Chapter 7
For a Topology of Dynamical Systems

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1 Introduction

Dynamical systems are mathematical objects meant to formally capture the dynamical features of deterministic systems. They are commonly defined as ordered pairs of the form $DS = (S, (f^t)_{t \in T})$, where S is a non-empty set of *states* or *points* called the *state space*, and $(f^t)_{t \in T}$ is a family of functions on S, indexed by T, called *state transitions*. For every $t \in T$, the state transition f^t is said to have *duration* t, where the *time set* T is usually taken to be a set of numbers, such as the reals $\mathscr{R}$, the non-negative reals $\mathscr{R}^0$, the integers $\mathscr{Z}$, or the non-negative integers $\mathscr{Z}^0$. Each state transition specifies the way its argument evolves in the time given by the corresponding duration. More specifically, it is required that the state transition of duration 0 is the identity map on S, while the composition of any two state transitions is identical to the state transition whose duration is the sum of their durations [3, 4].

So defined, dynamical systems suffice to model an extensive class of deterministic systems, ranging from classical pendulums to cellular automata. Nonetheless, it is possible to further generalize their definition as follows:

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Definition 7.1. Let S be a non-empty set, let $M = (T, +)$ be a monoid with identity 0 and let $(f^t)_{t \in T}$ be a family of functions on S indexed by T. The ordered pair $DS_M = (S, (f^t)_{t \in T})$ is a *dynamical system on M* if and only if:

$$\forall x \in S, \forall t, v \in T: \quad f^0(x) = x, \tag{7.1}$$

$$f^{t+v}(x) = f^t(f^v(x)). \tag{7.2}$$

The monoid M is here called the *time model* of the dynamical system, while T is called its *time set*. Giunti and Mazzola have shown that dynamical systems on monoids adequately capture all the fundamental notions of dynamical system theory, whereas no poorer mathematical structure would equally do the job. Notably, it is not possible to further reduce the algebraic properties of time models without compromising the ability of dynamical systems to model deterministic change [2].

By contrast, it appears remarkable that no similar constraint is imposed on state spaces. Most notably, these are not required to possess any topological structure, despite the fact that the evolution of a dynamical system is often analysed by looking at the topological features of its diagrammatical representation. To wit, a system whose evolution exhibits some periodicity can be represented by means of a closed line, while the evolution of a completely aperiodic system is modelled by an open curve. Two intersecting lines represent a system in which distinct inputs deliver the same output, and so on.

Similar considerations suggest that it is possible to recover the topological properties of state spaces from the dynamical properties of the corresponding systems, and possibly that any such topological property might supervene on some dynamical feature. The objective of this article is to make a preliminary step in this field, by examining whether some workable topology can actually be defined within the formal language of dynamical systems theory.

2 Outward Topologies

The following notions will be especially useful to us.

Definition 7.2. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on $M = (T, +)$. For any $x \in S$, the *orbit* of x is the set:

$$O(x) := \left\{ y \in S : \exists t \in T \, (y = f^t(x)) \right\}. \tag{7.3}$$

Definition 7.3. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on $M = (T, +)$. For any $x \in S$ and any $t \in T$, the *ϕ-interval* of x of duration t is the set of all states in the orbit of x from which $f^t(x)$ can be reached:

$$\phi_t(x) := \left\{ y \in S : y \in O(x) \wedge f^t(x) \in O(y) \right\}, \tag{7.4}$$

Definition 7.4. Let $DS_{M_1} = (S_1, (f^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $M_1 = (T_1, +)$, let $DS_{M_2} = (S_2, (f^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $M_2 = (T_2, \oplus)$, and let $\rho : T_1 \to T_2$ be a monoid isomorphism. The function $\psi : S_1 \to S_2$ is a *ρ-emulation* of DS_{M_1} in DS_{M_2} if and only if it is injective and

$$\forall x_1 \in S_1, \forall t_1 \in T_1 \quad (f^{\rho(t_1)}(\psi(x_1)) = \psi(f^{t_1}(x_1))). \tag{7.5}$$

The orbit of a point x intuitively encompasses all the states the system will eventually evolve into if initially set in that state, while the ϕ-interval of duration t of x is the subset of points in $O(x)$ from which $f^t(x)$ can be reached. The notion of ρ-emulation, instead, generalises the more common notion of emulation [1] to the case of dynamical systems on monoids with different time sets. We say that a dynamical system DS_{M_1} *emulates* a dynamical system DS_{M_2} just in case there is a ρ-emulation of the former into the latter. If that happens, the dynamics of DS_{M_1} is perfectly reproduced by DS_{M_2}, to the effect that for any state in DS_{M_1} there is a state in DS_{M_2} whose orbit has the same dynamical properties as the orbit of the former.

The easiest way to define a topology using the minimal vocabulary just provided is to simply take a neighborhood of a point to be a superset of its orbit, accordingly identifying open sets with sets that contain the orbits of all their elements. The class of open sets so obtained, supplemented with the empty set, could then be easily proved to generate a topology on the state space of the system, which we might call its *orbit topology*. Notably, orbit topologies are preserved by ρ-emulation, in the sense that every ρ-emulation of a dynamical system into another is a continuous function from the orbit topology of the former to the orbit topology of the latter.

This result is surely of interest, since it demonstrates that it is always possible to define a topology on the state space of a dynamical system starting from its sole dynamical properties. Furthermore, such a topology supervenes on the dynamical features of the system, to the effect that two dynamical systems cannot differ as to the former without exhibiting different dynamics. Nonetheless, orbit topologies have their downsides. To wit, owing to the fact that $x \in O(y)$ just in case $O(x) \subseteq O(y)$, no orbit topology can be Hausdorff, unless all points in the dynamical system are fixed points.

Our task will be, accordingly, to find a more appropriate way of identifying neighborhoods and open sets on the state space of a dynamical system. Let us begin, then, with the notion of open sets. We may try and refine the intuition underlying the orbit topology by requiring that an open set should include some ϕ-interval for any one of its elements. The class of sets satisfying this condition demonstrably generates a topology; however, that topology is of scarce interest, because it amounts to the discrete topology on the state space. A better option is obtained by excluding the ϕ-interval of null duration:

Definition 7.5. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on a monoid $M = (T, +)$ with identity 0. For any $X \subseteq S$, X is *outward open* (or *o-open*) if and only if:

$$\forall x \in X, \exists t \neq 0 \in T (\phi_t(x) \subseteq X). \tag{7.6}$$

Correspondingly, we define an *o-neighborhood* of a point as a superset of an o-open set including that point.

How is Definition 7.5 meant to provide a dynamical counterpart of the idea of open set? Notice that, for every point x in an o-open set X, there is a ϕ-interval $\phi_t(x)$ of non-null duration that is contained in X. Hence, there is no greatest ϕ-interval of x that is included in X. In plain words this means that, moving along the orbit of x, there is no last point one encounters before leaving X, i.e. before reaching a point in the orbit of x that does not belong to X. This is in perfect agreement with the intuition that an open set is one that does not include its own boundary.

To formally confirm the adequacy of the above definition, let us define:

Definition 7.6. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on $M = (T, +)$. DS_M is *ϕ-linear* if and only if

$$\forall x \in S, \forall t, v \in T\,(\phi_t(x) \subseteq \phi_v(x) \vee \phi_v(x) \subseteq \phi_t(x)). \tag{7.7}$$

This condition amounts to the request that $\subseteq$ be a linear order on the set of the ϕ-intervals of each point x. It is satisfied by all dynamical systems whose time model is a numerical set, along with the arithmetic operation of addition. It is therefore a plausible and relatively undemanding constraint to impose on a dynamical system. We can thus prove that:

Proposition 7.1. *Let $DS_M = (S, (f^t)_{t \in T})$ be a ϕ-linear dynamical system on a monoid $M = (T, +)$ with identity* 0 *and let Σ_o be the set of the o-open subsets of S. Then (S, Σ_o) is a topological space.*

Proof. To prove that (S, Σ_o) is a topological space, we need to show: (1) that S and $\emptyset$ are o-open; (2) that the union of any given collection of o-open subsets of S is o-open; and (3) that the intersection of any two o-open subsets of S is o-open.

To prove (1), it is sufficient to notice that S satisfies (7.6) by definition of dynamical system on a monoid, while $\emptyset$ satisfies it vacuously.

To prove (2), let $\Gamma \subseteq \Sigma_o$ be any collection of o-open subsets of S, and let Θ be the union of all the elements of Γ. If $\Theta = \emptyset$, then it is o-open, as just demonstrated. Otherwise, let $x \in \Theta$. By definition of Θ, there must exist some o-open set $X \in \Gamma$ such that $x \in X$. By (7.6) there is in consequence some $t \neq 0 \in T$ such that $\phi_t(x) \subseteq X \subseteq \Theta$. Since x was chosen arbitrarily among the elements of Θ, this shows that Θ is o-open.

To prove (3), let $X_1, X_2 \in \Sigma_o$. If $X_1 \cap X_2 = \emptyset$, then it is o-open. Otherwise, let $x \in X_1 \cap X_2$. Then, by (7.6) there exist $t \neq 0, v \neq 0 \in T$ such that $\phi_t(x) \subseteq X_1$ and $\phi_v(x) \subseteq X_2$. By ϕ-linearity, $\phi_t(x) \subseteq \phi_v$ or $\phi_v(x) \subseteq \phi_t$, so either $\phi_t(x) \subseteq X_1 \cap X_2$ or $\phi_v(x) \subseteq X_1 \cap X_2$. In either case, $X_1 \cap X_2$ is o-open. □

Let us call the topology so obtained the *outward topology* or *o-topology* of DS_M. It can be demonstrated that ϕ-linearity is a sufficient but not necessary condition for Σ_o to be an o-topology. However, dynamical systems that are not ϕ-linear are likely to lack any straightforward interpretation; therefore we can hereafter restrict our attention to ϕ-linear dynamical systems without any significant loss of generality.

It is also interesting to notice that every orbit is an o-open set, and that in consequence the o-topology of a dynamical system is identical to or finer than its orbit topology. Moreover, just like the latter, the o-topology is preserved by ρ-emulation.

Proposition 7.2. *Let* $DS_{M_1} = (S_1, (f^{t_1})_{t_1 \in T_1})$ *be a* ϕ*-linear dynamical system on a monoid* $M_1 = (T_1, +)$ *with identity* 0*, let* $DS_{M_2} = (S_2, (f^{t_2})_{t_2 \in T_2})$ *be a* ϕ*-linear dynamical system on a monoid* $M_2 = (T_2, \oplus)$ *with identity* θ*, and let* $\rho : T_1 \to T_2$ *be a monoid isomorphism. Let* Σ_{o_1} *be the collection of o-open sets of* DS_{M_1} *and let* Σ_{o_2} *be the collection of o-open sets of* DS_{M_2}*. Then every* ρ*-emulation of* DS_{M_1} *in* DS_{M_2} *is a continuous function of* (S_1, Σ_{o_1}) *in* (S_2, Σ_{o_2})*.*

Proof. Let ψ be a ρ-emulation of DS_{M_1} in DS_{M_2}, and let $X_2 \subseteq S_2$ be an arbitrary o-open subset of S_2. Because ψ is injective, ψ^{-1} is well-defined. So, let $X_1 = \psi^{-1}(X_2)$. To show that ψ is a continuous function, it will be sufficient to show that for any $x_1 \in X_1$ there exists some $t_1 \neq 0 \in T_1$ such that any element of $\phi_{t_1}(x_1)$ is in X_1, which means that any such element is the counterimage of some element of X_2 with respect to ψ.

So, let $x_1 \in X_1$ be chosen arbitrarily and let $x_2 = \psi(x_1)$. Because X_2 is open, there exists $t_2 \neq \theta \in T_2$ such that $\phi_{t_2}(x_2) \subseteq X_2$. For any $x_{i_2} \in \phi_{t_2}(x_2)$, it is then clear that $x_{i_2} \in X_2$ and therefore, by hypothesis, $\psi^{-1}(x_{i_2}) \in X_1$. Notably, $\psi^{-1}(f^{t_2}(x_2)) \in X_1$. Furthermore, since ρ is a monoid isomorphism, there exists exactly one $t_1 \neq 0 \in T_1$ such that $t_2 = \rho(t_1)$. Hence:

$$\psi^{-1}(f^{t_2}(x_2)) = \psi^{-1}(f^{\rho(t_1)}(\psi(x_1))) = \psi^{-1}(\psi(f^{t_1}(x_1)) = f^{t_1}(x_1). \tag{7.8}$$

Take now any $x_{i_1} \in S_1$ such that $x_{i_1} \in \phi_{t_1}(x_1)$. Clearly, there must exist $t_{i_1}, t_{j_1} \in T_1$ such that $f^{t_{i_1}}(x_1) = (x_{i_1})$ and $f^{t_{j_1}}(x_{i_1}) = f^{t_1}(x_1)$. Therefore, by (7.5):

$$\psi(x_{i_1}) = \psi(f^{t_{i_1}}(x_1)) = \psi(f^{t_{i_1}}(\psi^{-1}(x_2))) = f^{\rho(t_{i_1})}(\psi(\psi^{-1}(x_2))) = f^{\rho(t_{i_1})}(x_2), \tag{7.9}$$

and

$$f^{t_2}(x_2) = \psi(\psi^{-1}(f^{t_2}(x_2))) = \psi(f^{t_1}(x_1)) = f^{\rho(t_{j_1})}(\psi(x_{i_1})). \tag{7.10}$$

From (7.9), it follows that $\psi(x_{i_1}) \in O(x_2)$, while (7.10) entails that $f^{t_2}(x_2) \in O(\psi(x_{i_1}))$. Therefore, $\psi(x_{i_1}) \in \phi_{t_2}(x_2) \subseteq X_2$ and thus $x_{i_1} \in X_1$. □

3 Dynamical Topologies

O-topologies share the virtues of orbit topologies without suffering from analogous shortcomings. They are, accordingly, better candidates to examine the relation between the dynamical and the topological features of a system. Nonetheless, they are not without defects. Consider, for instance, the dynamical system $DS_M = (S, (f^t)_{t \in T})$ on $M = (T, +)$, where $S = \mathscr{R}$ is the set of the real numbers, $T = \mathscr{R}^0$

is the set of the non-negative real numbers, $+$ is the standard addition operation and, for any $t \in T$ and any $x \in S$, $f^t(x) = t + x$. It is easy to see that the set $X_1 = \{x \in S : x < 0\}$ is an o-open set, and so it is $X_2 = \{x \in S : 0 \leq x\}$. But since $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = \mathscr{R}$, it follows that $\mathscr{R}$ is disconnected, which is not what we would expect from the set of real numbers. For this reason, we need to identify a proper subclass of o-open sets, such that the topologies generated by that class avoid unpalatable consequences like the one just pointed out.

As we have noticed, o-open sets formally capture the idea that open sets are essentially unbounded, in the sense that there is no last point one needs to cross in order to step out of an open set; notably, this intuition is regimented through the requirement that, for any point in an o-open set, there is no greatest ϕ-interval belonging to that set. However, open sets are also intuitively unbounded in another sense, namely that there is no *first* point one encounters in an open set while stepping *inside* it. So, it seems that o-open sets only tell half of the dynamical story about open sets. The second half can be told with the aid of the following definitions:

Definition 7.7. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on $M = (T, +)$. For any $x \in S$ and any $t \in T$, the *β-interval* of x of duration t is the set of all states whose orbits contain x, and that can be reached from some state z such that $f^t(z) = x$:

$$\beta_t(x) := \left\{ y \in S : x \in O(y) \wedge \exists z \in S(x = f^t(z) \wedge y \in O(z)) \right\}. \tag{7.11}$$

Definition 7.8. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on a monoid $M = (T, +)$ with identity 0. For any $X \subseteq S$, X is *inward open* (or *i-open*) if and only if [1]:

$$\forall x \in X(\exists t \neq 0 \in T, \exists y \in S(f^t(y) = x) \rightarrow \exists v \neq 0 \in T(\emptyset \neq \beta_v(x) \subseteq X)). \tag{7.12}$$

Definition 7.9. Let $DS_M = (S, (f^t)_{t \in T})$ be a dynamical system on $M = (T, +)$. DS_M is *β-linear* if and only if

$$\forall x \in S, \forall t, v \in T(\beta_t(x) \subseteq \beta_v(x) \vee \beta_v(x) \subseteq \beta_t(x)). \tag{7.13}$$

It is clear that the notions of β-interval, i-open set and β-linearity are the duals of the notions of ϕ-interval, o-open set, and ϕ-linearity, respectively. It is thus no surprise that the class Σ_i of i-open sets of a β-linear dynamical system generates a topology on its state space, which we may call *inward topology*, and that all dynamical systems with numeric time models are β-linear. Furthermore, every ρ-emulation of a β-linear dynamical system into another is a continuous function from the inward topology of the former to the one of the latter. Proofs are similar to those of Propositions 7.1 and 7.2, respectively.

The o-open sets we are looking for are thus precisely the ones which are also i-open. We may call them *dynamically open*, or *d-open*. Given the above results, it

[1] Notice that, while for any $x \in X$ and any $t \neq 0 \in T$, the existence of $y = f^t(x) \in S$ is guaranteed by the definition of a dynamical system on a monoid, there is no similar guarantee that some $z \in S$ exists, for which $f^t(z) = x$. This explains why condition (7.12) below is comparatively stronger than the corresponding condition (7.6).

is elementary to prove that they generate a topology on every dynamical system that is both ϕ-linear and β-linear, and that such a topology is preserved by ρ-emulation. Let us label any such topology *dynamical*, or *d-topology*. It is immediate to see that d-topologies do not suffer from the type of shortcomings we saw to affect o-topologies. Most notably, the d-topology of the dynamical system considered at the beginning of this section is homeomorphic to the ordinary topology on the real numbers. It is thus reasonable to expect that d-open sets could adequately support a general examination of the way the dynamical features of a deterministic system naturally induce a topology on its state space.

4 Conclusion

Although no topological constraint is usually imposed on the state space of a dynamical system, there is prima facie evidence that its topological properties might naturally depend on the dynamical features of the system. This article has prepared the grounds for a systematic investigation of such dependence, by identifying in d-open sets promising candidates for the notion of a topology naturally induced by the underlying dynamics.

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