma 1.2.1. *At a general point $x \in X$, and a point $u \in \mathcal{U}_x$ corresponding to a standard rational curve ℓ with a marking at x, the tangent map τ_x is a holomorphic immersion at u. Assuming that $\tau_x(u) := [\alpha]$ is a smooth point of the VMRT $\mathcal{C}_x(X)$, we have $d\tau_x(u) : T_u(\mathcal{U}_x) \xrightarrow{\cong} T_{[\alpha]}(\mathcal{C}_x(X)) \subset T_{[\alpha]}(\mathbb{P}T_x(X)) \cong T_x(X)/\mathbb{C}\alpha$. More precisely, assuming for convenience that $\ell \subset X$ is embedded, we have $T_u(\mathcal{U}_x) =$*

$\Gamma(\ell, N_{\ell|X} \otimes \mathfrak{m}_x)$, *where* $\mathfrak{m}_x$ *is the maximal ideal sheaf at* x *on* ℓ, *and* $T_{[\alpha]}(\mathcal{C}_x(X)) = P_\alpha/\mathbb{C}\alpha$, *where* $P_\alpha = P_{\ell,x}$ *is the fiber at* x *of the positive part* $P_\ell \subset T(X)|_\ell$, *and for* $\nu \in T_u(\mathcal{U}_x) = \Gamma(\ell, N_{\ell|X} \otimes \mathfrak{m}_x)$, *we have* $d\tau_x(u)(\nu) = \partial_\alpha(\nu) + \mathbb{C}\alpha \in P_\alpha/\mathbb{C}\alpha \cong T_{[\alpha]}(\mathcal{C}_x(X))$.

We note that since $\nu(x) = 0$, the partial derivative $\partial_\alpha(\nu)$ is well-defined. Moreover, while the isomorphism $T_{[\alpha]}(\mathbb{P}T_x(X)) \cong T_x(X)/\mathbb{C}\alpha$ depends on the choice of $\alpha \in T_x(X)$ representing $[T_x(\ell)] \in \mathbb{P}T_x(X)$, there is a canonical isomorphism $T_{[\alpha]}(\mathbb{P}T_x(X)) \otimes L_{[\alpha]} \cong T_x(X)/\mathbb{C}\alpha$, where L denotes the tautological line bundle over $\mathbb{P}T_x(X)$, hence the formula for $d\tau_x(u)(\nu) \in T_{[\alpha]}(\mathcal{C}_x(X)) \subset T_{[\alpha]}(\mathbb{P}T_x(X))$ is independent of the choice of $\alpha \in T_x(\ell)$.

By Hwang-Mok [HM99a, HM01] at a general point $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ is birational (cf. (2.1)), and it is a morphism by Kebekus [Ke02]. Finally, Hwang-Mok [HM04b] proved that the tangent map is a birational finite morphism, hence $\tau_x : \mathcal{U}_x \to \mathcal{C}_x(X)$ is the normalization. There is a smallest subvariety $B' \supset B$ of X such that every member of K passing through any point $x \epsilon X - B'$ is a free rational curve immersed at the marked point x and $\tau_x : \mathcal{U}_x \to \mathcal{C}_x(X)$ is a birational finite morphism. We call $B' \subset X$ the enhanced bad locus of (X, K). In some cases, e.g., in the case of a projective submanifold $X \subset \mathbb{P}^n$ uniruled by projective lines it is easily seen (from the positivity of $T(X)|_\ell \subset T(\mathbb{P}^n)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ in the latter case) that at a general point $x \in X$ every projective line ℓ passing through x is a standard rational curve, and it has been thought for some time that this may actually be the case in general. Recently, Casagrande-Druel [CD12] found examples of uniruled projective manifolds $(X, \mathcal{K})$ equipped with minimal rational components in a more generalized sense (in the sense that the general VMRT $\mathcal{C}_x(X)$ is projective) on which the VMRT at a general point is actually singular, and Hwang-Kim [HK13b] has now obtained examples where $\mathcal{K}$ is a *bona fide* minimal rational component in the sense of (1.1). This means that for general results in the differential-geometric study of VMRT structures one has to deal with singularities which are smoothed out by normalization.

By the VMRT structure on $(X, \mathcal{K})$ we will mean the fibered space of VMRTs $\pi : \mathcal{C}(X) \to X, \mathcal{C}(X) \subset \mathbb{P}T(X)$. In the sequel we will speak of the VMRT structure $\pi : \mathcal{C}(X) \to X$ on X, being understood that we are talking about a subvariety $\mathcal{C}(X) \subset \mathbb{P}T(X)$ which projects onto a Zariski open subset of X. By the birationality of the tangent map we can now speak of the tautological foliation on a VMRT structure $\pi : \mathcal{C}(X) \to X$, and the extent to which the latter foliation is determined by the fibered space of VMRTs will play an important role in the rest of the article.

1.3 The Affine and Projective Second Fundamental Forms

The second fundamental form in affine or projective geometry will be essential in the geometric study of VMRTs. For generalities let V be a finite-dimensional complex vector space and denote by $\nu : V - \{0\} \to \mathbb{P}(V)$ the canonical projection onto the projective space $\mathbb{P}(V)$. For any subset $E \subset \mathbb{P}(V)$ we denote by $\widetilde{E} := \nu^{-1}(E)$ the affinization of E. In terms of the Euclidean flat connection on V, for a complex submanifold S on some open subset of V we have the second fundamental form $\sigma := \sigma_{S|V}$. If $A \subset \mathbb{P}(V)$ is a subvariety and $\eta \in \widetilde{A}$ is a smooth point we have the second fundamental form $\sigma_\eta := \sigma_{\widetilde{A}|V,\eta}$, $\sigma_\eta : S^2T_\eta(\widetilde{A}) \to V/T_\eta(A) := N_{\widetilde{A}|V,\eta}$. We have always $\sigma_\eta(\eta, \xi) = 0$ for any $\xi \in T_\alpha(\widetilde{A})$. Thus, considered as a vector-valued symmetric bilinear form, the kernel of σ_η always contains $\mathbb{C}\eta$. Passing to quotients we have the projective second fundamental form, denoted by $\sigma_{[\eta]}$: $S^2T_{[\eta]}(A) \to T_{[\eta]}(\mathbb{P}(V))/T_{[\eta]}(A) := N_{A|\mathbb{P}(V),[\eta]}$ which is equivalently defined by the canonical projective connection on $\mathbb{P}(V)$. (We will use the same notation σ for both the Euclidean and the projective second fundamental forms. The subscript, either $\eta \in \widetilde{\mathcal{A}}$ or $[\eta] \in \mathcal{A}$ will indicate which is meant.) Here $T_{[\eta]}(\mathbb{P}(V)) \cong V/\mathbb{C}\eta$, $T_{[\eta]}(A) \cong T_\eta(\widetilde{A})/\mathbb{C}\eta$, and the two normal spaces $N_{\widetilde{A}|V,\eta} \cong N_{A|\mathbb{P}(V),[\eta]}$ are naturally identified. The projective second fundamental form $\sigma_{[\eta]}$ is the differential at η of the Gauss map, hence the Gauss map is generically injective on $\mathcal{A}$ if and only if $\mathrm{Ker}\,\sigma_{[\eta]} = 0$ for a general point $[\eta]$ of each irreducible component of $\mathcal{A}$. We note that from projective geometry, the Gauss map on a nonlinear projective submanifold $A \subset \mathbb{P}(V)$ is always generically injective.

2 Analytic Continuation Along Minimal Rational Curves

2.1 Equidimensional Cartan-Fubini Extension

Let S be an irreducible Hermitian symmetric space. Denoting by $\mathcal{O}(1)$ the positive generator of the Picard group $\mathrm{Pic}(S) \cong \mathbb{Z}$, S admits an embedding $\theta : S \hookrightarrow \mathbb{P}(\Gamma(S, \mathcal{O}(1))^*)$ and as such S is uniruled by projective lines. When S is of rank ≥ 2, it is endowed with a particular type of G-structure. To explain this we start by recalling the notion of G-structures and flat G-structures. Let n be a positive integer, V be an n-dimensional complex vector space, and M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$.

Definition 2.1.1. Let $\mathrm{G} \subset \mathrm{GL}(V)$ be any complex Lie subgroup. A holomorphic G-structure is a G-principal subbundle $\mathcal{G}(M) \subset \mathcal{F}(M)$. An element of $\mathcal{G}_x(M)$ will be called a G-frame at x. For $\mathrm{G} \subsetneq \mathrm{GL}(V)$ we say that $\mathcal{G}(M)$ defines a holomorphic reduction of the tangent bundle to G. We say that a G-structure $\mathcal{G}(M)$ on M is flat if and only if there exists an atlas of charts $\{\varphi_\alpha : U_\alpha \to V\}$ such that the restriction $\mathcal{G}(U_\alpha)$ of $\mathcal{G}(M)$ to U_α is the product $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$ in terms of Euclidean coordinates on U_α given by the chart $\varphi_\alpha : U_\alpha \to V$.

As a first example we consider the hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ defined as the zero of a nondegenerate homogeneous quadratic polynomial. The projective second fundamental form σ of $Q^n \subset \mathbb{P}^{n+1}$ defines a section in $\Gamma(Q^n, S^2T^*_{Q^n} \otimes \mathcal{O}(2))$, $\mathcal{O}(2)$ being isomorphic to the normal bundle $N_{Q^n|\mathbb{P}^{n+1}}$ of $Q^n \subset \mathbb{P}^{n+1}$. The twisted symmetric bilinear form σ is everywhere nondegenerate, thereby equipping small open sets of Q^n with holomorphic metrics $\sum g_{\alpha\beta}(z)dz^\alpha \otimes dz^\beta$ in terms of local coordinates, unique up to multiplication by nowhere zero holomorphic functions. This gives a holomorphic conformal structure on Q^n. Here we have a reduction of the frame bundle to the complex conformal group $\mathrm{CO}(n;\mathbb{C}) = \mathbb{C}^*\cdot\mathrm{O}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$ being the complex orthogonal group with respect to a nondegenerate complex symmetric bilinear form.

Another example is the Grassmannian $G(p,q)$ of p-planes in a complex vector space $W_0 \cong \mathbb{C}^{p+q}$, where we have a tautological vector bundle F on $G(p,q)$ given by $F_x = E \subset W_0$ for $x = [E] \in G(p,q)$. Writing $V = W/F$, where $W = W_0 \times G(p,q)$ is a trivial vector bundle on $G(p,q)$, we have a canonical isomorphism $T_{G(p,q)} \cong U \otimes V$, $U = F^*$, yielding a Grassmann structure on $G(p,q)$. U and V are called the (semipositive) universal bundles on $G(p,q)$. Here, for a pq-dimensional manifold M on which the holomorphic tangent bundle $T(M) \cong A \otimes B$, where A resp. B is a holomorphic vector bundle of rank p resp. q, representing tangent vectors on X as matrices through tensor product decomposition, we have a reduction of the frame bundle from $\mathrm{GL}(pq,\mathbb{C})$ to the subgroup $H \subset \mathrm{GL}(pq,\mathbb{C})$ which is the image of $\mathrm{GL}(p,\mathbb{C})\times \mathrm{GL}(q,\mathbb{C})$ in $\mathrm{GL}(pq,\mathbb{C})$ under the homomorphism Φ given by $\Phi(C,D)(X) = CXD^t$. We refer the reader to Manin [Ma97] for Grassmann structures appearing in gauge field theory.

For generalities on Hermitian symmetric spaces we refer the reader to Wolf [Wo72]. Any irreducible Hermitian symmetric space S of the compact type and of rank ≥ 2 carries a canonical S-structure, which is a G-structure for some complex reductive linear subgroup $\mathrm{G} \subsetneq GL(T_0(S))$, as follows. Write $S = G/P$ as a complex homogeneous space, where G is a connected complex simple Lie group, and $P \subset G$ is a maximal parabolic subgroup. Let $P = L \cdot U$ be the Levi decomposition of P, where $U \subset P$ is the unipotent radical and $L \subset P$ is a Levi factor. Equip S with a canonical Kähler-Einstein metric g and write $S = G_c/K$, where G_c is the identity component of the isometry group of (S,g) and $K \subset G_c$ is the isotropy subgroup at a reference point $0 \in S$. Identifying $0 \in S$ with $eP \in G/P \cong S$, in the Levi decomposition $P = L\cdot U$, L can be identified with $K^{\mathbb{C}} \subset \mathrm{GL}(T_0(S))$ by means of Lie group homomorphism $\Phi : P \to \mathrm{GL}(T_0(S))$ given by $\Phi(\gamma) := d\gamma(0) \in \mathrm{GL}(T_0(S))$, $\Phi|_L : L \xrightarrow{\cong} K^{\mathbb{C}}$. On the other hand, $U = \exp(\mathfrak{m}^-) := M^-$, where $\mathfrak{m}^-$ is the Lie algebra of holomorphic vector fields on S vanishing to the order ≥ 2 at 0, thus $d\gamma(0)$ is the identity map on $T_0(S)$ whenever $\gamma \in M^-$. In other words, $U = M^- = \mathrm{Ker}(\Phi)$. Let η be a nonzero highest weight vector of the isotropy representation of $K^{\mathbb{C}}$ on $T_0(S)$. Since M^- acts trivially on $T_0(S)$, the G-orbit of $[\eta] \in \mathbb{P}T_0(S)$ gives a homogeneous holomorphic fiber subbundle $\mathcal{W} \subset \mathbb{P}T(S)$ whose fiber $\mathcal{W}_0$ over 0 is the $K^{\mathbb{C}}$-orbit of $[\eta]$, i.e., the highest weight orbit. Writing $V = T_0(S)$ and considering at $x \in S$ the set of all linear isomorphisms $\varphi : T_0(S) \xrightarrow{\cong} T_x(S)$ such that $\varphi(\widetilde{\mathcal{W}}_0) = \widetilde{\mathcal{W}}_x$, where $\widetilde{\mathcal{W}}_0$ consists of all nonzero highest weight vectors at 0, etc., we have a reduction of the frame bundle on S from $\mathcal{F}(S)$ to some $\mathcal{G}(S) \subsetneq \mathcal{F}(S)$ defining a G-structure with $\mathrm{G} = K^{\mathbb{C}}$. This canonical $K^{\mathbb{C}}$-structure is also called the canonical S-structure.

Flatness of the canonical $K^{\mathbb{C}}$-structure is not obvious. That this is the case is seen from the Harish-Chandra decomposition. The integrable almost complex structure on S is defined by $ad(j)$ of a certain element j in the one-dimensional center $\mathfrak{z}$ of the Lie algebra $\mathfrak{k}$ of K. Writing $\mathfrak{g}$ for the Lie algebra of G we have a decomposition $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, where $\mathfrak{k}^{\mathbb{C}}$ is the Lie algebra of $K^{\mathbb{C}} \cong L$, and $\mathfrak{m}^+$ resp. $\mathfrak{m}^-$ is the eigenspace of $ad(j)$ corresponding to the eigenvalue i resp. $-i$. Writing $M^+ := \exp(\mathfrak{m}^+)$, the mapping $M^+ \times K^{\mathbb{C}} \times M^- \mapsto G$ given by $(a, b, c) \mapsto abc$ is injective, leading to the identification of a Zariski open subset W of S with the vector space $\mathfrak{m}^+$ through the mapping $m^+ \mapsto \exp(m^+)P$, yielding Harish-Chandra coordinates $(z_1, \cdots, z_n)$, $n = \dim(S)$. The Abelian Lie subalgebra $\mathfrak{m}^+ \subset \mathfrak{g}$ is the Lie algebra of constant vector fields in the coordinates $(z_1, \cdots, z_n)$. The invariance of $\mathcal{W}$ under the vector group M^+ of Euclidean translations shows that $\mathcal{W}|_W = \mathcal{W}_0 \times W$, i.e., the $K^{\mathbb{C}}$-structure on S is flat.

The Harish-Chandra coordinates link immediately to the structure of minimal rational curves on S. A highest weight vector $\eta \in \widetilde{\mathcal{W}}_x$ yields readily a copy of $\mathfrak{sl}(2, \mathbb{C})$ which in standard notations is of the form $\mathbb{C}e_\rho \oplus \mathbb{C}[e_\rho, e_{-\rho}] \oplus \mathbb{C}e_{-\rho}$ in terms of root vectors $e_\rho \in \mathfrak{m}^+$, $e_{-\rho} \in \mathfrak{m}^-$ with respect to suitably chosen Cartan subalgebras, $[e_\rho, e_{-\rho}] \in \mathfrak{k}^{\mathbb{C}}$. Exponentiating one gets a copy of $\mathbb{P}SL(2, \mathbb{C})$, and the orbit of x under the latter group exhausts all the rational curves of degree 1 passing through W as x runs over W and $[\eta]$ runs over $\mathcal{W}_x$. Thus, intersections of minimal rational curves with a Harish-Chandra coordinate chart are given by affine lines ℓ such that $\mathbb{P}T_x(\ell) \in \mathcal{W}_x$ for $x \in \ell$. Moreover, $\mathcal{W}$ is nothing other than $\mathcal{C}(S) \subset \mathbb{P}T(S)$, the VMRT structure on S (cf. (1.2)). Examples of VMRTs $\mathcal{C}_0(S)$ are given in the case of hyperquadrics Q^n, $n \geq 3$, by $\mathcal{C}_0(Q^n) = Q^{n-2} \subset \mathbb{P}T_0(Q^n) \cong \mathbb{P}^{n-1}$, $\widetilde{\mathcal{C}}_0(Q^n) \cup \{0\}$ being the null-cone of the holomorphic conformal structure, and in the case of the Grassmannian $G(p, q)$; $p, q \geq 1$; by $\mathcal{C}_0(G(p, q)) = \varsigma(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}) \subset \mathbb{P}(\mathbb{C}^p \otimes \mathbb{C}^q)$, ς being the Segre embedding given by $\varsigma([u], [v]) = [u \otimes v]$, with the image being projectivizations of decomposable tensors.

The use of Harish-Chandra coordinates allows us to give a differential-geometric and complex-analytic proof (cf. Mok [Mk99]) of the following classical result of Ochiai on S-structures.

Theorem 2.1.1 (Ochiai [Oc70]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 equipped with the canonical S-structure. Let $U \subset S$ be a connected open subset and $f : U \xrightarrow{\cong} V \subset S$ be a biholomorphic map preserving the canonical S-structure. Then, there exists $F \in Aut(S)$ such that $F|_U \equiv f$. As a consequence, any simply connected compact complex manifold X admitting a flat S-structure must necessarily be biholomorphic to S.*

We refer the reader to [Mk99] and to Mok [Mk08b, (4.2), especially Lemma 4] for detailed discussions on Ochiai's Theorem from a geometric perspective. Denoting by $\mathcal{K}$ the minimal rational component of projective lines on S with the accompanying VMRT structure $\mathcal{C}(S) \subset \mathbb{P}T(S)$, the key issue is to show that f sends a connected open subset of a minimal rational curve onto an open subset of a minimal rational curve, i.e., writing $f_\sharp := [df]$, $\mathcal{F}_S$ for the tautological foliation on $\mathcal{S}$, $\mathcal{C}(U) := \mathcal{C}(S) \cap \mathbb{P}T(U)$, etc., we have to show that $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$. Granting this, taking U to be a Euclidean ball in Harish-Chandra coordinates so that $\ell \cap U$ is either empty or connected for $[\ell] \in \mathcal{K}$, and by $\mathcal{O} \subset \mathcal{K}$ the subset consisting of all $[\ell] \in \mathcal{K}$ such that $\ell \cap U \neq \emptyset$, $f : U \to S$ induces a holomorphic map $f^\sharp : \mathcal{O} \to \mathcal{K}$. Then, by the method of Mok-Tsai [MT92], Hartogs extension holds true for $\mathcal{O}$, and we conclude that $f^\sharp$ extends *meromorphically* to $\Phi : \mathcal{K} \dashrightarrow \mathcal{K}$. We extend f analytically to F beyond U by defining $F(x)$ to be the intersection of the lines $\Phi([\ell])$, as ℓ ranges over minimal rational curves passing through x. Arguing also with f^{-1} we get a birational extension of f to $F : S \dashrightarrow S$ which transforms minimal rational curves to minimal rational curves, and that is enough to imply that in fact $F \in \mathrm{Aut}(S)$, cf. [Mk99, (2.4)]. We have

Proof that $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$. We use Harish-Chandra coordinates. Restricted to such a Euclidean chart W, $\widetilde{\mathcal{C}}(S)|_W = \widetilde{\mathcal{C}}_x \times W$ for any $x \in W$, where $\widetilde{\mathcal{C}}_x := \widetilde{\mathcal{C}}_x(S)$. To prove $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$ it suffices to show $d^2f(\alpha,\alpha) \in \mathbb{C}df(\alpha)$ for $\alpha \in \widetilde{\mathcal{C}}_x$, $x \in U$. We may assume $df(x) = \mathrm{id}_{T_x(S)}$. For $\beta \in \widetilde{\mathcal{C}}_x$, we have $d^2f(\alpha,\beta) = \partial_\alpha(df(\widetilde{\beta}))$, where $\widetilde{\beta}$ stands for the constant vector field on U such that $\widetilde{\beta}(x) = \beta$. Since $\widetilde{\mathcal{C}}|_U = \widetilde{\mathcal{C}}_x \times U$, $\partial_\alpha(df(\widetilde{\beta}))$ is the tangent at β to some curve on $\widetilde{\mathcal{C}}_x$, hence $d^2f(\alpha,\beta) \in P_\beta = T_\beta(\widetilde{\mathcal{C}}_x)$, and by symmetry $d^2f(\alpha,\beta) \in P_\alpha \cap P_\beta$. To show $d^2f(\alpha,\alpha) \in \mathbb{C}\alpha$ note that for the second fundamental form σ of $\widetilde{\mathcal{C}}_x \subset T_x(S)$, we have $\mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, and it remains to show $d^2f(\alpha,\alpha) \in \mathrm{Ker}(\sigma_\alpha)$. Fix $\alpha \in \widetilde{\mathcal{C}}_x$ and let $\beta = \alpha(t), \alpha(0) = \alpha$, vary holomorphically on $\widetilde{\mathcal{C}}_x$ in the complex parameter t. Writing $\xi = \frac{d}{dt}\big|_{t=0}\alpha(t) \in P_\alpha$, from $d^2f(\alpha,\alpha(t)) \in P_\alpha$ it follows that $d^2f(\alpha,\xi) = \frac{d}{dt}\big|_{t=0}d^2f(\alpha,\alpha(t)) \in P_\alpha$. On the other hand, $\frac{d}{dt}\big|_{t=0}d^2f(\alpha(t),\alpha(t)) = 2d^2f(\alpha,\xi)$. Interpreting $d^2f(\beta,\beta) \in P_\beta$ as a vector field on $\widetilde{\mathcal{C}}_x$, we have $\nabla_\xi(d^2f(\beta,\beta)) \in P_\alpha$ for the Euclidean flat connection ∇ on $T_x(S)$, hence $\sigma_\alpha(\xi, d^2f(\alpha,\alpha)) = 0$. Varying $\xi \in P_\alpha$, we conclude that $d^2f(\alpha,\alpha) \in \mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, as desired. $\square$

A Harish-Chandra coordinate chart flattens the VMRT structure on S on the chart. Although the existence of such coordinates in the Hermitian symmetric case is a very special feature among uniruled projective manifolds, in (2.2) we will explain how the same argument applies in general on a uniruled projective manifold $(X,\mathcal{K})$ endowed with a minimal rational component. The gist of the matter is that, for the computation at a general point $x \in X$, and at a smooth point $[\alpha] \in \mathcal{C}_x(X)$, denoting by ℓ the standard minimal rational curve passing through x such that $T_x(\ell) = \mathbb{C}\alpha$, and assuming that ℓ is embedded for convenience, what one needs is simply a choice of holomorphic coordinates on a neighborhood U of x such that the positive part $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q = T(X)|_\ell$ is a constant vector subbundle of $T(X)|_\ell$ on $U \cap \ell$ in terms of the standard trivialization of $T(X)|_\ell$ induced by the holomorphic coordinates, and such choices of holomorphic coordinates exist in abundance.

For a uniruled projective manifold $(X,\mathcal{K})$ equipped with a minimal rational component, we introduced in Hwang-Mok [HM99a] differential systems on the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$ and on $\mathcal{K}$, and in [HM01] we gave a full proof of Cartan-Fubini extension using such differential systems. The machinery introduced was used at the same time to prove birationality of the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ at a general point $x \in X$ under the assumption that the Gauss map is generically injective, a result which was later on improved to yield that τ_x is a birational morphism, i.e., τ_x is the normalization map (cf. (1.1)). Restricting to Cartan-Fubini extension, we proved in [HM01]

Theorem 2.1.2 (Hwang-Mok [HM01]). *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two Fano manifolds of Picard number 1 equipped with minimal rational components. Assume that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$ and that furthermore the Gauss map is an immersion at a general point of each irreducible component of $\mathcal{C}_z(Z)$. Let $f : U \to V$ be a biholomorphic map from a connected open subset $U \subset Z$ onto an open subset $V \subset X$. If $f_\sharp = [df]$ sends each irreducible component of $\mathcal{C}(Z)|_U$ to an irreducible component of $\mathcal{C}(X)|_V$ biholomorphically, then f extends to a biholomorphic map $F : Z \to X$.*

We refer the reader to expositions on the differential systems in [HM99a, Mk08b] and [Hw12]. Here we will just describe briefly such distributions and their links to analytic continuation. Let $x \in X$ be a general point, and $\ell \subset X$ be a standard minimal rational curve passing through x, which we assume to be embedded for notational convenience. Let $u \in \mathcal{U}_x$ be the point corresponding to the minimal rational curve ℓ marked at x. Since ℓ is standard, there exists a neighborhood $\mathcal{O}$ of x in $\mathcal{U}$ such that the tangent map τ is a biholomorphism of $\mathcal{O}$ onto a complex submanifold $\mathcal{S}$ of some open subset of $\mathbb{P}T(X)$. Holomorphic distributions can now be defined on $\mathcal{S}$, as follows. $\varpi := \pi|_{\mathcal{S}} : \mathcal{S} \to X$ is a submersion, and the kernels of $d\varpi$ defines an integrable distribution $\mathcal{J} \subset T(\mathcal{S})$. In what follows on a coordinate chart $U \subset X$, $0 \in U$ a reference point, we consider the standard trivializations $T(U) \cong U \times T_0(U)$, $T(T(U)) \cong T(U) \times T(T_0(U)) \cong (U \times T_0(U)) \times (T_0(U) \times T_0(U))$, thus at $(x, \xi) \in T_x(U)$, $\eta \in T_0(U)$ is simultaneously used to describe two different vectors, viz., as coordinates for a tangent vector at x and as coordinates for a vector at $(x, \xi) \in T_x(U)$ tangent to $T_x(U)$. To avoid confusion we will write η for the first meaning, and η' for the second, thus η' is a "vertical" tangent vector. Writing $T_x(\ell) = \mathbb{C}\alpha$, and denoting the fibers $\varpi^{-1}(x)$ by $\mathcal{S}_x$, we have $T_{[\alpha]}(\mathcal{S}_x) = P'_\alpha/\mathbb{C}\alpha'$. Define now $\mathcal{P} \subset T(S)$ by $\mathcal{P}_{[\alpha]} = d\varpi^{-1}(P_\alpha)$. Then, $\mathcal{P} \subset T(\mathcal{S})$ is a holomorphic distribution of rank $2p + 1$, $p := \dim(\mathcal{C}_x(X))$, $\mathcal{P} \supset \mathcal{J}$.

On the other hand, writing $\mathcal{K}_{\mathrm{st}} \subset \mathcal{K}$ for the Zariski open subset consisting of standard minimal rational curves, we have on $\mathcal{K}_{\mathrm{st}}$ a holomorphic distribution $\mathcal{D} \subset T(\mathcal{K}_{\mathrm{st}})$, defined as follows. For $[\ell] \in \mathcal{K}$, $T_{[\ell]}(\mathcal{K}) = H^0(\ell, N_{\ell|X})$, where $N_{\ell|X}$ stands for the normal bundle of ℓ in X. When $[\ell] \in \mathcal{K}_{\mathrm{st}}$, we have $N_{\ell|X} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^q$, noting that $Q = \mathcal{O}(1)^p$ is the (strictly) positive part of the normal bundle $N_{\ell|X}$. $Q \subset N_{\ell|X}$ is characterized by the fact that $Q \otimes \mathcal{O}(-1)$ is spanned by $\Gamma(\ell, N_{\ell|X} \otimes \mathcal{O}(-1)) \cong \mathbb{C}^{2p}$, hence intrinsically defined, i.e., independent of the choice of Grothendieck decomposition. The assignment $[\ell] \mapsto \Gamma(\ell, \mathcal{O}(1)^p)$ defines a distribution $\mathcal{D}$ on $\mathcal{K}_{\mathrm{st}}$ of rank $2p$. We have

Proposition 2.1.1 (Hwang-Mok [HM01]). *Denoting by $\gamma : \mathcal{S} \to \mathcal{K}$ the canonical projection, we have $\mathcal{P} = d\gamma^{-1}(\mathcal{D})$. As a consequence $[\mathcal{F}, \mathcal{P}] \subset \mathcal{P}$, i.e., $\mathcal{F}$ lies on the Cauchy characteristic of the distribution $\mathcal{P}$. Moreover, assuming that for a general point $x \in X$, the projective second fundamental form σ on $\mathcal{C}_x$ is nondegenerate at a general smooth point $[\alpha] \in \mathcal{C}_x$. Then, $\mathcal{F} \subset P$ is exactly the Cauchy characteristic of $\mathcal{P}$.*

For the proof of the proposition we refer the reader to Hwang-Mok [HM99a, Corollary 3.1.5] and to Mok [Mk08b, (5.1), Proposition 5]. It suffices here to make a couple of remarks. First, given any holomorphic distribution $W \subset T(M)$ on a complex manifold, there is a holomorphic bundle homomorphism $\theta : \Lambda^2 W \to T(M)/W$ such that for any $\xi, \eta \in \Gamma(M, W)$ and for $x \in M$, we have $[\xi, \eta](x)$ mod $W = \theta(\xi, \eta)(x)$. We call θ the Frobenius form of $W \subset T(M)$. Denote now by φ the Frobenius form of $\mathcal{P}|_{\mathcal{S}} \subset T(\mathcal{S})$ and by ψ the Frobenius form of $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$. From the fact that $\gamma : \mathcal{S} \to \mathcal{K}_{\rm st}$ is a holomorphic submersion and from $\mathcal{P}|_{\mathcal{S}} = d\gamma^{-1}(\mathcal{D})$, for $\xi \in \mathcal{S}$ and $u, v \in \mathcal{S}_\xi$ it follows readily that $\psi(d\gamma(u), d\gamma(v)) = \beta(\varphi_\xi(u, v))$ where the bundle isomorphism $\beta : T(\mathcal{S})\big/\mathcal{P}|_{\mathcal{S}} \xrightarrow{\cong} \gamma^*(T(\mathcal{K}_{\rm st})/\mathcal{D})$ is naturally induced by $d\gamma$. From $\mathcal{F} = (d\gamma)^{-1}(0)$ it now follows readily that $[\mathcal{F}, \mathcal{P}] \subset \mathcal{P}$. The Frobenius form φ and equivalently the Frobenius form ψ can furthermore be computed in terms of the second fundamental forms σ of $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ for a general point $x \in X$, and the last statement of Proposition 2.1.1 follows from the computation.

Proposition 2.1.1 yields immediately that the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x$ is a birational map for a general point. Moreover, the Cartan-Fubini Extension Principle holds true. In fact, given $f : U \xrightarrow{\cong} V$ such that $f_\sharp(\mathcal{C}(X)|_U) = \mathcal{C}(Z)|_V$ as in the hypothesis of Theorem 2.1.2, denoting $\mathcal{P}$ by $\mathcal{P}(X)$ and the analogous distribution on $\mathcal{C}(Z)$ by $\mathcal{P}(Z)$, obviously $f_{\sharp *}(\mathcal{F}_X)$ lies on the Cauchy characteristic of $\mathcal{P}(Z)$, and from the characterization of the Cauchy characteristic as in Proposition 2.1.1 it follows that the local VMRT-preserving map $f : U \xrightarrow{\cong} V$ actually preserves the tautological foliation. That the foliation-preserving property implies the extendibility of f to a biholomorphism $F : X \xrightarrow{\cong} Z$ was established in [HM01] by a combination of techniques of analytic continuation in several complex variables and the deformation theory of rational curves.

On top of being intrinsic, the approach in [HM01] introduced into the subject differential systems on VMRT structures $\mathcal{C}(X) \subset \mathbb{P}T(X)$ and on minimal rational components $\mathcal{K}$. So far the distribution $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$ has not been much studied. Especially, when a uniruled projective manifold X carries some extra geometric structure, e.g., a contact structure, the distribution $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$ can be further enriched leading to an enhanced differential system on $\mathcal{K}_{\rm st}$, and it is tempting to believe that in certain cases this could lead to uniqueness or rigidity results concerning $\mathcal{K}$. The case where X carries a contact structure is especially interesting in view of the long-standing conjecture that a Fano contact manifold of Picard number 1 is homogeneous.

For applications of Cartan-Fubini extension in the equidimensional case we refer the reader to [HM01] and [HM04b]. Here we only note that in [HM04b] we obtained a new solution to the Lazarsfeld Problem, viz., proving that for $S := G/P$ a rational homogeneous space of Picard number 1 other than the projective space, any finite surjective holomorphic map $f : S \to X$ onto a projective manifold X

must necessarily be a biholomorphism. The proof there was based on Cartan-Fubini extension applied to VMRTs of the uniruled projective manifold X. Our original proof in [HM99b] was an application of our geometric theory of VMRTs at an early stage of its development relying heavily on Lie theory, especially on results concerning G-structures of Ochiai [Oc70] (Theorem 2.2.1 here) in the symmetric cases and those concerning differential systems on G/P of Yamaguchi [Ya93] in the non-symmetric cases. In [HM04b] the geometric theory on VMRTs was more self-contained, and we succeeded in entirely removing the detailed knowledge about G/P from the solution of Lazarsfeld's Problem.

2.2 Cartan-Fubini Extension in the Non-Equidimensional Case

Generalizing the arguments for the differential-geometric proof of Ochiai's Theorem, Hong-Mok [HoM10] established the following non-equidimensional Cartan-Fubini extension theorem.

Theorem 2.2.1 (Hong-Mok [HoM10, Theorem 1.1]). *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds equipped with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f : U \to X$ be a holomorphic embedding defined on a connected open subset $U \subset Z$. If f respects varieties of minimal rational tangents and is nondegenerate with respect to $(\mathcal{H}, \mathcal{K})$, then f extends to a rational map $F : Z \to X$.*

Here we say that f respects VMRTs if and only if $df(\widetilde{\mathcal{C}}_z(Z)) = \widetilde{\mathcal{C}}_{f(z)}(X) \cap df(T_z(Z))$, i.e., $f_\sharp(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(X) \cap f_\sharp(\mathbb{P}T_z(Z))$, where $f_\sharp$ is the projectivization of df. The holomorphic embedding $f : U \to X$ is said to be nondegenerate with respect to $(\mathcal{K}, \mathcal{H})$ if (a) its image $f(U)$ is not contained in the bad locus of $(X, \mathcal{K})$, and (b) at a general point $z \in U$ and a general smooth point $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$ such that the second fundamental form $\sigma(\eta, \xi)$ of $\widetilde{\mathcal{C}}_{f(z)}(X) \subset T_{f(z)}(X)$ at $df(\alpha)$, when restricted in ξ to the vector subspace $T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))) \subset T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X))$ and regarded as a family of linear maps on $T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X))$ in η, has common kernel $\mathbb{C}df(\alpha)$. Thus,

$$(\dagger) \quad \left\{\eta \in T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X)) : \sigma(\eta, \xi) = 0 \text{ for any } \xi \in T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z)))\right\} = \mathbb{C}df(\alpha)\,.$$

Alternatively (†) means that on $\mathcal{C}_{f(z)}(X)$, considering the projective second fundamental form $\sigma_{f_\sharp([\alpha])}$ of $f_\sharp(\mathcal{C}_z(Z)) \subset \mathcal{C}_{f_\sharp([\alpha])}(X)$ at $f_\sharp([\alpha])$, the common kernel of $\sigma(\,\cdot\,, \xi), \xi \in T_{f_\sharp([\alpha])}(\mathcal{C}_{f(z)}(X))$, reduces to 0. Using the arguments of analytic continuation along minimal rational curves as developed in Hwang-Mok [HM01], the key issue that we settled in Hong-Mok [HoM10] was to prove that f maps a germ of minimal rational curve on $(Z, \mathcal{H})$ into a germ of minimal rational curve on $(X, \mathcal{K})$. Equivalently we showed that the image $f_{\sharp *}(\mathcal{F}_Z)$ on $f_\sharp(\mathcal{C}(Z)|_U) \subset \mathcal{C}(X)$ agrees with the restriction of $\mathcal{F}_X$ to $f_\sharp(\mathcal{C}(Z)|_U)$ as holomorphic line subbundles of $T(\mathcal{C}(X))|_{f_\sharp(\mathcal{C}(Z)|_U)}$. We prove Theorem 2.2.1 along the line of a proof of Ochiai's Theorem (Theorem 2.1.1) using the Euclidean flat connection and Harish-Chandra coordinates as explained, showing that, in the case where Z and X are Hermitian symmetric and using Harish-Chandra coordinates, for the Hessian $d^2f(\xi, \eta)$ we have $d^2f(\alpha, \alpha) \in \mathbb{C}df(\alpha)$ when evaluated at a point $z \in Z$ and at a vector $\alpha \in \widetilde{\mathcal{C}}_z(Z)$ under the nondegeneracy condition on the second fundamental form as stated.

When Z and X are irreducible Hermitian symmetric spaces of the compact type and of rank ≥ 2 the proof is the same as in Mok [Mk99]. In general, one makes use of special coordinate systems, as follows. Let $z \in Z$ be a general point and $\alpha \in \widetilde{\mathcal{C}}_z(Z)$ be a smooth point such that $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$. Let $\ell \subset Z$ be the minimal rational curve with a marking at z, and assume that z is a smooth point of ℓ for convenience. Let $z' \in \ell$ be a smooth point close to z. Write $T_{z'}(\ell) = \mathbb{C}\alpha'$. Let $\mathcal{D} \subset \mathcal{C}_{z'}(Z)$ be a smooth neighborhood of $[\alpha']$ on $\mathcal{C}_{z'}(Z)$, and $\mathcal{O}$ be a neighborhood of 0 in $\mathbb{C}^p$ such that the assignment $t = (t_1, \cdots, t_p) \mapsto [\alpha'_t] \in \mathcal{D}$, $\alpha'_0 = \alpha'$, defines a biholomorphism from $\mathcal{O}$ onto $\mathcal{D}$. Consider now the family of minimal rational curves parametrized by $\mathcal{O}$ given by a holomorphic map $\Phi : \mathbb{P}^1 \times \mathcal{O} \to Z$ such that $\Phi(0, t) = z'$ for $t \in \mathcal{O}$, $\Phi(s, 0) \in \ell'_0 := \ell$ and such that, for $t \in \mathcal{O}$, $\varphi_t(s) := \Phi(s, t)$ parametrizes the minimal rational curve ℓ'_t passing through z' such that $T_{z'}(\ell'_t) = \mathbb{C}\alpha'_t$, $\alpha'_t \in \mathcal{D}$. We may assume that $z = \varphi_0(s_0)$ for some $s_0 \in \Delta$, $\Phi|_{\Delta^* \times \mathcal{O}}$ is an embedding and that $\varphi_t|_\Delta$ is an embedding for $t \in \mathcal{O}$.

Consider a holomorphic coordinate chart on a neighborhood U of z' in Z, $z \in U$, in which the minimal rational curves near $\ell'_0 = \ell$ passing through z' are represented on the chart as open subsets of lines through the origin. For each general point $w \in Z$, let $\mathcal{V}_w$ be the union of minimal rational curves passing through w. Thus, $\Sigma := \Phi(\Delta \times \mathcal{O}) \subset \mathcal{V}_{z'}$. Writing the parametrization as $\Phi(s, t) = s(\psi_1(t), \cdots, \psi_n(t)) = s\psi(t)$ in terms of the chosen Euclidean coordinates, observe that Σ is smooth along $\varphi_0(\Delta^*)$ and that for $w \in \varphi_0(\Delta^*)$ we have $T_w(\Sigma) = \mathrm{Span}\big\{(\psi(0), \frac{\partial \psi}{\partial t_1}(0), \cdots, \frac{\partial \psi}{\partial t_p}(0)\big\}$, which is independent of s. We call this the tangential constancy of Σ along the minimal rational curve ℓ. From the basics in the deformation theory of rational curves this implies that VMRTs are tangentially constant (in an obvious sense) along ℓ. The latter applies to all minimal rational curves at the same time when there is a coordinate system in which all minimal rational curves are represented by affine lines, which is in particular the case for Harish-Chandra coordinates in the Hermitian symmetric case, and that was the reason underlying the differential-geometric proof of Ochiai's Theorem.

In general for the computation at $z \in \ell \subset Z$, we have to resort to the special coordinates arising from some nearby point z' lying on ℓ, as described in the above. Compared to Riemannian geometry, the latter may be taken as an analogue of normal geodesic coordinates at z' in which minimal rational curves passing through z' appear as radial lines. Since other minimal rational curves intersecting with the chart need not be represented as affine lines, an elementary approximation argument was needed to carry through the proof, as was done in Hong-Mok [HoM10, Lemma 2.7].

Recently Hwang [Hw14b] has a generalized formulation of non-equidimensional Cartan-Fubini extension, as follows.

Theorem 2.2.2 (Hwang [Hw14b, Theorem 1.3]). *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f : U \to X$ be a holomorphic embedding defined on a connected open subset $U \subset Z$. Suppose $f_\sharp(\mathcal{C}(X)|_U) \subset \mathcal{C}(Z)$ and*

$$\left\{\eta \in T_{f_\sharp([\alpha])}(\mathcal{C}_{f(z)}(X)) : \sigma(\eta, \xi) = 0 \text{ for any } \xi \in T_{f_\sharp([\alpha])}(f_\sharp(\mathcal{C}_z(Z)))\right\} = 0\ ,$$

then f extends to a rational map $F : Z \to X$.

Hwang [Hw14b] made use of differential systems and Lie brackets of holomorphic vector fields more in the spirit of the proof in Hwang-Mok [HM01] and does not require the use of adapted coordinates, and the proof is therefore more intrinsic, although the original proof in Hong-Mok [HoM10] also applies to give the same statement. One motivation for the more generalized formulation is that even when dim(Z) = dim(X), Theorem 2.2.2 exceeds the equidimensional Cartan-Fubini extension in Theorem 2.1.2. The context applies, by the method of Hwang-Kim [HK13a] to equidimensional maps given by suitable double covers branched over Fano manifolds of Picard number 1 of large index.

3 Characterization and Recognition of Homogeneous VMRT Structures

3.1 Uniruled Projective Manifolds Equipped with Reductive Holomorphic G-Structures

Just as the flat Euclidean space (as a germ) is characterized among Riemannian manifolds by the vanishing of the curvature tensor, flat G-structures are characterized by the vanishing of certain structure functions (Guillemin [Gu65]). For $k \geq 1$, a G-structure $\mathcal{G}(X) \subset \mathcal{F}(X)$ is k-flat at x if and only if there exists a germ of biholomorphism $f : (X; x) \to (V; 0)$ such that $f_*\mathcal{G}$ is tangent to the flat G-structure $\mathcal{G}' = \mathrm{G} \times V$ along $\mathcal{G}'_0$ to the order $\geq k$. When a given G-structure on X is k-flat at every point, there is a naturally defined structure function c^k which measures the obstruction to $(k+1)$-flatness, which is a holomorphic 2-form taking values in some quotient bundles of tensor bundles of the form $T(X) \otimes S^kT^*(X)$. G acts on these quotient bundles. In the event that $\mathrm{G} \subsetneq \mathrm{GL}(V)$ is *reductive*, by identifying the latter quotient bundles with G-invariant vector subbundles of $T(X) \otimes S^kT^*(X)$, the structure functions concerned correspond to holomorphic sections θ_k of $\text{Hom}\big(\Lambda^2T(X), T(X) \otimes S^kT^*(X)\big)$. In this case for proving flatness it suffices to check the vanishing of a finite number of θ_k. Concerning uniruled projective manifolds endowed with reductive G-structures we have the following result of Hwang-Mok [HM97].

Theorem 3.1.1 (Hwang-Mok [HM97]). *Let X be a uniruled projective manifold admitting an irreducible reductive G-structure, $G \subsetneq GL(V)$. Then, X is biholomorphic to an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2.*

We refer the reader to Hwang-Mok [HM99a] and Mok [Mk08b, (4.3)] for discussions on G-structures on uniruled projective manifolds surrounding the above theorem, and will be contented here with some remarks on the proof of the theorem. When a G-structure $\mathcal{G} \subset \mathcal{F}(X)$ is defined we have an associated homogeneous holomorphic fiber subbundle $\mathcal{W} \subset \mathbb{P}T(X)$, where the fibers $\mathcal{W}_x \subset \mathbb{P}T_x(X)$ are highest weight orbits. The first step of the proof consists of showing that $\mathcal{W}$ agrees with the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$, and the proof is based on Grothendieck's classification of G-principal bundles on $\mathbb{P}^1$ (Grothendieck [Gro57]). The identification $\mathcal{C}(X) = \mathcal{W}$ implies that every minimal rational curve is standard, and that $\mathcal{C}_x(X)$ agrees with the VMRT of an irreducible Hermitian symmetric space S of the compact type of rank $r \geq 2$, i.e., the G-structure is an S-structure. After that it remains to check the vanishing of structure functions interpreted as elements $\theta_k \in \Gamma\big(X, \text{Hom}(\Lambda^2T(X), T(X) \otimes S^kT^*(X)\big)$. When θ_k is restricted to elements of the form $\alpha \wedge \xi$, where $\alpha \in \widetilde{\mathcal{C}}_x(X)$ and $\xi \in P_\alpha$, then $\theta_k(\alpha, \xi) = 0$ follows by restricting to standard minimal rational curves ℓ and checking degrees of summands in Grothendieck decomposition basing on $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, $T_x(\ell) = \mathbb{C}\alpha$ (assuming ℓ to be embedded to simplify the notation), noting that for $\xi \in P_\alpha, \alpha \wedge \xi \in \Lambda^2T(X)|_\ell$ belongs to a direct summand of degree 3, while all direct summands of $(T(X) \otimes S^kT^*(X))|_\ell$ are of degree ≤ 2. The flatness of the G-structure results from the fact that $\{\alpha \wedge P_\alpha : \alpha \in \widetilde{\mathcal{C}}_x(X)\}$ spans $\Lambda^2T_x(X)$, cf. Hwang-Mok [HM98, (5.1), Proposition 14].

3.2 Recognizing Rational Homogeneous Spaces of Picard Number 1 from the VMRT at a General Point

For a rational homogeneous space $S = G/P$ of Picard number 1, denoting by $\mathcal{O}(1)$ the positive generator of $\text{Pic}(S) \cong \mathbb{Z}$ we identify $S \subset \mathbb{P}(\Gamma(S, \mathcal{O}(1))^*)$ as a projective submanifold via the minimal embedding by $\mathcal{O}(1)$, and equip S with the minimal rational component on S consisting of projective lines on S. We are interested in characterizing a given uniruled projective manifold in terms of its VMRTs as projective submanifolds. Especially we have the following Recognition Problem for rational homogeneous spaces of Picard number 1 formulated in Mok [Mk08c].

Definition 3.2.1. Let $S = G/P$ be a rational homogeneous space of Picard number 1, and $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ be the VMRT of S at $0 = eP \in S$. For any uniruled projective manifold X of Picard number 1 equipped with a minimal rational component $\mathcal{K}$, we denote by $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ the VMRT of $(X, \mathcal{K})$ at a general point $x \in X$. We say that the Recognition Problem for S is solved in the affirmative if any uniruled projective manifold X of Picard number 1 must necessarily be biholomorphic to S whenever $\big(\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)\big)$ is projectively equivalent to $\big(\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)\big)$.

Cho, Miyaoka and Shepherd-Barron [CMS02] proved the characterization of the projective space $\mathbb{P}^n$ among uniruled projective manifolds by the fact that for a minimal rational curve ℓ we have $K_X^{-1} \cdot \ell = n + 1$, i.e., equipping X with some minimal rational component, the assumption $\mathcal{C}_x(X) = \mathbb{P}T_x(X)$ at a general point $x \in X$ implies that X is biholomorphic to $\mathbb{P}^n$. The purpose of the Recognition Problem was to deal with the characterization of $S = G/P$ different from a projective space, i.e., where $\mathcal{C}_0(S) \subsetneq \mathbb{P}T_0(S)$ at $0 = eP$. The following theorem gives our current state of knowledge on the Recognition Problem for rational homogeneous spaces of Picard number 1.

Theorem 3.2.1 (Mok [Mk08c], Hong-Hwang [HH08]). *Let G be a simple complex Lie group, $P \subset G$ be a maximal parabolic subgroup corresponding to a long simple root, and $S := G/P$ the corresponding rational homogeneous space of Picard number 1. Then, the Recognition Problem for S is solved in the affirmative.*

We refer the reader to Mok [Mk08b, (6.3)] for an exposition revolving around the Recognition Problem and the proof of Theorem 3.2.1. Here in its place we will explain the principle underlying our approach and give some highlights on how the principle applies in the proof of the theorem. The first geometric link between VMRT structures and differential geometry was the author's proof of the Generalized Frankel Conjecture (Mok [Mk88]) in Kähler geometry which characterizes compact Kähler manifolds of semipositive holomorphic bisectional curvature. In particular, if X is a Fano manifold of Picard number 1 admitting a Kähler metric g of semipositive holomorphic bisectional curvature, then by [Mk88] it must be biholomorphic to an irreducible Hermitian symmetric space S of the compact type. Regarding X as a uniruled projective manifold, it was proven that the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$ on X is invariant under holonomy of a metric g_t, $t > 0$, obtained from g by the Kähler-Ricci flow. (This was the core result of [Mk88] even though the term VMRT had not been introduced at that point.) In the absence of a Kähler metric with special properties, it was a challenge to introduce some notion of parallel transport that makes sense in the general setting of a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$ and hence with the associated VMRT structure $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T(X)$.

Our study of the Recognition Problem actually went back to the work Mok [Mk02] concerning the conjecture of Campana-Peternell [CP91] on compact complex manifolds with nef tangent bundle (cf. (3.3)), where we devised a method for reconstructing a Fano manifold X of Picard number 1 with nef tangent bundle from its VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ in the very special case where $\dim(\mathcal{C}_x(X)) = 1$. In the general case where the VMRT of $(X, \mathcal{K})$ at a general point $x \in X$ is congruent to that of the model, i.e., $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$, $S = G/P$, a priori there is a subvariety $E \subset X$, where $\mathcal{C}_y(X)$ is not known to be congruent to the model for $y \in E$. The key of the affirmative solution in the Hermitian symmetric case is a removable singularity theorem in codimension 1, viz., the assertion that for each irreducible component H of E which is a hypersurface of X, a general point y on H is a removable singularity for the VMRT structure. Note that in the Hermitian symmetric case the orbit of $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ is parametrized by an affine-algebraic variety $\mathcal{M} \subset \mathbb{C}^N$. In a neighborhood U of y in X, the VMRT structure can be described by a holomorphic map $\varphi : U - E \to \mathcal{M} \subset \mathbb{C}^N$, and saying that the S-structure has a removable singularity at y is the same as saying that the vector-valued holomorphic map has a removable singularity. Once we have proven a removable singularity theorem in codimension 1, it follows by Hartogs extension that the S-structure extends holomorphically to X, and we conclude that $X \cong S$ by Theorem 3.1.1, as desired.

In what follows, to simplify the notation we assume that standard rational curves are embedded. To prove the removable singularity theorem for VMRT structures in the Hermitian symmetric case we introduce a method of parallel transport of VMRTs. Note that every curve on X must intersect a hypersurface since X is of Picard number 1. Starting with a standard rational curve $\ell \not\subset E$ and considering the lifting $\widehat{\ell} \subset \mathbb{P}T(X)|_\ell$, we transport the second fundamental form $\sigma_{[\alpha]}$ of $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ at $[T_x(\ell)] := [\alpha]$ over $x \in \ell \cap (X - E)$ to the second fundamental form $\sigma_{[\beta]}$ of $\mathcal{C}_y(X) \subset \mathbb{P}T_y(X)$ at $[T_y(\ell)] := [\beta]$ over $y \in \ell \cap H$. Observing that the second fundamental form σ along $\widehat{\ell}$ gives a holomorphic section of a holomorphic vector bundle V (of rank $:= r$) which is *holomorphically trivial* [Mk08c, (6.1), Proposition 6], which results from the fact that ℓ is a standard rational curve, we have a *parallel transport* in the sense that $V|_{\widehat{\ell}} \cong \mathbb{C}^r \times \widehat{\ell}$, and that an element $\xi \in V_{[\alpha]}$ is transported to $\gamma([\beta]) \in V_{[\alpha]}$ for the unique holomorphic section $\gamma \in \Gamma(\widehat{\ell}, V)$ such that $\gamma([\alpha]) = \xi$. This already yields a removable singularity theorem for VMRT structures in the case of the hyperquadric, since the smooth hyperquadric $Q^{n-2} \cong \mathcal{C}_0(Q^n) \subset \mathbb{P}T_0(Q^n) \cong \mathbb{P}^{n-1}$ cannot be deformed to a singular hyperquadric unless its second fundamental form at a general point also degenerates. In the general case of S-structures more work is required, e.g., in the case where the VMRT $\mathcal{C}_0(S)$ is itself an irreducible Hermitian symmetric space of rank ≥ 2, parallel transport of the second fundamental form along $\widehat{\ell}$ implies a parallel transport of the VMRT structure of $\mathcal{C}_x(X)$, in a neighborhood of $[\alpha]$ to a neighborhood of $[\beta]$ on $\mathcal{C}_y(X)$. Here $\mathcal{C}_x(X)$ is regarded itself as a uniruled projective manifold equipped with the minimal rational component $\mathcal{H}$ consisting of projective lines in $\mathbb{P}T_x(X)$ lying on $\mathcal{C}_x(X)$, and the parallel transport of the VMRT structure of $(\mathcal{C}_x(X), \mathcal{H})$ is shown to be enough to force a removable singularity theorem of S-structures in codimension 1.

In [Mk08c] the argument was carried through also in the contact case, where on top of parallel transport of the second fundamental form we also introduced parallel transport of the third fundamental form, and resort to the work of Hong [Ho00] on the characterization of contact homogeneous spaces which replaces Theorem 3.1.1. When the VMRT of X at a general point is congruent to $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ for a Fano homogeneous contact manifold S of Picard number 1 other than an odd-dimensional projective space, the linear span of VMRTs on X defines a meromorphic distribution D of co-rank 1 on X, and parallel transport of the third fundamental form by the same principle as explained in the last paragraph is made possible by the splitting type $D|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^r \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)$. The argument was generalized in the other long-root cases by Hwang-Hong [HH08], in which analogues of the results of [Ho00] were obtained to solve the Recognition Problem in the affirmative for the remaining long-root cases.

Remarks.

(a) In [Mk88], in the event that $\mathcal{C}_x(X) = \mathbb{P}T_x(X)$ and the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x$ is not a biholomorphism at a general point, we proved that there exists a hypersurface $\mathcal{H} \subset \mathbb{P}T(X)$ which is invariant under holonomy of (X, g_t), $t > 0$ sufficiently small. A posteriori this situation does not occur.
(b) In Mok [Mk08b, (6.3), Conjecture 6], $S = G/P$ should have been "a Fano homogeneous space of Picard number 1" (instead of "a Fano homogeneous contact manifold of Picard number 1"). The Recognition Problem was expected to be always solved in the affirmative for $S = G/P$ of Picard number 1.

3.3 Rationale, Applications and Generalizations of the Recognition Problem

The original motivation of the Recognition Problem was an attempt to tackle the Campana-Peternell Conjecture in [CP91], which may be regarded as a generalization of the Hartshorne Conjecture, or as an algebro-geometric analogue of the Generalized Frankel Conjecture in Kähler geometry. It concerns projective manifolds X with nef tangent bundle. Especially, under the assumption that X is Fano and of Picard number 1, according to the Campana-Peternell Conjecture X is expected to be biregular to a rational homogeneous space, i.e., $X \cong G/P$, where G is a simple complex Lie group, and $P \subset G$ is a maximal parabolic subgroup. In Mok [Mk02] we took the perspective of reconstructing X from its VMRTs, and solved the problem in the very special case where X admits a minimal rational component $\mathcal{K}$ for which VMRTs are one-dimensional, and where in addition $b_4(X) = 1$. The latter topological condition was removed by Hwang [Hw07], leading to

Theorem 3.3.1 (Mok [Mk02], Hwang [Hw07]). *Let X be a Fano manifold of Picard number 1 with nef tangent bundle. Suppose X is equipped with a minimal rational component for which the variety of minimal rational tangents at a general point $x \in X$ is one-dimensional. Then, X is biholomorphic to the projective plane $\mathbb{P}^2$, the 3-dimensional hyperquadric Q^3, or the 5-dimensional Fano contact homogeneous space $K(G_2)$ of type G_2. In particular, X is a rational homogeneous space.*

Here we note that a weaker condition than the nefness of the tangent bundle was used in the proof of the theorem, viz., only the nefness of the restriction of the tangent bundle $T(X)$ to rational curves was used. The latter implies that deformation of rational curves is unobstructed, and hence the minimal rational component $\mathcal{K}$ is a projective manifold, and we have the universal family $\rho : \mathcal{U} \to \mathcal{K}$ accompanied by the evaluation map $\mu : \mathcal{U} \to X$ which is also a holomorphic submersion. When VMRTs are one-dimensional, under the nefness assumption one shows easily that the fibers $\mathcal{U}_x$ of $\mu : \mathcal{U} \to X$ are smooth rational curves. It was proven in [Mk02] that for the $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ the direct image of the relative tangent bundle T_ρ gives a rank-3 bundle which is stable, yielding by an application of the Bogomolov inequality that $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is of degree $d \leq 4$ under the additional assumption $b_4(X) = 1$, and for the equality case we resorted to the existence result of Uhlenbeck-Yau [UY86] on Hermitian-Einstein metrics on stable holomorphic vector bundles over projective manifolds to arrive at a contradiction, leaving behind the options $d = 1, 2, 3$. These options do exist as is given in the statement of Theorem 3.3.1. The Recognition Theorem now enters, allowing us to recover the hyperquadric Q^3 from the VMRT in case $d = 2$, and to recover the five-dimensional Fano contact homogeneous space $K(G_2)$ in case $d = 3$. They served as prototypes for the Recognition Problem for the Hermitian symmetric case and the Fano contact case as solved in the affirmative in Mok [Mk08c].

While Theorem 3.3.1 concerns a very special case of the Campana-Peternell Conjecture, it is worth noting that there is no assumption on the dimension of X itself. It is tempting to think that for VMRTs which are of dimension 2 one could identify the possible VMRTs for X of Picard number 1 and of nef tangent bundle, and recover X through the Recognition Problem. It appears for the time being conceptually difficult to devise a strategy for a solution of the Campana-Peternell Conjecture for Fano manifolds of Picard number 1 along the lines of thought in Theorem 3.3.1 since in the short-root case VMRTs are only almost homogeneous. Since a key element in the proof of Theorem 3.3.1 was a bound on the degree of the tangent map it would be meaningful to try to get an a priori bound of $\dim(X)$ in terms of dimensions of VMRTs under the nefness assumption on the tangent bundle.

Another reason for introducing the Recognition Problem was to give a conceptually unified proof of the deformation rigidity under Kähler deformation of rational homogeneous spaces $S = G/P$ of Picard number 1. The rigidity problem was taken up in Hwang-Mok [HM98], Hwang [Hw97], and Hwang-Mok [HM02, HM04a, HM05], according to an underlying classification scheme for $X = G/P$ in terms of complexity into the Hermitian symmetric case [HM98], the Fano contact case [Hw97], the remaining long-root cases [HM02], and the short-root cases [HM04a, HM05]. As it turned out, the case of the seven-dimensional Fano homogeneous contact manifold $\mathbb{F}^5$, i.e., the Chow component of minimal rational curves on the hyperquadric Q^5, was missed in Hwang [Hw97] and again in Hwang-Mok [HM02], and it was later found by Pasquier-Perrin [PP10] that $\mathbb{F}^5$ admits a deformation to a G_2-horospherical variety X^5. The corrected statement about the rigidity problem under Kähler deformation of rational homogeneous spaces $S = G/P$ of Picard number 1 is given by

Theorem 3.3.2. *Let $S = G/P$ be a rational homogeneous space of Picard number 1 other than the 7-dimensional Fano homogeneous contact manifold $\mathbb{F}^5$. Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of projective manifolds such that the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. Then, X_0 is also biholomorphic to S.*

There was a general scheme of proof adopted in the series of articles mentioned on the rigidity of $S = G/P$ under Kähler deformation. Note that X_0 is a uniruled projective manifold equipped with the minimal rational component $\mathcal{K}_0$ whose general point is a free rational curve of degree 1 with respect to the positive generator $\mathcal{O}(1)$ of $\mathrm{Pic}(X_0) \cong \mathbb{Z}$. The general scheme consists first of all of a proof that at a general point x_0 of the central fiber X_0, the VMRT $\mathcal{C}_{x_0}(X_0) \subset \mathbb{P}T_{x_0}(X_0)$ is projectively equivalent to the VMRT $\mathcal{C}_{x_t}(X_t) \subset \mathbb{P}T_{x_t}(X_t)$ at any point $x_t \in X_t$ for $t \neq 0$, i.e., projectively equivalent to the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ of the model manifold $S = G/P$ at the reference point $0 = eP$. We may call this the invariance of VMRTs (at a general point) under Kähler deformation. Shrinking Δ around 0 and rescaling the variable t if necessary, the comparison of VMRTs on different fibers was done by choosing $x_t = \sigma(t)$ for a holomorphic section $\sigma : \Delta \to \mathcal{X}$ where $\sigma(0)$ avoids the bad set B of $(X_0, \mathcal{K}_0)$. The VMRTs $\mathcal{C}_{\sigma(t)}(X_t)$, $t \in \Delta$, are images of the tangent map $\tau_{\sigma(t)} : \mathcal{U}_{\sigma(t)}(X_t) \dashrightarrow \mathcal{C}_{\sigma(t)}(X_t)$, where the set $\{\mathcal{U}_{\sigma(t)}(X_t) : t \in \Delta\}$ constitutes a regular family of projective manifolds. With some oversimplification in most long-root cases the latter fact allows us to deduce $\mathcal{U}_{\sigma(0)} \cong \mathcal{U}_0(S)$ by an inductive argument. For the inductive argument we recall that the tangent map $\tau_0 : \mathcal{U}_0(S) \xrightarrow{\cong} \mathcal{C}_0(S)$ is a biholomorphism and note that in the long-root case $\mathcal{U}_0(S) \cong \mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ is a Hermitian symmetric space of the compact type of rank ≤ 3. An inductive argument applies in the long-root case when $\mathcal{C}_0(S)$ is irreducible but there are cases where $\mathcal{C}_0(S)$ is reducible, as for example, the Grassmannian $G(p,q)$ of rank ≥ 2 where the VMRT is given by the Segre embedding $\varsigma : \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \to \mathbb{P}^{pq-1}$. In the latter case invariance of VMRTs under Kähler deformation was established in [HM98] by cohomological considerations in the deformation of projective subspaces of X_t associated to factors of $\mathcal{C}_0(G(p,q)) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$. After the invariance of VMRTs under deformation has been established, which in the short-root cases involves explicit descriptions of the VMRTs (which are only almost homogeneous) and their deformations, properly speaking the geometric theory of VMRTs enters in the study of the tangent map. In [HM98] it was proven in a general setting for uniruled projective manifolds of Picard number 1 that the distribution $\mathcal{W}$ spanned by VMRTs is not integrable, while sufficient projective-geometric conditions were given for the integrability of $\mathcal{W}$ forcing the tangent map τ_{x_0} to be an isomorphism at a general point $x_0 \in X_0$ in [HM98]. In the other long root cases the VMRTs of the model manifold are linearly degenerate, and we made use of the more general result that $\mathcal{W}$ must be bracket generating for a uniruled projective manifold X of Picard number 1 to reach the same conclusion in [HM02]. Here we say that a distribution $\mathcal{W}$ on X is bracket generating to mean that the tangent subsheaf generated by $\mathcal{W}$ from taking successive Lie brackets is the tangent sheaf. The same resoning was applied to the short-root cases in [HM04a] and [HM05]. In the cases of [HM05], the VMRTs of the model spaces are linearly nondegenerate.

For the long-root case to conclude $X_0 \cong S$ one makes use of results from differential systems, viz., Ochiai's theorem [Oc70] in the Hermitian symmetric case and Yamaguchi's result on differential systems [Ya93] for long-root cases other than the symmetric and contact cases. (In the contact case [Hw97] uses a simpler argument.) There is one short-root case of type F_4 in which the VMRT is linearly degenerate which was treated in [HM04a] along the line of [HM02], while the most difficult case of the rigidity problem was the short-root cases of the symplectic Grassmannian $S_{k,\ell}$ (cf. below) and the remaining F_4 short-root case, where the VMRTs are linearly nondegenerate. In the latter cases it was after establishing invariance of VMRTs that the real difficulty emerges, viz., the key issue was how one can recover S from its VMRTs. It was tempting to give a unified argument on rigidity under Kähler deformation (with one exception) as a consequence of (a) invariance of VMRTs under deformation and (b) an affirmative solution of the Recognition Problem. Such a unified scheme of proof would apply to the central fiber X_0 as a separate uniruled projective manifold equipped with the minimal rational component $\mathcal{K}$ without using the fact that it is the central fiber of a family such that the other fibers are biholomorphic to the model manifold $S = G/P$. In the exceptional case of the seven-dimensional Fano homogenous contact manifold $\mathbb{F}^5$ (cf.Theorem 3.3.2) rigidity under Kähler deformation fails precisely because (a) fails.

While the affirmative solutions of the Recognition Problem in the long-root cases give a unified explanation of the phenomenon of deformation rigidity (with one exception) in those cases, the same problem for the short-root cases remains unresolved. An important feature for the primary examples of symplectic Grassmannians $X = S_{k,\ell}$ is the existence of local differential-geometric invariants which cannot possibly be captured by the VMRT at a general point. To explain this we describe the symplectic Grassmannian. Consider a complex vector space W of dimension 2ℓ equipped with a symplectic form ω. Let k be an integer, $1 < k < \ell$, and consider the subset $S_{k,\ell} \subset \mathrm{Gr}(k, W)$ of k-planes in W isotropic with respect to ω. Let $x = [E] \in S_{k,\ell}$ be an arbitrary point. Suppose $A^{(k-1)}$ resp. $B^{(k+1)}$ are vector subspaces of W of dimension $k-1$ resp. $k+1$ such that $A^{(k-1)} \subset E \subset B^{(k+1)}$. Suppose furthermore that $B^{(k+1)}$ is isotropic with respect to ω. Let $\Gamma \subset \mathrm{Gr}(k, W)$ be the rational curve consisting of all k-planes F such that $A^{(k-1)} \subset F \subset B^{(k+1)}$. Then, $\omega|_F \equiv 0$ for every $[F] \in \Gamma$, hence $\Gamma \subset S_{k,\ell}$. For a minimal rational curve Γ as described we have $T_x(\Gamma) = \mathbb{C}\lambda$, where $\lambda \in \mathrm{Hom}(E, W/E) = T_x(S_{k,\ell})$ such that $\lambda|_{A^{(k-1)}} \equiv 0$ and $\mathrm{Im}(\lambda) = B^{(k+1)}/E$. The set of all $[T_x(\Gamma)] \in \mathbb{P}T_x(S_{k,\ell})$ thus described is given by $\mathcal{S}_x := \big\{[a \otimes b] : 0 \neq a \in E^*, 0 \neq b := \beta + E \in W/E, \omega|_{E+\mathbb{C}\beta}\big\} \equiv 0$. Thus, writing $Q_x = E^{\perp}/E$ we have $\mathcal{S}_x = \zeta(\mathbb{P}(E^*) \times \mathbb{P}(Q_x))$, where $\zeta : \mathbb{P}(E^*) \times \mathbb{P}(Q_x)) \to \mathbb{P}(E^* \otimes Q_x)$ is the Segre embedding. The assignment $E \to E^* \otimes Q_x$ defines a holomorphic distribution D on $S_{k,\ell}$ which is invariant under the symplectic group $\mathrm{Sp}(W, \omega) = \mathrm{Aut}(S_{k,\ell})$.

The minimal rational curves $\Gamma \subset S_{k,\ell}$ described in the above are special. In the definition of a minimal rational curve containing the k-plane $[E]$ in place of requiring ω to be isotropic on $B^{(k+1)}$ it suffices to have $B^{(k+1)} = E + \mathbb{C}\beta$, where $0 \neq \beta \in A^{\perp} - E, A = A^{(k-1)}$. In fact, writing $E = A^{(k-1)} + \mathbb{C}e$ and assuming $\beta \in W - E$, the condition that $E := A^{(k-1)} + \mathbb{C}\gamma$ is isotropic with respect to ω for any $\gamma \in \mathbb{C}e + \mathbb{C}\beta$ is equivalent to the requirement that $\beta \in A^{\perp}$. Thus, the VMRT $\mathcal{C}_x(S_{k,\ell})$ at x is given by $\mathcal{C}_x(S_{k,\ell}) = \big\{[a \otimes b] : 0 \neq a \in E^*, 0 \neq b := \beta + E \in W/E,\ \omega(\beta, \alpha) = 0$ for any α such that $a(\alpha) = 0\big\}$, and the locus of tangents of the 'special' minimal rational curves at x is given by $\mathcal{S}_x(S_{k,\ell}) = \mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}D_x$. The distribution $D \subsetneq T(S_{k,\ell})$ is not integrable. Observing that (W, ω) induces a symplectic form ϖ_x on $Q_x = E^{\perp}/E$, $\dim(Q_x) = 2(\ell - k)$, the Frobenius form $\varphi_x : \Lambda^2 D_x \to T_x(S_{k,\ell})/D$ is determined in a precise way by ϖ_x (cf. Hwang-Mok [HM05, Proposition 5.3.1]), so that for $0 \neq a, a' \in E^*$; $b, b' \in Q_x$, $\varphi_x(a \otimes b, a' \otimes b') \neq 0$ if and only if $\varpi_x(b, b') \neq 0$. The symplectic form ϖ on Q_x cannot be recovered from the VMRTs alone, in fact there exists a uniruled projective manifold Z of Picard number $\neq 1$ with isotrivial VMRTs $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ projectively equivalent to $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$ such that the analogous distribution D, is *integrable* (cf. Hwang [Hw12]). Here $\mathbb{P}D_z \subset \mathbb{P}T_z(Z)$ is retrieved from $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ as the linear span of the locus where the second fundamental form is degenerate.

It remains to recognize $X \cong S_{k,\ell}$ for a uniruled projective manifold X of Picard number 1 such that the VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ is projectively equivalent to $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$. While one of the original motivations of the Recognition Problem is to prove rigidity under Kähler deformation in a conceptually uniform manner this has so far not been possible for the short-root cases. Reasoning in the opposite direction, the method of proof of rigidity under Kähler deformation in Hwang-Mok [HM05] may give a hint to the solution of the Recognition Problem for $S_{k,\ell}$, which is conceptually an important problem in its own right. In Hwang-Mok [HM05] we obtained a foliation on the central fiber in the deformation problem by means of the Frobenius form $\varphi : \Lambda^2 D \to T(X)/D$ associated to the specific VMRT structure. Recall that in [HM05] we considered a regular family $\pi : \mathcal{X} \to \Delta$ of Kähler manifolds with $X_t := \pi^{-1}(t) \cong S_{k,\ell}$ for $t \neq 0$, and, using estimates of vanishing orders and dimensions of linear spaces of vector fields we studied also the possibility that the Frobenius form is a priori degenerate on $D \subset X$, leading to a meromorphic fibration on the central fiber X_0 by Grassmannians, from which we reached a contradiction by showing that the underlying space of such a meromorphic fibration must be *singular*. In the case of the Recognition Problem for the symplectic Grassmannian one has to first get local models for isotrivial VMRTs modeled on $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$ depending on the rank of some skew-symmetric bilinear form ϖ_x on a $2(\ell - k)$-dimensional vector space Q_x at a general point $x \in X$, where Q_x can be retrieved from the VMRT and ϖ_x is determined by the Frobenius form φ_x. The construction and parametrization of local models are by themselves a challenging problem requiring new ideas on the local study of differential systems arising from these specific VMRTs. In the case of $X = X_0$ being the central fiber of a regular projective family $\pi : \mathcal{X} \to \Delta$, there is a priori at most one model up to equivalence when the rank of ω is fixed at a general point, and the existence of such a model was finally shown to contradict the smoothness of X_0 unless the rank of ϖ_x is maximal at a general point, in which case we showed $X_0 \cong S_{k,\ell}$.

Regarding the exceptional case of $S = \mathbb{F}^5$ for rigidity under Kähler deformation of rational homogeneous spaces of Picard number 1 as stated in Theorem 3.3.1, Hwang [Hw14a] has established the following result giving two alternatives for the central fiber.

Theorem 3.3.3 (Hwang [Hw14a]). *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of projective manifolds over the unit disk Δ, and denote by $X_t := \pi^{-1}(t)$ the fiber over $t \in \Delta$. Suppose X_t is biholomorphic to $\mathbb{F}^5$ for each $t \in \Delta - \{0\}$, then the central fiber is biholomorphic to either $\mathbb{F}^5$ or to the G_2-horospherical variety X^5 in Pasquier-Perrin* [PP10].

Paradoxically, the failure of rigidity under Kähler deformation in the exceptional case of $S \cong \mathbb{F}^5$, coupled with Hwang's result above, lends credence to an important general principle in the geometric theory of VMRTs, viz., that in the case of a uniruled projective manifold X of Picard number 1, the underlying complex structure should be recognized by its VMRT at a general point. In [Hw14a], Theorem 3.3.3 was proven precisely by identifying two alternatives for a general VMRT on the central fiber and by recovering the complex structure of X_0 by means of solving the associated Recognition Problem in the affirmative.

For $S = \mathbb{F}_5 = G/P$ with a reference point $0 = eP \in S$, denoting by $D \subsetneq T(S)$ the unique G-invariant proper holomorphic distribution on S, to be called the minimal distribution, the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}D_0(S)$ is projectively equivalent to the image of $\eta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$, where, writing π_i; $i = 1, 2$; for the canonical projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto its i-th factor, η is the embedding given by $\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(2)$. Moreover, it can be proven that for the regular family $\pi : \mathcal{X} \to \Delta$, letting $\mathcal{D}^\flat$ be the distribution on $\mathcal{X}|_{\Delta^*}$ such that $\mathcal{D}^\flat|_{X_t} \subset T(X_t)$ agrees with the minimal distribution on X_t for $t \neq 0$, $\mathcal{D}^\flat$ extends across the central fiber to give a meromorphic distribution $\mathcal{D}$ on $\mathcal{X}$. In the proof of Theorem 3.3.3, taking a holomorphic section $\sigma : \Delta(\epsilon) \to \mathcal{X}$ for some $\epsilon > 0$ such that $\sigma(0)$ is a general point of X_0, and considering the regular family $\tau : \mathcal{E} \to \Delta(\epsilon)$ of projective submanifolds of $E_t := \sigma^*\mathcal{C}_{\sigma(t)}(X_t) \subset \sigma^*\mathbb{P}D_{\sigma(t)} \cong \mathbb{P}^5$, $\big(E_0 \subset \mathbb{P}^5\big)$ is proven to be projectively equivalent to either $\big(\eta(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^5\big)$ or to $\big(\Sigma \subset \mathbb{P}^5\big)$, where $\Sigma := \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-3))$ is a Hirzebruch surface embedded into $\mathbb{P}^5$ by some ample line bundle. In the latter case the VMRT as a projective submanifold at a general point of X_0 is projectively equivalent to the VMRT of the seven-dimensional G_2-horospherical variety X^5 in the notation of Pasquier-Perrin [PP10]. Hwang then solved the Recognition Problem for X^5 by resorting to the obstruction theory for the construction of an appropriate connection on the central fiber X_0. This involves G-structures for a certain non-reductive linear group and the proof in Hwang [Hw14a] is a *tour de force*.

4 Germs of Complex Submanifolds of Uniruled Projective Manifolds

4.1 An Overview

Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T_X$, be the associated VMRT structure. We are interested in studying germs of complex submanifolds of X in relation to the VMRT structure. Since very little on the topic has been discussed in earlier surveys, we will be more systematic here with the exposition. In Mok [Mk08a] we examined the question of characterizing Grassmannians $G(p', q') \subset G(p, q)$ of rank $r = \min(p', q') \geq 2$ realized as complex submanifolds by means of standard embeddings. The fundamental analytic tool was the non-equidimensional Cartan-Fubini extension developed in full generality in Hong-Mok [HoM10] as stated in Theorem 2.2.1, which was applied in [HoM10] to yield characterization theorems on standard embeddings for pairs of rational homogeneous spaces (X_0, X) of Picard number 1, $X_0 \subset X$, where X is defined by a Dynkin diagram marked at a long simple root, and X_0 is nonlinear and obtained from a marked sub-diagram. This result, and generalizations by Hong-Park [HoP11] to the short-root case and to the case of maximal linear subspaces, will be the focus in (4.2), where the geometric idea of parallel transport of VMRTs along minimal rational curves will be explained. In (4.3) we explain an application in Hong-Mok [HoM13] of the methods of (4.2) to homological rigidity of smooth Schubert cycles with a few identifiable exceptions, where the rigidity statement was reduced to the question whether local deformations of a smooth Schubert cycle $Z \subset X$ must be translates $\gamma(Z)$ of Z, $\gamma \in \mathrm{Aut}(X)$. The latter question was answered in the affirmative in Hong-Mok [HoM13] in most cases by using the rigidity of VMRTs as projective submanifolds under local deformation of Z coupled with the argument of parallel transport of VMRTs. We explain the complex-analytic argument of [HoM13] which deduces parallel transport of VMRTs along minimal rational curves in the case of homogeneous Schubert cycles from the compactness of the moduli of the homogeneous submanifold $\mathcal{C}_0(Z) \subset \mathcal{C}_0(X)$. In (4.4) we introduce the notion of sub-VMRT structures given by $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, for complex submanifolds S and discuss the rigidity result of Mok-Zhang [Mz15] for sub-VMRT structures which strengthens those in (4.2) from [HoM10] and [HoP11]. In (4.5) we formulate a rigidity result for sub-VMRT structures on general uniruled projective manifolds $(X, \mathcal{K})$, and formulate the Recognition Problem for the characterization of classes of uniruled projective subvarieties on $(X, \mathcal{K})$ in terms of sub-VMRT structures. We consider in (4.6) examples of sub-VMRT structures related to Hermitian symmetric spaces, and discuss analytic continuation of sub-VMRT structures for special classes of uniruled projective subvarieties.

It is worth noting that [Mk08a] originated from a method of proof of Tsai's Theorem [Ts93] concerning proper holomorphic maps between bounded symmetric domains of rank ≥ 2 in the equal rank case, and as such the study of sub-VMRT structures is at the same time a topic in local differential geometry in a purely transcendental setting. In (4.7) we explore various links of VMRT substructures to algebraic geometry, several complex variables, Kähler geometry and the geometry of submanifolds in Riemannian geometry, and describe some sources of examples, including those from holomorphic isometries of Mok [Mz15] and from the classification of sub-VMRT structures in the Hermitian symmetric case of Zhang [Zh14].

4.2 Germs of VMRT-Respecting Holomorphic Embeddings Modeled on Certain Pairs of Rational Homogeneous Spaces of Sub-Diagram Type and a Rigidity Phenomenon

Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be uniruled projective manifolds equipped with minimal rational components with positive-dimensional VMRTs. When Z is of Picard number 1, given a VMRT-respecting holomorphic embedding f of a connected open subset $U \subset Z$ into X satisfying some genericity condition and a nondegeneracy condition for the pair $\big(f_\sharp(\mathcal{C}_z(Z)) \subset \mathcal{C}_{f(z)}(X)\big)$ expressed in terms of second fundamental forms, non-equidimensional Cartan-Fubini extension (Theorem 2.2.1) gives an extension of $f(U)$ to a projective subvariety $Y \subset X$ such that $\dim(Y) = \dim(U)$. In the case of irreducible Hermitian symmetric spaces S of rank ≥ 2, for which Ochiai's Theorem on S-structures serves as a prototype for equidimensional Cartan-Fubini extension, and more generally in the case of rational homogeneous spaces $X = G/P$ of Picard number 1, one expects to be able to say more about the projective subvariety $Y \subset X$. This was first undertaken in Mok [Mk08a] in the special case of Grassmannians of rank ≥ 2. For a pair (X_0, X) of rational homogeneous spaces of Picard number 1 obtained from marked Dynkin diagrams, the works of Hong-Mok [HoM10] and Hong-Park [HoP11] yielded the following characterization theorem. Here we have a holomorphic equivariant embedding $\Phi : X_0 = G_0/P_0 \hookrightarrow G/P = X$ and a mapping $F : X_0 \to X$ is said to be a standard embedding if and only if $F = \gamma \circ \Phi$ for some $\gamma \in \mathrm{Aut}(X)$. We have the following result due to [HoM10] in the long-root case and [HoP11] in the short-root case.

Theorem 4.2.1 (Hong-Mok [HoM10, Theorem 1.2], Hong-Park [HoP11, Theorem 1.2]). *Let $X_0 = G_0/P_0$ and $X = G/P$ be rational homogeneous spaces associated to simple roots determined by marked Dynkin diagrams $(\mathcal{D}(G_0), \gamma_0)$, $(\mathcal{D}(G), \gamma)$ respectively. Suppose $\mathcal{D}(G_0)$ is obtained from a sub-diagram of $\mathcal{D}(G)$ with γ_0 being identified with γ. If X_0 is nonlinear and $f : U \to X$ is a holomorphic embedding from a connected open subset $U \subset X_0$ into X which respects VMRTs at a general point $x \in U$, then f is the restriction to U of a standard embedding of X_0 into X.*

For finite-dimensional complex vector spaces $E \cong E'$ and subvarieties $A \subset \mathbb{P}E$, $A' \subset \mathbb{P}E'$, we say that $\big(A \subset \mathbb{P}E\big)$ is projectively equivalent to $\big(A' \subset \mathbb{P}E'\big)$ if and only if there exists a projective linear isomorphism $\Psi : \mathbb{P}E \xrightarrow{\cong} \mathbb{P}E'$ such that $\Psi(A) = A'$. To compare to Theorem 2.2.1, in what follows we will write Z for X_0. Write $S := f(U)$, which is a complex submanifold of some open subset of X. Obviously for $z \in U$, $\big(\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)\big)$ is projectively equivalent to $\big(f_\sharp(\mathcal{C}_z(Z)) \subset f_\sharp(\mathbb{P}T_z(Z))\big)$, i.e., $\big(\mathcal{C}_{f(z)}(S) \subset \mathbb{P}T_{f(z)}(S)\big)$. Here and henceforth we write $\mathcal{C}_{f(z)}(S)$ for $\mathcal{C}_{f(z)}(X) \cap \mathbb{P}T_{f(z)}(S)$, which is the same as $f_\sharp(\mathcal{C}_z(Z))$ by the hypothesis that f respects VMRTs. (We will also write $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$.) Write $\Lambda_z = [df(z)] : \mathbb{P}T_z(Z) \xrightarrow{\cong} \mathbb{P}T_{f(z)}(S)$ for the projective linear isomorphism inducing the projective equivalence. In the long-root case, by the proof of Hong-Mok [HoM10, Proposition 3.4], denoting by $D(Z) \subset T(Z)$ the holomorphic distribution spanned by VMRTs, $\Lambda_z|_{\mathbb{P}D_z(Z)}$ can be extended to $\Phi_z = [d\gamma(z)] : \mathbb{P}T_z(X) \xrightarrow{\cong} \mathbb{P}T_{f(z)}(X)$ for some $\gamma \in \mathrm{Aut}(X)$ such that $\gamma(z) = f(z)$. Thus, given the germ of VMRT-respecting holomorphic map $f : U \to X$, $Z' := \gamma(Z)$ gives a rational homogeneous submanifold $Z' \subset X$ such that $\mathcal{C}_{f(0)}(Z') = \mathcal{C}_{f(0)}(S)$ and the proof of Theorem 4.2.1 consists of fitting S into the model Z'. (We remark that in the statement of [HoM10, Proposition 3.4], in place of $V = T_x(X)$ and $W := T_x(Z)$ one could have replaced V by the linear span $D_x(X)$ of $\widetilde{\mathcal{C}}_x(X)$ and W by the linear span $D_x(Z)$ of $\widetilde{\mathcal{C}}_x(Z)$. The arguments for the proof of [HoM10, Proposition 3.4] actually yield the following stronger statement, which we will use in what follows. If $\mathcal{B}' = \mathcal{C} \cap \mathbb{P}W'$ is another linear section such that $\big(\mathcal{B}' \subset \mathbb{P}W'\big)$ is projectively equivalent to $\big(\mathcal{B} \subset \mathbb{P}W\big)$, then there is $h \in P$ such that $\mathcal{B}' = h\mathcal{B}$. In other words, in place of $h \in \mathrm{Aut}(\mathcal{C}_x(X))$ the proof actually gives $h \in P$. The latter fact was used in [HoM10] in the process of fitting S into a model $Z' = \gamma(Z)$.)

For the proof of Theorem 4.2.1 the strategy was to compare the germ of manifold S at some base point with the model complex submanifold $Z' = \gamma(Z) \subset X$. In the ensuing discussion, replacing f by $\gamma^{-1} \circ f$ we will assume without loss of generality that $Z' = Z$, so that $\mathcal{C}_0(S) = \mathcal{C}_0(Z)$. Starting with the base point $0 = f(0) \in S$ and considering the union $\mathcal{V}_1$ of minimal rational curves on Z passing through 0, preservation of the tautological foliation, i.e., $f_{\sharp *}(\mathcal{F}_Z) = \mathcal{F}_X|_{\mathcal{C}(S)}$, as in the proof of non-equidimensional Cartan-Fubini extension (Theorem 2.2.1) implies that the germ $(S;0)$ contains $(\mathcal{V}_1;0)$. By the repeated adjunction of minimal rational curves, we have $0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_m = Z$. In order to prove that S is an open subset of Z it suffices to prove inductively that the germ $(S;0)$ contains $(\mathcal{V}_k;0)$ for each $k \geq 1$, $\mathcal{V}_0 = \{0\}$. For $k \geq 1$ the statement $(S;0) \supset (\mathcal{V}_k;0)$ is the same as saying that $\mathcal{C}_{x_{k-1}}(S) = \mathcal{C}_{x_{k-1}}(Z)$ for any $x_{k-1} \in \mathcal{V}_{k-1} \cap S$ sufficiently close to 0. The inductive argument was done in [Mk08a] and [HoM10] by means of parallel transport of VMRTs along minimal rational curves, as follows. Assume $(S;0) \supset (\mathcal{V}_k;0)$. If for a general point $x \in S \cap \mathcal{V}_k$, $x = f(z)$, we have $f_\sharp(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(Z)$, then $(S;x)$ contains the germ $(\mathcal{V}_1(x);0)$, where $\mathcal{V}_1(x)$ is the union of all rational curves on Z passing through $x = f(z) \in S \cap \mathcal{V}_k$. At a point $x = x_k \in S \cap (\mathcal{V}_k - \mathcal{V}_{k-1})$ we have a line ℓ joining some point $x' = x_{k-1} \in S \cap \mathcal{V}_{k-1}$ to x, and we know that $S = f(U)$ and Z share a nonempty connected open subset of the rational curve ℓ, which contains both x' and x. From $\mathcal{C}_{x'}(S) = \mathcal{C}_{x'}(Z)$ it follows that $\mathcal{C}_x(S)$ and $\mathcal{C}_x(Z)$ are tangent to each other at x. The argument of parallel transport of VMRTs along minimal rational curves consists of the following statement established in the long-root case by Hong-Mok [HoM10, Proposition 3.6] using the theory of Lie groups and representation theory.

Proposition 4.2.1. *Let (Z, X) be a pair of rational homogeneous spaces of Picard number 1 marked at a simple root, $X = G/P$, $0 = eP$. Suppose $\gamma \in P$ and $\mathcal{C}_0(\gamma(Z))$ and $\mathcal{C}_0(Z)$ are tangent to each other at a common smooth point $[\alpha] \in \mathcal{C}_0(\gamma(Z)) \cap \mathcal{C}_0(Z)$, then $\mathcal{C}_0(\gamma(Z)) = \mathcal{C}_0(Z)$.*

For a proof of Proposition 4.2.1 which works in both the long-root and the short-root cases we defer to (4.3). Returning to Theorem 4.2.1 for the short-root case, by Hong-Park [HoP11, Proposition 2.2] the existence of $\varphi \in P$ such that $Z' := \varphi(Z)$ is tangent to $S := f(U)$ at $f(0) = 0$ holds true in the short-root case excepting in the case of certain pairs (Z, X) where X is a symplectic Grassmannian and Z is a Grassmannian. For the short-root case other than the exceptional pairs (Z, X) Hong-Park proceeded along the line of Hong-Mok [HoM10], but for the verification of the nondegeneracy condition they resort to explicit description of VMRTs as projective submanifolds as given in Hwang-Mok [HM04a] and [HM05].

To describe the exceptional pairs (Z, X) recall that for $X = S_{k,\ell} \subset \mathrm{Gr}(k, W)$, there is an invariant distribution $D \subset T(X)$ given by $D = U \otimes Q$, where U and Q are homogeneous vector bundles over X with fibers $U_x = E^*$ and $Q_x = E^\perp/E$ at $x = [E] \in X$. Recall that $\mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}D_x(S_{k,\ell}) = \mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}(U_x \otimes Q_x) = \varsigma(\mathbb{P}U_x \times \mathbb{P}Q_x)$, where $\varsigma : \mathbb{P}U_x \times \mathbb{P}Q_x \to \mathbb{P}(U_x \otimes Q_x)$ denotes the Segre embedding given by $\varsigma([u], [q]) = [u \otimes q]$. Fix now isotropic subspaces $0 \neq F_1 \subset F_2$ and consider $Z \subset X$ consisting of all (isotropic) k-planes E such that $F_1 \subset E \subset F_2$. Then, $Z \subset X = S_{k,\ell} \subset \mathrm{Gr}(k, W)$ is a Grassmannian, and $T(Z) = U' \otimes Q'$, where $U'_x = (E/F_1)^* \subset U_x$, and $Q'_x = F_2/E \subset E^\perp/E = Q_x$. We have $\mathrm{rank}(U') = k - \dim F_1 := a$, $\mathrm{rank}(Q') = \dim F_2 - k := b$. Then, writing $x = [E] \in X$, for any a-plane $A_x \subset U_x$ and any b-plane $B_x \subset Q_x$, $\big(\varsigma(\mathbb{P}A_x \times \mathbb{P}B_x) \subset \mathbb{P}(A_x \otimes B_x)\big)$ is projectively equivalent to $\big(\varsigma(\mathbb{P}U'_x \times \mathbb{P}Q'_x) \subset \mathbb{P}(U_x \otimes Q_x)\big)$. Recall that (W, ω) induces a symplectic form ϖ_x on $Q_x = E^\perp/E$ (cf. (3.3)). By definition ϖ_x vanishes on $Q'_x = F_2/E$. If we choose now $B_x \subset Q_x$ to be such that $\varpi_x|_{B_x} \not\equiv 0$, then any projective linear isomorphism $\Lambda_x : \mathbb{P}(A_x \otimes B_x) \xrightarrow{\cong} \mathbb{P}(U'_x \otimes Q'_x)$ such that $\Lambda(\varsigma(\mathbb{P}A_x \times \mathbb{P}B_x)) = \varsigma(\mathbb{P}U'_x \times \mathbb{P}Q'_x)$ cannot possibly extend to an element of the parabolic subgroup P_x of G at x since ϖ_x is invariant under P_x.

For the solution of the special case (Z, X) above for the symplectic Grassmannian $X = S_{k,\ell}$ it is sufficient to embed X into the Grassmannian $\mathrm{Gr}(k, 2\ell)$ and solve the problem for $(Z, \mathrm{Gr}(k, 2\ell))$. The end result is that $S = F(U)$ is still an open subset of a sub-Grassmannian Z' in $\mathrm{Gr}(k, 2\ell)$ which lies on $X = S_{k,\ell}$. We observe that the proof yields the following. If we start with $f : (Z; 0) \to (S_{k,\ell}; 0)$ such that f respects VMRTs, and suppose $f_\sharp(\mathcal{C}_0(Z)) = \zeta(\mathbb{P}A_x \times \mathbb{P}B_x)$, then we have necessarily $\varpi_x|_{B_x} \equiv 0$.

Hong-Park [HoP11] considered in addition the cases involving maximal linear subspaces $Z = X_0$, as follows.

Theorem 4.2.2 (Hong-Park [HoP11, Theorem 1.3]). *Let $X = G/P$ be a rational homogeneous space associated to a simple root and let $Z \subset X$ be a maximal linear subspace. Let $f : U \to X$ be a holomorphic embedding from a connected open subset $U \subset Z$ into X such that $\mathbb{P}(df(T_z(U))) \subset \mathcal{C}_{f(z)}(X)$ for any $z \in U$. If there* *is a maximal linear space* Z_{max} *of* X *of dimension* $\dim(U)$ *which is tangent to* $f(U)$ *at some point* $x_0 = f(z_0), z_0 \in U$, *then* $f(U)$ *is contained in* Z_{max}, *excepting when* $(Z_{max}; X)$ *is given by* (a) X *is associated to* $(B_\ell, \alpha_i), 1 \le i \le \ell - 1$, *and* Z_{max} *is* $\mathbb{P}^{\ell-i}$; (b) X *is associated to* (C_ℓ, α_ℓ) *and* Z_{max} *is* $\mathbb{P}^1$; (c) X *is associated to* (F_4, α_1) *and* Z_{max} *is* $\mathbb{P}^2$.

The maximal linear subspaces $\Pi \subset \mathcal{C}_0(X)$ break into a finite number of isomorphism types under the action of the parabolic subgroup P on $\mathcal{C}_0(X)$ (cf. Landsberg-Manivel [LM03]). The assumption that there is a maximal linear space $Z_{\max}$ of X of dimension $\dim(U)$ which is tangent to $S := f(U)$ at some point $x_0 := f(z_0) \in S$ implies that $\mathbb{P}T_{x_0}(S) = \mathbb{P}T_{x_0}(Z_{\max})$, which forces $\mathbb{P}T_x(S) \subset \mathcal{C}_x(X)$ to be a maximal linear subspace of the same type for any $x = f(z) \in S$. Here the question is whether the tautological foliation $\mathcal{F}_X$ on $\mathbb{P}T(X)$, regarded as a holomorphic line subbundle, is tangent to $\mathbb{P}T(S)$ and hence restricts to $\mathcal{C}(X) \cap \mathbb{P}T(S) = \mathcal{C}(S)$. An affirmative answer to the latter question implies that S is an open set on a projective linear subspace obtained by adjoining projective lines at a single base point. Obviously $f_{\sharp *}(\mathcal{F}_Z)$ need not agree with $\mathcal{F}_X|_S$. In fact, trivially any immersion between projective spaces respects VMRTs.

Suppose $\alpha \in df(T_{z_0}(U))$. To show that $\mathcal{F}_X|_S$ defines a foliation on S, in place of requiring the nondegeneracy condition $\mathrm{Ker}\,\sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) = \mathbb{C}\alpha$ it suffices to have $\mathrm{Ker}\,\sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) \subset df(T_{z_0}(Z))$, which was checked to be the case for $Z = X_0$ being a maximal linear subspace, in which case $\mathrm{Ker}\,\sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) = df(T_{z_0}(Z))$ with the exceptions as stated in the theorem. For the exceptional cases (a)–(c), counter-examples had been constructed by Choe-Hong [CH04].

We note that Theorem 4.2.2, while formulated in terms of a holomorphic embedding $f : U \to X$, concerns only the germ of complex submanifold $(S; x_0)$, where $S = f(U)$. Here f plays only an auxilliary role. In fact, any germ of biholomorphism $h : (U; z_0) \to (S; x_0)$ satisfies the condition $\mathbb{P}(dh(T_z(U))) \subset \mathcal{C}_{h(z)}(X)$ whenever h is defined at $z \in U$. As such Theorem 4.2.2 may be regarded as a result on geometric substructures.

In the long-root and nonlinear case of admissible pairs (Z, X), in which case $\mathcal{C}_0(Z) \subset \mathbb{P}T_0(Z)$ is homogeneous and nonlinear, given a germ of VMRT-respecting holomorphic immersion $f : (Z; z_0) \to (X; x_0)$ the condition $\mathrm{Ker}\,\sigma_\alpha\big(\,\cdot\,, T_\alpha(\widetilde{\mathcal{C}}_{x_0}(Z))\big) \subset P_\alpha \cap df(T_{z_0}(Z))$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_{x_0}(X))$, forces $\mathrm{Ker}\,\sigma_\alpha\big(\,\cdot\,, T_\alpha(\widetilde{\mathcal{C}}_0(Z))\big) = \mathbb{C}\alpha$ since the projective second fundamental form on the nonlinear homogeneous projective submanifold $\mathcal{C}_{z_0}(Z) \subset \mathbb{P}T_{z_0}(Z)$ itself is everywhere nondegenerate. The same implication holds true in the short-root case provided that $[\alpha] \in \mathcal{C}_{x_0}(X) \cap \mathbb{P}(df(T_{z_0}(Z))$ is a general point of both $\mathcal{C}_{x_0}(X)$ and $f_\sharp(\mathcal{C}_{z_0}(Z))$. Thus, in the nonlinear case of admissible pairs (Z, X) one can examine the question of rational saturation for a germ of complex submanifold endowed with a certain type of geometric substructure (e.g., a sub-Grassmann structure) without the assumption of the existence of an underlying holomorphic map. We will show that this is indeed possible, and we will introduce a variation of the notion of nondegeneracy for that purpose. This will be taken up in (4.4) and (4.5).

4.3 Characterization of Smooth Schubert Varieties in Rational Homogeneous Spaces of Picard Number 1

Characterization of standard embeddings between certain pairs of rational homogeneous spaces (X_0, X) of Picard number 1 was achieved by means of non-equidimensional Cartan-Fubini extension and parallel transport of VMRTs along minimal rational curves. In Hong-Mok [HoM13] this approach was further adopted to deal with a problem of homological rigidity of smooth Schubert cycles on rational homogeneous spaces. Recall that a Schubert cycle $Z \subset X$ on a rational homogeneous space $X = G/P$ is one for which the G-orbit of $G \cdot [Z]$ in Chow(X) is projective. Suppose $S \subset X$ is a cycle homologous to the Schubert cycle Z, the homological rigidity problem is to ask whether S is necessarily equivalent to Z under the action of Aut(X). We note that by the extremality of the homology class of a Schubert cycle among homology classes of effective cycles, S is reduced and irreducible.

We reformulate the homological rigidity problem so as to relate it to the geometric theory of uniruled projective manifolds modeled on VMRTs, as follows. To start with, considering the irreducible component $\mathcal{Q}$ of Chow(X) containing the point $[S]$, there always exists a closed G-orbit in $\mathcal{Q}$, and as such it contains a point corresponding to some Schubert cycle, and thus $\mathcal{Q}$ must contain $[Z]$ itself by the uniqueness modulo G-action of Schubert cycles representing the same homology class. The homological rigidity problem would be solved in the affirmative if we established the local rigidity of Z. In Hong-Mok [HoM13] we dealt with the case of smooth Schubert cycles. When $X = G/P$ is a rational homogeneous space defined by a Dynkin diagram $\mathcal{D}(G)$ marked at a simple root, and $Z \subset X$ is defined by a marked sub-diagram, then $Z \subset X$ is a Schubert cycle (cf. [HoM13, §2, Example 1]). In [HoM13] we proved

Theorem 4.3.1 (Hong-Mok [HoM13, Theorem 1.1]). *Let $X = G/P$ be a rational homogeneous space associated to a simple root and let $X_0 = G_0/P_0$ be a homogeneous submanifold associated to a sub-diagram $\mathcal{D}(G_0)$ of the marked Dynkin diagram $\mathcal{D}(G)$ of X. Then, any subvariety of X having the same homology class as X_0 is induced by the action of $Aut_0(X)$, excepting when (X_0, X) is given by*

(a) $X = (C_n, \{\alpha_k\}),\ \Lambda = \{\alpha_{k-1}, \alpha_b\}, 2 \leq k < b \leq n$;
(b) $X = (F_4, \{\alpha_3\}),\ \Lambda = \{\alpha_1, \alpha_4\}$ *or* $\{\alpha_2, \alpha_4\}$;
(c) $X = (F_4, \{\alpha_4\}),\ \Lambda = \{\alpha_2\}$ *or* $\{\alpha_3\}$,

where Λ denotes the set of simple roots in $\mathcal{D}(G) \backslash \mathcal{D}(G_0)$ which are adjacent to the sub-diagram $\mathcal{D}(G_0)$.

When $X = G/P$ is defined by the marked Dynkin diagram $(\mathcal{D}(G), \gamma)$, where γ is a long simple root, it was established in [HoM13, Proposition 3.7] that any smooth Schubert cycle Z on X is a rational homogeneous submanifold corresponding to a marked sub-diagram of $(\mathcal{D}(G), \gamma)$, hence Theorem 4.3.1 in the long-root case exhausts all smooth Schubert cycles up to G-action. When γ is a short simple root, this need not be the case, and we refer the reader to [HoM13, Theorem 1.2] for results on the homological rigidity problem for the symplectic Grassmannian pertaining to smooth Schubert cycles which are not rational homogeneous submanifolds.

Parallel transport of VMRTs along a minimal rational curve was used in an essential way in [HoM13] for the proof of Theorem 4.3.1. Using a complex-analytic argument, we established in [HoM13, Proposition 3.3] a proof of Proposition 4.2.1 applicable to the case of Schubert cycles $Z \subset X = G/P$ corresponding to marked sub-diagrams as in Theorem 4.3.1, as follows. (We say that (Z, X) is of sub-diagram type.)

Proof of Proposition 4.2.1. Consider the point $[Z]$ in $\mathrm{Chow}(X)$ corresponding to the reduced cycle Z. Since $Z \subset X$ is a Schubert cycle, the G-orbit of $[Z]$ in $\mathrm{Chow}(X)$ is projective. From this and the fact that $Z \subset X$ is a rational homogeneous submanifold one deduces that the P-orbit of $[Z]$ in $\mathrm{Chow}(X)$ is also projective. Given this, the failure of Proposition 4.2.1 would imply the existence of a holomorphic family $\varpi : \mathcal{Q} \to \Gamma$ of projective submanifolds $Q_t \subset \mathcal{C}_0(X)$, $[Q_t] \in \mathcal{Q}$, parametrized by a *projective* curve Γ, such that all members Q_t contain $[\alpha]$ and such that they all share the *same tangent space* $V = T_{[\alpha]}(\mathcal{C}_0(Z))$. Consider the holomorphic section σ of $\varpi : \mathcal{M} \to \Gamma$ corresponding to the common base point $[\alpha] \in Q_t$ for all $t \in \Gamma$. The assumption that $T_{[\alpha]}(Q_t) = V$ for all $t \in \Gamma$ would imply that the normal bundle N of $\sigma(\Gamma)$ in $\mathcal{Q}$ is holomorphically trivial, which would contradict the negativity of the normal bundle resulting from the existence of a canonical map $\chi : \mathcal{Q} \to \mathcal{C}_0(X)$ collapsing $\sigma(\Gamma)$ to the single point $[\alpha]$ (cf. Grauert [Gra62]). This proves Proposition 4.2.1 by argument by contradiction. □

4.4 Sub-VMRT Structures Arising from Admissible Pairs of Rational Homogeneous Spaces of Picard Number 1 and a Rigidity Phenomenon

Consider the Grassmann manifold $G(p, q)$ of rank $r = \min(p, q) \geq 2$. We have $T(G(p, q)) = U \otimes V$ where U (resp. V) is a semipositive universal bundle of rank p (resp. q). Let $W \subset G(p, q)$ be an open subset and $S \subset W$ be a complex submanifold such that $T(S) = A \otimes B$, where $A \subset U|_S$ (resp. $B \subset V|_S$) is a holomorphic vector subbundle of rank p' (resp. q') such that $r' = \min(p', q') \geq 2$. Thus, by assumption S inherits a $G(p', q')$-structure, and the question we posed was whether S is an open subset of a projective submanifold $Z \subset G(p, q)$ such that Z is the image of $G(p', q')$ under a standard embedding. When the natural Grassmann structure (of rank $r' \geq 2$) on S is flat, then, given any $x_0 \in S$, there exists an open neighborhood $\mathcal{O}$ of x_0 on S, an open neighborhood U of $0 \in G(p', q')$, and a biholomorphic mapping $f : U \xrightarrow{\cong} \mathcal{O}$ such that f preserves $G(p', q')$-structures, equivalently $f_\sharp(\mathcal{C}(G(p', q'))|_U) = \mathcal{C}(S)$ where $\mathcal{C}(S) := \mathcal{C}(G(p, q)) \cap \mathbb{P}T(S)$. This is the situation dealt with in Mok [Mk08a] where it was proven that S is indeed an open subset of a sub-Grassmannian $Z \subset G(p,q)$ which is the image of $G(p',q')$ under a standard embedding. The question here is whether the flatness assumption is superfluous.

One can contrast Grassmann structures with holomorphic conformal structures. Let $n \geq 4$ and $(S;0)$ be an m-dimensional germ of complex submanifold on the hyperquadric Q^n, $3 \leq m < n$. We say that $(S;0)$ inherits a holomorphic conformal structure if and only if the restriction of the standard holomorphic conformal structure on Q^n to $(S;0)$ is nondegenerate. Here examples abound, indeed a *generic* germ of complex submanifold $(S;0)$ inherits such a structure. On a small coordinate neighborhood W of 0 the holomorphic conformal structure is given by the equivalence class (up to conformal factors) of a holomorphic metric g, i.e., a nondegenerate holomorphic covariant symmetric 2-tensor $g = \sum g_{\alpha\beta}(z)dz^\alpha \otimes dz^\beta$ in local coordinates, and the latter restricts to a holomorphic conformal structure on $(S;0)$ if and only if g is nondegenerate on $T_0(S)$. The case of the pair $(G(p',q');G(p,q))$ is very different. In the latter case, denoting by W a coordinate neighborhood of $0 \in G(p,q)$, it is a priori very special that $(S;0) \subset (W;0)$ inherits a $G(p',q')$-structure. Indeed, for a general $p'q'$-dimensional linear subspace $\Pi \subset T_0(G(p,q))$, the intersection $\mathcal{C}_0(G(p,q)) \cap \mathbb{P}(\Pi)$, if nonempty, is of codimension $pq - p'q'$ in $\mathcal{C}_0(G(p,q)) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, of dimension $p+q-2$. Thus the expected dimension of intersection is strictly less than that of a $G(p',q')$-structure, i.e., $p'+q'-2$, as soon as $(p',q') \neq (p,q)$. (For an example of low dimension, consider a *generic* four-dimensional complex submanifold $S \subset W \subset G(2,3)$. The intersection $\mathcal{C}_0(G(p,q)) \cap \mathbb{P}T_0(S)$ is expected to be of codimension 2 in $\mathcal{C}_0(G(2,3)) \cong \mathbb{P}^1 \times \mathbb{P}^2$, i.e., a curve, while it is a surface $\mathcal{C}_0(S) \cong \mathcal{C}_0(G(2,2)) \cong \mathbb{P}^1 \times \mathbb{P}^1$ when S inherits a $G(2,2)$-structure.) In view of the excessive intersection of VMRTs with projectivized tangent spaces it is perceivable that rigidity already follows from excessive intersection of VMRTs with projectivized tangent spaces and from the specific forms of the intersections, and that the flatness assumption is unnecessary.

In the case of Grassmann structures or other G-structures modeled on irreducible Hermitian symmetric spaces of rank ≥ 2, by Guillemin [Gu65] we could resort to proving the vanishing of a finite number of obstruction tensors to demonstrate flatness, but the method of G-structures is ill-adapted even for rational homogeneous spaces. In its place we examined the tautological foliation on VMRT structures and raised the question whether the restriction to $\mathbb{P}T(S)$ already defines a foliation. In the special and simpler case of linear model submanifolds such a question was formulated and solved by Hong-Park [HoP11] (Theorem 4.2.2 in the above). For the general formulation of rigidity phenomena on complex submanifolds we introduce the notion of admissible pairs of rational homogeneous spaces of Picard number 1, as follows (cf. Mok-Zhang [Mz15, Definition 1.1]). Recall that for a holomorphic immersion $\nu : M \to N$ between complex manifolds we write $\nu_\sharp$ for the projectivization of the differential $d\nu : T(M) \to T(N)$ of the map.

Definition 4.4.1. Let X_0 and X be rational homogeneous spaces of picard number 1, and $i : X_0 \hookrightarrow X$ be a holomorphic embedding equivariant with respect to a homomorphism of complex Lie groups Φ: $\text{Aut}_0(X_0) \to \text{Aut}_0(X)$. We say that $(X_0, X; i)$ is an admissible pair (of rational homogeneous spaces of Picard number 1) if and only if (a) i induces an isomorphism i_*: $H_2(X_0,\mathbb{Z}), \xrightarrow{\cong} H_2(X,\mathbb{Z})$, and (b) denoting by $\mathcal{O}(1)$ the positive generator of $Pic(X)$ and by ρ: $X \hookrightarrow \mathbb{P}(\Gamma(X,\mathcal{O}(1))*) := \mathbb{P}^N$ the first canonical projective embedding of X, $\rho\circ i$: $X_0 \hookrightarrow \mathbb{P}^N$ embeds X_0 as a (smooth) linear section of $\rho(X)$.

Next we introduce the notion of a sub-VMRT structure modeled on an admissible pair (X_0,X) of rational homogeneous spaces (cf. Mok-Zhang [Mz15, Definition 1.2]).

Definition 4.4.2. Let (X_0,X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be an open subset, and $S\subset W$ be a complex submanifold. Consider the fibered space $\pi : \mathcal{C}(X) \to X$ of varieties of minimal rational tangents on X. For every point $x\in S$ define $\mathcal{C}_x(S) := \mathcal{C}_x(X)\cap\mathbb{P}T_x(S)$ and write $\varpi : \mathcal{C}(S) \to S$ for $\varpi = \pi\big|_{\mathcal{C}(S)}$, $\varpi^{-1}(x) := \mathcal{C}_x(S)$ for $x \in S$. We say that $S \subset W$ inherits a sub-VMRT structure modeled on (X_0,X) if and only if for every point $x \in S$ there exists a neighborhood U of x on S and a trivialization of the holomorphic projective bundle $\mathbb{P}T(X)|_U$ given by $\Phi : \mathbb{P}T(X)|_U \xrightarrow{\cong} \mathbb{P}T_0(X)\times U$ such that (1) $\Phi(\mathcal{C}(X)|_U) = \mathcal{C}_0(X)\times U$ and (2) $\Phi(\mathcal{C}(S)|_U) = \mathcal{C}_0(X_0)\times U$.

The definition that $S \subset W$ inherits a sub-VMRT structure modeled on the admissible pair (X_0,X) can be reformulated as requiring

(†) For any $x \in X$ there exists a projective linear isomorphism $\Lambda_x : \mathbb{P}T_x(X) \xrightarrow{\cong} \mathbb{P}T_0(X)$ such that $\Lambda_x(\mathcal{C}_x(X)) = \mathcal{C}_0(X)$ and $\Lambda_x(\mathcal{C}_x(S)) = \mathcal{C}_0(X_0)$.

With an aim to generalize the characterization theorems of Hong-Mok [HoM10] and Hong-Park [HoP11] on standard embeddings between rational homogeneous spaces of Picard number 1 we introduced in Mok-Zhang [Mz15, Definition 5.1] the notion of rigid pairs (X_0,X), as follows.

Definition 4.4.3. For an admissible pair (X_0,X) of rational homogeneous spaces of Picard number 1, we say that (X_0,X) is rigid if and only if, for every complex submanifold S of some open subset of X inheriting a sub-VMRT structure modeled on (X_0,X), there exists some $\gamma \in \mathrm{Aut}(X)$ such that S is an open subset of $\gamma(X_0)$.

We are now ready to state one of the main results of Mok-Zhang [Mz15].

Theorem 4.4.1 (Mok-Zhang [Mz15, Main Theorem 1]). *Let (X_0,X) be an admissible pair of sub-diagram type of rational homogeneous spaces of Picard number 1 marked at a simple root. Suppose $X_0 \subset X$ is nonlinear. Then, (X_0,X) is rigid.*

Combining with Hong-Park [HoP11] on the maximal linear case, the question on rigidity of sub-VMRT structures modeled on admissible pairs (X_0,X) of rational homogeneous spaces of Picard number 1 is completely settled (cf. [Mz15, Corollary 1.1]).

Corollary 4.4.1. *An admissible pair (X_0,X) of rational homogeneous spaces of Picard number 1 of sub-diagram type is a rigid pair excepting when $X_0 \subset X$ is* *a non-maximal linear subspace, or when* $X_0 \subset X$ *is a maximal linear subspace* Z_{max} *given by* (a) X *is associated to* (B_ℓ, α_i), $1 \le i \le \ell - 1$, *and* Z_{max} *is* $\mathbb{P}^{\ell-i}$; (b) X *is associated to* (C_ℓ, α_ℓ) *and* Z_{max} *is* $\mathbb{P}^1$; *or* (c) X *is associated to* (F_4, α_1) *and* X_0 *is* $\mathbb{P}^2$.

For the study of rigidity of admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1, Zhang [Zh14, Main Theorem 2] classified all such pairs in the case where X is an irreducible Hermitian symmetric space of the compact type and $X_0 \subset X$ is nonlinear. (The linear cases for all rational homogeneous spaces X of Picard number 1 have been enumerated in Hong-Park [HoP11].) From the classification Zhang established

Theorem 4.4.2 (Zhang [Zh14, Main Theorem 2]). *An admissible pair* (X_0, X) *of irreducible Hermitian symmetric space of the compact type is non-rigid whenever* (X_0, X) *is degenerate for substructures.*

The key issue in the proof of Theorem 4.4.1 is to show that the tautological foliation $\mathcal{F}_X$ on $\mathcal{C}(X)$ is tangent to the total space $\mathcal{C}(S)$ of the sub-VMRT structure. Pick any $x \in S$ and any $[\alpha] \in \mathcal{C}_x(S)$, and denote by ℓ the minimal rational curve passing through x such that $T_x(\ell) = \mathbb{C}\alpha$. From the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ regarded as a holomorphic fiber bundle over S, there exists a holomorphic vector field θ on some neighborhood U of x on S such that $\theta(x) = \alpha$ and such that $\theta(y) \in \widetilde{\mathcal{C}}_y(S)$ whenever $y \in U$. The integral curve of θ passing through x gives a smooth holomorphic curve γ on S tangent to ℓ at x such that the lifting $\check{\gamma}$ of γ to $\mathbb{P}T(S)$ lies on $\mathcal{C}(S)$. At the point $[\alpha] \in \mathcal{C}_x(S)$ the difference between $T_{[\alpha]}(\check{\gamma})$ and $T_{[\alpha]}(\check{\ell})$, where $\check{\ell}$ denotes the tautological lifting $\check{\ell} \subset \mathcal{C}(X)$ of ℓ, gives a vector $\eta \in T_{[\alpha]}(\mathcal{C}_x(X))$. In view of the flexibility in the choice of γ, the vector η is only well-defined modulo $T_{[\alpha]}(\mathcal{C}_x(S)) \cong (P_\alpha \cap T_x(S))/\mathbb{C}\alpha$. Let $D(X) \subset T(X)$ be the G-invariant distribution spanned at each point $x \in X$ by $\widetilde{\mathcal{C}}_x(X)$. There is a vector-valued symmetric bilinear form $\tau_{[\alpha]} : S^2T_{[\alpha]}(\mathcal{C}_x(X)) \to T_{[\alpha]}(\mathbb{P}T_x(X))/(T_{[\alpha]}(\mathcal{C}_x(X)) + T_{[\alpha]}(\mathbb{P}(T_x(S) \cap D_x(X))$, given by $\tau_{[\alpha]} = \nu \circ \sigma_{[\alpha]}$, $\nu : T_{[\alpha]}(\mathbb{P}T_x(X))/T_{[\alpha]}(\mathcal{C}_x(X)) \to T_{[\alpha]}(\mathbb{P}T_x(X))/(T_{[\alpha]}(\mathcal{C}_x(X)) + T_{[\alpha]}(\mathbb{P}(T_x(S) \cap D_x(X)))$ being the canonical projection. Here $T_{[\alpha]}(\mathbb{P}T_x(X))/T_{[\alpha]}(\mathcal{C}_x(X))$ is the normal space of the inclusion $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ at $[\alpha] \in \mathcal{C}_x(S) \subset \mathcal{C}_x(X)$ and σ denotes the projective second fundamental form of the said inclusion. For both the second fundamental form σ and the variant τ, we use the same notation when passing to affinizations $\widetilde{\mathcal{C}}_x(X) \subset T_x(X)$. The context will make it clear which is meant.

Obviously $\tau_{[\alpha]}(\xi_1, \xi_2) = 0$ whenever $\xi_1, \xi_2 \in T_{[\alpha]}(\mathcal{C}_x(S))$. For the proof that $\mathcal{F}_X$ is tangent to $\mathcal{C}(S)$ it suffices to show that for $\eta \in T_{[\alpha]}(\mathcal{C}_x(X))$ as defined in the last paragraph we have actually $\eta \in T_{[\alpha]}(\mathcal{C}_x(S))$. In [Mz15] we show that $\tau_{[\alpha]}(\eta, \xi) = 0$ whenever $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ and derive the rigidity of the pairs (X_0, X) in Theorem 4.4.1 by checking that $\tau_{[\alpha]}(\eta, \xi) = 0$ for all $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ implies that $\eta \in T_{[\alpha]}(\mathcal{C}_x(S))$. We say in this case that $(\mathcal{C}_x(S), \mathcal{C}_x(X))$ is nondegenerate for substructures (cf. Definition 4.5.2 in the next subsection), noting that $T_x(S) \cap D_x(X)$ is the linear span of $\widetilde{\mathcal{C}}_x(S)$. The checking is derived from statements about the second fundamental form σ concerning nondegeneracy of Hong-Mok [HoM10] and the proof of Theorem 4.4.1 is completed by means of parallel transport of VMRTs along minimal rational curves and the standard argument of adjunction of minimal rational curves (cf. (4.2) and (4.3)). In the Hermitian symmetric case the proof that $\tau_{[\alpha]}(\eta, \xi) = 0$ for $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ results from a differential-geometric calculation with respect to the flat Euclidean connection in Harish-Chandra coordinates, and the general case is derived from adapted coordinates in the same setting as in [HoM10] as explained in (4.2).

Our arguments apply to uniruled projective manifolds to give a sufficient condition for a germ of complex submanifold to be rationally saturated, making it applicable to study sub-VMRT structures in general. This will be explained in (4.5).

For the formulation of sub-VMRT structures and nondegeneracy for substructures, one has to make use of the distribution on X spanned by VMRTs. In the event that the distribution $D(X) \subset T(X)$ spanned by VMRTs is linearly *degenerate*, the proof that $\mathcal{F}_X$ is tangent to $\mathcal{C}(S)$ relies on the fact that the kernel of the Frobenius form $\varphi : \Lambda^2 D(X) \to T(X)/D(X)$ contains the linear span of $\{\alpha \wedge P_\alpha : \alpha \in \widetilde{\mathcal{C}}_x(X)\}$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_x(X))$, a basic fact about distributions spanned by VMRTs that was established in Hwang-Mok [HM98, (4.2), Proposition 10].

4.5 Criteria for Rational Saturation and Algebraicity of Germs of Complex Submanifolds

Let now $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$. Using a generalization of the argument of Theorem 4.4.1 and the method of analytic continuation by the adjunction of (open subsets of) minimal rational curves of Hwang-Mok [HM01], Mok [Mk08a] and Hong-Mok [HoM10], Mok-Zhang [Mz15, Theorem 1.4] also obtained a general result on the analytic continuation of a germ of complex submanifold S on X when S inherits a certain geometric substructure. For its formulation, letting B' be the enhanced bad set of $(X, \mathcal{K})$ we consider a complex submanifold $S \subset W$ of an open subset $W \subset X - B'$. Writing $\mathcal{C}(S) := \mathcal{C}(X)|_S \cap \mathbb{P}T(S)$ and $\varpi := \pi|_{\mathcal{C}(S)} : \mathcal{C}(S) \to S$ we defined in [Mz15, Definition 5.1] the notion of a sub-VMRT structure, as follows.

Definition 4.5.1. We say that $\varpi := \pi|_{\mathcal{C}(S)} : \mathcal{C}(S) \to S$ is a sub-VMRT structure on $(X, \mathcal{K})$ if and only if (a) the restriction of ϖ to each irreducible component of $\mathcal{C}(S)$ is surjective, and (b) at a general point $x \in S$ and for any irreducible component Γ_x of $\mathcal{C}_x(S)$, we have $\Gamma_x \not\subset \mathrm{Sing}(\mathcal{C}_x(X))$.

Given a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ of $\pi : \mathcal{C}(X) \to X$, there is some integer $m \geq 1$ such that over a general point $x \in S$, $\mathcal{C}_x(S)$ has exactly m irreducible components and such that ϖ is a submersion at a general point χ_k of each irreducible component $\Gamma_{k,x}$ of $\mathcal{C}_x(S)$. We introduce now the notions of proper pairs of projective subvarieties and nondegeneracy for substructure for such pairs (cf [Mz15, Definitions 5.2 & 5.3]).

Definition 4.5.2. Let V be a Euclidean space and $\mathcal{A} \subset \mathbb{P}(V)$ be an irreducible subvariety. We say that $(\mathcal{B}, \mathcal{A})$ is a proper pair if and only if $\mathcal{B}$ is a linear section of $\mathcal{A}$, and for each irreducible component Γ of $\mathcal{B}$, $\Gamma \not\subset \mathrm{Sing}(\mathcal{A})$.

For a uniruled projective manifold X and a complex submanifold $S \subset W \subset X - B$ inheriting a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ as in Definition 4.5.1, at a general point $x \in S$, $(\mathcal{C}_x(S), \mathcal{C}_x(X))$ is a proper pair of projective subvarieties. We introduce now the notion of nondegeneracy for substructures for $(\mathcal{B}, \mathcal{A}; E)$. Here for convenience we assume that $\mathcal{A}$ is irreducible. When applied to sub-VMRT structures this means that the VMRT $\mathcal{C}_x(X)$ at a general point on the ambient manifold X is assumed irreducible.

Definition 4.5.3. Let V be a finite-dimensional vector space, $E \subset V$ be a vector subspace and $(\mathcal{B}, \mathcal{A})$ be a proper pair of projective subvarieties in $\mathbb{P}(V)$, $\mathcal{B} := \mathcal{A} \cap \mathbb{P}(E) \subset \mathcal{A} \subset \mathbb{P}(V)$. Assume that $\mathcal{A}$ is irreducible. Let $\xi \in \widetilde{\mathcal{B}}$ be a smooth point of both $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, and let $\sigma : S^2 T_\xi(\widetilde{\mathcal{A}}) \to V/T_\xi(\widetilde{\mathcal{A}})$ be the second fundamental form of $\widetilde{\mathcal{A}}$ in V with respect to the Euclidean flat connection on V. Write $V' \subset V$ for the linear span of $\widetilde{\mathcal{A}}$ and define $E' := E \cap V'$. Define by $\nu : V/T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ the canonical projection and $\tau : S^2 T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ by $\tau := \nu \circ \sigma$. For the proper pair $(\mathcal{B}, \mathcal{A})$, $\mathcal{B} = \mathcal{A} \cap \mathbb{P}(E)$, we say that $(\mathcal{B}, \mathcal{A}; E)$ is nondegenerate for substructures if and only if for each irreducible component Γ of $\mathcal{B}$ and for a general point $\chi \in \Gamma$, we have

$$\big\{\eta \in T_\chi(\widetilde{\mathcal{A}}) : \tau(\eta, \xi) = 0 \text{ for any } \xi \in T_\chi(\widetilde{\mathcal{B}})\big\} = T_\chi(\widetilde{\mathcal{B}}).$$

In case $E' = E \cap V'$ is the same as the linear span of $\widetilde{\mathcal{B}}$ we drop the reference to E, with the understanding that the projection map ν is defined by using the linear span of $\widetilde{\mathcal{B}}$ as E'. In the case of an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, writing $D(X) \subset T(X)$ for the G-invariant distribution spanned by VMRTs at each point $x \in X$, $D(X) \cap T(X_0)$ is the same as the G_0-invariant distribution on X_0 spanned by VMRTs. (When the Dynkin diagram is marked at a long simple root, $D(X)$ and $D(X_0)$ are the minimal nonzero invariant distributions, but the analogue fails for the short-root case.) In order to adapt the arguments for rational saturation to the general situation of sub-VMRT structures, we need to introduce an auxiliary condition on the intersection $\mathcal{C}(\mathcal{S}) = \mathcal{C}(X) \bigcap \mathbb{P}T(S)$, to be called Condition (T), which is automatically satisfied in the case of admissible pairs (X_0,X). We have

Definition 4.5.4. Let $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, be a sub-VMRT structure on $S \subset X - B'$ as in Definition 4.5.1. For a point $x \in S$, and $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$, we say that $(\mathcal{C}_x(S), [\alpha])$, or equivalently $(\widetilde{\mathcal{C}_x}(S), \alpha)$, satisfies Condition (T) if and only if $T_\alpha(\widetilde{\mathcal{C}_x}(S)) = T_\alpha(\widetilde{\mathcal{C}_x}(X)) \cap T_x(S)$. We say that $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) at x if and only if $(\widetilde{\mathcal{C}_x}(S), [\alpha])$ satisfies Condition (T) for a general point $[\alpha]$ of each irreducible component of $\mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$. We say that $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) if and only if it satisfies the condition at a general point $x \in S$.

Theorem 4.5.1 (Mok-Zhang [Mz15, Theorem 1.4]). *Let $(X, \mathcal{K})$ be a uniruled projective manifold X equipped with a minimal rational component $\mathcal{K}$ with asso-*

ciated VMRT structure given by $\pi : \mathcal{C}(X) \to X$. *Assume that at a general point* $x \in X$, *the VMRT* $\mathcal{C}_x(X)$ *is irreducible. Write* $B' \subset X$ *for the enhanced bad locus of* $(X, \mathcal{K})$. *Let* $W' \subset X - B'$ *be an open set, and* $S \subset W$ *be a complex submanifold such that, writing* $\mathcal{C}(S) := \mathcal{C}(X)|_S \cap \mathbb{P}T(S)$ *and* $\varpi := \pi\big|_{\mathcal{C}(S)}$, $\varpi : \mathcal{C}(S) \to S$ *is a sub-VMRT structure satisfying Condition (T) Suppose furthermore that for a general point x on S and for each of the irreducible components* $\Gamma_{k,x}$ *of* $\mathcal{C}_x(S)$, $1 \leq k \leq m$, *the inclusion* $\Gamma_{k,x} \subset \mathcal{C}_x(X)$ *at a general smooth point* χ_k *of* $\Gamma_{k,x}$ *is nondegenerate for substructures. Then, S is rationally saturated with respect to* $(X, \mathcal{K})$. *In other words, S is uniruled by open subsets of minimal rational curves belonging to* $\mathcal{K}$.

When X is of Picard number 1, by a line ℓ on X we mean a rational curve ℓ of degree 1 with respect to the positive generator of $\text{Pic}(X) \cong \mathbb{Z}$. We say that $(X, \mathcal{K})$ is a uniruling by lines to mean that members of $\mathcal{K}$ are lines. We prove for these uniruled projective manifolds a sufficient condition for the *algebraicity* of germs of sub-VMRT structures on them. Recall that a holomorphic distribution $\mathcal{D}$ on a complex manifold M is said to be bracket generating if and only if, defining inductively $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_{k+1} = \mathcal{D}_k + [\mathcal{D}, \mathcal{D}_k]$, we have $\mathcal{D}_m|_U = T(U)$ on a neighborhood U of a general point $x \in M$ for m sufficiently large. By a distribution we will mean a coherent subsheaf of the tangent sheaf. We have

Theorem 4.5.2 (Mok-Zhang [Mz15, Main Theorem 2]). *In the statement of Theorem 4.5.1 suppose furthermore that* $(X, \mathcal{K})$ *is a projective manifold of Picard number 1 uniruled by lines and that the distribution* $\mathcal{D}$ *on S defined at a general point* $x \in X$ *by* $\mathcal{D}_x := \text{Span}(\widetilde{\mathcal{C}}_x(S))$ *is bracket generating. Then, there exists an irreducible subvariety* $Z \subset X$ *such that* $S \subset Z$ *and such that* $\dim(Z) = \dim(S)$.

Thus, the subvariety $Z \subset X$, which is rationally saturated with respect to $\mathcal{K}$, is in particular uniruled by minimal rational curves belonging to $\mathcal{K}$. $Z \subset X$ is thus a uniruled projective subvariety. We say that S admits a projective-algebraic extension. Note that the hypothesis that $\mathcal{D}$ is bracket generating is trivially satisfied when $\mathcal{D}$ is linearly nondegenerate at a general point.

Modulo Theorem 4.5.1, which yields rational saturation for sub-VMRT structures under a condition of nondegeneracy for substructures, for the proof of Theorem 4.5.2 we reconstruct a projective-algebraic extension of S by a process of adjunction of minimal rational curves as in Hwang-Mok [HM98], Mok [Mk08a] and Hong-Mok [HoM10]. As opposed to the situation of these articles where the adjunction process is a priori algebraic, the major difficulty in the proof of Theorem 4.5.2 lies in showing that, starting from a *transcendental* germ of complex manifold $(S; x_0)$ on $X - B'$ equipped with a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ obtained from taking intersections with tangent subspaces, the process of adjoining minimal rational curves starting with those emanating from x_0 and tangent to S is actually *algebraic*. For this purpose we introduce a method of propagation of sub-VMRT structures along chains of special minimal rational curves and apply methods of extension of holomorphic objects in several complex variables coming from the Hartogs phenomenon (cf. Siu [Si74]), viz., we show that for the inductive process of propagation of the germ $(S; x_0)$ along chains of rational curves, the obstruction in essence lies on subvarieties of codimension ≥ 2 on certain universal families of chains of rational curves. Crucial to this process is a proof of the following "Thickening Lemma" which allows us to show that the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ can be propagated along a general member of certain algebraic families of standard rational curves which are defined inductively.

Proposition 4.5.1 (Mok-Zhang [[Mz15], Proposition 6.1]). *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component,* $\dim(X) := n$, *and $\varpi : \mathcal{C}(S) \to S$ be a sub-VMRT structure as in Theorem 1.4,* $\dim(S) := s$. *Let $[\alpha] \in \mathcal{C}(S)$ be a smooth point of both $\mathcal{C}(S)$ and $\mathcal{C}(X)$, $\varpi([\alpha]) := x$, and $[\ell] \in \mathcal{K}$ be the minimal rational curve (which is smooth at x) such that $T_x(\ell) = \mathbb{C}\alpha$, and $f : \boldsymbol{P}_\ell \to \ell$ be the normalization of ℓ, $\boldsymbol{P}_\ell \cong \mathbb{P}^1$. Suppose $(\mathcal{C}_x(s), [\alpha])$ satisfies Condition (T) in Definition 4.5.4. Then, there exists an s-dimensional complex manifold E, $\boldsymbol{P}_\ell \subset E$, and a holomorphic immersion $F : E \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$, and such that $F(E)$ contains an open neighborhood of x in S.*

In relation to Theorem 4.5.2 there is the problem of recognizing special classes of uniruled projective subvarieties, which we formulate as

Problem 4.5.1 (The Recognition Problem for Sub-VMRT Structures). Let X be a uniruled projective manifold endowed with a minimal rational component $\mathcal{K}$, and Φ be a class of projective subvarieties $Z \subset X$ which are rationally saturated with respect to $(X, \mathcal{K})$. Denote by B' the enhanced bad locus of $(X, \mathcal{K})$. We say that the Recognition Problem for the class $\Phi \subset \mathrm{Chow}(X)$ is solved in the affirmative if one can assign to each $x \in X - B'$ a variety of linear sections $\Psi_x \subset \mathrm{Chow}(\mathcal{C}_x(X))$ in such a way that a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ of $\pi : \mathcal{C}(X) \to X$ admits a projective-algebraic extension to a member Z of Φ if and only if $[\mathcal{C}_s(S)] \in \Psi_s$ for a general point $s \in S$.

As an example, let X be the Grassmannian $G(p, q)$; $p, q \geq 2$; identified as a submanifold of some $\mathbb{P}^N$ by means of the Plücker embedding. Suppose $2 \leq p' \leq p$, $2 \leq q' \leq q$ and let Φ to be the class of linear sections $Z = G(p, q) \cap \Pi$, $\Pi \subset \mathbb{P}^N$ a projective linear subspace, such that Z is the image of a standard embedding of $G(p', q')$ into $G(p, q)$. Then, by Theorem 4.5.1, the Recognition Problem is solved for Φ by taking Ψ_x at $x \in X$ to consist of linear sections $\varsigma(\mathbb{P}(U_x) \times \mathbb{P}(V_x)) \cap \mathbb{P}(U'_x \otimes V'_x)$ where $T(G(p, q)) \cong U \otimes V$, where ς denotes the Segre embedding, and $U'_x \subset U_x$ (resp. $V'_x \subset V_x$) runs over the set of p'-dimensional (resp. q'-dimensional) vector subspaces of U_x (resp. V_x). Another example is the Recognition of maximal linear subspaces. The result of Hong-Park [HoP11] (Theorem 4.2.1 here) says that maximal linear subspaces on $X = G/P$ can be recognized with only a few exceptions. By making use of a quantitative version of nondegeneracy for substructures the Recognition Problem for maximal linear subspaces can be solved in the affirmative on certain linear sections of rational homogeneous spaces. As an example we have the following result in the case of linear sections of Grassmannians taken from Mok-Zhang [Mz15, Corollary 9.1].

Proposition 4.5.2. *Consider the Grassmannian $G(p,q)$, $3 \le p \le q$, of rank $p \ge 3$. Let $Z \subset G(p,q)$ be a smooth linear section of codimension $\le p-2$, $\mathcal{H}$ be the space of projective lines on Z, and $E \subset Z$ be the bad locus of $(Z,\mathcal{H})$. Let $(S;x_0)$ be a germ of complex submanifold on $Z-E$ such that $\mathbb{P}T(S) \subset \mathcal{C}(Z)|_S$ and $\mathbb{P}T(S)$ contains a smooth point of $\mathcal{C}(Z)|_S$. Suppose $\mathbb{P}T_x(S) \subset \mathcal{C}_x(Z)$ is a maximal linear subspace for a general point $x \in S$. Then, $S \subset Z$ is a maximal linear subspace.*

By the very nature of the notion of sub-VMRT structures, viz., by taking linear sections with tangent subspaces, the Recognition Problem concerns primarily the recognition of a global linear section Z of a projective manifold X uniruled by projective lines from the fact that VMRTs of Z at a general point is a linear section of the VMRT of X with special properties. We may say that this amounts to recognizing certain global linear sections from sub-VMRTs which are special linear sections of VMRTs. In a direction beyond the current article, one could define higher order sub-VMRT structures by considering minimal rational curves which are tangent to the submanifold S to higher orders. There for instance one could raise the problem of recognizing the intersection of a sub-Grassmannians with a number of quadric hypersurfaces in terms of second order sub-VMRT structures.

In (4.6) we will discuss some concrete examples to which Theorems 4.5.1 and 4.5.2 apply.

4.6 Examples of Sub-VMRT Structures Related to Irreducible Hermitian Symmetric Spaces of the Compact Type

Theorem 4.5.1 gives sufficient conditions for proving that certain sub-VMRT structures are rationally saturated. In the event that the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) and it is furthermore linearly nondegenerate for substructures at a general point, it shows that the sub-VMRT structure arises from some uniruled projective subvariety. Here are some examples of sub-VMRT Structures to Which Theorems 4.5.1 and 4.5.2 apply

(a) Let X be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 other than a Lagrangian Grassmannian equipped with the minimal rational component $\mathcal{K}$ of projective lines. Let $[\alpha] \in \mathcal{C}_x(X)$ and consider $\mathcal{S}_{[\alpha]} := \mathcal{C}_x(X) \cap \mathbb{P}(P_\alpha)$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}_x}(X))$. Then $\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)$ is linearly nondegenerate. Note here that a Lagrangian Grassmannian is equivalently an irreducible symmetric space $G^{\mathrm{III}}(n,n)$, $n \ge 2$, of type III for which the VMRTs are Veronese embeddings $\nu : \mathbb{P}(V) \to \mathbb{P}(S^2V)$ for $V \cong \mathbb{C}^n$ for some $n \ge 2$, given by $\nu([v]) = [v \otimes v]$, in which case the analogue of $\mathcal{S}_{[\alpha]}$ is the single point $[\alpha]$ since the image of the Veronese embedding contains no lines. $\mathcal{S}_{[\alpha]}$ is the cone over a copy of the VMRT of $\mathcal{C}_{[\alpha]}(X)$. It can be proven that the proper pair $(\mathcal{S}_{[x]}, \mathcal{C}_x(X))$ is nondegenerate for substructures excepting in the cases where $X = Q^n$. $n \geqslant 3$, or $X = G(2,g)$, $g \geqslant 2$. Thus, by Theorem 4.5.1, excepting the latter cases, any complex submanifold $S \subset W$ on an open subset $W \subset X$

carrying a sub-VMRT structure $\varpi : \mathcal{C}_x(S) \to S$ with fibers $\big(\mathcal{C}_x(S) \subset \mathcal{C}_x(X)\big)$ projectively equivalent to $\big(\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)\big)$ is rationally saturated and in particular uniruled by projective lines. By the linear nondegeneracy of $\mathcal{S}_{[\alpha]}$ in $\mathbb{P}(P_\alpha)$, Theorem 4.5.2 applies to show that $S \subset W$ admits a projective-algebraic extension. As a model let $x \in X$ and consider the union $\mathcal{V}$ of all projective lines on X passing through x. Then $\mathcal{V} \subset X$ is a projective subvariety inheriting a sub-VMRT structure modeled on $\big(\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)\big)$.

We note that in the case where X is the Grassmannian $G(p,q)$ of rank $r = \min(p,q) \geq 2$, $T_{G(p,q)} \cong U \otimes V$, where U resp. V is a universal bundle of rank p resp. q, and $\mathcal{C}_x(X) = \varsigma\big(\mathbb{P}(U_x) \times \mathbb{P}(V_x)\big)$ for the Segre embedding ς, so that, writing $\alpha = u \otimes v$ we have $\mathcal{S}_{[\alpha]} = \mathbb{P}(\mathbb{C}u \otimes V_x) \cup \mathbb{P}(U_x \otimes \mathbb{C}v)$ is the union of two projective subspaces of dimension $p-1$ resp. $q-1$ intersecting at a single point. This gives an example of a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ with 2 irreducible components in each fiber $\mathcal{C}_x(S)$. Both components have to be taken into account at the same time in order to have linear nondegeneracy in $\mathbb{P}T_x(S)$ so that one can apply the last statement in Theorem 4.5.2 when $r \geqslant 3$.

(b) The following are particular cases of examples discussed in Mok-Zhang [Mz15,§9]. Embed the Grassmann manifold $G(p,q)$; $p,q \geq 2$; into the projective space by the Plücker embedding $\varphi : G(p,q) \to \mathbb{P}(\Lambda^p\mathbb{C}^{p+q}) := \mathbb{P}^N$ and thus identify $G(p,q)$ as a projective submanifold. Consider a smooth complete intersection $X = G(p,q) \cap (H_1 \cap \cdots \cap H_m)$ of codimension k, where for $1 \leq i \leq m$, $H_i \subset \mathbb{P}^N$ is a smooth hypersurface of degree k_i, $k := k_1 + \cdots + k_m$. Let δ be the restriction to X of the positive generator of $H^2(G(p,q),\mathbb{Z}) \cong \mathbb{Z}$. If $k \leq p+q-1$ then $c_1(X) = (p+q-k)\delta \geq \delta$ and X is Fano. If $k \leq p+q-2$ then $c_1(X) \geq 2\delta$. For the latter range X is uniruled by the minimal rational component $\mathcal{K}$ of projective lines on X and the associated VMRT $\mathcal{C}_x(X)$ of X at a general point of X is the intersection of $\mathcal{C}_x(G(p,q)) = \varsigma(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1})$, where ς stands for the Segre embedding, of codimension k, with k hypersurfaces in $\mathbb{P}T_x(X)$ of degrees $(1,\cdots,k_1;\cdots;1,\cdots,k_m)$, and $\dim(\mathcal{C}_x(X)) = (p-1)+(q-1)-k = (p+q-2)-k \geq 0$. Suppose now $2 \leq p' < p$, $2 \leq q' < q$, and suppose $X_0 := G(p',q') \cap (H_1 \cap \cdots \cap H_m)$ is also smooth. We have $c_1(X_0) = (p'+q'-k)\delta \geq 2\delta$ if and only if $k \leq p'+q'-2$, in which case the pair $\big(\mathcal{C}_x(X_0) \subset \mathcal{C}_x(X)\big)$ consists of projective submanifolds of $\mathbb{P}T(X)$ of the form $\big(\mathcal{C}_0(G(p',q')) \cap \mathcal{J} \subset \mathcal{C}_0(G(p,q)) \cap \mathcal{J}\big)$ for some subvariety $\mathcal{J} \subset \mathbb{P}T_0(X)$ of codimension k at a reference point $0 \in G(p',q') \subset G(p,q)$. Consider now a germ of complex submanifold $(S;x_0)$ on X, and assume that by intersecting with projectivized tangent spaces we have a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ over S, where over a general point $x \in S$, the pair $\big(\mathcal{C}_x(S) \subset \mathcal{C}_x(X)\big)$ is projectively equivalent to $\big(\mathcal{C}_0(G(p',q')) \cap \mathcal{J}_x \subset \mathcal{C}_0(G(p,q)) \cap \mathcal{J}_x\big)$ for some projective subvariety $\mathcal{J}_x \subset \mathbb{P}T_0(X)$ of codimension k, and where $\mathcal{C}_x(S) \subset \mathbb{P}T_x(S)$ is linearly nondegenerate. If $k \leq \min(p'-2, q'-2)$, we show that the sub-VMRT structure $\varpi : \mathcal{C}(S) \to \mathcal{S}$ satisfies the hypotheses of Theorem 4.5.2 and must hence extend to a projective subvariety $Z \subset X$, $\dim(Z) = \dim(S) = p'q' - k$ which is uniruled by projective lines. This gives examples of germs of complex submanifolds on a uniruled projective manifold with variable VMRTs for which the arguments for proving algebraicity of Theorem 4.5.2 are applicable to prove analytic continuation of S to a projective subvariety. We note also that nondegeneracy for substructures fails in general if we consider S as a germ of complex submanifold on $G(p,q)$ instead of X. In fact, it already fails in general for $S \subset G(p',q')$, $S := G(p',q') \cap H$ (hence a fortiori for $S \subset G(p,q)$) when H is a smooth hypersurface of $G(p,q)$).

Complete intersections give examples of complex submanifolds $Y \subset G(p,q)$ uniruled by projective lines for which the isomorphism type of $\mathcal{C}_y(Y) \subset \mathcal{C}_y(G(p,q))$ at a general point can be described, but this precise information is not necessary for the application of Theorem 4.5.2. The same argument in fact applies to any projective submanifold $Y \subset G(p,q)$ uniruled by projective lines under the assumption that $c_1(Y) = (p+q-k)\delta \geq 2\delta$.

4.7 Perspectives on Geometric Substructures

While a first motivation on the study of the geometric theory on uniruled projective manifolds was to tackle problems in algebraic geometry on such manifolds, as the theory was developing, it was clear that our theory carries a strong differential-geometric flavor. It was at least in part developed in a self-contained manner basing on the study of the double fibration arising from the universal family, the tautological foliation and associated differential systems, and the axiomatics of the theory were derived from the deformation theory of rational curves. Varieties of minimal rational tangents appear naturally as the focus of our study, in the context of the VMRT structure $\pi : \mathcal{C}(X) \to X$. While basic results in the early part of the theory, such as those on integrability issues concerning distributions spanned by VMRTs and on Cartan-Fubini extension, have led to solutions of a number of guiding problems, the theory also takes form on its own. It is legitimate to raise questions regarding the VMRTs themselves, such as the Recognition Problem and problems on the classification of isotrivial VMRT structures on uniruled projective manifolds.

One may develop the theory of sub-VMRT structures in analogy to the study of Riemannian submanifolds in Riemannian manifolds, where rationally saturated subvarieties may be taken as weak analogues to geodesic subspaces and, at least in cases of rational homogeneous spaces of Picard number 1, certain cycles such as (possibly singular) Schubert cycles can be taken as strong analogues. Concerning problems in algebraic geometry that may be treated by the study of sub-VMRT substructures, first of all it is natural to extend the characterization of smooth Schubert varieties in rational homogeneous spaces of Picard number 1 to the case of singular Schubert varieties, where one has to examine sub-VMRT structures $\varpi : \mathcal{C}(S) \to S$ with singular sub-VMRTs and to study parallel transport in such broader context. In view of the application of equidimensional Cartan-Fubini extension to prove rigidity of finite surjective holomorphic maps (Hwang-Mok

[HM01],[HM04b]), it is tempting to believe that non-equidimensional Cartan-Fubini extension can have implications for rigidity of certain non-equidimensional maps between uniruled projective varieties.

Cartan-Fubini extension can be taken as a generalization of Ochiai's Theorem in the context of S-structures (i.e., G-structures arising from irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2) from an entirely different angle, viz., from the perspectives of local differential geometry and several complex variables. The proof itself reveals the interaction of these aspects with algebraic geometry, notably with Mori's theory on rational curves. The study of sub-VMRT structures on a uniruled projective manifold in Mok [Mk08a] was first of all motivated by the desire to understand the heart of a rigidity phenomenon in several complex variables, viz., the rigidity of proper holomorphic maps between irreducible bounded symmetric domains of the same rank $r \geq 2$, established by Tsai [Ts93] by considering boundary values on certain product submanifolds, as was done in Mok-Tsai [MT92], and applying methods of Kähler geometry. Regarding a bounded symmetric domain Ω of rank ≥ 2 as an open subset of its dual Hermitian symmetric space S of the compact type by means of the Borel embedding, Ω carries a VMRT structure by restriction. The gist of the arguments of [Mk08a] consists of exploiting boundary values of the map, from which one shows that the mapping respects VMRTs because of properness and because of the decomposition of $\partial\Omega$ [Wo72] into the disjoint union of boundary faces of different ranks, and rigidity of the map results from non-equidimensional Cartan-Fubini extension as was later developed in full generality by Hong-Mok [HoM10]. For a proper holomorphic map $f : \Omega \to \Omega'$ where $r' = \text{rank}(\Omega') > \text{rank}(\Omega) = r$ the theory has yet to be further developed. In some very special cases they have led to VMRT-respecting holomorphic maps (Tu [Tu02]), but in general to a different form of geometric structures where VMRTs are mapped into vectors tangent to rational curves of degree $\leq r' - r + 1 < r'$, a context which was first discussed by Neretin [Ne99] in the case of classical domains of type I (which are dual to Grassmannians). The study of proper holomorphic mappings will remain a source of motivation for the further study of VMRTs or more general geometric substructures.

Another source of examples with sub-VMRT structures, somewhat surprisingly, is the study of holomorphic isometries of the complex unit ball into an irreducible bounded symmetric domain Ω of rank ≥ 2. Denote by $\Omega \subset S$ the Borel embedding of Ω as an open subset of its dual Hermitian symmetric space S of the compact type. There, the construction in Mok [Mz15] shows that, given a regular boundary point $q \in \text{Reg}(\partial\Omega)$, and denoting by $\mathcal{V}_q$ the union of minimal rational curves passing through q, the intersection $\Sigma := \mathcal{V}_q \cap \Omega$ is the image of a holomorphic isometric embedding of B^{p+1}, $p = \dim(\mathcal{C}_0(S))$. These were the examples which inherit, excepting in the case of Lagrangian Grassmannians, *singular* sub-VMRT structures $\varpi : \mathcal{C}(Z) \to Z$ which are nondegenerate for substructures excepting the cases of hypequadrics and rank-2 Grassmannias as in (4.5). The dimension $p + 1$ is the maximal possible dimension n for a holomorphic isometry $f : (B^n, g) \to (\Omega, h)$, where g resp. h are canonical Kähler-Einstein metrics normalized so that minimal disks are of Gaussian curvature -2, and, questions on uniqueness and rigidity in the case of $n = p + 1$ have led to the study of normal forms of tangent spaces of $T_x(Z)$ and interesting questions on the reconstruction of complex submanifolds from their sub-VMRT structures. Another exciting area where sub-VMRT structures enter is the study of geometric substructures on a quotient $X_\Gamma = \Omega/\Gamma$ of a bounded symmetric domains Ω by a torsion-free discrete subgroup $\Gamma \subset \text{Aut}(\Omega)$, where X_Γ is a quasi-projective manifold inheriting by descent an S-structure, which is equivalently a VMRT structure.

Taking VMRT structures as an area of research in its own right, it is necessary to examine more examples of interesting sub-VMRT structures. In the Hermitian symmetric case Zhang [Zh14] has now completely classified such pairs (X_0, X) and determined those which are nondegenerate for substructures. In addition, beyond the standard example of the holomorphic conformal structure on Q^n, $n \geq 3$, where germs of complex submanifolds with variable Bochner-Weyl curvature tensors abound, for the other admissible pairs in the Hermitian symmetric case where nondegeneracy for substructures fails, Zhang [Zh14] constructed examples of nonstandard complex submanifolds modeled on (X_0, X) (cf. Theorem 4.4.2 in the current article). At the same time, new cases (not of sub-diagram type) of admissible pairs (X_0, X) which are nondegenerate for substructures have been identified. These admissible pairs in the Hermitian symmetric case, said to be of special type, are not Schubert cycles and the argument of parallel transport fails (cf. Proposition 4.2.1 used in the sub-diagram cases). It will be interesting to classify in general admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1. Moreover, there are many interesting uniruled projective subvarieties such as Schubert cycles on $X = G/P$, for which the theory of sub-VMRT structures apply, and they may provide new sources for the study of the Recognition Problem for sub-VMRT structures and for formulating other geometric problems in the theory.

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Foliations, Shimura Varieties, and the Green-Griffiths-Lang Conjecture

Erwan Rousseau

Abstract Foliations have been recently a crucial tool in the study of the degeneracy of entire curves on projective varieties of general type. In this note, considering the Green-Griffiths locus, we explain how to deal with the case where there is no natural foliation to start with. As an application, we show that for most quotients of classical bounded symmetric domains, the Green-Griffiths locus is the whole variety.

Keywords Entire curves • Foliations • Shimura varieties

Mathematical Subject Classification: Primary: 32Q45, 32M15; Secondary: 37F75

1 Introduction

Foliations are known to play an important role in the study of subvarieties of projective varieties. One beautiful example is the proof of the following theorem of Bogomolov.

Theorem 1.1 ([Bog77]). *Let X be a projective surface of general type such that $c_1^2 > c_2$. Then X has only finitely many rational or elliptic curves.*

The numerical positivity $c_1^2 > c_2$ gives the existence of global symmetric tensors on X and reduces the problem to the study of rational or elliptic algebraic leaves of foliations on a surface of general type. This result fits into the general study of hyperbolic (in the sense of Kobayashi) properties of algebraic varieties as illustrated by the famous Green-Griffiths-Lang conjecture

Conjecture 1.2. Let X be a projective variety of general type. Then there exists a proper Zariski closed subset $Y \subsetneq X$ such that for all non-constant holomorphic curve $f : \mathbb{C} \to X$, we have $f(\mathbb{C}) \subset Y$.

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Although this is still largely open, even for surfaces, McQuillan has extended Bogomolov's theorem to transcendental leaves proving Green-Griffiths-Lang conjecture for surfaces of general type with positive second Segre number $c_1^2 - c_2$ [McQ0].

Due to ideas which can be traced back to Bloch, it is now classical that algebraic differential equations can be used to attack such problems as follows. Holomorphic maps $f : U \subset \mathbb{C} \to X$ canonically lift to projectivized jets spaces $f_{[k]} : U \to P(J_kX)$. Let A be an ample line bundle on X, $p_k : P(J_kX) \to X$ the natural projection and $B_{k,l} \subset P(J_kX)$ be the base locus of the line bundle $\mathscr{O}_{P(J_kX)}(l) \otimes p_k^*A^{-1}$. We set $B_k = \bigcap_{l\in\mathbb{N}} B_{k,l}$ and $GG = \bigcap_{k\in N} p_k(B_k)$, the Green-Griffiths locus. The strategy is based on the fundamental vanishing theorem

Theorem 1.3 (Green-Griffiths, Demailly, Siu-Yeung). *Let $f : \mathbb{C} \to X$ be a non-constant holomorphic curve. Then $f_{[k]}(\mathbb{C}) \subset B_k$ for all k. In particular, we have $f(\mathbb{C}) \subset GG$.*

On the positive side of this strategy, there is the recent result by Demailly which for our purposes can be stated as follows.

Theorem 1.4 ([De11]). *Let X be a projective variety of general type. Then for some $k \ggg 1$, $B_k \neq P(J_kX)$.*

In other words, all non-constant holomorphic curves $f : \mathbb{C} \to X$ in a projective variety of general type satisfy a non-trivial differential equation $P(f, f', \dots, f^k) \equiv 0$.

On the less optimistic side, it is recently shown in [DR] that one cannot hope $GG \neq X$ in general.

From [DR] Theorem 1.3, one can extract the following criterion

Theorem 1.5. *Let X be a projective variety endowed with a holomorphic foliation by curves $\mathscr{F}$. If the canonical bundle of the foliation $K_{\mathscr{F}}$ is not big then $GG = X$.*

This produces examples of (hyperbolic!) projective varieties of general type whose Green-Griffiths locus satisfies $GG = X$ (see [DR] for details).

Example 1.6. Let $X = C_1 \times C_2$ a product of compact Riemann surfaces with genus $g(C_i) \geq 2$. Then X is of general type, hyperbolic and $GG = X$.

Example 1.7. Let $X = \Delta \times \Delta/\Gamma$ be a smooth compact irreducible surface uniformized by the bi-disc. X is naturally equipped with 2 non-singular foliations $\mathscr{F}$ and $\mathscr{G}$ with Kodaira dimension $-\infty$. Then X is of general type, hyperbolic and $GG = X$.

Example 1.8. Let $X = \Delta^n/\Gamma$ be a (not necessarily compact) quotient of the polydisc by an arithmetic lattice commensurable with the Hilbert modular group. Then $\overline{X}$ any compactification satisfies $GG = \overline{X}$.

In all previous examples we have natural foliations on the manifold that one uses in a crucial way to obtain that the Green-Griffiths locus is the whole manifold. It is therefore tempting to think that the absence of these natural foliations could be an obstruction to the Green-Griffiths locus covering the whole manifold. In fact in [DR], using analytic techniques inspired by [Mok], we show that projective manifolds uniformized by bounded symmetric domains of rank at least 2 satisfy $GG = X$.

The goal of this note is to study the example of quotients of irreducible classical bounded symmetric domains, also in the non-compact case, using the arithmetic data of the Shimura varieties associated to these quotients.

2 Arithmetic Lattices in Classical Groups

Let D be an irreducible symmetric bounded domain in $\mathbb{C}^N$ and Γ a torsion-free lattice in the simple real Lie group $G = \mathrm{Aut}^0(D)$. Let $\overline{X}$ be a compactification of $X = D/\Gamma$. Let us recall the description of the irreducible classical bounded symmetric domains $D = G/K$:

- $D^I_{p,q} = \{Z \in M(p,q,\mathbb{C})/I_q - Z^*Z > 0\}$,
- $D^{II}_n = \{Z \in D^I_{n,n}/Z^t = -Z\}$,
- $D^{III}_n = \{Z \in D^I_{n,n}/Z^t = Z\}$,
- $D^{IV}_n = \{Z \in SL(2n,\mathbb{C})/Z^tZ = I_{2n}, Z^*J_nZ = J_n\}$.

The group G can be described as follows:

- $D^I_{p,q}$: G is the special unitary group of a hermitian form on $\mathbb{C}^{p+q}$ of signature (p,q), $q \leq p$.
- D^{II}_n: G is the special unitary group of a skew-hermitian form on $\mathbb{H}^n$.
- D^{III}_n: G is the special unitary group of a skew-symmetric form on $\mathbb{R}^{2n}$.
- D^{IV}_n: G is the special unitary group of a symmetric bilinear form on $\mathbb{R}^{n+2}$ of signature $(n,2)$.

Each of the $\pm$ symmetric/hermitian forms is isotropic, and if $t = \mathrm{rank}_{\mathbb{R}}(G)$, the maximal dimension of a totally isotropic subspace is $t = q, [\frac{n}{2}], n, 2$ in the cases I, II, III, and IV, respectively.

A deep theorem of Margulis states that if $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then any discrete subgroup $\Gamma \subset G$ is arithmetic. That means that there is an algebraic $\mathbb{Q}$-group $\mathbb{G}$ with $\Gamma \subset \mathbb{G}_{\mathbb{Q}} \subset \mathbb{G}_{\mathbb{R}} = G$ and a rational representation $\rho : \mathbb{G}_{\mathbb{Q}} \to GL(V_{\mathbb{Q}})$ such that $\rho^{-1}(GL(V_{\mathbb{Z}}))$ and Γ are commensurable.

So, from now on, we suppose that we are in the situation where $\mathbb{G}$ is a $\mathbb{Q}$-group such that $\mathbb{G}_{\mathbb{R}} = G$ is one of the above classical real Lie group and of real rank at least 2.

We shall prove that in this situation the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$ (except in a few cases that we cannot deal with yet described below).

Let us explain the idea to prove such a result. Recall that in the case of the polydisc $D = \Delta^r$, we used in an essential way the existence of natural holomorphic foliations on the manifold coming from factors of the polydisc.

So, to prove such a result for quotients of irreducible domains (e.g., Siegel modular varieties), we need to find something which replaces the existence of natural foliations. The idea is to use, instead of foliations on D, the existence of many embedded polydiscs. This is motivated by the polydisc theorem [Mok, ch.5, thm.1] which tells that there exists a totally geodesic submanifold E of D such that $(E, g_{|D})$ is isometric to a Poincaré polydisc (Δ^t, g_{Δ^t}), and $D = K.E$ where K is a maximal compact subgroup of G and g is the Bergman metric. If $M \subset G$ is the hermitian subgroup associated to E, to deduce that the restriction $\Gamma \cap M$ induces the existence of an algebraic subvariety we need some arithmetic conditions. In particular, it is sufficient that M comes from a reductive $\mathbb{Q}$-subgroup $\mathbb{M} \subset \mathbb{G}$ (see, for example, [hunt]). So, one reduces the problem to the existence of $\mathbb{Q}$-subgroups inducing a dense subset of subvarieties whose universal cover are polydiscs.

Let us recall the classification of classical $\mathbb{Q}$-groups of hermitian type. They are obtained by restriction of scalars $\mathbb{G} = Res_{k|\mathbb{Q}}\mathbb{G}'$ for an absolutely simple $\mathbb{G}'$ over k a totally real number field. The classification is the following (see [DWM] and [PR]):

(1) Unitary type:

- U.1) $\mathbb{G}' = SU(V, h)$, where V is an n-dimensional K-vector space, $K|k$ an imaginary quadratic extension, and h is a hermitian form. Then $G(\mathbb{R}) \cong \prod SU(p_\nu, q_\nu)$, where (p_ν, q_ν) are the signatures at infinite places.
- U.2) $\mathbb{G}' = SU(V, h)$ where D is a division algebra of degree $d \geq 2$, central simple over K with a $K|k$-involution and V is an n-dimensional right D-vector space with hermitian form h. Then $\mathbb{G}(\mathbb{R}) \cong \prod SU(p_\nu, q_\nu)$.

(2) Orthogonal type:

- O.1) $\mathbb{G}' = SO(V, h)$, V a k-vector space of dimension $n+2$, h a symmetric bilinear form such that at an infinite place ν, h_ν has signature $(n, 2)$ and $\mathbb{G}'_\nu(\mathbb{R}) \cong SO(n, 2)$.
- O.2) $\mathbb{G}' = SU(V, h)$, V a right D-vector space of dimension n, h is a skew-hermitian form, D is a quaternion division algebra, central simple over k, and at an infinite place ν, either
  - (i) $D_\nu \cong \mathbb{H}$ and $\mathbb{G}'_\nu(\mathbb{R}) \cong SO(n, \mathbb{H})$, or
  - (ii) $D_\nu \cong M_2(\mathbb{R})$ and $\mathbb{G}'_\nu \cong SO(2n-2, 2)$.

(3) Symplectic type:

- S.1) $\mathbb{G}' = Sp(2n, k)$ and $G'(\mathbb{R}) \cong Sp(2n, \mathbb{R})$
- S.2) $\mathbb{G}' = SU(V, h)$, where V is an n-dimensional right vector space over a totally indefinite quaternion division algebra, and h is a hermitian form on V. Then $\mathbb{G}'(\mathbb{R}) \cong Sp(2n, \mathbb{R})$.

We shall prove the following result.

Theorem 2.1. *Let $X = D/\Gamma$ be an irreducible arithmetic quotient of a bounded symmetric domain of real rank at least 2. If X is of type $U.1$, $O.1$ ($n \geq 4$), $O.2)(i)$, $S.1$, or $S.2$ then the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$.*

In other words, the only cases remaining are $U.2$ and $O.2)(ii)$.

3 The Case of Siegel Modular Varieties

An interesting case is the case of Siegel modular varieties corresponding to the symplectic case $S.1$ in the above classification. The proof of the following result will be very explicit giving the ideas of the general result.

Theorem 3.1. *Let $X = D/\Gamma$, where $D = D_n^{III}$ and $\Gamma \subset Sp(2n, \mathbb{R})$ commensurable with $Sp(2n, \mathbb{Z})$, $n \geq 2$, then the Green-Griffiths locus $GG(\overline{X}) = \overline{X}$.*

Proof. There is a totally geodesic polydisc $\Delta^n \hookrightarrow D$,

$$z = (z_1, \ldots, z_n) \to z^* = diag(z_1, \ldots, z_n)$$

of dimension n consisting of diagonal matrices $\{Z = (z_{ij})/z_{ij} = 0 \text{ for } i \neq j\} \subset D$. This corresponds to an embedding $Sl(2, \mathbb{R})^n \hookrightarrow Sp(2n, \mathbb{R})$: $M = (M_1, \ldots, M_n) \to M^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$, where $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, and $a^* = diag(a_1, \ldots, a_n)$ is the corresponding diagonal matrix.

More generally, taking $A \in Gl(n, \mathbb{R})$ one can consider the map $\Delta^n \hookrightarrow D_n^{III}$, given by

$$z = (z_1, \ldots, z_n) \to A^t z^* A.$$

In order to take quotients, one defines

$$\Gamma_A := \{M \in Sl(2, \mathbb{R})^n / \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix} M^* \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1} \in \Gamma\}.$$

Indeed we have a modular embedding

$$\varphi_A : \Delta^n / \Gamma_A \to X.$$

Considering a totally real number field $K/\mathbb{Q}$ of degree n with the embedding $K \hookrightarrow \mathbb{R}^n$, $\omega \to (\omega^{(1)}, \ldots, \omega^{(n)})$, the matrices $A = (\omega_i^{(j)})$ where $\omega_1, \ldots, \omega_n$ is a basis of K have the property that Γ_A is commensurable with the Hilbert modular group of K [Fr79].

These matrices A are obviously dense in $Gl_n(\mathbb{R})$.

Now, take a global jet differential of order k, $P \in H^0(\overline{X}, E_{k,m}^{GG} T_{\overline{X}}^*)$. Taking the pull-back, $\varphi_A^* P$ we obtain a k jet differential on a manifold uniformized by a polydisc. Therefore, from Example 1.8, we obtain that $\varphi_A(\Delta^n / \Gamma_A) \subset GG(\overline{X})$. By density, we finally get

$$GG(\overline{X}) = \overline{X}.$$

□

4 The Isotropic Case

The case of Siegel modular varieties is a particular case of the situation where $\mathbb{G}$ is isotropic and $\operatorname{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$.

Theorem 4.1. *If* $\operatorname{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$ *then the Green-Griffiths locus* $GG(\overline{X}) = \overline{X}$.

Proof. Let $\mathbb{G} = SU(D, h)$, where D is a division algebra over $\mathbb{Q}$ and h is a non-degenerate hermitian or skew hermitian form on D^m. $\operatorname{rank}_{\mathbb{Q}}(\mathbb{G})$ coincides with the Witt index of h, i.e., with the dimension of a maximal totally isotropic subspace in D^m see [PR]. Let $H_1' =< v >$ be a totally isotropic subspace of dimension 1 on D. Then we can find v' such that $H_1 =< v, v' >$ is a hyperbolic plane, i.e., with respect to a properly chosen basis $h_{|H}$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\operatorname{rank}_{\mathbb{Q}}(\mathbb{G}) \geq 2$ we can find a second hyperbolic plane $H_2 \subset H_1^{\perp}$. Then

$$N \cong SU(D, h_{|H_1}) \times SU(D, h_{|H_2})$$

is a subgroup of G. Therefore we have found a $\mathbb{Q}$-group $N = N_1 \times N_2 \subset \mathbb{G}$. $\Gamma_N := \Gamma \cap N_{\mathbb{Q}}$ is an arithmetic subgroup giving a subvariety

$$\varphi_N : X_{\Gamma_N} = D_N / \Gamma_N \hookrightarrow X_\Gamma := X,$$

whose universal cover D_N is a product $D_1 \times D_2$ corresponding to $N_1(\mathbb{R}) \times N_2(\mathbb{R}) \subset \mathbb{G}_{\mathbb{R}}$.

Now, take a global jet differential of order k, $P \in H^0(\overline{X}, E_{k,m}^{GG} T_{\overline{X}}^*)$. Taking the pull-back, $\varphi_N^* P$ we obtain a k jet differential on a manifold uniformized by a product. Therefore we obtain that $\varphi_N(X_{\Gamma_N}) \subset GG(\overline{X})$.

Consider $g \in \mathbb{G}_{\mathbb{Q}}$ then we have a subvariety

$$\varphi_{gNg^{-1}} : X_{\Gamma_{gNg^{-1}}} = g(D_N)/\Gamma_{gNg^{-1}} \hookrightarrow X_\Gamma := X.$$

Since G is connected $\mathbb{G}_{\mathbb{Q}}$ is dense in $G = \mathbb{G}_{\mathbb{R}}$ (see Theorem 7.7 in [PR]). We finally get

$$GG(\overline{X}) = \overline{X}.$$

5 Proof of Theorem 2.1

As the proof of the two previous results made clear, the key point is to find a product of $\mathbb{Q}$-groups in $\mathbb{G}$. If $\mathbb{G} = SU(V, h)$ is of type $U.1$, since G is simple, $\mathbb{G}$ is compact at all but one infinite place where $\mathbb{G}(K_v) = SU(p, q)$. We can diagonalize h and find as above two planes H_1, H_2 where h is of signature $(1,1)$. The corresponding $\mathbb{Q}$-group is $N_1 = SU(h_{|H_1}) \times SU(h_{|H_2})$ such that $N_1(\mathbb{R}) \cong SU(1,1) \times SU(1,1) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Of course, the same reasoning can be applied when $\mathbb{G}$ is of type $O.1$ (with $n \geq 4$) replacing the hermitian form by the symmetric bilinear form. This provides a $\mathbb{Q}$-subgroup N_2 such that $N_2(\mathbb{R}) \cong SO(1,2) \times SO(1,2) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Now suppose $\mathbb{G} = SU(V,h)$ is of type $O.2)i)$. Recall that the real rank is equal to $\mathrm{rank}_{\mathbb{R}}(G) = [\frac{n}{2}]$. So our hypothesis on the real rank implies $n \geq 4$. We can therefore find again two planes H_1, H_2 and a $\mathbb{Q}$-subgroup $N_3 = SU(h_{|H_1}) \times SU(h_{|H_2})$, such that $N_3(\mathbb{R}) \cong SO(2,\mathbb{H}) \times SO(2,\mathbb{H}) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ (modulo compact factors $SU(2)$).

Finally if $\mathbb{G} = SU(V,h)$ is of type $S.2$, the hypothesis on the real rank implies $n \geq 2$. So we can find two spaces H_1, H_2 one-dimensional over the quaternion algebra and consider as before the $\mathbb{Q}$-subgroup $N_4 = SU(h_{|H_1}) \times SU(h_{|H_2})$, such that $N_4(\mathbb{R}) = Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R}) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. □

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On the Structure of Codimension 1 Foliations with Pseudoeffective Conormal Bundle

Frédéric Touzet

Abstract Let X be a projective manifold equipped with a codimension 1 (maybe singular) distribution whose conormal sheaf is assumed to be pseudoeffective. By a theorem of Jean–Pierre Demailly, this distribution is actually integrable and thus defines a codimension 1 holomorphic foliation $\mathcal{F}$. We aim at describing the structure of such a foliation, especially in the non-abundant case: It turns out that $\mathcal{F}$ is the pull-back of one of the "canonical foliations" on a Hilbert modular variety. This result remains valid for "logarithmic foliated pairs."

Keywords Holomorphic foliations • Pseudoeffective line bundle • Abundance

Mathematical Subject Classification: 37F75

1 Introduction

In this paper, we are dealing with special class of codimension 1 holomorphic foliations. To this goal, let us introduce some basic notations.

Our main object of interest consists of a pair $(X, \mathcal{D})$ where X stands for a connected complex manifold (which will be fairly quickly assumed to be Kähler compact) equipped with a codimension 1 holomorphic (maybe singular) distribution $\mathcal{D} = \mathrm{Ann}\ \omega \subset TX$ given as the annihilator subsheaf of a twisted holomorphic non-trivial one form

$$\omega \in H^0(X, \Omega_X^1 \otimes L).$$

Without any loss of generalities, one can assume that ω is surjective in codimension 1, in other words that the vanishing locus Sing ω has codimension at least 2. This set will be also denoted by Sing $\mathcal{D}$.

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The line bundle L is usually called the **normal bundle** of $\mathcal{D}$ and denoted by $N_{\mathcal{D}}$. Here, we will be particularly interested in studying its dual $N_{\mathcal{D}}^*$, the **conormal bundle** of $\mathcal{D}$ in the case this latter carries some "positivity" properties.

We will also use the letter $\mathcal{F}$ (as foliation) instead of $\mathcal{D}$ whenever integrability holds, i.e: $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

In this context and unless otherwise stated, a **leaf** of $\mathcal{F}$ is an ordinary leaf of the induced *smooth* foliation on $X \setminus \mathrm{Sing}\, \mathcal{F}$.

Assuming now that X is **compact Kähler**, let us review some well-known situations where integrability automatically holds.

$\mathcal{D} = \mathcal{F}$ as soon as:

1. $N_{\mathcal{D}}^* = \mathcal{O}(D)$ where D is an integral effective divisor. This is related to the fact that $d\omega = 0$ (regarding ω as a holomorphic one form with D as zeroes divisor).
2. Kod $(N_{\mathcal{D}}^*) \geq 0$ where Kod means the Kodaira dimension. Actually, in this situation, one can get back to the previous one passing to a suitable branched cover.

In this setting, it is worth recalling the classical result of **Bogomolov Castelnuovo De Franchis** which asserts that one has always Kod $(N_{\mathcal{D}}^*) \leq 1$ and when equality holds, $\mathcal{D} = \mathcal{F}$ is a holomorphic fibration over a curve.

In fact, integrability still holds under weaker positivity assumptions on $N_{\mathcal{D}}^*$, namely if this latter is only supposed to be **pseudoeffective** (*psef*). This is a result obtained by Demailly in [de].

For the sequel, it will be useful to give further details and comments on this result.

First, the pseudoeffectiveness property can be translated into the existence of a singular metric h on $N_{\mathcal{D}}^*$ with plurisubharmonic (*psh*) local weights. This means that h can be locally expressed as

$$h(x, v) = |v|^2 e^{-2\varphi(x)}$$

where φ is a *psh* function.

The curvature form $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ is then well defined as a closed positive current representing the real Chern class $c_1(N_{\mathcal{D}}^*)$ and conversely any closed $(1, 1)$ positive current T such that $\{T\} = c_1(N_{\mathcal{D}}^*)$ (here, braces stand for "cohomology class") can be seen as the curvature current of such a metric.

What actually Demailly shows is the following vanishing property:

$$\nabla_{h^*}\omega = 0.$$

More explicitly, ω is closed with respect to the Chern connection attached to $N_{\mathcal{D}}$ endowed with the dual metric h^*.

This somehow generalizes the well-known fact that every holomorphic form on compact Kähler manifold is closed.

Locally speaking, this equality can be restated as

$$2\partial\varphi \wedge \omega = -d\omega \tag{1}$$

if one looks at ω as a suitable local defining one form of the distribution.

Equality (1) provides three useful information:

The first one is the sought integrability: $\omega \wedge d\omega = 0$, hence $\mathcal{D} = \mathcal{F}$.

One then obtains taking the $\overline{\partial}$ operator, that T is **$\mathcal{F}$-invariant** (or invariant by holonomy of the foliation), in other words, $T \wedge \omega = 0$.

One can also derive from (1) the closedness of η_T, where η_T is a globally defined positive $(1, 1)$ form which can be locally expressed as

$$\eta_T = \frac{i}{\pi} e^{2\varphi} \omega \wedge \overline{\omega}.$$

In some sense, η_T defines the dual metric h^* on $N_{\mathcal{D}}$ whose curvature form is $-T$ and is canonically associated with T up to a positive multiplicative constant.

Note that, as well as T, the form η_T has no reasons to be smooth. Nevertheless, it is well defined as a current and taking its differential makes sense in this framework.

By closedness, one can also notice that η_T is indeed a **closed positive $\mathcal{F}$ invariant current**.

Now, the existence of two such invariant currents, especially η_T which has locally bounded coefficients, may suggest that this foliation has a nice behavior, in particular from the dynamical viewpoint, but also, as we will see later, from the algebraic one. Then it seems reasonable to describe them in more detail.

As an illustration, let us give the following (quite classical) example which also might be enlightening in the sequel.

Let $S = \frac{\mathbb{D}\times\mathbb{D}}{\Gamma}$ be a surface of general type uniformized by the bidisk. Here, $\Gamma \subset \operatorname{Aut} \mathbb{D}^2$ is a cocompact irreducible and torsion free lattice which lies in the identity component of the automorphism group of the bidisk.

The surface S is equipped with two "tautologically" defined minimal foliations by curves $\mathcal{F}_h$ and $\mathcal{F}_v$ obtained, respectively, by projecting the horizontal and vertical foliations of the bidisk (recall that "minimal" means that every leaf is dense and here, this is due to the lattice irreducibility).

Both of these foliations have a *psef* and more precisely a semipositive conormal bundle. For instance, if one considers $\mathcal{F} = \mathcal{F}_h$, a natural candidate for η_T and T is provided by the transverse Poincaré's metric $\eta_{\text{Poincaré}}$, that is the metric induced by projecting the standard Poincaré's metric of the second factor of $\mathbb{D} \times \mathbb{D}$.

In this example, it is straightforward to check that $T = \eta_T$ is $\mathcal{F}$ invariant. We have also $T \wedge T = 0$. This implies that the numerical dimension $\nu(N^*_{\mathcal{D}})$ is equal to one. By minimality, and thanks to Bogomolov's theorem this equality certainly does not hold for Kod $N^*_{\mathcal{D}}$. In fact, this is not difficult to check that this latter is negative.

Maybe this case is the most basic one where abundance does not hold for the conormal. One can then naturally ask whether this foliation is in some way involved as soon as the abundance principle fails to be true. It turns out that (at least when the ambient manifold X is projective) the answer is yes, as precised by the following statement:

Theorem 1. *Let* $(X, \mathcal{F})$ *be a foliated projective manifold by a codimension* 1 *holomorphic foliation* $\mathcal{F}$*. Assume moreover that* $N^*_{\mathcal{F}}$ *is pseudoeffective and* $Kod\, N^*_{\mathcal{F}} = -\infty$*, then one can conclude that* $\mathcal{F} = \Psi^*\mathcal{G}$ *where* Ψ *is a morphism of analytic varieties between* X *and the quotient* $\mathfrak{H} = \mathbb{D}^n/\Gamma$ *of a polydisk,* $(n \geq 2)$*, by an irreducible lattice* $\Gamma \subset (Aut\, \mathbb{D})^n$ *and* $\mathcal{G}$ *is one of the* n *tautological foliations on* $\mathfrak{H}$*.*

Remark 1.1. Note that the complex dimension n of $\mathfrak{H}$ may differ (and will differ in general) from that of X. However, one can ask if both coincide whenever the foliation $\mathcal{F}$ is regular. These considerations are motivated by the description of regular foliations on surface of general type due to Marco Brunella in [brsurf], section 7. These latters have pseudoeffective conormal bundle and split into two cases:

- Either $\mathcal{F}$ is a fibration,
- either $\mathcal{F}$ is a transversely hyperbolic minimal foliation on a surface S whose universal covering $\tilde{S}$ is a fibration by disks over the disk, each fiber being a leaf of the lift foliation $\tilde{\mathcal{F}}$.

This last situation (extended to singular foliations) is the one described in Theorem 1 above (this point will be justified in Sect. 5). Brunella has conjectured that $\tilde{S}$ is the bidisk. If so, it is then readily seen that $\mathcal{F}$ is the projection of the horizontal or vertical foliation. To the best of our knowledge, this problem is still unsolved, and one can only assert that $\mathcal{F}$ comes from one of the tautological foliations on a (maybe higher dimensional) polydisk.

One can extend the previous theorem for "logarithmic foliated pairs" in the following sense:

Theorem 2. *The same conclusion holds replacing the previous assumptions on* $N^*_{\mathcal{F}}$ *by the same assumptions on the logarithmic conormal bundle* $N^*_{\mathcal{F}} \otimes \mathcal{O}(H)$ *where* $H = \sum_i H_i$ *is reduced and normal crossing divisor, each component of which being invariant by the foliation and replacing* $\mathfrak{H}$ *by its Baily–Borel compactification* $\overline{\mathfrak{H}}^{BB}$*.*

Remark 1.2. In the logarithmic case, compactification is needed because, unlike to the non-logarithmic setting, there may exist among the components of H some special ones arising from the possible existence of cusps in $\mathfrak{H}$ (this will be discussed in Sect. 10).

Let us give the outline of the proof. For the sake of simplicity, we will take only in consideration the non-logarithmic situation corresponding to Theorem 1. The logarithmic case follows *mutatis mutandis* the same lines although we have to overcome quite serious additional difficulties.

The first step consists in performing and analyzing the divisorial Zariski decomposition of the pseudoeffective class $c_1(N^*_{\mathcal{F}})$ using that we have at our disposal the two positive closed invariant currents T, η_T and their induced cohomology class $\{T\}$ and $\{\eta_T\}$.

Actually, the existence of such currents yields strong restriction on the Zariski decomposition. Without entering into details, this basically implies that this latter shares the same nice properties than the classical two dimensional Zariski decomposition.

Once we have obtained the structure of this decomposition, one can solve by a fix point method (namely, the Shauder–Tychonoff theorem applied in a suitable space of currents) the equation

$$T = \eta_T + [N] \tag{2}$$

where $[N]$ is some "residual" integration current. More precisely, among the closed positive currents representing $c_1(N^*_{\mathcal{F}})$, there exists one (uniquely defined) which satisfies the equality (2) (with η_T suitably normalized).

This equality needs some explanations. Indeed, it really holds when the positive part in the Zariski decomposition is non-trivial, that is when $N^*_{\mathcal{F}}$ has positive numerical dimension. From now on, we focus only on this case, regardless for the moment of what occurs for vanishing Kodaira dimension.

We also have to be precise what is N: it is just the negative part in the Zariski decomposition and it turns out that N is a $\mathbb{Q}$ effective $\mathcal{F}$ invariant divisor.

Then, outside Supp N, near a point where the foliation is assumed to be regular and defined by $dz = 0$, (2) can be locally expressed as

$$\Delta_z \varphi(z) = e^{2\varphi(z)}$$

where φ is a suitably chosen local potential of T depending only on the transverse variable z and Δ_z is the Laplacean with respect to the single variable z. This is the equation giving metric of constant negative curvature, which means that $\mathcal{F}$ falls into the realm of the so-called transversely projective foliations and more accurately the transversely hyperbolic ones. Equivalently, those can be defined by a collection of holomorphic local first integral valued in the disk such that the glueing transformation maps between these first integrals are given by disk automorphisms. In that way, it gives rise to a representation

$$\rho : \pi_1(X \setminus \mathrm{Supp}\, N) \to \mathrm{Aut}\, \mathbb{D}.$$

Remark 1.3. We need to remove Supp N because the transversely hyperbolic structure may degenerate on this hypersurface (indeed, on a very mildly way as recalled in Sect. 3).

Let G be the image of the representation: it is the monodromy group of the foliation and faithfully encodes the dynamical behavior of $\mathcal{F}$. More precisely, one can show that there are two cases to take into consideration:

Either G is a cocompact lattice and in that case, $\mathcal{F}$ is a holomorphic fibration, Kod $N^*_{\mathcal{F}} = 1$ and coincides with the numerical dimension.

Either G is dense (w.r.t the ordinary topology) in Aut $\mathbb{D}$ and the foliation is quasi-minimal (i.e: all leaves but finitely many are dense). Moreover, Kod $N^*_{\mathcal{F}} = -\infty$. Of course, we are particularly interested in this last situation.

To reach the conclusion, the remaining arguments heavily rely on a nice result of Kevin Corlette and Carlos Simpson (see [corsim]) concerning rank two local systems on quasiprojective manifolds, especially those with Zariski dense monodromy in $SL(2, \mathbb{C})$.

Roughly speaking, their theorem gives the following alternative:

(1) Either such a local system comes from a local system on a Deligne–Mumford curve $\mathcal{C}$.
(2) Either it comes from one of the tautological rank two local systems on a what the authors call a "polydisk Shimura DM stack" $\mathfrak{H}$ (this latter can be thought as some kind of moduli algebraic space uniformized by the polydisk, in the same vein than Hilbert Modular varieties).

Remark 1.4. In the setting of transversely hyperbolic foliations, as we do not really deal with rank 2 local but rather with projectivization of these, one can get rid of the DM stack formalism, only considering the target spaces $\mathcal{C}$ or $\mathfrak{H}$ with their "ordinary" orbifold structure (which consists of finitely many points). Moreover, in the full statement of Corlette-Simpson, one needs to take into account some quasiunipotency assumptions on the monodromy at infinity. Actually, these assumptions are fulfilled in our situation and are related to the rationality of the divisorial coefficients in the negative part N.

This paper is a continuation of the previous work [to] where one part of the strategy outlined above has been already achieved (namely, the construction of special metrics transverse to the foliation).

Note also that the description of transversely projective foliations on projective manifolds was recently completed in [lptbis]. However, there are several significant differences between our paper and *loc.cit* (and in fact, the techniques involved in both papers are not the same):

- In our setting, the transverse projective structure is not given a priori,
- [lptbis] also deals with irregular transversely projective foliations (ours turn out to be regular singular ones).
- When the monodromy associated with the transverse projective structure is Zariski dense in $PSL(2, \mathbb{C})$, the factorization theorem D, item (2) and (3) of [lptbis] can be made more precise for the class of foliations considered here. Namely, pull-back of a Ricatti foliation over a curve (item (2)) is replaced by pull-back of a foliation by points on a curve, that is $\mathcal{F}$ is a fibration, and analogously, item (3) can be reformulated changing "pull-back between representations" by "pull-back between foliations," as precised by Theorems 1 and 2 above.

Remark 1.5. To finish this introduction, maybe it is worth noticing an analogy (at least formal) between our results and the description of singular reduced foliations on surfaces not satisfying the abundance principle with respect to the foliated **canonical bundle** $K_{\mathcal{F}}$ and which has been achieved in [brsub] (the final and one of the most impressive steps in the Brunella–Mendes–McQuillan's classification of singular foliations on projective surfaces).

Indeed, it turns out that this class of foliations satisfied the assumptions of Theorem 2. However, this is not a priori obvious and can be actually deduced from the existence of the so-called Monge–Ampère transverse foliation (see [brsub], section 4), a highly non-trivial result!

2 Positive Currents Invariant by a Foliation

In this short section, we review some definitions/properties (especially intersection ones) of positive current which are invariant by holonomy of a foliation. They will be used later.

2.1 Some Basic Definitions

Let $\mathcal{F}$ be a codimension 1 holomorphic foliation on a complex manifold X defined by $\omega \in H^0(X, \Omega^1_X \otimes L)$.

Definition 2.1 (see also [brbir, to]). Let T be a positive current of bidegree $(1,1)$ defined on X. T is said to be **invariant by** $\mathcal{F}$, or $\mathcal{F}$-invariant, or invariant by holonomy of $\mathcal{F}$ whenever

(1) T is closed
(2) T is directed by the foliation, that is $T \wedge \omega = 0$.

Notice that when T is a non-vanishing smooth semi-positive $(1,1)$ form, we fall into the realm of the so-called **transversely hermitian foliations**.

Definition 2.2 (see[to]). Let $x \in X$ be a singular point of $\mathcal{F}$. We say that x is an elementary singularity if there exists near x a non-constant first integral of $\mathcal{F}$ of the form

$$f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$$

where $f_i \in \mathcal{O}_{X,x}$ and λ_i is a positive real number .

Equivalently, $\frac{df}{f}$ is a local defining logarithmic form of $\mathcal{F}$ with positive residues.

Definition 2.3. Let T be a closed positive $(1,1)$ current on X. Then T is said to be **residual** if the Lelong number $\lambda_H(T)$ of T along any hypersurface H is zero.

Let X be compact Kähler, and T a $(1,1)$ positive current on X, and $\{T\} \in H^{1,1}(X, \mathbb{R})$ its cohomology class . By definition, this latter is a *pseudoeffective* class and admits, as such, a **divisorial Zariski decomposition** (a fundamental result due to [bo, na], among others):

$$\{T\} = N + Z$$

where N is the "negative part," an $\mathbb{R}$ effective divisor, whose support is a finite union of prime divisor $N_1, ..N_p$ such that the family $\{N_1, .., N_p\}$ is exceptional (in the sense of [bo]) and Z, the "positive part" is *nef* in codimension 1 (or modified *nef*). This implies in particular that the restriction of Z to any hypersurface is still a pseudoeffective class.

(For a precise definition and basic properties of Zariski decomposition useful for our purpose, see [to] and the references therein).

2.2 Intersection of $\mathcal{F}$-Invariant Positive Currents

Let X be a Kähler manifold equipped with a codimension 1 holomorphic foliation $\mathcal{F}$ which carries a $(1,1)$ closed positive T current invariant by holonomy.

Proposition 2.4. *Assume that near every point, T admits a local potential locally constant on the leaves. Let S be an other closed positive current invariant by $\mathcal{F}$, then* $\{S\}\{T\} = 0$

Proof. Let $(U_i)_{i\in I}$ be an open cover of X such that $T = \frac{i}{\pi}\partial\bar{\partial}\varphi_i$, φ_i *psh* locally constant on the leaves. On intersections, one has $\partial\varphi_i - \partial\varphi_j = \omega_{ij}$ where ω_{ij} is a holomorphic one form defined on $U_i \cap U_j$ vanishing on the leaves (i.e., a local section of the conormal sheaf $N^*_{\mathcal{F}}$ considered as an invertible subsheaf of Ω^1_X). Let η a $(1,1)$ be closed smooth form such that $\{\eta\} = \{T\}$. One can then write on each U_i (up refining the cover) $\eta = \frac{i}{\pi}\partial\bar{\partial}u_i$, $u_i \in \mathcal{C}^\infty(U_i)$, such that $\partial u_i - \partial u_j = \omega_{ij}$. As a consequence, the collection of $\frac{i}{\pi}\partial u_i \wedge S$, with S any closed invariant current, gives rise to a globally well-defined $(1,2)$ current whose differential is $\eta \wedge S$. This proves the result. □

Remark 2.5. As pointed out by the referee, this proposition still holds true without requiring positivity assumptions on S or T.

Remark 2.6. One can slightly improve this statement replacing X by $X \setminus S$ where S is an analytic subset of codimension ≥ 3 (it will be useful in the sequel).

To this goal, we use that $H^1(U_i \setminus S, N^*_{\mathcal{F}}) = 0$ (for a suitable cover). This means that one can find on each U_i an holomorphic one form Ω_i such that $\partial\varphi_i - \partial\varphi_j = \omega_{ij} + \Omega_i - \Omega_j$. Indeed, take an open cover (V_k) of $U_i \setminus S$ such that $\partial\varphi_i = \partial\varphi_k + \xi_k$ where φ_k is *psh* and ξ_k a holomorphic one form both defined on V_k. By vanishing of the cohomology, one has $\xi_k - \xi_l = \omega_k - \omega_l$ where ω_k is an holomorphic form vanishing on the leaves. Then the $\xi_k - \omega_k$'s glue together on $U_i \setminus S$ into a holomorphic one form whose extension through S is precisely Ω_i. Taking η and S as previously, one obtains that the current defined locally by $(\partial u_i - \Omega_i) \wedge S$ is globally defined. Then, the same conclusion follows taking the $\bar{\partial}$ differential instead of the differential and applying the $\partial\bar{\partial}$ lemma. □

Proposition 2.7. *Assume that the singular points of $\mathcal{F}$ are all of elementary type (as defined above). Let S, T be $\mathcal{F}$-invariant positive currents. Assume moreover that*

S is residual, then the local potentials of S can be chosen to be constant on the leaves and in particular

(1) $\{S\}\{T\} = 0$
(2) $\{S\}$ *and* $\{T\}$ *are collinear assuming in addition that* T *is also residual.*

Proof. By Proposition 2.4, it suffices to show that one can choose the local potentials of T to be locally constant on the leaves to prove the first item.

Let $x \in \operatorname{Sing} \mathcal{F}$ and F be an elementary first integral of X near x. One can assume that

$$F = \prod_{i=1}^{p} f_i^{\lambda_i}$$

with $\lambda_1 = 1$. Following the main result of [pa2], the (multivaluate) function F has connected fibers on $U \setminus Z(F)$ where U is a suitable arbitrarily small open neighborhood of x and $Z(F) = \{\prod f_i = 0\}$. Let $\mathcal{T} \subset U$ be a holomorphic curve transverse to $\mathcal{F}$ which cuts out the branch $f_1 = 0$ in x_1, a regular point of $\mathcal{F}$ and such that $f_{1|\mathcal{T}}$ sends biholomorphically $\mathcal{T}$ onto a disk $\mathbb{D}_r = \{|z| < r\}$. Let G be the closure in $S^1 = \{z = 1\}$ of the group generated by $\{e^{2i\pi\lambda_k}\},\ k = 1, \ldots, p$; thus, either G is the whole S^1 or is finite.

Up shrinking U, one can assume that, on U, $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ where φ is *psh*. Let $\varphi_1 = \varphi_{|\mathcal{T}}$ and $T_1 = \frac{i}{\pi}\partial\overline{\partial}\varphi_1$.

Set

$$H = \{h \in \operatorname{Aut}(\mathcal{T}) | h(z) = gz, g \in G\}.$$

(here, and in the sequel we identify $\mathcal{T}$ and $\mathbb{D}_r$). The fibers connectedness property remains clearly valid if one replaces U by the saturation of $\mathcal{T}$ by $\mathcal{F}$ in U (still called U).

Thus T_1 is H-invariant, that is $h^*T_1 = T_1$ for very $h \in H$. One can also obtain a subharmonic function ψ_1 H-invariant when averaging φ_1 with respect to the Haar measure dg on the compact group G [ra, th 2.4.8]:

$$\psi_1(z) = \int_G \varphi_1(gz)dg$$

Note that $\frac{i}{\pi}\partial\overline{\partial}\psi_1 = T_1$; then, by H invariance, ψ_1 uniquely extends on U as a *psh* function ψ locally constant on the leaves and such that

$$T = \frac{i}{\pi}\partial\overline{\partial}\psi.$$

Indeed, this equality obviously holds on $U \setminus Z(F)$ (where the foliation is regular) and then on U as currents on each side do not give mass to $Z(F)$. This proves the first part of the proposition.

The assertion (2) is a straightforward consequence of the Hodge's signature theorem if one keeps in mind that $\{S\}^2 = \{T\}^2 = \{S\}\{T\} = 0$. □

Remark 2.8. Actually, the same conclusion (and proof) holds under the weaker assumption that every singular point of $\mathcal{F}$ contained in Supp $S \cap$ Supp T is of elementary type, instead of requiring all the singular points to be elementary.

3 Foliations with Pseudoeffective Conormal Bundle

Throughout this section, X is a compact Kähler manifold carrying a codimension 1 (maybe singular) holomorphic foliation whose conormal bundle $N^*_{\mathcal{F}}$ is pseudoeffective (*psef* for short). The study of such objects has been undertaken in [to] and in this section, we would like to complete the results already obtained.

3.1 Some Former Results

Before restating the main theorem of [to] and other useful consequences, let us recall what are the basic objects involved (see [to], INTRODUCTION for the details).

Let $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ be a positive $(1,1)$ current representing $c_1(N^*_{\mathcal{F}})$. One inherits from T a canonically defined (up to a multiplicative constant) $(1,1)$ form η_T with L^∞_{loc} coefficients which can be locally written as

$$\eta_T = \frac{i}{\pi}e^{2\varphi}\omega \wedge \overline{\omega}$$

where ω is a holomorphic one form which defines $\mathcal{F}$ locally. It turns out that

$$d\eta_T = 0$$

in the sense of currents (lemma 1.6 in [to]). We can then deduce that T and η_T are both **positive $\mathcal{F}$-invariant currents**, in particular the hypersurface Supp N (or equivalently the integration current $[N]$) is invariant by $\mathcal{F}$. Moreover, using that η_T represents a *nef* class, one can show that the positive part Z is a non-negative multiple of $\{\eta_T\}$ (Proposition 2.14 of [to]). Then, in case that $Z \neq 0$, one can normalize η_T such that $\{\eta_T\} = Z$.

Here is the main result of [to] (Théorème 1 p.368):

Theorem 3.1. *Let $\{N\} + Z$ be the Zariski decomposition of the pseudoeffective class $c_1(N^*_{\mathcal{F}})$, then there exists a unique $\mathcal{F}$-invariant positive $(1,1)$ current T with minimal singularities (in the sense of [bo]) such that*

(1) $\{T\} = c_1(N_{\mathcal{F}}^*)$
(2) $T = [N]$ *if* $Z = 0$ ***(euclidean type)***.
(3) $T = [N] + \eta_T$ *if* $Z \neq 0$ ***(hyperbolic type)***

Actually, the existence of such a current provides a transverse invariant metric for $\mathcal{F}$ (with mild degeneracies on Supp $N \cup$ Sing $\mathcal{F}$), namely η_T, whose curvature current is $-T$. This justifies the words "euclidean" and "hyperbolic."

To each of these transversely invariant hermitian structure, one can associate the **sheaf of distinguished first integrals** $\mathcal{I}^\varepsilon$ $\varepsilon = 0, 1$ depending on whether this structure is euclidean or hyperbolic.

It will be also useful to consider the sheaf $\mathcal{I}^\varepsilon_{d\log}$ derived from $\mathcal{I}^\varepsilon$ by taking logarithmic differentials.

These two locally constant sheaves have been introduced in [to] (Définition 5.2) respectively, as "faisceau des intégrales premières admissibles" and "faisceau des dérivées logarithmiques admissibles." Then, we will not enter into further details. Strictly speaking, they are only defined on $X \setminus$ Supp N .

As a transversely homogeneous foliation, $\mathcal{F}$ admits a developing map ρ defined on any covering $\pi : X_0 \to X \setminus$ Supp N where $\pi^*\mathcal{I}^\varepsilon$ becomes a constant sheaf [to, section 6, and the references therein].

Recall that ρ is just a section of $\pi^*\mathcal{I}^\varepsilon$ (and in particular a holomorphic first integral of the pull-back foliation $\pi^*\mathcal{F}$). One also recovers a representation

$$r : \pi_1(X \setminus \text{Supp } N) \to \mathfrak{I}^\varepsilon$$

uniquely defined up to conjugation associated with the locally constant sheaf $\mathcal{I}^\varepsilon$ and taking values in the isometry group $\mathfrak{I}^\varepsilon$ of U^ε, $U^0 = \mathbb{C},\ U^1 = \mathbb{D}$.

One can now restate theorem 3 p.385 of [to].

Theorem 3.2. *The developing map is complete. That is,* ρ *is surjective onto* $\mathbb{C}$ *(euclidean case) or* $\mathbb{D}$ *(hyperbolic case).*

One can naturally ask if the fibers or ρ are connected, a question raised in [to]. A positive answer would give useful information on the dynamic of $\mathcal{F}$. This problem is settled in the next section.

3.2 Connectedness of the Fibers

Let $\pi : X_0 \to X \setminus$ Supp N be a covering such that the developing map ρ is defined on X_0.

Let K be an invariant hypersurface of $\mathcal{F}$: In a sufficiently small neighborhood of $K \cup$ Sing $\mathcal{F}$, we get a well-defined logarithmic 1 form η_K defining the foliation and which is a (semi) local section of $\mathcal{I}^\varepsilon_{d\log}$ and such that $K \cup$ Sing $\mathcal{F}$ is contained in the polar locus of η_K (Proposition 5.1 of [to]). The main properties of η_K are listed below and are directly borrowed from *loc.cit.* They will be used repeatedly and implicitly in the sequel.

- The residues of η_K are non-negative real numbers. To be more precise, the germ of η_K at $x \in K \cup \operatorname{Sing} F$ can be written as

$$\eta_K = \sum_{i=1}^{n_x} \lambda_i \frac{df_i}{f_i}.$$

where the $f_i \in \mathcal{O}_{X,x}$ are pairwise irreducible and the λ_i's are ≥ 1.
In particular, the multivaluate section $e^{\int \eta_K} = \prod f_i^{\lambda_i}$ of $\mathcal{I}^{\varepsilon}$ is an elementary first integral of $\mathcal{F}$.
- If $f_i = 0$ is a local equation of a local branch N_{f_i} of Supp N, then $\lambda_i = \lambda_{f_i}(N_{f_i}) + 1$ where $\lambda_{f_i}(N_{f_i})$ stands for the weight of N_{f_i} in N and $\lambda_i = 1$ otherwise. Thus, e^{η_K} is univaluate and reduced (as a germ of $\mathcal{O}_{X,x}$) if and only if $x \notin \operatorname{Supp} N$.
- The local separatrices at x (i.e., the local irreducible invariant hypersurfaces containing x) are exactly those given by $f_i = 0$.
- x is a regular point of $\mathcal{F}$ if and only if $n_x = 1$ and f_1 is submersive at x.

Our aim is to show that the fibers of ρ are connected if one allows to identify certain components contained in the same level of ρ. On the manifold X, this corresponds to identify some special leaves in a natural way. To clarify this, one may typically think of the fibration defined by the levels of $f = xy$ on $\mathbb{C}^2$. All the fibers are connected. However, if we look at the underlying foliation, the fiber $f^{-1}(0)$ consists in the union of two leaves that must be identified if one wants to make the space of leaves Hausdorff. Observe also that, when blowing up the origin, these two special leaves intersect the exceptional divisor E on the new ambient manifold $\tilde{\mathbb{C}^2}$. Coming back to our situation, one may think that leaves which "intersect" the same components of Supp N must be identified. This indeed occurs if we replace the general X_0 by the basic previous example $\tilde{\mathbb{C}^2} \setminus E$.

More precisely, we proceed as follows: For $p \in X_0$, denote by $\mathcal{C}_p$ the connected component of $\rho^{-1}(\rho(p))$ containing p. Let $p, q \in X_0$. Define the binary relation $\mathcal{R}$ by $p\mathcal{R}q$ whenever

(1) $\mathcal{C}_p = \mathcal{C}_q$
or
(2) $\rho(p) = \rho(q)$ and there exists a connected component H of Supp N such that on any sufficiently small neighborhood of H, $(\eta_H)_\infty \cap \pi(\mathcal{C}_p) \neq \emptyset$ and $(\eta_H)_\infty \cap \pi(\mathcal{C}_q) \neq \emptyset$.

We will denote by $\overline{\mathcal{R}}$ the equivalence relation generated by $\mathcal{R}$ and $A_{\overline{\mathcal{R}}}$ and the saturation of $A \subset X_0$ by $\overline{\mathcal{R}}$.

Remark 3.3. The saturation $(\mathcal{C}_p)_{\overline{\mathcal{R}}}$ of a connected component of a fiber of ρ is generically reduced to itself. In fact, as a consequence of the above description of η_K, this holds outside a countable set of components which projects by π onto a finite set of leaves of $\mathcal{F}$ whose cardinal is less or equal to the number of irreducible components of $(\eta_K)_\infty$ with $K = \operatorname{Supp} N$.

Consider now the quotient space $X_0/\overline{\mathcal{R}}$ endowed with the quotient topology; as an obvious consequence of our construction, ρ factorize through a surjective continuous map $\overline{\rho} : X_0/\overline{\mathcal{R}} \to U^\varepsilon$.

Using that the singularities of $\mathcal{F}$ are elementary and [to] (Lemme 6.1, corollaire 6.2), we get the

Lemma 3.4. *The saturation $V_{\overline{\mathcal{R}}}$ of an open set V is open in X_0.*

Remark 3.5. When X_0 is Galoisian, any deck transformation τ of X_0, r is compatible with the equivalence relation above in the following sense: there exists a unique set theoretically automorphism $\overline{\tau}$ of $X_0/\overline{\mathcal{R}}$ such that $\rho \circ \tau = \overline{\rho} \circ \overline{\tau} \circ \varphi_{\overline{\mathcal{R}}}$ where $\varphi_{\overline{\mathcal{R}}} : X_0 \to X_0/\overline{\mathcal{R}}$ denotes the canonical projection. Observe also that $\overline{\tau}$ is a homeomorphism, by definition of the quotient topology.

The following result somewhat teaches us that, modulo these identification, the fibers of ρ are connected.

Theorem 3.6. *The map $\overline{\rho} : X_0/\overline{\mathcal{R}} \to U^\varepsilon$ is one to one.*

As a straightforward consequence, we obtain the

Corollary 3.7. *Suppose that the negative part N is trivial, then the fibers of ρ are connected.*

The proof is in some sense close to that of the completeness of ρ which makes use of a metric argument (see [to]). In this respect, it will be useful to recall the following definition (see [mo]).

Definition 3.8. Let $\mathcal{F}$ be a regular transversely hermitian codimension 1 holomorphic foliation on a complex manifold X. Let h be the metric induced on $N_{\mathcal{F}}$ by the invariant transverse hermitian metric. Let g be an hermitian metric on X and g_N the metric induced on $N_{\mathcal{F}}$ by g via the canonical identification $T_{\mathcal{F}}{}^{\perp} \simeq N_{\mathcal{F}}$.

The metric g is said to be **bundle-like** ("quasi-fibrée" in French terminology) if $g_N = h$.

Remark 3.9. Because of the "codimension 1" assumption on $\mathcal{F}$, one can notice that, when $\mathcal{F}$ is transversely hermitian, any hermitian metric on X is conformally equivalent to a bundle-like one .

The following property is used in [to] (Lemme 6.3):

Proposition 3.10 (cf [mo], Proposition 3.5 p.86). *Let $\mathcal{F}$ be a transversely hermitian foliation equipped with a bundle like metric g. Let $\gamma : I \to X$ be a geodesic arc (with respect to g). Assume that γ is orthogonal to the leaf passing through $\gamma(t_0)$ for some $t_0 \in I$. Then, for every $t \in I$, γ remains orthogonal to the leaf passing through $\gamma(t)$.*

Remark 3.11. In our setting, it is also worth recalling (*loc.cit*) *that $\rho(\gamma)$ is a geodesic arc of U^ε endowed with its natural metric ds^ε depending on the metric type of $N_{\mathcal{F}}$ (euclidean or hyperbolic).*

Proof of Theorem 3.6. It is the consequence of the following sequence of observations. Consider firstly a bundle-like metric g defined on the complement of $E = \text{Supp } N \cup \text{Sing } \mathcal{F}$ in X. For $p \in X \setminus E$ and $\varepsilon > 0$, $B(p, \varepsilon)$ stands for the closed geodesic ball of radius ε centered at p. This latter is well defined if ε is small enough (depending on p).

According to [to], prop 5.1, there exists a logarithmic 1 form η with positive residues defining the foliation near $E = \text{Supp } N \cup \text{Sing } \mathcal{F}$ (keep in mind that η is just a semi-local section of the sheaf $\mathcal{I}^{\varepsilon}_{d\log}$ mentioned above). Let $C_1, \ldots, C_p$ be the connected component of E. Pick one of these, says C_1. Following [pa2], there exists an arbitrary small connected open neighborhood U_1 of C_1 such that the fibers of $f_1 = \exp \int \eta$ (defined as a multivaluate function) on $U_1 \setminus (\eta)_\infty$ are connected and one can also assume that the polar locus η_∞ is connected. This latter can be thought as the "singular fiber" of f_1. Actually, assumptions of theorem B of [pa2] are fulfilled: this is due to the fact that the residues of η are positive equal to 1 on the local branches of $(\eta)_\infty$ intersecting C_1 not contained in C_1, noticing moreover that this set S^1 of local branches is **non-empty** (see [to], Proposition 5.1 and 5.2), a fundamental fact for the following. Let $D_1 \subset C_1$ be the union of irreducible codimension 1 components; in other words, D_1 is the connected component of Supp N contained in C_1. Remark that $S_1 = f_1^{-1}(0)$ when extending f_1 through $S_1 \setminus D_1$. Observe also that S_1 is not necessarily connected, as the "negative" component D_1 has been removed. This justifies the definition of the previous equivalence relation $\overline{\mathcal{R}}$.

Let $W_1 \Subset V_1 \Subset U_1$ be some open neighborhood of C_1. One can perform the same on each component and we thus obtain open set $W_i \Subset V_i \Subset U_i$ $i = 1 \ldots p$ with f_i, S_i similarly defined.

Let $U = \bigcup_i U_i$, $V = \bigcup_i V_i$, $W = \bigcup_i W_i$. Up to shrinking U_i, V_i and W_i, one can assume that the following holds:

(1) $U_i \cap U_j = \emptyset$, $i \neq j$
(2) There exists $\alpha > 0$ such that for every $m \in X \setminus \overline{W}$, $B(m, \alpha)$ is well defined.
(3) Let $x \in V_i \setminus \text{Supp } N$, and $f_{i,m}$ a determination of f_i at m, then the image of $U_i \setminus \text{Supp } N$ by f_i contains the closed geodesic ball (euclidean or hyperbolic) of radius α centered at $f_{i,m}(m)$, with $f_{i,m}$ any germ of determination of f_i in m.

Indeed, one can exhibit such V_i (the only non-trivial point) in the following way. Pick $x \in S_i$ such that $\mathcal{F}$ is smooth on a neighborhood of x and take a small holomorphic curve

$$\gamma : \{|z| < \varepsilon\} \to U_i, \quad \gamma(0) = x$$

transverse to $\mathcal{F}$ such that property (3) holds for every $y \in \text{Im } \gamma$ (this means that α can be chosen independently of y). This is possible because f_i (more exactly any of its determinations) is submersive near x. By [to] (Corollaire 6.2), the saturation of $\mathcal{C} = \text{Im}(\gamma)$ by $\mathcal{F}_{|U_i}$ is also an open neighborhood of C_i and one can take V_i equal to it. Note also that the property (3.2) mentioned above does not depend on the choice of the determination, as others are obtained by left composition through a global isometry of U^{ε}.

Lemma 3.12. *The quotient space $X_0/\overline{\mathcal{R}}$ is Hausdorff (with respect to the quotient topology).*

Proof. Suppose that $X_0/\overline{\mathcal{R}}$ is not Hausdorff. Using Lemma 3.4 and Remark 3.3, one can then find two points p, q belonging to the same fiber of ρ such that $\{p\}_{\overline{\mathcal{R}}} \neq \{q\}_{\overline{\mathcal{R}}}$ and a sequence $(p_n),\ p_n \in X_0$ converging to p such that

(1) $\{p_n\}_{\overline{\mathcal{R}}}$ coincide with the connected component of $\rho^{-1}(\rho(p_n))$ containing p_n and such that $\{p_n\}_{\overline{\mathcal{R}}}$ does not intersect the singular locus of the pull-back foliation $\pi^*\mathcal{F}$ (in other words, ρ is submersive near every point of $\{p_n\}_{\overline{\mathcal{R}}}$).
(2) For every neighborhood $\mathcal{V}$ of q, there exists n such that $\{p_n\}_{\overline{\mathcal{R}}} \cap \mathcal{V} \neq \emptyset$. Equivalently, up passing to a subsequence, one can find $(q_n),\ q_n \in \{p_n\}_{\overline{\mathcal{R}}}$ converging to q. In particular, $\lim_{n\to+\infty} \rho(q_n) = \rho(q) = \rho(p)$.

Without any loss of generality, we may also assume that ρ is submersive near p and q.

Let $\gamma_n : [0,1] \to \{p_n\}_{\overline{\mathcal{R}}}$ be a continuous path joining p_n to q_n.

Let $t \in [0,1]$ and $p_n(t) = \rho(\gamma_n(t))$. In $[0,1]$, consider the open sets $O_n^1 = p_n^{-1}(U \setminus \overline{W})$ and $O_n^2 = p_n^{-1}(V)$.

One can choose n big enough such that the following constructions make sense.

If $p_n(t) \notin \overline{W}$, there exists a unique geodesic arc in $B(p_n(t), \alpha)$ orthogonal to the leaves which lifts in X_0 to a path joining $\gamma_n(t)$ to a point $r_n(t)$ such that $\rho(r_n(t) = \rho(p)$.

In that way, one can define a map

$$\alpha_n : O_n^1 \to X_0/\overline{\mathcal{R}}$$

setting $\alpha_n(t) = \{r_n(t)\}_{\overline{\mathcal{R}}}$. Note that α_n is *locally constant*.

Now suppose that $p_n(t) \in V_i$ for some i and denote by π^{-1} the local inverse of the covering map π sending $p_n(t)$ to $\gamma_n(t)$. Keeping track of the conditions imposed on $V_i \Subset U_i$, there exists a path in $U_i \setminus \operatorname{Supp} N$ starting from $p_n(t)$ such that the analytic continuation of $\rho \circ \pi^{-1}$ has $\rho(p)$ as ending value. This path lifts in X_0 to a path joining $\gamma_n(t)$ to a point $s_n(t)$.

Remark 3.13. Note that the point $s_n(t)$ depends on the choice of this path. However, as a consequence of the connectedness of the fibers of f_i, the equivalent class $\{s_n(t)\}_{\overline{\mathcal{R}}}$ *is uniquely defined.*

This allows us to define a map

$$\beta_n : O_2^n \to X_0/\overline{\mathcal{R}}$$

setting $\beta_n(t) = \{s_n(t)\}_{\overline{\mathcal{R}}}$ which is also obviously locally constant.

Moreover, by Remark 3.13, α_n and β_n coincide on $O_1^n \cap O_2^n$.

We have thus constructed a *locally constant* map from $[0,1]$ to $X_0/\overline{\mathcal{R}}$ sending 0 to $\{p\}_{\overline{\mathcal{R}}}$ and 1 to $\{q\}_{\overline{\mathcal{R}}}$. This provides the sought contradiction. ◻

End of the Proof of the Theorem 3.6. By Lemma 3.4, the map $\overline{\rho}$ is a local homeomorphism. Moreover, $X_0/\overline{\mathcal{R}}$ is Hausdorff. Hence, $X_0/\overline{\mathcal{R}}$ carries, via $\overline{\rho}$, a uniquely defined connected Riemann surface structure which makes $\overline{\rho}$ analytic.

Moreover the pull-back by $\overline{\rho}$ of the metric ds^ε is complete (this is a straightforward consequence of the proof of Theorem 3.2). Hence, it turns out that $\overline{\rho}$ is indeed an analytic covering map. The space U^ε being simply connected, one can conclude that $\overline{\rho}$ is actually a biholomorphism. □

This allows us to recover (modulo further information postponed in the sequel (see Sect. 5) a result due to Carlos Simpson who has proved in [sim1] (among other things).

Theorem 3.14. *Let ω be a holomorphic one form on X Kähler compact and let $\tau : \tilde{X} \to X$ be a covering map such that*

$$g : \tilde{X} \to \mathbb{C}$$

obtained by integration of ω is well defined. Then we get the following alternative:

(1) ω factors through a surjective morphism $X \to B$ on a compact Riemann surface B,
(2) the fibers of g are connected.

Proof. Let $\kappa(N^*_{\mathcal{F}})$ be the Kodaira dimension of the conormal bundle of the foliation $\mathcal{F}$ defined by ω.

If $\kappa((N^*_{\mathcal{F}})) = 1$, one can apply the Castelnuovo–De Franchis theorem on a suitable ramified cyclic covering and show that $\mathcal{F}$ define a fibration over an algebraic curve B (see [Reid]); moreover (*loc.cit*, see also [brme]), this dimension can't exceed one. Then one can assume that $\kappa((N^*_{\mathcal{F}})) = 0$; this is equivalent to say that $c_1(N^*_{\mathcal{F}})$ has zero numerical dimension (see Theorem 4, Sect. 5) and this implies that $U^\varepsilon = U^0 = \mathbb{C}$. Let $\mathcal{I}$ be the sheaf on X whose local sections are primitives of ω. In restriction to the complement of Supp N we thus have $\mathcal{I} = \mathcal{I}^0$.

Keeping the previous notations, one can take X^0 equal to $\tilde{X} \setminus g^{-1}(H)$, setting $H =$ Supp N. In particular, one can conclude when N is trivial (Corollary 3.7).

We will denote by g^0 the restriction of g to X^0. This is nothing but the developing map considered previously, in particular, the image of g^0 is $\mathbb{C}$. Let $A = \{a \in \mathbb{C}$ such that $g^{-1}(a) \cap \tau^{-1}(H) \neq \emptyset\}$. Following Theorem 3.6, the fiber $g^{-1}(x)$ is connected whenever $x \notin A$. We want to show that the same holds true if $x \in A$. Actually this easily results from the definition of $\overline{\mathcal{R}}$, Theorem 3.6 and the following fact: for every point $y \in H$, there exists a continuous path $\gamma : [0, 1] \to H$ such that $\gamma(0) = y$ and $\gamma(1)$ belongs to a local separatrix which is not a local branch of H [to, Proposition 5.1 and 5.2].

4 Uniformization and Dynamics

We keep the same assumptions and notations of the previous section, in particular, unless otherwise stated, X still denotes compact Kähler manifold carrying a codimension 1 foliation such that $N^*_{\mathcal{F}}$ is pseudoeffective.

The Hermitian transverse structure of the foliation $\mathcal{F}$, as explicited in [to] and recalled in Sect. 3, may degenerate on the negative part with very "mild" singularities, as described in *loc.cit.*

Hence, one can guess that this kind of degeneracy will not deeply affect the dynamics of leaves in comparison with the usual case: transversely Riemannian foliations on (real) compact manifolds. Let us recall some properties in this setting (see [mo]).

- The manifold is a disjoint union of minimals (recall that a minimal is a closed subset of X saturated by the foliation and minimal with respect to these properties).
- When the real codimension $\mathcal{F}$ is equal to 2, a minimal $\mathcal{M}$ falls into one of the following three types:

(1) $\mathcal{M}$ is a compact leaf.
(2) $\mathcal{M}$ coincides with the whole manifold.
(3) $\mathcal{M}$ is a real hypersurface.

Now, let's come back to our situation.

Recall that $\mathfrak{I}^\varepsilon$ denotes the isometry group of U^ε, $\varepsilon \in \{0, 1\}$ (keep in mind that $U^0 = \mathbb{C}$ and $U^1 = \mathbb{D}$). The locally constant sheaf $\mathcal{I}^\varepsilon$ induces a representation

$$r : \pi_1(X \setminus \operatorname{Supp} N) \to \mathfrak{I}^\varepsilon.$$

Let G be the image of the representation r and $\overline{G}$ its closure (with respect to the usual topology) in $\mathfrak{I}^\varepsilon$.

Let $\rho : X_0 \to U^\varepsilon$ be the associated developing map. For the sake of convenience, we will assume that the covering $\pi : X_0 \to X \setminus \operatorname{Supp} N$ is Galoisian and as such satisfies the following equivariance property

$$\rho \circ \gamma : r(\gamma) \circ \rho$$

where γ in the right side is an element of $\pi_1(X \setminus \operatorname{Supp} N)$, in the left one is the corresponding deck transformation.

Definition 4.1. The group G is called the monodromy group of the foliation $\mathcal{F}$.

We would like to recover the aforementioned dynamical properties for regular foliations. As we deal with singular foliations, we will have to modify slightly the usual definition of a leaf, taking into account the identifications made in a previous section.

For this purpose let $c \in U^{\varepsilon}$ and recall that $\rho^{-1}(c) = (\rho^{-1}(c))_{\overline{\mathcal{R}}}$, that is coincide with its saturation by the equivalence relation $\overline{\mathcal{R}}$ (Theorem 3.6). Let $F_c = \pi(\rho^{-1}(c))$; as a consequence of the previous construction, we have $F_c = F_{c'}$ if and only if there exists $g \in G$ such that $c' = g(c)$ (see Remark 3.5). Let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ be the connected components of Supp N such that $(\eta_{\mathcal{C}_i})_\infty \cap F_c \neq \emptyset$ and let $\mathcal{L}_c = F_c \cup_i \mathcal{C}_i$. Of course, this family of components depends on the choice of c.

Definition 4.2. A subset of X of the form $\mathcal{L}_c$ is called **modified leaf** of the foliation $\mathcal{F}$.

Definition 4.3. A subset $\mathcal{M}$ is called **modified minimal** of the foliation whenever $\mathcal{M}$ is the closure of a modified leaf and is minimal for the inclusion.

By Remark 3.3, all modified leaves, except a finite number, are indeed ordinary leaves.

The family $\{\mathcal{L}_c\}$ defines a partition of X. Let $X/\mathcal{F}$ be the associated quotient space which, in some sense, represents the space of leaves. One obtains a canonical bijection

$$\Psi : X/\mathcal{F} \to U^{\varepsilon}/G$$

induced by $\mathcal{L}_c \to c$.

One can translate the previous observations into the following statement.

Theorem 4.4. *The map Ψ is a homeomorphism w.r.t. the quotient topology on each space.*

By compactness of X, we obtain the

Corollary 4.5. *The group G is cocompact, i.e: U^{ε}/G is a (non-necessarily Hausdorff) compact space for the quotient topology.*

With this suitable definition of the leaves space $X/\mathcal{F}$, one can observe that the dynamic behavior of $\mathcal{F}$ is faithfully reflected in those of the group G. The next step is then to describe its topological closure $\overline{G}$.

Assume firstly that $U^{\varepsilon} = \mathbb{D}$ and G is affine (i.e: G fix a point in $\partial\mathbb{D}$). We want to show that this situation can't happen.

In this case, G does not contain any non-trivial elliptic element. In particular, this means that $F = e^{\int \eta_H}$ is a well-defined (univaluated) *holomorphic* first integral of the foliation on a neighborhood V of $H = \operatorname{supp} N$. Furthermore, one can notice that the zero divisor of the differential dF coincides with the negative part N. Using again that G is affine, one eventually obtains that $\mathcal{F}$ is defined by a section of $N^*_{\mathcal{F}} \otimes \mathcal{O}(-N) \otimes E$ without zeroes in codimension 1 where E is a flat line bundle (take the half plane model instead of $\mathbb{D}$). This contradicts the fact that the positive part Z is non-trivial.

Once this case has been eliminated, the remaining ones can be easily described as follows (taking into account that G is cocompact):

Proposition 4.6. *Up to conjugation in $\mathfrak{I}^{\varepsilon}$, the topological closure $\overline{G}$ has one of the following forms:*

(1) $G = \overline{G}$ and then is a cocompact lattice.
(2) The action of G is minimal and

(a) either $\varepsilon = 0$ and $\overline{G}$ contains the translation subgroup of $\mathfrak{I}^0$,
(b) either $\varepsilon = 1$ and $\overline{G} = \mathfrak{I}^1$.

(3) $\varepsilon = 0$ and $\overline{G}$ contains the subgroup of real translation, $T = \{t_\alpha, \alpha \in \mathbb{R}\}(t_\alpha(z) = z + \alpha)$ and more precisely

(a) either $\overline{G} = \langle T, t\rangle$,
(b) either $\overline{G} = \langle T, t, s\rangle$

with $t(z) = z + ai$ for some real $a \neq 0$ and $s(z) = -z$.

When considering the foliation viewpoint, this latter description leads to the

Theorem 3. *The topological closure of every modified leaf is a modified minimal, the collection of those modified minimals forms a partition of X, moreover*

(1) in case (1) of Proposition 4.6, modified leaves coincide with their closure and $\mathcal{F}$ defines, via Ψ, a holomorphic fibration over the curve U^{ε}/G;
(2) in case (2) of Proposition 4.6, X is the unique modified minimal;
(3) in case (3) of Proposition 4.6, every modified minimal is a real analytic hypersurface.

Remark 4.7. It is worth noticing that the existence of a closed modified leaf implies that the other ones are automatically closed (case 1 of the previous theorem).

Corollary 4.8. *Assume that $\mathcal{F}$ is not a fibration, then the family of irreducible hypersurfaces invariant by $\mathcal{F}$ is exceptional and thus has cardinal bounded by the picard number $\rho(X)$.*

Proof. Assume that there exists a non-exceptional family of invariant hypersurfaces. One can then find a connected hypersurface $K = K_1 \cup K_2 \cup \ldots \cup K_r$ invariant by $\mathcal{F}$ such that the family of irreducible components $\{K_1, \ldots, K_r\}$ is not an exceptional one. By virtue of Proposition 5.2 of [to], K is necessarily a modified leaf and $\mathcal{F}$ is tangent to a fibration by Remark 4.7. □

5 Kodaira Dimension of the Conormal Bundle

As previously, $\mathcal{F}$ is a codimension 1 holomorphic foliation on a compact Kähler manifold X such that $N^*_{\mathcal{F}}$ is pseudoeffective. We adopt the same notations as before.

In [to] (see Remarque 2.16 in *loc.cit*), it has been proved that for $\alpha = c_1(N^*_{\mathcal{F}})$, the numerical dimension $\nu(\alpha)$ takes value in $\{0, 1\}$ depending on the metric type of $N_{\mathcal{F}}$; 0 corresponds to the Euclidean one and 1 to the hyperbolic one (see Theorem 3.1). Moreover, we always have

$$\kappa(N^*_{\mathcal{F}}) \leq \nu(\alpha)$$

where the left member represents the Kodaira dimension of $N^*_{\mathcal{F}}$. This is a common feature for line bundles (see [na]) that can be directly verified in our case. Indeed, we always have $\kappa(N^*_{\mathcal{F}}) \leq 1$ (cf. [Reid]) and when $\alpha = \{N\}$ (in particular, $N = \sum_i \lambda_i D_i$ is a $\mathbb{Q}$ effective divisor), $[N]$ is the only positive current which represents $\{N\}$. This latter property implies that $\kappa(N^*_{\mathcal{F}}) \leq \kappa(N) = 0$.

Among the different cases described in Theorem 3, we would like to characterize those for which abundance holds, that is $\kappa(N^*_{\mathcal{F}}) = \nu(\alpha)$.

Theorem 4. *Let* $\alpha = c_1(N^*_{\mathcal{F}})$. *Then,* $\nu(\alpha) = \kappa(N^*_{\mathcal{F}})$ *in both cases:*

(1) $\varepsilon = 0$ *(Euclidean type).*
(2) $\varepsilon = 1$ *(Hyperbolic type) and G is a lattice.*

Moreover, in the remaining case $\varepsilon = 1$ *and G dense in* $\Im^1$*, we have* $\kappa(N^*_{\mathcal{F}}) = -\infty$ *and* $\nu(\alpha) = 1$.

Proof. Let us firstly settle the case $\varepsilon = 1$ and $\overline{G} = \Im^1$.

Assume that $\kappa(N^*_{\mathcal{F}}) \geq 0$. Consider in first place the simplest situation $h^0(N^*_{\mathcal{F}}) \neq 0$; this means that the foliation is defined by a holomorphic 1 form ω. Taking local primitives of ω, we can equip the foliation with a transverse projective structure different from the hyperbolic one (attached to $\varepsilon = 1$). By means of the Schwarzian derivative, one can thus construct, as usual, a quadratic differential of the form

$$F\omega \otimes \omega$$

where F is a meromorphic first integral of $\mathcal{F}$. For obvious dynamical reasons, F is automatically constant and this forces local first integrals $g \in \mathcal{I}^1$ to be expressed in the form $g = e^{\lambda f}$ (modulo the left action of $\Im^1$). In particular the monodromy group G must be Abelian and this leads to a contradiction.

In the more general setting, the same conclusion holds with the same arguments, considering a suitable ramified covering $\pi : X_1 \to X$ such that the pull-back foliation is defined by a holomorphic form.

Concerning the case 2) of Theorem 4, we know that $\mathcal{F}$ is a fibration over the compact Riemann surface $S = \mathbb{D}/G$. For $n \gg 0$ one can then find two independent holomorphic sections s_1, s_2 of ${K_S}^{\otimes n}$ (with respect to the hyperbolic orbifold structure). One must take care that the map Ψ is not necessarily a morphism in the orbifold sense. Indeed, the local expression of Ψ along Supp N is given, up to left composition by an element of $\Im^\varepsilon$, as a multivaluate section of $\mathcal{I}^\varepsilon$ of the form $f_1^{\lambda_1} \dots f_r^{\lambda_r}$ (see Remark 3.3). Hence, it fails to be an orbifold map through Supp N unless the λ_i are integers. However, the fact that they are > 1 (as pointed out in *loc.cit*) guarantees that s_1 and s_2 lift (via Ψ) to 2 independent *holomorphic* sections of ${N^*_{\mathcal{F}}}^{\otimes n}$, whence $\kappa(N^*_{\mathcal{F}}) = 1$.

The remaining case (1) is a bit more delicate to handle. Here, the Kodaira dimension $\kappa(N^*_{\mathcal{F}})$ is directly related to the linear part G_L of G. Namely, $\kappa(N^*_{\mathcal{F}}) = 0$ if and only if G_L is finite. This occurs in particular when $\mathcal{F}$ is a fibration.

One can then eliminate this case and suppose from now on that the generic leaf of $\mathcal{F}$ is not compact. Assume firstly that N is an effective integral divisor (a priori, it is only $\mathbb{Q}$ effective). This means that the foliation is defined by a twisted one form $\omega \in H^0(X, \Omega^1 \otimes E)$ with value in a numerically trivial line bundle E, that is $E \in \mathrm{Pic}^\tau(M) := \{L \in \mathrm{Pic}\ (M) | c_1(L) = 0 \in H^2(M, \mathbb{R})\}$. Note that the zeroes divisor of ω is precisely N and the theorem will be established once we will prove that E is actually a torsion line bundle.

To this aim, remark that the dual E^* is an element of the so-called Green–Lazarsfeld subset

$$S^1(X) = \{L \in \mathrm{Pic}^\tau \text{ such that } H^1(X, L) \neq 0\}.$$

In the sequel, it will be useful to consider $H^1(X, \mathbb{C}^*)$ as the parameterizing space of rank one local systems and to introduce the set

$$\Sigma^1(X) = \{\mathcal{L} \in H^1(X, \mathbb{C}^*) \text{ such that } H^1(X, \mathcal{L}) \neq 0\}.$$

When $\mathcal{L}$ is unitary, $S^1(X)$ and $\Sigma^1(X)$ are related as follows:

$$\mathcal{L} \in \Sigma^1(X) \text{ if and only if } L \in S^1(X) \cup -S^1(X)$$

where $L = \mathcal{L} \otimes \mathcal{O}_X$ is the numerically trivial line bundle defined by $\mathcal{L}$ (see [beau], a) Proposition 3.5).

When X is projective, Simpson has shown that the isolated points of $\Sigma^1(X)$ are torsion characters [sim]. This has been further generalized by Campana in the Kähler setting [camp].

We proceed by contradiction. Assume that $\kappa(N^*_{\mathcal{F}}) \neq 0$, in other words, E is not torsion.

This can be translated into the existence of a flat unitary connection

$$\nabla_u : E \to \Omega^1(E)$$

whose monodromy representation has infinite image H in the unitary group $U(1)$. As mentioned before, H is nothing but the linear part G_L of the monodromy group G attached to the foliation.

Keeping track of Simpson's result, $\mathcal{E} = \ker \nabla_u$ is not isolated in $\Sigma^1(X)$, hence, following [beau], one can conclude that the existence of a holomorphic fibration with connected fibers over a compact Riemann surface of genus $g \leq 1$ such that $\mathcal{E} \in p^*(H^1(B, \mathbb{C}^*)$, up replacing X by a finite étale covering.

From $\nabla_u \omega = 0$ and the fact that E comes from a line bundle on B, one can conclude that ω restricts to a holomorphic closed one form on open sets $p^{-1}(V)$, V simply connected open set in B.

Let F be a smooth fiber of p and $\Gamma < \mathrm{Aut}(H^1(F, \mathbb{C}))$ the monodromy group associated with the Gauss–Manin connection. Let D be the line spanned by $\omega_{|F}$ in $H^1(F, \mathbb{C})$ (recall that the foliation is not tangent to the fibration).

One can observe that D is globally Γ invariant (actually, H is just the induced action of Γ on D). By [del], Corollaire 4.2.8, or the fact that the connected component of $\overline{\Gamma}^{Zar}$ is semi-simple (see *loc.cit*), one can then infer that H is finite.

The fact that N is in general only effective over $\mathbb{Q}$ does not really give troubles. One can easily reduce this case to the previous one, working on a suitable ramified covering. □

Remark 5.1. Deligne's results quoted above are stated in an algebraic setting but the semi-simplicity of $\overline{\Gamma}^{Zar}$ remains true in the Kähler realm, as pointed out in [camp].

6 The Non-abundant Case

We are dealing with the non-abundant case: $\kappa(N^*_{\mathcal{F}}) = -\infty < \nu(N^*_{\mathcal{F}}) = 1$. From now on, we will also assume that the ambient manifold X is **projective**.

As a consequence of Theorem 4, we know that $\mathcal{F}$ is a **quasi-minimal** foliation (i.e: all but finitely many leaves are dense).

The foliation $\mathcal{F}$ falls into the general setting of transversely projective foliations. For the reader convenience, we now recall some definitions/properties concerning these latters.

6.1 Transversely Projective Foliations, Projective Triples

We follows closely the presentation of [lope] (see also [lptbis]) and we refer to *loc.cit* for precise definitions. A transversely projective foliation $\mathcal{F}$ on a complex manifold X is the data of (E, ∇, σ) where

- $E \to X$ is a rank 2 vector bundle,
- ∇ a flat meromorphic connection on E and
- $\sigma : X \to P = \mathbb{P}(E)$ a meromorphic section generically transverse to the codimension one Riccati foliation $\mathcal{R} = \mathbb{P}(\nabla)$ and such that $\mathcal{F} = \sigma^*\mathcal{R}$.

Let us assume (up to birational equivalence of bundles) that $E = X \times \mathbb{P}^1$ is the trivial bundle (this automatically holds when X is projective by GAGA principle), σ is the section $\{z = 0\}$ so that the Riccati equation writes

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

with ω_0 defining $\mathcal{F}$. Setting $z = \frac{z_2}{z_1}$ we get the $\mathfrak{sl}_2$-connection (i.e., trace free)

$$\nabla : Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto dZ + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} Z$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} := \begin{pmatrix} -\frac{1}{2}\omega_1 & -\omega_2 \\ \omega_0 & \frac{1}{2}\omega_1 \end{pmatrix}.$$

Note that

$$\nabla \cdot \nabla = 0 \Leftrightarrow \Omega \wedge d\Omega = 0 \Leftrightarrow \begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \omega_1 \wedge \omega_2 \end{cases}$$

Change of triples arise from birational gauge transformations of the bundle fixing the zero section

(3) $$\frac{1}{z} = a\frac{1}{z'} + b$$

where a, b are rational functions on X, $a \not\equiv 0$:

$$\begin{cases} \omega_0' = a\omega_0 \\ \omega_1' = \omega_1 - \frac{da}{a} + 2b\omega_0 \\ \omega_2' = \frac{1}{a}\left(\omega_2 + b\omega_1 + b^2\omega_0 - db\right) \end{cases}$$

A projective structure is thus the data of a triple $(\omega_0, \omega_1, \omega_2)$ up to above equivalence.

Remark 6.1. Once ω_0 and ω_1 are fixed, ω_2 is completely determined.

Definition 6.2. The transversely projective foliation $\mathcal{F}$ has regular singularities if the corresponding connection has at worst regular singularities in the sense of [Deligne].

Let us make this more explicit in the case we are concerned with.

Pick a point $x \in X$ and assume first that $x \notin \operatorname{Supp} N$. Over a small neighborhood of x, consider the Ricatti equation $dz + df$ where f is a distinguished first integral of $\mathcal{F}$.

Consider now the hypersurface $H = \operatorname{Supp} N$ along which $\mathcal{F}$ is defined by the logarithmic closed form η_H. Let us define the Ricatti equation over a neighborhood of $\mathcal{H}$ as

(4) $$\frac{dz}{z+1} + \eta_C.$$

Note that one recovers the original foliation on the section $z = 0$.

Let $\{F = 0\}$ be the local equation at p of the polar locus of η_C corresponding to residues equal to one. Performing the birational transformation of $\mathbb{P}^1$ bundle $z \to Fz$, one obtains the new Ricatti equation $dz + F\eta_C + z(\eta_C - \frac{dF}{F})$ without pole on $\{F = 0\}$.

This latter property enables us to glue together these local models with gluing local bundle automorphisms of the previous form (3). Note that these transformations preserve the section $\{z = 0\}$. In this way, we have obtained a $\mathbb{P}^1$ bundle equipped with a Riccati foliation (in this setting, this means a foliation transverse to the general fiber) which induced the sought foliation $\mathcal{F}$ on a meromorphic (actually holomorphic) section. By [gro], this $\mathbb{P}^1$ is actually the projectivization of a rank 2 vector bundle.

Without additional assumptions on the complex manifold X, nothing guarantees that this Riccati foliation $\mathbb{P}^1$ bundle is the projectivization of a rank 2 meromorphic flat connection.

However, when X is projective, one can assume that (birationally speaking) this $\mathbb{P}^1$ bundle is trivial and that the resulting Riccati foliation is then defined by a global equation

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2 \tag{5}$$

hence defined as the projectivization of the rank two meromorphic flat connection.

$$\nabla \ : \ Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto dZ + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} Z.$$

It is worth noticing that Δ has regular singular poles, thanks to the local expression given by (4).

Let $(\nabla)_\infty$ be this polar locus. From flatness, we inherit a representation of monodromy

$$r_\nabla : \pi_1(X \setminus (\nabla)_\infty) \to SL(2, \mathbb{C})$$

whose projectivization gives the original representation

$$r : \pi_1(X \setminus \operatorname{Supp} N) \to PSL(2, \mathbb{C})$$

attached to the transversely hyperbolic foliation $\mathcal{F}$. In particular, one can assume that the image of r_∇ **lies in** $SL(2, \mathbb{R})$ (if one takes the Poincaré upper half plane model $\mathbb{H}$ instead of the disk).

As we are dealing with regular singularities, note also that one can recover ∇ from r_∇ (Riemann–Hilbert correspondence).

Remark 6.3. As we work up to birational equivalence of connection, $(\nabla)_\infty$ does not necessarily coincide with Supp N. We may have added some "fakes" poles around which the local monodromy (associated with ∇) is $\pm Id$.

Let U be a dense Zariski open subset of X which does not intersect $(\nabla)_\infty$ and denote by $\rho_U \ : \ \pi_1(U) \ \to \ SL(2, \mathbb{R})$ the monodromy representation of the flat connection ∇ restricted to U. We want to show that the representation ρ_∇ does not come from a curve in the following sense:

Theorem 6.4. *Let $\mathcal{C}$ be a quasiprojective curve and let*

$$\rho_{\mathcal{C}} : \pi_1(\mathcal{C}) \to PSL(2, \mathbb{C})$$

a representation. Then there is no morphism

$$\varphi : U \to \mathcal{C}$$

such that ρ_U factors projectively to $\rho_{\mathcal{C}}$ through φ.

Proof. Up removing some additional points in $\mathcal{C}$ one can realize $\rho_{\mathcal{C}}$ as the monodromy projectivization of a regular singular rank 2 connection $\nabla_{\overline{\mathcal{C}}}$ defined on the projective closure $\overline{\mathcal{C}}$. Let $\mathcal{R}_X$ be the Riccati foliation defined by (5) and $\mathcal{R}_{\overline{\mathcal{C}}}$ the one defined by $\mathbb{P}(\nabla_{\overline{\mathcal{C}}})$. They respectively lie on $\mathbb{P}^1$ bundles E_X, $E_{\overline{\mathcal{C}}}$ over X and $\overline{\mathcal{C}}$.

We proceed by contradiction. The existence of the morphism ϕ can be translated into the existence of a rational map of $\mathbb{P}^1$ bundles $\Psi : E_X \dashrightarrow E_{\overline{\mathcal{C}}}$ such that

$$\mathcal{R}_X = \Psi^* \mathcal{R}_{\overline{\mathcal{C}}}.$$

In particular, we obtain that ${\Psi_X}^* \mathcal{R}_{\overline{\mathcal{C}}} = \mathcal{F}$, where Ψ_X denotes the restriction of Ψ to X (identified with the section $\{z = 0\}$). The foliation $\mathcal{F}$ being a quasi-minimal foliation, this implies that Ψ_X is dominant (otherwise, $\mathcal{F}$ would admit a rational first integral). Moreover, as the image of r lies in $PSL(2, \mathbb{R})$, one can assume that the same holds for $\rho_{\overline{\mathcal{C}}}$ (this is true up to finite index). In particular none of the leaves $\mathcal{R}_{\overline{\mathcal{C}}}$ is dense (the action of $PSL(2, \mathbb{R})$ on $\mathbb{P}^1$ fix a disk). This contradicts the quasi-minimality of $\mathcal{F}$. □

We are now ready to give the proof of the main Theorem 1 of the introduction.

Proof of Theorem 1. By [to], iv) of Proposition 2.14, the defining form η_H has *rational* residues. As a by-product, the local monodromy of the rank two local system defined by $\nabla = 0$ around $(\nabla)_\infty$ is finite (and more precisely takes values in a group of roots of unity). Keeping in mind Theorem 6.4, one can now apply the main theorem of [corsim] (rigid case in the alternative) which asserts that there exists an algebraic morphism Φ between $X \setminus (\nabla)\infty$ and the orbifold space $\mathfrak{H}$ such that $r = \Phi^* \rho_i$, $1 \leq i \leq p = \mathrm{Dim}\ \mathfrak{h}$ where ρ_i is the ith tautological representation $\Gamma = \pi_1^{orb}(X) \to \Gamma_i$ defined by the projection onto the ith factor. Actually these representations are related to one another by field automorphisms of $\mathbb{C}$.

Note that there may exist other "tautological" representations of Γ of unitary type (hidden behind the arithmetic nature of Γ) but they are not relevant in our case, as r_∇ takes values in $SL(2, \mathbb{R})$ (see [corsim] for a thorough discussion).

Actually, the theorem of Corlette and Simpson only claims a priori the existence of a morphism which factorizes representations and where foliations are not involved, hence we have to provide an additional argument which allows passing from representation to foliations, as stated by Theorem 1.

To achieve this purpose, consider the n codimension 1 foliations $\mathcal{F}_j, j = 1, \ldots, n$ on $\mathfrak{H}$. These are obtained from the codimension 1 foliations $dz_j = 0$ on the polydisk $\mathbb{D}^p$ after passing to the quotient (this makes sense as Γ acts diagonally on $\mathbb{D}^p$). These p foliations give rise by pull-back p transversely hyperbolic foliations $\Phi^*\mathcal{F}_j$ on $X \setminus (\nabla)_\infty$. Note that $\Phi^*\mathcal{F}_i$ has the same monodromy representation than $\mathcal{F}$ but might a priori differ from this latter.

Remark that one can recover Φ from the datum of $\mathcal{F}_j$ by choosing in a point $x \in X$ n germs of distinguished first integrals $Z(\mathcal{F}_j)$ $, j = 1, \ldots p$ attached to $\mathcal{F}_j$ and taking their analytic continuation along paths in $X \setminus (\nabla)_\infty$ starting at x. More precisely, the analytic continuation of the $p - uple$ $(Z(\mathcal{F}_1), \ldots, Z(\mathcal{F}_p))$ defines a map $X \setminus (\nabla)_\infty$ to $\mathfrak{H}$ which is nothing but Φ.

Actually, Φ extends across $(\nabla_\infty) \setminus$ Supp N (as an orbifold morphism). This is just a consequence of the Riemann extension theorem, keeping in mind that distinguished first integrals take values in $\mathbb{D}$ and that the local monodromy around extra poles of $(\nabla)_\infty$ is projectively trivial. By the same kind of argument, one can extend Φ through Supp N using that the finiteness of the local monodromy around this latter. The only difference with the previous case lies in the fact that Φ extends as an analytic map between X and the underlying analytic space of the orbifold $\mathfrak{H}$, but not as a map in the orbifold setting (this phenomenon occurs when this local monodromy is not projectively trivial).

To get the conclusion, we are going to modify Φ into an algebraic map

$$\Psi : X \to \mathfrak{H}$$

in order to have $\mathcal{F} = \Psi^*\mathcal{F}_i$.

For convenience, $\mathfrak{H}$ will be regarded as a quotient of the product $\mathbb{H}^p$ of p copies of the upper Poincaré's upper half plane.

Let $r_i : \pi_1(X \setminus \text{supp } N) \to PSL(2, \mathbb{R})$ the monodromy representation associated with $\Phi^*\mathcal{F}_i$. One knows that r and r_i coincide up to conjugation in $PSL(2, \mathbb{C})$. In other words, there exists $\alpha \in PSL(2, \mathbb{C})$ such that for every loop γ of $\pi_1(X \setminus \text{supp } N)$, $r_i(\gamma) = \alpha r(\gamma)\alpha^{-1}$.

As the image of r is dense in $PSL(2, \mathbb{R})$, the conjugating transformation α lies in the normalizer of $PSL(2, \mathbb{R})$ in $PSL(2, \mathbb{C})$ which is the group generated by $PSL(2, \mathbb{R})$ and the inversion $\tau(z) = \frac{1}{z}$. We then have two cases to examine, depending on whether one can choose $\alpha \in PSL(2, \mathbb{R})$ or not.

- If $\alpha \in PSL(2, \mathbb{R})$, take p germs of distinguished first integrals associated to $\Phi^*\mathcal{F}_j j \neq i$ and $\mathcal{F}$ in $x \in X \setminus (\nabla)_\infty$ and consider their analytic continuation along paths. This induces an analytic orbifold map $X \setminus (\nabla)_\infty$ to $\mathfrak{H}$ which analytically extends to the whole manifold X using the same lines of argumentation as previously. This defines the sought morphism Ψ.
- Otherwise, one can assume that $\alpha = \tau$. We do the same work as before except that we replace the germ of distinguished first integral f_x of $\mathcal{F}$ by $\overline{\tau \circ f_x}$. We thus obtain a well-defined map

$$X \setminus (\nabla)_\infty \to \mathfrak{H}$$

which is no longer holomorphic (it's antiholomorphic on the "i^{th}" component). Let $\varphi : \mathbb{H}^p \to \mathbb{H}^p$ defined by $\varphi(z_1, \dots, z_p) = (z_1, \dots, z_i, \overline{\tau}(z_i), z_{i+1}, \dots, z_p)$ and $\Gamma' = \varphi\Gamma\varphi^{-1}$. As φ is an isometry, Γ' is still a lattice. We then obtain, passing to the quotient, a map

$$\Psi_2 : \mathfrak{H} \to \mathfrak{H}' = \mathbb{H}^p / \Gamma'$$

One can now easily check that $\Psi = \Psi_2 \circ \Psi_1$ (more exactly its extension to the whole X) has the required property by changing the complex structure on $\mathfrak{H}$ as described above. □

7 The Logarithmic Setting

We begin this section by collecting some classical results about logarithmic 1 form for further use.

7.1 Logarithmic One Forms

We follow closely the exposition of [brme]. Let X be a complex manifold and H a *simple normal crossing hypersurface* which decomposes into irreducible components as $H = H_1 \cup \dots H_p$. A *logarithmic 1 form* on X with poles along H is a meromorphic 1 form ω with polar set $(\omega)_\infty$ such that ω and $d\omega$ have at most simple poles along H. In other words, if $f = 0$ is a local reduced equation of H, then $f\omega$ and $fd\omega$ are holomorphic. This latter properties are equivalent to say that $f\omega$ and $df \wedge d\omega$ are holomorphic. By localization on arbitrarily small open subset, one obtains a locally free sheaf on X denoted by $\Omega^1_X(\log H)$.

If $(z_1, \dots, z_n)$ is a local coordinate system around $x \in X$ such that H is locally expressed as $z_1 \dots z_k = 0$, then every section of $\Omega^1_X(\log H)$ around x can be written as

$$\omega = \omega_0 + \sum_{i=1}^{k} g_i \frac{dz_i}{z_i} \qquad (6)$$

where ω_0 is a holomorphic 1 form and each g_i is a holomorphic function.

Equivalently, one can locally write ω as

$$\omega = \omega_i + g_i \frac{dz_i}{z_i}$$

where g_i is holomorphic and ω_i is a local section of $\Omega^1_X(\log H)$ whose polar set does not contain $H_i = \{z_i = 0\}$.

Despite that this decomposition fails to be unique, we have a well-defined map (the so-called *Residue map*):

$$\mathrm{Res} : \Omega^1_X(\log H) \to \bigoplus_{i=1}^{p} \mathcal{O}_{H_i}$$

summing on i the maps

$$\mathrm{Res}_{H_i} : \omega \to g_{i|H_i}.$$

We will also denote by Res the induced morphism

$$H^0(\Omega^1_X(\log H) \otimes L) \to \bigoplus_{i=1}^{p} H^0(H_i, \mathcal{O}_{L_i})$$

for global logarithmic form twisted by a line bundle L (setting $\mathcal{O}(L_i) := \mathcal{O}(L_{|H_i})$).

7.2 A Class of Twisted Logarithmic Form

From now on, we will suppose that there exists on X **Kähler** compact, with fixed Kähler form θ a globally defined twisted logarithmic form $\omega \in H^0(\Omega^1_X(\log H) \otimes L)$, assuming moreover that the dual bundle L^* is **pseudoeffective**.

Let h be a (maybe singular) hermitian metric on L^* with positive curvature current $\Theta_h \geq 0$. In a local trivialization $L_{|U} \simeq U \times \mathbb{C}$, we thus have

$$h(x, v) = |v|^2 e^{-2\varphi(x)}.$$

Hence,

$$T = \frac{i}{\pi} \partial\overline{\partial}\varphi$$

is a positive current representing $c_1(L^*) = -c_1(L)$ with divisorial Zariski decomposition

$$\{T\} = \{N\} + Z.$$

In the sequel we will assume that $\mathrm{Res}_{H_i}(\omega) \neq 0$ for every component H_i of H.

The following lemma is straightforward and will be used repeatedly in the remainder of this section.

Lemma 7.1. *Let $\omega \in H^0(\Omega^1_X(\log H) \otimes L)$ be a twisted logarithmic form whose polar locus is exactly H. Assume that $L^* \in Pic(X)$ is pseudoeffective and denote by N its negative part. Set $L_i = L_{|H_i}$ ($H = H_1 \cup H_2 \ldots \cup H_p$).*

Then

- *either L_i is the trivial line bundle,*
- *either $L_i{}^*$ is not pseudoeffective and $H_i \subset Supp\, N$.*

As a by-product, one obtains the

Lemma 7.2. *Let $\mathcal{C}$ be a connected component of H, then the following alternative holds:*

a) $\mathcal{C}$ is contained in $Supp\, N$.
b) For any irreducible component $H_{\mathcal{C}}$ of $\mathcal{C}$, $H_{\mathcal{C}} \cap Supp\, N = \emptyset$ and $Z.\{H_{\mathcal{C}}\} = 0$. Such a $\mathcal{C}$ is then called a ***non-exceptional component****.*

From now on, we fix a current T representing $c_1(L^*)$.

By definition of the negative part, T has non-vanishing Lelong's numbers along Supp N.

In particular, the local potentials φ of T are necessarily equal to $-\infty$ on connected component of H satisfying the item a) in Lemma 7.2.

Assume now that $H_{\mathcal{C}} \subset \mathcal{C}$ fulfills the properties of point (2). In this case, L is trivial restricted to $H_{\mathcal{C}}$ (Lemma 7.1). Then,

(1) either the restrictions of φ on $H_{\mathcal{C}}$ are local *pluriharmonic functions*,
(2) either these restrictions are identically $-\infty$.

Following the previous numerotation, $H_{\mathcal{C}}$ is said to be of **type** (1) **or** (2). Note that, by connectedness, every irreducible component of $\mathcal{C}$ has the same type. This allows us to define **connected components of** H **of type** (1) **and** (2) with respect to T, noticing that any connected component belongs to one (and only one) of these types.

Definition 7.3. Let T be a closed positive current such that $\{T\} = c_1(N^*_{\mathcal{F}})$.

The boundary divisor H is said to be of *type* (1) (resp. *type* (2)) with respect to T if each of its connected components is of type (1) (resp. of type (2)) and when both situations occur, H is said to be of *mixed type* (with respect to T).

Lemma 7.4. *Let $\mathcal{C} \subset H$ be a connected component of type* (1)*. Assume that $Z \neq 0$. Then there exists a non-negative real number λ and an integral effective divisor D, $Supp\, D = \mathcal{C}$ such that $\{D\} = \lambda Z$.*

Proof. By pluriharmonicity of the local $\varphi_{|H}$, h restricts on H to a unitary flat metric g. We denote by ∇_{g^*} the $(1,0)$ part of the Chern connection associated with the dual metric g^*. This expresses locally as

$$\nabla_{g^*} = \partial\varphi_{|H} + d$$

and $\partial\varphi_{|H}$ is a *holomorphic* 1 form.

Fix $j_0 \in \{1,\dots,p\}$. The norm $a_{j_0j_0} = \{\mathrm{Res}_{H_{j_0}}(\omega), \{\mathrm{Res}_{H_{j_0}}(\omega)\}_{g*}$ defined on H_{j_0} is locally given by

$$e^{2\varphi_{|H}}|g_{j_0}|^2$$

which is obviously *plurisubharmonic*. This implies that $a_{j_0j_0}$ is indeed *constant*. Similarly, one can deduce that $a_{jj_0} = \{\mathrm{Res}_{H_j}(\omega), \mathrm{Res}_{H_{j_0}}(\omega)\}_{g*}$ is constant along every connected component of $H_{j_0} \cap H_j$.

Locally, one can write

$$\omega = \omega = \omega_0 + \sum_{j=1}^{p} g_j \frac{df_j}{f_j}$$

where f_{j_0} is the local expression of a global section of $\mathcal{O}(H_{j_0})$ vanishing on H_{j_0}. Consider a smooth metric on (H_{j_0}) with local weight ϕ. One can then easily check that the local forms

$$\omega_\phi = \omega_0 + \sum_{j \neq j_0} g_i \frac{dz_j}{z_j} + 2g_{j_0}\partial\phi$$

glue together along H_{j_0} in a section ω_{j_0} of $\Omega^1_{H_{j_0}}(\log D_{j_0}) \otimes \mathcal{E}^\infty \otimes L_{j_0}$ where $D_{j_0} = \bigcup_{j\neq j_0} H_{j_0} \cap H_j$ is a simple normal crossing hypersurface on H_{j_0} and $\mathcal{E}^\infty$ is the sheaf of germs of smooth functions in H_{j_0}.

The hermitian product

$$\{\omega_i, \mathrm{Res}_{H_{j_0}}(\omega)\}_{g*} = e^{2\varphi_{|H}}\overline{g_{j_0}} \wedge \omega_\phi$$

is well defined as a current on H_{j_0}. One may compute its differential with respect to the $\overline{\partial}$ operator. Using that

$$\frac{1}{2i\pi}\overline{\partial}(\frac{dz_i}{z_j}) = [z_j = 0] \text{ (the integration current along } z_j = 0)$$

and that the a_{jj_0}'s are constant, one easily gets that

$$\frac{1}{2i\pi}\overline{\partial}(\{\omega_i, \mathrm{Res}_{H_{j_0}(\omega)}\}_{g^*})$$

represents in cohomology the intersection class

$$(\sum_j a_{jj_0}\{H_j\})\{H_{j_0}\}.$$

In particular Stokes theorem yields that $(\sum_j a_{jj_0}\{H_j\}\{H_{j_0}\})\{\theta\})^{n-2} = 0$. Summing up on all j_0, we obtain the following vanishing formula

$$(\sum_{i,j} a_{ij}\{H_i\}\{H_j\})\{\theta\}^{n-2} = 0.$$

Thanks to Cauchy–Schwartz's inequality and the fact that $\{H_i\}\{H_j\}\{\theta\}^{n-2} \geq 0$ if $i \neq j$, one can promptly deduce that

$$\langle I\Lambda, \Lambda\rangle \geq 0$$

where I is the intersection matrix $\{H_i\}\{H_j\}\{\theta\}^{n-2}$, $\langle\rangle$ is the standard Hermitian product on $\mathbb{C}^p$, and Λ denotes the vector $\begin{pmatrix} \sqrt{a_{11}} \\ \vdots \\ \sqrt{a_{pp}} \end{pmatrix}$.

In particular, I is non-negative and from elementary bilinear algebra, one can produce an effective integral divisor D whose support is $\mathcal{C}$ and such that

$$\{D\}^2\{\theta\}^{n-2} = 0.$$

Keeping in mind that we have also $Z^2\{\theta\}^{n-2} \geq 0$ and $Z\{D\} = 0$, one concludes by Hodge's signature theorem that $\{D\}$ is a multiple of Z.

Let $\{\omega, \omega\}_{h^*}$ be the norm of ω with respect to the dual metric h^*. This is a well-defined (a priori non-closed) positive current on $X \setminus H$ which locally expressed as

$$\eta = \frac{i}{\pi} e^{2\varphi}\omega \wedge \overline{\omega}$$

(By a slight abuse of language, we also denote by ω the restriction of the global twisted 1 form ω to a trivializing open set).

Following [de], we inherit from the dual metric h^* a Chern connection whose $(1,0)$ part ∂_{h^*} acts on ω as

$$\partial_{h^*}\omega = \partial\omega + 2\partial\varphi \wedge \omega$$

on a trivializing chart.

Note that the resulting $(2,0)$ form (well defined outside H) has L^1_{loc} coefficients.

Theorem 5. *Let X be a compact Kaḧler manifold and ω a non-trivial section of $\Omega^1_X(\log H) \otimes L$ with L^* pseudoeffective. Assume that either $Z = 0$ (T is then unique and equal to $[N]$), either $Z > 0$, and H is of type (2), then, outside the polar locus H, the following identity holds (in the sense of current):*

$$\partial_{h^*}\omega = 0. \qquad (7)$$

In particular, ω is Frobenius integrable ($\omega \wedge d\omega = 0$); moreover, η and $T = \frac{i}{\pi}\partial\bar{\partial}\varphi$ are closed positive $\mathcal{F}$-invariant currents, respectively, on $X \setminus H$ and X, where $\mathcal{F}$ is the codimension 1 foliation defined by ω.

Remark 7.5.
- When $H = \emptyset$, this theorem has been established in [de] and this statement is just a generalization to the logarithmic setting of the invariance properties of currents recalled in Sect. 3.1.
- By invariance of H, it is enough to prove the invariance of T on $X \setminus H$ to get the invariance on the whole X (see [to], Remarque 2.2).

The proof of Theorem 5 is divided into two parts, each one corresponding to a particular "numerical property" of the positive part Z and the type of the boundary hypersurface H.

7.3 The Case $Z = 0$

In this situation N is a $\mathbb{Q}$ effective divisor.

Assume firstly that N is an integral effective divisor.

We have

$$\omega \in H^0(X, \Omega^1_X(\log H)(-N) \otimes E)$$

with E a numerically trivial line bundle.

Consider the canonical injection

$$\iota : \Omega^1_X(\log H)(-N) \otimes E \hookrightarrow \Omega^1_X(\log H) \otimes E$$

where the first term is regarded as the sheaf of logarithmic 1 forms with respect to H whose zeroes divisor is contained in N.

Let ∇_E be the unique unitary connection on E and ∇_{E_i} its restriction on H_i. By flatness of E_i, one can infer that

$$\nabla_{E_i} \mathrm{Res}_{H_i}(\iota(\omega)) = 0$$

As a by-product, and using that $\bar{\partial}$ commutes with ∇_E, one obtains that

$$\xi = \nabla_E \omega \in H^0(X, \Omega^2_X \otimes E)$$

In particular, one obtains that the differential d of the well-defined $(2,1)$ current $(\nabla_E \omega) \wedge \overline{\Omega} \wedge \theta^{n-2}$ is a *positive* $(2,2)$ form, namely $-\xi \wedge \bar{\xi} \wedge \theta^{n-2}$. By Stokes theorem, this latter is necessarily identically zero. In other words,

$$\nabla_E\, \iota(\omega) = 0. \tag{8}$$

The proof above is the same of that presented in [brbir], p.81 (see also [brme] and the references therein). We have just replaced L *trivial* by L *numerically trivial*.

For some suitable open covering $\mathcal{U}$, ω is defined by the data of local logarithmic one forms $\omega_U \in H^0(U, \Omega^1_X \otimes \log H)$, $U \in \mathcal{U}$ with the glueing condition $f_U\omega_U = g_{UV}f_V\omega_V$, $g_{UV} \in \mathcal{O}^*(U \cap V)$ such that $|g_{UV}| = 1$ and $f_U = 0$ is a suitably chosen local equation of N.

The vanishing property (8) can be rephrased as

$$\partial(f_U\omega_U) = 0 \text{ for every } U$$

which finally gives (7), using that $T = [N]$, the integration current on N.

Note that ∂_{h^*} expresses locally as $\partial + \frac{df_U}{f_U}$ is indeed a *meromorphic logarithmic flat* connection on L.

Consider now the general case where N is only effective over $\mathbb{Q}$. By ramified covering trick and after suitable blowing-up, we obtain a Kähler manifold $\hat{X}$, $\dim \hat{X} = \dim X$ equipped with a generically finite morphism $\psi : \hat{X} \to X$ such that $\psi^*(N)$ is an *integral* effective divisor and $\hat{H} = \psi^{-1}(H)$ a normal crossing hypersurface. Let us give some details about this construction (already implicitly used in Sect. 5): first, select $F \in \mathrm{Pic}(X)$ and a positive integer m such that mN is integral and $mF = mN$. Consider $V = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(F))$. Let $Y \subset V$ be the image of X under the multivaluate map $X \ni x \to [1 : \sqrt[m]{s(x)}]$ where s is a section of mF vanishing exactly on mN. Y is then a singular ramified covering of X whose embedded desingularization gives the sought $\hat{X}$ (one of the connected components of the strict transform). It is well known that the Kähler property remains stable under projective bundle and blowing-up of smooth centers (see, for instance, [voi], I.3), thus ensuring that $\hat{X}$ is still Kähler.

One can then assert that $\pi^*\omega \in H^0(\hat{X}, \Omega^1_{\hat{X}}(\log\ \hat{H}) \otimes \hat{L})$. where $\hat{L} \in \mathrm{Pic}\ X$ is numerically equivalent to $\pi^*(N)$. Let $\nabla_{\hat{h}}$ be the logarithmic connection on $\hat{L}$ defined by a metric $\hat{h}$ whose curvature current is $-[\psi^*N]$. In other words, $\nabla_{\hat{h}}$ is the pull-back by ψ of the logarithmic connection ∇_{h^*}. By the computation performed above, we have $\nabla_{\hat{h}}\psi^*\omega = 0$. For obvious functoriality reasons, this can be restated as $\psi^*(\nabla_{h^*}\omega) = 0$, which eventually leads to $\nabla_{h^*}\omega = 0$. □

7.4 The Case H of Type (2)

For $j = 1, \dots p$, let us fix a smooth metric g_j on $\mathcal{O}(H_i)$ with local weight ϕ_j. For $\varepsilon > 0$, we thus obtain a smooth metric on $\mathcal{O}(H_j)$ with local weight $\phi_{j_\varepsilon} = \mathrm{Log}\,(|f|^2 + \varepsilon e^{2\phi_j})$.

The $(1, 1)$ positive form $\eta := \frac{i}{\pi}e^{2\varphi}\omega\wedge\overline{\omega}$ is well defined as a current on the whole $X \setminus \mathrm{Supp}\ H$, but failed to be integrable along the boundary divisor H. To cure this, take a smooth section Ω of $\Omega^{1,0}_X \otimes L$ (a sheaf in the $\mathcal{C}^\infty$ category), and consider

$$\Delta = \frac{i}{\pi} e^{2\varphi} \Omega \wedge \overline{\Omega}$$

which is well defined as a positive current on the whole X (making again the confusion between Ω and its writing on a trivializing chart).

Let us compute $\overline{\partial}\Delta$ as a current. This latter decomposes as:

$$\pi \overline{\partial} \Delta = A + B$$

where

$$A = ie^{2\varphi} \overline{\partial} \Omega \wedge \overline{\Omega}$$

and

$$B = i((\overline{\partial} e^{2\varphi})\Omega \wedge \overline{\Omega} - e^{2\varphi} \Omega \wedge \overline{\partial \Omega})$$

each of the summand being a well-defined current on X.

An easy calculation yields

$$i\pi\, \partial\overline{\partial}\,(\Delta) = i\partial A - 2e^{2\varphi} \Omega \wedge \overline{\Omega} \wedge \partial\overline{\partial}\,\varphi - e^{2\varphi}(2\partial\varphi \wedge \Omega + \partial\Omega) \wedge \overline{(2\partial\varphi \wedge \Omega + \partial\Omega)}$$

$$+ \overline{\partial}(e^{2\varphi} \Omega \wedge \partial\overline{\Omega}) + e^{2\varphi} \partial\overline{\Omega} \wedge \overline{\partial\overline{\Omega}}$$

Hence, by Stokes theorem,

$$(9) \quad \langle i\pi \partial\overline{\partial}\Delta \wedge \theta^{n-2}, 1\rangle = 0 = \langle T_1 \wedge \theta^{n-2}, 1\rangle + \langle T_2 \wedge \theta^{n-2}, 1\rangle + \langle \gamma \wedge \theta^{n-2}, 1\rangle$$

where

$$T_1 = -2e^{2\varphi} \Omega \wedge \overline{\Omega} \wedge \partial\overline{\partial}\,\varphi$$

and

$$T_2 = -e^{2\varphi}(2\partial\varphi \wedge \Omega + d\Omega) \wedge \overline{(2\partial\varphi \wedge \Omega + d\Omega)}$$

are both positive $(2, 2)$ currents (which make sense, according to [de]) and

$$\gamma = e^{2\varphi} \partial\overline{\Omega} \wedge \overline{\partial\overline{\Omega}}.$$

Actually, one performs the same computation as in [de] except that the final expression contains the residual term γ arising from the holomorphicity defect of Ω.

Let $\mathcal{U}$ be an open cover of X such that each $U \in \mathcal{U}$ is equipped with local coordinates $(z_1, .., z_n)$ in such a way that $\omega_{|U}$ can be written as

$$\omega_{|U} = \omega_0 + \sum_{j=1}^{p} g_j \frac{df_j}{f_j} \tag{10}$$

where $\{f_j = 0\}$ is a local equation of H_j and f_j coincides with one of the coordinates z_k whenever $H_j \cap U \neq \emptyset$.

For $\varepsilon > 0$, set

$$\omega_\varepsilon^U = \omega_0 + \sum_{j=1}^{p} g_j \varphi_{j,\varepsilon} \frac{df_j}{f_j}$$

where $\varphi_{j,\varepsilon} = \frac{|f_j|^2}{|f_j|^2 + \varepsilon e^{2\phi_j}}$. One can remark that for each j, the $\varphi_{j,\varepsilon}$'s glue together on overlapping charts and thus define a global smooth function of X. One can see ω_ε^U as a smooth section of $\Omega_X^1 \otimes L$ which approximates ω (on U) in the following sense:

(1) $\lim_{\varepsilon \to 0} \omega_\varepsilon^U = \omega$ weakly as a current.
(2) For every $K \Subset U \setminus H$, ω_ε^U converges uniformly to ω on K when ε goes to 0.

Before mentioning other properties, we introduce the following notation:

Let T a current of bidegree (p, q) defined on U with L^1_{loc} coefficient written as

$$T = \sum_{I,J} F_{I,J} dz_I \wedge d\overline{z_J}$$

in standard multiindices notation with respect to the coordinate system $(z_1, ..z_n)$.

We define

$$\|T\|_K = \sup_{I,J} \|F_{I,J}\|_{L^1(K)}$$

the L^1 norm being evaluated with respect to the Lebesgue measure $d\mu = |dz_1 \wedge d\overline{z_1} \dots .dz_n \wedge d\overline{z_n}|$. Now, the key point is to notice that

$$\sup_{\varepsilon > 0} \|\partial(\overline{\omega_\varepsilon^U}) \wedge \overline{\partial(\overline{\omega_\varepsilon^U})}\|_K < +\infty \tag{11}$$

for every $K \Subset U$.

Indeed, the choice of $\varphi_{i,\varepsilon}$ allows us to "get rid" of the non-integrable terms $\frac{dz_i}{z_i} \wedge \frac{d\overline{z_i}}{\overline{z_i}}$ which could appear after passing to the limit.

Moreover, $\overline{\partial\omega_\varepsilon^U} \wedge \overline{\partial\overline{\omega_\varepsilon}}$ converges uniformly to 0 on every $K \Subset U \setminus H$.

Let (ψ_U) be a partition of unity subordinate to the open cover $\mathcal{U}$. Set

$$\Omega_\varepsilon = \sum_U \psi_U \omega_\varepsilon^U.$$

Note that this defines a smooth section of $\Omega^1_X \otimes L$ whose restriction to U shares the same properties than $\omega_\varepsilon{}^U$.

In particular, we get

$$\lim_{\varepsilon\to 0} e^{2\varphi}\partial\overline{\Omega_\varepsilon} \wedge \overline{\partial\overline{\Omega_\varepsilon}} = 0 \text{ in the weak sense.} \tag{12}$$

Actually this is obvious on $K \Subset X$ (where the convergence is even uniform) and this extends to the whole X, thanks to (11) and the fact that $\varphi = -\infty$ on $U \cap H$ and is upper-semicontinuous (as a plurisubharmonic function).

In order to conclude, replace Ω by Ω_ε in (9) and $\gamma,\ T_1,\ T_2$ by the corresponding $\gamma_\varepsilon,\ T_{1\varepsilon},\ T_{2\varepsilon}$, keeping in mind that $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$, and that $T_{1\varepsilon}$ and $T_{2\varepsilon}$ are positive current which converges on $X \setminus H$, respectively, to

$$S_1 = -2e^{2\varphi}\omega \wedge \overline{\omega} \wedge \partial\overline{\partial}\varphi$$

and

$$S_2 = -e^{2\varphi}(2\partial\varphi \wedge \omega + d\omega) \wedge \overline{(2\partial\varphi \wedge \omega + d\omega)}$$

Passing to the limit when ε goes to 0, one can infer that $S_1 = S_2 = 0$, whence the result. □

Corollary 7.6. *Let X be a compact Kähler manifold and $\omega \in H^0(\Omega^1_X(\log H) \otimes L)$ a non-trivial twisted logarithmic 1 form with poles on a simple normal crossing hypersurface H. Assume that L^* is pseudo-effective. Then ω is integrable and there exists an $\mathcal{F}$ invariant positive current representing L^*, where $\mathcal{F}$ is the foliation defined by ω. In particular, the integration current $[N]$ is $\mathcal{F}$ invariant, where N is the negative part in the Zariski's divisorial decomposition of $c_1(L^*)$.*

Proof. One can assume that ω has non-trivial residues along the irreducible components of H. The only remaining case to deal with is $Z \neq 0$ and among the connected components of $\mathcal{C}$, there exists at least one of type (1) with respect to a given closed positive current T representing $c_1(N^*_{\mathcal{F}})$. By making use of Lemma 7.4, one can change T into another representing positive current T' such that H becomes type (2) with respect to T'.

8 Pseudoeffective Logarithmic Conormal Bundle and Invariant Metrics

Let $(X, \mathcal{D})$ be a pair consisting of a compact Kähler manifold X equipped with a codimension 1 holomorphic distribution $\mathcal{D}$. We assume that there exists a $\mathcal{D}$-invariant hypersurface H (i.e: H is tangent to $\mathcal{D}$) such that

$$L = N^*_{\mathcal{D}} \otimes \mathcal{O}(H)$$

is *psef*.

From the normal crossing and invariance conditions, $\mathcal{D}$ is defined by a twisted logarithmic one form $\omega \in H^0(X, \Omega^1_X(\log H) \otimes L)$ with non-vanishing residues along H (see, for instance, Lemma 3.1 in [brme]).

As a consequence of Corollary 7.6, $\mathcal{D}$ is **integrable** and we will call $\mathcal{F}$ the corresponding foliation.

The following definitions are borrowed from the terminology used in the study of log singular varieties, even if the setting is somewhat different.

Definition 8.1. Let $\mathcal{F}$ be a foliation like above. Let $H = H_1 \cup \dots . \cup H_p$ the decomposition of H into irreducible components.

$\mathcal{F}$ is said to be of

(1) **KLT** type (with respect to H) if there exist p rational numbers $0 \leq a_i < 1$, $i = 1, \dots, p$ such that

$$c_1(N^*_{\mathcal{F}}) + \sum_i a_i \{H_i\}$$

is a *pseudoeffective* class.

(2) **strict Log-Canonical** type (with respect to H) if the condition above is not satisfied.

8.1 Existence of Special Invariant Transverse Metrics for KLT Foliations

Here, we assume that the foliation $\mathcal{F}$ is of KLT type. This means (see definition above) that $c_1(N^*_{\mathcal{F}}) + \sum_i \lambda_i \{H_i\}$ is *pseudoeffective* where the λ_i's are real numbers such that $0 \leq \lambda_i < 1$.

Definition 8.2. Set $\Lambda = (\lambda_1, \dots, \lambda_p)$ and $H_\Lambda = \sum \lambda_i H_i$.

Λ will be called a **KLT datum**.

By definition of pseudoeffectiveness, there exists on $N^*_{\mathcal{F}}$ a singular metric h which locally can be expressed as

$$h(x, v) = |v|^2 e^{-2\varphi(x) + \sum 2\lambda_i \log |f_i|}$$

where φ is a locally defined *psh* function such that

$$T_\Lambda = \frac{i}{\pi} \partial\bar{\partial} \varphi$$

is a globally defined positive current representing $c_1(N^*_{\mathcal{F}}) + \{D\}$, the $\{f_i = 0\}$'s are local reduced equation of H_i and ω is a local defining one form of $\mathcal{F}$. On $N_{\mathcal{D}}$, the dual metric h^* can be represented by the positive $(1, 1)$ form

$$\eta_{T_\Lambda} = \frac{i}{\pi} e^{2\varphi - 2\sum_i \lambda_i \log|f_i|} \omega \wedge \overline{\omega}$$

with locally integrable coefficients, which is unique up to multiplication by a positive constant once T_Λ has been fixed (cf [to] Remarque 1.4, Remarque 1.5).

Let $\mathcal{C}_{\mathcal{F}}$ be the cone of closed positive current T_Λ such that $\{T_\Lambda\} = c_1(N^*_{\mathcal{F}}) + \{H_\Lambda\}$.

We are going to establish a similar statement to that of Theorem 3.1.

Theorem 8.3. *Let $\{N_\Lambda\} + Z_\Lambda$ be the Zariski decomposition of the pseudoeffective class $c_1(N^*_{\mathcal{F}}) + \{H_\Lambda\}$ and $T_\Lambda \in \mathcal{C}_{\mathcal{F}}$.*

*Then, both T_Λ and η_{T_Λ} are **closed $\mathcal{F}$ invariant positive currents**. Moreover, T_Λ can be chosen in such a way that*

*(1) $T_\Lambda = [N_\Lambda]$ if $Z_\Lambda = 0$ (**euclidean type**).*
*(2) $T_\Lambda = [N] + \eta_{T_\Lambda}$ if $Z_\Lambda \neq 0$ (**hyperbolic type**) (η_{T_Λ} being normalized such that $\{\eta_{T_\Lambda}\} = Z_\Lambda$).*

Proof. Let T_Λ in $\mathcal{C}_{\mathcal{F}}$. The closed positive current $S = T_\Lambda + \sum_i (1-\lambda_i)[H_i]$ represents $c_1(N^*_{\mathcal{F}}) + \{H\}$. As a by-product of Theorem 5, we obtain that

$$(2\partial\varphi - 2\sum \lambda_i \frac{df_i}{f_i}) \wedge \omega = -d\omega = 0$$

which implies the closedness and $\mathcal{F}$ invariance of T_Λ and η_{T_Λ}.

The remainder follows the same line of argumentation as in [to] (which corresponds to $\Lambda = 0$). From $\mathcal{F}$-invariance of T_Λ, one can deduce that $[N_\Lambda]$ is also $\mathcal{F}$-invariant and that the same holds for any closed positive current representing Z_Λ. Note that the theorem is proved when $Z_\Lambda = 0$.

We investigate now the nature of the singularities of the foliations. Outside H, η_T has a local expression of the form $ie^\Psi \omega \wedge \overline{\omega}$ with Ψ *psh* and thus admits local *elementary first integrals* (Théorème 2 of [to]). Near a point on H, the same holds replacing η_T by $\pi^*\eta_T$ where π is a suitable local branched covering over H (we use that $\lambda_i < 1$) and one can then again conclude that $\mathcal{F}$ admits a local elementary first integral (this property is clearly invariant under ramified covering).

By Proposition 2.7, $\{\eta_{T_\Lambda}\}\{W\} = 0$ for any closed $\mathcal{F}$ invariant positive current W. In particular, this holds whenever $W = \eta_{T_\lambda}$ or $\{W\} = Z_\Lambda$. By Hodge's signature theorem, one can then infer that Z_Λ is a multiple of $\{\eta_{T_\Lambda}\}$ and after normalization, that $\{\eta_{T_\Lambda}\} = Z_\Lambda$ whenever $Z_\Lambda \neq 0$. In this case, we have

$$\{T_\Lambda\} = \{N_\Lambda\} + \{\eta_{T_\Lambda}\}.$$

From now on, we assume that $Z_\Lambda \neq 0$. By the normalization process above, η_T is then uniquely determined by the datum of $T_\Lambda \in \mathcal{C}_{\mathcal{F}}$.

Let $\mathcal{C}_{Z_\Lambda}$ be the cone of closed positive currents T such that $\{T\} = Z_\Lambda$.

By Banach–Alaoglu's theorem, $\mathcal{C}_{Z_\Lambda}$ is compact with respect to the weak topology on current.

From (8.1), one inherits a map

$$\beta : \mathcal{C}_{Z_\Lambda} \to \mathcal{C}_{Z_\Lambda}$$

defined by $\beta(T) = \eta_{T+[N_\Lambda]}$. Remark that the point (2) of the theorem is equivalent to the existence of a fix point for β. This fix point will be produced by the Leray–Schauder–Tychonoff's theorem once we have proved that β is *continuous*. This can be done following the same approach than [to] (Démonstration du lemme 3.1, p.371). We briefly recall the idea (see *loc.cit* for details): Let $(T_n) \in \mathcal{C}_{Z_\Lambda}{}^{\mathbb{N}}$ converging to T such that the sequence $\beta(T_n)$ is convergent. One has to check that

$$\lim_{n\to+\infty} \beta(T_n) = \beta(T).$$

Locally, one can write $T_n = \frac{i}{\pi}\partial\overline{\partial}\varphi_n$ and $\beta(T_n) = \frac{i}{\pi}e^{2\varphi_n}\frac{\omega\wedge\overline{\omega}}{\prod_i |f_i|^{2\lambda_i}}$. By plurisubharmonicity, one can suppose, up extracting a subsequence, that (φ_n) is uniformly bounded from above on compact sets and converges in L^1_{loc} to a *psh* function φ which necessarily satisfies

$$\frac{i}{\pi}\partial\overline{\partial}\varphi = T.$$

By Lebesgue's dominated convergence, one obtains that $e^{2\varphi_n}\frac{\omega\wedge\overline{\omega}}{\prod_i |f_i|^{2\lambda_i}}$ converges in L^1_{loc} to $e^{2\varphi}\frac{\omega\wedge\overline{\omega}}{\prod_i |f_i|^{2\lambda_i}}$. This obviously implies that

$$\lim_{n\to+\infty} \beta(T_n) = \beta(T).$$

□

The main information provided by the proof above are:

- The $\mathcal{F}$-invariant positive current η_{T_Λ} represents the positive part Z_Λ when this latter is non 0.
- For every $\mathcal{F}$-invariant positive current S, one has $\{\eta_{T_\Lambda}\}\{S\} = 0$, in particular $\{\eta_{T_\Lambda}\}^2 = 0$.

This is enough to get an analogous statement to that of [to], Proposition 2.14 and Corollaire 2.15.

Proposition 8.4. *Let $\mathcal{F}$ be a KLT foliation with KLT datum $\Lambda = (\lambda_1, \ldots, \lambda_p)$. Let Ξ be a $(1,1)$ positive current $\mathcal{F}$-invariant (for instance, $\Xi = T_\Lambda$). Let's consider the Zariski's decomposition.*

$$\alpha = \{N(\alpha)\} + Z(\alpha)$$

with $\alpha = \{\Xi\}$ (e.g: $\alpha = c_1(N^*_{\mathcal{F}}) + \{H_\Lambda\}$).

Then the following properties hold:

a) *the components D_i of the negative part $N(\alpha)$ are hypersurfaces invariant by the foliation; in particular, $Z(\alpha)$ can be represented by an $\mathcal{F}$ invariant closed positive current.*
b) *$Z(\alpha)$ is a multiple of $\{\eta_T\}$.*
c) *$Z(\alpha)$ is nef and $Z(\alpha)^2 = 0$.*
d) *The decomposition is orthogonal: $\{N(\alpha)\}Z(\alpha) = 0$. More precisely, for every component D_i of $N(\alpha)$, one has $\{D_i\}Z(\alpha) = 0$.*
e) *In $H^{1,1}(M, \mathbb{R})$, the $\mathbb{R}$ vector space spanned by the components $\{D_i\}$ of $\{N(\alpha)\}$ intersects the real line spanned by $\{\eta_T\}$ only at the origin.*
f) *The decomposition is rational if X is projective and α is a rational class (e.g., $\alpha = c_1(N^*_{\mathcal{F}}) + \{H_\Lambda\}$).*
g) *Every $(1, 1)$ closed positive current which represents α is necessarily $\mathcal{F}$-invariant.*
h) *Let A be a hypersurface invariant by the foliation and $A_1, \ldots, A_r$ its irreducible components; the family $\{A_1, \ldots, A_r\}$ is exceptional if and only if the matrix $(m_{ij}) = (\{A_i\}\{A_j\}\{\theta\}^{n-2})$ is negative, where θ is a Kähler form on X.*

Proof. This is essentially a consequence of the Hodge's signature theorem (see [to] for the details).

In this setting one can introduce, exactly on the same way as [to] Définition 5.2 (see also Sect. 3), the locally constant sheaves $\mathcal{I}^\varepsilon$ and $\mathcal{I}^\varepsilon_{d\log}$ attached to the transverse metric structure given by Theorem 8.3. Strictly speaking, those sheaves are only defined on the complement of $H \cup \operatorname{Supp} N_\Lambda$.

Let $\mathcal{P}$ be the set of prime divisor of X. To each $P \in \mathcal{P}$, one can associate non-negative numbers λ_D, ν_D defined by the decompositions

$$N_\Lambda = \sum_{D \in \mathcal{P}} \nu_D D.$$

$$H_\Lambda = \sum_{D \in \mathcal{P}} \lambda_D D.$$

Following again exactly the same lines of argumentation of [to] p.381–384, one can establish the following proposition similar to proposition 5.1 and proposition 5.2 of *loc.cit*:

Proposition 8.5. *Let $K_1, \ldots, K_r$ be a finite family of irreducible hypersurfaces invariant by $\mathcal{F}$. Set $K = K_1 \cup K_2 \ldots \cup K_r$. Then there exists on a sufficiently small connected neighborhood of $K \cup$ Sing $(\mathcal{F})$ a uniquely defined section η_K of $\mathcal{I}^\varepsilon_{d\log}$ such that the polar locus $(\eta_K)_\infty$ contains $K \cup$ Sing $(\mathcal{F})$ with the following additional properties:*

(1) *For every $i = 1, \ldots, r$, the residue of η_K along K_i is equal to $1 + \nu_{K_i} - \lambda_{K_i}$ and equal to 1 on the other components of $(\eta_K)_\infty$. In particular the residues of η_K are **non-negative** real numbers.*

(2) *If K is connected, $\{K_1, \ldots ., K_r\}$ is an exceptional family if and only if the germ of $(\eta_K)_\infty$ along K does not coincide with K.*

Remark 8.6. As the foliation only admits singularities of elementary type, the set of separatrices at any point $x \in K \cup \operatorname{Sing} \mathcal{F}$ coincide locally with the poles of η_K (see [to], Remarque 4.1).

8.2 Existence of Special Invariant Transverse Metrics for Strict Log-Canonical Foliations

Let $\mathcal{F}$ be a strict Log-Canonical foliation. As usual, $\{N\}+Z$ will denote the Zariski's divisorial decomposition of $c_1(N^*_\mathcal{F}) + \{H\}$.

8.2.1 The Case Z=0

In this situation, $c_1(N^*_\mathcal{F}) \otimes \mathcal{O}(H)$ is represented by a unique positive current, namely $T = [N]$.

Let H_b be the union of non-exceptional components (see Lemma 7.2). H_b will be called the **boundary part** of H. Note that H_b is non-empty, otherwise the foliation would be a KLT one. On $X \setminus H_b$, $\mathcal{F}$ admits an invariant transverse metric η. The semi-positive $(1,1)$ form η involved in the statement of Theorem 5 is well defined as a current on $X \setminus H_b$. Indeed, it can be expressed locally as

$$\eta = \frac{i}{\pi} e^{2\psi} \frac{\tilde{\omega} \wedge \overline{\tilde{\omega}}}{|F|^2 \prod_i |f_i|^{2\mu_i}}$$

where the $f_i = 0$'s are defining equations of the components H_i of H contained in Supp N, with $0 \leq \mu_i < 1$, $F = 0$ is a defining equation of H_b, ψ is a plurisubharmonic function such that $\frac{i}{\pi} \partial\overline{\partial} \psi = [N] - \sum_i \mu_i [H_i]$ and $\tilde{\omega}$ is a local generator of $N^*_\mathcal{F}$ (i.e. a local defining holomorphic 1 form of $\mathcal{F}$ without zeroes divisor). On a neighborhood V_x of a point $x \in H_b$ where $z_1, ..z_{n_x} = 0$ is a local equation of H_b, the closedness property (7) shows that the foliation is defined by a closed logarithmic form of the form

$$\omega = \sum_{i=1}^{n_x} \alpha_i \frac{dz_i}{z_i}$$

and that

$$\eta = \frac{i}{\pi} \omega \wedge \overline{\omega}. \tag{13}$$

(for suitable choice of z_i's) where the α_i's are non-zero complex numbers.

Because of this local expression, one can observe that $\operatorname{Sing} \mathcal{F}_b := \operatorname{Sing} \mathcal{F} \setminus H_b$ is a *compact* analytic subset of $X \setminus H_b$. Contrarily to the KLT case, this current η does not extend to X for the simple reason that it cannot be extended through H_b.

The archetypal example is provided by logarithmic foliations on the projective space $\mathbb{P}^n$, that is foliations defined by a logarithmic form ω. Such a form is automatically closed whenever the polar locus $(\omega)_\infty$ is normal crossing and thus induces the euclidean transverse metric $\eta = \frac{i}{\pi}\omega \wedge \overline{\omega}$, only defined outside the polar locus. In this example, it is also worth noticing that H_b is ample, a phenomenon which does not occur for KLT foliations: in the KLT situation, the class $\{D\}$ of any divisor whose support is $\mathcal{F}$-invariant has numerical dimension at most 1, thanks to Proposition 8.4. Another important difference lies in the fact that the number of $\mathcal{F}$-invariant hypersurfaces is arbitrarily high (compare with Proposition 9.8).

Coming back to our setting, η defined an euclidean transverse structure to which one can associate as before a locally constant sheaf of distinguished first integrals $\mathcal{I}^0$ and the corresponding sheaf of logarithmic differentials $\mathcal{I}^0_{d\log}$. These latters are defined a priori on the complement of $\operatorname{Supp} N \cup H$. Near a point $x \in H_b$, the foliation admits as first integral the multivaluate section $\sum_i \alpha_i \log z_i$ of $\mathcal{I}^0$ whereas these multivaluate sections take the form of elementary first integral $\prod f_i^{\lambda_i}$, $\lambda_i > 0$ at a neighborhood of $x \in \operatorname{Supp} N$.

Thanks to this euclidean structure, one can easily exhibit a section η_K of $\mathcal{I}^0$ on a neighborhood of $K \cup \operatorname{Sing} \mathcal{F}_b$ where $K = \operatorname{Supp} N$ whose polar locus $(\eta_K)_\infty$ contains $\operatorname{Supp} N$.

Lemma 8.7. *$(\eta_K)_\infty$ does not coincide with K along K.*

Proof. Suppose that equality holds. Then

$$\Big(\sum_i \lambda_i \{N_i\}\Big)^2 = 0$$

where the N_i's stand for the irreducible components of $\operatorname{Supp} N$ and $\lambda_i > 0$ is the residue of η_N along N_i. Moreover, $F = |e^{\int \eta_N}|$ is a function defined near $\operatorname{Supp} N$, constant on the leaves and such that the family $\{F < \varepsilon\}$ forms a basis of neighborhoods of $\operatorname{Supp} N$.

Let $\varepsilon > 0$ and $\psi : \mathbb{R} \to [0, \infty)$ continuous non-identically zero such that ψ vanishes outside $(0, \varepsilon)$. Set $g = \psi \circ F$ (this makes sense as ε is chosen small enough). This guarantees that $g\eta$ is an $\mathcal{F}$ invariant closed positive current S such that $\{S\}\{N_i\} = \{S\}^2 = 0$, thanks to Remark 2.8. By Hodge's signature theorem, $\sum_i \lambda_i \{N_i\}$ is proportional to $\{S\}$. This latter being a *nef* class (S has no positive Lelong numbers), we get the sought contradiction. □

Definition 8.8. Let V be an open neighborhood of H_b; a hermitian metric g defined on $V \setminus H_b$ is said to be *complete at infinity* if g is the restriction of a complete metric on $X \setminus H_b$.

Lemma 8.9. *There exists a bundle like metric on $X \setminus (H_b \cup \mathrm{Supp}\ N \cup \mathrm{Sing}\ \mathcal{F})$ inducing the invariant transverse metric η which is complete at infinity.*

Proof. Keeping the notation above, for $x \in H_b$, one puts on V_x the metric with poles $g_x = i \sum_{l=1}^{n_x} \frac{dz_l}{z_l} \wedge \frac{d\overline{z_l}}{\overline{z_l}} + h_x$ where h_x is a smooth hermitian metric on V_x. Let h be a smooth hermitian metric on $X \setminus H_b$ which is the restriction of a hermitian metric on X. By partition of unity subordinate to the cover of X determined by $X \setminus H_b$ and the collection of V_x, these models glue together in a complete hermitian metric defined on $X \setminus H_b$. Thanks to the expression of η given in (13), this latter is conformally equivalent on $X \setminus (H_b \cup \mathrm{Supp}\ N \cup \mathrm{Sing}\ \mathcal{F})$ to a metric g fulfilling the conclusion of the lemma. □

8.2.2 The Case $Z \neq 0$

As we shall see later, this situation is described by the Theorem 2 of the introduction. We first collect some observations useful in the sequel.

Lemma 8.10. *Let $\mathcal{C}$ be a connected component of H which is not contained in Supp N, and let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be its irreducible components.*

Then, the intersection form $\{\mathcal{C}_i\}\{\mathcal{C}_j\}\{\theta\}^{n-2}$ is negative definite and in particular $\{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ forms an exceptional family.

Proof. Suppose by contradiction that the intersection matrix is not negative. Then, thanks to Remark 7.2 and Hodge's signature theorem, one can exhibit r non-negative real numbers $\mu_1, \ldots, \mu_r$ with at least one positive, such that $\sum_{i=1}^{r} \mu_i \{\mathcal{C}_i\}$ is collinear to Z.

In particular, one can represent Z by the integration current $\lambda[D]$, where $\lambda > 0$ and $D = \sum_{i=1}^{r} \mu_i \mathcal{C}_i$. With the help of Lemma 7.4, this implies that for any other connected component $\mathcal{C}'$ of H not supported on Supp N, one can extract an effective divisor D' supported exactly on $\mathcal{C}'$ such that $\{D'\}$ and Z are collinear. Then $\mathcal{F}$ turns out to be a KLT foliation: a contradiction. □

Like the previous situation, we will denote by H_b the union of these connected components and will call it the *boundary component*.

Lemma 8.11. *Let T a closed positive current representing Z. Then T has no Lelong numbers along H_b and the local psh potentials of T are necessarily equal to $-\infty$ on H.*

Proof. The fact that the local potentials are $-\infty$ on H_b is a consequence of Lemma 7.4 together with Lemma 8.10. In particular, T is necessarily invariant according to Theorem 5. Let T be a positive closed current, $\{T\} = Z$. Assume by contradiction that T has some non-vanishing Lelong numbers on a connected component $\mathcal{C}$ of H_b. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the irreducible components of $\mathcal{C}$, and λ_i the Lelong number of T along $\mathcal{C}_i$. We first claim that there doesn't exist two indices $i \neq j$ such that $\lambda_i = 0$, $\lambda_j \neq 0$, and $\mathcal{C}_i \cap \mathcal{C}_j \neq 0$. Otherwise this would imply that $Z\{\mathcal{C}_i\}\{\theta^{n-2}\} > 0$, a contradiction with Lemma 7.2. One can then easily deduce that $\lambda_i \neq 0$ for each i. Let

$$T = \sum_{i=1}^{+\infty} \mu_i [K_i] + R$$

the Siu's decomposition of T.

Remark that from the $\mathcal{F}$-invariance of T, one can infer that each hypersurface K_i is also $\mathcal{F}$ invariant whenever $\mu_i \neq 0$ and that the same holds for the residual part R.

By virtue of Theorem 5, $\mathcal{F}$ admits in a neighborhood of $\mathcal{C}$ an invariant current $\eta = \eta_T$ which can be locally written

$$\eta = \frac{i}{\pi} e^{2\psi} \frac{\omega \wedge \overline{\omega}}{\prod_j |f_j|^{2\alpha_j}}$$

with $f_j = 0$ a reduced equation of $\mathcal{C}_j$, ω a defining form of $\mathcal{F}$, $0 \leq \alpha_j < 1$, and ψ a *psh* function.

This implies (see the proof of Theorem 8.3) that the singularities of the foliation near $\mathcal{C}$ are of elementary type, more precisely of the form

$F = u{f_1}^{\beta_1} \ldots {f_r}^{\beta_r}$

where u is some unit and the β_i's are non-negative real numbers. Suppose that for some i, $K_i \nsubseteq \mathcal{C}$. By remark 4.1 of [to], $K_i \cap \mathcal{C} = \emptyset$ and in particular $\{K_i\}\{\mathcal{C}_j\} = 0$ for every j. Moreover, $\{R\}\{\mathcal{C}_j\} = 0$ according to Remark 2.8. Hence, the equality

$$Z\{C_j\} = 0$$

can be reformulated as

$$(\sum_i \lambda_i \{\mathcal{C}_i\})\{\mathcal{C}_j\} = 0$$

which contradicts Lemma 8.10. □

Remark 8.12. By Theorem 5, $\eta = \eta_T$ is well defined as a closed positive current on $X \setminus H_b$ but a priori does not extend through H_b as a current. We will actually show that this extension really holds but this requires further analysis and the proof is postponed to the last section. Moreover, we will explain how we can ensure that the invariant transverse metric associated with η is hyperbolic, a key property to reach the conclusion of Theorem 2.

9 Settling the Case of KLT Foliations

In this section, we aim at discussing abundance properties of the logarithmic conormal bundle $N^*_{\mathcal{F}} \otimes \mathcal{O}(H)$ for KLT foliations (defined in the previous chapter). The ambient manifold is again assumed to be *Kähler compact*.

Here, Kod(L) stands for the *Kodaira dimension* of a line bundle L and $\nu(L)$ for its *numerical Kodaira dimension*.

One firstly recalls the statement of Bogomolov's theorem (see [brme] and the references therein).

Proposition 9.1. *Let X be a compact Kähler manifold X and $H \subset X$ a normal crossing hypersurface. Assume that for some $L \in Pic(X)$, there exists a non-trivial twisted logarithmic form $\omega \in H^0(X, \Omega^1_X(\log H) \otimes L)$. Then $Kod(L^*) \leq 1$; moreover, when equality holds, ω is integrable and the corresponding foliation is a meromorphic fibration.*

Corollary 9.2. *Let $\mathcal{F}$ be a codimension 1 foliation on a compact Kähler manifold X. Let H be a normal crossing hypersurface invariant by $\mathcal{F}$. Then $Kod(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) \leq 1$ and $\mathcal{F}$ is a meromorphic fibration when equality holds.*

By virtue of Corollary 7.6 and item 8.4) of Proposition 8.4, this upper bound remains valid in the numerical setting:

Proposition 9.3. *Let $\mathcal{F}$ be a codimension 1 foliation on a projective manifold X. Let H be a normal crossing hypersurface invariant by $\mathcal{F}$. Then $\nu(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) \leq 1$, where ν stands for the numerical dimension.*

Definition 9.4. Let $\Lambda = (\lambda_1, \dots, \lambda_p)$ be a KLT datum, $A \subset X$ a prime divisor, and $\mathcal{C}$ a connected component of H.

- We will denote by $m(A)$ the multiplicity of A along H_Λ: $m(A) = \lambda_i$ if $A = H_i$, $m(A) = 0$ otherwise.
- $\mathcal{C}$ is said to be exceptional if its irreducible components form an exceptional family.

Lemma 9.5. *Let $\mathcal{C}$ be a connected component of H, then the following properties are equivalent*

- *$\mathcal{C}$ is exceptional.*
- *$\mathcal{C}$ is contained in Supp N.*

Proof. Assume that $\mathcal{C}$ is exceptional and suppose by contradiction that $\mathcal{C}$ is not contained in Supp N. This implies, using Lemma 7.1, that $\mathcal{C} \cap H$ is empty and $L = N^*_{\mathcal{F}} \otimes \mathcal{O}(H)$ is trivial in restriction to each irreducible component of $\mathcal{C}$. The foliation $\mathcal{F}$ is defined by a logarithmic form $\omega \in H^0(X, \Omega(\log H) \otimes L)$ whose residues along the irreducible components of $\mathcal{C}$ are nowhere vanishing. Pick a point $x \in \mathcal{C}$ which meets n_p branches of $\mathcal{C}$ and such that $\eta_{\mathcal{C}_\infty}$ has extra components (not supported in $\mathcal{C}$) at p (this is made possible by Proposition 8.5) and consider a local holomorphic coordinate patch $z = (z_1, \dots, z_n)$, $z(x) = 0$ such the equation of $\mathcal{C}$ is given by $z_1 \dots z_{n_x} = 0$ and such that the logarithmic form $\eta_{\mathcal{C}}$ is expressed near x as

$$\eta_{\mathcal{C}} = \sum_{i=1}^{n_x} \alpha_i \frac{dz_i}{z_i} + \frac{df}{f}$$

where $f = 0$ is a reduced equation of the additional poles and the α_i's are *positive* real numbers. On the other hand,

$$\omega = \omega_0 + \sum_{i=1}^{n_x} g_i \frac{dz_i}{z_i}$$

in a trivializing neighborhood of p where ω_0 is a holomorphic 1 form and the g_i's are units. As ω and $\eta_{\mathcal{C}}$ define the same foliation, there exists a meromorphic function U such that $U\eta_{\mathcal{C}} = \omega$. One can then easily check that $U = fV$ with V holomorphic and this forces the residue g_i to vanish on $\{z_i = 0\} \cap \{f = 0\}$: a contradiction. The converse implication is obvious. □

9.1 The Case $Z \neq 0$

Let $\{N\} + Z$ be the Zariski decomposition of $c_1(N^*_{\mathcal{F}}) + \{H\}$ and assume that $Z \neq 0$.

Fix an arbitrarily small $\varepsilon > 0$ and consider an arbitrary non-exceptional component $\mathcal{C}$ of H. Up renumerotation of indices, one can write $\mathcal{C} = H_1 \cup \ldots \cup H_l,\ l \leq p$. Keeping track of Proposition 8.4, one can extract from $\mathcal{C}$, using Hodge's signature theorem, a $\mathbb{Q}$ effective divisor $D = \sum_{i=1}^{l} \lambda_i H_i$ with the following properties:

(1) For every $1 \leq i \leq l$, one has $0 < \lambda_i < 1$.
(2) there exists a positive integer p such that $\lambda_1 = \frac{1}{p}$.
(3) there exists a real number λ, $0 < \lambda < \varepsilon$ such that $\{D\} = \lambda Z$.

Combining these observations with Lemma 9.5, one obtains the following statement:

Proposition 9.6. *Let $\mathcal{F}$ be a KLT foliation and assume that $Z \neq 0$.*

Then one can find a KLT datum Λ which enjoys the following properties:

(1) $Supp(N_\Lambda) = Supp(N)$.
(2) Z_Λ is a non-trivial multiple of Z.
(3) For any non-exceptional connected component $\mathcal{C} \subset H$, there exist a prime divisor $H_{\mathcal{C}} \subset \mathcal{C}$ and a positive integer p such that $m(H_{\mathcal{C}}) = 1 - \frac{1}{p}$.

We are now ready to prove the main theorem of this section (which corresponds to Theorem 2 of the introduction).

Theorem 6. *Let $\mathcal{F}$ be a KLT foliation on X projective. Assume moreover that $Z \neq 0$. Then,*

(1) Either $\mathcal{F}$ is tangent to the fibers of a holomorphic map $f : X \to S$ onto a Riemann surface and the abundance principle holds:

$$Kod(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = \nu(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = 1.$$

This case occurs in particular whenever there exists at least one non-exceptional component $\mathcal{C} \in H$,

(2) *either* $\mathcal{F} = \Psi^*\mathcal{G}$ *where* Ψ *is a morphism of analytic varieties between* X *and the quotient* $\mathfrak{H} = \mathbb{D}^n/\Gamma$ *of a polydisk,* ($n \geq 2$) *by an irreducible lattice* $\Gamma \subset (Aut\,\mathbb{D})^n$ *and* $\mathcal{G}$ *is one of the n tautological foliations on* $\mathfrak{H}$.

Proof. Let Λ be a KLT datum satisfying the properties given by the previous proposition and consider the transversely hyperbolic structure attached to this latter. The strategy remains essentially the same as in the proof of Theorem 1. In particular, the first step consists in relating the monodromy of the foliation inherited from the hyperbolic transverse structure with its dynamical properties. Assume firstly that each component of H is exceptional and therefore it is contained in Supp N. In this case, we do exactly the same and prove the same as the situation "$N^*_{\mathcal{F}}$ pseudoeffective," starting from the fibers connectedness of the developing map studied in paragraph 3.2, then establishing the same results proved in Sects. 5 and 6 which finally leads to the same conclusion. One more time, the projectivity assumption is only needed to get item (2) which corresponds to the non-abundant situation

$$\mathrm{Kod}(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = -\infty,\ \nu(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = 1.$$

One needs to modify slightly the previous construction when there exist non-exceptional components. Indeed, on such a component $\mathcal{C}$, the hyperbolic transverse metric is likely to degenerate; moreover, $\mathcal{C}$ is not connected by any exterior leaf in the sense that every separatrix of $\mathcal{F}$ passing through $x \in \mathcal{C}$ is a local branch of $\mathcal{C}$ (consequence of Proposition 8.5). In Sect. 3, one had to remove supp (N) (where the transverse metric also degenerates) to define properly the developing map ρ. However, if one looks thoroughly at the arguments developed in the same section, the fact that ρ is both complete and has "connected fibers" (in the sense of Theorem 3.6) heavily relies on the existence of connecting exterior leaf in supp N. If one comes back to our current setting, the developing map is now defined on some covering of $X \setminus (\mathrm{supp}N \cup H)$. The point is that one can extend the basis of this covering and in that way get rid of the inexistence of connecting leaves. This can be done as follows: Let $\{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ be the set of non-exceptional components. For each $\mathcal{C}_i$, select a prime divisor $H_{\mathcal{C}_i} \subset \mathcal{C}$ such that $m(H_{\mathcal{C}_i}) = 1 - \frac{1}{p_i}$ for some positive integer p_i. By Proposition 8.5, $\mathcal{F}$ admits locally around each $H_{\mathcal{C}_i}$ and away from Sing $\mathcal{F}$ a (multivaluate) section of $\mathcal{I}^1$ of the form $f_i^{\frac{1}{p_i}}$ where $f_i = 0$ is a suitable reduced equation of $H_{\mathcal{C}_i}$. Set $\tilde{H}_i = H_{\mathcal{C}_i} \setminus \mathrm{Sing}\,\mathcal{F}$ and $A = \bigcup \tilde{H}_i$ (note that Sing $\mathcal{F} \cap H_{\mathcal{C}_i}$ is precisely the union of intersection loci of $H_{\mathcal{C}_i}$ with the other components of $\mathcal{C}_i$).

Using this observation, one can construct an infinite Galoisian ramified covering

$$\pi : X_0 \to (X \setminus (H \cup \mathrm{Supp}\,N)) \cup A.$$

such that $\pi^*\mathcal{I}^1$ becomes a constant sheaf and which ramified exactly over each $H_{\mathcal{C}_i}$ with order p_i. In particular the developing map ρ (defined as a section of $\pi^*\mathcal{I}^1$) is submersive along $\pi^{-1}(A)$. This allows us to prove that ρ is complete (its image is the whole Poincaré disk) using exactly the same arguments as [to], théorème 3.

Keeping the same notations as Sect. 3.2, one defines a binary relation $\mathcal{R}$ on X_0 setting $p\mathcal{R}q$ if

(1) $\mathcal{C}_p = \mathcal{C}_q$
or
(2) $\rho(p) = \rho(q)$ and there exists a connected component K of Supp N such that $(\eta_K)_\infty \cap \pi(\mathcal{C}_p) \neq \emptyset$ and $(\eta_K)_\infty \cap \pi(\mathcal{C}_q) \neq \emptyset$ near K
or
(3) There exists $i \in \{1, \ldots, r\}$ such $\rho(p) = \rho(q) \in H_{\mathcal{C}_i}$

We denote by $\overline{\mathcal{R}}$ the equivalence relation generated by $\mathcal{R}$.

Similarly to Proposition 3.2, one can prove that the map $\overline{\rho} : X/\overline{\mathcal{R}}$ onto $\mathbb{D}$ is one to one. The notion of modified leaf remains the same as Definition 4.2 except that we include now as modified leaves each non-exceptional component. Taking into account this slight modification the statement of Theorem 3 remains valid. In particular, one can conclude that $\mathcal{F}$ is a fibration provided there exists at least one non-exceptional component.

Following the same line of argumentation as in the beginning of the proof of Theorem 4, one can also conclude that $\mathrm{Kod}(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = 1$. □

9.2 The Case $Z = 0$

Here, we aim at proving the analogue of Theorem 4 item 1), namely

Theorem 9.7. *Let $\mathcal{F}$ be a KLT foliation on X Kähler. Assume moreover that $Z = 0$. Then,*

$$Kod(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = \nu(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = 0.$$

Proof. In this situation, N is an effective divisor over $\mathbb{Q}$. From the fact that the integration current $[N]$ is the only closed positive current which represents the class $\{N\}$, one easily infers that H is contained in Supp N. Let N_i, $i = 1, \ldots, l$ be the irreducible components of Supp N. One can then exhibit a KLT datum $\Lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{Q}^p$ such that

$$c_1(N^*_{\mathcal{F}}) + \{H_\Lambda\} = \{N_\Lambda\}$$

where $N_\Lambda \leq N$ is $\mathbb{Q}$ effective. This KLT datum induces a transverse euclidean metric in the sense of Theorem 8.3. Let $G \subset \Im(\mathbb{C})$ be the monodromy group attached to this transverse structure (recall that is defined as the image of the representation $r : \pi_1(X \setminus \mathrm{Supp}\, N) \to \Im(\mathbb{C})$) associated with the locally constant sheaf $\mathcal{I}^0$. One proceeds on the same way that the proof of Theorem 4 ($\varepsilon = 0$): When the generic leaf of $\mathcal{F}$ is compact, the linear part of G is finite and abundance holds for $N^*_{\mathcal{F}} \otimes \mathcal{O}(H)$. Otherwise, by performing a suitable covering ramified over Supp N, one can suppose, after desingularization, that the foliation is given by a *holomorphic* section of $\Omega^1_X \otimes L$ (thanks to the fact that $N_\lambda - H_\lambda = \sum_i \mu_i N_i$ where each μ_i is a rational > -1) where L is a flat line bundle such that Kod $(N^*_{\mathcal{F}} \otimes \mathcal{O}(H)) = 0$ whenever L is torsion. This actually holds for exactly the same reasons that *loc.cit*. □

Note also that the statement of Corollary 4.8 remains valid:

Proposition 9.8. *Let $\mathcal{F}$ be a KLT foliation on a compact Kähler manifold X. Assume that $\mathcal{F}$ is not a fibration, then the family of irreducible hypersurfaces invariant by $\mathcal{F}$ is exceptional and thus has cardinal bounded from above by the Picard number $\rho(X)$.*

10 Dealing with Strict Log-Canonical Foliations

10.1 The Case Z = 0

The next result shows that the abundance principle holds true for this class of foliations.

Theorem 7. *Let $\mathcal{F}$ be a strict log-canonical foliation on a projective manifold X. Assume moreover that $Z = 0$. Then the Kodaira dimension of $N^*_{\mathcal{F}} \otimes \mathcal{O}(H)$ is equal to zero.*

Proof. We will make use of information collected in Sect. 8.2.1. We will also closely follow the presentation made in [coupe] as well as some results of *loc.cit* (especially paragraph 3.2, section 4, and the references therein). From the existence of the euclidean transverse structure (namely, the locally constant sheaf $\mathcal{I}^0$), one inherits a representation

$$\varphi : \Gamma \to \Im(\mathbb{C})$$

setting $\Gamma = \pi_1(X \setminus (\text{Supp}\, N \cup H))$.

If γ belongs to Γ, then we can write $\varphi(\gamma)(z) = \varphi_L(\gamma) + \tau(\gamma)$ where

$$\varphi_L : \Gamma \to S^1 \subset \mathbb{C}^* = GL(1, \mathbb{C})$$

is a homomorphism and

$$\tau : \Gamma \to \mathbb{C}$$

is a 1-cocycle with values in $\mathbb{C}_{\varphi_L}$, that is $\mathbb{C}$ with its structure of Γ module induced by the linear representation φ_L.

As usual, we will call the image G of φ the monodromy group of the foliation. Clearly, abundance holds whenever the linear part $G_L = \mathrm{Im}(\varphi_L)$ of G is finite.

As previously, one can define a developing map ρ attached to $\mathcal{I}^0$, that is ρ is a section of $\pi^*\mathcal{I}^0$ where $\pi : X_0 \to X \setminus (\mathrm{Supp}\, N \cup H))$ is a suitable Galoisian covering.

One can then define the same equivalence relation $\overline{\mathcal{R}}$ as in Sect. 3.2 and conclude similarly that the induced map

$$\overline{\rho} : X_0/\overline{\mathcal{R}} \to \mathbb{C}$$

is one to one and onto. Actually the proof proceeds in the same way, using Lemma 8.7 and replacing compactness of X by the existence of a complete bundle-like metric, as stated by Lemma 8.9.

Like in Sect. 4, this enables us to claim that G faithfully encodes the dynamic of $\mathcal{F}$ and in particular that G_L is finite whenever the topological closure of leaves are hypersurfaces of X.

Henceforth, we proceed by contradiction assuming $|G_L| = +\infty$. One can notice in addition that small loops around the components of H_b give rise to infinite additive monodromy. In other words, the subgroup of translations of G is also infinite. In particular, G has no fixed points. At the cohomological level, this means that

$$H^1(\Gamma, \mathbb{C}_{\varphi_L}) \neq 0.$$

From a result directly borrowed from [coupe] (Theorem 4.1), one can deduce that there exists a morphism f of $X \setminus (\mathrm{Supp}\, N \cup H))$ onto an orbifold curve C and a representation φ_C of the orbifold fundamental group of C in $\Im(\mathbb{C})$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Im(\mathbb{C}) \\ \downarrow{\scriptstyle f_*} & \nearrow{\scriptstyle \varphi_C} & \\ \pi_1^{orb}(C) & & \end{array}$$

Up performing some suitable blowing-up at infinity, one can assume that f extends to a holomorphic fibration

$$\tilde{f} : X \to \overline{C}.$$

Looking at the diagram above, one can observe that the sheaf $\mathcal{I}^0$ extends through the neighborhood of a generic fiber $\tilde{F}$ as a constant sheaf. In particular, the foliation admits a *holomorphic* first integral on such a neighborhood. This latter is necessarily constant on $\tilde{F}$ by compactness of the fibers. Hence, one obtains that $\mathcal{F}$ is tangent to the fibration, whence the contradiction. □

10.2 The Case Z ≠ 0

This last possible situation requires some little bit more involved analysis.

Let $\mathcal{C} \subset H_b$ be a boundary connected component and $D_1, \dots, D_l$ its irreducible component admitting, respectively, the reduced equations $f_1 = 0, .., f_l = 0$.

On a neighborhood of D_k the foliation is represented by a twisted one form ω^k valued in a line bundle trivial on D_k, which locally expresses as

$$\omega^k = \omega_0 + \sum_{i=1}^{l} \lambda_i \frac{df_i}{f_i}$$

where ω_0 is holomorphic and for each i, λ_i is a non-vanishing holomorphic functions on $f_i = 0$.

For notational convenience, we will assume that $D_k \cap D_j$ is connected (maybe empty) for every k, l. Note that the quotient of residues $\lambda_{kj} = \frac{\lambda_k}{\lambda_j}$ on the non empty intersections $D_k \cap D_j$ is intrinsically defined as a non-zero complex number and also that $\lambda_{lk} = \frac{1}{\lambda_{kl}}$ .Setting $\lambda_{kl} = 0$ when $D_k \cap D_l = \emptyset$, this leads to the following intersection property:

$$\text{For every } k, \sum_{j=1}^{l} \lambda_{kj} D_j.D_k = 0.$$

Lemma 10.1. *Let $k \in \{1, \dots, l\}$. Then, there exist $p \geq 2$ integers $\{i_1, \dots, i_p\} \in \{1, \dots, l\}$ such that*

(1) $k = i_1 = i_p$
(2) $D_{i_j} \cap D_{i_{j-1}} \neq \emptyset$ *for every* $2 \leq j \leq p$.
(3) $\prod_{j=1}^{p-1} \lambda_{i_j i_{j+1}} \neq 1$.

Proof. Assume the contrary. This means that there exist l non-zero complex numbers λ_1, λ_l such that $\lambda_{ij} = \frac{\lambda_i}{\lambda_j}$ as soon as $D_i \cap D_j \neq \emptyset$. Hence, for every j, we obtain that

$$(\sum_{i=1}^{l} \lambda_i \{D_i\})\{D_j\} = 0$$

which contradicts the negativity of the intersection matrix $\{D_i\}\{D_j\}\{\theta^{n-2}\}$. □

10.2.1 Reduction to Dimension 2

We begin by the following local statement.

Proposition 10.2. *Let $\mathcal{F}$ be a foliation defined on $(\mathbb{C}^n, 0)$ by some meromorphic form $\omega = \omega_0 + \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$, with ω_0 holomorphic, $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Then there exists a germ of submersive holomorphic map $u = (u_1, u_2)$ of $(\mathbb{C}^n, 0)$ onto $(\mathbb{C}^2, 0)$ such that $\mathcal{F} = u^*\mathcal{F}'$ where $\mathcal{F}'$ is a foliation on $(\mathbb{C}^2, 0)$ defined by $\omega' = \omega_0' + \lambda_1 \frac{du_1}{u_1} + \lambda_2 \frac{du_2}{u_2}$. Moreover, there exist units v_i such that $u_i = z_i v_i$.*

Proof. Suppose that $n > 2$. Consider the holomorphic 1 form $\Omega = z_1 z_2 \omega = z_1 z_2 \omega_0 + \lambda_1 z_2 dz_1 + \lambda_2 z_1 dz_2$. Note that the vanishing locus Sing (Ω) of Ω is precisely the codimension 2 analytic subset $\{z_1 = 0\} \cap \{z_2 = 0\}$. One can write $\omega_0 = \sum_i a_i dz_i$ where the a_i's are holomorphic functions and remark that $Z = \frac{\partial}{\partial z_n} - \frac{z_1 a_n}{\lambda_1} \frac{\partial}{\partial z_1}$ is a non-vanishing vector field belonging to the kernel of Ω and tangent to Sing (Ω). Consequently, there exists a diffeomorphism $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ preserving the axis $\{z_1 = 0\}$ and $\{z_2 = 0\}$ such that $\phi_* Z = \frac{\partial}{\partial z_n}$. One can thus assume that $Z = \frac{\partial}{\partial z_n}$. Keeping in mind that Ω is Frobenius integrable ($\Omega \wedge d\Omega = 0$), this easily implies, up multiplying by a suitable unit, that Ω does not depend on the z_n variable. We then obtain the result by induction on n. □

Pick a point $x \in \mathcal{C}$ contained in the pure codimension 2 locus of Sing $\mathcal{F}$. i.e: x belongs to some intersection $D_i \cap D_j$ and $x \notin D_k$ for each $k \notin \{i, j\}$.

Following the previous proposition, in a suitable local coordinates system $z = (z_1, \ldots, z_n)$, $z(x) = 0$, $D_i = \{z_1 = 0\}$, $D_j = \{z_2 = 0\}$ the foliation $\mathcal{F}$ is defined by a holomorphic one form only depending on the (z_1, z_2) variable

$$\Omega = \lambda_1 z_2 dz_1 + \lambda_2 z_1 dz_2 + z_2 z_1 \omega_0$$

with λ_1, λ_2 two non-zero complex numbers whose quotient $\frac{\lambda_1}{\lambda_2}$ is precisely λ_{ij} and ω_0 an holomorphic one form.

Thanks to Theorem 5 and Sect. 8.2.2, $\mathcal{F}$ admits near p two invariant currents T and η (recall that this latter is a priori only defined in the complement of $D_i \cup D_j = \{z_1 z_2 = 0\}$).

It is worth keeping in mind that the local potentials of T are automatically equal to $-\infty$ on $D_i \cup D_j$ despite the fact that T has no non-vanishing Lelong numbers along $D_i \cup D_j$.

We are therefore reduced to study a germ foliation in $(\mathbb{C}^2, 0)$ defined by a logarithmic form

$$\omega = \omega_0 + \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$$

with non-vanishing residues λ_1, λ_2 such that there exists in addition a *psh* function ψ satisfying the following properties:

(1) $T = \frac{i}{\pi} \partial\overline{\partial} \psi$ is an $\mathcal{F}$ invariant current. Moreover $\psi = -\infty$ on the axis $z_1 z_2 = 0$ but T has no atomic part (i.e: Lelong numbers) along $\{z_1 z_2 = 0\}$.
(2) $\eta = \frac{i}{\pi} e^{2\psi} \omega \wedge \overline{\omega}$ is an $\mathcal{F}$ invariant current in the complement of $\{z_1 z_2 = 0\}$.

Lemma 10.3. *Let $\mathcal{F}$ be a germ a foliation in* $(\mathbb{C}^2, 0)$ *satisfying the above properties. Then $\mathcal{F}$ is linearizable; more precisely, there exists a germ a biholomorphism $\Phi \in$ Diff$(\mathbb{C}^2, 0)$ preserving each axis such that $\Phi^*\mathcal{F}$ is defined by the form $\lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$. Moreover, $\lambda = \frac{\lambda_1}{\lambda_2}$ is* ***positive real number****.*

Proof. Putting the two properties mentioned in (10.2.1) together, one can infer that T is a positive current without atomic part on the separatrices $\{z_1 z_2 = 0\}$ which is non-zero on every neighborhood of x. According to [br], this implies that $\frac{\lambda_1}{\lambda 2}$ is a real number. One has to discuss the following cases which might occur and which depend on the value of $\lambda = \frac{\lambda_1}{\lambda_2}$.

- $\lambda < 0$. One knows (see [brbir]) that $\mathcal{F}$ is linearizable; indeed, either $-\lambda \notin \mathbb{N}^* \cup \frac{1}{\mathbb{N}^*}$ and then belongs to the Poincaré domain, either the foliation singularity admits a Poincaré Dulac's normal form which is linearizable thanks to the existence of two separatrices. In particular, one can assume that $\mathcal{F}$ is defined by $\omega = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$, a closed logarithmic 1 form. By assumptions, there exists ψ *psh* such that $\eta = \frac{i}{\pi} e^{2\psi} \omega \wedge \overline{\omega}$ is closed as a current. This easily implies that ψ is actually constant on the leaves.
  Note also that $F = z_1 {z_2}^{\lambda}$ is a multivaluate first integral of $\mathcal{F}$ with connected fibers on $U \setminus \{z_2 = 0\}$ where U is a suitable neighborhood of the origin and that $F(U \setminus \{z_2 = 0\} = \mathbb{C}$ (thanks to $\lambda < 0$). Therefore, there exists a subharmonic function $\tilde{\psi}$ defined on $\mathbb{C}$ such that $\tilde{\psi} \circ F = \psi$. Keeping in mind that a bounded subharmonic function on $\mathbb{C}$ is actually constant and applying this principle to $e^{\tilde{\psi}}$, one obtains that ψ is constant, which obviously contradicts property (10.2.1).
- $\lambda > 0$. The linearizability of the foliation is equivalent to that of the germ of holonomy diffeomorphism

$$h(z) = e^{2i\pi\lambda} z + \ hot, \ \lambda = \frac{\lambda_1}{\lambda_2}$$

  evaluated on a transversal $\mathcal{T} = (\mathbb{C}, 0)$ of $\{z_1 = 0\}$ (see [MM]). This one does not hold automatically and is related to diophantine property of λ (Brujno's conditions).
  Assume that h is non-linearizable and let $\mathbb{D}_r = \{|z| < r\}$, $r > 0$ small enough. We want to show that this leads to a contradiction with the existence of the invariant current η. Following Perez-Marco [pe], there exists a totally invariant compact subset K_r with empty interior (the so-called hedgehog) which verifies the following properties:

  (1) K_r contains 0, is connected, is totally invariant by h: $h(K_r) = K_r$ and is maximal with respect to these properties.
  (2) $K_r \cap \partial\mathbb{D}_r \neq \emptyset$.

  Let $0 < r < r'$ for which the above properties hold.
  The following argument is adapted from [brsurf, preuve du lemme 11, p.590].

Let γ be a rectifiable curve contained in $\mathcal{T} \setminus \{0\}$. Its length with respect to the metric form $g = e^{\varphi}\frac{|dz|}{|z|}$ is given by

$$l_g(\gamma) = \int_\gamma \frac{e^{\varphi(z)}}{|z|} d\mathcal{H}$$

where $\mathcal{H}$ denotes the dimension 1 Hausdorff measure.
Because that η is invariant by the foliation, one gets

$$l_g(\gamma) = l_g(h \circ \gamma).$$

Using the existence and properties of K_r, $K_{r'}$ stated above, one can easily produce a sequence of rectifiable curves

$$\gamma_n : [0, 1] \to \{r \leq |z| \leq r'\}$$

with $|\gamma_n(0)| = r, |\gamma_n(1)| = r'$ and such that $l_g(\gamma_n)$ converges to 0. Up extracting a subsequence, γ_n converge, with respect to the Hausdorff distance, to a compact connected subset K not reduced to a point contained in a polar subset, which provides, like in [brsurf], *loc.cit* the sought contradiction. □

Remark 10.4. We have shown that every singularity of $\mathcal{F}$ is actually elementary.

10.2.2 Dulac's Transform and Extension of the Positive Current η_T Through the Boundary Components

Let $\mathcal{F}$ be a foliation given on a neighborhood of $0 \in \mathbb{C}^2$ by the 1 logarithmic form

$$\omega = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}, \ \lambda_1\lambda_2 > 0 \text{ (Siegel domain)}$$

Let T_1 be the transversal $\{z_2 = 1\}$ and T_2 the transversal $\{z_1 = 1\}$. The foliation induces a multivaluate holonomy map, the *Dulac's transform*, defined as

$$\begin{aligned} d_\lambda : \quad & T_1 \to T_2 \\ & z_1 \to z_2 = {z_1}^\lambda \end{aligned}$$

where $\lambda = \frac{\lambda_1}{\lambda_2}$.

One now makes use of property 10.1 stated in Lemma 10.1 above. Let D be a component of $\mathcal{C}$. Up renumerotation, one can assume that $D = D_1$ and that there exists $D_2, \ldots, D_p$. such that $D_i \cap D_{i-1} \neq \emptyset$ for every $2 \leq i \leq p$, $D_1 \cap D_p \neq \emptyset$ and $\prod_{j=1}^{p-1} \lambda_{i_j i_{j+1}} \neq 1$.

For every $i \in \{1, \ldots, p\}$, let $T_i \simeq (\mathbb{C}, 0)$ be a germ a transversal to $\mathcal{F}$ in a regular point m_i of D_i.

Equip (T_1, m_1) with a holomorphic coordinate z, $z(m_1) = 0$. Denote by S_η the invariant positive current η_T restricted to T_1. For any Borel subset B of T_1, let $\nu_B(S_\eta)$ be the mass of S_η on B, i.e: $\nu_B(S_\eta) = \int_B \frac{i}{\pi} \frac{e^{2\varphi(z)} dz \wedge d\bar{z}}{|z|^2}$. Note that this mass is finite whenever B is compact subset of $T_1 \setminus \{m_1\}$.

When composing Dulac's transforms, one gets a multivaluate holonomy transformation of T_1 which can be written as $h(z) = z^r U(z)$ where $r = \lambda_{12}\lambda_{23} \ldots .. \lambda_{p1}$ is positive real number strictly greater than 1 (up reversing the order of composition) and U is a bounded unit on each angular sector such that $\lim_{z \to 0} U(z)$ exists and equal to 1 up doing a change of variables $z \to \lambda z$. Up iterating h and thus replacing r by r^n, one can also assume that $r > 2$.

By a slight abuse of notation, we will identify h with one of its determinations on a sector of angle $< 2\pi$, that is $S_{\theta_1,\theta_2} = \{z \in T_1 | \theta_1 < \arg z < \theta_2,\ 0 < \theta_2 - \theta_1 < 2\pi\}$.

Consider $\rho > 0$ small enough and for every integer $n > 0$, let $A_n \in T_1$ be the annulus $\{\frac{\rho^{r^{n+1}}}{2} < |z| < 2\rho^{r^n}\}$.

For $\theta = \theta_2 - \theta_1 \in]0, \pi[$, consider the annular sector $A_{n,\theta_1,\theta_2} = S_{\theta_1,\theta_2} \cap A_n$.

One can easily check that $h(A_{n,\theta_1,\theta_2})$ contains an annular sector of the form $A_{n+1,\theta_1',\theta_2'}$ with $\theta_2' - \theta_1' > 2(\theta_2 - \theta_1)$.

From these observations and the fact that S_η is invariant by h, one deduces that $\nu_{A_n}(S_\eta) \leq \frac{1}{2^{n-1}} \nu_{A_1}(S_\eta)$. As $\{0 < |z| < \rho\} = \bigcup_n A_n$, S_η has finite mass on $T_1 \setminus \{0\}$.

Consequently, the closed positive current η_T well defined only a priori on $X \setminus H_b$ extends trivially on the whole X as a closed positive current. This extension will be also denoted by η_T.

10.2.3 Existence of an Invariant Transverse Hyperbolic Metric, Monodromy Behavior at Infinity and Consequences on the Dynamics

Every singularity of the foliation has elementary type and then, according to Proposition 2.7 and Remark 2.6,

$$\{\eta_T\}\{S\} = 0$$

for every $\mathcal{F}$ invariant closed positive current S.

The positive part Z of $c_1(N_{\mathcal{F}}^* \otimes \mathcal{O}(H))$ is represented by an $\mathcal{F}$ invariant positive current, one can then make use of Hodge's index theorem to conclude that

$$\{\eta_T\} = Z$$

up normalization of η_T by a multiplicative positive number.

This last property enables us to give an analogous statement to that of Proposition 8.4:

Proposition 10.5. *Let $\mathcal{F}$ be a strict Log-Canonical foliation on a Kähler manifold X. Suppose moreover that $Z \neq 0$. Let Ξ be a $(1,1)$ positive current $\mathcal{F}$-invariant.*

$$\alpha = \{N(\alpha)\} + Z(\alpha)$$

with $\alpha = \{\Xi\}$ (for instance, $\alpha = c_1(N^*_{\mathcal{F}}) + \{H\}$).

Then the following properties hold:

(1) the components D_i of the negative part $N(\alpha)$ are hypersurfaces invariant by the foliation; in particular, $Z(\alpha)$ can be represented by an $\mathcal{F}$ invariant closed positive current.
(2) $Z(\alpha)$ is a multiple of $\{\eta_T\}$.
(3) $Z(\alpha)$ is nef and $Z(\alpha)^2 = 0$.
(4) The decomposition is orthogonal: $\{N(\alpha)\}Z(\alpha) = 0$. More precisely, for every component D_i of $N(\alpha)$, one has $\{D_i\}Z(\alpha) = 0$.
(5) In $H^{1,1}(M,\mathbb{R})$, the $\mathbb{R}$ vector space spanned by the components $\{D_i\}$ of $\{N(\alpha)\}$ intersects the real line spanned by $\{\eta_T\}$ only at the origin.
(6) The decomposition is rational if X is projective and α is a rational class (e.g: $\alpha = c_1(N^*_{\mathcal{F}}) + \{H\}$).
(7) Every $(1,1)$ closed positive current which represents α is necessarily $\mathcal{F}$-invariant.
(8) Let A be a hypersurface invariant by the foliation and $A_1,\dots,A_r$ its irreducible components; the family $\{A_1,\dots,A_r\}$ is exceptional if and only if the matrix $(m_{ij}) = (\{A_i\}\{A_j\}\{\theta\}^{n-2})$ is negative, where θ is a Kähler form on X.

Doing the same than in the proof of Proposition 8.3, one can prove that there exists a closed positive current T representing $c_1(N^*_{\mathcal{F}} \otimes \mathcal{O}(H))$ such that

$$T = [N] + \eta_T \tag{14}$$

As previously, this means that N_F comes equipped with a transverse hyperbolic metric with degeneracies on $K = \operatorname{supp} N \cup H_b$. This provides a representation

$$r : \pi_1(X \setminus K) \to \operatorname{Aut}(\mathbb{D}).$$

Write N as a positive linear combination of prime divisors

$$N = \sum_{i=1}^{p} N_i$$

As previously, one denotes by $\mathcal{I}^1$ the corresponding locally constant sheaf of distinguished first integrals. Like in the previous cases, there are multivaluate sections of $\mathcal{I}^1$ around Supp N given by elementary first integrals $f_1^{\mu_1} \dots f_p^{\mu_p}$ with f_i a local section of $\mathcal{O}(-N_i)$ and μ_i non-negative real number such that $\mu_i = \lambda_i + 1$ if $N_i \leq H$ and $\mu_i = \lambda_i$ otherwise (see item (1) of Proposition 8.5). By the point (10.5) of the proposition above, the exponents μ_i's are indeed rational whenever X is projective. In particular, the monodromy around the local branches of Supp N is finite Abelian.

It remains to analyze the behavior of such first integral near the boundary component H_b. Roughly speaking, one may think of the leaves space of $\mathcal{F}$ as a (non-Hausdorff) orbifold hyperbolic Riemann surface with finitely many orbifolds points (corresponding to the components of the negative part) and cusps (corresponding to the components of H_b). This is made more precise by the following statement:

Lemma 10.6. *Let $\varphi \in PSL(2,\mathbb{C})$ be a homography such that $\varphi(\mathbb{D}) = \mathbb{H}$. Then, around each $x \in H_b$, there exist a local coordinate system $z = (z_1,\dots,z_n)$ and positive real numbers $\alpha_1,\dots,\alpha_p$ such that H_b is defined by $z_1\dots.z_p = 0$ and*

$$f(z) = -i(\sum_{j=1}^{p} \alpha_j \log z_j)$$

is a multivaluate section of $\varphi_*(\mathcal{I}^1)$. In particular, the foliation is defined on a neighborhood of x by the logarithmic form with positive residues

$$\omega = \sum_{i=1}^{p} \alpha_i \frac{dz_i}{z_i} \tag{15}$$

Proof. Assume firstly that $x \notin$ Sing $\mathcal{F}$.

The identity (14) can be locally expressed by an equation involving only one variable z (which parameterizes the local leaves space and such that $H_b = \{z = 0\}$), namely

$$\Delta\varphi(z) = \frac{e^{2\varphi}}{|z|^2}.$$

which is nothing but the equation of negative constant curvature metric near the cuspidal point $z = 0$. In particular, the foliation admits as first integrals multivaluate sections of $\phi(\mathcal{I}^1)$ of the form $f(z) = -i\alpha \log z + h(z)$ where h is holomorphic and $\alpha > 0$. Up right composition by a diffeomorphism of $(\mathbb{C}, 0)$, one can then assume that

$$f(z) = -i\alpha \log z \tag{16}$$

(the transverse invariant hyperbolic metric is then given by $\eta_T = \frac{i}{\pi}\frac{dz\wedge d\bar{z}}{|z|^2(\log|z|)^2}$). The general case easily follows from the Abelianity of the local monodromy on the complement of the normal crossing divisor $\{z_1\dots z_p = 0\}$. □

Lemma 10.7. *The representation ρ has dense image.*

Proof. Let ω be the logarithmic form given by (15). Note that ω is nothing but the meromorphic extension of a section of $-id(\varphi_*(\mathcal{I}^1))$ near H_b and is uniquely defined up to multiplication by a real positive scalar.

Consequently, the monodromy representation $\varphi_* r$ restricted to $U \setminus \mathcal{C}$, where U is a small neighborhood of a connected component $\mathcal{C}$ of H_b takes values in the affine group Aff of $PSL(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$ and the image $G_{\mathcal{C}}$ contains non-trivial translations (thanks to the existence of non-trivial residues). Moreover, $G_{\mathcal{C}}$ does not contain only translations, otherwise this would mean that ω extends as a logarithmic form on the whole U with poles on $\mathcal{C}$ and this obviously contradicts Lemma 8.10. Denote by G the image of $\varphi_* r$. The observations easily imply the following alternative:

(1) Either $G \subset \mathrm{Aff}$,
(2) either the topological closure $\overline{G}$ of G is $PSL(2,\mathbb{R})$.

By an argument already used in Sect. 4, the first case forces the positive part Z to vanish and then yields a contradiction.

The monodromy group of the foliation is thus dense in $\mathrm{Aut}(\mathbb{D})$. By using a bundle-like metric complete at infinity in the spirit of Lemma 8.9 (here, it is convenient to choose local model of the form $g_x = i\sum_{l=1}^{n_x} \frac{dz_l \wedge \{d\overline{z_l}}{|z_l|^2(\log|z_l|)^2} + h_x$ near $x \in H_b$). Following the same line of argumentation as in Sect. 10.1, one can show that the dynamical behavior of $\mathcal{F}$ is completely encoded by G and in particular that $\mathcal{F}$ is a **quasi-minimal foliation**. □

Remark 10.8. As an illustration of this monodromy behavior, it may be relevant to keep in mind the example of modular Hilbert modular foliations (see [pemen] for a thorough discussion on this subject). In this case, one may think of H_b as the exceptional divisors arising from the resolution of cusps by toroidal compactification (see [eh]). The affine monodromy near H_b is here related to the isotropy groups of the cusps (the maximal parabolic subgroup of the lattice defining the Hilbert modular surface) which decomposes as additive monodromy *around* H and multiplicative monodromy *along* H.

Remark 10.9. Like in the non-logarithmic and KLT case, the local monodromy at infinity remains quasi-unipotent. The new phenomenon, here, is that it is no more finite monodromy around the components of H_b (because of the log terms in the local expression of distinguished first integrals).

10.2.4 Proof of the Main Theorem

It is namely the Theorem 2 of the introduction. Recall that $\mathcal{F}$ is a strict Log-Canonical foliation with non-vanishing positive part Z. From now on, we will assume that the ambient manifold X is **projective**.

Thanks to the transverse hyperbolic structure together with the quasi-minimality of $\mathcal{F}$, one can conclude by the use of the Schwarzian derivative "trick" (see the proof of Theorem 4) that $\kappa := \mathrm{Kod}(N_{\mathcal{F}}^{*} \otimes \mathcal{O}(H)) = -\infty$. Indeed, if $\kappa \geq 0$, up doing a suitable ramified covering, one can assume that the foliation is defined by a logarithmic form ω with normal crossing poles, hence closed.

Thanks to the moderate growth of distinguished first integrals, $\mathcal{F}$ is a transversely projective foliations with regular singularities and the corresponding Riccati foliation does not factor through a Ricatti foliation over a curve for exactly the same dynamical reasons as those already observed in Sect. 6.1.

One can then similarly claim that there exists Ψ a morphism of analytic varieties between $X \setminus H_b$ and the quotient $\mathfrak{H} = \mathbb{D}^n / \Gamma$ of a polydisk ($n \geq 2$) by an irreducible lattice $\Gamma \subset (\mathrm{Aut}\, \mathbb{D})^n$ such that $\mathcal{F} = \Psi^{*}\mathcal{G}$ where $\mathcal{G}$ is one of the n tautological foliations on $\mathfrak{H}$.

To get the final conclusion, one extends Ψ to an algebraic morphism

$$\overline{\Psi} \to \overline{\mathfrak{H}}^{BB}$$

thanks to Borel's extension theorem [bor].

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Erratum

Foliation Theory in Algebraic Geometry

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The original version of the book contained errors which have been corrected in an updated version.
The corrections are given below:

Chapter 1
In the original version of Chapter 1, in page 4, fourth paragraph should be read as below:

Given a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on a variety X, we denote by $\mathscr{F}^*$ the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. If r is the generic rank of $\mathscr{F}$, then we denote by $\det(\mathscr{F})$ the sheaf $(\wedge^r \mathscr{F})^{**}$. For $m \in \mathbb{N}$, we denote by $\mathscr{F}^{[m]}$ the sheaf $(\mathscr{F}^{\otimes m})^{**}$. If $\mathscr{G}$ is another sheaf of $\mathscr{O}_X$-modules on X, then we denote by $\mathscr{F}[\otimes]\mathscr{G}$ the sheaf $(\mathscr{F} \otimes \mathscr{G})^{**}$.

On page 12, under section 3.3, "Hör 13" in second paragraph should be read as "Hör 14". Also, "Hör 13" in the reference list is now changed as "Hör 14".

The online version of the original chapter can be found under
http://dx.doi.org/10.1007/978-3-319-24460-0_1

Chapter 2
The second affiliation of F. Bogomolov is now updated as "National Research University Higher School of Economics, Russian Federation, AG Laboratory, HSE, 7 Vavilova str., Moscow 117312, Russia"

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On page 37, line 6, “I.3.4” should be read as “1.3.3”.

On page 46, line 5, “II.1.1” should be read as “2.1.1”.

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Chapter 5
On page 108, line 16, “$\mathcal{C}_x(X)$” has been changed as “$\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$”.

On page 123, last seven lines have been replaced as:

In the other long root cases the VMRTs of the model manifold are linearly degenerate, and we made use of the more general result that $\mathcal{W}$ must be bracket generating for a uniruled projective manifold X of Picard number 1 to reach the same conclusion in [HM02]. Here we say that a distribution $\mathcal{W}$ on X is bracket generating to mean that the tangent subsheaf generated by $\mathcal{W}$ from taking successive Lie brackets is the tangent sheaf. The same resoning was applied to the short-root cases in [HM04a] and [HM05]. In the cases of [HM05], the VMRTs of the model spaces are linearly nondegenerate.

On page 134, section 4.4.1, “i” in the second line has been italicized.

On page 135, “i” in the first line has been italicized.

On page 138, in fourth paragraph, third from the last line “$\mathcal{C}\mathcal{S}$” has been changed as “$\mathcal{C}(\mathcal{S})$”.

On page 138, Definition 4.5.4 has been updated with the below text:

Definition 4.5.4. Let $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, be a sub-VMRT structure on $S \subset X - B'$ as in Definition 4.5.1. For a point $x \in S$, and $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$, we say that $(\mathcal{C}_x(S), [\alpha])$, or equivalently $(\widetilde{\mathcal{C}_x}(S), \alpha)$, satisfies Condition (T) if and only if $T_\alpha(\widetilde{\mathcal{C}_x}(S)) = T_\alpha(\widetilde{\mathcal{C}_x}(X)) \cap T_x(S)$. We say that $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) at x if and only if $(\widetilde{\mathcal{C}_x}(S), [\alpha])$ satisfies Condition (T) for a general point $[\alpha]$ of each irreducible component of $\mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$. We say that $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) if and only if it satisfies the condition at a general point $x \in S$.

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